Mathematical formulae

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Contents

Chapter 1

Some basics

1.1 Kronecker and Levi-Civita symbols

$$\delta_{ij} = 1 \ \text{ if } \ i = j, \ 0 \ \text{ otherwise} \tag{1.1}$$

$$\begin{aligned} \epsilon_{ijk} &= 1 \ \text{ if } \ ijk \ \text{ are an even permutation of } \ 123 \\ &= -1 \ \text{ if } \ ijk \ \text{ are an odd permutation of } \ 123 \\ &= 0 \ \text{ if } \ i = j \ \text{ or } \ i = k \ \text{ or } \ j = k \end{aligned}$$

$$\sum_{i,j} \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{1.2}$$

$$\sum_i \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.3}$$

1.2 Taylor series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n!}{(n-r)!r!}x^r + ... \tag{1.4}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{1.5}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \tag{1.6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{1.7}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ... \tag{1.8}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \tag{1.9}$$

1.3 Trigonometric relations

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B && (1.10a)\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B && (1.10b)
\end{aligned}
$$

$$
\begin{aligned}
2\sin A \sin B &= \cos(A-B) - \cos(A+B) && (1.11a)\\
2\cos A \cos B &= \cos(A+B) + \cos(A-B) && (1.11b)\\
2\sin A \cos B &= \sin(A+B) + \sin(A-B) && (1.11c)
\end{aligned}
$$

$$
\begin{aligned}
\sin 2A &= 2\sin A \cos A && (1.12a)\\
\cos 2A &= \cos^2 A - \sin^2 A && (1.12b)\\
\sin \frac{1}{2}A &= \sqrt{\frac{1}{2}(1-\cos A)} && (1.12c)\\
\cos \frac{1}{2}A &= \sqrt{\frac{1}{2}(1+\cos A)} && (1.12d)\\
\sin^2 A &= \frac{1}{2}(1-\cos 2A) && (1.12e)\\
\cos^2 A &= \frac{1}{2}(1+\cos 2A) && (1.12f)
\end{aligned}
$$

1.4 Hyperbolic trigonometric relations

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 && (1.13a)\\
\tanh^2 x + \mathrm{sech}^2 x &= 1 && (1.13b)\\
\mathrm{ctnh}^2 x - \mathrm{csch}^2 x &= 1 && (1.13c)\\
& && (1.13d)
\end{aligned}
$$

$$
\begin{aligned}
\sinh(ix) &= i\sin x && (1.14a)\\
\cosh(ix) &= \cos x && (1.14b)\\
& && (1.14c)
\end{aligned}
$$

1.5 Logarithms

The logarithm to base a is defined by

$$x = a^{\log_a x}$$

for any nonnegative x.

$$\begin{aligned}
\log_a(1) &= 0 \\
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a(x/y) &= \log_a(x) - \log_a(y) \\
\log_a(x^p) &= p\log_a(x)
\end{aligned}$$

The above relations hold for any base.

Taking $a = e$ we see that

$$x = e^{\log_e x} = e^{\ln x}$$

where $\ln = \log_e$ is the natural logarithm, i.e. logarithm to base e.

So

$$e^{\ln x} = a^{\log_a x} = e^{\ln a \log_a x}$$

and

$$\log_a x = \frac{\ln x}{\ln a}.$$

More generally

$$\log_b x = \frac{\log_a x}{\log_a b},$$

where a is an arbitrary base.

1.6 Solution of cubic equations

Write the cubic equation as

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0. \tag{1.15}$$

Define

$$\begin{aligned}
q &= \frac{a_1}{3} - \frac{a_2^2}{9} && (1.16)\\
r &= \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27} && (1.17)\\
h &= q^3 + r^2 && (1.18)
\end{aligned}$$

For $h > 0$ there are one real root and a pair of complex conjugate roots, for $h = 0$ all roots are real, and at least two are equal, and for $h < 0$ all roots are real.

The three roots are

$$\begin{aligned}
z_1 &= s_1 + s_2 - \frac{a_2}{3}, && (1.19)\\
z_2 &= -\frac{s_1 + s_2}{2} - \frac{a_2}{3} + i\frac{\sqrt{3}}{2}(s_1 - s_2), && (1.20)\\
z_3 &= -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - i\frac{\sqrt{3}}{2}(s_1 - s_2), && (1.21)
\end{aligned}$$

where

$$s_1 = (r + \sqrt{h})^{1/3}, \quad s_2 = (r - \sqrt{h})^{1/3}. \tag{1.22}$$

Note

$$z_1 + z_2 + z_3 = -a_2, \tag{1.23}$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = a_1, \tag{1.24}$$

$$z_1 z_2 z_3 = -a_0. \tag{1.25}$$

1.7 Combinatorial factors

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n, k \text{ integers} \tag{1.26}$$

Binomial series

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k = 1 + ax + \frac{a(a-1)}{2!} x^2 + \ldots \tag{1.27}$$

Here a is an arbitrary complex number and $\binom{a}{k} \equiv \frac{a(a-1)(a-2)..(a-k+1)}{k!}$.

The binomial distribution $B_p(n, N)$ gives the probability of obtaining n successes in N trials when a single trial yields success with probability p.

$$B_p(n, N) = \binom{N}{n} p^n (1-p)^{N-n}. \tag{1.28}$$

1.8 Matrices

The Kronecker product of a matrix $\mathbf{A}$ with dimensions $m \times n$ and a matrix $\mathbf{B}$ with dimensions $p \times q$ is a matrix $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ with dimensions $mp \times nq$. The elements of the matrices are related by

$$\mathbf{C}_{\alpha\beta} = \mathbf{A}_{ij} \mathbf{B}_{kl}$$

where

$$\begin{aligned} \alpha &= p(i-1) + k, \\ \beta &= q(j-1) + l. \end{aligned}$$

Given α, β we have the inverse relations

$$\begin{aligned} i &= \mathrm{int}(\alpha/p) + 1, \\ j &= \mathrm{int}(\beta/q) + 1, \\ k &= \alpha - p(i-1), \\ l &= \beta - q(j-1). \end{aligned}$$

Chapter 2

Poisson distribution

The Poisson distribution is
$$P(n,\bar{n}) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}.$$
This gives the probability of observing n events when the mean number observed is $\bar{n}$. For large $\bar{n}$ the Poisson distribution becomes Gaussian. To see this assume $x = n$ is a real variable, consider $\ln P$ and use the Stirling formula $n! \simeq \sqrt{2\pi n} n^n e^{-n}$ to get
$$\ln P = x\ln(\bar{n}/x) + (x - \bar{n}) - \ln\sqrt{2\pi x}.$$
Then put $y = x - \bar{n}$, assume $y \ll \bar{n}$, and use $\ln\sqrt{2\pi x} \simeq \ln\sqrt{2\pi\bar{n}}$ to get
$$\ln P = -\frac{y^2}{2\bar{n}} - \ln\sqrt{2\pi\bar{n}}.$$
Therefore
$$P = \frac{1}{\sqrt{2\pi\bar{n}}} e^{-y^2/2\bar{n}} = \frac{1}{\sqrt{2\pi\bar{n}}} e^{-(x-\bar{n})^2/2\bar{n}}$$
which is a Gaussian with mean $\bar{n}$ and standard deviation $\sigma_n = \sqrt{\bar{n}}$.

