Ex

Find the absolute max and min values of $f(x) = x^{5/3}$ on $-1 \leq x \leq 8$

$$f'(x) = \frac{5}{3} x^{2/3} = 0 \Rightarrow x = 0, \quad f(0) = 0$$

$$f(-1) = -1, \quad f(8) = \sqrt[3]{8^5} = \sqrt[3]{(2^3)^5} = 2^{\cancel{\frac{3\cdot 5}{3}}} = 2^5$$

**Def.** A point $x_0 \in D_f$ is a critical point of $f$ if $f'(x_0) = 0$.

Absolute max (min) is either at a critical pt. or at an end pt.

Ex Find the critical pts for $f(x) = x^2 - 6x + 7$

$$f'(x) = 2x - 6 = 2(x-3) = 0 \Rightarrow x = 3, \quad f(3) = 3^2 - 6\cdot 3 + 7 = -2$$

$x = 3$ is critical pt.

Ex Find the critical pts for $f(x) = 6x^2 - x^3$

$$f'(x) = 12x - 3x^2 = 3x(4 - x) = 0 \Rightarrow \boxed{x = 0, 4}; \quad f(0) = 0, \quad f(4) = \dots$$

Ex Find the critical pts for $f(x) = x(4-x)^3$, $f(x) = (x-1)^2(x-3)^2$,

$f(x) = x^2 + 2/x$, $f(x) = \sqrt{2x - x^2}$

Ex

$$f(x) = x(4-x)^3 \Rightarrow f'(x) = (4-x)^3 + 3x(4-x)^2 \cdot (-1) = (4-x)^2(4-x-3x)$$
$$= (4-x)^2(4-4x) = 4(4-x)^2(1-x)$$

$x = 1, 4$, critical pts.

Ex $$f(x) = (x-1)^2(x-3)^2 \Rightarrow f'(x) = 2(x-1)(x-3)^2 + 2(x-1)^2(x-3)$$
$$= 2(x-1)(x-3)(x-3+x-1)$$
$$= 4(x-1)(x-3)(x-2) \Rightarrow x = 1, 2, 3 \text{ crt. pts.}$$

Ex

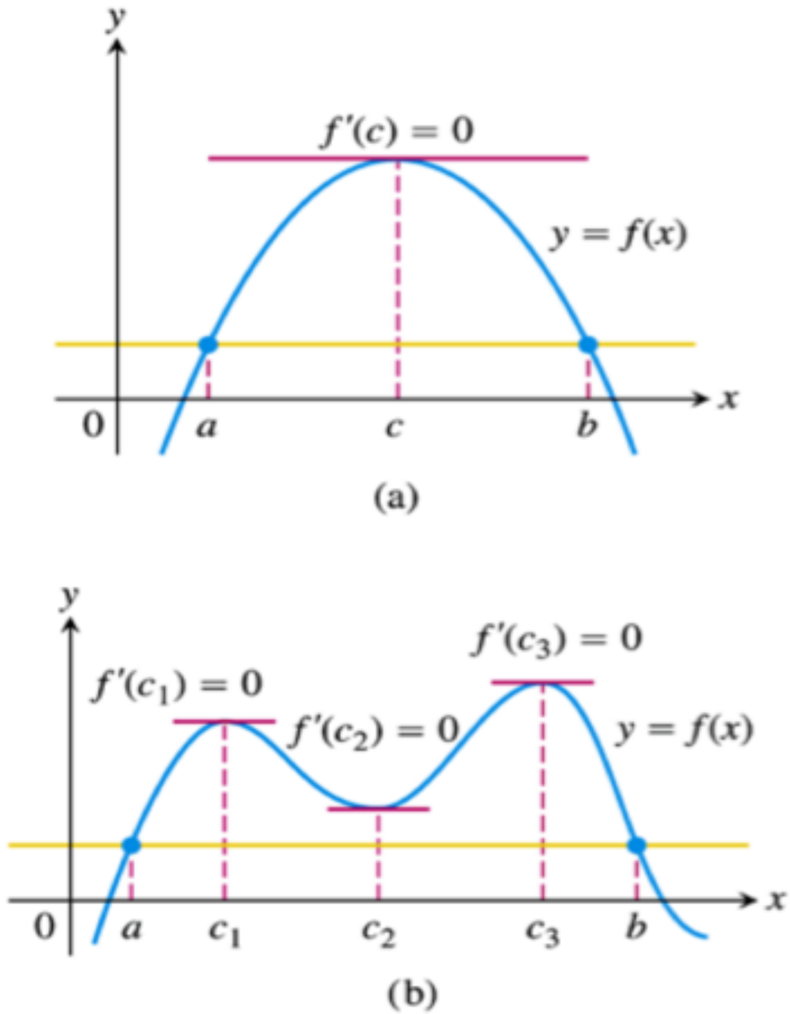
$$f(x) = x^2 + \frac{2}{x} \Rightarrow f'(x) = 2x - \frac{2}{x^2} = \frac{2(x^3-1)}{x^2}$$ , $x = 1$, crt. pt.

