Chapter 2

First Order DE’s

First Order DE's

Standard Form:

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$y' = f(x, y)$$

Interpretation: The form shows the slope of the tangent at each point of the xy-plane. Thus the standard form of the first order differential equation give the slope field. Solving the equation gives the original curve family having the slope field expressed by the differential equation.

Preliminary Concepts

First-order differential equation:

$$F(x, y, y') = 0$$

involves a first but no higher derivatives

$$\text{e.g., } y'(x) - 2\cos x = 4$$

$$y \;:\; \text{dependent variable (function of } x\text{)}$$

$$x \;:\; \text{independent variable}$$

$$F(x, y, y') = 0 \quad \Rightarrow \quad y(x);\; \textbf{solution}$$

First-Order DE

Derivative form:

$$a_0(x)\frac{dy}{dx} + a_1(x)y = g(x)$$

Differential form:

$$[a_1(x)y - g(x)]dx + a_0(x)dy = 0$$

General form:

$$\frac{dy}{dx} = F(x, y) \quad \text{or} \quad F\left(x, y, \frac{dy}{dx}\right) = 0$$

General and Particular Solutions

$$y' + y = 2$$

General solution: $y(x) = 2 + ce^{-x}$ where c is arbitrary constant

$$y'(x) = -ce^{-x}$$

To verify that this equation substitute $y(x)$ & $y'(x)$ into the DE

$$-ce^{-x} + 2 + ce^{-x} = 2$$

Particular solutions:

$$c = 0 \quad \Rightarrow \quad y = 2$$

$$c = 1 \quad \Rightarrow \quad y = 2 + e^{-x}$$

$$c = 2 \quad \Rightarrow \quad y = 2(1 + e^{-x})$$

Exact Equations

A differential equation $$M(x,y)dx + N(x,y)dy = 0.$$

is exact if there exists a function $g(x,y)$ such that

$$d[g(x,y)] = M(x,y)dx + N(x,y)dy.$$

Test for exactness: If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then

$$M(x,y)dx + N(x,y)dy = 0$$

is exact iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Exact Equations

How to obtain an exact equation from a function?

$$F(x, y) = c$$

$$\frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy = 0$$

$$M(x, y)dx + N(x, y)dy = 0$$

Method of Solution

To solve an exact equation, first solve the equations for $F(x, y)$.

$$F(x,y) = \int M(x,y)dx + k(y) \qquad or \qquad F(x,y) = \int N(x,y)dy + n(x)$$

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + k' = N(x,y) \qquad \frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[\int N(x,y)dy\right] + n' = M(x,y)$$

$$\text{Find } k'(y), k(y) \qquad \text{Find } n'(x), n(x)$$

The solution to the exact equation is then given implicitly by $F(x, y) = c$, where c represents an arbitrary constant.

ex. 1: Make an exact ODE using $u(x,y) = x + x^2y^3$

$$u(x,y) = x + x^2y^3 = c$$

then

$$du = (1 + 2xy^3)dx + 3x^2y^2dy = 0$$

$$y' = dy/dx = -(1 + 2xy^3)/3x^2y^2$$

ex. 2: Solve $\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))dy = 0$

here $\quad \mathrm{M} = \cos(x+y)$ & $N = 3y^2 + 2y + \cos(x+y)$

Since $\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x} \Rightarrow \therefore$ equation is exact.

$$u(x,y) = \int \cos(x+y)\,dx + k(y) = \sin(x+y) + k(y)$$

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk(y)}{dy} = \mathrm{N} = 3y^2 + 2y + \cos(x+y)$$

$$\frac{dk(y)}{dy} = 3y^2 + 2y \Rightarrow k(y) = y^3 + y^2 + c$$

$$\boxed{u(x,y) = \sin(x+y) + y^3 + y^2 + c}$$

ex. 2: alternative sol.'n by grouping method

Since $\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x} \Rightarrow \therefore$ equation is exact. Then grouping similar terms

$$[\cos(x+y)\,dx + \cos(x+y)\,dy] + 3y^2dy + 2ydy = 0$$

$$d(\sin(x+y) + y^3 + y^2) = d(c)$$

İgnoring differentiation operation for both side, remaining is answer

$$\boxed{\sin(x+y) + y^3 + y^2 = c}$$

ex. 3: Solve $y' = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$

@ first rewrite in differential form as: $(2xy^3 + 2)dx + (3x^2y^2 + 8e^{4y})dy = 0$

Here $\mathrm{M} = 2xy^3 + 2$ & $N = 3x^2y^2 + 8e^{4y}$

Since $\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x} \Rightarrow \therefore$ equation is exact.

