Linear Programming

(5531)

Lecture 06 Duality Theory / Primal and Dual Simplex Methods

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November 5, 2020

Agenda

- Duality Theory (Chapter 5)
- Primal and Dual Simplex Methods (Chapters 5 and 6)

Duality Theory

Primal Dual Pairs

- LP problems come in primal-dual pairs
- The original LP is called **primal** *problem*, and associated with the LP is called its **dual** *problem*
- Example:

(Primal)

$$\begin{aligned} \text{maxmize} \quad & 4x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(Dual)

$$\begin{aligned} \text{minimize} \quad & y_1 + 3y_2 \\ \text{subject to} \quad & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

- The dual of the dual problem is the primal problem
- Every feasible solution for one of these two LPs (primal-dual pair) gives a **bound** on the optimal objective function value for the other

Conversion between Primal and Dual Problems

Maximization Problem	**Minimization Problem**
Constraint	*Variable*
≤	≥ 0
≥	≤ 0
=	Free
Variable	*Constraint*
≥ 0	**≥**
≤ 0	**≤**
Free	=

The Constraints for maximization problem and variables in the minimization are in *opposite direction*

If the constraint for the minimization problem is **≥,** the corresponding variables for the maximization problem is in the *same direction* **≥**

(*Primal*)

$$
\begin{aligned}
\max\ & c_1^T x_1 + c_2^T x_2 + c_3^T x_3 \\
\text{s.t. } & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \le b_1 \\
& A_{12}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \\
& A_{13}x_1 + A_{32}x_2 + A_{33}x_3 \ge b_3 \\
& x_1 \ge 0 \\
& x_2 \le 0 \\
& x_3 \ \text{free}
\end{aligned}
$$

(*Dual*)

$$
\begin{aligned}
\min\ & y_1^T b_1 + y_2^T b_2 + y_3^T b_3 \\
\text{s.t. } & y_1 \ge 0 \\
& y_2 \ \text{free} \\
& y_3 \le 0 \\
& y_1^T A_{11} + y_2^T A_{21} + y_3^T A_{31} \ge c_1^{\mathrm{T}} \\
& y_1^T A_{12} + y_2^T A_{22} + y_3^T A_{32} \le c_2^T \\
& y_1^T A_{13} + y_2^T A_{23} + y_3^T A_{33} = c_3^T
\end{aligned}
$$

The Lower Bounds of Objective Value

- Any primal feasible solution provides a lower bound on the optimal objective function value
- Example

$$\begin{array}{lrl} \text{maxmize} & 4x_1 + \quad x_2 + 3x_3 & \\ \text{subject to} & x_1 + 4x_2 \qquad & \leq 1 \\ & 3x_1 - \quad x_2 + x_3 & \leq 3 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

- ,where the feasible solution (x_1,x_2,x_3) = (1, 0, 0) with the objective value *4* is a *lower* bound of the optimal objective function value
- Another feasible solution (x_1,x_2,x_3) = (0, 0, 3) with objective value *9* is another lower bound

The Upper Bounds of Objective Value

$$\begin{aligned} \text{maxmize} \quad & 4x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 4x_2 \le 1 \\ & 3x_1 - x_2 + x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

- To find the upper bound, we multiple the first constraint by 2 and the second constraint by 3:

$$\begin{array}{rcr} 2\times(x_1+4x_2) & \le & 2\times(1) \\ +\,3\times(3x_1 - x_2 + x_3) & \le & 3\times(3) \\ \hline 11x_1+5x_2+3x_3 \le 11 & & \end{array}$$