Ex

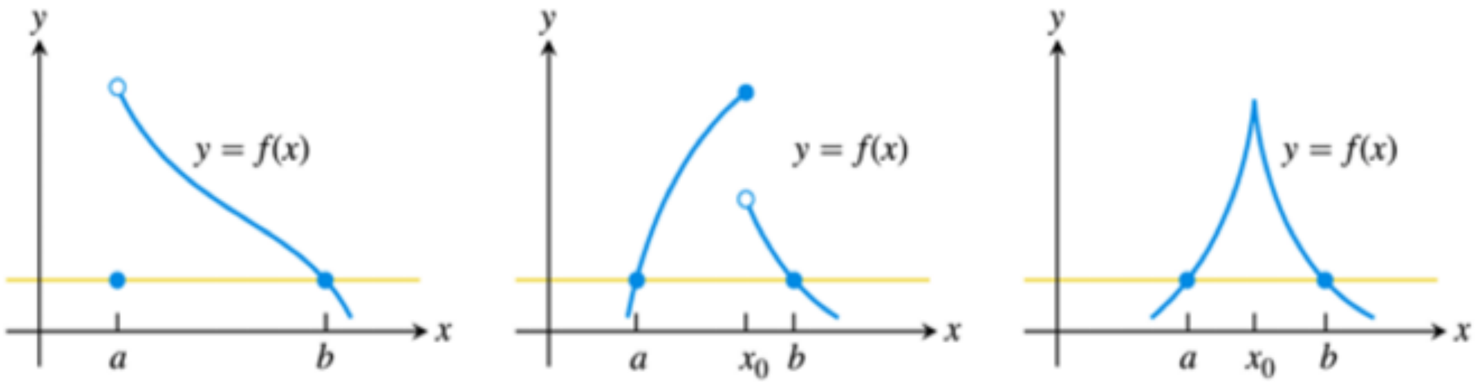
$$f(x) = \sqrt{2x - x^2} = (2x - x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2-2x)$$

$$f'(x) = \frac{2(1-x)}{2\sqrt{2x-x^2}}$$ , $x = 1$, crt pt.

**THEOREM -Rolle's Theorem** Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.



## Rolle's Theorem do not hold for the following situations



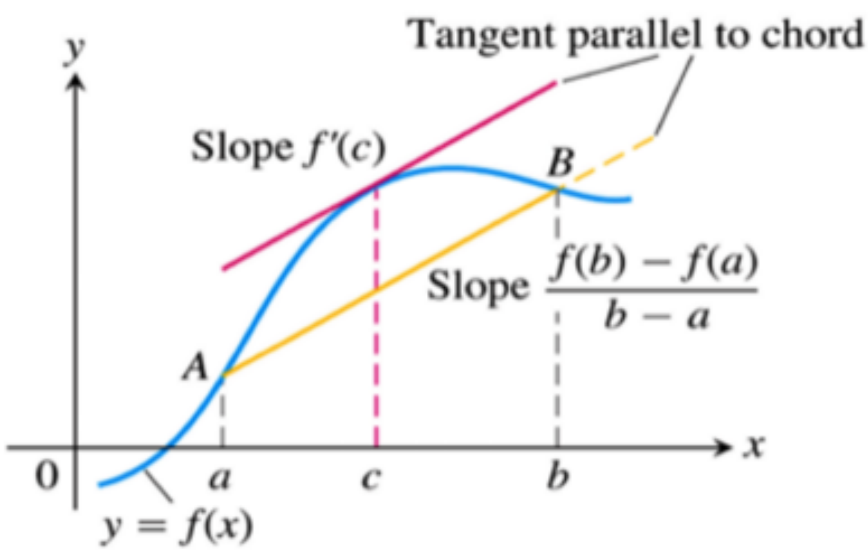

(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

**THEOREM** **The Mean Value Theorem** Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$



Ex Verify that $f(x) = x^3 + x - 1$ satisfies the hypothesis of the MVT on [0, 2]. Find all such c € (0, 2).

$$f'(c) = \frac{f(b) - f(a)}{b - a} \equiv \text{slope}$$

$$\frac{f(2) - f(0)}{2 - 0} = \frac{2^3 + 2 - \cancel{1} + \cancel{1}}{2} = 5 = f'(c)$$

$$f'(x) = 3x^2 + 1 \Rightarrow f'(c) = 3c^2 + 1$$

$$5 = 3c^2 + 1 \Rightarrow c = \pm 2/\sqrt{3}$$

Ex Verify that $f(x) = x + \frac{1}{x}$ satisfies the hypothesis of the MVT on [1/2, 2]. Find all such c € (1/2, 2).

