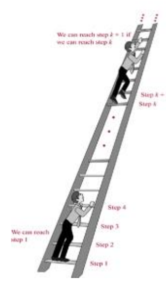
CSE2023 Discrete Computational Structures

Lecture 14

5.1 Mathematical induction



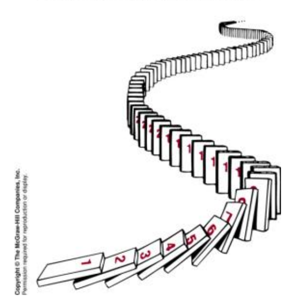
- Want to know whether we can reach *every* step of this ladder
  - We can reach *first* rung of the ladder
  - If we can reach *a particular* run of the ladder, then we can reach *the next* run
- **Mathematical induction**: show that p(n) is true for **every** positive integer n

Mathematical induction

- Two steps
  - **Basis step**: show that p(1) is true
  - **Inductive step**: show that for all positive integers k, if p(k) is true, then p(k+1) is true. That is, we show **p(k)→p(k+1)** for all positive integers k
- The assumption p(k) is true is called the **inductive hypothesis**
- Proof technique:

$$[p(1) \wedge \forall k(p(k) \rightarrow p(k+1))] \rightarrow \forall n p(n)$$

Analogy



Example

- Show that $1+2+\ldots+n=n(n+1)/2$, if n is a positive integer
  - Let p(n) be the proposition that $1+2+\ldots+n=n(n+1)/2$
  - **Basis step**: p(1) is true, because $1=1*(1+1)/2$
  - **Inductive step**: Assume p(k) is true for an arbitrary k. That is, $1+2+\ldots+k=k(k+1)/2$

    We must show that $1+2+\ldots+(k+1)=(k+1)(k+2)/2$

    From **p(k)**, $1+2+\ldots+k\mathbf{+(k+1)}=(k(k+1)/2)\mathbf{+(k+1)}=(k+1)(k+2)/2$

    which means p(k+1) is true
  - We have completed the basic and inductive steps, so by mathematical induction we know that p(n) is true for all positive integers n. That is $1+2+\ldots+n=n(n+1)/2$

Example

- Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using mathematical induction
- 1=1, 1+3=4, 1+3+5=9, 1+3+5+7=16, 1+3+5+7+9=25
- It is reasonable to conjecture the sum of first n odd integers is n^2, that is, $1+3+5+\ldots+(2n-1)=n^2$
- We need a method to prove whether this conjecture is correct or not

Example

- Let p(n) denote the proposition
- Basic step: $p(1)=1^2=1$
- Inductive steps: Assume that p(k) is true, i.e., $1+3+5+\ldots+(2k-1)=k^2$

  We must show $1+3+5+\ldots+(2k+1)=(k+1)^2$ is true for p(k+1)

  Thus,$1+3+5+\ldots+(2k-1)\mathbf{+(2k+1)}=k^2\ \mathbf{+(2k+1)}=(k+1)^2$ which means p(k+1) is true

  (Note p(k+1) means $1+3+5+\ldots+(2k+1)=(k+1)^2$)
- We have completed both the basis and inductive steps. That is, we have shown p(1) is true and $p(k)\rightarrow p(k+1)$
- Consequently, p(n) is true for all positive integers n

Example

- Use mathematical induction to show that

  $1+2+2^2+\ldots+2^n=2^{n+1}-1$
- Let p(n) be the proposition: $1+2+2^2+\ldots+2^n=2^{n+1}-1$
- Basis step: $p(0)=2^{0+1}-1=1$
- Inductive step: Assume p(k) is true, i.e., $1+2+2^2+\ldots+2^k=2^{k+1}-1$

