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Ph779 Advanced quantum computing

Homework Set 1
Due Thursday, February 4th

This assignment is 18 points total.

Note: Some of the problems on this assignment are based off of problems from previous semesters of Physics 779 and/or Physics 545, and were originally written by M. Saffman and/or S. Kolkowitz. Please do not search for solutions online, and do not post this assignment or the solutions online. Those of you who have previously taken Physics 545 will occasionally have a chance to revisit some of your favorite homework problems from days gone by...

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1) (4 points) a) Given a qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ write down the density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ as a 2×2 matrix.

b) Show that the density operator can be written in the form

$$\hat{\rho} = \frac{1}{2}(\hat{I} + \mathbf{m} \cdot \hat{\boldsymbol{\sigma}}).$$

Find $\mathbf{m}$ in terms of a and b. Here $\mathbf{m} = m_x\hat{x} + m_y\hat{y} + m_z\hat{z}$ is a vector and $\boldsymbol{\sigma} = \hat{\sigma}_x\hat{x} + \hat{\sigma}_y\hat{y} + \hat{\sigma}_z\hat{z}$.

c) The spin flip transformation $\hat{\rho} \to \hat{\tilde{\rho}} = \hat{\sigma}_y\hat{\rho}^*\hat{\sigma}_y$ gives a density matrix $\hat{\tilde{\rho}} = \frac{1}{2}(\hat{I} + \tilde{\mathbf{m}} \cdot \hat{\boldsymbol{\sigma}})$. Find the (simple) relation between $\tilde{\mathbf{m}}$ and $\mathbf{m}$.

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2) (4 points) The square of the fidelity between two states is

$$F_{\rm sq}(\rho, \rho_0) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho_0\sqrt{\rho}}\right]\right)^2.$$

Show that when $\rho = |\psi\rangle\langle\psi|$ and $\rho_0 = |\psi_0\rangle\langle\psi_0|$ are pure states and commute then

$$F_{\rm sq}(\rho, \rho_0) = |\langle\psi|\psi_0\rangle|^2.$$

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3) (4 points) Qubit decoherence may be modeled as an exponential decay with time constants T_1 for the population and T_2 for the coherence. It is often the case that

$$\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_\phi}$$

where $1/T_\phi$ is a dephasing rate that leads to loss of coherence without changing the population. The evolution of the density matrix can be written as

$$\rho = \begin{pmatrix} 1 - \rho_{11} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{pmatrix} \to \rho' = \begin{pmatrix} 1 - \rho_{11}e^{-t/T_1} & \rho_{01}e^{-t\left(\frac{1}{2T_1}+\frac{1}{T_\phi}\right)} \\ \rho_{01}^*e^{-t\left(\frac{1}{2T_1}+\frac{1}{T_\phi}\right)} & \rho_{11}e^{-t/T_1} \end{pmatrix}. \tag{1}$$

Find the Krauss operators corresponding to this decoherence channel. Write your answers in terms of I, X, Y, Z and T_1, T_ϕ.

Hint: We saw in class that longitudinal decay alone, i.e. finite T_1, implies a finite $T_2 = 2T_1$. This problem is more complicated since we have $T_2 < 2T_1$ due to the presence of T_ϕ. One way to proceed is as follows. We know that T_1 alone changes the diagonal elements of the density operator. You can verify that the Kraus operators that change the upper and lower diagonal elements of the density operator are

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\begin{aligned}
A_1 &= \begin{pmatrix} 0 & \sqrt{a} \\ 0 & 0 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{b} \end{pmatrix}
\end{aligned}
$$

for some constants a, b. There must be a third Kraus operator A_3 in order to satisfy the normalization condition $\sum_i A_i^\dagger A_i = I$. Using the normalization condition find A_3 in terms of a, b. Then to answer the question express A_1, A_2, A_3 in terms of Pauli operators and fix the constants a, b to match Eq. (1).

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4) (6 points) Consider a qubit with states $|g\rangle, |e\rangle$ having energies U_g, U_e and energy separation $U_e - U_g = \hbar\omega_q$. Assume $U_e > U_g$. The qubit is illuminated with radiation at frequency ω where $\omega - \omega_q = \Delta$, and assume $\Delta > 0$. The radiation couples the two levels through the electric dipole Hamiltonian $H_{E1} = -\hat{d}E = -\hat{d}\left(\frac{\mathcal{E}^*}{2}e^{-\imath\omega t} + \frac{\mathcal{E}}{2}e^{\imath\omega t}\right)$. For this problem we will neglect the polarization of the field, assume $\mathcal{E} = \mathcal{E}^*$, and use the Rabi frequency $\Omega = \mathcal{E}d_{eg}/\hbar$ with $d_{eg} = \langle e|\hat{d}|g\rangle$ to characterize the strength of the qubit-radiation coupling.

a) Assume at $t = 0$ the qubit state is $|\psi\rangle = |g\rangle$. Give a formula for the probability to be in state $|e\rangle$ as a function of time t using the Schrödinger equation solution (Rabi oscillations).

b) Assume at $t = 0$ the qubit state is $|\psi\rangle = |g\rangle$. Give a formula for the probability to be in state $|e\rangle$ as a function of time t using first order time dependent perturbation theory and the rotating wave approximation.

c) Same as b) but do not make the rotating wave approximation.

d) Compare the results found in a), b), and c) for short times such that $\Omega t \ll 1, \Delta t \ll 1$, and $\omega_q t \ll 1$. If the results are not all the same explain why.