Linear Programming

(5531)

Week 03 Simplex Methods

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Introduction of Simplex Methods

- An iterative approach to solve general LP problems
- All variables are either **Basic Variables (*B*)** or **Non-basic Variables (*N*)**
  - Another name is known as the *dependent varia*ble or *independent variable*
  - IF $x_b > 0$, then x is called a *nondegenerate basic feasible solution*. If at leas one element of x_b is zero, *then* x is call *degenerate basic feasible solution*
  - In general, the number of basic feasible solutions is less than or equal to $\binom{n+m}{m}$
- Algebraically:

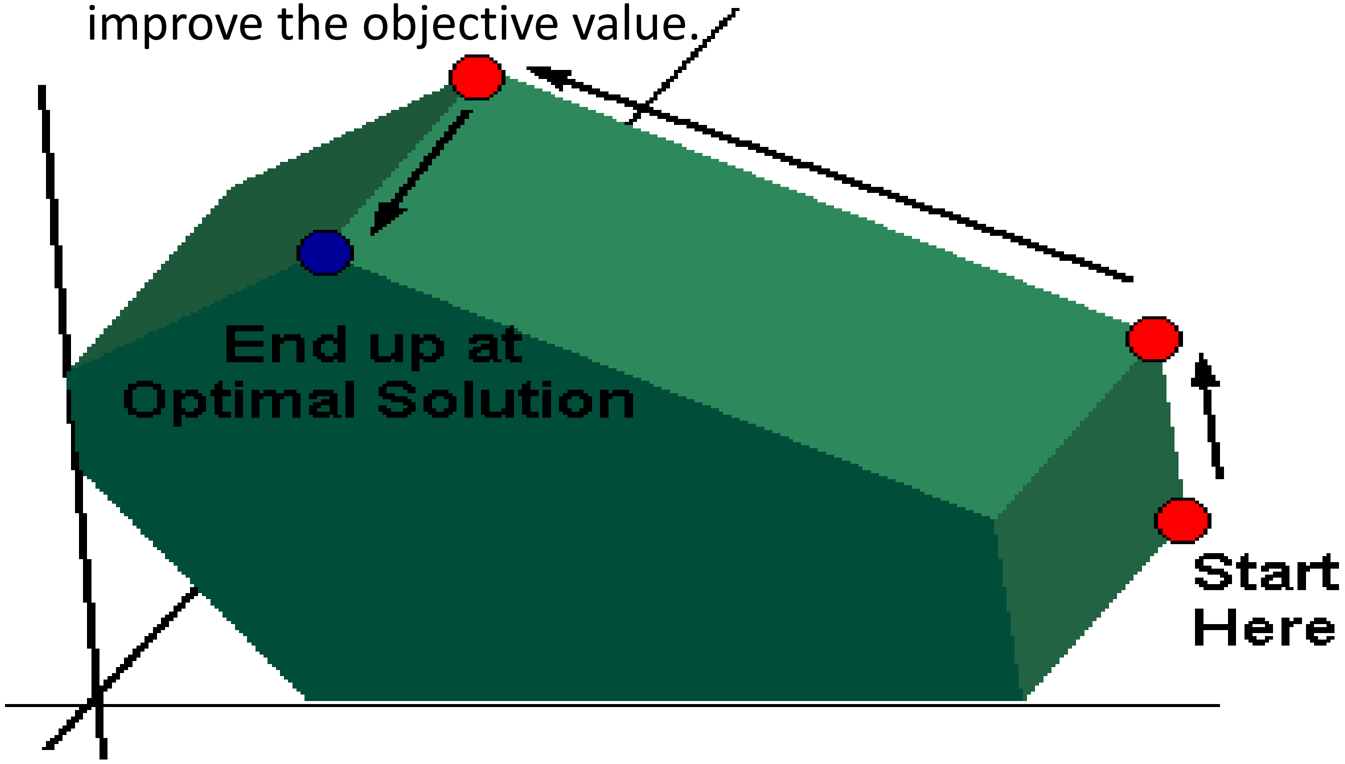
  Set **Non-basic Variable** to one of it's **bound**, and solve for **Basic Variables** to satisfy **constraints**
- Geometrically:
  - The set of feasible solution is a **polyhedral**
  - **Basic feasible solutions (BFSs)** are **extreme points** of the polyhedron set
  - Simplex algorithm only consider to BFSs (i.e., extreme points)

