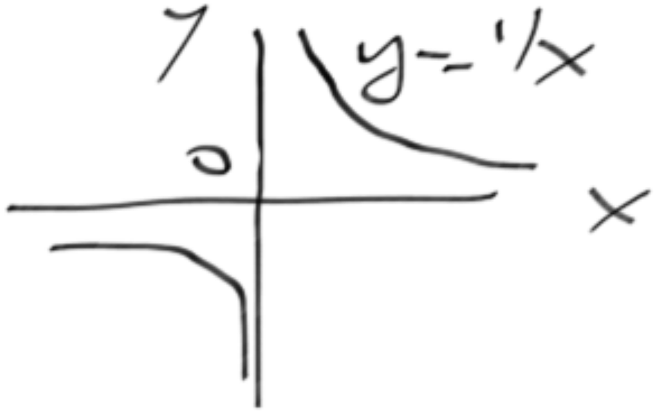
Sec. 2.6 Limits at Infinity

$$\lim_{x\to\infty} f(x) = L \quad , \quad \lim_{x\to -\infty} f(x) = M$$

Ex

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



by change of variables

$$t = \frac{1}{x}, \quad \text{As } x\to\infty,\ t\to 0.$$

$$\lim_{x\to\infty} \frac{1}{x} \equiv \lim_{t\to 0} t = 0$$

Ex

$$\lim_{x\to\infty} \frac{-x^3+2x-7}{3x^3+x^2-1} = \frac{\cancel{x^3}\left(-1+\frac{2}{x}-\frac{7}{x^3}\right)}{\cancel{x^3}\left(3+\frac{1}{x}-\frac{1}{x^3}\right)}$$

Technique: divide by the largest power of x

$$= -1/3$$

Ex

$$\lim_{x\to\infty} \frac{x^2+1}{x^{3/2}} \equiv \lim_{x\to\infty} \frac{x^2+1}{x^{1.5}} = \infty$$

Ex

$$\lim_{x\to\infty} \frac{x^2+5}{x^3-x^2-2} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x}}^{0} + \cancel{\frac{5}{x^3}}^{0}}{1-\frac{1}{x}-\frac{2}{x^3}}$$

$$= 0$$

Ex

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0,$$

By the sandwich thm.

$$\lim_{x\to\pm\infty} \underbrace{\frac{-1}{x}}_{\downarrow\, 0} \leq \underbrace{\frac{\sin x}{x}}_{\text{has to be zero!}} \leq \underbrace{\frac{1}{x}}_{\downarrow\, 0}$$

Ex

$$\lim_{\theta \to \infty} \sin\theta,$$ limit does not exist!

Ex

$$\lim_{x \to \infty} x \sin\frac{1}{x} = \lim_{x \to \infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

$$\theta = \frac{1}{x}$$

As $x \to \infty$, $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Definition: A horizontal line y=c is a horizontal asymptote of f(x) if

$$\lim_{x \to \pm\infty} f(x) = c$$

Definition: A vertical line x=a is a vertical asymptote of f(x) if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or}$$

