Partial Fractions

Ex $\quad I = \int \frac{(7x+2)\,dx}{x^2+x-2} = \int \frac{A}{x+2}\,dx + \int \frac{B\,dx}{x-1}$

$$\frac{7x+2}{x^2+x-2} = \frac{7x+2}{(x-1)(x+2)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$-1 \quad 2$

$(x-1) \qquad (x+2)$

$$7x+2 = A(x-1) + B(x+2)$$

$$x=1: \quad 9 = 3B \Rightarrow B = 3$$

$$x=-2: \; -12 = -3A \Rightarrow A = 4$$

$$I = \int \frac{4}{x+2}\,dx + \int \frac{3}{x-1}\,dx$$

$$= 4\ln(x+2) + 3\ln(x-1) + C$$

$= \ln(x+2)^4 + \ln(x-1)^3$

Ex $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$

$$\begin{array}{r|l} \cancel{x^2}+1 & x^2-1 \\ \hline \mp \cancel{x^2} \mp 1 & 1 \\ \hline 2 & \end{array}$$

$$= \int \left(1 + \frac{2}{x^2-1}\right) dx = \int dx + 2\int \frac{dx}{x^2-1}$$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1)$$

$x=1$: $1 = 2A \implies A = 1/2$

$x=-1$: $1 = -2B \implies B = -1/2$

$$= \int dx + 2\cdot\frac{1}{2}\int\frac{dx}{x-1} - 2\cdot\frac{1}{2}\int\frac{dx}{x+1}$$

$$= x + \ln|x-1| - \ln|x+1| + C = x + \ln\left|\frac{x-1}{x+1}\right| + C$$

Repeated real roots:

Ex $$I = \int \frac{dx}{x(x+1)^2}$$

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$(x+1)^2 \qquad x(x+1) \qquad x$

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=0: \quad 1 = A$

$x=-1: \quad 1 = -C \Rightarrow C = -1$

$x=1: \quad 1 = A(2)^2 + B\cdot 2 + C \Rightarrow B = -1$

$$I = \int \frac{dx}{x} - \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$

$x+1 = u$

$dx = du$

$$I = \ln|x| - \ln|x+1| - \int u^{-2}\,du = \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C$$

Complex roots:

Ex $\quad I = \displaystyle\int \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$(x^2+1) \qquad\qquad (x+1)$

$$1 = A(x^2+1) + (Bx+C)(x+1)$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$0 \cdot x^2 + 0x + 1 = (A+B)x^2 + (B+C)x + (A+C)$$

$A+B=0 \Rightarrow A=-B$

$B+C=0 \Rightarrow B=-C$

$A+C=1 \Rightarrow A=C$

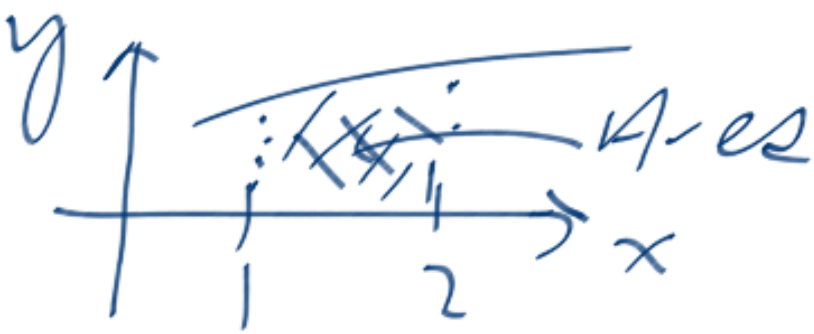
$2A=1 \Rightarrow A=1/2, \; C=1/2, \; B=-1/2$

$$I = \frac{1}{2}\int\frac{dx}{x+1} - \frac{1}{2}\cdot\frac{1}{2}\int\frac{\overset{du}{2x\,dx}}{\underset{u}{x^2+1}} + \underbrace{\int\frac{dx}{x^2+1}}_{\text{Arctan}\,x}$$

$$I = \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \text{Arctan}\,x + C_1$$

$$I = \ln\frac{(x+1)^{1/2}}{(x^2+1)^{1/4}} + \text{Arctan}\,x + C_1$$

Ex Find the area of the region between $y = \dfrac{7x+3}{x^2+x}$ and the x-axis, for $1 \leq x \leq 2$.



$$\frac{7x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$(x+1) \qquad x$$

$$7x+3 = A(x+1) + Bx$$

$x=0$: $+3 = A$

$x=-1$: $-4 = -B \Rightarrow B = 4$

$$A = \int_1^2 \frac{7x+3}{x(x+1)}\,dx = \int_1^2 \frac{3\,dx}{x} + \int_1^2 \frac{4\,dx}{x+1}$$

$$= 3\ln|x| + 4\ln|x+1| \Big|_1^2$$

$$= \ln x^3(x+1)^4 \Big|_1^2 = \ln 2^3\cdot 3^4 - \ln 2^4$$

$$= \ln \frac{\cancel{2^3}\cdot 3^4}{2^{\cancel{4}}} = \ln \frac{81}{2}$$

Ex $\displaystyle\int \frac{x^4+81}{x(x^2+9)^2}\,dx$

$$\frac{x^4+81}{x(x^2+9)^2} = \underset{(x^2+9)^2}{\frac{A}{x}} + \underset{x(x^2+9)}{\frac{Bx+C}{x^2+9}} + \underset{x}{\frac{Dx+E}{(x^2+9)^2}}$$

$$x^4+81 = A(x^2+9)^2 + (Bx+C)\,x(x^2+9) + (Dx+E)\,x$$

$$x^4+81 = A(x^2+9)^2 + Bx^2(x^2+9) + Cx(x^2+9) + Dx^2 + Ex$$