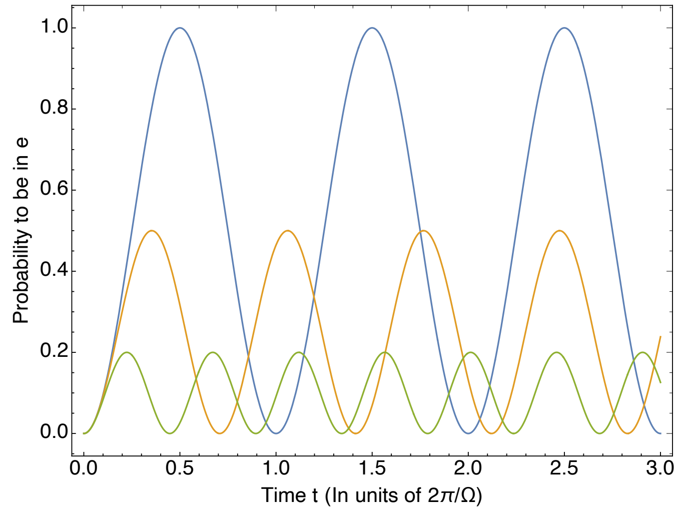
Coherence, relaxation, and Rabi flopping

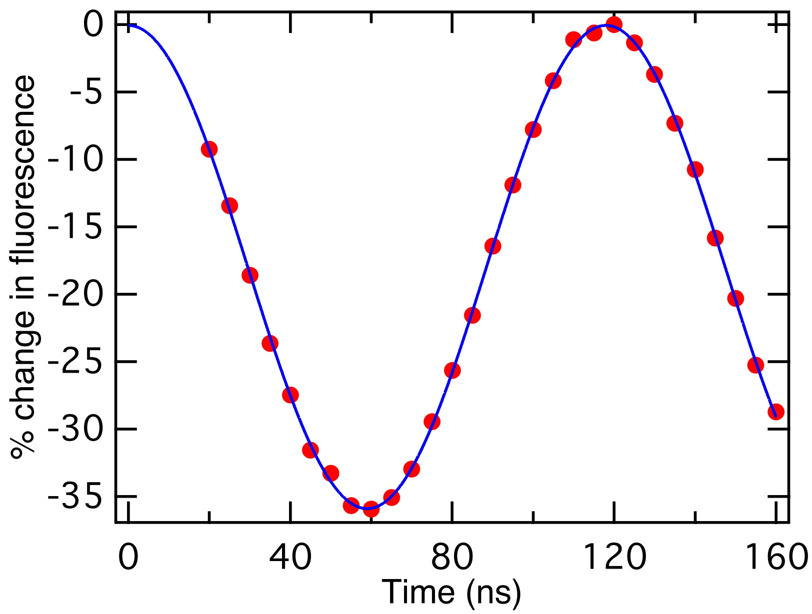
Last lecture:

- Density matrix refresher
- Time evolution of open quantum systems
- Kraus operators
- Lindblad equation

Topics for today:

- Distance measures
- Master equation for relaxation
- T1 and T2
- Bloch equations
- Rabi oscillations





Distance measures

Due to interactions with the environment, errors in applied control fields, or imperfect state preparation and readout, a quantum system will almost never end up in exactly the state desired. It is therefore important to quantify how close the state $\hat{\rho}$ is to a desired state $\hat{\rho}_o$.

There is no universal answer to this question, as it depends on what $\hat{\rho}$ will be used for. Nevertheless, it is useful to define figures of merit to quantify and compare quantum processes.

Question for the class:

What are some ways by which $\hat{\rho}$ might differ from $\hat{\rho}_o$?

Distance measures

The most common measure you are likely to encounter is the trace overlap, also called the fidelity:

$$F(\hat{\rho}, \hat{\rho}_o) = \mathrm{Tr}\left[\sqrt{\sqrt{\hat{\rho}}\hat{\rho}_o\sqrt{\hat{\rho}}}\right]$$

$$F(\hat{\rho}_o, \hat{\rho}_o) = 1,\ F(\hat{\rho}, \hat{\rho}_o) = F(\hat{\rho}_o, \hat{\rho})$$

The square of the fidelity is also used:

$$F_{sq}(\hat{\rho}, \hat{\rho}_o) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\hat{\rho}}\hat{\rho}_o\sqrt{\hat{\rho}}}\right]\right)^2$$

$F_{sq}(\hat{\rho}, \hat{\rho}_o)$ corresponds to the probability of observing $\hat{\rho}$, but $F(\hat{\rho}, \hat{\rho}_o)$ is used more commonly.