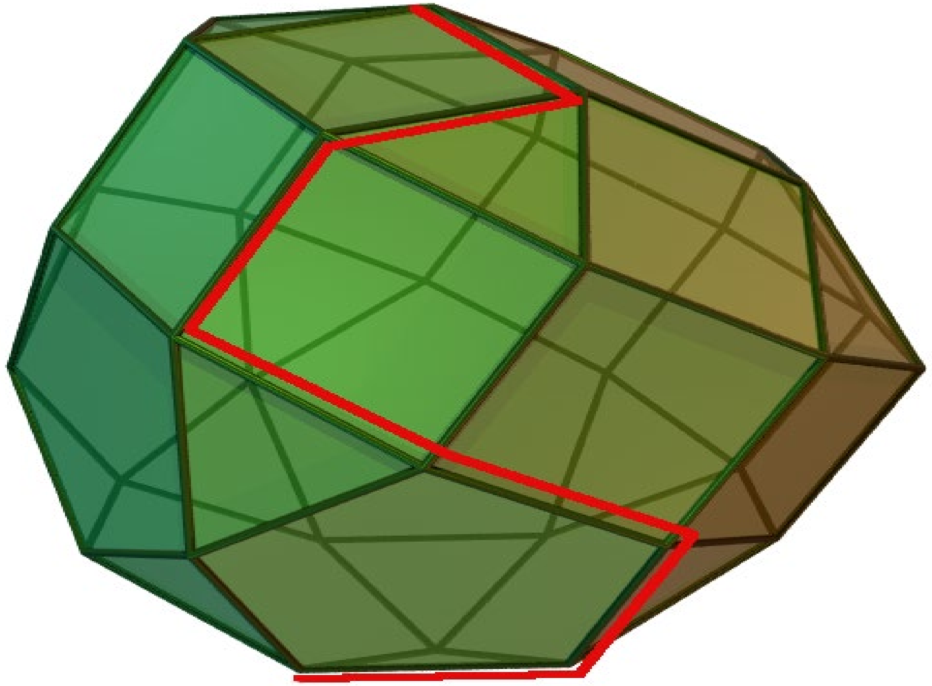
(Digression) Extreme Points

- **Definition (Extreme point)**: If there do not exist vectors x_1, x_2 in X, $x_1 \neq x$ and $x_2 \neq x$, and $x \in X$, and a scalar a in (0,1), such that $x = a\, x_1 + (1\text{-}a)\, x_2$, then we call x is an extreme point of X
- **Examples:**
  - Vertexes of a straight line
  - Vertexes of a rectangular
  - Any point lied on the perimeter of a circle

Graphical Illustration of Simplex Methods

At every step (iteration), the basic solution move to an adjacent vertex, and ideally, the new solution will improve the objective value.





images sources Keely L. Croxton and Rex Kincaid

Simplex Algorithm Steps and Notations

1. Convert LP to **Standard form**:

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i=1,2,\ldots,m \\ & && x_j \geq 0 \quad j=1,2,\ldots,n \end{aligned}$$

2. Add **slack variables**:

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$\omega_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad i=1,2,\ldots,m$$

3. Find an initial solution

4. Write the **dictionary**:

$$\zeta = \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j$$

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad i \in B$$

5. Simplex **pivoting**:

- Select a nonbasic variable, x_k, entering basis:

  pick k from $\{j \in N, \ \bar{c}_j > 0\}$

- Select a basic basis , x_l, leaving basis (**ratio test**):

  pick l from $\{i \in B: \ \bar{a}_{ik} > 0 \text{ and } \bar{b}_i / \bar{a}_{ik} \text{ is minimal}\}$

- Determine value for the entering variable:

$$x_k = \min_{\forall i \in B, \bar{a}_{ik} > 0} \frac{\bar{b}_i}{\bar{a}_{ik}}$$

Standard Form

- Requirements:
  - Maximize the objective function
  - All constraints are $\leqq$
  - All variables are non-negative
- Can every LP convert to standard form?

