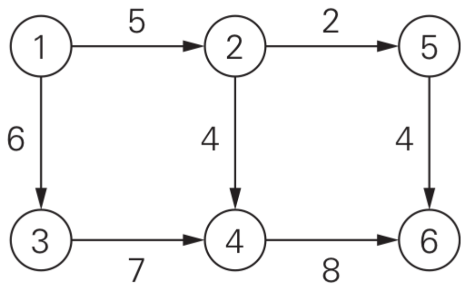
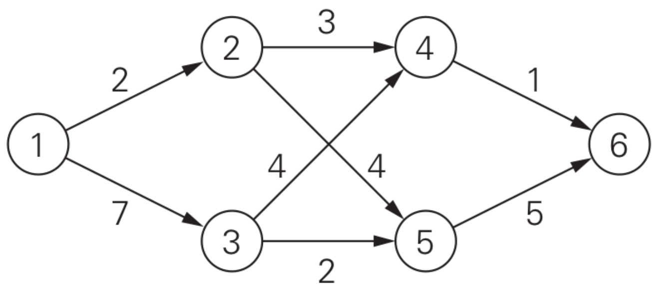
10.2. **2.** Apply the shortest-augmenting path algorithm to find a maximum flow and a minimum cut in the following networks.

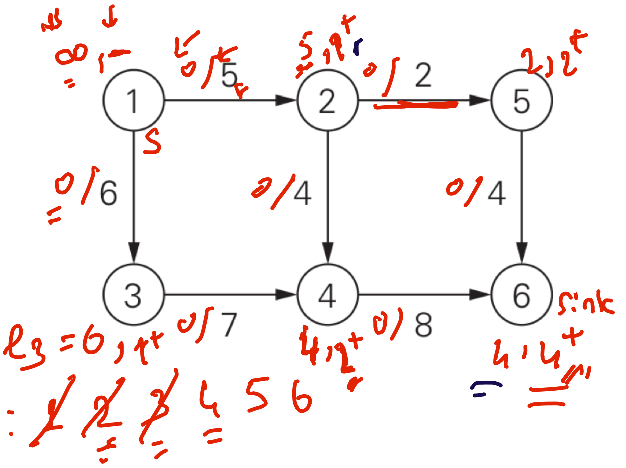
a.



b.



a)



