Marmara University, **2021**

Probability and Statistics

Subject 10

Inference from Small Samples

Mujdat Soyturk, Ph.D.
Associate Professor

Contents

Most parts of the slides are derived from the textbook: "Mendenhall, Beaver, Beaver, *Introduction to Probability and Statistics*, 14th Ed., Brooks/Cole, Cengage Learning, 2013"

Introduction

- When the sample size is small, the estimation and testing procedures of Chapter 8 are not appropriate.
- There are equivalent small sample test and estimation procedures for
  - μ, the mean of a normal population
  - $\mu_1-\mu_2$, the difference between two population means
  - σ^2, the variance of a normal population
  - The ratio of two population variances.

The Sampling Distribution of the Sample Mean

- When we take a sample from a normal population, the sample mean $\overline{x}$ has a normal distribution for any sample size ***n***, and ***z*** has a standard normal distribution.

$$z = \frac{\overline{x}-\mu}{\sigma/\sqrt{n}}$$

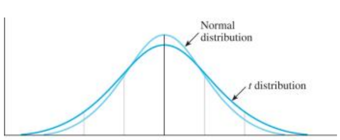
- But if σ is unknown, and we must use ***s*** to estimate it, the resulting statistic **is not normal**.

$$\frac{\overline{x}-\mu}{s/\sqrt{n}} \text{ is not normal!}$$

Student's *t* Distribution

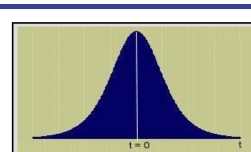
- Fortunately, this statistic does have a sampling distribution that is well known to statisticians, called the **Student's *t* distribution,** with ***n*-1 degrees of freedom.**

$$t = \frac{\overline{x}-\mu}{s/\sqrt{n}}$$



- We can use this distribution to create estimation testing procedures for the population mean μ.

Properties of Student's *t*



- **Mound-shaped** and symmetric about 0.
- **More variable than *z***, with "heavier tails"

- Shape depends on the sample size *n* or the **degrees of freedom, *n*-1.**
- As ***n*** increases the shapes of the ***t*** and ***z*** distributions become almost identical.

MY APPLET

Using the t-Table

- Table 4 gives the values of t that cut off certain critical values in the tail of the t distribution.
- Index ***df*** and the appropriate tail area α to find t_α, the value of t with area α to its right.

Using the t-Table

- Table 4 gives the values of t that cut off certain critical values in the tail of the t distribution.
- Index ***df*** and the appropriate tail area α to find t_α the value of t with area α to its right.

df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$
1	3.078	6.314	12.706	31.821
2	1.886	2.920	4.303	6.965
3	1.638	2.353	3.182	4.541
4	1.533	2.132	2.776	3.747
5	1.476	2.015	2.571	3.365
6	1.440	1.943	2.447	3.143
7	1.415	1.895	2.365	2.998
8	1.397	1.860	2.306	2.896
9	1.383	1.833	2.262	2.821
10	1.372	1.812	2.228	2.764
11	1.363	1.796	2.201	2.718
12	1.356	1.782	2.179	2.681
13	1.350	1.771	2.160	2.650
14	1.345	1.761	2.145	2.624
15	1.341	1.753	2.131	2.602

For a random sample of size n = 10, find a value of t that cuts off .025 in the right tail.

Row = df = n –1 = 9

Column subscript = α = .025

$\mathbf{t_{.025} = 2.262}$

Small Sample Inference for a Population Mean μ

The basic procedures are the same as those used for large samples. For a test of hypothesis:

Test $H_0: \mu = \mu_0$ versus H_a : one or two tailed using the test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

using p-values or a rejection region based on a t-distribution with $df = n - 1$.

Small Sample Inference for a Population Mean μ

For a 100(1-α)% confidence interval for the population mean μ :

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is the value of t that cuts off area $\alpha/2$ in the tail of a t-distribution with $df = n - 1$.

