MATH 2055

Second Order Homogeneous LDEs

Examples of Second Order DEs

Examples of second-order differential equations

Differential equation	Physical problem	Quantities	Constitutive law
$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = 0$	One-dimensional heat flow	T = temperature A = area k = thermal conductivity Q = heat supply	Fourier $q = -k\, dT/dx$ q = heat flux
$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$	Axially loaded elastic bar	u = displacement A = area E = Young's modulus b = axial loading	Hooke $\sigma = E\, du/dx$ σ = stress
$S\frac{d^2w}{dx^2} + p = 0$	Transversely loaded flexible string	w = deflection S = string force p = lateral loading	

Examples of Second Order DEs

Examples of second-order differential equations

Differential equation	Physical problem	Quantities	Constitutive law
$\frac{d}{dx}\left(AD\frac{dc}{dx}\right)+Q=0$	One-dimensional diffusion	c = iron concentration A = area D = diffusion coefficient Q = ion supply	Fick $q = -D\,dc/dx$ q = ion flux
$\frac{d}{dx}\left(A\gamma\frac{dV}{dx}\right)+Q=0$	One-dimensional electric current	V = voltage A = area γ = electric conductivity Q = electric charge supply	Ohm $q = -\gamma\,dV/dx$ q = electric charge flux
$\frac{d}{dx}\left(A\frac{D^2}{32\mu}\frac{dp}{dx}\right)+Q=0$	Laminar flow in pipe (Poiseuille flow)	p = pressure A = area D = diameter μ = viscosity Q = fluid supply	$q = -(D^2/32\mu)\,dp/dx$ q = volume flux q = mean velocity

Wronskian

- A set of n functions $y_1(x), y_2(x), \cdots, y_n(x)$, is said to be ***linearly dependent*** (LD) over an interval I if there exist n constants $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

- Otherwise the set of functions is said to be ***linearly independent*** (LI)

Wronskian

A set of n functions $y_1(x), y_2(x), \cdots, y_n(x)$, is ***linearly independent*** over an interval I if and only if the determinant (***Wronski determinant***, or ***Wronskian***)

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix} \neq 0$$

example

Check linear dependency of set of fn's given below $\cos x\,, \sin x$

Sol'n:

$$W(\cos x\,, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$
$$= \cos^2 x + \sin^2 x$$
$$= 1 \neq 0$$

$\therefore \cos x\,, \sin x$ are LI

Recall that if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2 by 2 matrix, its determinant is the product of the main (upper-left to lower right) diagonal minus the product of the other diagonal.

That is, det = ad - bc. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

example

The solution of the $$y'' - 2y'' - 15 = 0,$$

$$y = C_1 e^{5x} + C_2 e^{-3x}$$

is composed of the two component functions,

$$y_1(x) = e^{5x} \quad \& \quad y_2(x) = e^{-3x}$$

$$W(e^{5x}, e^{-3x}) = \begin{vmatrix} e^{5x} & e^{-3x} \\ 5e^{5x} & -3e^{-3x} \end{vmatrix} = -3e^{2x} - 5e^{2x} = -8e^{2x} \neq 0 \; \forall \; x$$

It is impossible to turn e^{5x} into e^{-3x} by multiplying one or the other by a constant. These two component functions are linearly independent.

example

The functions $y_1(x) = e^{4x} \quad \& \quad y_2(x) = 5e^{4x}$

are **not** linearly independent. The Wronskian is

$$W(e^{4x}, 5e^{4x}) = \begin{vmatrix} e^{4x} & 5e^{4x} \\ 4e^{4x} & 20e^{4x} \end{vmatrix} = 20e^{8x} - 20e^{8x} = 0 \; \forall \; x$$

Note that one function can be made into the other by multiplying by a constant, e.g.

$$y_1(x) = \frac{1}{5} y_2(x) \quad \text{or} \quad y_2(x) = 5y_1(x)$$

example

The functions

$$(x) = x^2 + 2x, \quad y_2(x) = 3x + 1, \quad y_3(x) = 2x^2 + x - 1$$

are **not** linearly independent. Since the Wronskian is

$$W(y_1, y_2, y_3) = \begin{vmatrix} x^2 + 2x & 3x + 1 & 2x^2 + x - 1 \\ 2x + 2 & 3 & 4x + 1 \\ 2 & 0 & 4 \end{vmatrix}$$

$$W(y_1, y_2, y_3) = 2 \begin{vmatrix} 3x + 1 & 2x^2 + x - 1 \\ 3 & 4x + 1 \end{vmatrix} + 4 \begin{vmatrix} x^2 + 2x & 3x + 1 \\ 2x + 2 & 3 \end{vmatrix}$$

$$W(y_1, y_2, y_3) = 0 \; \forall \; x$$

Existence and Uniqueness Theorem

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

$$\mathrm{y} = c_1 y_1 + c_2 y_2 \qquad (2)$$

$$y(x_0) = k_0, y'(x_0) = k_1 \qquad (3)$$

If $p(x)$ and $q(x)$ are **continuous** function on some open interval I and x_0 is in I, then the initial value problem consisting of (**1**) and (**3**) has a **unique** solution $y(x)$ on the interval I.

Linear Dependence and Independence of Sol'n

- Suppose that (1) has continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of (1) on I are linear dependent on I if and only if their Wronskian W **is zero** at some x_0 in I.
- Furthermore, if $W = 0$ for $x = x_0$, then $W = 0$ on I; hence if there is an x_1 in I at which W **is not zero**, then y_1, y_2 are liner independent on I .

Linear Dependence and Independence of Sol'n

Typically, solutions to linear, homogeneous, autonomic *n*th-order differential equations appear in the following ways:

- As functions of the form $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$, where $r_1, r_2, \ldots, r_n$ are all different real numbers. These will always be **linearly independent**.
- As functions of the form $e^{rx}, xe^{rx}, x^2 e^{rx}$, ... These will also be **linearly independent**. (We have not seen a case like this yet. We will.)
- As trigonometric functions $\sin(bx), \cos(bx)$ or $e^{ax}\sin(bx), e^{ax}\cos(bx)$. These will also be **linearly independent**.

A General Solution of (1) includes All Sol'ns

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

- **Theorem 3 (Existence of a general solution)**

  If $p(x)$ and $q(x)$ are continuous on an open interval I, then (1) has a general solution on I.

A General Solution of (1) includes All Sol’ns

- **Theorem 4 (General solution)**

  Suppose that (1) has continuous coefficients $p(x)$ and $q(x)$ on some open interval I. Then every solution $y = y(x)$ of (1) is of the form

$$y = c_1 y_1(x) + c_2 y_2(x)$$

  where y_1, y_2 form a basis of solutions of (1) on I and c_1, c_2 are suitable constants. Hence (1) does not have singular solutions (i.e., solutions not obtainable from a general solution)

Exercises - Solve

Check if the given pair of functions are linearly dependent or not.

(a) $f(t) = e^t, \quad g(t) = e^{-t}$

(b) $f(t) = \sin\omega t, \quad g(t) = \cos\omega t$

(c) $f(t) = t + 1, \quad g(t) = 4t + 4$

(d) $f(t) = 2t, \quad g(t) = |t|$

the Wronskian

Suppose $y_1(t), y_2(t)$

are two solutions of $y'' + p(t)y' + q(t)y = 0$

Then

I. We have either $\mathrm{W}(y_1, y_2) \equiv 0$ or $\mathrm{W}(y_1, y_2)$ never zero;

II. If $\mathrm{W}(y_1, y_2) \neq 0$, then $\mathrm{y} = c_1 y_1 + c_2 y_2$ is the general solution.

They are also called to form a fundamental set of solutions.

As a consequence, for any IC's $y(t_0) = y_0, \quad y'(t_0) = y'_0,$

there is a unique set of (c_1, c_2) that give a unique solution.