- x_1,x_2 and x_3 are nonnegative. In addition, 4 < 11, 1 < 5, and 3 = 3

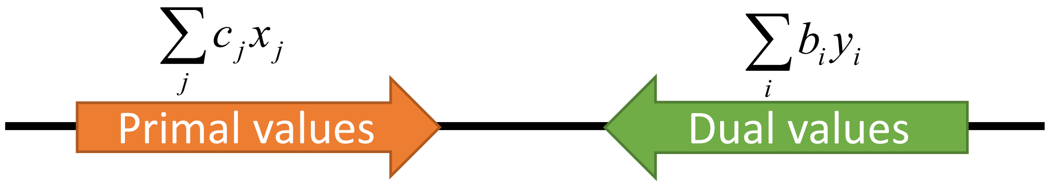
$$4x_1 + x_2 + 3x_3 \le 11x_1 + 5x_2 + 3x_3 \le 11$$

- Thus, **11 is a upper bound** of the optimal objective function value
- The optimal objective function value is between *9* and *11*

Important Duality Theorems

- Weak duality theorem
- Strong duality theorem
- Complementary slackness

Weak Duality Theorem



- **Theorem 5.1** If $(x_1, x_2, \ldots, x_n)$ is **feasible** for the primal and $(y_1, y_2, \ldots, y_m)$ is **feasible** for the dual, then:

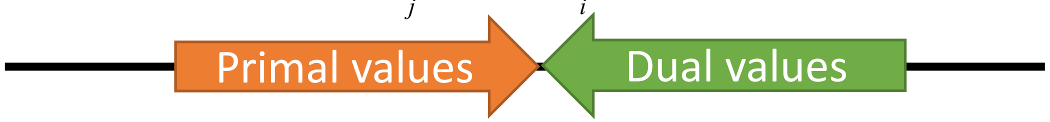
$$\sum_j c_j x_j \le \sum_i b_i y_i$$

Proof:

$$\begin{aligned}
\sum_j c_j x_j &\le \sum_j \Big(\sum_i y_i a_{ij}\Big) x_j \qquad \Big(\because \sum_i y_i a_{ij} \ge c_j\Big) \\
&= \sum_j \sum_i y_i a_{ij} x_j \\
&= \sum_i \sum_j (a_{ij} x_j) y_i \\
&\le \sum_i b_i y_i \qquad \Big(\because \sum_j a_{ij} x_j \le b_i\Big)
\end{aligned}$$

Strong Duality Theorem

$$\sum_j c_j x^*_j = \sum_i b_i y^*_i$$



- **Theorem 5.2** If the primal problem has an **optimal** solution,

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*)$$

- then the dual also has an **optimal** solution,

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*)$$

- such that:

$$\sum_j c_j x^*_j = \sum_i b_i y^*_i$$

The Proof of Strong Duality Theorem

$In \quad primal, \quad the \quad optimal \quad dictionary \quad can \quad written \quad as:$

$$\zeta = \zeta^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} d_i^* \omega_i$$

$$Let \quad d_i^* = -y_i^*, \quad and \quad \omega_i = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$Then, \quad \zeta = \zeta^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j)$$

$$\zeta = \zeta^* - \sum_{i=1}^{m} b_i y_i^* + \sum_{j=1}^{n} (c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*) x_j$$

$$Since \quad \zeta = \sum_{j=1}^{n} c_j x_j \quad (by \quad definition \quad of \quad primal \quad objective \quad function)$$

$The \quad following \quad equations \quad should \quad hold:$

$$\zeta^* = \sum_{i=1}^{m} b_i y_i^* \qquad (I)$$

$$c_j = c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*, \qquad j = 1,2,\dots,n \qquad (II)$$

$$Since \quad c_j^* \le 0 \quad (Optimality \quad condition \quad for \quad primal)$$

$$Thus, \quad c_j \le \sum_{i=1}^{m} a_{ij} y_i^*, \qquad j = 1,2,\dots,n \qquad (y_i^* \quad is \quad a \quad feasible \quad solution \quad for \quad dual)$$

$$Also, \zeta^* = \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^* \qquad (The \quad objective \quad value \quad for \quad dual \quad is \quad same \quad with \quad primal)$$