$$\lim_{x \to a^+} f(x) = \pm\infty$$

Ex

$$f(x) = \frac{x+1}{x+3}, \quad x=-3 \text{ is the vertical asymptote.}$$

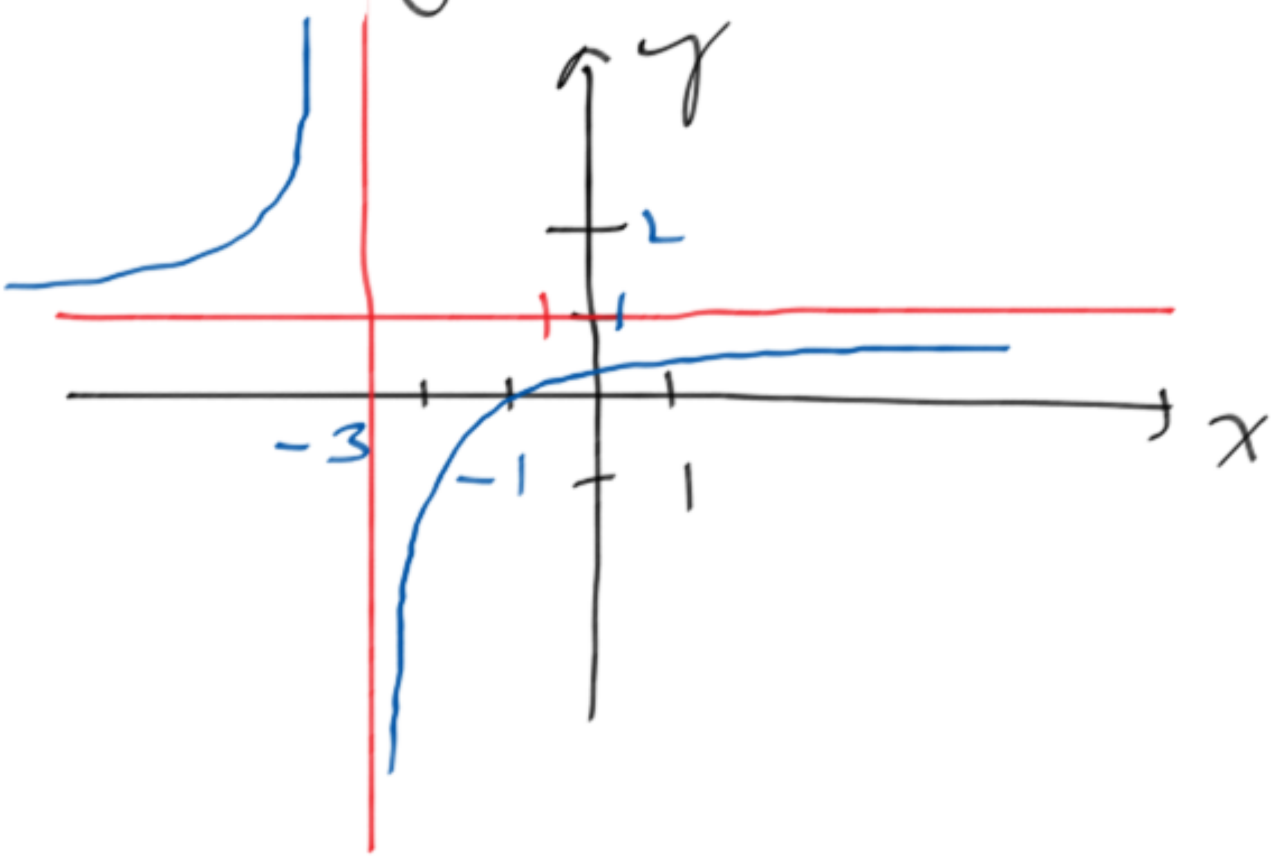
$-3 \to$

$$\frac{x+1}{x+3}$$

$$\begin{array}{r|l} \cancel{x}+1 & x+3 \\ \cline{2-2} -\cancel{x}-3 & 1 \\ \cline{1-1} -2 & \end{array}$$

