Review of Integration

Ex $$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \quad , \quad n \neq -1$$

Ex $$\int \sec^2 x \, dx = \tan x + c \quad , \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

Ex Find the solution of the diff. eq.

$$\frac{dy}{dx} = \sqrt{x}\,(x+1)$$

which passes through $(1,2)$

$$\int dy = \int \sqrt{x}\,(x+1)\,dx$$

$$y = \int (x^{3/2} + x^{1/2})\,dx$$

$$y(x) = \frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + c$$

$$y(1) = \frac{2}{5}(1)^{5/2} + \frac{2}{3}(1)^{3/2} + c = 2$$

$$c = 14/15$$

$$y(x) = \frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + \frac{14}{15}$$

Substitutions – examples

Ex $$I = \int \underbrace{\left(\tfrac{1}{2}x^2 - 4\right)}_{}^{2/5} x\,dx = \int u^{2/5}\,du = \frac{5}{7}\left(\tfrac{1}{2}x^2-4\right)^{7/5} + C$$

$u = \frac{1}{2}x^2 - 4$

$du = x\,dx$

Ex $$I = \int x^5\sqrt{1-x^3}\,dx = \int x^3\sqrt{1-x^3}\,x^2\,dx$$

$u = 1 - x^3$ | $x^3 = 1-u$

$du = -3x^2\,dx$

$-\frac{1}{3}\,du = x^2\,dx$

$$I = \int x^3\sqrt{1-x^3}\,x^2\,dx = \int (1-u)\,u^{1/2}\left(-\frac{du}{3}\right)$$
$$= -\frac{1}{3}\int \left(u^{1/2} - u^{3/2}\right) du$$

Substitution in definite integrals

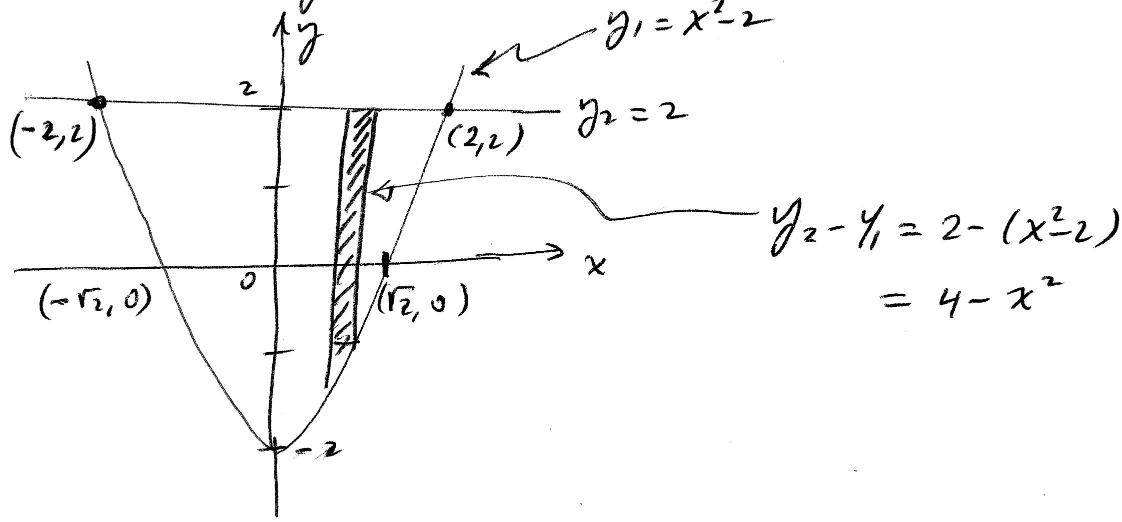
Ex $$I = \int_0^{\pi/4} \sin x\sqrt{\cos x}\,dx = -\int u^{1/2}\,du = -\frac{2}{3}\cos^{3/2} x \Big|_0^{\pi/4}$$