Illustration of Theorem 2

Example 1: $$y'' + w^2 y = 0$$

has sol'ns $$y_1 = \cos wx\,, \quad y_2 = \sin wx$$

$$W(\cos wx\,, \sin wx) = 1 \Longrightarrow y = c_1 \cos wx + c_2 \sin wx$$

Example 2: $$y'' - 2y' + y = 0,$$

has sol'ns $$y_1 = e^x, \quad y_2 = xe^x$$

$$W(e^x, xe^x) = e^{2x}, \Longrightarrow y = (c_1 + c_2 x)e^x$$

the Wronskian

Abel's Theorem

Let y_1, y_2 be two (linearly independent) solutions to

$$y'' + p(t)y' + q(t)y = 0$$

on an open interval I. Then, the Wronskian $W(y_1, y_2)$ on I is given by

$$W(y_1, y_2)(t) = Ce^{\int -p(t)\, dt}$$

for some constant C depending on y_1, y_2, but independent on t in I.

the Wronskian

Example: Given

$$t^2y'' - t(t + 2)y' + (t + 2)y = 0$$

Find $W(y_1, y_2)$ without solving the equation.

Sol'n: We first find the $p(t)$

$$p(t) = -t(t+2)$$

which is valid for $t \neq 0$. By Abel's Theorem, we have

$$W(y_1, y_2)(t) = Ce^{\int -p(t)\,dt} = Ce^{\int t(t+2)\,dt} = Ce^{t+2\ln|t|} = t^2Ce^t$$

the Wronskian

Example: For the equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

given one solution $y_1 = {}^1/_t$, find a second linearly independent solution.

Sol'n: By Abel's Theorem, and choose $C = 1$, we have

$$W(y_1, y_2)(t) = Ce^{\int -p(t)\,dt} = Ce^{\int -\frac{3t}{2t^2}\,dt} = t^{-3/2}$$

By definition of the Wronskian,

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = \frac{y_2'}{t} + \frac{y_2}{t^2} = t^{-3/2}$$

Solve this for y_2 (taking $C = 0$): $y_2 = \frac{2}{3}\sqrt{t}$

Exercises

Exercise: If y_1, y_2 are two solutions of

$$ty'' + 2y' + te^t y = 0$$

and $W(y_1, y_2)(1) = 2$, find $W(y_1, y_2)(5)$.

Exercise: If $W(f, g) = 3e^{4t}$, and $f = e^{2t}$, find g.

Exercise: Consider the equation

$$t^2 y'' - t(t + 2)y' + (t + 2)y = 0, t > 0$$

Given $y_1 = t$, find the general solution.

Exercise: Given the equation $t^2 y'' - (t - \frac{3}{16})\, y = 0, \quad t > 0$ and

$y_1 = t^{1/4} e^{2\sqrt{t}}$, find y_2.

Linear Second Order DEs

The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

if one or more of $p(t), q(t), r(t)$ is not constant, the above equation has *variable coefficients.*

The *constant coefficient* linear second order differential equation is

$$ay'' + by' + cy = g(t)$$

where a, b, c are all constants.

Linear Second Order DEs

Initially we will make our life easier by looking at differential equations

with $g(t) = 0$.

When $g(t) = 0$ we call the differential equation **homogeneous** (**HDE**),

when $g(t) \neq 0$ we call the differential equation **non-homogeneous**.

Example

$$y(x) = 6\cos(4x) - 17\sin(4x)$$

is a solution of $\quad y'' + 16\mathrm{y} = 0$

$$y' = -24\sin(4x) - 68\cos(4x)$$
$$y'' = -96\cos(4x) + 272\sin(4x)$$

By substitution: $\quad y'' + 16y = 0$

$$F(x, y, y', y'') = 0$$
$$F(x, y(x), y'(x), y(x)'') = 0$$

Example

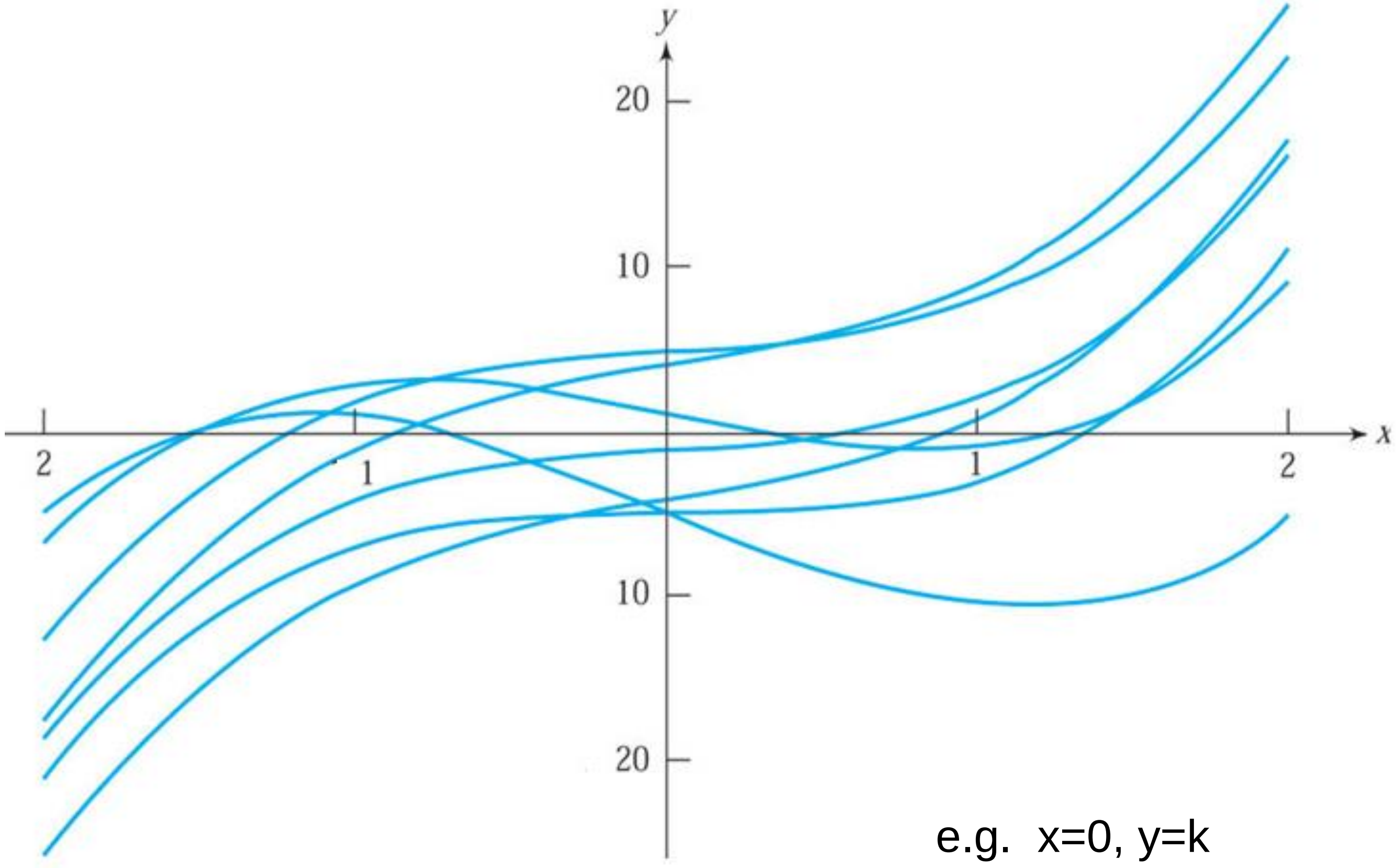
Consider the simple, 2nd-order LDE

$$y'' - 12x = 0$$

$$\begin{aligned} y'' - 12x = 0 \quad &\Longrightarrow \quad && y'' = 12x \\ y' = \int y''(x)dx \quad &= \int 12x dx \quad && = 6x^2 + C \\ y = \int y'(x)dx \quad &= \int (6x^2 + C)dx \quad && = 2x^3 + Cx + K \end{aligned}$$

To determine C and K, we need **two** initial conditions, one specify **a point** lying on the **solution curve** and the other its **slop**e at that point, e.g. $y(0) = K$, $y'(0) = C$