$$u(x,y) = \int (3x^2y^2 + 8e^{4y})\, dy + k(x) = x^2y^3 + 2e^{4y} + k(x)$$

$$\frac{\partial u}{\partial x} = 2xy^3 + \frac{dk(x)}{dx} = \mathrm{M} = 2xy^3 + 2$$

$$\frac{dk(x)}{dx} = 2 \Rightarrow k(x) = 2x + c \Rightarrow \boxed{u(x,y) = x^2y^3 + 2e^{4y}+2x+c}$$

ex. 3: alternative sol.'n by grouping method

Since $\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x} \Rightarrow \therefore$ equation is exact. Then grouping similar terms

$$[2xy^3dx + 3x^2y^2dy] + 2dx + 8e^{4y}dy = 0$$

$$d(x^2y^3 + 2x + 2e^{4y}) = d(c)$$

İgnoring differentiation operation for both side, remaining is answer

$$\boxed{x^2y^3 + 2x + 2e^{4y} = \text{c}}$$

Examples and Exercises

Make an exact differential equation from the functions

$$f(x, y) = x^2 y + 6y - y^3$$

$$u(x, y) = x + x^2 y^3.$$

Check for the exactness and solve

$(y^2 - 2x)dx + (2x + 1)dy = 0,$		$ans:$	
$(3x^2 y - 1)dx + (x^3 + 6y - y^{-2})dy = 0,$	$y(0) = -1,$	$ans:$	$x^3 y - x + 3y^2 + y^{-1} = 2$
$2xy - 9x^2 + (2y + x^2 + 1)y' = 0$	$y(0) = -3,$	$ans:$	$x^2 y - x^3 + y^2 + y = 6$
$(1 + t^2)y' + 4ty = (1 + t^2)^{-2},$	$y(0) = 1$	$ans:$	
$y' = \dfrac{2x - e^x \sin y}{e^x \cos y + 1}$	$y(0) = 0$	$ans:$	$e^x \sin y + y - x^2 = 0$

Integrating Factors (IF)

If an equation is inexact (not exact), it is possible to transform Such equation into an exact differential equation by a judicious multiplication.

A function $\mu(x, y)$ (Greek letter "mu") is an *integrating factor* for an inexact equation if the equation

$$\mu(x, y)[M(x, y)dx + N(x, y)dy] = 0$$

is an exact equation.

If $\mu = \mu(x)$ only

Starting with $M(x,y)dx + N(x,y)dy = 0$, multiply by $\mu(x)$:

$$\mu(x)[M(x,y)dx + N(x,y)dy] = 0$$

İf the above is an exact equation :

$$[\mu(x) \cdot M(x,y)]_y = [\mu(x) \cdot N(x,y)]_x$$

$$\mu(x)M_y + \frac{\partial\mu(x)}{\partial y}M(x,y) = \mu(x)N_x + \frac{d\mu(x)}{dx}N(x,y) \Rightarrow \therefore \frac{\partial\mu(x)}{\partial y} = 0$$

This is a separable differential equation… so separate:

$$\frac{d\mu(x)}{dx} = \mu(x)\frac{M_y - N_x}{N}$$

If $\mu = \mu(x)$ only

Integrating both sides, we have

$$\int \frac{d\mu}{\mu(x)} = \int \frac{M_y - N_x}{N} dx$$

After integration, we have

$$\ln \mu(x) = \int \frac{M_y - N_x}{N} dx + C.$$

Here, we only need one form of the antiderivative, so we let $C = 0$. Taking base-e on both sides,

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}}$$

If $\mu = \mu(y)$ only

Starting with $M(x,y)dx + N(x,y)dy = 0$, multiply by $\mu(y)$:

$$\mu(y)[M(x,y)dx + N(x,y)dy] = 0$$

İf the above is an exact equation :

$$[\mu(y) \cdot M(x,y)]_y = [\mu(y) \cdot N(x,y)]_x$$

$$\mu(y)M_y + \frac{d\mu(y)}{dy}M(x,y) = \mu(y)N_x + \frac{\partial\mu(y)}{\partial x}N(x,y) \Rightarrow \therefore \frac{\partial\mu(y)}{\partial x} = 0$$

This is a separable differential equation… so separate:

$$\frac{d\mu(y)}{dy} = \mu(y)\frac{N_x - M_y}{M}$$

If $\mu = \mu(y)$ only

Integrating both sides, we have

$$\int \frac{d\mu}{\mu(y)} = \int \frac{N_x - M_y}{M} dy$$

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$$\ln \mu(y) = \int \frac{N_x - M_y}{M} dy + C.$$