$l_2 = \min\{l_1, r_{12}\} = 5$ where

$r_{12} = u_{12} - x_{12}$

$l_3 = \min\{l_1, r_{13}\} = 6$

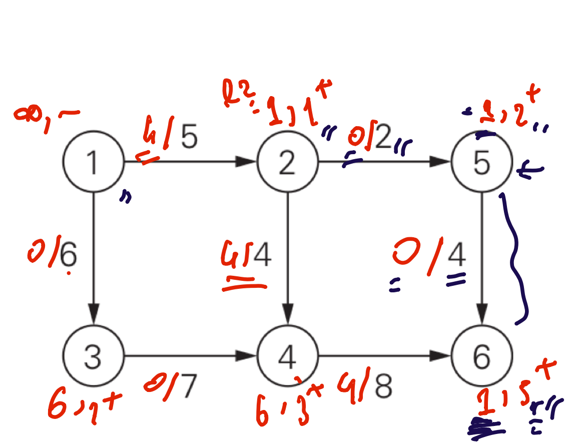
$r_{13} = u_{13} - x_{13} = 6 - 0$

$r_{13} = 6$

Q: ~~1~~ ~~2~~ ~~3~~ 4 5 6

Augmenting Path: 1 → 2 → 4 → 6 →

Augment the flow by 4



$l_2 = \min\{l_1, r_{12}\} = 1$

$r_{12} = u_{12} - x_{12} = 5 - 4 = 1$

$l_6 = \min\{l_5, r_{56}\} = 1$

$r_{56} = u_{56} - x_{56} = 4 - 0 = 4$

$l_5 = \min\{l_2, r_{25}\} = 1$

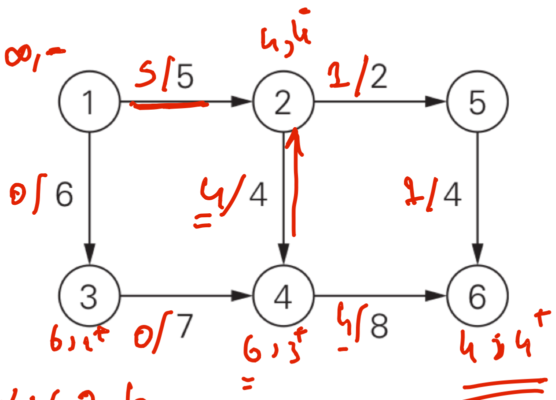
$r_{25} = 2 - 0 = 2$

Q: ~~1~~ ~~2~~ ~~3~~ ~~5~~ 4 6

Augmenting path: 1 → 2 → 5 → 6

Augment the flow by 1

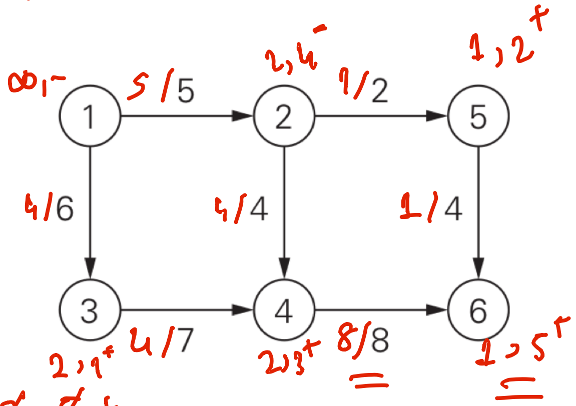
$l_2 = \min\{l_4, x_{42}\} = 4$
$\quad\quad\quad\quad\quad\quad \| \quad\quad \|$
$\quad\quad\quad\quad\quad\quad 6 \quad\quad 4$



Q: ~~1~~ ~~3~~ ~~4~~ 2 6

Augment path : 1 → 3 → 4 → 6

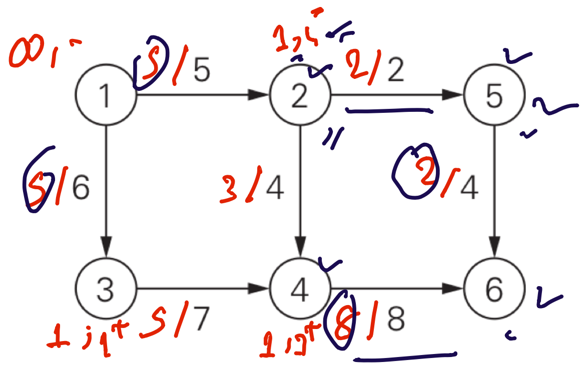
Augment by 4



Q: ~~1~~ ~~3~~ ~~4~~ ~~2~~ ~~5~~ 6

Augmenting path : 1 → 3 → 4 ← 2 → 5 → 6

Augment the flow by 1



Q: ~~1~~ ~~3~~ ~~4~~ 2

The maximum flow is 10

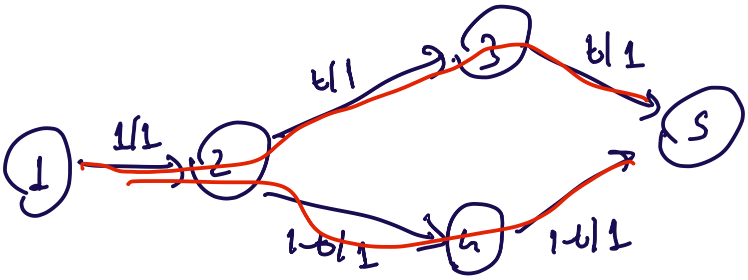
The minimum cut is {(2,5), (4,6)}

10.2. **3. a.** Does the maximum-flow problem always have a unique solution? Would your answer be different for networks with different capacities on all their edges?