Complementary Slackness Theorem

- **Theorem 5.3** Suppose that $x = (x_1, x_2, \ldots, x_n)$ is a primal feasible solution and $y = (y_1, y_2, \ldots, y_m)$ is a dual feasible solution. Let $(\omega_1, \omega_2, \ldots, \omega_m)$ be the corresponding primal slack variables, and $(z_1, z_2, \ldots, z_n)$ be the corresponding dual slack variables. Then x and y are optimal for their respective problems *if only if*:

$$x_j z_j = 0, \qquad for \quad j = 1,2,\ldots,n$$

$$\omega_i y_i = 0, \qquad for \quad i = 1,2,\ldots,m$$

The Proof of Complementary Slackness Theorem

In the proof of weak duality we have:

$$\sum_j c_j x_j \le \sum_j (\sum_i y_i a_{ij}) x_j = \sum_i \sum_j (a_{ij} x_j) y_i \le \sum_i b_i y_i$$

Strong duality tells us that if x and y are optimal then $\sum_j c_j x_j = \sum_i b_i y_i$

This means that equality must hold for each of the above inequalities

Consider $\sum_j c_j x_j = \sum_j (\sum_i y_i a_{ij}) x_j$, we know that $x_j \ge 0$ and $c_j \le \sum_i y_i a_{ij}$

In order for the sums to be equal, we must have either $x_j = 0$ or $c_j = \sum_i y_i a_{ij}$

, the second equality would mean that $z_j = c_j - \sum_i y_i a_{ij} = 0$

Therefore, $x_j z_j = 0, \quad for \quad j = 1,2,\dots,n$

Similar analysis, we can show $\omega_i y_i = 0, \quad for \quad i = 1,2,\dots,m$

Summary of LP Fundamental Theorem and Duality Theorems

- Every LP problem is either having an optimal solution, infeasible or unbounded (*fundamental theorem of LP*)
- If the primal problem has an optimal solution, the dual problem also has an optimal solution (*strong duality theorem*)
- If the primal problem has an unbounded solution, then the dual problem is infeasible
  - By the *weak duality theorem*, a dual feasible solution would give a bound on the primal (dual) problem
- It is possible both primal and dual are simultaneously infeasible
- If the primal problem is infeasible, what about the dual?

Example of Both Primal and Dual Infeasible

(P)

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 - x_2 \le -1 \\ & -x_1 + x_2 \le -1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & -y_1 - y_2 \\ s.t. \quad & y_1 - y_2 \ge 1 \\ & -y_1 + y_2 \ge 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

The Primal and Dual Simplex Methods

Primal and Dual Simplex Methods

- In some cases, solving a dual dictionary is easier than solving original primal dictionary
- Example:

$$\begin{aligned} \text{maxmize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & -2x_1 - x_2 \le 4 \\ & -2x_1 + 4x_2 \le -8 \\ & -x_1 + 3x_2 \le -7 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- The initial dictionary is:

$$\begin{aligned} \zeta &= -x_1 - x_2 \\ \hline \omega_1 &= 4 + 2x_1 + x_2 \\ \omega_2 &= -8 + 2x_1 - 4x_2 \\ \omega_3 &= -7 + x_1 - 3x_2 \end{aligned}$$

All coefficients are negative, which variable should enter the basis?

This dictionary is not feasible, finding the initial feasible solution is not as easy as just setting nonbasic variables to zeros.