Here, we only need one form of the antiderivative, so we let $C = 0$. Taking base-e on both sides,

$$\boxed{\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}}$$

Integrating Factors

The famous formulas to obtain an integrating factor are as follows:

If $\mu = \mu(x)$ only $\Longrightarrow$ $\dfrac{1}{N(x,y)}\left[\dfrac{\partial M(x,y)}{\partial y} - \dfrac{\partial N(x,y)}{\partial x}\right] \equiv g(x)$ $\Longrightarrow$ $\mu(x) = e^{\int g(x)dx}$

If $\mu = \mu(y)$ only $\Longrightarrow$ $\dfrac{1}{M(x,y)}\left[\dfrac{\partial N(x,y)}{\partial x} - \dfrac{\partial M(x,y)}{\partial y}\right] \equiv f(y)$ $\Longrightarrow$ $\mu(y) = e^{\int f(y)dy}$

If $\mathrm{M} = yf(xy)$ and $\mathrm{N} = xg(xy)$ $\Longrightarrow$ $\mu(x,y) = \dfrac{1}{xM - yN}$

or try $\mu(x,y) = x^a y^b$

ex. 4: Find the general solution of $y' + \frac{2}{x}y = x$

Solution: First, rewrite the eqn.

$$(x^2 - 2y)dx - xdy = 0$$

$$M_y = -2 \neq N_x = -1 \Longrightarrow \text{not exact}$$

Assume that $\mu = \mu(y)$ only, then

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{-1-(-2)}{x^2-2y} dy} \Longrightarrow \mu \neq \mu(y)$$

Let's check $\mu = \mu(x)$ only, then

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{-2-(-1)}{-x} dx} = e^{\int \frac{dx}{x}} = x$$

ex. 4: continued

$$x[(x^2 - 2y)dx - xdy = 0]$$

$$M_y = -2x = N_x \Longrightarrow \text{exact}$$

by grouping method,

$$x^3dx - (2xydx + x^2dy) = 0$$

$$d\left(\frac{1}{4}x^4 - x^2y\right) = d(c)$$

$$\boxed{\frac{1}{4}x^4 - x^2y = c}$$

ex. 5: Solve the IVP: $y' = \frac{y^2+2xy}{x^2}$, $y(0) = 5$

Solution: First, rewrite the eqn.

$$(y^2 + 2xy)dx - x^2dy = 0$$

$$M_y = 2(x + y) \neq N_x = -2x \Longrightarrow \text{not exact}$$

Assume that $\mu = \mu(x)$ only, then

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{2(x+y)-(-2x)}{-x^2} dx} = e^{\int \frac{2(2x+y)}{-x^2} dx} \Longrightarrow \mu \neq \mu(x)$$

Let's check $\mu = \mu(y)$ only, then

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{-2x-2(x+y)}{y^2+2xy} dy} = y^{-2}$$

ex. 5: continued

$$y^{-2}[(y^2 + 2xy)dx - x^2dy = 0]$$

$$M_y = -2xy^{-2} = N_x \Longrightarrow \text{exact}$$

by grouping method,

$$dx + (2xy^{-1}dx - x^2y^{-2}dy) = 0$$

$$d(x + x^2y^{-1}) = d(c)$$

$$\boxed{x + x^2y^{-1} = c}$$

excercises

$$(x - xy) - y' = 0 \qquad Ans: \quad y = 1 - ce^{\frac{-x^2}{2}}$$

$$(3x - 2xy)dx - dy = 0 \qquad y(0) = 5 \qquad Ans: \quad y = \frac{3}{2} + \frac{7}{2}e^{-x^2}$$

$$2y^2 - 9xy + (3xy - 6x^2)y' = 0 \qquad Ans: \quad x^2y^3 - 3x^3y^2 = c$$

$$y' = -\frac{(1 + ye^{xy})}{(1 + xe^{xy})} \qquad Ans: \quad x + y + e^{xy} = c$$

$$y' = \frac{y - 1}{x - 3} \qquad y(-1) = 0 \qquad Ans: \quad y = \frac{x + 1}{4}$$

Group of terms	Integrating factor $I(x, y)$	Exact differential $dy(x, y)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{xy}$	$\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln\dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2 + y^2}$	$\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\arctan\dfrac{y}{x}\right)$
$y\,dx + x\,dy$	$\dfrac{1}{xy}$	$\dfrac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\dfrac{1}{(xy)^n}, \quad n > 1$	$\dfrac{y\,dx + x\,dy}{(xy)^n} = d\left[\dfrac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dy + x\,dx$	$\dfrac{1}{x^2 + y^2}$	$\dfrac{y\,dy + x\,dx}{x^2 + y^2} = d\left[\dfrac{1}{2}\ln(x^2 + y^2)\right]$
$y\,dy + x\,dx$	$\dfrac{1}{(x^2 + y^2)^n}, \quad n > 1$	$\dfrac{y\,dy + x\,dx}{(x^2 + y^2)^n} = d\left[\dfrac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$ (a, b constants)	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

Separable Equations

A differential equation is *separable* if it can be written (perhaps after some algebraic manipulation) as

$$y' = \frac{dy}{dx} = g(x)\, h(y)$$

in which the derivative equals a *product* of a function just of *x* and a function just of *y*.