Distance measures

$$F_{sq}(\hat{\rho}, \hat{\rho}_o) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\hat{\rho}}\hat{\rho}_o\sqrt{\hat{\rho}}} \right] \right)^2$$

For a pure state $\hat{\rho}_o = |\psi_o\rangle\langle\psi_o|$ this simplifies to

$$F_{sq}(\hat{\rho}, \hat{\rho}_o) = \mathrm{Tr}\left[\sqrt{\sqrt{\hat{\rho}}\hat{\rho}_o\sqrt{\hat{\rho}}} \right]^2 = \langle\psi_o|\hat{\rho}|\psi_o\rangle$$

For 2 pure states $\hat{\rho}_o = |\psi_o\rangle\langle\psi_o|$, $\hat{\rho} = |\psi\rangle\langle\psi|$ this simplifies even further to

$$F_{sq}(\hat{\rho}, \hat{\rho}_o) = \mathrm{Tr}\left[\sqrt{\sqrt{\hat{\rho}}\hat{\rho}_o\sqrt{\hat{\rho}}} \right]^2 = |\langle\psi_o|\psi\rangle|^2$$

Distance measures

Another useful distance measure is the trace distance

$$D(\hat{\rho}, \hat{\rho}_o) = \frac{1}{2}\mathrm{Tr}\left[\sqrt{(\hat{\rho} - \hat{\rho}_o)^\dagger(\hat{\rho} - \hat{\rho}_o)}\right]$$

Note that $D(\hat{\rho}_o, \hat{\rho}_o) = 0$, and $1 - D(\hat{\rho}, \hat{\rho}_o)$ gives a measure of the similarity between the states that can be compared to the fidelity.

The trace distance is an alternative to fidelity that is sensitive to phase errors that may not be picked up by fidelity.

Qubit relaxation

We will now consider a simple example of an application of the Lindblad form of the master equation that is particularly relevant to this course, that of qubit relaxation.

A qubit prepared in $|e\rangle$ is almost always out of thermal equilibrium with its surrounding environment. It will eventually decay/thermalize.

Question for the class:

Does a superconducting qubit in $|e\rangle$ with a 1 GHz splitting between $|g\rangle$ and $|e\rangle$ have more or less energy than a qubit in equilibrium with its environment?

How about the optical transition of an atom with a 430 THz splitting? The 9192631770 Hz hyperfine transition of Cs?

Qubit relaxation

For the time being we will assume our qubit is an atom that only experiences spontaneous emission out of the excited state. For spontaneous emission the decay is exponential:

$$\rho_{ee}(t) = \rho_{ee}(0)e^{-\gamma t}$$

For short times we therefore have:

$$\rho_{ee}(\delta t) = \rho_{ee}(0) - \rho_{ee}(0)\gamma\delta t + \mathcal{O}(\delta t^2)$$

For a qubit conservation of probability requires:

$$\rho_{gg}(\delta t) = \rho_{gg}(0) + \rho_{ee}(0)\gamma\delta t + \mathcal{O}(\delta t^2)$$

We can now apply the master equation, but what is the Lindblad or "jump" operator for relaxation?

Qubit relaxation

Relaxation of the excited state to the ground state corresponds to the operator $|g\rangle\langle e|$. So $\hat{L} \propto |g\rangle\langle e|$.

To give us the correct probability of a decay for an infinitesimally short time, we require (see last lecture):

$$\gamma\delta t = \hat{A}_j\hat{\rho}\hat{A}_j^\dagger = \hat{L}_j\hat{\rho}\hat{L}_j^\dagger\delta t$$

$$\implies \hat{L} = \sqrt{\gamma}\,|g\rangle\langle e|$$

Qubit relaxation

The master equation is:

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \sum_{j \neq 0} \hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2}\hat{L}_j^\dagger \hat{L}_j \hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_j^\dagger \hat{L}_j}$$

$$\hat{L} = \sqrt{\gamma}\,|g\rangle\langle e|$$

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L}$$

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma\,|e\rangle\langle g|\hat{\rho}|g\rangle\langle e| - \frac{1}{2}\gamma\,|e\rangle\langle g||g\rangle\langle e|\hat{\rho} - \frac{1}{2}\gamma\,|\hat{\rho}|e\rangle\langle g||g\rangle\langle e|$$

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma\left(\rho_{ee}|g\rangle\langle g| - \frac{1}{2}|e\rangle\langle e|\hat{\rho} - \frac{1}{2}\hat{\rho}|e\rangle\langle e|\right)$$