An Example

- Original LP:

$$\min \ -3x_1 - 2x_2$$

s.t.

$$x_1 + 2x_2 \quad <= 5$$
$$-2x_1 - 4x_2 >= -1$$
$$x_1 >= 0,\ x_2 <= 0$$

- LP in Standard Form:

$$\max \ 3x_1 - 2y_2$$

s.t.

$$x_1 - 2y_2 \quad <= 5$$
$$2x_1 - 4y_2 \quad <= 1$$
$$x_1 >= 0,\ y_2 \ >= 0$$

- If x_2 is unrestricted, what would be the corresponding standard form?

Strict Inequality (< or >)

- How to handle strict inequalities (< or >) in LP?
- For example, $x_1 > 5$
- We may approximate $x_1 > 5$ by $x_1 >= 5 + e$, where e = 10^-6

Adding Slack Variables

1. LP in Standard Form:

$$\max\ 3x_1 - 2y_2$$
s.t.
$$x_1 - 2y_2 <= 5$$
$$2x_1 - 4y_2 <= 1$$
$$x_1 >= 0,\ y_2 >= 0$$

2. Add slack variables (ω_1 & ω_2) to constraints, and change constraints to equations :

$$\max\ 3x_1 - 2y_2$$
s.t.
$$x_1 - 2y_2 + \omega_1 = 5$$
$$2x_1 - 4y_2 + \omega_2 = 1$$
$$x_1 >= 0,\ y_2 >= 0$$
$$\omega_1 >= 0,\ \omega_2 >= 0$$

3. Rewrite the problem:

$$\max\ 3x_1 - 2y_2$$
s.t.
$$\omega_1 = 5 - x_1 + 2y_2$$
$$\omega_2 = 1 - 2x_1 + 4y_2$$
$$x_1 >= 0,\ y_2 >= 0$$
$$\omega_1 >= 0,\ \omega_2 >= 0$$

Dictionary

- Induce a notation, ζ, representing to the objective function value:

$$\zeta = 3x_1 + 2y_2$$

- The system of equations is so call a **Dictionary**:

$$\zeta = 3x_1 + 2y_2$$
$$\omega_1 = 5 - x_1 + 2y_2$$
$$\omega_2 = 1 - 2x_1 + 4y_2$$

Dictionaries, Basic Variables, Nonbasic Variables and Basic Feasible Solutions

- The dictionary is a system of equations. For example :

If the LP is:

$$\begin{aligned} \text{maximize} \quad & 5x_1+4x_2+3x_3 \\ \text{subject to} \quad & \\ & 2x_1+3x_2+x_3 \le 5 \\ & 4x_1+x_2+2x_3 \le 11 \\ & 3x_1+4x_2+2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

, then the initial dictionary is:

$$\begin{aligned} \zeta &= 5x_1+4x_2+3x_3 \\ \omega_1 &= 5-2x_1-3x_2-x_3 \\ \omega_2 &= 11-4x_1-x_2-2x_3 \\ \omega_3 &= 8-3x_1-4x_2-2x_3 \end{aligned}$$

- **Basic variables**: variables appear on the left in a dictionary except the first row (ω_1, ω_2 and ω_3)
- **Nonbasic variables**: variables on the right (x_1, x_2 and x_3)

- **Basic Feasible Solutions (BFSs)**
  - Nonbasic variables are set to zeros: $x_1=0$, $x_2=0$, $x_3=0$
  - Basic variables can be calculated by replacing nonbasic variables with zeros in the equations: $\omega_1 = 5$, $\omega_2 = 11$, $\omega_3 = 8$

Simplex Pivot (1/7)

$$\zeta = \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j$$

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad i \in B$$

- Finding a better solution: If the current BFS is not optimal, then choose an **entering variable** which can be any nonbasic variable with $\bar{c}_j \geq 0$
- Let the entering variable be x_k. It will enter the basis and force another variable to leave the basis
- Preserving feasibility: The **leaving variable** is chosen so that the resulting basic solution is in fact a BFS
- The value of x_k will increase from 0 to some positive value while all the other nonbasic variables remain at zero
- How much can x_k be increased?

