CSE2023 Discrete Computational Structures

Lecture 2

1.3 Propositional equivalences

- Replace a statement with another statement with the same truth value
- For efficiency (speed-up) or implementation purpose (e.g., circuit design)

Tautology and contradiction

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- A compound proposition:
  - **Tautology**: always true
  - **Contradiction**: always false
  - **Contingency**: neither a tautology nor a contradiction

Logical equivalence

- p ≡ q (p⟺q): the compound propositions p and q are logically equivalent if p ↔ q is a tautology
- Can use truth table to determine whether two propositions are equivalent or not

Example

- Show that ┐(p v q) and ┐p ^ ┐ q are equivalent

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	p→q	┐p˅q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example

$$\neg(p \vee q) \qquad \neg p \wedge \neg q$$

$$p \rightarrow q \qquad \neg p \vee q$$

$$p \vee (q \wedge r) \qquad (p \vee q) \wedge (p \vee r)$$

Example

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

De Morgan's laws

TABLE 2 De Morgan's Laws.
$\neg(p \land q) \equiv \neg p \lor \neg q$
$\neg(p \lor q) \equiv \neg p \land \neg q$

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Example

- Express the negation of "Heather will go to the concert or Steve will go to the concert"
- Negation:

  Heather will not go to the concert AND Steve will not go to the concert.

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De Morgan's law: general form

- The first example above is known as the De Morgan's law

$$\neg(p_1 \lor p_2 \lor \cdots \lor p_n) \equiv (\neg p_1 \land \neg p_2 \land \cdots \land \neg p_n)$$

$$\neg(p_1 \land p_2 \land \cdots \land p_n) \equiv (\neg p_1 \lor \neg p_2 \lor \cdots \lor \neg p_n)$$

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Logical equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

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TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg (p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditionals.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing new logical equivalences

- Show $\neg (p \to q) \equiv p \wedge \neg q$

  $\neg (p \to q) \equiv \neg(\neg p \vee q)$

  $\equiv \neg(\neg p) \wedge \neg q$

  $\equiv p \wedge \neg q$

Constructing new logical equivalences

- Show $\neg (p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

  $\neg (p \vee (\neg p \wedge q)) \equiv \neg p \wedge (\neg(\neg p \wedge q))$

  $\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$

  $\equiv \neg p \wedge (p \vee \neg q)$

  $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$

  $\equiv F \vee (\neg p \wedge \neg q)$

  $\equiv \neg p \wedge \neg q$

Limitations of proposition logic

- Proposition logic cannot adequately express the meaning of statements
- Suppose we know

  "*<u>Every</u>* computer connected to the university network is functioning property"
- No rules of propositional logic allow us to conclude

  "MATH3 is functioning property"

  where MATH3 is one of the computers connected to the university network

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Example

- Cannot use the rules of propositional logic to conclude from

  "CS2 is under attack by an intruder"

  where CS2 is a computer on the university network

  to conclude the truth

  "*<u>There is</u>* a computer on the university network that is under attack by an intruder"

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1.4 Predicate and quantifiers

- Can be used to express the meaning of a wide range of statements
- Allow us to reason and explore relationship between objects
- **Predicates**: statements involving variables, e.g., "x > 3", "x=y+3", "x+y=z", "computer x is under attack by an intruder", "computer x is functioning property"

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Example: x > 3

- The variable x is the subject of the statement
- **Predicate** "is greater than 3" refers to a property that the subject of the statement can have
- Can denote the statement by p(x) where p denotes the predicate "is greater than 3" and x is the variable
- p(x): also called the value of the **propositional function** p at x
- Once a value is *assigned* to the variable x, p(x) becomes a proposition and has a truth value

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Example

- Let p(x) denote the statement "x > 3"
  - p(4): setting x=4, thus p(4) is true
  - p(2): setting x=2, thus p(2) is false
- Let a(x) denote the statement "computer x is under attack by an intruder". Suppose that only CS2 and MATH1 are currently under attack
  - a(CS1)? : false
  - a(CS2)? : true
  - a(MATH1)?: true

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N-ary Predicate

- A statement involving n variables, $x_1, x_2, \ldots, x_n$, can be denoted by $p(x_1, x_2, \ldots, x_n)$
- $p(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** p at the n-tuple $(x_1, x_2, \ldots, x_n)$
- p is also called **n-ary predicate**