Small Sample Hypothesis Test for μ

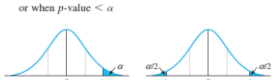
1. Null hypothesis: $H_0 : \mu = \mu_0$
2. Alternative hypothesis:

One-Tailed Test	Two-Tailed Test
$H_a : \mu > \mu_0$ (or, $H_a : \mu < \mu_0$)	$H_a : \mu \neq \mu_0$

3. Test statistic: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
4. Rejection region: Reject H_0 when

One-Tailed Test	Two-Tailed Test
$t > t_\alpha$ (or $t < -t_\alpha$ when the alternative hypothesis is $H_a : \mu < \mu_0$)	$t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

or when p-value $< \alpha$



Example

A sprinkler system is designed so that the average time for the sprinklers to activate after being turned on is no more than 15 seconds. A test of 6 systems gave the following times:

17, 31, 12, 17, 13, 25

Is the system working as specified? Test using α =0.05

H_0 : $\mu = 15$ (working as specified) or $\mu \le 15$

H_a : $\mu > 15$ (not working as specified)

Example

Data: 17, 31, 12, 17, 13, 25

First, calculate the sample mean and standard deviation, using your calculator or the formulas in Chapter 2.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{115}{6} = 19.167$$

$$s = \sqrt{\frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}} = \sqrt{\frac{2477 - \frac{115^2}{6}}{5}} = 7.387$$

Example

Data: 17, 31, 12, 17, 13, 25

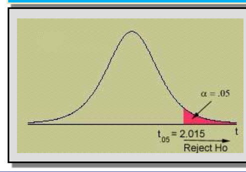
Calculate the test statistic and find the rejection region for α =0.05

Test statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{19.167 - 15}{7.387/\sqrt{6}} = 1.38$$

Degrees of freedom:

$$df = n - 1 = 6 - 1 = 5$$



Rejection Region: Reject H_0 if $t > 2.015$. If the test statistic falls in the rejection region, its *p*-value will be less than α=0.05

Conclusion

Data: 17, 31, 12, 17, 13, 25

Compare the observed test statistic to the rejection region, and draw conclusions.

$H_0: \mu = 15$

$H_a: \mu > 15$

Test statistic: $t = 1.38$

Rejection Region:

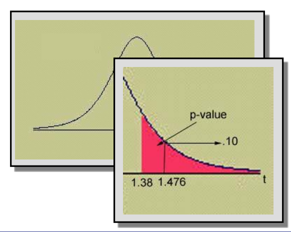
Reject H_0 if $t > 2.015$.

Conclusion: For our example, t = 1.38 does not fall in the rejection region and H_0 is not rejected. There is insufficient evidence to indicate that the average activation time is greater than 15.

Approximating the *p*-value

You can only approximate the *p*-value for the test using Table 4.

df	$t_{.100}$	$t_{.050}$
1	3.078	6.314
2	1.886	2.920
3	1.638	2.353
4	1.533	2.132
5	1.476	2.015



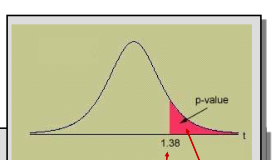
Since the observed value of t = 1.38 is smaller than $t_{.10}$ = 1.476,

p-value > 0.10

The exact *p*-value

You can get the exact *p*-value using some calculators or a computer.

p-value = .113 which is greater than .10 as we approximated using Table 4.