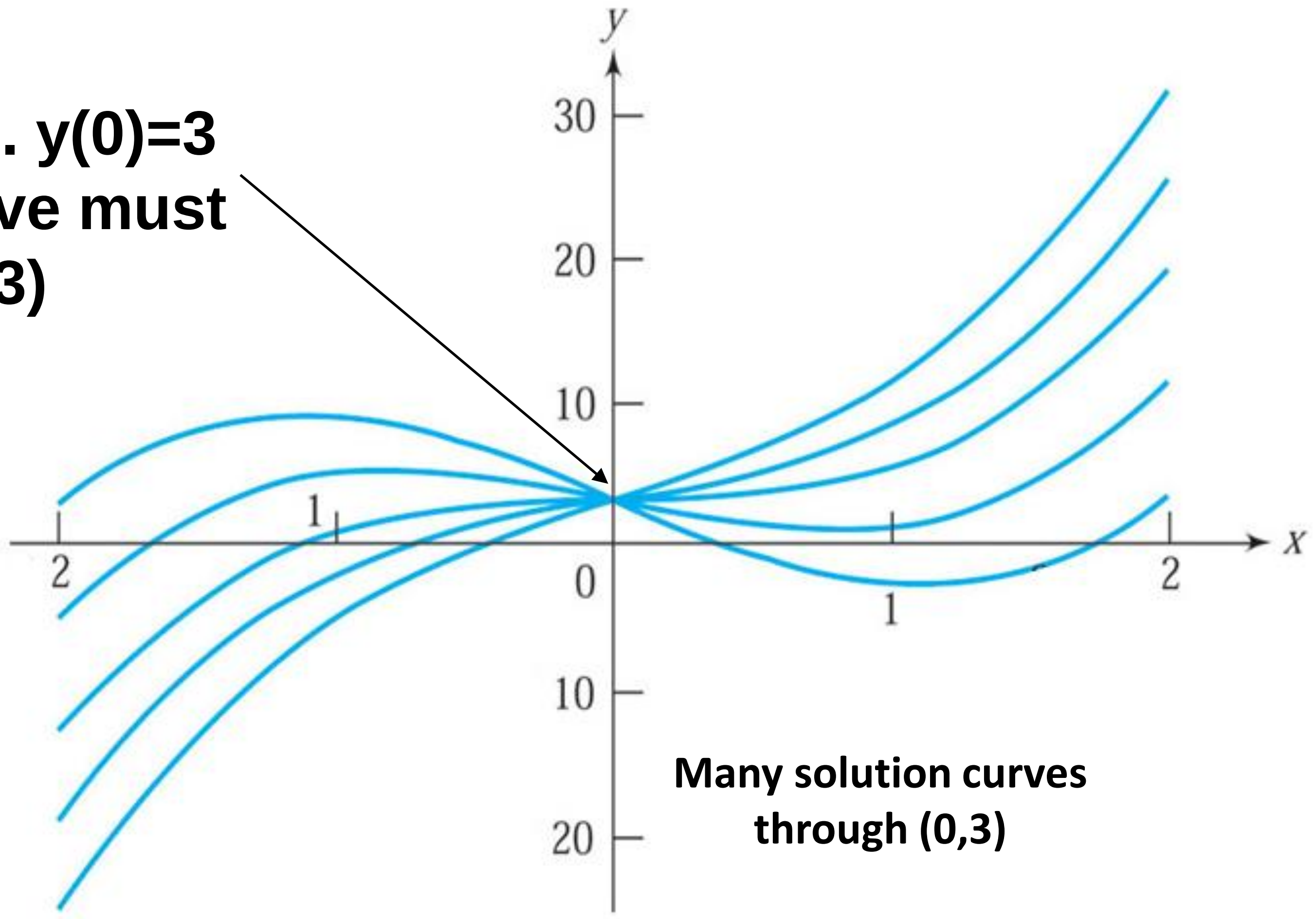
Example - continued



Graphs of y = 2x³ + C x +K for various values of C and K.

Example - continued

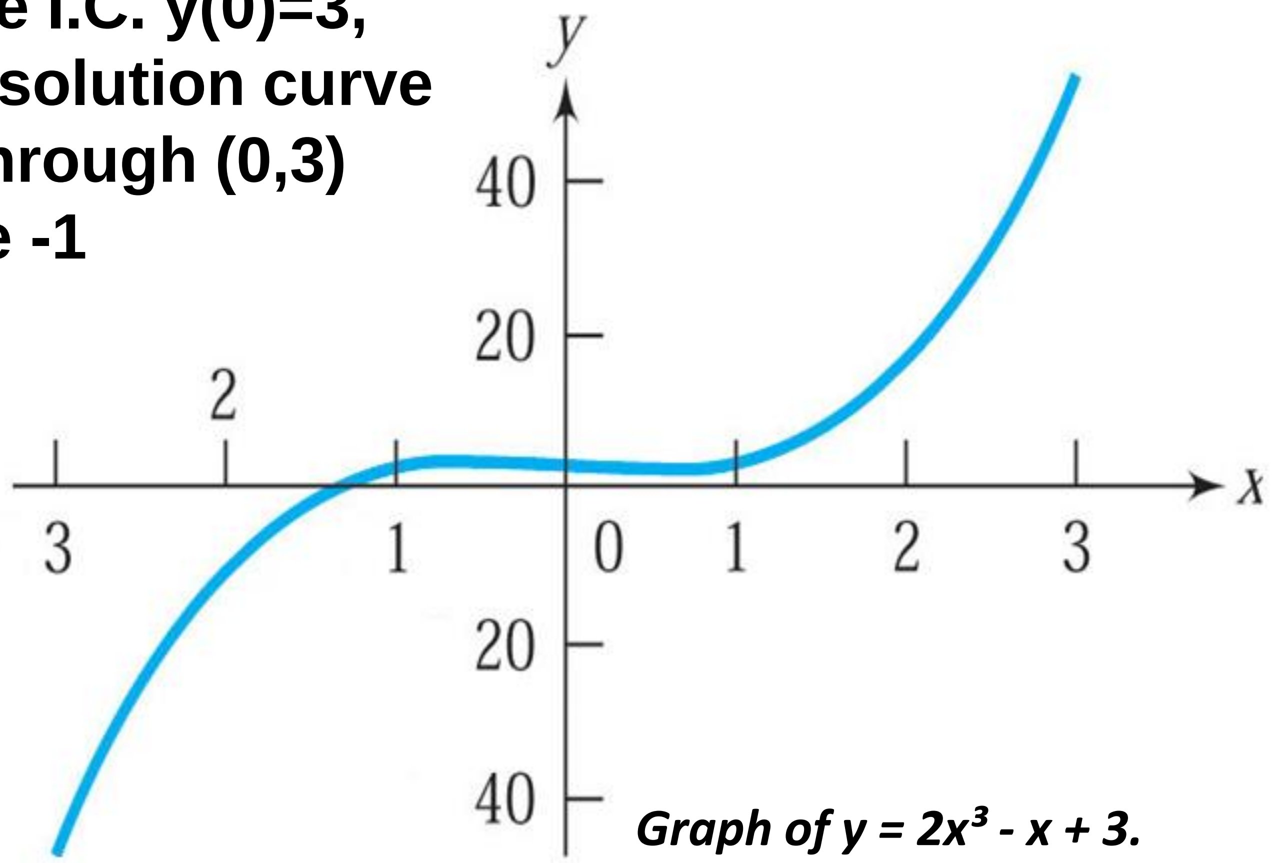
To satisfy the I.C. y(0)=3
The solution curve must pass through (0,3)



Graphs of $y = 2x^3 + C x + 3$ for various values of C.

Example - continued

To satisfy the I.C. $y(0)=3$, $y'(0)=-1$, the solution curve must pass through (0,3) having slope -1



Graph of $y = 2x^3 - x + 3$.

Linear Differential Operators

Consider the linear differential equation of the n-th order

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x)$$

Using the differential operator D this equation can be written as

$$\mathrm{L}(D)y(x) = F(x)$$

where $\mathrm{L}(D)$ is the differential polynomial equal to

$$\mathrm{L}(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$$

In other words, the operator $\mathrm{L}(D)$ is an algebraic polynomial, in which the differential operator D plays the role of a variable.