Negative Transpose Property

- Each *primal dictionary* has an associated *dual dictionary* that is simply the **negative transpose** of the primal dictionary
- Example:

$$
\begin{aligned}
\zeta &= \quad\;\; - \;\; x_1 - \;\; x_2 \\
\hline
\omega_1 &= \;\;\, 4 + 2x_1 + \;\; x_2 \\
\omega_2 &= -8 + 2x_1 - 4x_2 \\
\omega_3 &= -7 + \;\; x_1 - 3x_2
\end{aligned}
$$

- The negative transpose of the primal dictionary:

$$
\begin{bmatrix} 0 & -1 & -1 \\ 4 & 2 & 1 \\ -8 & 2 & -4 \\ -7 & 1 & -3 \end{bmatrix}
\rightarrow
\begin{bmatrix} 0 & -4 & 8 & 7 \\ 1 & -2 & -2 & -1 \\ 1 & -1 & 4 & 3 \end{bmatrix}
$$

Primal and Dual Dictionaries

- Introduce slack variables of $(\omega_1, \omega_2, \omega_3)$ for primal and (z_1, z_2) for dual
- Initial primal and dual dictionaries are:

(P)

$$
\begin{array}{rl}
\zeta = & \quad - x_1 - x_2 \\
\hline
\omega_1 = & 4 + 2x_1 + x_2 \\
\omega_2 = & -8 + 2x_1 - 4x_2 \\
\omega_3 = & -7 + x_1 - 3x_2
\end{array}
$$

(D)

$$
\begin{array}{rl}
-\xi = & \quad -4y_1 + 8y_2 + 7y_3 \\
\hline
z_1 = & 1 - 2y_1 - 2y_2 - y_3 \\
z_2 = & 1 - y_1 + 4y_2 + 3y_3
\end{array}
$$

Dual Simplex Method

- *Ideas:*
  - The dual simplex method starts from a **dual feasible solution**, and then pivots toward **primal feasibility** while remaining dual solutions are feasible
  - The optimal solution is obtained when **both primal and dual solutions are feasible**

Procedure of Dual Simplex Method

- **Step1 (Optimality test):** Check with the primal dictionary. If the basic solution is primal feasible, then the solution is optimal
- **Step 2 (Select an entering variable for dual):** Select a nonbasic dual variable with the largest coefficient (or any nonbasic dual variable with positive coefficient) to enter the basis
- **Step 3 (Determining a leaving variable for dual):** Perform ratio test to determine a leaving variable
- **Step 4 (Determining entering and leaving variables for primal):** The entering and leaving variables for primal are just the complementary to the selective dual variables
- **Step 5 (Updating primal and dual dictionaries):** Perform row operations for both primal and dual dictionaries

Example

- Given primal and dual dictionaries:

(P)

$$\begin{array}{l} \zeta = \quad - x_1 - x_2 \\ \hline \omega_1 = 4 + 2x_1 + x_2 \\ \omega_2 = -8 + 2x_1 - 4x_2 \\ \omega_3 = -7 + x_1 - 3x_2 \end{array}$$

(D)

$$\begin{array}{l} -\xi = -4y_1 + 8y_2 + 7y_3 \\ \hline z_1 = 1 - 2y_1 - 2y_2 - y_3 \\ z_2 = 1 - y_1 + 4y_2 + 3y_3 \end{array}$$

- Step 1 (Optimality test): Since the primal is infeasible, the dual solution is not optimal
- Step 2 (Entering variable): Select y_2 entering the basis, since its objective coefficient is the largest in the dual dictionary
- Step 3 (Leaving variable): Since the coefficient of y_2 in the second constraint is negative, we can only select z_1 leaving basis
- Step 4 (Entering and leaving variables for primal): ω_2 is the complementary to y_2 and x_1 is complementary to z_1, we will choose ω_2 leaving the basis and x_1 entering the basis in the primal dictionary
- Step 4 (Updating dictionaries): The new dictionaries are:

(P)

$$\begin{array}{l} \zeta = -4 - 0.5\omega_2 - 3x_2 \\ \hline \omega_1 = 12 + \omega_2 + 5x_2 \\ x_1 = 4 + 0.5\omega_2 + 2x_2 \\ \omega_3 = -3 + 0.5\omega_2 - x_2 \end{array}$$