One-Sample T: Times

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Test of mu = 15 vs > 15

                                                 95%
                                               Lower
Variable  N     Mean  StDev  SE Mean    Bound     T      P
Times     6  19.1667 7.3869   3.0157  13.0899  1.38  0.113
```


Testing the Difference between Two Means

As in Chapter 9, independent random samples of size n_1 and n_2 are drawn from populations 1 and 2 with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

Since the sample sizes are small, the two populations must be normal.

To test:

- **H_0: $\mu_1 - \mu_2 = D_0$ versus H_a: one of three**

where D_0 is some hypothesized difference, usually 0.

Testing the Difference between Two Means

The test statistic used in Chapter 9

$$z \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

does not have either a *z* or a *t* distribution, and cannot be used for small-sample inference.

- We need to make one more assumption, that **the <u>population variances</u>, although unknown, <u>are equal</u>. (Both populations have exactly the same shape.)**

Testing the Difference between Two Means

Instead of estimating each population variance separately, we estimate the common variance with

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

And the resulting test statistic,

$$t = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

has a *t* distribution with n_1+n_2-2 ***degrees of freedom***.

Estimating the Difference between Two Means

You can also create a 100(1-α)% confidence interval for μ_1-μ_2.

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$\text{with } s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Remember the three assumptions:

1. **Original populations normal**
2. **Samples random and independent**
3. **Equal population variances.**

Test of Hypothesis for the Difference Between Two Population Means: Independent Random Samples

1. Null hypothesis: $H_0 : (\mu_1 - \mu_2) = D_0$, where D_0 is some specified difference that you wish to test. For many tests, you will hypothesize that there is no difference between μ_1 and μ_2; that is, $D_0 = 0$.
2. Alternative hypothesis:

One-Tailed Test	Two-Tailed Test
$H_a : (\mu_1 - \mu_2) > D_0$ [or $H_a : (\mu_1 - \mu_2) < D_0$]	$H_a : (\mu_1 - \mu_2) \neq D_0$

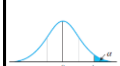
3. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where

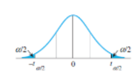
$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

4. Rejection region: Reject H_0 when

One-Tailed Test	Two-Tailed Test
$t > t_\alpha$ [or $t < -t_\alpha$ when the alternative hypothesis is $H_a : (\mu_1 - \mu_2) < D_0$]	$t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

or when *p*-value < α





Example

Two training procedures are compared by measuring the time that it takes trainees to assemble a device. A different group of trainees are taught using each method. Is there a difference in the two methods? Use α = 0.01.

Time to Assemble	Method 1	Method 2
Sample size	10	12
Sample mean	35	31
Sample Std Dev	4.9	4.5

$$H_0 : \mu_1 - \mu_2 = 0$$
$$H_a : \mu_1 - \mu_2 \neq 0$$

Test statistic:

$$t = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Example

Solve this problem by approximating the *p*-value using Table 4.

Time to Assemble	Method 1	Method 2
Sample size	10	12
Sample mean	35	31
Sample Std Dev	4.9	4.5

Calculate:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$
$$= \frac{9(4.9^2) + 11(4.5^2)}{20} = 21.942$$

Test statistic:

$$t = \frac{35 - 31}{\sqrt{21.942\left(\frac{1}{10} + \frac{1}{12}\right)}}$$
$$= 1.99$$

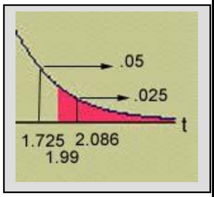
Example

$$p\text{-value}: P(t > 1.99) + P(t < -1.99)$$

$$P(t > 1.99) = \frac{1}{2}(p\text{-value})$$

$df = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$

df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	df
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20



.025 < ½(*p*-value) < .05

.05 < *p*-value < .10

Since the *p*-value is greater than $\alpha = .01$, H_0 is not rejected. There is insufficient evidence to indicate a difference in the population means.

Testing the Difference between Two Means

How can you tell if the equal variance assumption is reasonable?

Rule of Thumb:

If the ratio, $\dfrac{\text{larger } s^2}{\text{smaller } s^2} \le 3$,

the equal variance assumption is reasonable.

If the ratio, $\dfrac{\text{larger } s^2}{\text{smaller } s^2} > 3$,