Linear Differential Operators

We denote by D the simplest differential operator, that is,

$$D = \frac{d}{dx} \quad \text{or} \quad D = \frac{d}{dt}$$

D acting on a function y, "returns" the first derivative of this function

$$Dy(x) = y'(x)$$

Higher order derivatives can be written in terms of D, that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

where D^2 is just the composition of D with itself. Similarly,

$$\frac{d^ny}{dx^n} = D^ny$$

some properties of the operator $\mathrm{L}(D)$

- The operator L(D) is linear:

$$\mathrm{L}(D)[c_1 y_1 + c_2 y_2] = c_1 \mathrm{L}(D) y_1 + c_2 \mathrm{L}(D) y_2$$

In the case of several operators $\mathrm{L}(D)$, $\mathrm{M}(D)$ and $\mathrm{N}(D)$ (the degree of the differential polynomials can be different), the following properties also hold:

- Commutative law of addition:

$$\mathrm{L}(D) + \mathrm{M}(D) = \mathrm{M}(D) + \mathrm{L}(D)$$

some properties of the operator $\mathrm{L}(D)$

- Associative law of addition:

$$[\mathrm{L}(D)+\mathrm{M}(D)]+\mathrm{N}(D)=\mathrm{L}(D)+[\mathrm{M}(D)+\mathrm{N}(D)]$$

For two operators L(D) and M(D), one can also define the multiplication operation:

$$[\mathrm{L}(D)\cdot \mathrm{M}(D)]y(x)=\mathrm{L}(D)\cdot[\mathrm{M}(D)y(x)]$$

It is important to note that the multiplication operation is commutative for differential operators with constant coefficients, that is for the operators of the form

$$\mathrm{L}(D)=a_0D^n+a_1D^{n-1}++a_{n-1}D+a_n$$

where $a_0,\cdots,a_n$ are constant numbers.

some properties of the operator $\mathrm{L}(D)$

- Commutative law of multiplication:

$$\mathrm{L}(D) \cdot \mathrm{M}(D) = \mathrm{M}(D) \cdot \mathrm{L}(D)$$

- Associative law of multiplication:

$$[\mathrm{L}(D) \cdot \mathrm{M}(D)] \cdot \mathrm{N}(D) = \mathrm{L}(D) \cdot [\mathrm{M}(D) \cdot \mathrm{N}(D)]$$

- Distributive law of multiplication over addition:

$$\mathrm{L}(D) \cdot [\mathrm{M}(D) + \mathrm{N}(D)] = \mathrm{L}(D) \cdot \mathrm{M}(D) + \mathrm{L}(D) \cdot \mathrm{N}(D)$$

- We also mention another useful property of the operator

$$D^m D^n = D^{m+n}$$

example

$$y'' + 4y' + 3y = 0$$

We use the convention and rewrite the DE

$$\begin{aligned} Dy &= y' \\ D^2y &= y'' \Longrightarrow \\ D^ny &= y^{(n)} \end{aligned} \qquad \begin{aligned} D^2y + 4Dy + 3y &= 0 \\ (D^2 + 4D + 3)y &= 0 \end{aligned} \qquad \Longrightarrow L(D) = D^2 + 4D + 3$$

symbolically, Since formerly

$$D^2 + 4D + 3 = (D + 3)(D + 1)$$

we see that:

$$L(D) = (D + 3)(D + 1)$$

example

$$y'' + y' - 2y = 0$$

rewrite the DE

$$\begin{aligned} D^2y + Dy - 2y &= 0 \\ (D^2 + D - 2)y &= 0 \end{aligned} \quad \Longrightarrow L(D) = D^2 + D - 2$$

$$D^2 + D - 2 = (D + 2)(D - 1)$$

$$L(D) = (D + 2)(D - 1)$$

example

$$y''' + 8y = 0$$

rewrite the DE

$$D^3y + 8y = 0$$
$$(D^3 + 8)y = 0$$
$$\Longrightarrow L(D) = D^3 + 8$$

$$D^3 + 8 = (D + 2)(D^2 - 2D + 4)$$

$$L(D) = (D + 2)(D^2 - 2D + 4)$$

example

$$y^{IV} - 2y'' + y = 0$$

rewrite the DE

$$\begin{aligned} D^4 y - 2D^2 y + y &= 0 \\ (D^4 - 2D^2 + 1)y &= 0 \end{aligned} \quad \Longrightarrow \quad L(D) = D^4 - 2D^2 + 1$$

$$D^4 - 2D^2 + 1 = (D^2 - 1)^2$$

$$L(D) = (D^2 - 1)^2$$

Solving Second Order, Linear, Homogeneous ODE with Constant Coefficients

Consider the sol'n of the type $\quad y(t) = e^{rt}$

Substituting in $\quad ay'' + by' + cy = 0$

We have $\quad a(r^2 e^{rt}) + b(re^{rt}) + ce^{rt} = 0$

$$ar^2 + br + c = 0 \quad \text{as} \quad e^{rt} \neq 0$$

Solving Second Order, Linear, Homogeneous ODE with Constant Coefficients

$$ar^2 + br + c = 0$$

Above equation is typically called the **characteristic (auxiliary, m) equation**

This will be a quadratic equation and so we should expect two roots, r_1 & r_2. Once we have these two roots we have two solutions to the differential equation.

$$y_1(t) = e^{r_1 t} \quad \& \quad y_2(t) = e^{r_2 t}$$

Characteristic Equation Cases

Case 1: Distinct Real Roots, $r_1 \neq r_2$

$$\boldsymbol{a^2 - 4b > 0} \quad \Longrightarrow \quad y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: Double roots, $r_1 = r_2 = r$

$$\boldsymbol{a^2 - 4b = 0} \quad \Longrightarrow \quad y = (c_1 + c_2 x) e^{r_1 x}$$

Case 3: Complex root, $r_{1,2} = \alpha \mp i\beta$

$$\boldsymbol{a^2 - 4b < 0} \quad \Longrightarrow \quad y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

Characteristic Equation – Distinct Real Roots

Example: Find two sol'ns to $y'' - 9y = 0$.

Sol'n: The characteristic equation is

$$r^2 - 9 = 0 \Rightarrow r_{1,2} = \pm 3$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = e^{3t} \quad \& \quad y_2(t) = e^{-3t}$$

The general solution is $y_c(t) = c_1 e^{3t} + c_2 e^{-3t}$

Characteristic Equation – Distinct Real Roots

Example: Find two sol'ns to

$$6y'' - y' - 2y = 0$$

Sol'n: The characteristic equation is

$$6r^2 - r - 2r = 0 \Rightarrow (2r+1)(3r-2) = 0 \Rightarrow r_{1,2} = \frac{-1}{2}, \frac{2}{3}$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = e^{-(1/2)t} \quad \& \quad y_2(t) = e^{(2/3)t}$$

$$y_c(t) = c_1 e^{-(1/2)t} + c_2 e^{(2/3)t}$$

Characteristic Equation – Distinct Real Roots

Example: Find two sol’ns to

$$y'' + 5y' + 4y = 0$$

Sol’n: The characteristic equation is

$$6r^2 - r - 2r = 0 \Rightarrow (r+1)(r+4) = 0 \Rightarrow r_{1,2} = -1, -4$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = e^{-t} \quad \& \quad y_2(t) = e^{-4t}$$