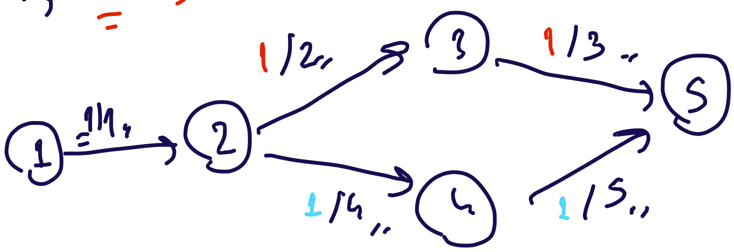
b. Answer the same questions for the minimum-cut problem of finding a cut of the smallest capacity in a given network.

a) Maximum flow problem may have more than one optimal solution

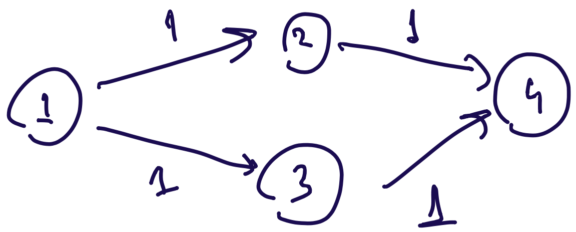
$0 \le t \le 1$



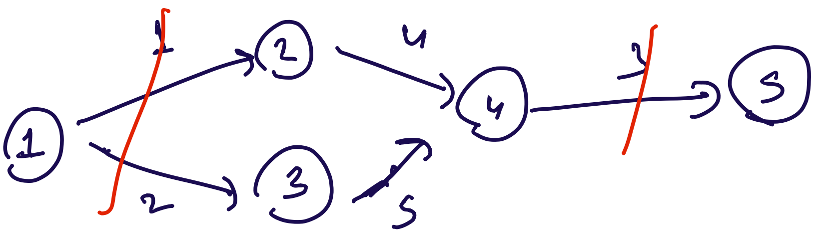
→ $t=0$, $t=1$



b)



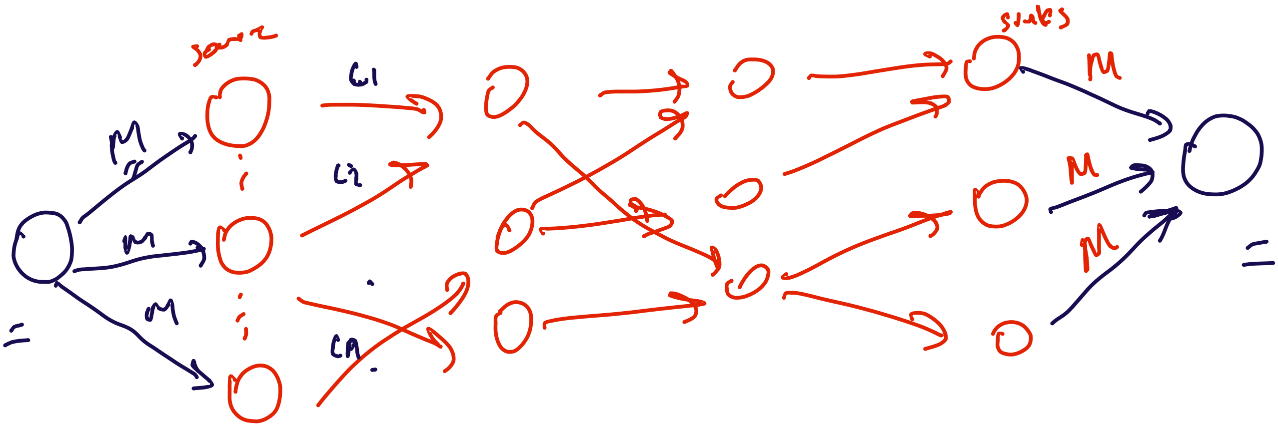
Four minimum cuts: {(1,2),(1,3)}
{(1,2),(3,4)}
{(2,4),(1,3)}
{(2,4),(3,4)}