$$f(x)=\frac{x+1}{x+3}=1-\frac{2}{x+3}$$

$y=1$, horizontal asymptote

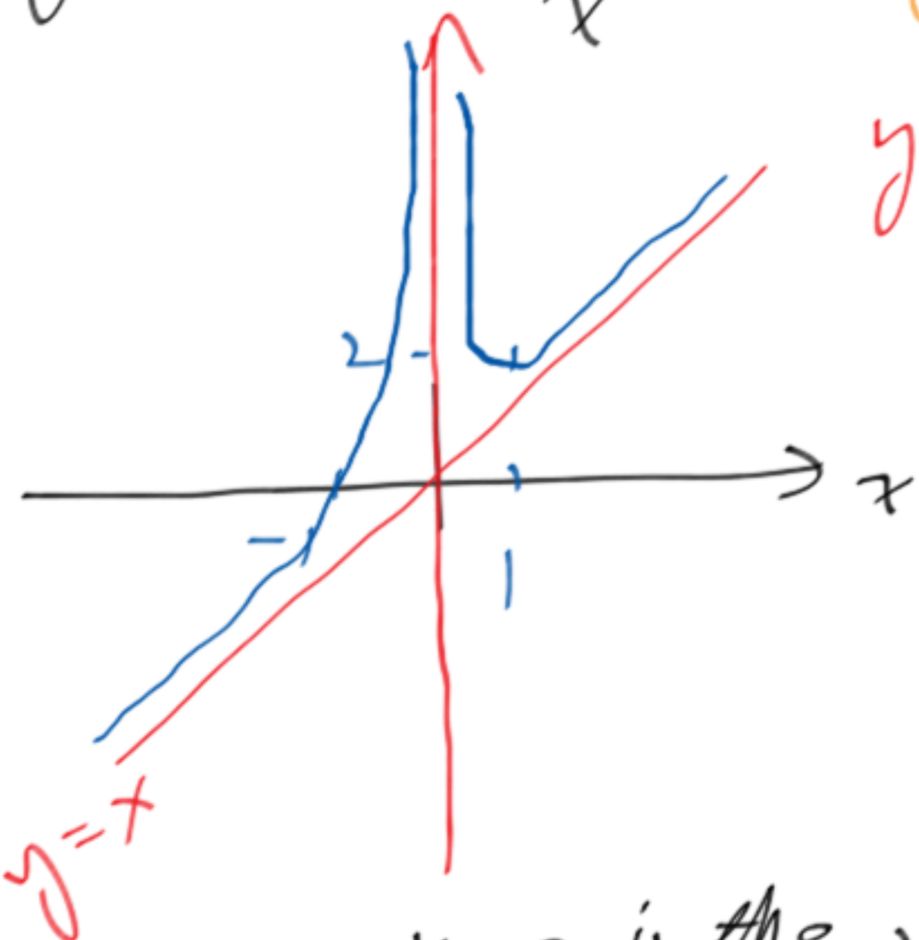


Oblique Asymptote

$$\frac{p(x)}{q(x)}, \quad \deg p = \deg q + 1$$

Ex

$$y(x) = \frac{x^3+1}{x^2} = x + \frac{1}{x^2}$$



$y=x$ is the oblique asymptote.

