Simplifications
of
Context-Free Grammars

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A Substitution Rule

Equivalent grammar

$$S \to aB$$
$$A \to aaA$$
$$A \to abBc$$
$$B \to aA$$
$$B \to b$$

Substitute
$$B \to b$$

$$S \to aB \mid ab$$
$$A \to aaA$$
$$A \to abBc \mid abbc$$
$$B \to aA$$

$$S \to aB \mid ab$$

$$A \to aaA$$

$$A \to abBc \mid abbc$$

$$B \to aA$$

Substitute

$$B \to aA$$

$$S \to \cancel{aB} \mid ab \mid aaA$$

$$A \to aaA$$

$$A \to \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general: $A \to xBz$

$$B \to y_1$$

Substitute $B \to y_1$

$$A \to xBz \mid xy_1z$$

equivalent grammar

Nullable Variables

$\varepsilon -\text{production}:$ $\quad X \rightarrow \varepsilon$

Nullable Variable: $\quad Y \Rightarrow \ldots \Rightarrow \varepsilon$

Example:

$$S \rightarrow aMb$$

$$M \rightarrow aMb$$

$$M \rightarrow \varepsilon$$

Nullable variable

$\varepsilon -\text{production}$

Removing $\varepsilon-\text{productions}$

$$S \to aMb$$

$$M \to aMb$$

$$\cancel{M \to \varepsilon}$$

Substitute $M \to \varepsilon$

$$S \to aMb \mid ab$$

$$M \to aMb \mid ab$$

After we remove all the $\varepsilon-\text{productions}$
all the nullable variables disappear.
(except for the start variable.)

Unit-Productions

Unit Production: $X \rightarrow Y$

(A single variable in both sides)

Example:

$$S \rightarrow aA$$

$$A \rightarrow a$$

$$\boxed{A \rightarrow B}$$

$$\boxed{B \rightarrow A}$$

Unit Productions

$$B \rightarrow bb$$

Removal of unit productions:

$$S \to aA$$
$$A \to a$$
$$\cancel{A \to B}$$
$$B \to A$$
$$B \to bb$$

Substitute

$$A \to B$$

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to A \mid B$$
$$B \to bb$$

Unit productions of form $X \rightarrow X$

can be removed immediately.

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$B \rightarrow A \mid \cancel{B}$$

$$B \rightarrow bb$$

Remove

$$B \rightarrow B$$

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$B \rightarrow A$$

$$B \rightarrow bb$$

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$\cancel{B \rightarrow A}$$

$$B \rightarrow bb$$

Substitute

$$B \rightarrow A$$

$$S \rightarrow aA \mid aB \mid aA$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Remove repeated productions

$$S \to aA \mid aB \mid \cancel{aA}$$

$$A \to a$$

$$B \to bb$$

Final grammar

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to bb$$

Useless Productions

$$S \to aSb$$

$$S \to \varepsilon$$

$$S \to A$$

$$A \to aA$$ Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \rightarrow A$$

$$A \rightarrow aA$$

$$A \rightarrow \varepsilon$$

$$B \rightarrow bA$$ Useless Production

Not reachable from S

In general:

If there is a derivation

$$S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w \quad \in L(G)$$

consists of terminals

Then, variable A is useful.

Otherwise, variable A is useless.

A production $A \to x$ is useless
if any of its variables is useless

$S \to aSb$

$S \to \varepsilon$ Productions

Variables $S \to A$ useless

useless $A \to aA$ useless

useless $B \to C$ useless

useless $C \to D$ useless

Removing Useless Variables and Productions

Example Grammar:

$$S \rightarrow aS \mid A \mid C$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$C \rightarrow aCb$$

First: Find all variables that can produce strings with only terminals or ε

(possible useful variables)

$$S \to aS \mid A \mid C$$

$$A \to a$$

$$B \to aa$$

$$C \to aCb$$

Round 1: $\{A, B\}$

(the right hand side of production that has only terminals)

Round 2: $\{A, B, S\}$

(the right hand side of a production has terminals and variables of previous round)