Qubit relaxation

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma \left(\rho_{ee} |g\rangle\langle g| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} - \frac{1}{2} \hat{\rho} |e\rangle\langle e| \right)$$

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{ge}^* & \rho_{ee} \end{pmatrix} = \rho_{gg} |g\rangle\langle g| + \rho_{ge} |g\rangle\langle e| + \rho_{ge}^* |e\rangle\langle g| + \rho_{ee} |e\rangle\langle e|$$

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma \left(\rho_{ee} |g\rangle\langle g| - \frac{1}{2} \left[|e\rangle\langle g| \rho_{ge}^* + |e\rangle\langle e| \rho_{ee} \right] - \frac{1}{2} \left[|g\rangle\langle e| \rho_{ge} + |e\rangle\langle e| \rho_{ee} \right] \right)$$

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma \begin{pmatrix} \rho_{ee} & -\rho_{ge}/2 \\ -\rho_{ge}^*/2 & -\rho_{ee} \end{pmatrix}}$$

Qubit relaxation

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma \begin{pmatrix} \rho_{ee} & -\rho_{ge}/2 \\ -\rho_{ge}^{*}/2 & -\rho_{ee} \end{pmatrix}}$$

Unsurprisingly, if we ignore the coherent time evolution, this has given us the dynamics we started out with as an assumption:

$$\frac{d\rho_{gg}}{dt} = \gamma \rho_{ee} \qquad \rho_{gg}(t) = 1 - \rho_{ee}(0)e^{-\gamma t}$$

$$\frac{d\rho_{ee}}{dt} = -\gamma \rho_{ee} \qquad \rho_{ee}(t) = \rho_{ee}(0)e^{-\gamma t}$$

The population decays from the excited to ground state at the decay rate γ.

Qubit relaxation

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \gamma \begin{pmatrix} \rho_{ee} & -\rho_{ge}/2 \\ -\rho_{ge}^*/2 & -\rho_{ee} \end{pmatrix}}$$

$$\frac{d\rho_{gg}}{dt} = \gamma \rho_{ee} \qquad \rho_{gg}(t) = 1 - \rho_{ee}(0)e^{-\gamma t}$$

$$\frac{d\rho_{ee}}{dt} = -\gamma \rho_{ee} \qquad \rho_{ee}(t) = \rho_{ee}(0)e^{-\gamma t}$$

But we have also learned something interesting…

$$\frac{d\rho_{ge}}{dt} = -\frac{\gamma}{2}\rho_{ge}$$

The coherence decays from the excited to ground state at half the population decay rate $\gamma/2$.

Qubit relaxation

The qubit lifetime, called the T_1 time, is given by:

$$T_1 = \frac{1}{\gamma}$$

The qubit coherence time, called the T_2 time, is given by:

$$T_2 = \frac{2}{\gamma} = 2T_1$$

In a real situation there may be additional dephasing mechanisms that cause the coherence to decay faster without affecting the populations so that, more generally:

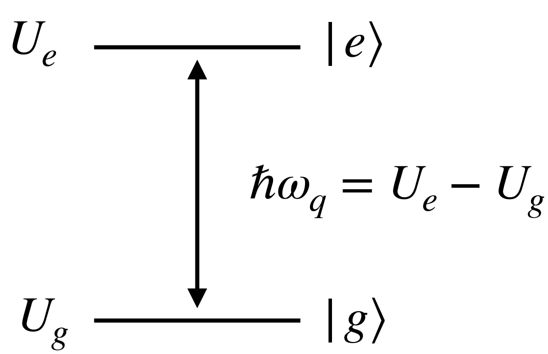
$$T_2 \leq 2T_1$$

Bloch equations

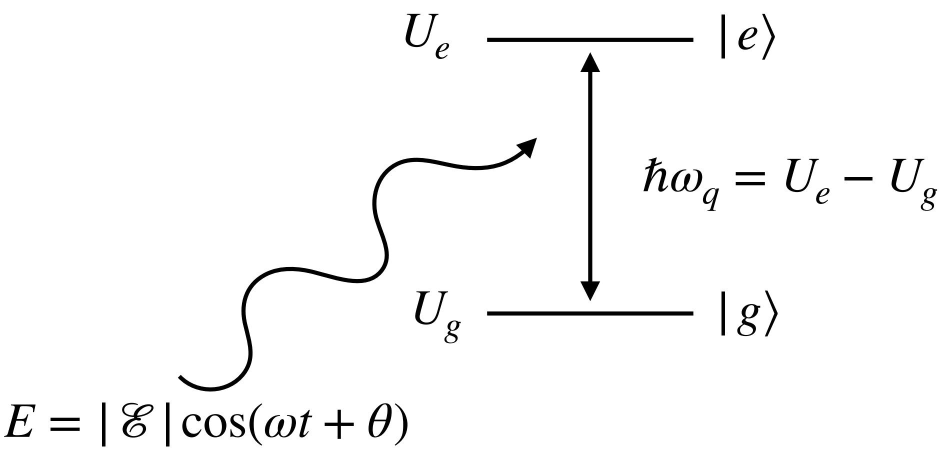
We turn now to a consideration of what happens in the combined presence of coherent dynamics and dissipation/relaxation. We consider the case where we wish to coherently control the state of a qubit using an external control field.

<u>Question for the class:</u>

If I have a qubit with an energy splitting $\hbar\omega_q$ prepared in $|g\rangle$, what should I do to transfer it into $|e\rangle$?



Bloch equations



$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = \begin{bmatrix} U_g & 0 \\ 0 & U_e \end{bmatrix}$$

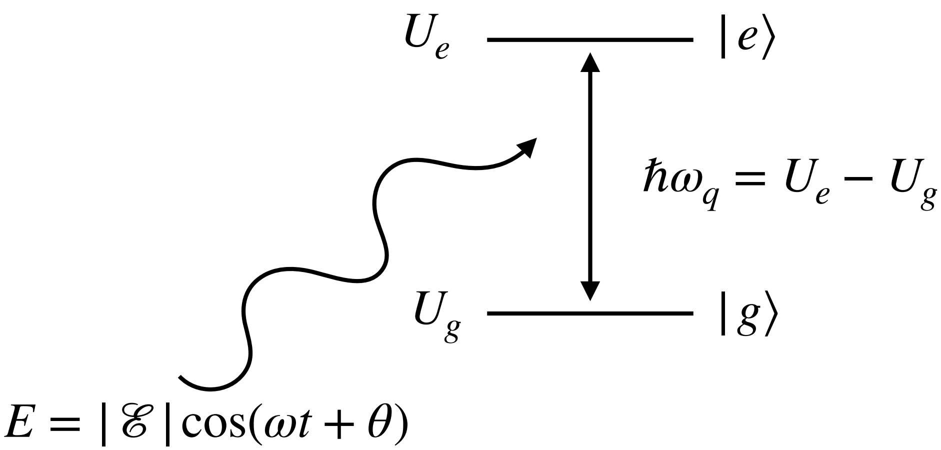
$$\hat{H}_1 = \begin{bmatrix} 0 & -\mathbf{E}\cdot\mathbf{d}_{ge} \\ -\mathbf{E}\cdot\mathbf{d}_{ge} & 0 \end{bmatrix}$$

$$\mathbf{d}_{ge} = \langle g|\hat{\mathbf{d}}|e\rangle = \mathbf{d}^*_{eg}$$

Ignore vector aspects:

$$\hat{H}_1 = -E\begin{bmatrix} 0 & d^* \\ d & 0 \end{bmatrix} \qquad \hat{H} = \begin{bmatrix} U_g & -Ed^* \\ -Ed & U_e \end{bmatrix}$$

Bloch equations



$$\hat{H} = \begin{bmatrix} U_g & -Ed^* \\ -Ed & U_e \end{bmatrix}$$

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}]$$

$$\frac{d\rho_{gg}}{dt} = -\frac{i}{\hbar}E(d\rho_{ge} - d^*\rho_{eg})$$

$$\frac{d\rho_{ee}}{dt} = -\frac{i}{\hbar}E(d^*\rho_{eg} - d\rho_{ge})$$

$$\frac{d\rho_{eg}}{dt} = -i\omega_q\rho_{eg} - \frac{i}{\hbar}Ed^*(\rho_{ee} - \rho_{gg})$$

Note that $\rho_{eg} = \rho_{ge}^*$, so we only need 3 equations not 4.