The $FWHM$ of the distribution is found from
$$e^{-(\bar{n}+FWHM/2-\bar{n})^2/2\bar{n}} = \frac{1}{2}$$
so
$$FWHM = (8\ln(2)\,\bar{n})^{1/2} \simeq 2.355\sqrt{\bar{n}}.$$

Chapter 3

Curvilinear coordinates

3.1 3D Cartesian (x, y, z)

$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$

line element $d\mathbf{s} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

volume element $d\mathbf{r} = d^3r = dxdydz$

$$\nabla\psi = \hat{x}\frac{\partial\psi}{\partial x} + \hat{y}\frac{\partial\psi}{\partial y} + \hat{z}\frac{\partial\psi}{\partial z} \tag{3.1a}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{3.1b}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{3.1c}$$

$$\nabla\times\mathbf{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{3.1d}$$

3.2 3D cylindrical (ρ, ϕ, z)

line element $d\mathbf{s} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$

volume element $d\mathbf{r} = d^3r = \rho\, d\rho\, d\phi\, dz$

$$\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{3.2a}$$

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z \tag{3.2b}$$

$$\begin{aligned}
\nabla\psi &= \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z} && \text{(3.3a)}\\
\nabla^2\psi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} && \text{(3.3b)}\\
\nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} && \text{(3.3c)}\\
\nabla\times\mathbf{A} &= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right) && \text{(3.3d)}
\end{aligned}$$

3.3 3D spherical (r, θ, ϕ)

line element $d\mathbf{s} = dr\hat{r} + rd\phi\hat{\phi} + r\sin\phi d\theta\hat{\theta}$

volume element $d\mathbf{r} = d^3r = r^2\sin\theta dr d\theta d\phi$

$$\begin{aligned}
r &= \sqrt{x^2+y^2+z^2}, \quad \theta = \cos^{-1}\left(\frac{z}{r}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) && \text{(3.4a)}\\
x &= r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta && \text{(3.4b)}
\end{aligned}$$

$$\begin{aligned}
\nabla\psi &= \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} && \text{(3.5a)}\\
\nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} && \text{(3.5b)}\\
\nabla\cdot\mathbf{A} &= \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)A_r + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} + \left(\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cot\theta}{r}\right)A_\theta && \text{(3.5c)}\\
\nabla\times\mathbf{A} &= \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\mathbf{A}_\phi) - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right) && \text{(3.5d)}
\end{aligned}$$

3.4 Vector identities

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) && \text{(3.6a)}\\
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} && \text{(3.6b)}\\
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) && \text{(3.6c)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})_i &= \sum_{j,k}\epsilon_{ijk}A_jB_k && \text{(3.7)}\\
\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) &= \sum_{i,j,k}\epsilon_{ijk}C_iA_jB_k && \text{(3.8)}
\end{aligned}$$

Derivatives of products

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \tag{3.9a}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B} \tag{3.9b}$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{3.9c}$$

$$\nabla\cdot(\psi\mathbf{A}) = \mathbf{A}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{A} \tag{3.10a}$$
$$\nabla\times(\psi\mathbf{A}) = \nabla\psi\times\mathbf{A} + \psi\nabla\times\mathbf{A} \tag{3.10b}$$

Second derivatives

$$\nabla\times\nabla\psi = 0 \tag{3.11a}$$
$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{3.11b}$$
$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{3.11c}$$

$$\nabla\cdot\mathbf{r} = 3, \quad \nabla\times\mathbf{r} = 0 \tag{3.12a}$$
$$\nabla\cdot\hat{\mathbf{r}} = \frac{2}{r}, \quad \nabla\times\hat{\mathbf{r}} = 0 \tag{3.12b}$$

For any constant vector $\mathbf{A}$,

$$\mathbf{A} = \nabla\times\left(\frac{\mathbf{A}\times\mathbf{r}}{2}\right)$$

Chapter 4

Complex numbers

Define $z = x + iy$. Then

$$\begin{aligned} z^* &= x - iy && (4.1a)\\ |z|^2 &= zz^* = x^2 + y^2 && (4.1b)\\ |z| &= \sqrt{x^2+y^2} && (4.1c)\\ \arg(z) &= \tan^{-1}(y/x) && (4.1d) \end{aligned}$$

It follows that

$$\begin{aligned} (z+w)^* &= z^* + w^* && (4.2a)\\ (zw)^* &= z^* w^* && (4.2b)\\ (z/w)^* &= z^*/w^* && (4.2c)\\ (z^*)^* &= z && (4.2d)\\ |z^*| &= |z| && (4.2e)\\ 1/z &= z^*/(zz^*) = z^*/|z|^2 && (4.2f) \end{aligned}$$

$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \qquad (4.3)$$

Euler and DeMoivre:

$$\begin{aligned} e^{\imath\theta} &= \cos(\theta) + i\sin(\theta) && (4.4a)\\ (\cos(\theta) + i\sin(\theta))^n &= \cos(n\theta) + i\sin(n\theta) && (4.4b) \end{aligned}$$

$$\begin{aligned} (e^{\imath z})^* &= e^{-\imath z^*}, && (4.5a)\\ \left(e^{\imath\theta}\right)^* &= e^{-\imath\theta}, \quad \theta \text{ real} && (4.5b) \end{aligned}$$

$z = a + ib = re^{\imath\theta}$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$.

Logarithm:

$$\log(z) = \log(re^{\imath\theta}) = \ln r + i(\theta + 2\pi k) \tag{4.6}$$

with k an integer.