$u = \cos x$

$du = -\sin x\,dx$

The Area Under a Curve

Ex Find the area of the region enclosed by $y = x^2 - 2$ and $y = 2$.



$$A = 2\int_0^2 (y_2 - y_1)\,dx$$

$$= 2\int_0^2 (4 - x^2)\,dx$$

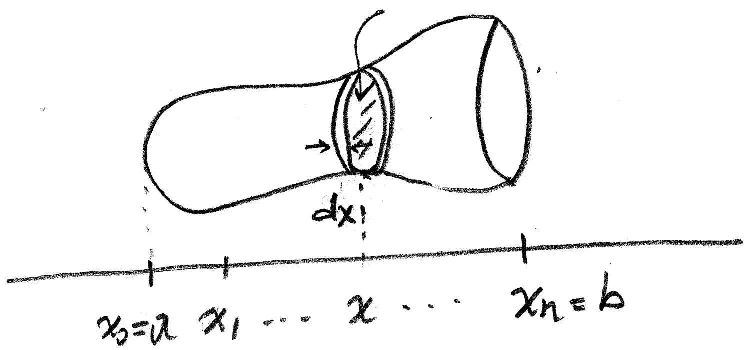
$$= 2\left[4x - \frac{1}{3}x^3\right]_0^2$$

$$= 2\left(8 - \frac{8}{3}\right) = 32/3.$$

The Volume Problem

Volumes using cross-sections

$A(x)$ = cross-sectional area at x



$$V \sim \sum_{i=1}^{n} A(x_i)\,\Delta_i x$$

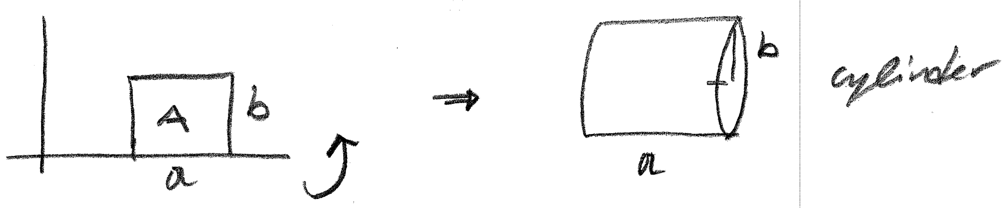
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta_i x$$

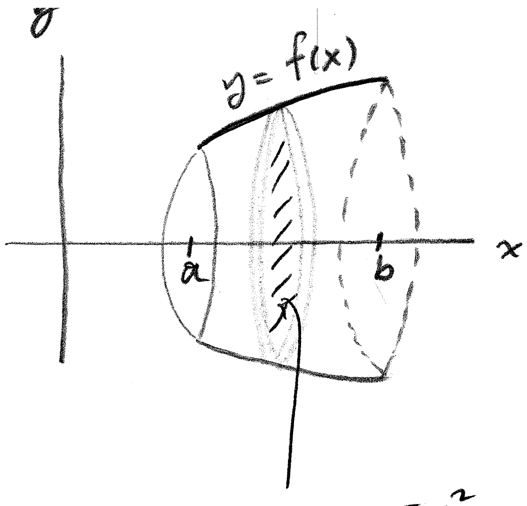
$$V = \int_a^b A(x)\,dx$$

Volume of a solid of revolution:

The **DISK** method

A solid generated by rotating an area in the plane about a fixed line is called a solid of revolution.