Method of Solution: Integrate both sides will give the solution of such type of equations.

Separable Equations

For y such that $G(y) \neq 0$, write the differential form

$$\frac{1}{G(y)}\,dy = F(x)\,dx$$

In this equation, we say that the variables have been separated. Integrate

$$\int \frac{1}{G(y)}\,dy = \int F(x)\,dx$$

ex. 6: Solve the IVP $y' = y^2 e^{-x}$, $y(0) = 4$

- first write $$\frac{dy}{dx} = y^2 e^{-x}$$
- If $y \neq 0$, this has the differential form $$\frac{1}{y^2} dy = e^{-x} dx$$
- The variables have been separated. Integrate $$\int \frac{1}{y^2} dy = \int e^{-x} dx$$

or $$-\frac{1}{y} = -e^{-x} + c$$

in which c is a constant of integration.

- Solve for y to get $$y(x) = \frac{1}{e^{-x} - c}$$

This is a solution of the differential equation for any number c.

ex. 6: continued

- Choose c so that

$$y(1) = \frac{1}{1-c} = 4$$

- Solve this equation for c to get

$$c = \frac{3}{4}$$

- The solution of the initial value problem is

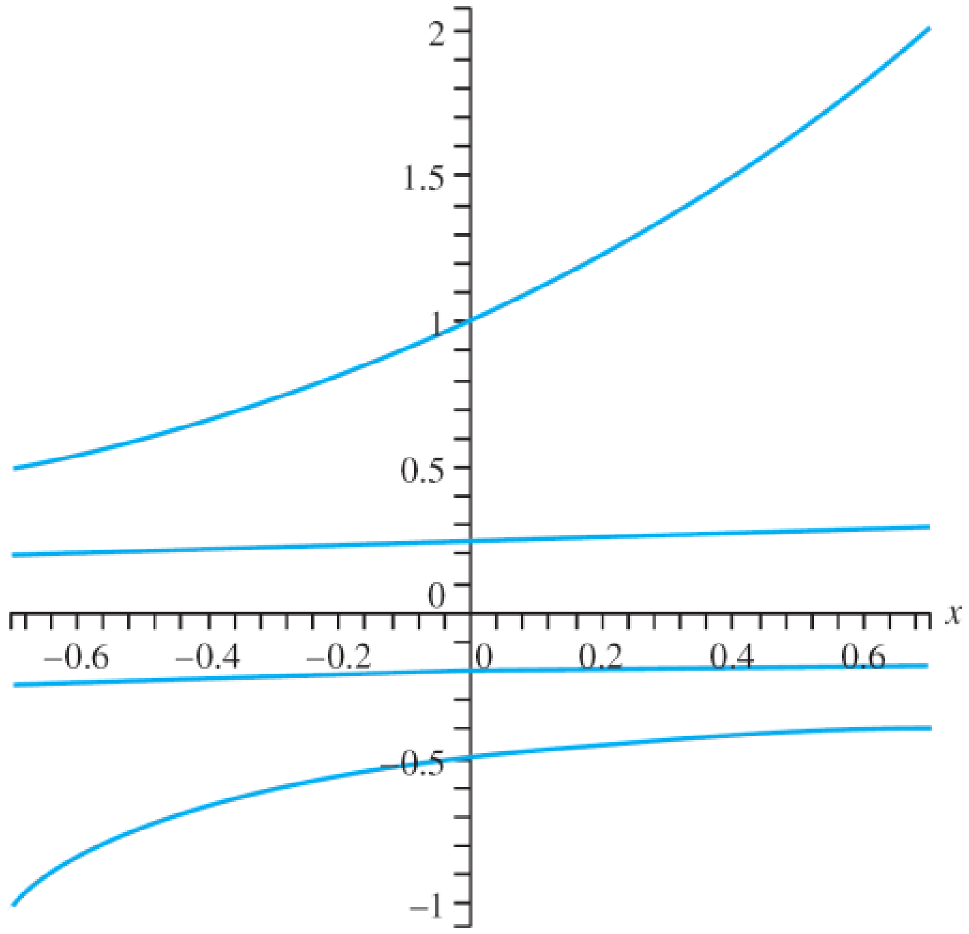
$$y(x) = \frac{4}{4e^{-x} + 3}$$

ex. 6: continued

$$y(x) = \frac{1}{e^{-x} - 3} \quad , \quad y(x) = \frac{1}{e^{-x} + 3}$$

$$y(x) = \frac{1}{e^{-x} - 6} \quad , \quad y(x) = \frac{1}{e^{-x}} = e^{x}$$

