CSE2023 Discrete Computational Structures

Lecture 6



## 1.8 Proof methods and strategy

$$((p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q)$$
$$\leftrightarrow ((p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q))$$

- **Proof by cases**: $p_i \rightarrow q$ for i=1,2,…,n
- When it is not possible to consider all cases at the same time
- **Exhaustive proof**: some theorems can be proved by examining a relatively small number of examples



## Example

- Prove $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
- Proof by exhaustion as we only need to verify n=1,2,3 and 4.
- For n=1, $(n+1)^3=8 \geq 3^1=3$
- For n=2, $(n+1)^3=27 \geq 3^2=9$
- For n=3, $(n+1)^3=64 \geq 3^3=27$
- For n=4, $(n+1)^3=125 \geq 4^3=64$



## Example

- An integer is a perfect power if it equals $n^a$, where a is an integer greater than 1
- Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9
- Can prove this fact by examining positive integers n not exceeding 100
  - First check whether n is a perfect power, and then check whether n+1 is a perfect power

## Example

- For positive integers
  - The squares ≤ 100: 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100
  - The cubes ≤ 100: 1, 8, 27, and 64
  - The $4^{th}$ powers $n^4$ ≤ 100: 1, 16, and 81
  - The $5^{th}$ powers $n^5$ ≤ 100: 1 and 32
  - The $6^{th}$ powers $n^6$ ≤ 100: 1 and 64
  - Look at the list of perfect powers, we see that the pair of n=8 and n+1=9 is the only two consecutive powers ≤ 100

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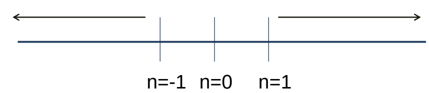
## Proof by cases

- Prove that if n is an integer, then $n^2 \geq n$
- We prove this by 3 cases:
  - n=0: trivial case as $0^2 \geq 0$
  - n≥1: If n≥1 then n·n ≥ n·1 and thus $n^2 \geq n$
  - n≤-1: If n ≤ -1 then $n^2 \geq 0 > n$ and thus $n^2 \geq n$

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## Example

- Show that |xy|=|x||y| for real numbers

  $$((p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q) \leftrightarrow ((p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q))$$

- x≥0, y≥0: xy ≥0 |xy|=xy=|x||y|
- x≥0, y<0: xy<0 |xy|=-xy=x(-y)=|x||y|
- x<0, y ≥0:xy<0 |xy|=-xy=(-x)y=|x||y|
- x<0, y<0: xy>0 |xy|=xy=(-x)(-y)=|x||y|

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## Example

- Formulate a conjecture about the decimal digits that occur at the final digit of the squares of an integer and prove the result
- The smallest perfect squares are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225 and so on
- Note that the digits that occur at the final digit of a squares are: 0, 1, 4, 5, 6, and 9 (and no 2, 3, 7, and 8) → conjecture

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## Example

- We can express an integer n as 10a+b were a and b are positive integers and $0 \le b \le 9$
- $n^2=(10a+b)^2=100a^2+20ab+b^2=10(10a^2+2b)+b^2$, so the final digit is the final digit of $b^2$
- Note also that the final digit of $(10-b)^2=100-20b+b^2$. Thus, we only consider 6 cases
- Case 1: if final digit of n is 1 or 9 (or b), then the last digit of $n^2$ is 1

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## Example

- Case 2: if the final digit of n is 2 or 8, then the final digit of $n^2$ is 4
- Case 3: if the final digit of n is 3 or 7, then the final digit of $n^2$ is 9
- Case 4: if the final digit of n is 4 or 6, then the final digit of $n^2$ is 6
- Case 5: if the final digit of n is 5, then the final digit of $n^2$ is 5
- Case 6: if the final digit of n is 0, then the final digit of $n^2$ is 0

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## Example

- Show that there are no solutions in integers x and y of $x^2+3y^2=8$
- $x^2>8$ when $|x|\ge 3$, and $3y^2>8$ when $|y|\ge 2$. The only values for x are -2,-1,0,1,2 and for y are -1, 0, 1
- So, possible values for $x^2$ are, 0, 1, and 4. The possible values for $3y^2$ are 0 and 3
- No pair of x and y can be solution

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