Two cuts: {(1,2),(1,3)}
{(4,5)}

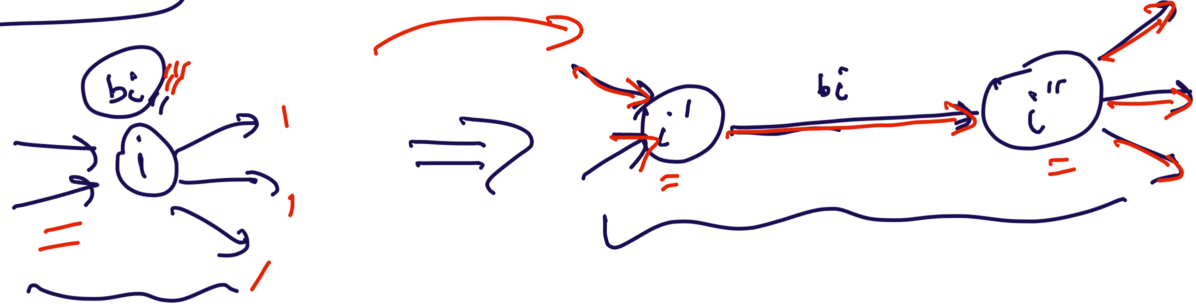
10.2. **4. a.** Explain how the maximum-flow problem for a network with several sources and sinks can be transformed into the same problem for a network with a single source and a single sink.

b. Some networks have capacity constraints on the flow amounts that can flow through their intermediate vertices. Explain how the maximum-flow problem for such a network can be transformed to the maximum-flow problem for a network with edge capacity constraints only.



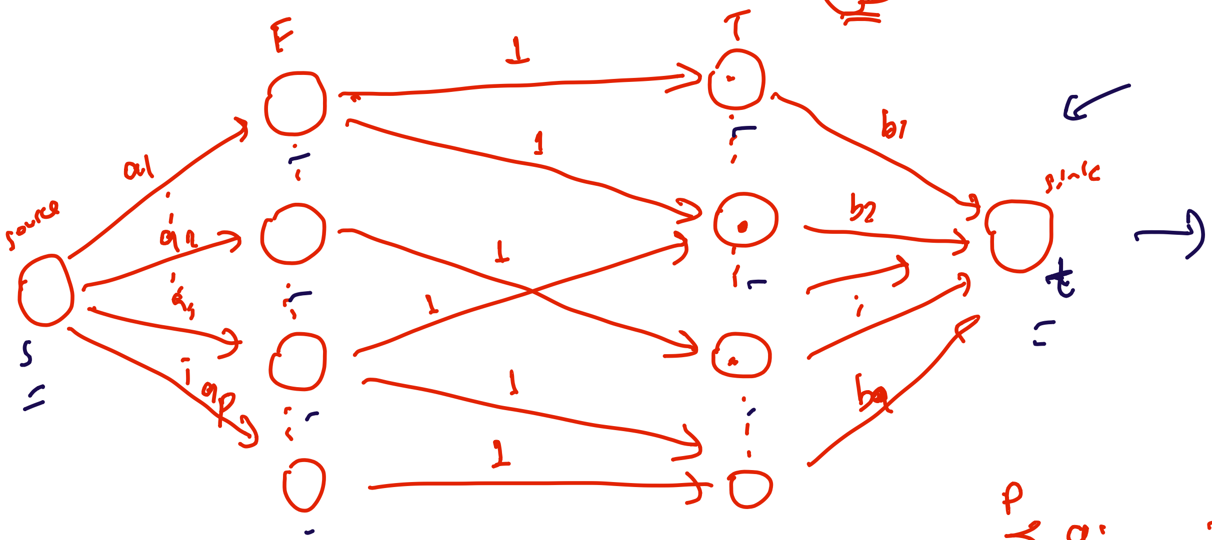
$$\boxed{M \geq \sum_{i=1}^{n} c_n}$$

b)



→ All edges entering and leaving i in the original network should enter i' and leave i'', respectively.