are particular solutions corresponding to $c = \pm 3$, 6, & 0. Particular solutions are also called *integral curves* of the differential equation.



ex. 7: solve $x^2y' = 1 + y$

- first write $$\frac{dy}{dx} = \frac{1+y}{x^2}$$
- If $y \neq -1$ and $x \neq 0$, this has the differential form $$\frac{1}{1+y}dy = \frac{1}{x^2}dx$$
- The variables have been separated. Integrate $$\int \frac{1}{1+y}dy = \int \frac{1}{x^2}dx$$

or $$\ln|1+y| = -\frac{1}{x} + c$$

in which c is a constant of integration.

- Solve for y to get $$y = -1 + be^{-1/x}$$

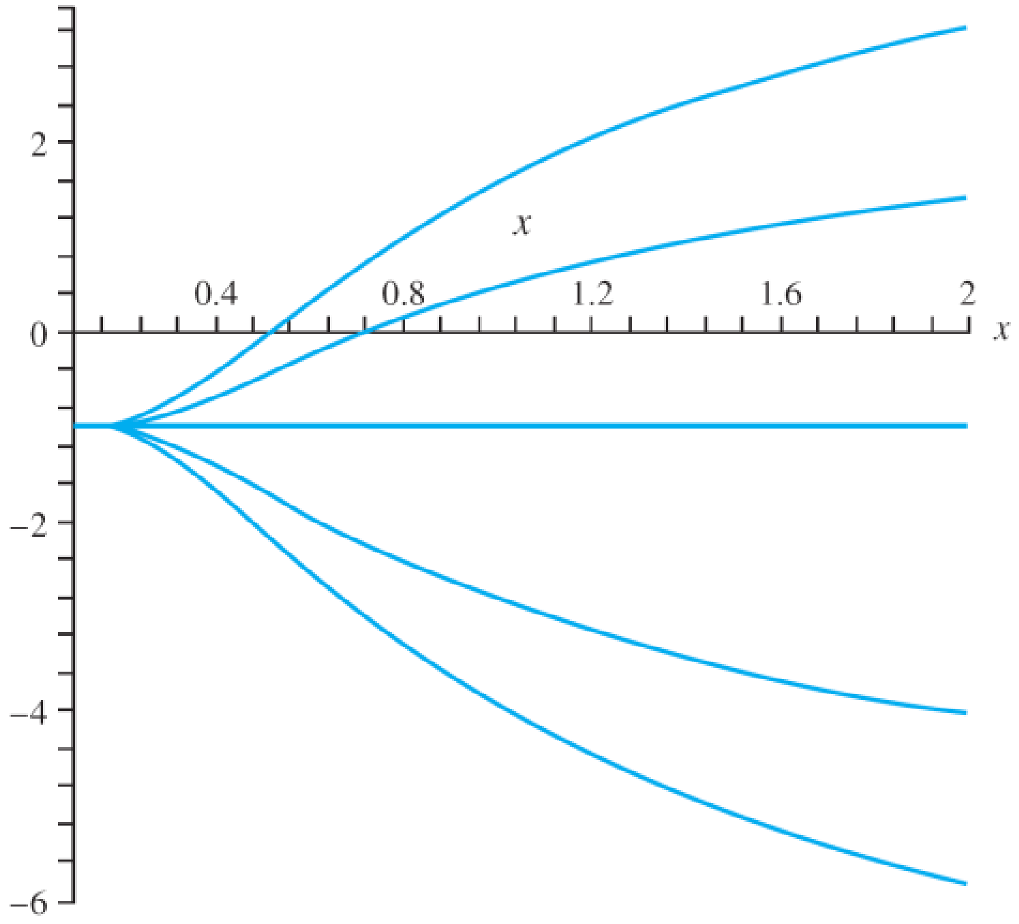
This is a solution of the differential equation for any number where we have written $b = e^c$.

ex. 7: continued

- Now notice that the differential equation also has the singular solution $y = -1$, which was disallowed in the separation of variables process when we divided by $y + 1$. We can include this singular solution in the solution by separation of variables by allowing $b = 0$, which gives $y = -1$.
- We therefore have the general solution

$$y = -1 + be^{-1/x}$$

in which b can be any real number, including zero. This expression contains all solutions. Integral curves (graphs of solutions) corresponding to $b = 0, 4, 7, -5, \& -8$.