Chapter 5

Useful integrals

$$\int_a^b dx\, f(x)g'(x) = fg|_a^b - \int_a^b dx\, f'g \tag{5.1}$$

Indefinite integrals:

$$\int dx \frac{1}{1+x^2} = \tan^{-1}(x) \tag{5.2a}$$

$$\int dx \frac{1}{(1+x^2)^2} = \frac{1}{2}\left[\tan^{-1}(x) + \frac{x}{1+x^2}\right] \tag{5.2b}$$

$$\int dx \frac{x}{1+x^2} = \frac{1}{2}\ln(1+x^2) \tag{5.2c}$$

$$\int dx \frac{x}{(1+x^2)^2} = -\frac{1}{2(1+x^2)} \tag{5.2d}$$

$$\int dx \frac{x^2}{1+x^2} = x - \tan^{-1}(x) \tag{5.2e}$$

$$\int dx \frac{x^2}{(1+x^2)^2} = \frac{1}{2}\tan^{-1}(x) - \frac{x}{2(1+x^2)} \tag{5.2f}$$

$$\int dx\, e^{-ax} = -\frac{1}{a}e^{-ax} \tag{5.3a}$$

$$\int dx\, xe^{-ax} = -\frac{ax+1}{a^2}e^{-ax} \tag{5.3b}$$

$$\int dx\, x^2e^{-ax} = -\frac{a^2x^2+2ax+2}{a^3}e^{-ax} \tag{5.3c}$$

Some trigonometric integrals:

$$\int dx\ \sin^2 x = \frac{x}{2} - \frac{\sin x\cos x}{2} \tag{5.4}$$

$$\int dx\ \cos^2 x = \frac{x}{2} + \frac{\sin x\cos x}{2} \tag{5.5}$$

$$\tag{5.6}$$

Assorted integrals:

$$\int dx\ \frac{1}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\ln|a+be^{px}| \tag{5.7}$$

$$\int_0^\infty dx\ e^{-x}\ln x = -C, \tag{5.8}$$

where

$$C = \lim_{p\to\infty}\left[-\ln p + 1 + \frac{1}{2} + + \frac{1}{3} + ... + \frac{1}{p}\right] = 0.5772157 \tag{5.9}$$

is Euler's constant.
Definite integrals:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \tag{5.10}$$

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \frac{\sqrt{\pi}}{a^{1/2}}, \qquad Re[a] > 0 \tag{5.11a}$$

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{2a^{3/2}}, \qquad Re[a] > 0 \tag{5.11b}$$

$$\int_{-\infty}^{\infty} dx\, x^4 e^{-ax^2} = \frac{3\sqrt{\pi}}{4a^{5/2}}, \qquad Re[a] > 0 \tag{5.11c}$$

$$\int_{-\infty}^{\infty} dx\, x^6 e^{-ax^2} = \frac{15\sqrt{\pi}}{8a^{7/2}}, \qquad Re[a] > 0 \tag{5.11d}$$

$$\int_{-\infty}^{\infty} dx\, x^p e^{-ax^2} = \frac{1+(-1)^p}{2a^{\frac{p+1}{2}}}\Gamma\left(\frac{p+1}{2}\right), \qquad Re[a] > 0,\ Re[p] > -1, \tag{5.11e}$$

see Sec. 6.1 for the Γ function.

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12a}$$

$$\int_{-\infty}^{\infty} dx\, xe^{-ax^2+bx} = \frac{\sqrt{\pi}b}{2a^{3/2}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12b}$$

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2+bx} = \frac{\sqrt{\pi}(2a+b^2)}{4a^{5/2}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12c}$$

$$\int_{-\infty}^{\infty} dx\, x^3 e^{-ax^2+bx} = \frac{\sqrt{\pi}(6ab+b^3)}{8a^{7/2}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12d}$$

$$\int_{-\infty}^{\infty} dx\, x^4 e^{-ax^2+bx} = \frac{\sqrt{\pi}(12a^2+12ab^2+b^4)}{16a^{9/2}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12e}$$

$$\int_{-\infty}^{\infty} dx\, x^6 e^{-ax^2+bx} = \frac{\sqrt{\pi}(120a^3+180a^2b^2+30ab^4+b^6)}{64a^{13/2}}e^{b^2/(4a)}, \qquad Re[a] > 0 \tag{5.12f}$$

The following are formally divergent but useful

$$\int_{-\infty}^{\infty} dx\ e^{\imath a x^2} = \sqrt{\frac{i\pi}{a}}, \quad a \text{ real} \tag{5.13a}$$

$$\int_{-\infty}^{\infty} dx\ e^{\imath\left(a x^2 + b x\right)} = \sqrt{\frac{i\pi}{a}} e^{-ib^2/(4a)}, \quad a \text{ real} \tag{5.13b}$$

$$\int_{-\infty}^{\infty} dx \frac{1}{1+bx^2} = \frac{\pi}{\sqrt{b}} \tag{5.14a}$$

$$\int_{-\infty}^{\infty} dx \frac{e^{-ax^2}}{1+bx^2} = \frac{\pi}{\sqrt{b}} e^{a/b} \mathrm{Erfc}(\sqrt{a/b}), \text{ where } \mathrm{Erfc}(x) = 1 - \mathrm{Erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \tag{5.14b}$$

$$\int_{-\infty}^{\infty} dx \frac{\sin(ax)}{ax} = \frac{\pi}{a} \tag{5.15a}$$

$$\int_{-\infty}^{\infty} dx \left[\frac{\sin(ax)}{ax}\right]^2 = \frac{\pi}{a} \tag{5.15b}$$

$$\int_{-\infty}^{\infty} dx \left[\frac{\sin(ax)}{ax}\right]^3 = \frac{3\pi}{4a} \tag{5.15c}$$

$$\int_0^{\infty} dx\, e^{-ax} = \frac{1}{a}, \quad Re[a] > 0 \tag{5.16a}$$

$$\int_0^{\infty} dx\, x e^{-ax} = \frac{1}{a^2}, \quad Re[a] > 0 \tag{5.16b}$$

$$\int_0^{\infty} dx\, x^2 e^{-ax} = \frac{2}{a^3}, \quad Re[a] > 0 \tag{5.16c}$$

$$\int_0^{\infty} dx\, x^3 e^{-ax} = \frac{6}{a^4}, \quad Re[a] > 0 \tag{5.16d}$$

$$\int_0^{\infty} dx\, x^4 e^{-ax} = \frac{24}{a^5}, \quad Re[a] > 0 \tag{5.16e}$$

$$\int_0^{\infty} dx\, x^5 e^{-ax} = \frac{120}{a^6}, \quad Re[a] > 0 \tag{5.16f}$$

$$\int_0^{\infty} dx\, x^6 e^{-ax} = \frac{720}{a^7}, \quad Re[a] > 0 \tag{5.16g}$$

$$\int_0^{\infty} dx\, x^p e^{-ax} = \frac{\Gamma(p+1)}{a^{p+1}}, \quad Re[a] > 0,\ Re[p] > -1, \tag{5.16h}$$

see Sec. 6.1 for the Γ function.