$y = f(x) > 0$ on $[a, b]$

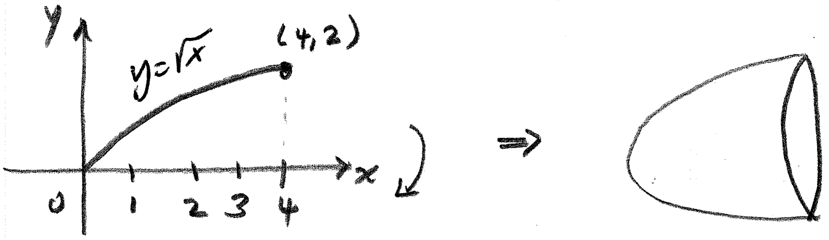
The region between y and the x-axis is rotated about the x-axis. | $a \leq x \leq b$

$$V = \int A(x)\,dx = \int_a^b \pi y^2\,dx$$

Similarly, if the region between the curve $x = g(y)$ and the y-axis, for $c \leq y \leq d$, is rotated about the y-axis,

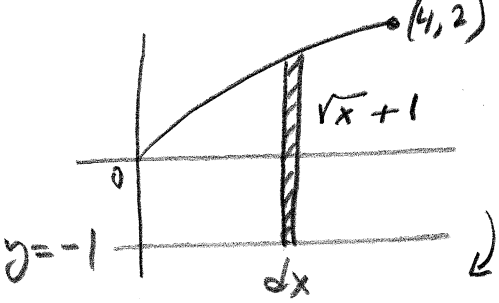
$$V = \int_c^d \pi x^2\,dy$$

Ex Find the volume of the solid of revolution formed by rotating $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x-axis

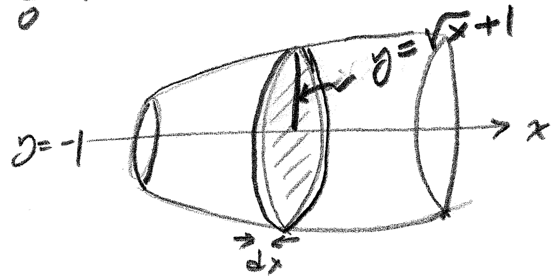


$$V = \int_0^4 \pi y^2\,dx = \pi \int_0^4 x\,dx = \frac{\pi}{2} x^2 \Big|_0^4 = 8\pi$$

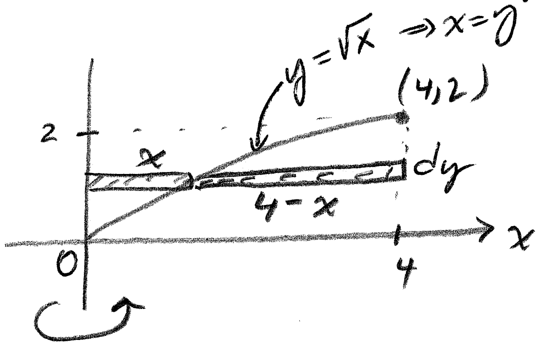
Ex Find the volume ... rotating about the line $y=-1$.



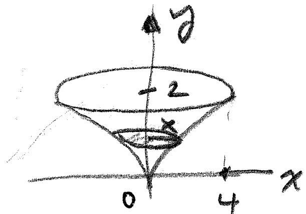
$$V=\int_0^4 \pi y^2\,dx = \pi\int(\sqrt{x}+1)^2\,dx$$



Ex ... about the y-axis



$$V=\int_0^2 \pi x^2\,dy$$

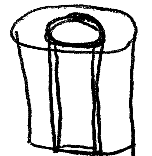


$$= \pi\int_0^2 (y^2)^2\,dy = \pi\int_0^2 y^4\,dy$$

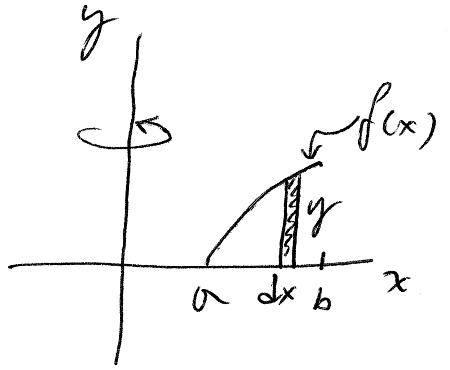
$$= \pi \left.\frac{y^5}{5}\right|_0^2$$

$$= 32\pi/5$$

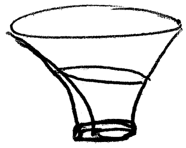
The Shell Method



The region between $y = f(x)$ and the y-axis, $a \leq x \leq b$, is rotated about the y-axis