***ex.* 8:** Solve the IVP $y' = y\frac{(x-1)^2}{y+3}$; $y(3) = -1$

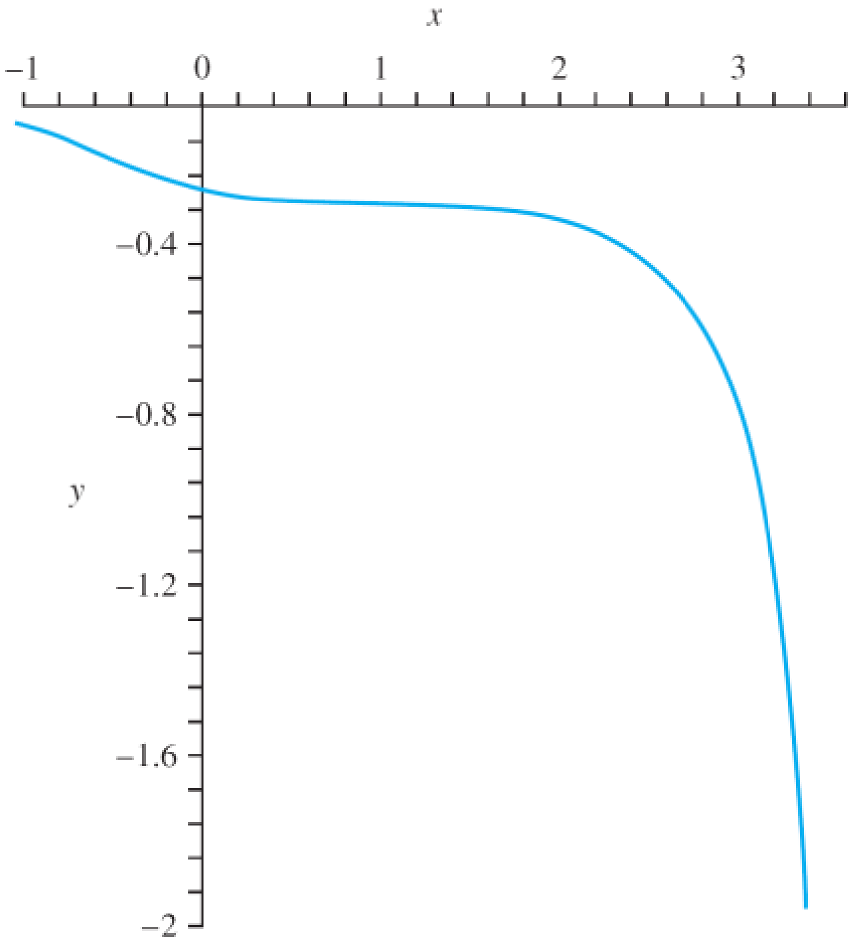
$$\frac{y+3}{y}dy = (x-1)^2 dx$$

$$\int \left(1 + \frac{3}{y}\right) dy = \int (x-1)^2 dx$$

$$y + 3\ln|y| = \frac{1}{3}(x-1)^3 + c$$

$$-1 = \frac{1}{3}(2^3) + c$$

$$y + 3\ln|y| = \frac{1}{3}(x-1)^3 - \frac{11}{3}$$



ex.'s

ex.	9	10
	$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{y}}{\mathrm{x}}$	$\frac{\mathrm{dz}}{\mathrm{dt}}=\frac{1}{\mathrm{ze^t}}$
Soln:	$\frac{dy}{dx}=\frac{y}{x}$ $\frac{dy}{y}=\frac{dx}{x}$ $\int\frac{dy}{y}=\int\frac{dx}{x}$ $\ln\lvert y\rvert=\ln\lvert x\rvert+c$	$\frac{dz}{dt}=\frac{1}{ze^t}$ $zdz=\frac{dt}{e^t}$ $\int zdz=\int\frac{dt}{e^t}$ $\frac{z^2}{2}=-e^{-t}+c$

Exercises

$\dfrac{dp}{dt} = \dfrac{\sqrt{1-p^2}}{t}$		ans:	$p = \sin(c + \ln t)$
$y' = xy^2$	$y(0) = 2$		$y = 2(C - x^2)^{-1}$
$y' = x + xy$	$y(0) = 0$		$y = e^{0.5x^2} - 1$
$y' + 2xy = 3x$	$y(0) = 5$		$y = \dfrac{1}{2}\left(1 + 7e^{-x^2}\right)$

Homogeneous Equations (of degree zero)

Form of Equation: $\frac{dy}{dx} = f(x, y)$

$f(x, y)$ is a function homogeneous of degree zero, i.e.,

$$f(tx, ty) = t^0 f(x, y) = f(x, y)$$

A function homogeneous of degree *n* can be defined as

$$f(tx, ty) = t^n f(x, y).$$

Method of Solution: Substituting $y = vx \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, reduces the equation in form separable in variables v and x. Integrate both sides and then substituting the value of v will give the solution.