$$y_c(t) = c_1 e^{-t} + c_2 e^{-4t}$$

Characteristic Equation – Distinct Real Roots

Example: Solve the following IVP

$$y'' + 11y' + 24y = 0, \quad y(0) = 0, \quad y'(0) = -7$$

Sol'n: The characteristic equation is

$$r^2 + 11r + 24 = 0 \Rightarrow (r+8)(r+3) = 0 \Rightarrow r = -8, -3$$

The general solution and its derivative is

$$y_c(t) = c_1 e^{-3t} + c_2 e^{-8t}$$

$$y'_c(t) = -3c_1 e^{-3t} - 8c_2 e^{-8t}$$

Characteristic Equation – Distinct Real Roots

Putting the initial conditions, we have the following system of equations

$$0 = y_c(0) = c_1 + c_2$$

$$-7 = {y'}_c(0) = -3c_1 - 8c_2$$

Solving we get

$$c_1 = \frac{-7}{5} \quad \& \quad c_2 = \frac{7}{5}$$

Thus the solution is

$$y_c(t) = \frac{-7}{5}\, e^{-3t} + \frac{7}{5} e^{-8t}$$

Characteristic Equation – Distinct Real Roots

Example: Solve the following IVP

$$y'' + y' - 2y = 0, y(0) = 4, y'(0) = -5$$

Solution: The characteristic equation is

$$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = 1, -2$$

The general solution and its derivative is

$$y_c(t) = c_1 e^t + c_2 e^{-2t}$$

$$y'_c(t) = c_1 e^t - 2c_2 e^{-2t}$$

Characteristic Equation – Distinct Real Roots

Putting the initial conditions, we have the following system of equations

$$4 = y_c(0) = c_1 + c_2$$

$$-5 = y'_c(0) = c_1 - 2c_2$$

Solving we get

$$c_1 = 1 \quad \& \quad c_2 = 3$$

Thus the solution is

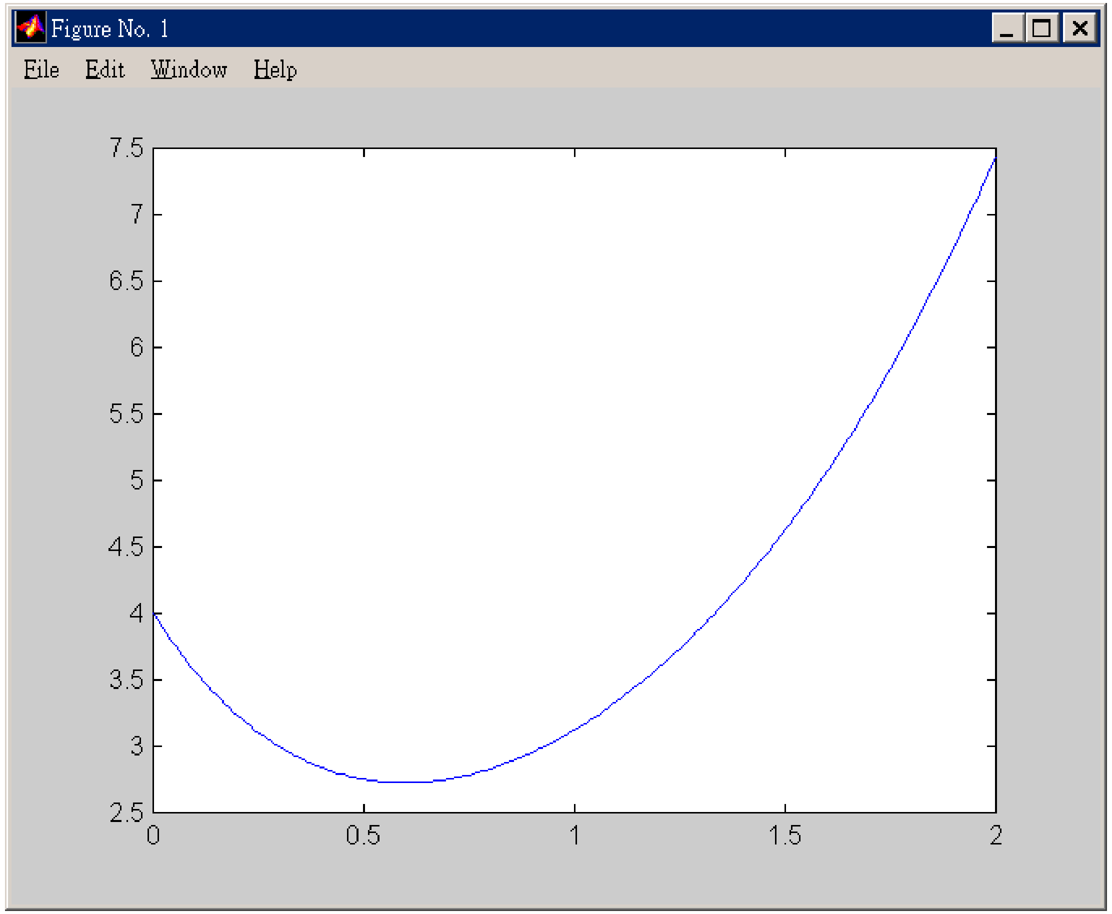
$$y_c(t) = e^t + 3e^{-2t}$$

Characteristic Equation – Distinct Real Roots

Plot Particular Solution

MATLAB Code:

```
t=[0:0.01:2];
y=exp(t)+3*exp(-2*t);
plot(t,y)
```



Reduction of Order

Let $y(x) = y_1(x)$ be the known solution of second order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Assume $y_2 = u(x)y_1(x)$ is the other solution. Then

$$a_2(x)y_2''(x) + a_1(x)y_2'(x) + a_0(x)y_2(x) = 0$$

$$a_2(x)\{u(x)y_1(x)\}'' + a_1(x)\{u(x)y_1(x)\}' + a_0(x)u(x)y_1(x) = 0$$

Reduction of Order

$$a_2\,(x)\{u''(x)y_1(x) + 2u'(x){y_1}'(x) + u(x){y_1}''(x)\} + a_1(x)\,\{u'(x)y_1(x)$$

$$+\,u(x){y_1}'(x)\} + a_0(x)u(x)y_1(x) = 0$$

$$u''(x)a_2(x)y_1(x) + u'(x)\{2a_2(x){y_1}'(x) + a_1(x)y_1(x)\}$$

$$+\,u(x)\{a_2(x){y_1}''(x) + a_1(x){y_1}'(x) + a_0\,(x)y_1(x)\} = 0$$

Since

$$a_2\,(x){y_1}''(x) + a_1(x){y_1}'(x) + a_0(x)y_1(x) = 0$$

Reduction of Order

therefore

$$u''(x)a_2(x)y_1(x) + u'(x)\{2a_2(x)y_1{}'(x) + a_1(x)y_1(x)\} = 0$$

$$u''(x)a_2(x)y_1(x) = -u'(x)\{2a_2(x)y_1{}'(x) + a_1(x)y_1(x)\}$$

$$\frac{u''(x)}{u'(x)} = -\left(\frac{2y_1{}'(x)}{y_{1(x)}} + \frac{a_1(x)}{a_2(x)}\right) \implies u(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}dx}}{y_1^2} dx$$

$$y_2 = u(x)y_1(x) = y_2 = y_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}dx}}{y_1^2} dx$$

Characteristic Equation – Repeated Roots

Example: Solve the following IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 1$$

Sol'n: The characteristic equation is

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2,2$$

The general solution is

$$y_c(t) = c_1 e^{2t} + c_2 e^{2t}$$

But it is written as

$$y_c(t) = (c_1 + c_2)e^{2t} = Ce^{2t} \quad \text{where} \quad C = c_1 + c_2$$

Characteristic Equation – Repeated Roots

Therefore we have to use Reduction of Order

$$y_2 = u(t)y_1(t) = u(t)e^{2t}$$

its derivatives are

$$y'_2(t) = 2u(t)e^{2t} + u'(t)e^{2t}$$
$$y''_2(t) = 4u(t)e^{2t} + 4u'(t)e^{2t} + u''(t)e^{2t}$$

Substituting in DE

$$\underbrace{4u(t)e^{2t} + 4u'(t)e^{2t} + u''(t)e^{2t}}_{y''} \quad \underbrace{-4(2u(t)e^{2t} + u'(t)e^{2t})}_{-4y'} \quad \underbrace{+4u(t)e^{2t}}_{+4y} \quad = 0$$

$$= 0$$

Characteristic Equation – Repeated Roots

$$u''(t)e^{2t} = 0 \quad \Longrightarrow \quad u''(t) = 0$$

$$u'(t) = c_2 \quad \Longrightarrow \quad u(t) = c_1 + c_2 t$$

Number of repeated roots	Multiply known sol’n by
2	$c_1 + c_2 t$
3	$c_1 + c_2 t + c_3 t^2$
4	$c_1 + c_2 t + c_3 t^2 + c_4 t^3$