$\Rightarrow$



washer $\Rightarrow$ thin cylindrical shell of area ΔA

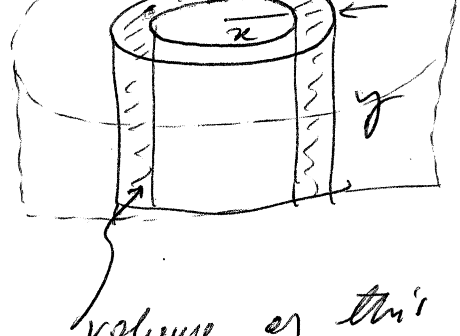
$$\Delta A = \pi\underbrace{(x+\Delta x)}_{r_2}^2 - \pi\underbrace{x}_{r_1}^2$$

$$= \pi(r_2^2 - r_1^2)$$

$$= 2\pi \underbrace{(r_2 - r_1)}_{\Delta x} \underbrace{\frac{(r_2 + r_1)}{2}}_{\text{avg } r \sim x}$$

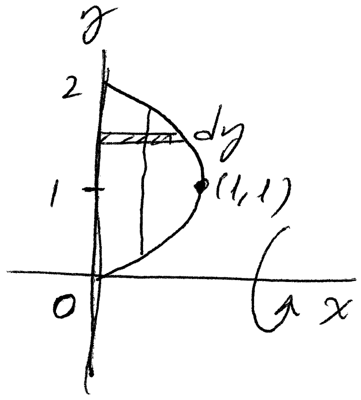
$$= 2\pi x \Delta x$$

$$V = \int_a^b 2\pi x y \, dx$$



volume of this shell is $2\pi x y \, dx$

Ex Find the volume obtained by rotating the region between $x = 2y - y^2$ and the y-axis about the x-axis



$$V = \int 2\pi x y \, dy$$

$$= 2\pi \int_0^2 (2y - y^2) y \, dy$$

$$= 2\pi \int_0^2 (2y^2 - y^3) \, dy$$

$$= 2\pi \left[\frac{2}{3} y^3 - \frac{1}{4} y^4 \right]_0^2$$

$$= 2\pi \left[\frac{2}{3} \cdot 8 - \frac{1}{4} \cdot 16 \right] = 8\pi/3$$

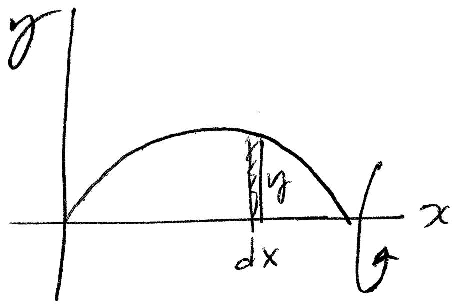
Sln. with Disk Method

$$V = \pi \int_0^1 (y_2^2 - y_1^2) \, dx \qquad y = ??!!$$

Ex Consider one arch of the sine curve

$y = \sin x \,, \quad 0 \leq x \leq \pi$

(a) Rotate about the x-axis

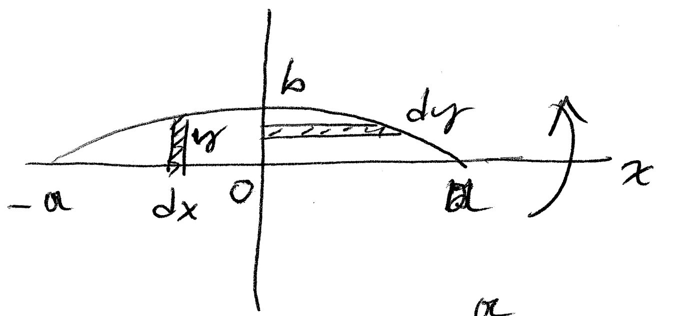


$$V = \int_0^{\pi} \pi y^2 dx = \pi \int_0^{\pi} \sin^2 x \, dx$$

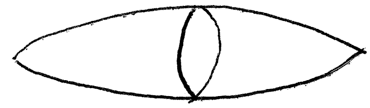
$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) dx = \frac{\pi^2}{2}$$

