The Ratio and the Root Test

THEOREM —The Ratio Test Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

THEOREM —The Root Test Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \rho = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

the test is inconclusive since $\rho = 1$.

$\sum \frac{1}{n}$ is a p-series with $p=1$, hence, it diverges.

Ex $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$

$$\rho=\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}=\lim_{n\to\infty}\frac{\cancel{n!}}{(n+1)}\cdot\frac{1}{\cancel{n!}}=\lim_{n\to\infty}\frac{1}{n+1}=0<1$$

Since $\rho=0<1$, $\sum\frac{1}{n!}$ is convergent by the ratio test.

Ex $\displaystyle\sum_{n=1}^{\infty}n^3e^{-n}$

$$\rho=\lim_{n\to\infty}\frac{(n+1)^3\cdot e^{-n-1}}{n^3e^{-n}}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\cdot\frac{1}{e}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{1}{e}=\frac{1}{e}$$

$e=2.71$, $\frac{1}{e}=\frac{1}{2.71}<1$, it converges by the ratio test.

Ex $\displaystyle\sum_{n=1}^{\infty}\frac{nx^{2n}}{2^n}$, the series contains only positive terms no matter what x is, due to x^{2n}.

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)x^{2n+2}}{2^{n+1}}}{\frac{nx^{2n}}{2^n}}=\lim_{n\to\infty}\frac{(n+1)\cancel{x^{2n}}\cdot x^2}{2\cdot\cancel{2^n}}\cdot\frac{\cancel{2^n}}{n\cancel{x^{2n}}}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\frac{x^2}{2}=\frac{x^2}{2}$$

$$\frac{x^2}{2}<1\Rightarrow x<\pm\sqrt{2},\quad \underbrace{-\sqrt{2}}_{D}<\underbrace{x}_{C}<\underbrace{\sqrt{2}}_{D}$$

Ex $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{\cancel{(n+1)}}{(n+1)^n} \cdot \frac{\cancel{n!}}{\cancel{n+1}} \cdot \frac{n^n}{\cancel{n!}}$$

$$\rho = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1} = \frac{1}{e} < 1$$

$y = \left(\frac{x}{x+1}\right)^x$

conv. by ratio test

$$\ln y = x \ln \frac{x}{x+1} = \frac{\ln \frac{x}{x+1}}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln \frac{x}{x+1}}{1/x} = \lim_{x\to\infty} \frac{\frac{x+1}{x}\left(\frac{x+1-x}{(x+1)^2}\right)}{-1/x^2}$$ (L'Hosp.)

$$= \lim_{x\to\infty} \frac{-x^2}{x(x+1)} = \lim_{x\to\infty} - \frac{x^2/x^2}{\frac{x^2}{x^2}+\frac{x}{x^2}} = -1$$

$$\lim_{x\to\infty} \ln y = -1$$

$$\lim_{x\to\infty} y = e^{-1}$$

Ex

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \qquad \rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

converges by n-root test.

Ex

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{2}$$

$$= \frac{1}{2} \lim_{n\to\infty} n^{2/n} = \frac{1}{2} < 1 \text{, converges}$$

Ex

$$y = x^{2/x}$$

$$\ln y = \frac{2}{x} \ln x$$

$$\lim_{x\to\infty} \ln y = 2 \lim_{x\to\infty} \frac{\ln x}{x} = 2 \lim_{x\to\infty} \frac{1/x}{1} = 0, \qquad \text{(L'Hospital)}$$

$$\lim_{x\to\infty} y = e^0 = 1$$

$$\lim_{x\to\infty} x^{2/x} = 1$$

Ex $$\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right) \qquad \rho = \lim_{n\to\infty} \sqrt[n]{\sin^n(1/\sqrt{n})} = \lim_{n\to\infty} \sin(1/\sqrt{n}) = 0$$

converges.

Ex $$\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1} \qquad \rho = \lim_{n\to\infty} \left\{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}\right\}^{1/(n+1)}$$

$$\rho = \lim_{n\to\infty} \ln\left(e^2+\frac{1}{n}\right) = \ln e^2 = 2\ln e$$

$$= 2 > 1$$

Diverges by n^{th} root test.

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\sum_{n=1}^{\infty} \frac{n+2}{3^n}$
3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$
5. $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$
6. $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$
7. $\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!\,3^{2n}}$
8. $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$
10. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$
11. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$
12. $\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$
13. $\sum_{n=1}^{\infty} \frac{8}{(3+(1/n))^{2n}}$
14. $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$
15. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

    (*Hint:* $\lim_{n\to\infty}(1+x/n)^n = e^x$)
16. $\sum_{n=2}^{\infty} \frac{1}{n^{1+n}}$