Simplex Pivot (2/7)

- **Ratio test**:
  - Choose leaving variable x_l such that $\bar{b}_i / \bar{a}_{ik}$ is minimal (or $\bar{a}_{ik} / \bar{b}_i$ is maximal)
  - Therefore we can obtain the largest increase in entering variable:

$$x_k = \min_{\forall i \in B} \frac{\bar{b}_i}{\bar{a}_{ik}}$$

  - *Note:* The choice of entering and leaving variables may not be unique. So the simplex algorithm is really a family of algorithms, and each algorithm has its own pivot rules to select entering and leaving variables
- After choosing the entering and leaving variables, solve the leaving variable's equation for the entering variable, and then replace the entering variable at other equations

Simplex Pivot - Example (3/7)

- Given the LP problem:

$$
\begin{aligned}
&\textit{maximize} \quad 5x_1 + 4x_2 + 3x_3 \\
&\textit{subject to} \\
&2x_1 + 3x_2 + x_3 \le 5 \\
&4x_1 + x_2 + 2x_3 \le 11 \\
&3x_1 + 4x_2 + 2x_3 \le 8 \\
&x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Simplex Pivot - Example (4/7)

- The initial dictionary is:

$$
\begin{array}{l}
\zeta = 5x_1 + 4x_2 + 3x_3 \\
\hline
\omega_1 = 5 - 2x_1 - 3x_2 - x_3 \\
\omega_2 = 11 - 4x_1 - x_2 - 2x_3 \\
\omega_3 = 8 - 3x_1 - 4x_2 - 2x_3
\end{array}
$$

- Initial feasible solution is:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \omega_1 = 5,\ \omega_2 = 11,\ \omega_3 = 8$$

- What is the objective value?
- Is this an optimal solution?
- If not, which variable should enter basis and which variable should leave basis?

Simplex Pivot - Example (5/7)

- Dictionary:

$$\begin{array}{l} \zeta = 5x_1 + 4x_2 + 3x_3 \\ \hline \omega_1 = 5 - 2x_1 - 3x_2 - x_3 \\ \omega_2 = 11 - 4x_1 - x_2 - 2x_3 \\ \omega_3 = 8 - 3x_1 - 4x_2 - 2x_3 \end{array}$$

- Choose x_1 entering basis (5 > 4 > 3)
- Choose ω_1 leaving basis (5/2 < 8/3 < 11/4)
- Row operations:

Step1. Move x_1 to the left side and ω_1 to the right side in the second row

$$x_1 = \frac{5}{2} - \frac{1}{2}\omega_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

Step2. Replace x_1 with $\frac{5}{2} - \frac{1}{2}\omega_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$ for the other equations

Simplex Pivot - Example (6/7)

- After x_1 entering basis and ω_1 leaving basis, the new dictionary becomes:

$$\zeta = 12.5 - 2.5\,\omega_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5\omega_1 - 1.5x_2 - 0.5x_3$$
$$\omega_2 = 1 + 2\omega_1 + 5x_2$$
$$\omega_3 = 0.5 + 1.5\omega_1 + 0.5x_2 - 0.5x_3$$

- The new solution is:

$$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, \omega_1 = 0, \omega_2 = 1, \omega_3 = \frac{1}{2}$$

- The objective value has improved from zero to $\underline{12.5}$

Simplex Pivot - Example (7/7)

- After x_3 entering basis and ω_3 leaving basis, the new dictionary becomes:

$$\zeta = 13 - \omega_1 - 3x_2 - \omega_3$$
$$x_1 = 2 - 2\omega_1 - 2x_2 + \omega_3$$
$$\omega_2 = 1 + 2\omega_1 + 5x_2$$
$$x_3 = 1 + 3\omega_1 + x_2 - 2\omega_3$$

- The new solution is:

$$x_1 = 2, x_2 = 0, x_3 = 1, \omega_1 = 0, \omega_2 = 1, \omega_3 = 0$$

- The objective value has improved from zero to $\underline{13}$
- Can we further improve the objective value ?