Bloch equations

We will now add decay processes in. We do so using the master equation formalism, but will skip over the somewhat tedious process of finding and plugging in the appropriate Lindblad operators, and instead will add them in by hand.

$$\frac{d\rho_{gg}}{dt} = \frac{\rho_{ee}}{T_1} - \frac{i}{\hbar}E(d\rho_{eg}^* - d^*\rho_{eg})$$

$$\frac{d\rho_{ee}}{dt} = -\frac{\rho_{ee}}{T_1} - \frac{i}{\hbar}E(d^*\rho_{eg} - d\rho_{eg}^*)$$

$$\frac{d\rho_{eg}}{dt} = -\left(\frac{1}{T_2} + i\omega_q\right)\rho_{eg} - \frac{i}{\hbar}Ed^*(\rho_{ee} - \rho_{gg})$$

Bloch equations

We will now add in our oscillating field:

$$E = |\mathscr{E}|\cos(\omega t + \theta) = \frac{\mathscr{E}}{2}e^{-i\omega t} + c.c.$$

But before we do so, we make a clever substitution of variables that is called "entering the rotating frame":

$$\rho_{eg} = \tilde{\rho}_{eg}e^{-i\omega t}$$

Plugging in gives:

$$\frac{d\rho_{gg}}{dt} = \frac{\rho_{ee}}{T_1} - \frac{i}{2\hbar}(\mathscr{E}d\tilde{\rho}_{eg}^* - \mathscr{E}^*d^*\tilde{\rho}_{eg} + \mathscr{E}^*d\tilde{\rho}_{eg}^*e^{i2\omega t} - \mathscr{E}d^*\tilde{\rho}_{eg}e^{-i2\omega t})$$

$$\frac{d\rho_{ee}}{dt} = -\frac{\rho_{ee}}{T_1} + \frac{i}{2\hbar}(\mathscr{E}d\tilde{\rho}_{eg}^* - \mathscr{E}^*d^*\tilde{\rho}_{eg} + \mathscr{E}^*d\tilde{\rho}_{eg}^*e^{i2\omega t} - \mathscr{E}d^*\tilde{\rho}_{eg}e^{-i2\omega t})$$

$$\frac{d\tilde{\rho}_{eg}}{dt} = -\left(i\Delta - \frac{1}{T_2}\right)\tilde{\rho}_{eg} - i\frac{d}{2\hbar}\left(\mathscr{E} + \mathscr{E}^*e^{2i\omega t}\right)(\rho_{ee} - \rho_{gg}),\ \Delta = \omega - \omega_q$$

Bloch equations

These equations do not have exact analytical solutions. However, in the often physically relevant limit $|\Delta|, \Omega \ll \omega, \omega_q$, where $\Omega = \dfrac{d\mathcal{E}}{\hbar}$, they simplify.

We can neglect all the fast oscillating terms at frequency 2ω, known as the Rotating Wave Approximation (RWA), to arrive at the Bloch equations:

$$\boxed{\begin{aligned}
\frac{d\rho_{gg}}{dt} &= \frac{\rho_{ee}}{T_1} - \frac{i}{2}(\Omega\tilde{\rho}_{eg}^* - \Omega^*\tilde{\rho}_{eg}) \\
\frac{d\rho_{ee}}{dt} &= -\frac{\rho_{ee}}{T_1} + \frac{i}{2}(\Omega\tilde{\rho}_{eg}^* - \Omega^*\tilde{\rho}_{eg}) \\
\frac{d\tilde{\rho}_{eg}}{dt} &= -\left(i\Delta - \frac{1}{T_2}\right)\tilde{\rho}_{eg} - i\frac{\Omega}{\hbar}(\rho_{ee} - \rho_{gg})
\end{aligned}}$$

Bloch equations

If we introduce the "population inversion" $w = \rho_{ee} - \rho_{gg}$, we can reduce the Bloch equations down to two:

$$\frac{dw}{dt} = -\frac{(1+w)}{T_1} + i(\Omega\tilde{\rho}^*_{eg} - \Omega^*\tilde{\rho}_{eg})$$

$$\frac{d\tilde{\rho}_{eg}}{dt} = -\left(i\Delta - \frac{1}{T_2}\right)\tilde{\rho}_{eg} - i\frac{\Omega}{\hbar}w$$

It is also often relevant to add in some population inversion w_o at no drive $(\Omega = 0)$:

$$\frac{dw}{dt} = -\frac{(w-w_o)}{T_1} + i(\Omega\tilde{\rho}^*_{eg} - \Omega^*\tilde{\rho}_{eg})$$

$$\frac{d\tilde{\rho}_{eg}}{dt} = -\left(i\Delta - \frac{1}{T_2}\right)\tilde{\rho}_{eg} - i\frac{\Omega}{\hbar}w$$

Bloch equations

As we will see, the coherent dynamics evolve on a frequency scale set by $\Omega = \dfrac{d\mathscr{E}}{\hbar}$.