10-2. **10.** *Dining problem* Several families go out to dinner together. To increase their social interaction, they would like to sit at tables so that no two members of the same family are at the same table. Show how to find a seating arrangement that meets this objective (or prove that no such arrangement exists) by using a maximum-flow problem. Assume that the dinner contingent has p families and that the ith family has a_i members. Also assume that q tables are available and the jth table has a seating capacity of b_j. [Ahu93]



→ If the maximum flow value is equal to $\sum_{i=1}^{p} a_i$, the problem has a solution.

Otherwise, the problem does not have a solution.

$V = \{s, t\} \cup \{u_i \mid 1 \le i \le p\} \cup \{v_j \mid 1 \le j \le q\}$

$E = \{(s, u_i) \mid 1 \le i \le p\} \cup \{(u_i, v_j) \mid 1 \le i \le p,\ 1 \le j \le q\} \cup \{(v_j, t) \mid 1 \le j \le q\}$

$c(s, u_i) = a_i$,

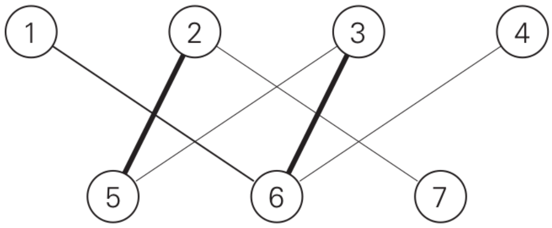
$c(u_i, v_j) = 1$,

$c(v_j, t) = b_j$,

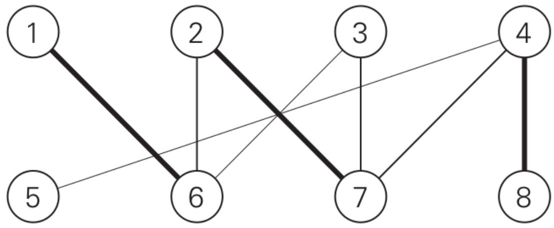
10.3.

1. For each matching shown below in bold, find an augmentation or explain why no augmentation exists.

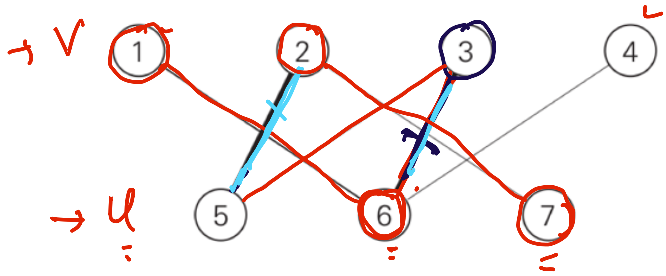
a.



b.



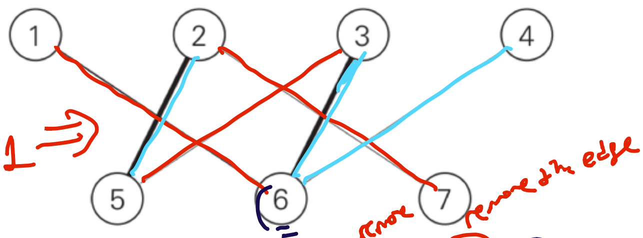
a)