Chapter 6

Special functions

6.1 Gamma function

Definition:

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \tag{6.1}$$

$$\Gamma(n) = \int_0^\infty dx\ x^{n-1} e^{-x} = \int_0^1 dx\ \left(ln\frac{1}{x} \right)^{n-1} \tag{6.2}$$

$$\tag{6.3}$$

Properties:

$$\Gamma(z) = (z-1)\Gamma(z-1) \tag{6.4a}$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)} \tag{6.4b}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{6.4c}$$

$$\tag{6.4d}$$

Values:

$$\Gamma(1/2) = \sqrt{\pi} \tag{6.5a}$$

If z is a positive integer $z = n = 1, 2, 3...$ then

$$\Gamma(n) = (n-1)! \tag{6.6a}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi}\frac{(2n-1)!!}{2^n} \tag{6.6b}$$

$$\Gamma\left(\frac{1}{2} - n\right) = \sqrt{\pi}\frac{(-1)^n 2^n}{(2n-1)!!} \tag{6.6c}$$

where $n!! = n(n-2)..5\,3\,1$, $n > 0$ odd, $n(n-2)...4\,2$, $n > 0$ even, and $-1!! \equiv 1, 0!! \equiv 1$.

6.2 Riemann zeta function

Definition:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{6.7}$$

Properties:

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos(\pi z/2)\Gamma(z)\zeta(z) \tag{6.8a}$$

Values:

$$\zeta(2) = \frac{\pi^2}{6} \tag{6.9a}$$

$$\zeta(3) = 1.2020569032... \tag{6.9b}$$

6.3 Bessel and Hankel functions

Differential equation:

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2 - \nu^2) f = 0. \tag{6.10}$$

Solutions are the Bessel functions of the first kind $J_\nu(x)$, Bessel functions of the second kind $Y_\nu(x)$, and the Hankel functions $H_\nu(x)$.

Power Series

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2j} \tag{6.11a}$$

$$J_{-\nu}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{-\nu+2j} \tag{6.11b}$$

Bessel functions of the second kind (Neumann functions):

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{6.12}$$

Bessel functions of the third kind (Hankel functions):

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ &= \frac{i}{\sin\nu\pi}\left[e^{-\imath\nu\pi} J_\nu(x) - J_{-\nu}(x)\right] \end{aligned} \tag{6.13a}$$

$$\begin{aligned} H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \\ &= \frac{-i}{\sin\nu\pi}\left[e^{\imath\nu\pi} J_\nu(x) - J_{-\nu}(x)\right] \end{aligned} \tag{6.13b}$$

$$J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right] \tag{6.14a}$$

$$J_{-\nu}(x) = \frac{1}{2}\left[e^{\imath\nu\pi}H_\nu^{(1)}(x) + e^{-\imath\nu\pi}H_\nu^{(2)}(x)\right] \tag{6.14b}$$

Generating function:

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \quad n \text{ takes on integer values} \tag{6.15a}$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k} \quad \text{for } n \text{ integer} \tag{6.15b}$$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi d\theta\, \cos(n\theta - x\sin\theta) \quad \text{for } n \text{ integer} \tag{6.15c}$$

Orthogonality:

$$\int_0^\infty dx\, x J_\nu(ax) J_\nu(bx) = \frac{1}{a}\delta(|a| - |b|) \tag{6.16}$$

Recurrence relations (valid for both $J_\nu(x)$ and $H_\nu(x)$):

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x) \tag{6.17a}$$

$$\frac{2\nu}{x}J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x) \tag{6.17b}$$

$$\frac{dJ_\nu(x)}{dx} = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) \tag{6.17c}$$

$$\frac{dJ_\nu(x)}{dx} = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x) \tag{6.17d}$$

$$\frac{dJ_\nu(x)}{dx} = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right] \tag{6.17e}$$

Symmetries:

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for } n \text{ integer.} \tag{6.18a}$$

Particular values:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \tag{6.19a}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \tag{6.19b}$$

Asymptotic values:

For $x \ll 1$

$$\begin{aligned}
J_\nu(x) &\rightarrow \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu && (6.20a)\\
Y_0(x) &\rightarrow \frac{2}{\pi}\left[\ln(x/2) + 0.5772...\right] && (6.20b)\\
Y_\nu(x) &\rightarrow -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu && (6.20c)
\end{aligned}$$

For $x \gg 1$

$$\begin{aligned}
J_\nu(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \nu\pi/2 - \pi/4\right) && (6.21a)\\
Y_\nu(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \nu\pi/2 - \pi/4\right) && (6.21b)\\
H_\nu^{(1)}(x) &\rightarrow \sqrt{\frac{2}{\pi x}}e^{\imath(x-\nu\pi/2-\pi/4)} && (6.21c)\\
H_\nu^{(2)}(x) &\rightarrow \sqrt{\frac{2}{\pi x}}e^{-\imath(x-\nu\pi/2-\pi/4)} && (6.21d)
\end{aligned}$$

Sums:

$$\begin{aligned}
e^{\imath z\cos\theta} &= \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{\imath n\theta} \quad n \text{ takes on integer values} && (6.22a)\\
1 &= \sum_{n=-\infty}^{\infty} J_n(x) \quad n \text{ takes on integer values} && (6.22b)\\
J_n(x+y) &= \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y) && (6.22c)
\end{aligned}$$

6.4 Delta function

Definition:

$$\begin{aligned}
\delta(x) &= 0 \quad x \neq 0, && (6.23a)\\
&= \infty \quad x = 0, && (6.23b)\\
\int_{-\infty}^{\infty} dx\, f(x)\delta(x) &= f(0) && (6.23c)
\end{aligned}$$

Properties:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, \delta(x) &= 1 && (6.24a)\\
\int_{-\infty}^{\infty} dx\, f(x)\delta(x-x_0) &= f(x_0) && (6.24b)
\end{aligned}$$

$$\begin{aligned}
\delta(x-x_0) &= \delta(x_0-x) && (6.25a)\\
\delta(ax) &= \frac{1}{|a|}\delta(x) && (6.25b)\\
\delta(g(x)) &= \sum_j \frac{1}{|g'(x_j)|}\delta(x-x_j), \qquad \text{where } x_j \text{ are the roots of } g && (6.25c)\\
\frac{d^n}{dx^n}\delta(x) &= \frac{(-1)^n n!}{x^n}\delta(x) && (6.25d)
\end{aligned}$$

Representations:

$$\begin{aligned}
\delta(x) &= \lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2} && (6.26a)\\
&= \lim_{\sigma\to 0}\frac{1}{\pi}\frac{\sigma}{x^2+\sigma^2} && (6.26b)\\
&= \lim_{\sigma\to 0}\frac{1}{\pi x}\sin(x/\sigma) && (6.26c)\\
&= \lim_{\sigma\to 0}\frac{\sigma}{2\pi}\frac{\sin^2\frac{x}{2\sigma}}{(x/2)^2} && (6.26d)
\end{aligned}$$

$$\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{i\pi\varepsilon}}e^{\imath\frac{x^2}{\varepsilon}} \tag{6.27}$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{\imath kx} = \int_{-\infty}^{\infty}dk\,e^{\imath 2\pi kx} \tag{6.28}$$