*ex.*11: Solve $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

$$f(x,y) = \frac{y-x}{y+x}$$

$$f(tx,ty) = \frac{y-x}{y+x}$$

$$y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{y-x}{y+x} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1}$$

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1} \quad \Longrightarrow \quad x\frac{dv}{dx} = \frac{-1-v^2}{v+1}$$

$$\int \frac{(v+1)dv}{1+v^2} = -\int \frac{dx}{x}$$

$$tan^{-1}v + \frac{1}{2}\ln|1+v^2| = -\ln|x| + c$$

$$tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| = -\ln|x| + c$$

*ex.*12: Solve $\frac{dy}{dx} = \frac{y}{x}$

$$f(x, y) = \frac{y}{x}$$

$$f(tx, ty) = \frac{ty}{tx} = \frac{y}{x}$$

$$y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v \quad \Longrightarrow \quad x\frac{dv}{dx} = 0 \quad \Longrightarrow \quad \frac{dv}{dx} = 0$$

Integrating we get

$$v = c$$

or

$$\frac{y}{x} = c\ , y = cx$$

Exercises

$$y' = \frac{x}{y}$$

ans:

$$y(0) = 2$$
$$y(0) = 0$$
$$y(0) = 5$$

First Order Linear Equations

Form of Equation: $$\frac{dy}{dx} + P(x)y = Q(x)$$

Method of Solution: The integrating factor is $\mu = e^{\int p(x)dx}$. Multiplying the equation with this factor then solving this gives the solution y.

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}Q(x)$$

$$ye^{\int P(x)dx} = \int \left(e^{\int P(x)dx}Q(x)\right)dx + c$$

*ex.*13: Solve $\dfrac{dy}{dx} + 3y = x$

$p(x) = 3, q(x) = x$

Substituting values in

$$ye^{\int 3dx} = \int \left(xe^{\int 3dx}\right)dx \quad \Rightarrow \quad ye^{3x} = \int (xe^{3x})dx$$

$$ye^{3x} = \frac{xe^{3x}}{3} + \frac{e^{3x}}{9} + c \quad \Rightarrow \quad y = \frac{1}{9}(3x+1) + ce^{-3x}$$

$ex.14$: Solve $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

$$p(x) = -\frac{1}{x}, q(x) = 0.$$

Substituting values in

$$ye^{\int p(x)dx} = \int \left(e^{\int p(x)dx} q(x)\right) dx$$

$$ye^{\int -\frac{1}{x}dx} = \int 0 \, dx$$

$$ye^{-\ln|x|} = c \quad \Longrightarrow \quad y = cx$$

Exercise: Solve

$$\frac{dp}{dt} + pt = 3$$

Bernoulli Equations

A **Bernoulli differential equation** can be written in the following standard form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n, n \in \mathbb{R}\,, n \neq 0,1$$

where $n \neq 1$ (the equation is thus nonlinear).

Bernoulli Equations

Method of Solution:

The substitution $z = y^{1-n}$ transform the Bernoulli's equation into a linear equation in z.

This linear equation, after solving and back substituting the value of z gives the solution of the Bernoulli's equation.

and can be solved using the integrating factor method.

Bernoulli Equations

Dividing the above standard form by y^n gives:

$$\frac{1}{y^n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x)$$

Bernoulli Equations

Dividing the above standard form by y^n gives:

$$\frac{1}{y^n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x)$$

Bernoulli Equations

$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x)$$

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

$$\frac{dz}{dx} + p_1(x)z = q_1(x)$$

$$p_1(x) = (1-n)p(x), \qquad q_1(x) = (1-n)q(x)$$

$ex.1$: Solve $\dfrac{dy}{dx} + 3y = xy^2$

$$n = 2 \Longrightarrow z = y^{1-2} = \frac{1}{y} \Longrightarrow y = \frac{1}{z} \Longrightarrow \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

$$\frac{dz}{dx} - 3z = -x \quad \Longrightarrow \begin{matrix} p(x) = -3 \\ q(x) = -x \end{matrix}$$

