# Linear Programming

## (5531)

### Lecture 07 Steepest Edge Rule / Sensitivity Analysis

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# Agenda

- Steepest edge rule (Ch8.8)
- Sensitivity analysis (Ch7.1)

# Steepest Edge Rule

- *Motivations:*
  - When selecting an entering variable, the largest-coefficient rule only considers the objective coefficient but *doesn't consider the step size*
  - Remembering the issue of using the largest-coefficient rule for solving the *Klee-Minty problem*
  - *How to consider both the objective coefficient and step size for selecting the entering variable?*
  - To answer the question, we may measure the **increasing rate**
  - The idea of the *steepest edge rule* is to select an entering variable with the **largest increasing rate** of the objective function among **both basic and nonbasic variables**
  - This rate increase is measured for in the space of nonbasic variables

# Procedure

- Fix a nonbasic index *j* ∈ *N*. If $x_j$ enters the basis, the step direction is:

$$\Delta x = \begin{bmatrix} \Delta x_B \\ \Delta x_N \end{bmatrix} = \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix}$$

- As we know, the objective function is:

$$f(x) = c^T x = c_B^T x_B + c_N^T x_N$$

- The derivative of *f(x)* in the direction of *Δx* is:

$$\frac{\partial f}{\partial \Delta x} = c^T \frac{\Delta x}{\|\Delta x\|} = \frac{c_B^T \Delta x_B + c_N^T \Delta x_N}{\|\Delta x\|}$$

  - ,where

$$\begin{aligned} c_B^T \Delta x_B + c_N^T \Delta x_N &= c_j - c_B^T B^{-1} N e_j \\ &= \left(c_N - \left(B^{-1}N\right)^T c_B\right)_j \\ &= -z_j^* \end{aligned}$$

  - and:

$$\|\Delta x\|^2 = \|\Delta x_B\|^2 + 1 = \|B^{-1}Ne_j\|^2 + 1$$

- What is the problem here?
  - To compute $B^{-1}Ne_j$ for every $j \in N$ is time consuming
- To avoid the computational expense, we will compute $B^{-1}Ne_j$ for every $j \in N$ once at the start and then update the norms of these vectors at each iteration

- Let $$v_k = \left\| B^{-1} N e_k \right\|^2, \quad k \in N$$
- Let $\tilde{B} = BE$ is a new basis matrix, and we knew that:
$$E^{-1} = I - \frac{(\Delta x_B - e_i)e_i^T}{\Delta x_i}$$
- Then new value can be computed as:
$$\begin{aligned} \tilde{v}_k &= a_k^T \tilde{B}^{-T} \tilde{B}^{-1} a_k \\ &= a_k^T B^{-T} E^{-T} E^{-1} B^{-1} a_k \\ &= a_k^T B^{-T} \left( I - \frac{e_i(\Delta x_B - e_i)^T}{\Delta x_i} \right) \left( I - \frac{(\Delta x_B - e_i)e_i^T}{\Delta x_i} \right) B^{-1} a_k \end{aligned}$$

- (Continue) In the old iteration, we had:

$$v = B^{-T} e_i$$

- ,and

$$w = B^{-T} \Delta x_B$$

- The new value for $v$ can be obtained by:

$$\tilde{v}_k = a_k^T B^{-T} \left( I - \frac{e_i (\Delta x_B - e_i)^T}{\Delta x_i} \right) \left( I - \frac{(\Delta x_B - e_i) e_i^T}{\Delta x_i} \right) B^{-1} a_k$$

$$\tilde{v}_k = v_k - 2 \frac{a_k^T v (w - v)^T a_k}{\Delta x_i} + \left( a_k^T v \right)^2 \frac{\left\| \Delta x_B - e_i \right\|^2}{\left( \Delta x_i \right)^2}$$

# Homework. Steepest Edge Rule

- (Continuous the previous homework) Implement the steepest edge rule
- (Optional) Solve Klee-Minty problem instances ($n$ >> 10) using CPLEX callable library:

  **(a)** How many iterations are performed if using the largest coefficient rule to select the entering variable?

  **(b)** How many iterations are performed if using the steepest edge rule?

# Sensitivity Analysis

# Sensitivity Analysis

- Why need sensitivity (or postoptimality) analysis?
  - Most real-world problems are uncertain. Various possible data scenarios need to consider when making decisions. Also, data can fluctuate and changing from day to day, such as stock prices
  - The robustness of optimal solution needs to be verified via answering the following questions:
    - What if an objective coefficient is changed?
    - What if a right-hand-side value is changed?
    - What if a constraint coefficient is changed?
    - What if a new constraint is added?
    - What if a new variable is added?