If $u_n(x)$ form a complete set then

$$\delta(x-x') = \sum_n u_n(x)u_n^*(x') \tag{6.29}$$

Multiple dimensions:
Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \tag{6.30a}$$

Spherical coordinates:

$$\begin{aligned}
\delta(\mathbf{r}-\mathbf{r}') &= \frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') && (6.31a)\\
&= \frac{1}{r^2}\delta(r-r')\frac{\delta(\theta-\theta')}{\sin\theta}\delta(\phi-\phi') && (6.31b)\\
\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) &= -4\pi\delta(\mathbf{r}-\mathbf{r}') && (6.31c)
\end{aligned}$$

Chapter 7

Fourier transforms

7.1 Transforms on the full line

Transform pair (independent variables x and k):

$$f(x) = \mathcal{F}^{-1}[g(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, g(k)e^{\imath kx} \tag{7.1a}$$

$$g(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)e^{-\imath kx} \tag{7.1b}$$

Transform pair (independent variables t and ν):

$$f(t) = \mathcal{F}^{-1}[g(\nu)] = \int_{-\infty}^{\infty} d\nu\, g(\nu)e^{-\imath 2\pi\nu t} \tag{7.2a}$$

$$g(\nu) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} dt\, f(t)e^{\imath 2\pi\nu t} \tag{7.2b}$$

Plancherel:

$$\int_{-\infty}^{\infty} dx\, f_1^*(x)f_2(x) = \int_{-\infty}^{\infty} dk\, g_1^*(k)g_2(k) \tag{7.3a}$$

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |g(k)|^2 \tag{7.3b}$$

$$\tag{7.3c}$$

Convolution:

$$f_1(x) * f_2(x) \equiv \int_{-\infty}^{\infty} dx'\, f_1(x')f_2(x-x') \tag{7.4a}$$

$$\mathcal{F}[f_1(x) * f_2(x)] = \sqrt{2\pi}\, g_1(k)g_2(k) \tag{7.4b}$$

Correlation:

$$f_1(x) \star f_2(x) \equiv [f_1(-x)]^* * f_2(x) = \int_{-\infty}^{\infty} dx'\, [f_1(-x)]^* f_2(x-x') \tag{7.5a}$$

$$\mathcal{F}[f_1(x) \star f_2(x)] = \sqrt{2\pi}\, [g_1(k)]^* g_2(k) \tag{7.5b}$$

Multiple dimensions:

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{k}\, g(\mathbf{k}) e^{\imath \mathbf{k}\cdot\mathbf{r}} \tag{7.6a}$$

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{r}\, f(\mathbf{r}) e^{-\imath \mathbf{k}\cdot\mathbf{r}} \tag{7.6b}$$

Quantum mechanics version:

$$f(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{p}\, g(\mathbf{p}) e^{(\imath/\hbar)\mathbf{p}\cdot\mathbf{r}} \tag{7.7a}$$

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{r}\, f(\mathbf{r}) e^{-(\imath/\hbar)\mathbf{p}\cdot\mathbf{r}} \tag{7.7b}$$

Transform pairs:

When $f(x) \leftrightarrow g(k)$ then:

$$f(x - x_0) \leftrightarrow e^{-\imath k x_0} g(k) \tag{7.8a}$$

$$e^{\imath k_0 x} f(x) \leftrightarrow g(k - k_0) \tag{7.8b}$$

$$\frac{df(x)}{dx} \leftrightarrow ikg(k) \tag{7.8c}$$

$$xf(x) \leftrightarrow i\frac{dg(k)}{dk} \tag{7.8d}$$

$$x\frac{df(x)}{dx} \leftrightarrow -g(k) - k\frac{dg(k)}{dk} \tag{7.8e}$$

$$\delta(x - x_0) \leftrightarrow \frac{1}{\sqrt{2\pi}} e^{-\imath k x_0} \tag{7.9a}$$

$$e^{\imath k_0 x} \leftrightarrow \sqrt{2\pi}\,\delta(k - k_0) \tag{7.9b}$$

$$e^{-ax^2+bx} \leftrightarrow \frac{1}{\sqrt{2a}} e^{(b-ik)^2/(4a)} \tag{7.9c}$$

$$xe^{-ax^2+bx} \leftrightarrow \frac{b-ik}{(2a)^{3/2}} e^{(b-ik)^2/(4a)} \tag{7.9d}$$

$$x^2 e^{-ax^2+bx} \leftrightarrow \frac{2a + (b-ik)^2}{(2a)^{5/2}} e^{(b-ik)^2/(4a)} \tag{7.9e}$$

$$\frac{1}{1+x^2} \quad \leftrightarrow \quad \sqrt{\frac{\pi}{2}}e^{-|k|} \tag{7.10a}$$

$$\frac{1}{(1+x^2)^2} \quad \leftrightarrow \quad \sqrt{\frac{\pi}{8}}(1+|k|)e^{-|k|} \tag{7.10b}$$

7.2 Transforms on the half line

For functions defined on the half line there are sin and cos transform pairs:

$$f(x) = \mathcal{F}_c^{-1}[g(k)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, g(k)\cos(kx) \tag{7.11a}$$

$$g(k) = \mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dx\, f(x)\cos(kx) \tag{7.11b}$$

$$f(x) = \mathcal{F}_s^{-1}[g(k)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, g(k)\sin(kx) \tag{7.12a}$$

$$g(k) = \mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dx\, f(x)\sin(kx) \tag{7.12b}$$

Chapter 8

Orthogonal polynomials

8.1 Hermite polynomials

Differential equation:

$$\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + 2nf = 0. \tag{8.1}$$

Particular solutions for n integer are the Hermite polynomials $H_n(x)$.