(D)

$$\begin{array}{l} -\xi = 4 - 12y_1 - 4z_1 + 3y_3 \\ \hline y_2 = 0.5 - y_1 - 0.5z_1 - 0.5y_3 \\ z_2 = 3 - 5y_1 - 2z_1 + y_3 \end{array}$$

- **Second iteration:** y_3 enters basis and y_2 leaves basis in the dual dictionary. ω_3 leaves basis and ω_2 enters basis in primal
- The new dictionaries are:

(P)

$$
\begin{aligned}
\zeta &= -7 - \omega_3 - 4x_2 \\
\hline
\omega_1 &= 18 + 2\omega_3 + 7x_2 \\
x_1 &= 7 + \omega_3 + 3x_2 \\
\omega_2 &= 6 + 2\omega_3 + 2x_2
\end{aligned}
$$

(D)

$$
\begin{aligned}
-\xi &= 7 - 18y_1 - 7z_1 - 6y_2 \\
\hline
y_3 &= 1 - 2y_1 - z_1 - 2y_2 \\
z_2 &= 4 - 7y_1 - 3z_1 - 2y_2
\end{aligned}
$$

- The primal dictionary is feasible (surely, the dual is feasible). Thus, the optimal solution is obtained

Primal Simplex Method

- Similarly, the primal simplex method starts with primal feasibility, and then pivots toward dual feasibility while remaining primal feasible
- An optimal solution is obtained when both primal and dual are feasible

Procedure

- **Step1 (Optimality test):** Check with the dual dictionary. If the basic solution is dual feasible, then the solution is optimal
- **Step 2 (Select an entering variable for primal):** Select a nonbasic primal variable with the largest coefficient (or any nonbasic dual variable with positive coefficient) to enter the basis
- **Step 3 (Determining a leaving variable for primal):** Perform ratio test to determine a leaving variable
- **Step 4 (Determining entering and leaving variables for dual):** The entering and leaving variables for dual are just the complementary to the selective primal variables
- **Step 5 (Updating primal and dual dictionaries):** Perform row operations for both primal and dual dictionaries

The Primal and Dual Simplex Method in Matrix Notations

- The following slides use slightly different notations to illustrate the primal and dual simplex method:
  - For primal, the notations of slack variables become:

    $(x_1, x_2, \ldots, x_n, \omega_1, \omega_1, \ldots, \omega_n) \rightarrow (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m})$
  - For dual, the notations of slack(or surplus) variables become:

    $(z_1, z_2, \ldots, z_n, y_1, y_1, \ldots, y_m) \rightarrow (z_1, z_2, \ldots, z_n, z_{n+1}, z_{n+2}, \ldots, z_{n+m})$

- LP in matrix form: $$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$
- The constraints are $[B \quad N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b$, and then rewrite the vector of basic variables as:

$$x_B = B^{-1}b - B^{-1}Nx_N$$

- The objective function is $c^T x = \begin{bmatrix} c_B^T & c_N^T \end{bmatrix}\begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^T x_B + c_N^T x_N$, and then substitute x_B in the objective function:

$$\zeta = c_B^T x_B + c_N^T x_N = c_B^T (B^{-1}b - B^{-1}Nx_N) + c_N^T x_N = c_B^T B^{-1}b - ((B^{-1}N)^T c_B - c_N)^T x_N$$

- The basic feasible solution is obtained by setting $x^*_N = 0$ and $x^*_B = B^{-1}b$, and objective value is $\zeta^* = c_B^T B^{-1} b$
- The next dictionary is: , and associated dual dictionary is:

$$\begin{aligned} \zeta &= \zeta^* - z_N^{*T} x_N \\ x_B &= x_B^* - B^{-1}Nx_N \end{aligned} \qquad\qquad \begin{aligned} -\xi &= \zeta^* - (x_B^*)^T z_B \\ z_N &= z_N^* + (B^{-1}N)^T z_B \end{aligned}$$