However, eventually T_1 and T_2 will lead to loss of coherence, and the system will settle into a steady state that is a balance of the drive and the dissipation. Setting the time derivatives equal to zero and solving we find the stationary solutions:

$$w = w_o \frac{1 + \Delta^2 T_2^2}{1 + \Delta^2 T_2^2 + T_1 T_2 |\Omega|^2}$$

$$\tilde{\rho}_{eg} = -iw \frac{\Omega T_2}{2} \frac{1 - i\Delta T_2}{1 + \Delta^2 T_2^2}$$

Bloch equations, relaxation, and Rabi oscillations

The Bloch equations describe the dynamics of a qubit in the presence of a near-resonance driving field with decoherence processes. In the absence of the drive, they reduce to the qubit relaxation we already considered (with a different and potentially shorter coherence time T_2.)

In the absence of relaxation and dephasing, the Bloch equations reduce to the solution to the Schrodinger equation for a driven qubit. In this case we arrive at the well known Rabi oscillations or Rabi flopping.

Rabi oscillations

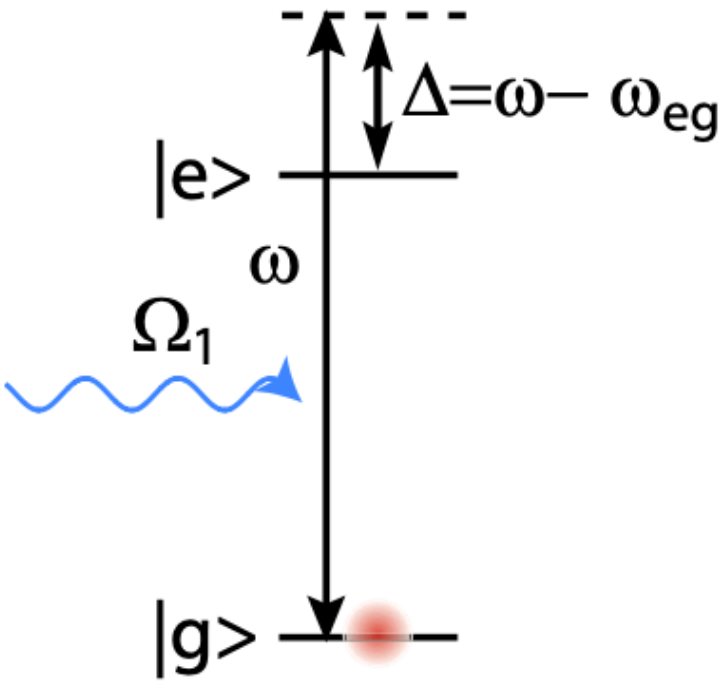
If T_1 and T_2 go to infinity, and if we make the simplifying assumption that the population starts in the ground state at *t*=0, we find:

$$\rho_{gg} = \cos^2\left(\frac{\Omega' t}{2}\right) + \frac{\Delta^2}{|\Omega|^2 + \Delta^2}\sin^2\left(\frac{\Omega' t}{2}\right)$$