Generating function:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \tag{8.2a}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{8.2b}$$

Recurrence relations:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{8.3a}$$

$$H_n'(x) = 2nH_{n-1}(x) \tag{8.3b}$$

Particular values:

$$H_n(-x) = (-1)^n H_n(x) \tag{8.4}$$

$$H_0(x) = 1 \tag{8.5a}$$

$$H_1(x) = 2x \tag{8.5b}$$

$$H_2(x) = 4x^2 - 2 \tag{8.5c}$$

$$H_3(x) = 8x^3 - 12x \tag{8.5d}$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \tag{8.5e}$$

$$H_5(x) = 32x^5 - 160x^3 + 120x \tag{8.5f}$$

Orthogonality relations:

$$u_n(x) = \sqrt{\frac{1}{\pi^{1/2} n! 2^n a}} H_n(x/a) e^{-x^2/(2a^2)} \quad a \text{ is a scaling parameter which can be 1} \tag{8.6}$$

$$\int_{-\infty}^{\infty} dx\, u_m(x)u_n(x) = \delta_{mn} \tag{8.7a}$$

$$\int_{-\infty}^{\infty} dx\, u_n(x)u_m'(x) = \frac{1}{a}\sqrt{\frac{n+1}{2}}\delta_{m,n+1} - \frac{1}{a}\sqrt{\frac{n}{2}}\delta_{m,n-1} \tag{8.7b}$$

$$\int_{-\infty}^{\infty} dx\, u_m(x)xu_n(x) = a\sqrt{\frac{n+1}{2}}\delta_{m,n+1} + a\sqrt{\frac{n}{2}}\delta_{m,n-1} \tag{8.7c}$$

$$\int_{-\infty}^{\infty} dx\, u_m(x)x^2u_n(x) = a^2\frac{2n+1}{2}\delta_{mn} + a^2\frac{\sqrt{(n+2)(n+1)}}{2}\delta_{m,n+2} + a^2\frac{\sqrt{n(n-1)}}{2}\delta_{m,n-2} \tag{8.7d}$$

Delta function expansion:

$$\delta(x-y) = \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{2^n n!} \tag{8.8}$$

Fourier transform:

$$\mathcal{F}[H_n(x/a)e^{-x^2/(2a^2)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, H_n(x/a)e^{-x^2/(2a^2)}e^{-\imath kx} = (-i)^n aH_n(ak)e^{-a^2k^2/2} \tag{8.9a}$$

$$\mathcal{F}[u_n(x/a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, u_n(x/a)e^{-\imath kx} = (-i)^n u_n(ak) \tag{8.9b}$$

8.2 Laguerre polynomials

Differential equation:

$$x\frac{d^2f}{dx^2} + (1-x)\frac{df}{dx} + nf = 0. \tag{8.10}$$

Particular solutions for n integer are the Laguerre polynomials $L_n(x)$.

Generating function:

$$\frac{1}{1-z}e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x)z^n \tag{8.11a}$$

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \tag{8.11b}$$

Recurrence relations:

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{8.12a}$$

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x) \tag{8.12b}$$

Particular values:

$$\begin{aligned} L_0(x) &= 1 & (8.13a)\\ L_1(x) &= 1-x & (8.13b)\\ L_2(x) &= \frac{1}{2}(x^2-4x+2) & (8.13c)\\ L_3(x) &= \frac{1}{6}(-x^3+9x^2-18x+6) & (8.13d)\\ L_4(x) &= \frac{1}{24}(x^4-16x^3+72x^2-96x+24) & (8.13e) \end{aligned}$$

Orthogonality relations:

$$\begin{aligned} \int_0^\infty dx\, L_m(x)L_n(x)e^{-x} &= \delta_{mn} & (8.14a)\\ \int_0^\infty dx\, x\,[L_n(x)]^2\, e^{-x} &= (2n+1) & (8.14b) \end{aligned}$$

8.3 Associated Laguerre polynomials

Differential equation:

$$x\frac{d^2 f}{dx^2}+(k+1-x)\frac{df}{dx}+nf=0. \qquad (8.15)$$

Particular solutions for n, k integer are the associated Laguerre polynomials $L_n^k(x)$.

Generating function:

$$\begin{aligned} \frac{1}{(1-z)^{k+1}}e^{-xz/(1-z)} &= \sum_{n=0}^\infty L_n^k(x)z^n & (8.16a)\\ \frac{1}{1-z}e^{\frac{-xz+u}{1-z}} &= \sum_{k=0}^\infty\sum_{n=k}^\infty \frac{L_n^k(x)z^n u^k}{k!} & (8.16b)\\ L_n^k(x) &= \frac{x^{-k}e^x}{n!}\frac{d^n}{dx^n}(x^{n+k}e^{-x}) & (8.16c)\\ L_n^k(x) &= (-1)^k\frac{d^k}{dx^k}L_{n+k}(x) & (8.16d) \end{aligned}$$

Recurrence relations:

$$\begin{aligned} L_n^k(x) &= L_{n-1}^k(x)+L_n^{k-1}(x) & (8.17a)\\ (n+1)L_{n+1}^k(x) &= (2n+k+1-x)L_n^k(x)-(n+k)L_{n-1}^k(x) & (8.17b)\\ x\frac{dL_n^k(x)}{dx} &= nL_n^k(x)-(n+k)L_{n-1}^k(x) & (8.17c) \end{aligned}$$

Particular values:

$$
\begin{aligned}
L_0^k(x) &= 1 && (8.18a)\\
L_1^k(x) &= 1+k-x && (8.18b)\\
L_2^k(x) &= \frac{1}{2}(x^2-2(k+2)x+(k+1)(k+2)) && (8.18c)\\
L_3^k(x) &= \frac{1}{6}(-x^3+3(k+3)x^2-3(k+3)(k+2)x+(k+3)(k+2)(k+1)) && (8.18d)\\
L_n^k(0) &= L_n^k(0) = \binom{n+k}{n} = \frac{(n+k)!}{n!k!} && (8.18e)
\end{aligned}
$$

Orthogonality relations:

$$
\begin{aligned}
\int_0^\infty dx\, L_m^k(x)L_n^k(x)e^{-x}x^k &= \frac{(n+k)!}{n!}\delta_{mn} && (8.19a)\\
\int_0^\infty dx\, \left[L_n^k(x)\right]^2 e^{-x}x^{k+1} &= \frac{(n+k)!}{n!}(2n+k+1) && (8.19b)
\end{aligned}
$$

8.4 Legendre polynomials

Differential equation:

$$(1-x^2)\frac{d^2f}{dx^2}-2x\frac{df}{dx}+l(l+1)f=0. \qquad (8.20)$$

Particular solutions for l integer are the Legendre polynomials $P_l(x)$.

Generating function:

$$
\begin{aligned}
\frac{1}{\sqrt{t^2-2xt+1}} &= \sum_{l=0}^{\infty}P_l(x)t^l && (8.21a)\\
P_l(x) &= \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l && (8.21b)
\end{aligned}
$$

Recurrence relations:

$$
\begin{aligned}
(l+1)P_{l+1}(x) &= (2l+1)xP_l(x)-lP_{l-1}(x) && (8.22a)\\
lP_l(x) &= xP_l'(x)-P_{l-1}'(x) && (8.22b)\\
lP_l(x) &= lxP_{l-1}(x)+(x^2-1)P_{l-1}'(x) && (8.22c)\\
(l+1)P_l(x) &= P_{l+1}'(x)-xP_l'(x) && (8.22d)
\end{aligned}
$$

Particular values:

$$P_l(-x)=(-1)^lP_l(x) \qquad (8.23)$$

$$
\begin{aligned}
P_l(1) &= 1 && (8.24a)\\
P_l(-1) &= (-1)^l && (8.24b)\\
P_l(0) &= \frac{\sqrt{\pi}}{\Gamma(\frac{1-l}{2})\Gamma(1+\frac{l}{2})} && (8.24c)
\end{aligned}
$$

$$P_0(x) = 1 \tag{8.25a}$$
$$P_1(x) = x \tag{8.25b}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{8.25c}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{8.25d}$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{8.25e}$$

Orthogonality relations:

$$u_l(x) = \sqrt{\frac{2l+1}{2}} P_l(x) \tag{8.26}$$

$$\int_{-1}^{1} dx\, u_l(x) u_{l'}(x) = \delta_{ll'} \tag{8.27a}$$
$$\sum_{l=0}^{\infty} u_l(x) u_l(x') = \delta(x - x') \tag{8.27b}$$

Sums:

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{8.28}$$

where $\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)$.