$x=0$ is the vertical asymptote

Ex

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right) =$$

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} =$$

$$\lim_{x\to\infty}\frac{\cancel{9x^2}+x-\cancel{9x^2}}{\sqrt{9x^2+x}+3x} =$$

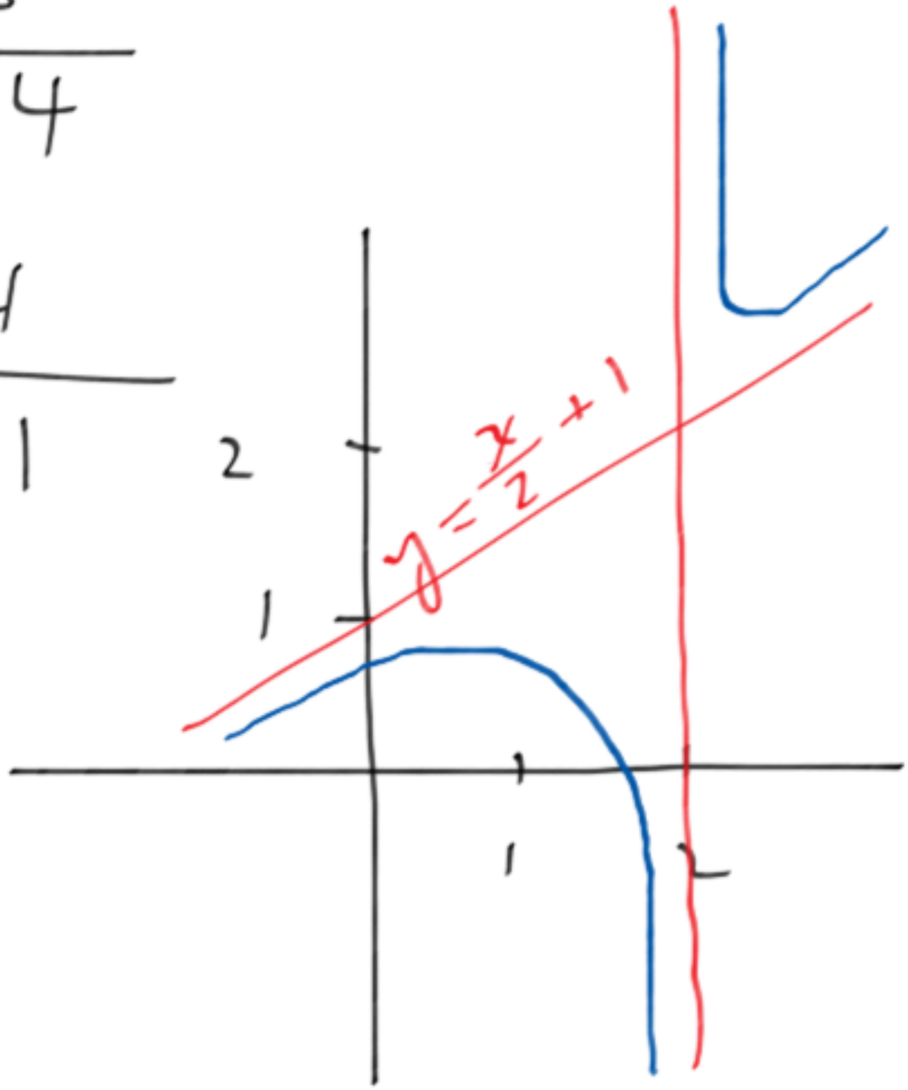
$$\lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{9+\underset{\substack{\uparrow\\0}}{1/x}}+3\right)} = \frac{1}{6}$$

$y=\frac{1}{6}$ is the horizontal asymptote

Ex

$$\lim_{x\to+\infty}\frac{x^2-3}{2x-4}$$

$$\begin{array}{r|l} x^2-3 & 2x-4 \\ \cline{2-2} -(x^2-2x) & \frac{1}{2}x+1 \\ \cline{1-1} 2x-3 & \\ -(2x-4) & \\ \cline{1-1} 1 & \end{array}$$



$$\lim_{x\to+\infty}\frac{x^2-3}{2x-4}=\lim_{x\to+\infty}\Big(\underbrace{\tfrac{1}{2}x+1}_{\text{oblique asymt.}}+\frac{1}{2x-4}\Big)$$

$x=2$ is the vertical asymp.

oblique asymt.

$y=\frac{1}{2}x+1$

Ex $\lim_{x\to\infty} \frac{\sin 2x}{x} = 2\lim_{x\to\infty} \frac{\sin 2x}{2x} = 0$

by the sandwich thm.

$$\frac{-1}{2x} \le \frac{\sin 2x}{2x} \le \frac{1}{2x}$$

$$\lim_{x\to\infty}: \quad 0 \qquad 0 \qquad 0$$

Ex $\lim_{\theta\to-\infty} \frac{\cos\theta}{3\theta} = \frac{1}{3}\lim_{\theta\to-\infty} \frac{\cos\theta}{\theta} = 0$

by the sandwich thm.

$$\frac{-1}{\theta} \le \frac{\cos\theta}{\theta} \le \frac{1}{\theta}$$

$$\lim_{\theta\to-\infty}: \quad 0 \qquad 0 \qquad 0$$

Ex

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x} =$$

$$\lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{7}{x}-5\frac{\sin x}{x}} = \frac{1}{2}$$

$0 \qquad 0 \qquad 0$

Ex

$$\lim_{x\to -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} =$$

$$\lim_{x\to -\infty} \frac{-4/x^3+3}{\sqrt{1+9/x^3}} = 3$$

Homework: $\lim_{x\to +\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = ?$