use an alternative test statistic.

Testing the Difference between Two Means

If the population variances cannot be assumed equal, the test statistic

$$t \approx \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

$$df \approx \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

has an approximate *t* distribution with degrees of freedom given above. This is most easily done by computer.

The Paired-Difference Test

- Sometimes the assumption of independent samples is intentionally violated, resulting in a **matched-pairs** or **paired-difference test**.
- By designing the experiment in this way, we can eliminate unwanted variability in the experiment by analyzing only the differences,

$$d_i = x_{1i} - x_{2i}$$

to see if there is a difference in the two population means, $\mu_1 - \mu_2$.

Example

One Type A and one Type B tire are randomly assigned to each of the rear wheels of five cars. Compare the average tire wear for types A and B using a test of hypothesis.

Car	1	2	3	4	5
Type A	10.6	9.8	12.3	9.7	8.8
Type B	10.2	9.4	11.8	9.1	8.3

$$H_0: \mu_1 - \mu_2 = 0$$
$$H_a: \mu_1 - \mu_2 \ne 0$$

But the samples are not independent. The pairs of responses are linked because measurements are taken on the same car.

The Paired-DifferenceTest

To test $H_0: \mu_1 - \mu_2 = 0$ we test $H_0: \mu_d = 0$

using the test statistic

$$t = \frac{\bar{d} - 0}{s_d / \sqrt{n}}$$

where n = number of pairs, $\bar{d}$ and s_d are the mean and standard deviation of the differences, d_i.

Use the p-value or a rejection region based on a t-distribution with $df = n - 1$.

The Paired-DifferenceTest

1. Null hypothesis: $H_0 : \mu_d = 0$
2. Alternative hypothesis:

One-Tailed Test	Two-Tailed Test
$H_a : \mu_d > 0$ (or $H_a : \mu_d < 0$)	$H_a : \mu_d \neq 0$

3. Test statistic: $t = \dfrac{\bar{d} - 0}{s_d/\sqrt{n}} = \dfrac{\bar{d}}{s_d/\sqrt{n}}$

where n = Number of paired differences
$\bar{d}$ = Mean of the sample differences
s_d = Standard deviation of the sample differences

$$= \sqrt{\frac{\Sigma(d_i - \bar{d})^2}{n-1}} = \sqrt{\frac{\Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n}}{n-1}}$$

4. Rejection region: Reject H_0 when

One-Tailed Test	Two-Tailed Test
$t > t_\alpha$ (or $t < -t_\alpha$ when the alternative hypothesis is $H_a : \mu_d < 0$)	$t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

or when p-value $< \alpha$

Example

$$\mathrm{H_0} : \mu_1 - \mu_2 = 0$$
$$\mathrm{H_a} : \mu_1 - \mu_2 \neq 0$$

Car	1	2	3	4	5
Type A	10.6	9.8	12.3	9.7	8.8
Type B	10.2	9.4	11.8	9.1	8.3
Difference	.4	.4	.5	.6	.5

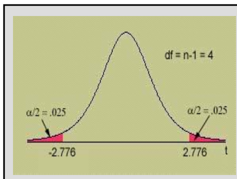
$$\text{Calculate } \bar{d} = \frac{\sum d_i}{n} = .48$$

$$s_d = \sqrt{\frac{\sum d_i^2 - \frac{(\sum d_i)^2}{n}}{n-1}} = .0837$$

$$\text{Test statistic:}\quad t = \frac{\bar{d} - 0}{s_d/\sqrt{n}} = \frac{.48 - 0}{.0837/\sqrt{5}} = 12.8$$

Example

Car	1	2	3	4	5
Type A	10.6	9.8	12.3	9.7	8.8
Type B	10.2	9.4	11.8	9.1	8.3
Difference	.4	.4	.5	.6	.5



Rejection region: Reject H_0 if $t > 2.776$ or $t < -2.776$.

Conclusion: Since $t = 12.8$, H_0 is rejected. There is a difference in the average tire wear for the two types of tires.

Some Notes

You can construct a 100(1-α)% confidence interval for a paired experiment using

$$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$$

Once you have designed the experiment by pairing, you MUST analyze it as a paired experiment. If the experiment is not designed as a paired experiment in advance, do not use this procedure.

Inferences Concerning a Population Variance

- Sometimes the primary parameter of interest is not the population mean μ but rather the population variance σ^2. We choose a random sample of size n from a normal distribution.
- Previously we have used $s^2 = \dfrac{\Sigma(x_i - \bar{x})^2}{n-1}$

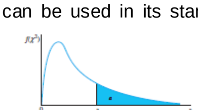
as an unbiased estimator of the population variance.

- But, <u>how close</u> or <u>far</u> from the target parameter σ^2 is our estimator s^2 likely to be?
- Use repeated random sampling.

Inferences Concerning a Population Variance

- The distribution begins at $s^2 = 0$.
- The mean is equal to the σ^2.
- Its shape is nonsymmetric.
- Its shapes changes depending on sample size n and σ^2.
- We can standardize the sampling distributions.
- The sample variance s^2 can be used in its standardized form:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

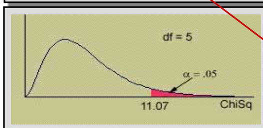