Substituting values in

$$ze^{-\int 3dx} = \int (-xe^{-\int 3dx})dx \quad \Longrightarrow \quad \frac{e^{-3x}}{y} = -\int (xe^{-3x})dx$$

$$ye^{3x} = 1/(\frac{xe^{3x}}{3} + \frac{e^{3x}}{9} + c_1) \quad \Longrightarrow \quad y = \frac{9e^{-3x}}{[3xe^{3x} + e^{3x} + c]}$$

where $c = 3c_1$

*ex.*2: Solve $\dfrac{dy}{dx} + xy = xy^2$

$$z = y^{1-2} = \frac{1}{y} \Longrightarrow y = \frac{1}{z}, \qquad \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

$$\frac{dz}{dx} - xz = -x \qquad \Longrightarrow \begin{aligned} p(x) &= -x \\ q(x) &= -x \end{aligned}$$

Substituting values in

$$ze^{-\int xdx} = \int\left(-xe^{-\int xdx}\right)dx \quad \Longrightarrow \quad ye^{\frac{-x^2}{2}} = \int\left(-xe^{\frac{-x^2}{2}}\right)dx$$

$$ye^{\frac{-x^2}{2}} = e^{\frac{-x^2}{2}} + \mathrm{c} \quad \Longrightarrow \quad y = 1 + \mathrm{c}e^{0.5x^2}$$

Exercises: Solve

$$y' - \frac{y}{x} = xy^2 \qquad \text{ans:} \qquad \frac{1}{y} = -\frac{x^2}{3} + x$$

$$y' + \frac{y}{x} = y^2 \qquad y(0) = 2 \qquad \frac{1}{y} = x(C - \ln x)$$

$$y' + \frac{y}{3} = e^x y^4 \qquad y(0) = 0 \qquad \frac{1}{y^3} = e^x(C - 3x)$$

$$xy' + y = xy^3 \qquad y(0) = 5 \qquad y^2 = \frac{1}{2x + Cx^2}$$

Exercises: Solve

$$y' + \frac{2}{x}y = -x^2y^2\cos x$$ ans: $$\frac{1}{y} = x^2(\sin x + C)$$

$$2y' + y\tan x = \frac{(4x+5)^2}{\cos x}y^3 \quad y(0) = 2$$ $$\frac{1}{y^2} = \frac{(4x+5)^3}{12\cos x} + \frac{C}{\cos x}$$

$$xy' + y = x^2y^2\ln x \quad y(0) = 0$$ $$\frac{1}{xy} = C + x(1 - \ln x)$$

$$y' = y\cot x + y^3\operatorname{cosec} x \quad y(0) = 5$$ $$y^2 = \frac{\sin^2 x}{2\cos x + C}$$

Orthogonal Trajectories

Given a one-parameter family of curves $F(x, y, c) = 0$.
A curve that intersects each member of the family at right angles (orthogonally) is called an orthogonal trajectory of the family.

The one-parameter families $F(x, y, c) = 0$ and $G(x, y, k) = 0$ are orthogonal trajectories if each member of one family is an orthogonal trajectory of the other family.

A procedure for finding a family of orthogonal trajectories $G(x, y, k) = 0$ for a given family of curves $F(x, y, c) = 0$ is as follows:
Step 1. Determine the differential equation for the given family $F(x, y, c) = 0$.
Step 2. Replace y_0 in that equation by $-1/y_0$; the resulting equation is the differential equation for the family of orthogonal trajectories.
Step 3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Example

Find family of curves orthogonal to one parameter family of quadratic parabolas $y = cx\char`^2$.

Solution:

$$y = cx^2$$

$$F(x, y, c) = y - cx^2 = 0$$

$$\frac{y}{x^2} = c \Rightarrow \frac{x^2y' - 2xy}{x^4} = 0 \Rightarrow y'_{old} = \frac{2y}{x}.$$

Now the slope of the family of new curves is

$$y'_{new} = \frac{-1}{y'_{old}} = \frac{-1}{\frac{2y}{x}} = \frac{-x}{2y}.$$

$$2yy' = -x$$

$$2\int yy'dy = -\int xdx$$

$$y^2 = -\frac{x^2}{2} + c$$

is required family of curves.

Exercises

1. Find the family of orthogonal trajectories of:

$$y = Cx^2 + 2$$

Answer: $x^2 + 2y^2 - 8y = C$

2. Find the orthogonal trajectories of the family of parabolas with vertical axis and vertex at the point $(-1, 3)$.

Answer: $(x+1)^2 + 2(y-3)^2 = C$