  It follows

  $(1+2+2^2+\ldots+2^k)\mathbf{+2^{k+1}}=(2^{k+1}-1)\mathbf{+2^{k+1}}=2*2^{k+1}-1=2^{k+2}-1$ which means p(k+1): $1+2+2^2+\ldots+2^{k+1}=2^{k+2}-1$ is true
- We have completed both the basis and inductive steps. By induction, we show that $1+2+2^2+\ldots+2^n=2^{n+1}-1$

Example

- In the previous step, p(0) is the basis step as the theorem is true ∀n p(n) for all non-negative integers
- To use mathematical induction to show that p(n) is true for n=b, b+1, b+2, … where b is an integer other than 1, we show that p(b) is true, and then p(k)→p(k+1) for k=b, b+1, b+2, …
- Note that b can be negative, zero, or positive

Example

- Use induction to show $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1}-a}{r-1}$ if $r \neq 1$
- Basis step: p(0) is true as $\frac{ar^1-a}{r-1} = \mathrm{a}$
- Inductive step: assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$ if $r \neq 1$

$$\sum_{j=0}^{k+1} ar^j = a + ar + \ldots + ar^k + ar^{k+1}$$
$$= \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$$
$$= \frac{ar^{k+1}-a+ar^{k+2}-ar^{k+1}}{r-1}$$
$$= \frac{ar^{k+2}-a}{r-1}$$

- So p(k+1) is true. By induction, p(n) is true for all nonnegative integers

Example

- Use induction to show that $n<2^n$ for $n>0$
- Basis step: p(1) is true as $1<2^1=2$
- Inductive step: Assume p(k) is true, i.e., $\mathbf{k<2^k}$

  We need to show $\mathbf{k+1<2^{k+1}}$

  $k+1<2^k+1\leq 2^k+2^k=2^{k+1}$ Thus p(k+1) is true
- We complete both basis and inductive steps, and show that p(n) is true for all positive integers n

Example

- Use induction to show that $2^n<n!$ for $n \geq 4$
- Let p(n) be the proposition, $2^n<n!$ for $n \geq 4$
- Basis step: p(4) is true as $2^4=16<4!=24$
- Inductive step: Assume p(k) is true, i.e., $2^k<k!$ for $k \geq 4$. We need to show that $2^{k+1}<(k+1)!$ for $k \geq 4$

  $2^{k+1} = 2\ 2^k<2\ k!<(k+1)\ k! = (k+1)!$

  This shows p(k+1) is true when p(k) is true
- We have completed basis and inductive steps. By induction, we show that p(n) is true for $n \geq 4$

Example

- Show that n^3-n is divisible by 3 when n is positive
- Basis step: p(1) is true as 1-1=0 is divisible by 3
- Inductive step: Suppose **p(k)= k^3-k** is true, we must show that $(k+1)^3$-(k+1) is divisible by 3

  $(k+1)^3$-(k+1)=$k^3+3k^2+3k+1-(k+1)=(k^3-k)+3(k^2+k)$

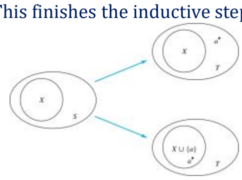
  As both terms are divisible by 3, $(k+1)^3$-(k+1) is divisible by 3
- We have completed both the basis and inductive steps. By induction, we show that n^3-n is divisible by 3 when n is positive

Example

- Show that if S is a finite set with n elements, then S has 2^n subsets
- Let p(n) be the proposition that a set with n elements has 2^n subsets
- Basis step: p(0) is true as a set with zero elements, the empty set, has exactly 1 subset
- Inductive step: Assume p(k) is true, i.e., S has 2^k subsets if |S|=k.