$M_0 = \{(2,5), (3,6)\}$

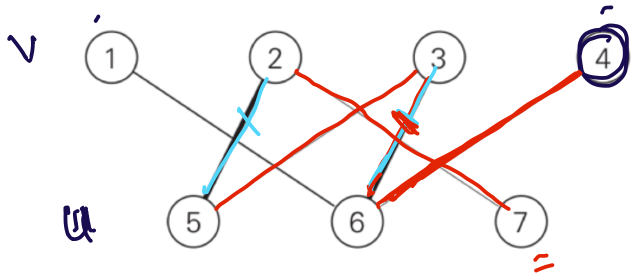
$M = \{(1,6), (2,7), (3,5)\}$

Augmentation 1 ⇒

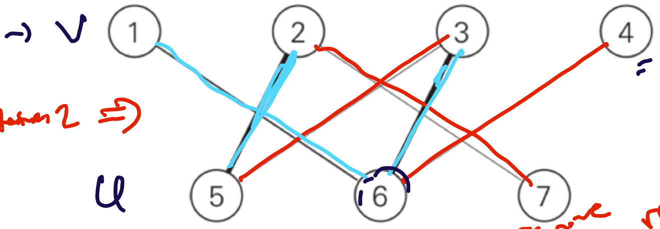


Augmenting path : 1 6 3 5 2 7 (remove, remove the edge; add, add, add the edge)

$M = \{(2,5), (3,6)\}$

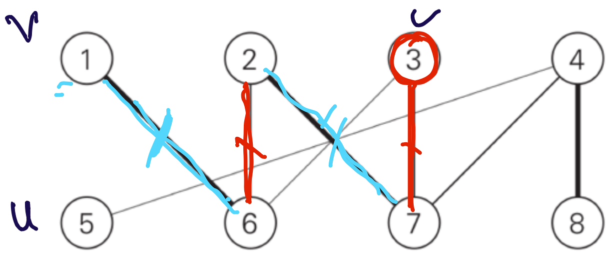


Augmentation 2 ⇒



$M = \{(2,7), (3,5), (4,6)\}$

Augmenting path : 4 6 3 5 2 7 (add, remove, add, remove, add) b)

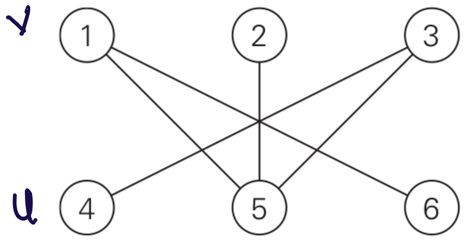


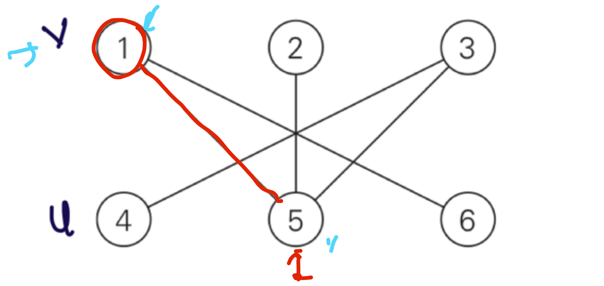
$M = \{(1,6), (2,7), (4,8)\}$

No augmentation of the matching

10.3.

2. Apply the maximum-matching algorithm to the following bipartite graph:

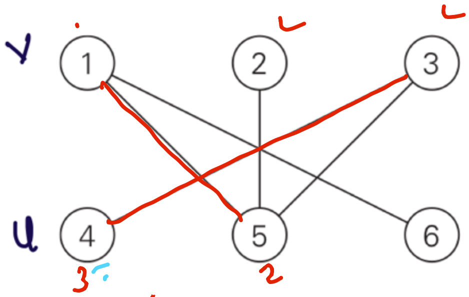




Q: ~~1~~ 2 3

Augment from 5 until a free vertex in V is reached

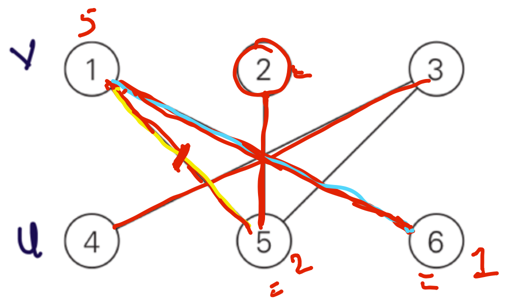
Augmenting path: $\underbrace{15}_{\text{add}}$