$$f'(c) = \frac{f(2) - f(1/2)}{2 - \frac{1}{2}} = \frac{2 + \frac{1}{2} - \frac{1}{2} - 2}{3/2} = 0$$

$$f'(x) = 1 - \frac{1}{x^2} \Rightarrow f'(c) = 1 - \frac{1}{c^2}$$

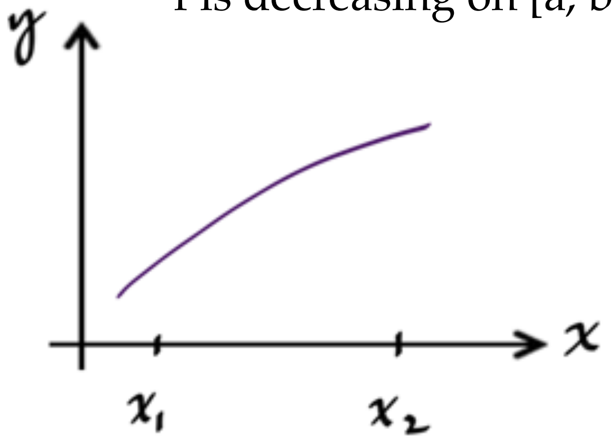
$$0 = f'(c) = 1 - \frac{1}{c^2} \Rightarrow c = \pm 1$$

# Increasing and decreasing functions (monotonic)

Suppose that f is continuous on [a, b] and diff on (a, b), then,

f is increasing on [a, b] if $f' > 0$ on (a, b)

f is decreasing on [a, b] if $f' < 0$ on (a, b)




f is increasing

$f(x_2) > f(x_1)$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Alternatively,

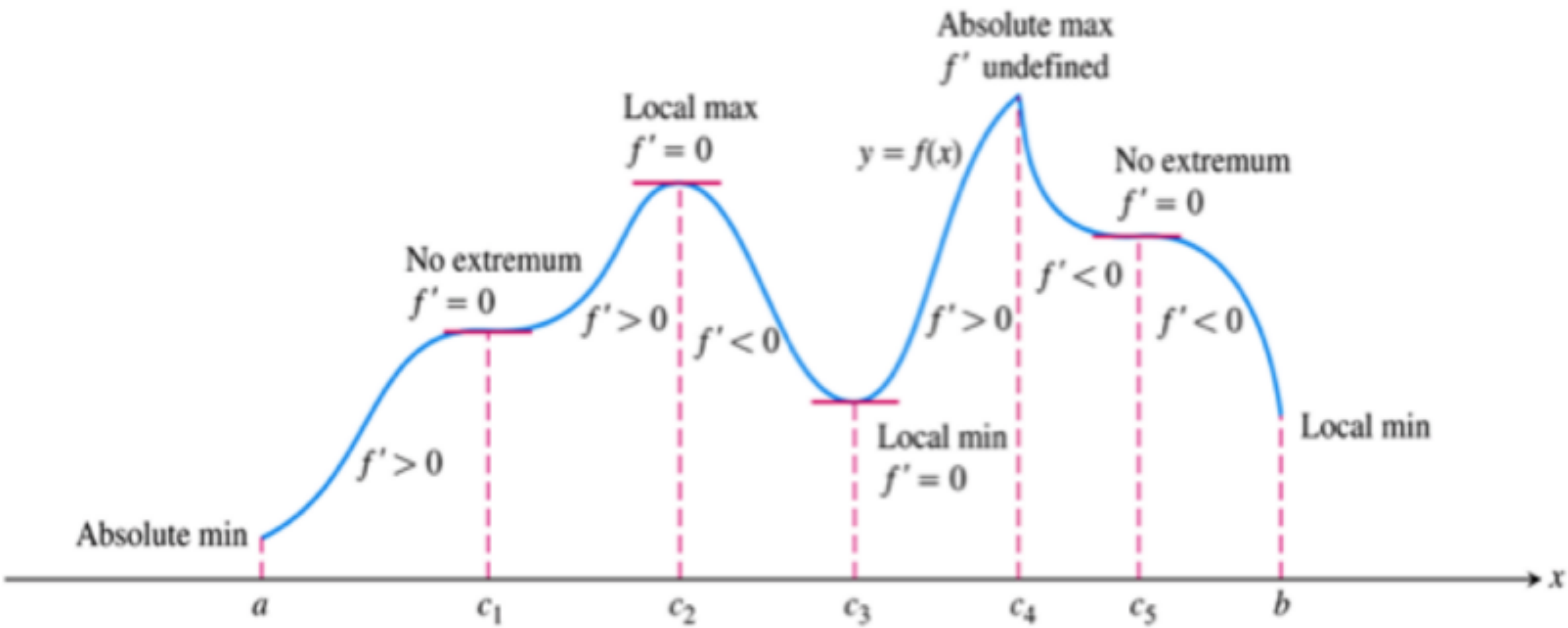
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for some } c$$

$f'(c) > 0 \Rightarrow f(x_2) > f(x_1)$ if $x_2 > x_1$.

### First Derivative Test for Local Extrema

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across $c$ from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.




The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

Ex. Find the critical points of $f(x) = x^3 - 3x + 3$ and identify the interval on which f is increasing and on which f is decreasing

$f'(x) = 3x^2 - 3 = 3(\underbrace{x^2-1}_{0})$, $x = \pm 1$, crt. pts

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y' | + | + | 0 | − | 0 | + | + |
| y | -16 | 1 | 5 | 3 | 1 | 5 | |

$y'$ +, increasing
$y'$ −, decreasing.