then

$$y_2(t) = u(t)e^{2t} = (c_1 + c_2 t)e^{2t}$$

or we can write directly according to table

$$y_2(t) = c_1 e^{2t} + c_2 t e^{2t}$$

Characteristic Equation – Repeated Roots

The general solution and its derivative is

$$y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$$

$$y'_c(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$$

Putting the initial conditions, we have the following system of equations

$$3 = y_c(0) = c_1$$

$$1 = y'_c(0) = 2c_1 + c_2$$

Solving we get

$$c_1 = 3 \quad \& \quad c_2 = -5$$

Thus the solution is

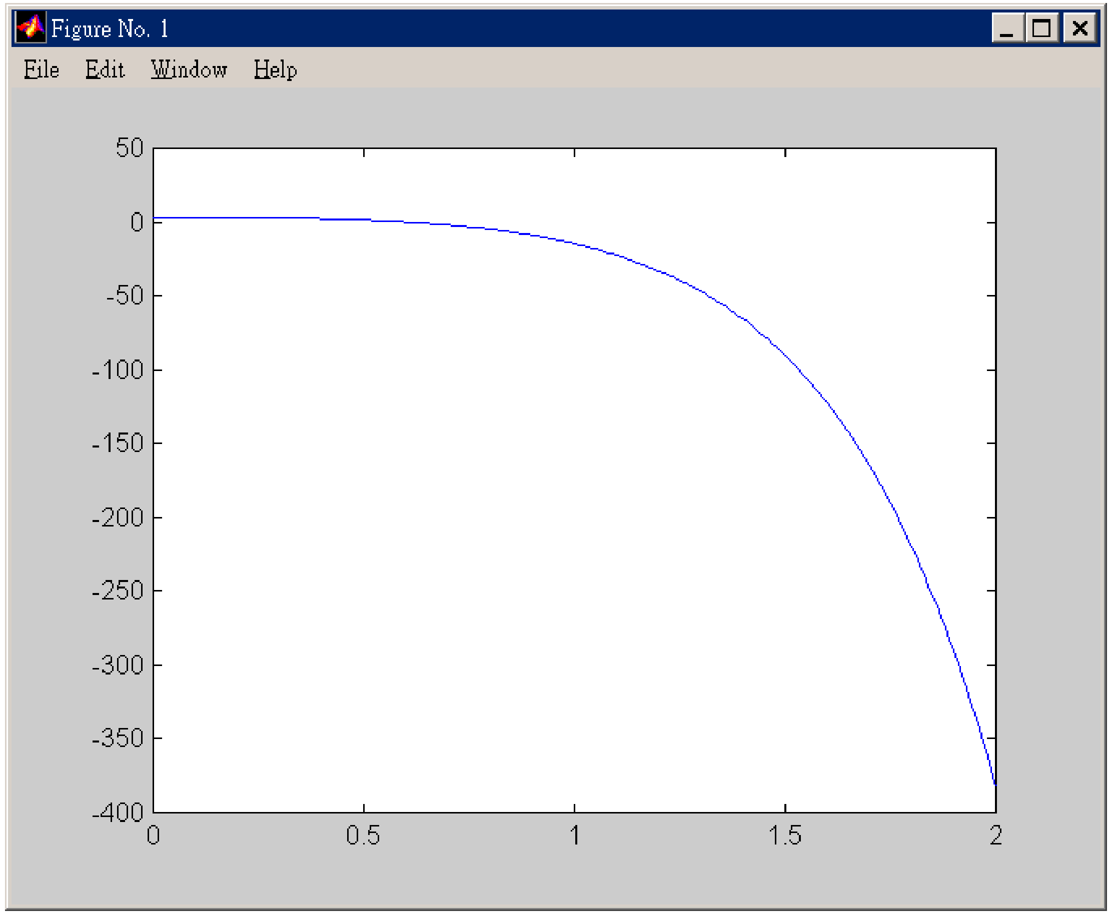
$$y(t) = (3 - 5t)e^{2t}$$

Characteristic Equation – Repeated Roots

Plot Particular Solution

MATLAB Code:

```
t=[0:0.01:2];
y=(3-5*t).*exp(2*t);
plot(t,y)
```



Example

What is the correct form of the solution of

$$y'' - 6y' + 9y = 0$$

Solution: The characteristic equation is

$$r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r_{1,2} = 3,3$$

The general solution is

$$y_c = (C_1 + C_2 x)e^{3x}$$

Example

Solve $$y''' + 3y'' + 3y' + y = 0$$

Solution: The auxiliary polynomial is

$$r^3 + 3r^2 + 3r + 1 = 0 \quad \Longrightarrow \quad (r+1)^3 = 0 \quad \Longrightarrow \quad r_{1,2,3} = -1$$

Thus, is the root of this polynomial, with multiplicity 3. The individual solutions are

$$y_1 = e^{-x}, \quad y_2 = xe^{-x} \quad \& \quad y_3 = x^2 e^{-x}$$

The general solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{-x}$$

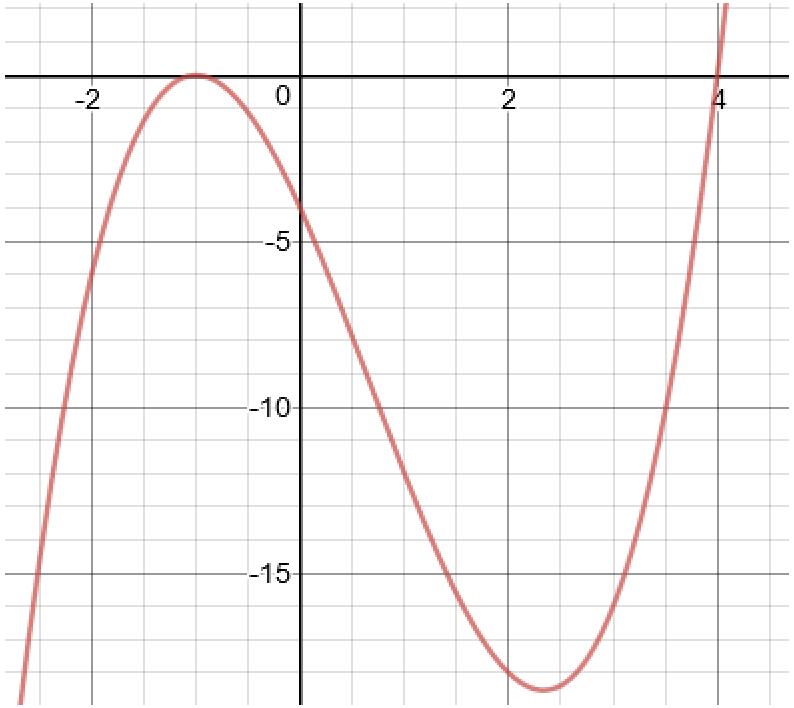
Example

Solve $$y''' - 2y'' - 7y' - 4y = 0$$

Solution: The auxiliary polynomial is

$$r^3 - 2r^2 - 7r - 4 = 0$$

It's difficult to factor a cubic, so we graph it to locate its roots: The graph appears to pass through *r* = 4, and glance the *r*-axis at *r* = –1, which suggests a root of multiplicity 2.



The possible factorization is

$$(r+1)^2(r-4) = 0$$

(You should expand this to verify that this is true.)

Example - continued

The general solution is $y = C_1 e^{4x} + [C_2 + C_3 x]e^{-x}$

We now check that the individual solutions are linearly independent by finding the Wronskian:

$$W(e^{4x}, e^{-x}, xe^{-x}) = \begin{vmatrix} e^{4x} & e^{-x} & xe^{-x} \\ 4e^{4x} & -e^{-x} & (1-x)e^{-x} \\ 16e^{4x} & e^{-x} & (x-2)e^{-x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & x \\ 4 & -1 & (1-x) \\ 16 & 1 & (x-2) \end{vmatrix} e^{2x}$$

$$= e^{2x}\left[\begin{vmatrix} -1 & (1-x) \\ 1 & (x-2) \end{vmatrix} - 1\begin{vmatrix} 4 & (1-x) \\ 16 & (x-2) \end{vmatrix} + x\begin{vmatrix} 4 & -1 \\ 16 & 1 \end{vmatrix}\right]$$

$$= 25e^{2x} \neq 0$$

Thus, the three individual solutions are linearly independent.