Rotating about y-axis : $V = 2\pi \int_0^{\pi} xy \, dx = 2\pi \int_0^{\pi} x \sin x \, dx$

Ex The ellipsoid of revolution is the solid obtained by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis



$\Rightarrow$

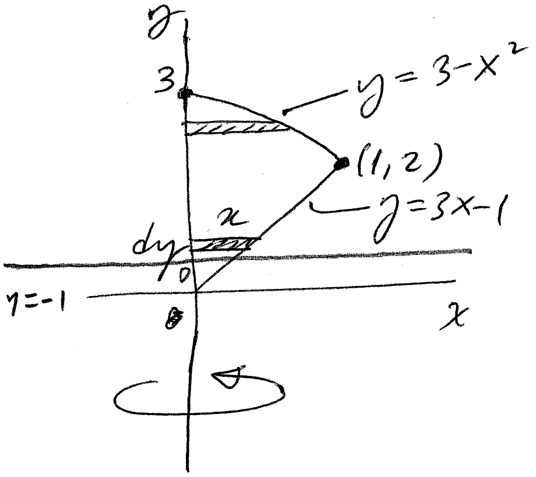


$$V = 2\int_0^{a} \pi y^2 dx = 2\pi \int_0^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2\pi b^2 \left(x - \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{4}{3}\pi a b^2$$

if $a = b$, we get a sphere.

Ex The ice-cream cone problem



pts of intersections:

$$3-x^2 = 3x-1$$

$$x^2+3x-4=0$$

$$(x-1)(x+4)=0$$

$\boxed{x_1=1}$ $\Rightarrow$ gives $y=2$

$x_2=-4$

$$V = \int_{-1}^{2} \pi x_1^2\,dy + \int_{2}^{3} \pi x_2^2\,dy$$

$$= \pi\int_{-1}^{2} \left(\frac{1+y}{3}\right)^2 dy + \pi\int_{2}^{3} (3-y)\,dy$$

$$= \frac{\pi}{3}\int_{-1}^{2} (1+2y+y^2)\,dy + \pi\int_{2}^{3} (3-y)\,dy$$

$$= \frac{\pi}{3}\left(y+y^2+\frac{y^3}{3}\right)\Big|_{-1}^{2} + \pi\left(3y-\frac{y^2}{2}\right)\Big|_{2}^{3}$$

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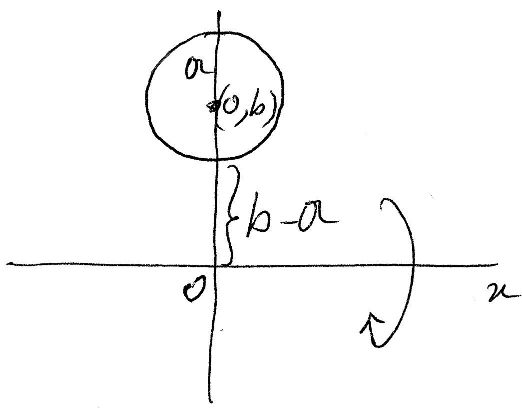
Ex The torus: rotate the circle with center $(0,b)$, radius a, $a<b$, about the x-axis.



