Math

No:

Full Name :

Math 104 3rd Midterm Exam
(29 April 2019, 19:00-20:00)

IMPORTANT

1. Write down your name and surname on top of each page. **2.** The exam consists of 4 questions, some of which have multiple parts. **3.** Read each question carefully and put your answers neatly on the answer sheets. Simplify your answers. **4.** Show all your work. Correct answers without justification will not get credit. **5.** Unless otherwise specified, you may use any method from classwork to solve the problems. **6.** Calculators are not allowed. **7.** All cell phones and electronic devices are to be kept shut and out of sight. All cell phones are to be left on the instructor's desk prior to the exam.

Q1	Q2	Q3	Q4	TOT
20 pts	20 pts	35 pts	25 pts	100 pts

Q1. Determine whether the given sequence converges or diverges. If it converges, find the limit:

$\{n \sin(\pi/n)\}$

$\infty \cdot 0$ $\qquad$ $0/0$

$$\lim_{x\to\infty} x \sin\frac{\pi}{x} = \lim_{x\to\infty} \frac{\sin \pi/x}{1/x}$$

$$\underset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\cos \pi/x \cdot (-\pi/x^2)}{-1/x^2}$$

$$= \lim_{x\to\infty} \pi \cos\frac{\pi}{x} = \boxed{\pi}$$

OR

$$\lim_{x\to\infty} x \sin\frac{\pi}{x} = \lim_{x\to\infty} \pi \frac{\sin \pi/x}{\pi/x}$$

$$= \pi \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \pi \cdot 1 = \boxed{\pi}$$

by $\theta = \pi/x$

Q2. Determine whether the following series converges or diverges:

$$\sum_{n=0}^{\infty} \frac{(Arctann)^2}{n^2+1}$$

Method I: Integral Test

$$\int \frac{(\text{Arctan}x)^2}{x^2+1}dx = \int u^2 du = \frac{u^3}{3}+C$$

$u = \text{Arctan}x$

$du = \frac{dx}{x^2+1}$

$$= \frac{1}{3}(\text{Arctan}x)^3 + C$$

$$\int_0^\infty \frac{(\text{Arctan}x)^2}{x^2+1}dx = \frac{1}{3}\lim_{b\to\infty}(\text{Arctan}x)^3\Big|_0^b$$

$$= \frac{1}{3}\left(\left(\frac{\pi}{2}\right)^3 - 0\right) = \frac{\pi^3}{24}$$

The integral converges, so the series converges

OR Method II: Limit Comparison Test

with $\sum \frac{1}{n^2}$, p-series, $p=2>1$, convergent

$$\lim_{n\to\infty} \frac{\frac{(\text{Arctan}n)^2}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\left\{\underbrace{\frac{n^2}{n^2+1}}_{\to 1}\cdot\underbrace{(\text{Arctan}n)^2}_{\to (\frac{\pi}{2})^2}\right\}$$

$$= \frac{\pi^2}{4} \neq 0, \infty$$

$\therefore$ The given series converges.

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Q3. Determine whether the following series converge or diverge:

a) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$

Comparison Test

$$\frac{(\ln n)^2}{n} > \frac{1}{n} \quad \text{for } n \geq 4$$

$\sum \frac{1}{n}$ is the harmonic series, divergent

$\therefore$ The given series diverges by comparison.

(You can also solve this question by using the Integral Test)

b) $\sum_{n=1}^{\infty} \frac{n^2+2n+6}{8n^7+n-8}$

Limit Comparison Test, write $\sum \frac{1}{n^5}$ p-series, $p=5>1$, convergent

$$\lim_{n\to\infty} \frac{\frac{n^2+2n+6}{8n^7+n-8}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^5(n^2+2n+6)}{8n^7+n-8}$$

$$= \lim_{n\to\infty} \frac{n^7+2n^6+6n^5}{8n^7+n-8} = \lim_{n\to\infty} \frac{\cancel{n^7}(1+2/n+6/n^2)}{\cancel{n^7}(8+1/n^6-8/n^7)}$$

$$= \frac{1}{8} \neq 0, \infty$$

$\therefore$ The series converges

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Q4. Given the power series $\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n (x+5)^n$

(a) Find the radius of convergence.
(b) Find the interval of convergence.

Generalized Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{\left(\frac{1}{4}\right)^{n+1}(x+5)^{n+1}}{\left(\frac{1}{4}\right)^n (x+5)^n} \right| = \lim_{n\to\infty} \left| \frac{x+5}{4} \right|$$

$$= \frac{|x+5|}{4}$$

$\rho < 1 \Rightarrow$ series converges
$\rho > 1 \Rightarrow$ " diverges
$\rho = 1 \Rightarrow$ no info

$\rho < 1 \Rightarrow \frac{|x+5|}{4} < 1 \Rightarrow -4 < x+5 < 4$
$-9 < x < -1$

$\rho > 1 \Rightarrow x > -1$ or $x < -9$

$\rho = 1 \Rightarrow x = -1$ or -9

$x = -1 \Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot 4^n = \sum 1$
$= 1 + 1 + \cdots,$
diverges

$x = -9 \Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n (-4)^n = \sum (-1)^n,$
diverges by oscillation

$\therefore$ Interval of convergence is $\boxed{-9 < x < -1}$

Radius of convergence $= \boxed{4}$