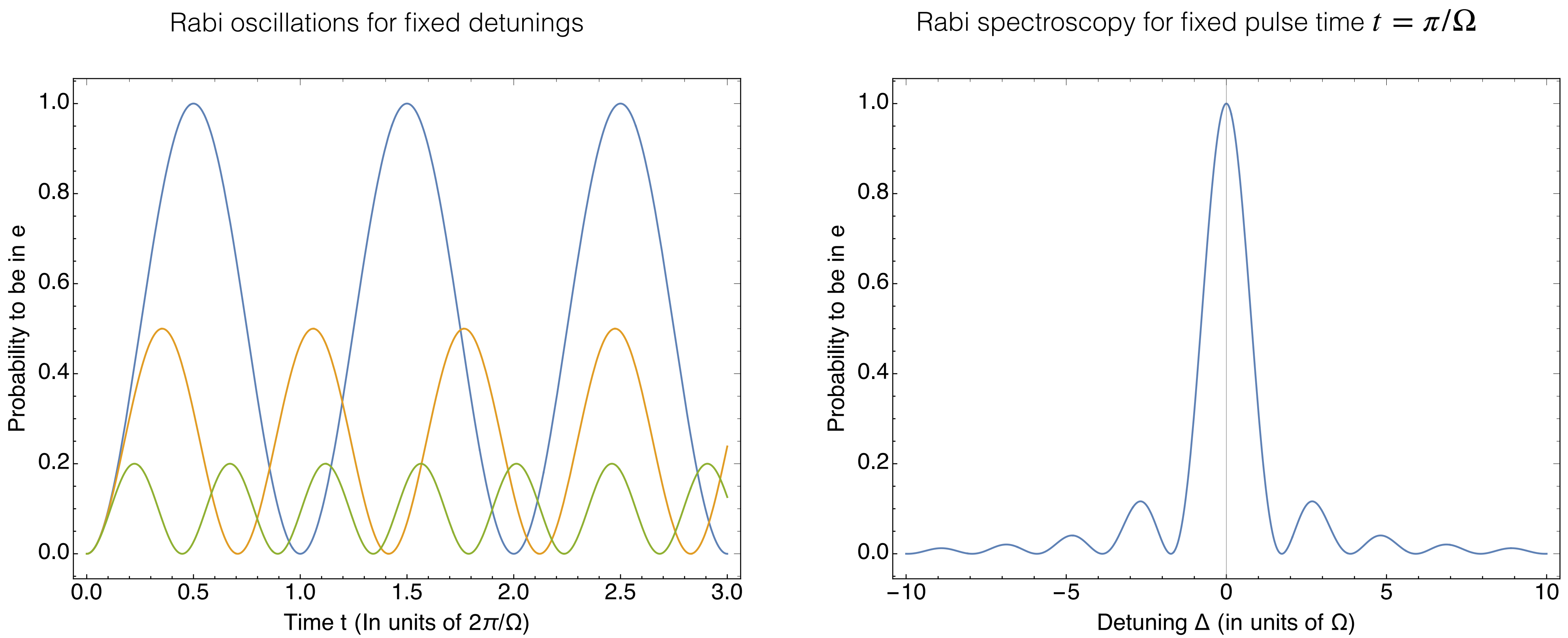
$$\rho_{ee} = \frac{|\Omega|^2}{|\Omega|^2 + \Delta^2}\sin^2\left(\frac{\Omega' t}{2}\right)$$

$$\Omega' = \sqrt{|\Omega|^2 + \Delta^2}$$

(effective Rabi frequency)