# Primal Feasibility and Dual Feasibility

- The answer to these post-optimality questions always concerns with **primal feasibility** and **dual feasibility**
- That is, will data changes impact the feasibility of the optimal dictionary?

$$\zeta = \boxed{\zeta^*} - \boxed{z^*_N}^T x_N$$
$$x_B = \boxed{x^*_B} - B^{-1} N x_N$$

, and its solutions are computed by:

$$\zeta^* = c_B{}^T B^{-1} b$$
$$x^*_B = B^{-1} b$$
$$z^*_N = (B^{-1}N)^T c_B - c_N$$

# Chart of Possibilities

- For each cell, indicate whether it is "always" true, "sometimes" true, or "never" true:

| *If changing...* | *Primal feasible* | *Dual feasible* | *Both* | *Neither* |
|---|---|---|---|---|
| *Objective coefficient* | Always | Sometimes | | |
| *RHS* | Sometimes | Always | | |
| *Constraint coefficient* | | | Sometimes | |
| *Add constraints* | | | Sometimes | |
| *Add non-negative variables* | | | Sometimes | |

$$x_B^* = B^{-1}b$$
$$z_N^* = (B^{-1}N)^T c_B - c_N$$

# Changing the Objective Coefficients

- What if we change the objective coefficients from *c* to $\tilde{c}$ ?
- Recall,

$$\zeta^* = c_B{}^T B^{-1} b$$
$$x^*{}_B = B^{-1} b$$
$$z^*{}_N = (B^{-1} N)^T c_B - c_N$$

- Changing from *c* to $\tilde{c}$ requires us to recompute $\zeta^*$ *and* $z^*{}_N$
- We don't need to recompute $x^*{}_B$ . Why?
- Since the basis remains the same, the primal dictionary is still feasible

# Changing the Right Hand Side Values

- What if we change the right hand side values from *b* to $\tilde{b}$?
- Recall,

$$\zeta^* = c_B{}^T B^{-1} b$$

$$x^*{}_B = B^{-1} b$$

$$z^*{}_N = (B^{-1}N)^T c_B - c_N$$

- Changing the RHS values requires to recompute $\zeta^*$ *and* $x^*{}_B$
- The $z^*{}_N$ remains the same
- Since the dual dictionary is feasible, we can start from dual simplex method

# Changing Others

- What if changing both objective coefficient and RHS values?
- What if the entries in *B* and *N* change?
- In both cases, $\zeta^*, x^*_B, z^*_N$ all need to be recalculated
- As long as the new basis matrix *B* is nonsingular, we can make a new dictionary that preserves the old classification into basic and nonbasic variables

# Primal Feasibility and Dual Feasibility

- If the change does not affect primal and dual feasibilities, then the current basis is still optimal
- If the change does not affect primal feasibility, then we can use primal simplex to solve the problem
- If the change does not affect dual feasibility, then can use dual simplex to solve the problem

# Maintaining the Current Optimal Basis throughout Changes

- For the change on objective coefficients or RHS values, we often consider the range of changes to maintain the current basis optimal
- In such a case, we don't need to resolve the problem
- The range is only valid when **one** objective value or RHS value is changed

# Ranging

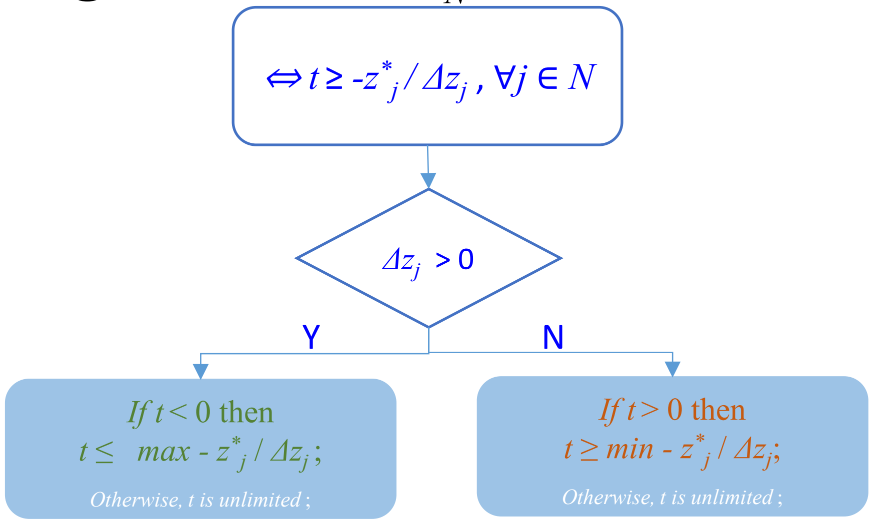
$$\overbrace{\text{Mix - } \Delta z_j / z_j^* \leq 0 \leq \text{Max - } \Delta z_j / z_j^*}^{t}$$