Euler Formula - Proof

$$e^{ix} = \cos x + i \sin x$$

Maclaurin Series

$$\begin{cases} e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{cases}$$

Euler Formula - Proof

$$e^{ix} = \cos x + i \sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + i \sin x$$

Euler Formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} = -1$$

$$e^{i2\pi} = 1$$

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = 0 + i$$

$$e^{-i\pi/2} = -1$$

Complex Exponential Function

$$z = s + it$$

$$e^{z} = e^{s+it} = e^{s}e^{it} = e^{s}(\cos t + i \sin t)$$

$$\bar{z} = s - it$$

$$e^{\bar{z}} = e^{s-it} = e^{s}e^{-it} = e^{s}(\cos t - i \sin t)$$

Characteristic Equation – Complex Roots

$$r_1 = \alpha + i\beta \quad \& \quad r_2 = \alpha - i\beta$$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$

$$y = e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$y = e^{\alpha x}[(c_1 + c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x]$$

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Characteristic Equation – Complex Roots

Example*:* Solve the following IVP

$$y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -8$$

Sol'n: The characteristic equation is

$$r^2 - 4r + 9 = 0 \Rightarrow r_{1,2} = 2 \pm \sqrt{5}i.$$

The general solution and its derivative is

$$y_c(t) = c_1 e^{2t} \cos\sqrt{5}t + c_2 e^{2t} \sin\sqrt{5}t$$

Putting the initial conditions, we have the following system of equations

$$0 = y_c(0) = c_1$$

Characteristic Equation – Complex Roots

so

$$y_c(t) = c_2 e^{2t} \sin \sqrt{5}t$$

$$y'_c(t) = 2c_2 e^{2t} \sin \sqrt{5}t + \sqrt{5}c_2 e^{2t} \cos \sqrt{5}t$$

$$y'_c(0) = \sqrt{5}c_2 \Rightarrow -8 = \sqrt{5}c_2 \Rightarrow c_2 = \frac{-8}{\sqrt{5}}$$

Thus the solution is

$$y(t) = \frac{-8}{\sqrt{5}} e^{2t} \sin \sqrt{5}t$$

Characteristic Equation – Complex Roots

Example*:* Solve the following IVP

$$y'' + 0.2y' + 4.01y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

Sol'n: The characteristic equation is

$$r^2 + 0.2r + 4.01 = 0 \Rightarrow r_{1,2} = -0.1 \pm 2i$$

The general solution and its derivative is

$$y_c(t) = e^{-0.1t}(A\cos 2t + B\sin 2t)$$

Putting the initial conditions, we have the following system of equations

$$0 = y_c(0) = A$$

Characteristic Equation – Complex Roots

so

$$y_c(t) = Be^{-0.1t} \sin 2t$$

$$y'_c(t) = e^{-0.1t}(-0.1B \sin 2t + 2B \cos 2t)$$

$$2 = y'_c(0) = 2B \Rightarrow B = 1$$

Thus the solution is

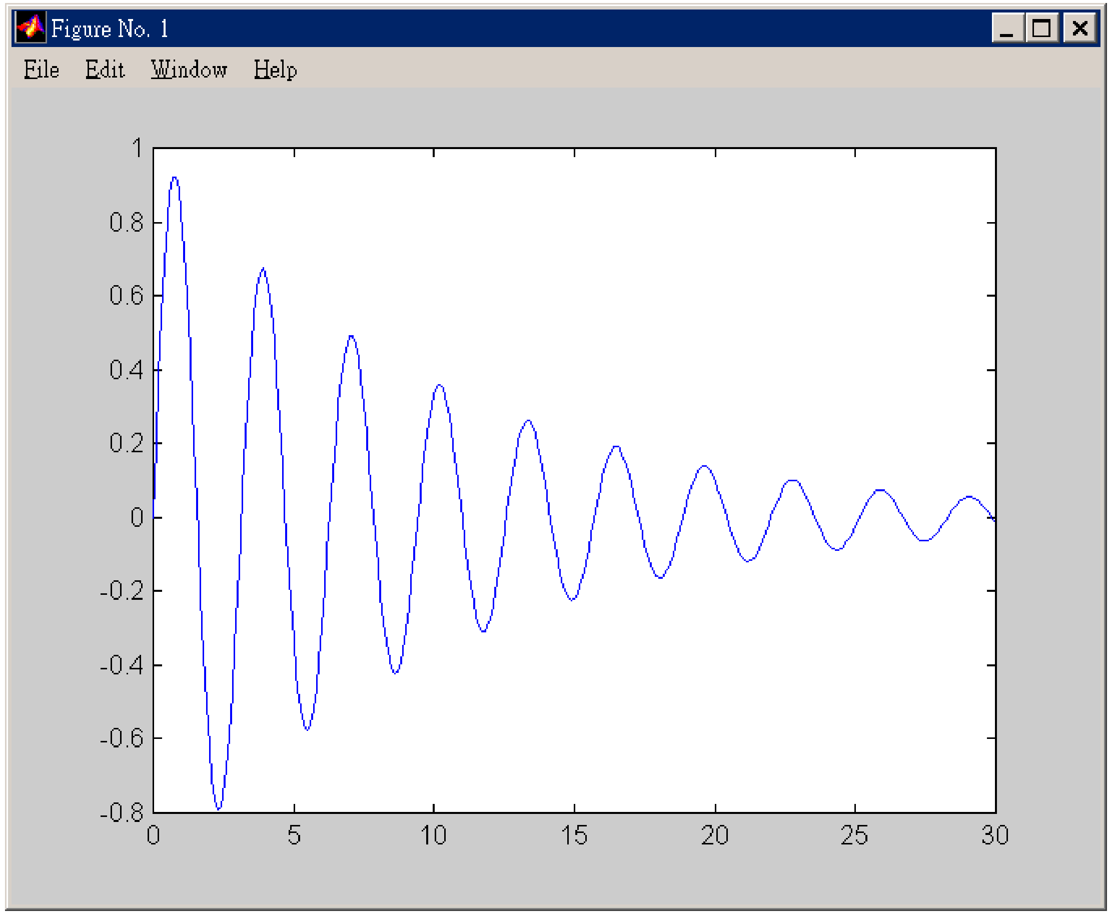
$$y(t) = e^{-0.1t} \sin 2t$$

Characteristic Equation – Complex Roots

Plot Particular Solution

MATLAB Code

```
x=[0:0.1:30];
y=exp(-0.1*t).*sin(2*t);
plot(t,y)
```



Example

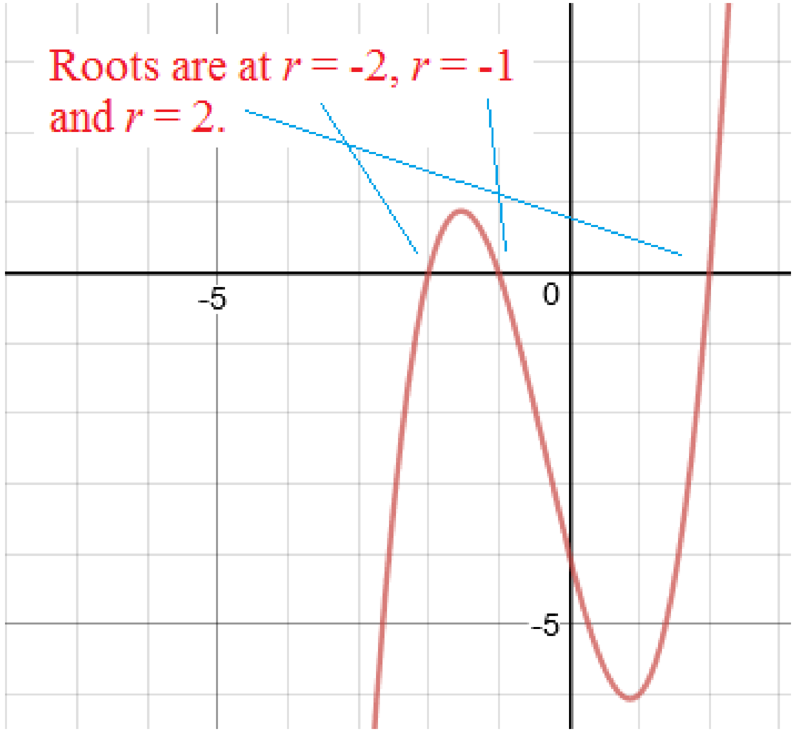
Find the general solution of

$$y''' + y'' - 4y' - 4y = 0$$

Solution: The auxiliary polynomial is

$$r^3 + r^2 - 4r - 4 = 0$$

To locate roots, we graph it:



Thus, we conclude that the general solution is

$$y = C_1 e^{-2x} + C_2 e^{-x} + C_3 e^{2x}$$

Example

Solve $$y^{iv} + 8y'' + 16y = 0.$$

Solution: The auxiliary polynomial is

$$r^4 + 8r^2 + 16 = 0 \quad \Longrightarrow \quad (r^2+4)^2 = 0 \quad \Longrightarrow \quad \begin{aligned} r_{1,2} &= \pm 2i \\ r_{3,4} &= \pm 2i \end{aligned}$$

Thus, both complex and conjugate each of multiplicity 2. The individual solutions are

$$\begin{aligned} y_1 &= \cos 2x \quad & y_3 &= x \cos 2x \\ y_2 &= \sin 2x \quad & y_4 &= x \sin 2x \end{aligned}$$

The general solution is

$$y = (C_1 + C_3 x)\cos 2x + (C_2 + C_4 x)\sin 2x$$

Example

Solve $$y^{iv} + 4y''' + 14y'' + 20y' + 25y = 0$$

Solution: The auxiliary polynomial is

$$r^4 + 4r^3 + 14r^2 + 20r + 25 = 0$$

$$(r^2 + 2r + 5)^2 = 0 \quad \Longrightarrow \quad \begin{array}{l} r_{1,2} = -1 \pm 2i \\ r_{3,4} = -1 \pm 2i \end{array}$$

The general solution is:

$$y = e^{-x}[(C_1 + C_3 x)\cos 2x + (C_2 + C_4 x)\sin 2x]$$

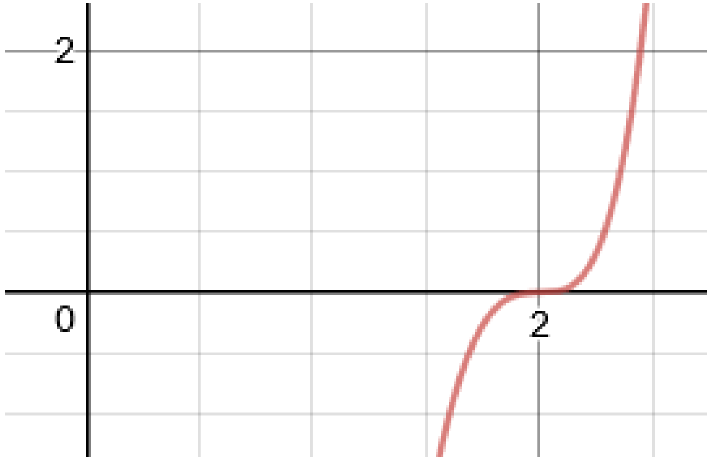
There is no general way to factor quartic polynomials. The above polynomial was factored using Wolframalpha.

Example

Solve $$y^{v} - 2y^{iv} - 5y''' - 2y'' + 52y' - 56y = 0$$

Solution: The m-eqn is $r^5 - 2r^4 - 5r^3 - 2r^2 + 52r - 56$

Its graph is shown at the right.



Note that there appears to be a root at 2, and that it passes tangentially through the horizontal axis, so the root probably has multiplicity 3.

However, we need to actually verify this using **synthetic division**.

Example - continued

$$
\begin{array}{r|rrrrrr}
2 & 1 & -2 & -5 & -2 & 52 & -56 \\
  &   & 2 & 0 & -10 & -24 & 56 \\
\hline
  & 1 & 0 & -5 & -12 & 28 & \boxed{0}
\end{array}
$$

A remainder of 0 indicates that 2 is a root

$$
\begin{array}{r|rrrrr}
2 & 1 & 0 & -5 & -12 & 28 \\
  &   & 2 & 4 & -2 & -28 \\
\hline
  & 1 & 2 & -1 & -14 & \boxed{0}
\end{array}
$$

Repeat the process with the new coefficients

Again, a remainder of 0 indicates that 2 is a root one more time

$$
\begin{array}{r|rrrr}
2 & 1 & 2 & -1 & -14 \\
  &   & 2 & 8 & 14 \\
\hline
  & 1 & 4 & 7 & \boxed{0}
\end{array}
$$

Repeat yet again

Remainder 0 shows that 2 is a root a third time. Thus, 2 is a root of multiplicity 3

Example - continued

After showing that 2 is a root of multiplicity 3, the coefficients of the remaining factors are 1, 4 and 7.

$$r^5 - 2r^4 - 5r^3 - 2r^2 + 52r - 56 = 0 \Rightarrow (r-2)^3(r^2 + 4r + 7) = 0$$

Using the quadratic formula on the factor

$$r^2 + 4r + 7 = 0 \Rightarrow r_{4,5} = -2 \pm i\sqrt{3}$$

The general solution is

$$y = (C_1 + C_2 x + C_3 x^2)e^{2x} + e^{-2x}\left(C_4 \cos\sqrt{3}x + C_5 \sin\sqrt{3}x\right)$$

Summary of Cases I–III

Case	Roots of m eqn.	Basis of DE	General Solution of DE
I	Distinct real m_1, m_2	$e^{m_1 x},\ e^{m_2 x}$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
II	Real double root $m_{1,2} = a$	e^{ax}, xe^{ax}	$y = (c_1 + c_2 x)e^{ax}$
III	Complex conjugate $m_{1,2} = \alpha \pm i\beta$	$e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$	$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

Exercises: Solve the following IVPs

$$y'' + 3y' - 10y = 0, \quad y(0) = 4, \quad y'(0) = -2$$

Ans:

$$y = \frac{2}{7}e^{-5t}(9e^{7t} + 5)$$

$$3y'' - 2y' - 8y = 0, \quad y(0) = -6, \quad y'(0) = -18$$

$$y = \frac{1}{5}\left(9e^{-4t/3} - 39e^{2t}\right)$$

$$4y'' - 5y' = 0, \quad y(-2) = 0, \quad y'(-2) = 7$$

$$y = \frac{28}{5}\left(e^{5(t+2)/4} - 1\right)$$

$$y'' - 8y' + 17y = 0, \quad y(0) = -4, \quad y'(0) = -1$$

$$y = e^{4t}(15\sin t - 4\cos t)$$

Exercises: Solve the following IVPs

$$4y'' - 24y' + 37y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$$

Ans:

$$y = e^{3(t-\pi)}\left(\sin\frac{t}{2} + 6\cos\frac{t}{2}\right)$$

$$y'' + 16y = 0, \quad y\left({}^{\pi}/_{2}\right) = -1, \quad y'\left({}^{\pi}/_{2}\right) = 4$$

$$y = (\sin t - \cos t)$$

$$16y'' - 40y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = -{}^{9}/_{4}$$

$$y = -3e^{5t/4}(2x - 1)$$

$$y'' + 14y' + 49y = 0, \quad y(-4) = -1, \quad y'(-4) = 5$$

$$y = -e^{-7(t+4)}(2x + 9)$$

Exercises: Solve the following IVPs

$$y'' - 6y' - 2y = 0 \quad \text{ans:} \quad y = e^{3t}\left(c_1 e^{-\sqrt{11}t} + c_2 e^{\sqrt{11}t}\right)$$

$$4y'' + 4y' - 3y = 0\,, \quad \text{ans:} \quad y = c_1 e^{t/2} + c_2 e^{-3t/2}$$

$$2y'' - 9y' = 0\,, \quad \text{ans:} \quad y = c_1 e^{9t/2} + c_2$$

$$y'' + 4y' + 4y = 0, \quad \text{ans:} \quad y = e^{-2t}(c_1 + c_2 t)$$

$$9y'' - 30y' + 25y = 0\,, \quad \text{ans:} \quad y = e^{5t/3}(c_1 + c_2 t)$$

$$y'' - 2y' + 2y = 0\,, \quad \text{ans:} \quad y = e^{t}(c_1 \cos t + c_2 \sin t)$$

$$4y'' + 4y' + 10y = 0\,, \quad \text{ans:} \quad y = e^{-t/2}(c_1 \cos t + c_2 \sin t)$$