Rabi oscillations and Rabi spectroscopy



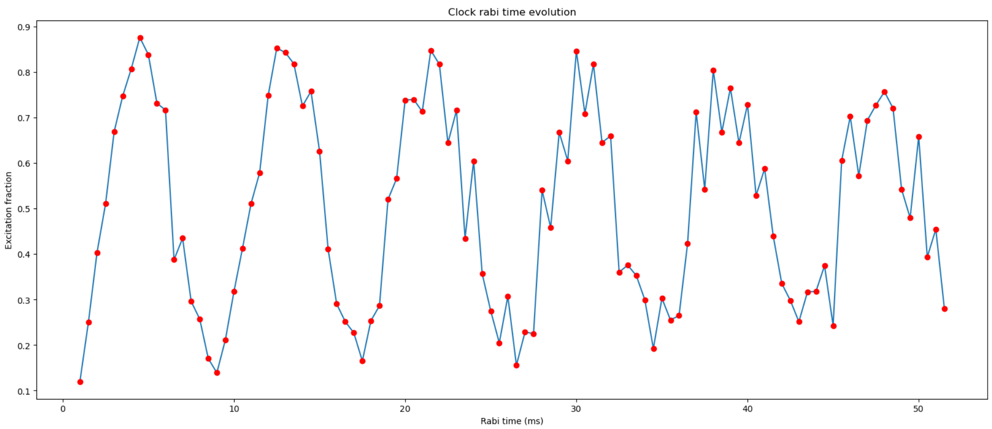
$\Delta = 0$

$\Delta = \Omega$

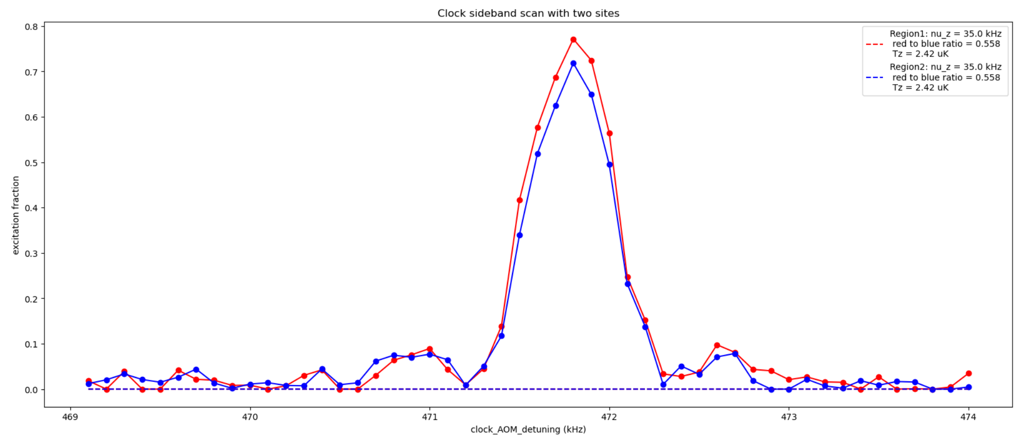
$\Delta = 2\Omega$

Rabi oscillations and Rabi spectroscopy

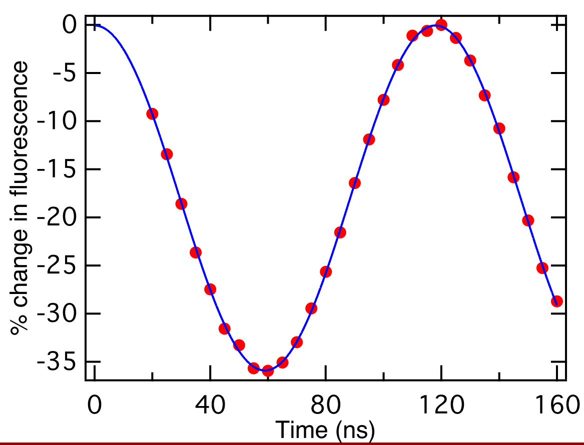
Rabi oscillations for fixed detunings with strontium atoms on optical clock transition:



Rabi spectroscopy for fixed pulse time $t = \pi/\Omega$ with strontium atoms on optical clock transition:



Rabi oscillations for fixed detunings with single NV center on electronic spin transition:



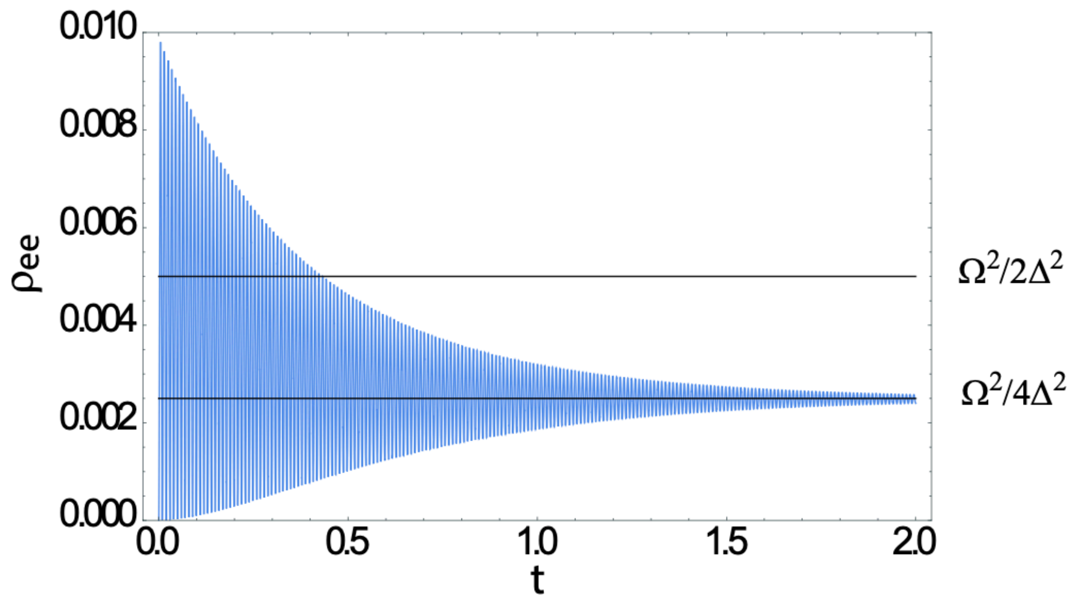
Comparison between Bloch and Rabi equations

When T_1 and T_2 go to infinity, the master equation approach and the Schrodinger equation are identical, and the Bloch and Rabi equations give the same result. However, when that is not the case, there can be significantly different results.

For large detuning at long times:

$$\langle P_e \rangle = \frac{1}{2} \frac{|\Omega|^2}{|\Omega|^2 + \Delta^2} \approx \frac{|\Omega|^2}{2\Delta^2}$$

$$\rho_{ee} = \frac{|\Omega|^2}{4\Delta^2}$$



$\Omega = 2\pi, \Delta = 2\pi \times 10, T_1 = 2.5, T_2 = 5$