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Quantifiers

- Express the extent to which a predicate is true
- In English, *all*, *some*, *many*, *none*, *few*
- Focus on two types:
  - **Universal**: a predicate is true for **every** element under consideration
  - **Existential**: a predicate is true for there is **one or more** elements under consideration
- **Predicate calculus**: the area of logic that deals with predicates and quantifiers

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Universal quantifier $\forall$

- "p(x) for all values of x in the domain"
  $\forall x \quad p(x)$
- Read it as "for all x p(x)" or "for every x p(x)"
- A statement $\forall x \quad p(x)$ is false if and only if p(x) is not always true
- An element for which p(x) is false is called a **counterexample** of $\forall x \quad p(x)$
- A single counterexample is all we need to establish that $\forall x \quad p(x)$ is not true

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Example

- Let p(x) be the statement "x+1>x". What is the truth value of $\forall x \quad p(x)$?
  - Implicitly assume the domain of a predicate is not empty
  - Best to avoid "for any x" as it is ambiguous to whether it means "every" or "some"
- Let q(x) be the statement "x<2". What is the truth value of $\forall x \quad q(x)$ where the domain consists of all real numbers?

Example

- Let p(x) be "$x^2>0$". To show that the statement $\forall x \quad p(x)$ is false where the domain consists of all integers
  - Show a counterexample with x=0
- When all the elements can be listed, e.g., x_1, x_2, …, x_n, it follows that the universal quantification $\forall x \quad p(x)$ is the same as the conjunction $p(x_1)$ ^$p(x_2)$ ^…^ $p(x_n)$

Example

- What is the truth value of $\forall x \quad p(x)$ where p(x) is the statement "$x^2 < 10$" and the domain consists of positive integers not exceeding 4?

  $\forall x \quad p(x)$ is the same as p(1)^p(2)^p(3)^p(4)

Existential quantification $\exists$

- "There exists an element x in the domain such that p(x) (is true)"
- Denote that as $\exists x \quad p(x)$ where $\exists$ is the existential quantifier
- In English, "for some", "for at least one", or "there is"
- Read as "There is an x such that p(x)", "There is at least one x such that p(x)", or "For some x, p(x)"

Example

- Let p(x) be the statement "x>3". Is $\exists x \quad p(x)$ true for the domain of all real numbers?
- Let q(x) be the statement "x=x+1". Is $\exists x \quad p(x)$ true for the domain of all real numbers?
- When all elements of the domain can be listed, , e.g., x_1, x_2, …, x_n, it follows that the existential quantification is the same as disjunction $p(x_1) \vee p(x_2) \vee \ldots \vee p(x_n)$

Example

- What is the truth value of $\exists x \quad p(x)$ where p(x) is the statement "$x^2 > 10$" and the domain consists of positive integers not exceeding 4?

  $\exists x \quad p(x)$ is the same as $p(1) \vee p(2) \vee p(3) \vee p(4)$

Uniqueness quantifier $\exists! \quad \exists_1$

- There exists a unique x such that p(x) is true

  $\exists! \quad p(x)$
- "There is exactly one", "There is one and only one"

Quantifiers with restricted domains

- What do the following statements mean for the domain of real numbers?

$$\forall x < 0, x^2 > 0 \quad \text{same as} \quad \forall x(x < 0 \rightarrow x^2 > 0)$$
$$\forall y \neq 0, y^3 \neq 0 \quad \text{same as} \quad \forall y(y \neq 0 \rightarrow y^3 \neq 0)$$
$$\exists z > 0, z^2 = 2 \quad \text{same as} \quad \exists z(z > 0 \wedge z^2 = 2)$$

Be careful about → and ∧ in these statements

Precedence of quantifiers

- $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus

$\forall x\ \ p(x) \vee q(x) \equiv (\forall x\ \ p(x)) \vee q(x)$ rather than $\forall x\ \ (p(x) \vee q(x))$

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Binding variables

- When a quantifier is used on the variable x, this occurrence of variable is **bound**
- If a variable is not bound, then it is **free**
- All variables occur in propositional function of predicate calculus must be bound or set to a particular value to turn it into a proposition
- The part of a logical expression to which a quantifier is applied is the **scope** of this quantifier

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Example

What are the scope of these expressions?
Are all the variables bound?

$$\exists x(x+y=1)$$
$$\exists x(p(x) \wedge q(x)) \vee \forall x R(x)$$
$$\exists x(p(x) \wedge q(x)) \vee \forall y R(y)$$

The same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap

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