torus

$$x^2+(y-b)^2=a^2$$

$$V = 2\pi\int_{b-a}^{b+a} x\,y\,dy = 2\cdot 2\pi\int_{b-a}^{b+a} y\sqrt{a^2-(y-b)^2}\,dy$$

$$y-b=u$$

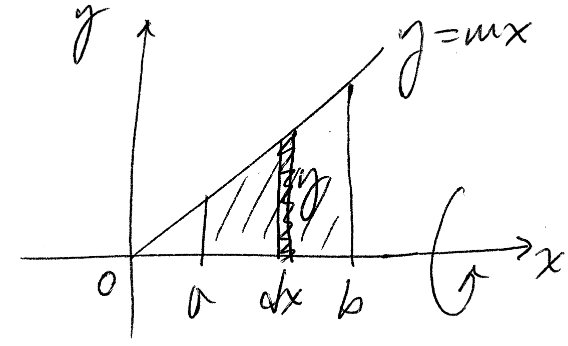
$$dy=du$$

$$V = 4\pi\int_{-a}^{+a}(u+b)\sqrt{a^2-u^2}\,du$$

$$= \underbrace{4\pi\int_{-a}^{a} u\sqrt{a^2-u^2}\,du}_{0 \text{ by symmetry}} + 4\pi b\underbrace{\int_{-a}^{+a}\sqrt{a^2-u^2}\,du}_{\text{semi-circle area}}$$

$$4\pi b\cdot\frac{\pi a^2}{2} = 2\pi^2 a^2 b$$

Ex Frustum of a cone



(a) About x-axis

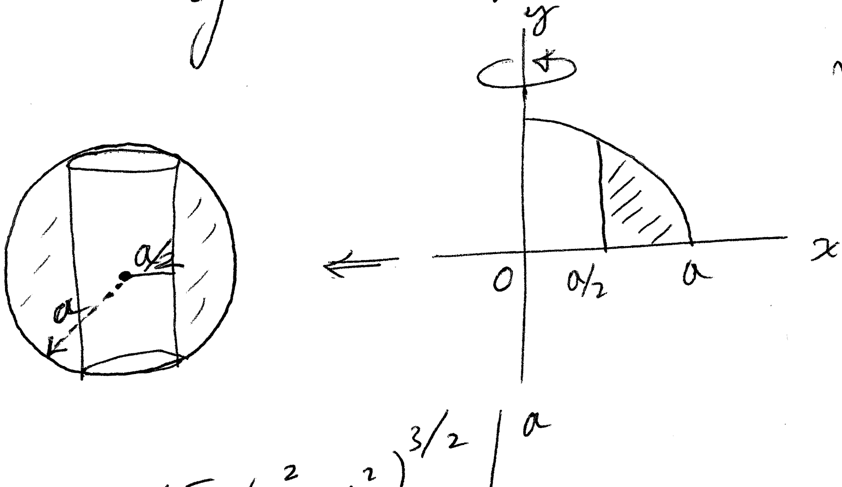
$$V = \int_0^b \pi y^2 dx$$

$$= \pi \int_0^b m^2 x^2 dx$$

$$= \frac{m^2 \pi}{3}(b^3 - a^3)$$

if $a = 0$, we get the volume of a cone

Ex A hole of diameter a is drilled symmetrically through a sphere of radius a, the axis of the hole being the diameter of the sphere. Find the remaining volume



$$V = 2 \cdot \int_{a/2}^{a} 2\pi x y \, dx$$

$$= 4\pi \int_{a/2}^{a} x \sqrt{a^2 - x^2} \, dx$$

$$= 2\pi \int_{a/2}^{a} \sqrt{a^2 - x^2} \, 2x dx$$

$u = a^2 - x^2$

$-du = 2x dx$

$$= -2\pi \int u^{1/2} du = -2\pi \cdot \frac{2}{3} u^{3/2}$$

$$= -\frac{4\pi}{3}(a^2 - x^2)^{3/2} \Big|_{a/2}^{a}$$

$$V = -\frac{4\pi}{3}(a^2 - x^2)^{3/2} \Big|_{a/2}^{a}$$

$$= \frac{\sqrt{3}}{2} \pi a^3$$