Example

- Let T be a set with k+1 elements. So, T=SU{a}, and |S|=k, |U|=k+1
- For each subset X of S, there are exactly two subsets of T, i.e., X and X U{a}
- Because there are 2^k subsets of S, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive step



Example

- Use mathematical induction to show one of the De Morgan's law: $\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$ where A_1, A_2, …, A_n are subsets of a universal set U, and n≥2
- Basis step: $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ (proved Section 2.2, page 131)
- Inductive step: Assume $\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$ is true for k≥2

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{(\bigcap_{j=1}^{k} A_j) \cap A_{k+1}} = \overline{(\bigcap_{j=1}^{k} A_j)} \cup \overline{A_{k+1}}$$
$$= (\bigcup_{j=1}^{k} \overline{A_j}) \cup \overline{A_{k+1}} = \bigcup_{j=1}^{k+1} \overline{A_j}$$

Axioms for the set of positive integers

- See appendix 1
- Axiom 1: The number 1 is a positive integer
- Axiom 2: If n is a positive integer, then n+1, the successor of n, is also a positive integer
- Axiom 3: Every positive integer other than 1 is the successor of a positive integer
- Axiom 4: **Well-ordering property** Every non-empty subset of the set of positive integers has a least element

17

Why mathematical induction is valid?

- From mathematical induction, we know p(1) is true and the proposition p(k)→p(k+1) is true for all positive integers
- To show that p(n) must be true for all positive integers, assume that there is at least one positive integer such that p(n) is false
- Then the set S of positive integers for which p(n) is false is non-empty
- By well-ordering property, S has a least element, which is demoted by m
- We know that m cannot be 1 as p(1) is true
- Because m is positive and greater than 1, m-1 is a positive integer

18

Why mathematical induction is valid?

- Because m-1 is less than m, it is not in S
- So p(m-1) must be true
- As the conditional statement p(m-1)→p(m) is also true, it must be the case that p(m) is true
- This contradicts the choice of m
- Thus, p(n) must be true for every positive integer n

19

Template for inductive proof

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

5.2 Strong induction and well-ordering

- **Strong induction**: To prove p(n) is true for all positive integers n, where p(n) is a propositional function, we complete two steps
- Basis step: we verify that the proposition p(1) is true
- Inductive step: we show that the conditional statement $(p(1) \wedge p(2) \wedge \ldots \wedge p(k)) \rightarrow p(k+1)$ is true for all positive integers k

Strong induction

- Can use all k statements, p(1), p(2), …, p(k) to prove p(k+1) rather than just p(k)
- Mathematical induction and strong induction are equivalent
- Any proof using mathematical induction can also be considered to be a proof by strong induction (induction → strong induction)
- It is more awkward to convert a proof by strong induction to one with mathematical induction (strong induction → induction)

Strong induction

- Also called the **second principle of mathematical induction** or **complete induction**
- The principle of mathematical induction is called **incomplete induction**, a term that is somewhat misleading as there is nothing incomplete
- Analogy:
  – If we can reach the first step
  – For every integer k, if we can reach all the first k steps, then we can reach the k+1 step

Example

- Suppose we can reach the 1^{st} and 2^{nd} rungs of an infinite ladder
- We know that if we can reach a rung, then we can reach two rungs higher
- Can we prove that we can reach every rung using the principle of mathematical induction? or strong induction?

Example – mathematical induction

- Basis step: we verify we can reach the 1st rung
- Attempted inductive step: the inductive hypothesis is that we can reach the k-th rung
- To complete the inductive step, we need to show that we can reach k+1-th rung based on the hypothesis
- However, no obvious way to complete this inductive step (because we do not know from the given information that we can reach the k+1-th rung from the k-th rung)

25

Example – strong induction

- Basis step: we verify we can reach the 1st rung
- Inductive step: the inductive hypothesis states that we can reach each of the first k rungs
- To complete the inductive step, we need to show that we can reach k+1-th rung
- We know that we can reach 2nd rung.
- We note that we can reach the (k+1)-th rung from (k-1)-th rung we can climb 2 rungs from a rung that we already reach
- This completes the inductive step and finishes the proof by strong induction

26

Which one to use

- Try to prove with mathematical induction first
- Unless you can clearly see the use of strong induction for proof

27