- The range of changing a objective coefficient is computed as follows:
- Suppose *c* is changed to $c + t\Delta c$
- $z^*_N$ is increased by $t\Delta z_N$
  ,where $\Delta z_N = (B^{-1}N)^T \Delta c_B - \Delta c_N$
- The current basis will remain dual feasible as long as: $z^*_N + t\Delta z_N \geq 0$
- For $t > 0$, the $z^*_N + t\Delta z_N \geq 0$ is valid if:

$$t \leq \left( \max_{j \in N} - \frac{\Delta z_j}{z^*_j} \right)^{-1}$$

- For $t < 0$ , the $z^*_N + t\Delta z_N \geq 0$ is valid if :

$$t \geq \left( \min_{j \in N} - \frac{\Delta z_j}{z^*_j} \right)^{-1}$$



# Example

$$\begin{aligned}
\text{maximize} \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 \le 5 \\
& 4x_1 + x_2 + 2x_3 \le 11 \\
& 3x_1 + 4x_2 + 2x_3 \le 8 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

- The optimal dictionary is:

$$\begin{aligned}
\xi &= 13 - 3x_2 - x_4 - x_6 \\
\hline
x_3 &= 1 + x_2 + 3x_4 - 2x_6 \\
x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\
x_5 &= 1 + 5x_2 + 2x_4
\end{aligned}$$

- The original coefficients are: $c = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix}^T$
- If we wish to change coefficient for $x_1$, then we put: $\boxed{\Delta c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T}$
- Based on the current solution (optimal dictionary), we have:

$$\Delta c_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \Delta c_N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Next, we need to recompute $\Delta z_N$:

$$\Delta z_N = (B^{-1}N)^T \Delta c_B - \Delta c_N = \begin{bmatrix} -1 & 2 & -5 \\ -3 & 2 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

- To ensure dual feasible ($z^*_N + t\Delta z_N \geq 0$), the following conditions must be true:

$$3 + 2t \geq 0, \qquad 1 + 2t \geq 0, \qquad and \qquad 1 - t \geq 0$$

- To hold these three inequalities, the value of *t* is between:

$$-\frac{1}{2} \leq t \leq 1$$

# Resolve the Problem Starting from Previous Basis

- What if the change exceeds the range?
  - If the change is relatively small, it is reasonable to expect the new optimal solution to be "near" the previous optimal solution
  - Solving the new problem using the previous basis may be easier than solving the problem from the scratch
  - It may only take a few pivots (using the primal simplex method. Why?) to reach the new optimal solution when starting from the previous optimal basis

# Dual Variable Solution ≡ Shadow Price

- Dual variable solutions tell us the *change in the objective value per unit change in the corresponding constraint bound* if the current basis remains optimal
- Can be used to identify the most critical constraint
- For this reason, the dual prices are also known as **shadow prices** or **marginal values**

- If the shadow price is zero, increasing the corresponding RHS value in primal will not lead to an increase in the objective value
- If the shadow price is great than zero, that means that **adding one unit increase** of the corresponding RHS value will increase the objective value by the **shadow price**

# Example

$$(\text{Primal})$$
$$\begin{aligned} \max \quad & 300x_1 + 500x_2 \\ \textit{subject} \quad & \textit{to} \\ & x_1 \le 40 \\ & 2x_2 \le 120 \\ & 3x_1 + 2x_2 \le 180 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$(\textit{Dual})$$
$$\begin{aligned} \min \quad & 40y_1 + 120y_2 + 180y_3 \\ \textit{subject} \quad & \textit{to} \\ & y_1 + 3y_3 \ge 300 \\ & 2y_2 + 2y_3 \ge 500 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

- The dual solutions are $y_1^* = 0, y_2^* = 150, y_3^* = 100$
- If we increase RHS value for the first constraint, the objective value won't change. However, if we increase one unit of the second RHS value, the objective values will increase by 150

# Homework

- Exercises 7.1 & 7.2 (sensitivity analysis)
- (Optional) Read the following paper

  Richard E. Wendell, (2004) Tolerance Sensitivity and Optimality Bounds in Linear Programming. Management Science 50(6):797-803.

  https://doi.org/10.1287/mnsc.1030.0221