8.5 Associated Legendre polynomials

Differential equation:

$$(1 - x^2)\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + \left[l(l+1) - \frac{m^2}{1 - x^2}\right] f = 0. \tag{8.29}$$

Particular solutions for l, m integer with $-l \leq m \leq l$ are the associated Legendre polynomials $P_l^m(x)$.

Generating function:

$$\frac{1}{\sqrt{t^2 - 2t(x + z\sqrt{1 - x^2}) + 1}} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(x) \frac{(-1)^m t^l z^m}{m!} \tag{8.30a}$$
$$P_l^m(x) = (-1)^m \frac{(1 - x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l, \ \text{all } m \tag{8.30b}$$
$$P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \qquad m > 0 \tag{8.30c}$$
$$\tag{8.30d}$$

Recurrence relation (there are many more):

$$(2l+1)xP_l^m(x) = (l-m+1)P_{l+1}^m(x) + (l+m)P_{l-1}^m(x) \tag{8.31a}$$

Particular values:

$$P_l^0(x) = P_l(x) \tag{8.32a}$$
$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{8.32b}$$
$$P_l^l(x) = (-1)^l (2l-1)!!(1-x^2)^{l/2} \tag{8.32c}$$
$$P_{l+1}^l(x) = (2l+1)xP_l^l(x) \tag{8.32d}$$

$$P_0^0(x) = 1 \tag{8.33a}$$
$$P_1^{-1}(x) = \frac{\sqrt{1-x^2}}{2}, \quad P_1^0(x) = x, \quad P_1^1(x) = -\sqrt{1-x^2} \tag{8.33b}$$
$$P_2^{-2}(x) = \frac{1-x^2}{8}, \quad P_2^{-1}(x) = \frac{x\sqrt{1-x^2}}{2}, \quad P_2^0(x) = \frac{3x^2-1}{2}, \tag{8.33c}$$
$$P_2^1(x) = -3x\sqrt{1-x^2}, \quad P_2^2(x) = -3(x^2-1) \tag{8.33d}$$

Orthogonality relations:

$$u_l^m(x) = \sqrt{\frac{2l+1}{2}} P_l^m(x) \tag{8.34}$$

$$\int_{-1}^1 dx\, u_l^m(x) u_{l'}^m(x) = \frac{(l+m)!}{(l-m)!}\delta_{l,l'} \tag{8.35a}$$
$$\int_{-1}^1 dx\, u_l^m(x) u_l^{m'}(x) \frac{1}{1-x^2} = \frac{2l+1}{2}\frac{(l+m)!}{m(l-m)!}\delta_{m,m'} \tag{8.35b}$$

8.6 Spherical Harmonics

Differential equation:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] f = 0 \tag{8.36}$$

Particular solutions are $Y_{lm}(\theta,\phi)$ with integer $l \ge 0$, $-l \le m \le l$. The notation $Y_l^m(\theta,\phi) = Y_{lm}(\theta,\phi)$ is also used.

Definition:

$$Y_{lm}(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta) e^{\imath m\phi} \tag{8.37}$$

Symmetries:

$$\begin{aligned}
Y_{l,-m}(\theta,\phi) &= (-1)^m [Y_{lm}(\theta,\phi)]^* && (8.38a)\\
Y_{lm}(-\theta,\phi) &= (-1)^m Y_{lm}(\theta,\phi) && (8.38b)\\
Y_{lm}(\theta,-\phi) &= (-1)^m Y_{l,-m}(\theta,\phi) && (8.38c)\\
Y_{lm}(-\theta,-\phi) &= Y_{l,-m}(\theta,\phi) && (8.38d)
\end{aligned}$$

$$\begin{aligned}
Y_{lm}(\pi-\theta,\pi+\phi) &= (-1)^l Y_{lm}(\theta,\phi) && (8.39a)\\
Y_{lm}(\pi-\theta,\phi) &= (-1)^{l+m} Y_{lm}(\theta,\phi) && (8.39b)\\
Y_{lm}(\theta,\pi+\phi) &= (-1)^m Y_{lm}(\theta,\phi) && (8.39c)
\end{aligned}$$

Integrals over the entire solid angle with $d\Omega = \sin\theta d\theta d\phi$

$$\begin{aligned}
\int d\Omega\,[Y_{lm}(\theta,\phi)]^* Y_{l'm'}(\theta,\phi) &= \delta_{ll'}\delta_{mm'} && (8.40a)\\
\int d\Omega\, Y_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) &= (-1)^{m'}\delta_{l,l'}\delta_{-m,m'} && (8.40b)\\
\int d\Omega\, Y_{l_1m_1}Y_{l_2m_2}Y_{l_3m_3} &= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix} && (8.40c)\\
&= (-1)^{m_3}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} C^{l_30}_{l_10l_20} C^{l_3m_3}_{l_1-m_1l_2-m_2} && (8.40d)\\
\langle l_3m_3|Y_{l_2m_2}|l_1m_1\rangle &= (-1)^{l_1+l_2-l_3}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} C^{l_30}_{l_10l_20} C^{l_3m_3}_{l_1m_1l_2m_2} && (8.40e)
\end{aligned}$$

Sums:

$$\begin{aligned}
\sum_{m=-l}^{l} |Y_{lm}(\theta,\phi)|^2 &= \frac{2l+1}{4\pi} && (8.41a)\\
\sum_{m=-l}^{l} m|Y_{lm}(\theta,\phi)|^2 &= 0 && (8.41b)\\
\sum_{m=-l}^{l} m^2|Y_{lm}(\theta,\phi)|^2 &= \frac{(2l+1)(l+1)l}{8\pi}\sin^2\theta && (8.41c)
\end{aligned}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}[Y_{lm}(\theta,\phi)]^* Y_{lm}(\theta',\phi') = \delta(\cos\theta-\cos\theta')\delta(\phi-\phi') \qquad (8.42)$$

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} {Y_{lm}}^*(\theta_2,\phi_2) Y_{lm}(\theta_1,\phi_1) \qquad (8.43)$$

where $\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$.