Optimality Condition (1/2)

- **Theorem**

The current solution is optimal if $\bar{c}_j \leq 0$ for all $j \in N$

- **Proof:**

The current dictionary is a set of equations satisfied by every feasible solution to the LP. In particular, every feasible solution to the LP satisfies $\zeta = \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j$. By nonnegativity, no value of x_j can be less than zero, so the best objective value we can hope for is ζ . The current BFS achieves this objective value and is therefore optimal.

Optimality Condition – An Example (2/2)

- Can you find an independent variable from the objective function with an increasing in its value can produce a corresponding increasing in ζ ?

$$\zeta = 13 - \omega_1 - 3x_2 - \omega_3$$
$$x_1 = 2 - 2\omega_1 - 2x_2 + \omega_3$$
$$\omega_2 = 1 + 2\omega_1 - 5x_2$$
$$x_3 = 1 + 3\omega_1 + x_2 - 2\omega_3$$

Summary of Simplex Algorithm

1. Convert LP to standard form:

maximize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,2,\ldots,m$

$$x_j \ge 0 \quad j = 1,2,\ldots,n$$

2. Add slack variables:

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$\omega_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad i = 1,2,\ldots,m$$

3. Find an initial solution

4. Write down dictionary:

$$\zeta = \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j$$

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad i \in B$$

5. Pick a nonbasic variable entering basis:

$$k \text{ from } \{j \in N : \bar{c}_j > 0\}$$

6. Pick a basic variable leaving basis and determine value for the entering variable:

$$x_k = \min_{\forall i \in B,\, \bar{a}_{ik} > 0} \bar{b}_i / \bar{a}_{ik}$$

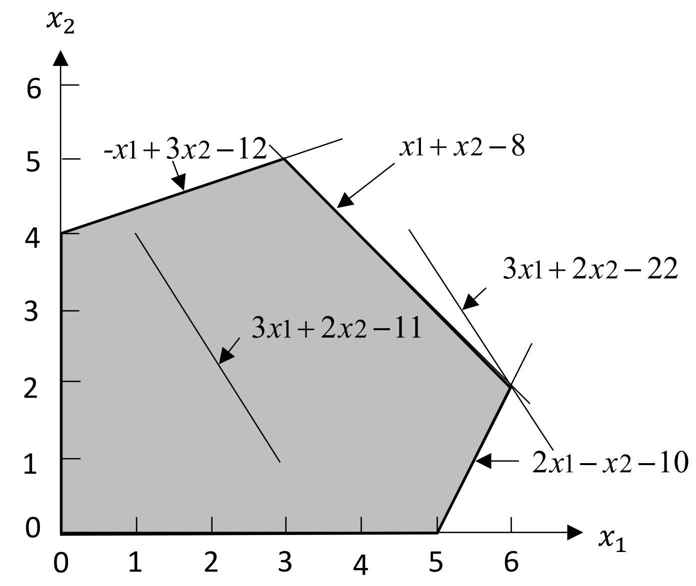
7. Row operations

9. If the current solution is optimal ($\bar{c}_j \le 0$ for all $j \in N$) then stop; otherwise go back to step 5

Example

(LP)

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \le 12 \\ & x_1 + x_2 \le 8 \\ & 2x_1 - x_2 \le 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$



- Standard form
- Slack variable
- Initial feasible solution
- Dictionary / basic variable / nonbasic variable
- The optimal solution

Homework

- (Handwriting exercises)
  - Draw the feasible region for exercise 2.2.
  - Show how the simplex algorithm would pivot to find the optimal solution. Write down each dictionary, show basic and nonbasic variables, variable values, and objective function value.
  - Show the vertex corresponding to the dictionary and the path followed by the solution.
  - Exercise 2.15
  - Exercise 2.18

Next Week

- We will discuss some special cases when doing simplex pivot. Remember the largest possible increase in the entering variable is:

$$x_k = \min_{\forall i \in B, \bar{a}_{ik} > 0} \bar{b}_i / \bar{a}_{ik}$$

- What happens if $\bar{b}_i = 0$?