CSE2023 Discrete Computational Structures

Lecture 15

5.2 Strong induction and well-ordering

- Use strong induction to show that if **n** is an integer greater than 1, then **n** can be written as the product of primes
- Let p(n) be the proposition that n can be written as the product of primes
- Basis step: p(2) is true as 2 can be written as the product of one prime, itself
- Inductive step: Assume p(k) is true with the assumption that p(j) is true for $j \le k$

Proof with strong induction

- That is, j ($j \le k$) can be written as a product of primes
- To complete the proof, we need to show p(k+1) is true (i.e., k+1 can be written as a product of primes)

Proof with strong induction

- There are two cases: when k+1 is prime or composite
  - If k+1 is prime, we immediately see that p(k+1) is true
  - If k+1 is composite and can be written as a product of two positive integers **a** and **b**, with $2 \le a \le b < k+1$
- By inductive hypothesis, both **a** and **b** can be written as product of primes

Proof with strong induction

- Thus, if k+1 is composite, it can be written as the product of primes, namely, the primes in the factorization of **a** and those in the factorization of **b**

Proof with induction

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps
- First use mathematical induction for proof
- **Basis step**: Postage of 12 cents can be formed using 3 4-cent stamps
- **Inductive step**: The inductive hypothesis assumes p(k) is true
- That is, we need to sure p(k+1) is true when k≥12

Proof induction

(p(k+1) is true when k≥12)

- Suppose that **at least one 4-cent stamp** is used to form postage of **k** cents *(k can be combination of 4-cent and 5-cent stamps)*
- We can replace this 4-cent stamp with 5-cent stamp to form postage of **k+1** cents
- If **no 4-cent stamps** are used, we can form postage of **k** cents using **only 5-cent stamps**
- As k≥12, we need at least three 5-cent stamps to form postage of k cents *(in this case k must be 15 or more)*
- So, we can replace 3 **5-cent** stamps with 4 **4-cent** stamps for **k+1** cents
- As we have completed basis and inductive steps, we know p(n) is true for n≥12

Proof with strong induction

- Use strong induction for proof
- In the basis step, we show that p(12), p(13), p(14) and p(15) are true
- In the inductive step, we show that how to get postage of **k+1** cents for k≥15 from postage of **k-3** cents
- **Basis step**: we can form postage of 12, 13, 14, 15 cents using 3 **4-cent stamps**, 2 **4-cent**/1 **5-cent** stamps, 2 **5-cent**/1 **4-cent** stamps, and 3 **5-cent** stamps. So p(12), p(13), p(14), p(15) are true

Proof with strong induction

- **Inductive step**: The inductive hypothesis is the statement p(j) is true for **12≤j ≤k**, where k is an integer with **k≥15**. We need to show **p(k+1)** is true
- We can **assume p(k-3) is true because k-3 ≥12**, that is, we can form postage of k-3 cents using just 4-cent and 5-cent stamps
- To form postage of k+1 cents, we need only add another 4-cent stamp to the stamps we used to form postage of k-3 cents. That is, we show p(k+1) is true
- As we have completed basis and inductive steps of a strong induction, we show that p(n) is true for n≥12
- There are other ways to prove this

Proofs using well-ordering property

- Validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of the set of integers, the well-ordering property
- **Well-order property**: every non-empty set of non-negative integers has a least element
- The well-ordering property can be used directly in proofs
- The **well-ordering property**, the **principle of mathematical induction**, and **strong induction** are all equivalent