Rank l spherical tensor

$$C_{lm}(\theta,\phi) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{lm}(\theta,\phi) \tag{8.44}$$

and

$$\begin{aligned} P_l(\cos\gamma) &= \sum_{m=-l}^{l} C^*_{lm}(\theta_2,\phi_2)C_{lm}(\theta_1,\phi_1) \\ &= \sum_{m=-l}^{l} (-1)^m C_{l-m}(\theta_2,\phi_2)C_{lm}(\theta_1,\phi_1) \\ &= (\mathbf{C}_l(1)\cdot\mathbf{C}_l(2)) \end{aligned} \tag{8.45}$$

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}{}^*(\theta_2,\phi_2)Y_{lm}(\theta_1,\phi_1)$$

Solutions of the equation $\nabla^2 f(r,\theta,\phi)=0$ are of the form (solid harmonics)

$$r^l Y_{lm}(\theta,\phi) \tag{8.46a}$$

or

$$\frac{1}{r^{l+1}}Y_{lm}(\theta,\phi) \tag{8.46b}$$

Chapter 9

Angular momentum algebra

9.1 Clebsch-Gordan coefficients and symmetry properties

There exist a number of explicit expressions for the Clebsch-Gordan coefficents. An expression due to Wigner is

$$C^{c\gamma}_{a\alpha b\beta} = \delta_{\gamma,\alpha+\beta}\Delta(abc)\left[\frac{(c+\gamma)!(c-\gamma)!(2c+1)}{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!}\right]^{1/2} \times \sum_z \frac{(-1)^{b+\beta+z}(c+b+\alpha-z)!(a-\alpha+z)!}{z!(c-a+b-z)!(c+\gamma-z)!(a-b-\gamma+z)!} \tag{9.1}$$

where

$$\Delta(abc) = \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}$$

and in the summation z assumes all integer values for which the factorial arguments are nonnegative. The Clebsch-Gordan coefficients satisfy the following unitarity relations

$$\sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2} C^{j'm'}_{j_1 m_1 j_2 m_2} = \delta_{jj'}\delta_{mm'} \tag{9.2a}$$

$$\sum_{jm} C^{jm}_{j_1 m_1 j_2 m_2} C^{jm}_{j_1 m_1' j_2 m_2'} = \delta_{m_1 m_1'}\delta_{m_2 m_2'}. \tag{9.2b}$$

There are a large number of symmetry relations involving permutations of indices.Some of them are:

$$C^{c\gamma}_{a\alpha b\beta} = (-1)^{a+b-c} C^{c\gamma}_{b\beta a\alpha} = (-1)^{a-\alpha}\sqrt{\frac{2c+1}{2b+1}} C^{b-\beta}_{a\alpha c-\gamma} = (-1)^{a-\alpha}\sqrt{\frac{2c+1}{2b+1}} C^{b\beta}_{c\gamma a-\alpha} = (-1)^{b+\beta}\sqrt{\frac{2c+1}{2a+1}} C^{a-\alpha}_{c-\gamma b\beta} = (-1)^{b+\beta}\sqrt{\frac{2c+1}{2a+1}} C^{a\alpha}_{b-\beta c\gamma} \tag{9.3}$$

$$C^{c\gamma}_{a\alpha b\beta} = (-1)^{a+b-c} C^{c-\gamma}_{a-\alpha b-\beta} \tag{9.4}$$

When one of the momenta is zero:

$$C^{00}_{a\alpha b\beta} = (-1)^{a-\alpha}\frac{\delta_{ab}\delta_{\alpha-\beta}}{\sqrt{2a+1}} \tag{9.5a}$$

$$C^{c\gamma}_{a\alpha 00} = \delta_{ac}\delta_{\alpha\gamma} \tag{9.5b}$$

When the third momentum is the maximum possible:

$$C^{a+b\alpha+\beta}_{a\alpha b\beta} = \left[\frac{(2a)!(2b)!(a+b+\alpha+\beta)!(a+b-\alpha-\beta)!}{(2a+2b)!(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!}\right]^{1/2} \tag{9.6}$$

When all the $m's$ are zero:

$$C^{a+b0}_{a0b0} = \frac{(a+b)!}{a!b!}\left[\frac{(2a)!(2b)!}{(2a+2b)!}\right]^{1/2} \tag{9.7a}$$

$$C^{a-b0}_{a0b0} = (-1)^b\frac{a!}{b!(a-b)!}\left[\frac{(2b)!(2a-2b+1)!}{(2a+1)!}\right]^{1/2} \tag{9.7b}$$

In particular when $b=1$

$$C^{j\pm 10}_{j010} = \pm\sqrt{\frac{j_{\max}}{2j+1}} \tag{9.8}$$

where $j_{\max}$ is the larger of $j, j\pm 1$.

The coefficients for the cases when one of the momenta is 1/2 or 1 are often needed in atomic calculations. The $b=1/2$ cases are:

$$C^{a+1/2\alpha\pm 1/2}_{a\alpha 1/2\pm 1/2} = \sqrt{\frac{a\pm\alpha+1}{2a+1}} \tag{9.9a}$$

$$C^{a-1/2\alpha\pm 1/2}_{a\alpha 1/2\pm 1/2} = \mp\sqrt{\frac{a\mp\alpha}{2a+1}} \tag{9.9b}$$

The $b=1$ cases are:

$$C^{a+1\alpha}_{a\alpha 10} = \sqrt{\frac{(a+\alpha+1)(a-\alpha+1)}{(2a+1)(a+1)}},\quad C^{a+1\alpha\pm 1}_{a\alpha 1\pm 1} = \sqrt{\frac{(a\pm\alpha+1)(a\pm\alpha+2)}{2(2a+1)(a+1)}} \tag{9.10a}$$

$$C^{a\alpha}_{a\alpha 10} = \frac{\alpha}{\sqrt{a(a+1)}},\quad C^{a\alpha\pm 1}_{a\alpha 1\pm 1} = \mp\sqrt{\frac{(a\pm\alpha+1)(a\mp\alpha)}{2a(a+1)}} \tag{9.10b}$$

$$C^{a-1\alpha}_{a\alpha 10} = -\sqrt{\frac{(a+\alpha)(a-\alpha)}{a(2a+1)}},\quad C^{a-1\alpha\pm 1}_{a\alpha 1\pm 1} = \sqrt{\frac{(a\mp\alpha-1)(a\mp\alpha)}{2a(2a+1)}} \tag{9.10c}$$

The 3j symbols are given by

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{2a+c+\gamma}\frac{1}{\sqrt{2c+1}}C^{c\gamma}_{a-\alpha b-\beta}. \tag{9.11}$$