Q: ~~2~~ ~~3~~ 5

Augment from 4

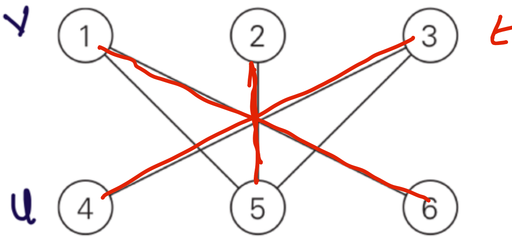
Augmenting path: $\underbrace{34}_{\text{add}}$



Q: ~~2~~ ~~5~~ ~~1~~

Augment from 6

Augmenting path: $\underbrace{25}_{\text{add}}\ \overset{\text{remove}}{}\ \underbrace{16}_{\text{add}}$



No free vertices

Maximum matching (here there is a perfect matching)

M: {(1,6),(2,5),(3,4)}

10.4. **1.** Consider an instance of the stable marriage problem given by the following ranking matrix:

	A	B	C
α	1, 3	2, 2	3, 1
β	3, 1	1, 3	2, 2
γ	2, 2	3, 1	1, 3

For each of its marriage matchings, indicate whether it is stable or not. For the unstable matchings, specify a blocking pair. For the stable matchings, indicate whether they are man-optimal, woman-optimal, or neither. (Assume that the Greek and Roman letters denote the men and women, respectively.)

There are total 3! = 6 one-one matchings of two disjoint 3-element sets

1)

men \ women	A	B	C
α	$\boxed{1, 3}$	2, 2	3, 1
β	3, 1	$\boxed{1, 3}$	2, 2
γ	2, 2	3, 1	$\boxed{1, 3}$

$\{(\alpha, A), (\beta, B), (\gamma, C)\}$

→ No blocking pair, therefore, it is stable

→ man-optimal

2)

	A	B	C
α	$\boxed{1, 3}$	2, 2	3, 1
β	3, 1	1, 3	$\boxed{2, 2}$
γ	2, 2	$\boxed{3, 1}$	1, 3

$\{(\alpha, A), (\beta, C), (\gamma, B)\}$ is unstable

(γ, A) is a blocking pair

3)

	A	B	C
α	1, 3	$\boxed{2, 2}$	3, 1
β	$\boxed{3, 1}$	1, 3	2, 2
γ	2, 2	3, 1	$\boxed{1, 3}$

$\{(\alpha, B), (\beta, A), (\gamma, C)\}$ is unstable

(β, C) is a blocking pair

4)

	A	B	C
α	1, 3	$\boxed{2, 2}$	3, 1
β	3, 1	1, 3	$\boxed{2, 2}$
γ	$\boxed{2, 2}$	3, 1	1, 3

$\{(\alpha, B), (\beta, C), (\gamma, A)\}$ is stable

Neither a man-optimal nor a woman optimal

5)

	A	B	C
α	1, 3	2, 2	$\boxed{3, 1}$
β	$\boxed{3, 1}$	1, 3	2, 2
γ	2, 2	$\boxed{3, 1}$	1, 3

$\{(\alpha, C), (\beta, A), (\gamma, B)\}$ is stable

woman-optimal

6)

	A	B	C
α	1, 3	2, 2	$\boxed{3, 1}$
β	3, 1	$\boxed{1, 3}$	2, 2
γ	$\boxed{2, 2}$	3, 1	1, 3

$\{(\alpha, C), (\beta, B), (\gamma, A)\}$ is unstable

(α, B) is a blocking pair