Example of the Primal Simplex Method

- LP:

$$\begin{aligned} \text{maximize}\quad & 4x_1 + 3x_2 \\ \text{subject to}\quad & x_1 - x_2 \le 1 \\ & 2x_1 - x_2 \le 3 \\ & x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- The matrix A is:

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

- The initial basic and nonbasic indices are:

$$B = \{3,4,5\} \quad \text{and} \quad N = \{1,2\}$$

- , and their corresponding matrices of A are:

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

- The initial values of basic primal variables are:

$$x_B^* = b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

- The initial nonbasic dual variables are:

$$z_N^* = -c_N = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

The First Iteration

- Step 1. Check optimality and found some z_N^* are negative, the current solution is not optimal
- Step 2. Select entering variable. z_N^* is the most negative nonbasic dual variable, the entering index is j = 1
- Step 3. Calculate step direction for primal $\Delta x_B = B^{-1}Ne_j = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
- Step 4. Ratio test

$$t = \left(\max\left\{ \tfrac{1}{1}, \tfrac{2}{3}, \tfrac{0}{5} \right\} \right)^{-1} = 1$$

- Step 5. Select leaving variable: the first ratio is the maximum and this ratio corresponds to basis index 3, thus, i = 3
- Step 6. Calculate step direction for dual $\Delta z_N = -\left(B^{-1}N\right)^T e_i = -\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The First Iteration

- Step 7. Step size for dual $s = \frac{z_j^*}{\Delta z_j} = \frac{-4}{-1} = 4$

- Step 8. Update primal and dual basic variables

$$x_1^* = 1, \qquad x_B^* = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

$$z_3^* = 4, \qquad z_N^* = \begin{bmatrix} -4 \\ -3 \end{bmatrix} - 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

- Step 9. Update basic of nonbasic indices

$$B = \{1,4,5\} \quad \text{and} \quad N = \{3,2\}$$

- Corresponding these new sets of basic and nonbasic indices, the basic and nonbasic are:

$$B = \begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix} \end{array} \qquad N = \begin{array}{c} \begin{array}{cc} 3 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \end{array}$$

- , and the new basic primal variable and nonbasic dual variables are:

$$x_B^* = \begin{bmatrix} x_1^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \qquad z_N^* = \begin{bmatrix} z_3^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

The Second Iteration

- Step 1. Check optimality and found z_2^* is negative, the current solution is not optimal
- Step 2. Select entering variable. Since $z_2^* = -7$, we select the entering index is $j = 2$
- Step 3. Step direction for primal $\Delta x_B = B^{-1}Ne_j = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
- Step 4. Ratio test $t = \left(\max\left\{ \frac{-1}{1}, \frac{1}{1}, \frac{1}{5} \right\} \right)^{-1} = 1$
- Step 5. Select leaving variable: the second ratio is the maximum and this ratio corresponds to basis index 4, thus, $i = 4$

- Step 6. Calculate step direction for dual

$$\Delta z_N = -\left(B^{-1}N\right)^T e_i$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & 1 & \\ & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

- Step 7. Step size for dual

$$s = \frac{z_j^*}{\Delta z_j} = \frac{-7}{-1} = 7$$

- Step 8. Update basic primal variables and nonbasic dual variables

$$x_2^* = 1, \qquad x_B^* = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$z_4^* = 7, \qquad x_N^* = \begin{bmatrix} 4 \\ -7 \end{bmatrix} - 7 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

- Step 9. The new basic and nonbasic indices are $B = \{1,2,5\}$ and $N = \{3,4\}$

, their corresponding basic or nonbasic submatrices are:

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

, and the new basic primal and nonbasic dual variables are:

$$x_B^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad z_N^* = \begin{bmatrix} z_3^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}$$