This process can be generalized

Then, remove productions that use variables other than $\{A, B, S\}$

$$S \to aS \mid A \mid \cancel{C}$$

$$A \to a$$

$$B \to aa$$

$$\cancel{C \to aCb}$$

$$\Rightarrow$$

$$S \to aS \mid A$$

$$A \to a$$

$$B \to aa$$

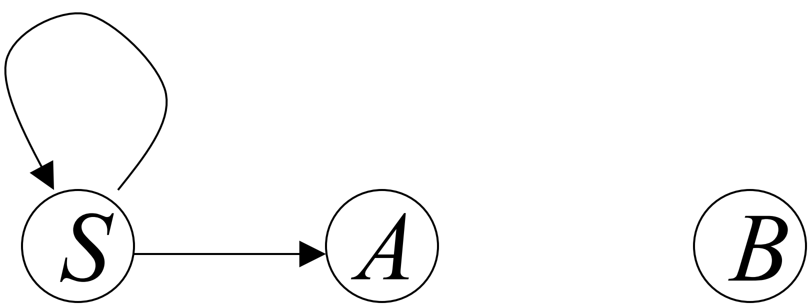
Second: Find all variables reachable from S

Use a Dependency Graph where nodes are variables

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

$$B \rightarrow aa$$



unreachable

Keep only the variables reachable from S

Final Grammar

$$S \rightarrow aS \mid A$$
$$A \rightarrow a$$
$$\cancel{B \rightarrow aa}$$

$$S \rightarrow aS \mid A$$
$$A \rightarrow a$$

Contains only useful variables

Removing All

Step 1: Remove Nullable Variables

Step 2: Remove Unit-Productions

Step 3: Remove Useless Variables

This sequence guarantees that unwanted variables and productions are removed.

Normal Forms for Context-free Grammars

Chomsky Normal Form

Each production has form:

$$A \to BC \qquad \text{or} \qquad A \to a$$

variable variable terminal

(these may not be the start variables)

($S \to \varepsilon$ is allowed where S is the start variable)

Examples:

$$S \to AS$$

$$S \to a$$

$$A \to SA$$

$$A \to b$$

Chomsky
Normal Form

$$S \to AS$$

$$S \to AAS$$

$$A \to SA$$

$$A \to aa$$

Not Chomsky
Normal Form

Conversion to Chomsky Normal Form

Example:

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

Not Chomsky Normal Form

We will convert it to Chomsky Normal Form

Introduce new variables for the terminals:

$$T_a, T_b, T_c$$

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to ABT_a$$

$$A \to T_a T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

Introduce new intermediate variable V_1 to break first production:

$$\boxed{S \to ABT_a}$$

$$A \to T_a T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} S &\to AV_1 \\ V_1 &\to BT_a \end{aligned}}$$

$$A \to T_a T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Longrightarrow$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aV_2$$
$$V_2 \to T_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to AV_1$$

$$V_1 \to BT_a$$

$$A \to T_aV_2$$

$$V_2 \to T_aT_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

In general:

From any context-free grammar
not in Chomsky Normal Form,
we can obtain:

an equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

Nullable variables

Unit productions

(Useless variables optional)

Then, for every symbol a :

New variable: T_a

Add production $T_a \rightarrow a$

In productions with length at least 2
replace a with T_a

Productions of form $A \rightarrow a$
do not need to change!

Replace any production $A \to C_1 C_2 \cdots C_n$

with
$$
\begin{aligned}
A &\to C_1 V_1 \\
V_1 &\to C_2 V_2 \\
&\cdots \\
V_{n-2} &\to C_{n-1} C_n
\end{aligned}
$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

Observations

- Chomsky normal forms are good for parsing and proving theorems.

- It is easy to find the Chomsky normal form for any context-free grammar.

$\mathcal{E}$

Greinbach Normal Form

All productions have form:

$$A \to a\, V_1 V_2 \cdots V_k \qquad\qquad k \geq 0$$

symbol

variables

Examples:

$$S \to cAB$$

$$A \to aA \mid bB \mid b$$

$$B \to b$$

Greinbach
Normal Form

$$S \to abSb$$

$$S \to aa$$

Not Greinbach
Normal Form

Conversion to Greinbach Normal Form:

$$S \to abSb$$

$$S \to aa$$

$$S \to aT_bST_b$$

$$S \to aT_a$$

$$T_a \to a$$

$$T_b \to b$$

Greinbach

Normal Form

Observations

- Greinbach normal forms are very good for parsing strings (better than Chomsky Normal Forms).

- However, it is difficult to find the Greinbach normal of a grammar.