The Third Iteration

- Step 1. Check optimality and found z_3^* is negative, the current solution is not optimal
- Step 2. Select entering variable. Since $z_3^* = -10$, we select the entering index is $j = 3$
- Step 3. Step direction for primal $\Delta x_B = B^{-1}Ne_j = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$
- Step 4. Ratio test $t = \left(\max \left\{ \frac{-1}{2}, \frac{-2}{1}, \frac{2}{4} \right\} \right)^{-1} = 2$
- Step 5. Select leaving variable: the third ratio is the maximum and this ratio corresponds to basis index 5, thus, $i = 5$

- Step 6. Calculate step direction for dual

$$\Delta z_N = -\left(B^{-1}N\right)^T e_i$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Step 7. Step size for dual

$$s = \frac{z_j^*}{\Delta z_j} = \frac{-10}{-2} = 5$$

- Step 8. Update basic primal variables and nonbasic dual variables

$$x_3^* = 2, \qquad x_B^* = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

$$z_5^* = 5, \qquad x_N^* = \begin{bmatrix} -10 \\ 7 \end{bmatrix} - 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- Step 9. The new basic and nonbasic indices are $B = \{1,2,3\}$ and $N = \{5,4\}$

, their corresponding basic or nonbasic submatrices are:

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

, and the new basic primal and nonbasic dual variables are:

$$x_B^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \qquad z_N^* = \begin{bmatrix} z_5^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The Fourth Iteration

- Step 1. Since x_N^* are all nonnegative, the current solution is optimal. The optimal objective function value is

$$\zeta^* = 4x_1^* + 3x_2^* = 31$$

Summary of Primal and Dual Simplex Methods

Primal Simplex	Dual Simplex
Suppose $x_B^* \geq 0$	Suppose $z_N^* \geq 0$
while $(z_N^* \ngeq 0)$ {	while $(x_B^* \ngeq 0)$ {
pick $j \in \{j \in N : z_j^* < 0\}$	pick $i \in \{i \in B : x_i^* < 0\}$
$\Delta x_B = B^{-1} N e_j$	$\Delta z_N = -(B^{-1}N)^T e_i$
$t = \left(\max_{i \in B} \frac{\Delta x_i}{x_i^*} \right)$	$s = \left(\max_{j \in N} \frac{\Delta z_j}{z_j^*} \right)$
pick $i \in \operatorname{argmax}_{i \in B} \frac{\Delta x_i}{x_i^*}$	pick $j \in \operatorname{argmax}_{j \in N} \frac{\Delta z_j}{z_j^*}$
$\Delta z_N = -(B^{-1}N)^T e_i$	$\Delta x_B = B^{-1} N e_j$
$s = \frac{z_j^*}{\Delta z_j}$	$t = \frac{x_i^*}{\Delta x_i}$
$x_j^* \leftarrow t$	$x_j^* \leftarrow t$
$x_B^* \leftarrow x_B^* - t\Delta x_B$	$x_B^* \leftarrow x_B^* - t\Delta x_B$
$z_i^* \leftarrow s$	$z_i^* \leftarrow s$
$z_N^* \leftarrow z_N^* - s\Delta x_N$	$z_N^* \leftarrow z_N^* - s\Delta x_N$
$B \leftarrow B \setminus \{i\} \cup \{j\}$	$B \leftarrow B \setminus \{i\} \cup \{j\}$
}	}

Choose Primal Simplex Method, Dual Simplex Method, or Other Approach?

- If all b_i >= 0 and all c_N <= 0, then the initial dictionary is optimal
- If all b_i >= 0 and some c_N <= 0, then the initial dictionary is primal feasible. We can start solving the LP with the primal simplex method
- If some b_i >= 0 and all c_N <= 0, then the initial dictionary is dual feasible. We can solve the LP with the dual simplex method
- If both b and c have negative components, then the initial dictionaries of primal and dual are infeasible. We need to employ the two-phase method

Homework

- (Continuous the previous homework) Print out the solutions of primal variables, primal slack variables, dual variables, and dual slack variables for each simplex iteration
- Exercise 5.7
- Exercise 6.6