- which has a Chi-Square distribution with n - 1 degrees of freedom.

Inference Concerning a Population Variance

Table 5 gives both upper and lower critical values of the chi-square statistic for a given *df*.

TABLE 5 (continued) $\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$	*df*
2.70554	3.84146	5.02389	6.63490	7.87944	1
4.60517	5.99147	7.37776	9.21034	10.5966	2
6.25139	7.81473	9.34840	11.3449	12.8381	3
7.77944	9.48773	11.1433	13.2767	14.8602	4
9.23635	11.0705	12.8325	15.0863	16.7496	5
10.6446	12.5916	14.4494	16.8119	18.5476	6
12.0170	14.0671	16.0128	18.4753	20.2777	7



For example, the value of chi-square that cuts off .05 in the upper tail of the distribution with *df* = 5 is χ^2 =11.07.

Inference Concerning a Population Variance

To test $H_0: \sigma^2 = \sigma_0^2$ versus H_a : one or two tailed we use the test statistic

$\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$ with a rejection region based on a chi-square distribution with $df = n-1$.

Confidence interval:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{(1-\alpha/2)}}$$

Inference Concerning a Population Variance

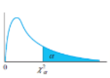
1. Null hypothesis: $H_0 : \sigma^2 = \sigma_0^2$
2. Alternative hypothesis:

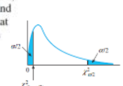
One-Tailed Test	Two-Tailed Test
$H_a : \sigma^2 > \sigma_0^2$ (or $H_a : \sigma^2 < \sigma_0^2$)	$H_a : \sigma^2 \neq \sigma_0^2$

3. Test statistic: $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$
4. Rejection region: Reject H_0 when

One-Tailed Test	Two-Tailed Test
$\chi^2 > \chi^2_\alpha$ (or $\chi^2 < \chi^2_{(1-\alpha)}$ when the alternative hypothesis is $H_a : \sigma^2 < \sigma_0^2$), where χ^2_α and $\chi^2_{(1-\alpha)}$ are, respectively, the upper- and lower-tail values of χ^2 that place α in the tail areas	$\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 < \chi^2_{(1-\alpha/2)}$, where $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$ are, respectively, the upper- and lower-tail values of χ^2 that place $\alpha/2$ in the tail areas

or when *p*-value < α





Example

A cement manufacturer claims that his cement has a compressive strength with a standard deviation of 10 kg/cm² or less. A sample of *n* = 10 measurements produced a mean and standard deviation of 312 and 13.96, respectively.

A test of hypothesis:

H_0: $\sigma^2 \le 10^2$ (claim is correct)

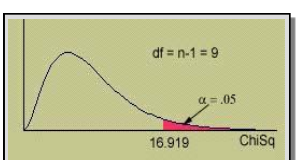
H_a: $\sigma^2 > 10^2$ (claim is wrong)

uses the test statistic:

$$\chi^2 = \frac{(n-1)s^2}{10^2} = \frac{9(13.96^2)}{100} = 17.5$$

Example

Do these data produce sufficient evidence to reject the manufacturer's claim? Use α = .05.

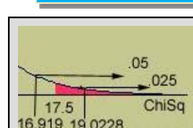


Rejection region: Reject H_0 if a χ^2 > 16.919 (α = .05).

Conclusion: Since χ^2 = 17.5, H_0 is rejected. The standard deviation of the cement strengths is more than 10.

Approximating the *p*-value

$$p\text{-value}: P(\chi^2 > 17.5) \text{ with } df = n-1 = 9$$



$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$	*df*
2.70554	3.84146	5.02389	6.63490	7.87944	1
4.60517	5.99147	7.37776	9.21034	10.5966	2
6.25139	7.81473	9.34840	11.3449	12.8381	3
7.77944	9.48773	11.1433	13.2767	14.8602	4
9.23635	11.0705	12.8325	15.0863	16.7496	5
10.6446	12.5916	14.4494	16.8119	18.5476	6
12.0170	14.0671	16.0128	18.4753	20.2777	7
13.3616	[illegible]	[illegible]	20.0902	21.9550	8
14.6837	16.9190	19.0228	21.6660	23.5893	9

.025 < *p*-value < .05

Since the *p*-value is less than α = .05, H_0 is rejected. There is sufficient evidence to reject the manufacturer's claim.

Inference Concerning Two Population Variances

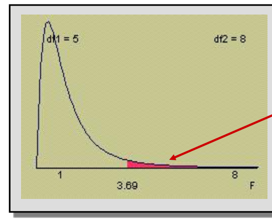
- We can make inferences about the ratio of two population variances in the form a ratio. We choose two independent random samples of size n_1 and n_2 from normal distributions.
- If the two population variances are equal, the statistic

$$F = \frac{s_1^2}{s_2^2}$$

has an F distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$ degrees of freedom.

Inference Concerning Two Population Variances

Table 6 gives only upper critical values of the F statistic for a given pair of df_1 and df_2.



For example, the value of F that cuts off .05 in the upper tail of the distribution with df_1 = 5 and df_2 = 8 is F =3.69.

Inference Concerning Two Population Variances

To test

$H_0 : \sigma_1^2 = \sigma_2^2$ versus

H_a : one or two tailed

we use the test statistic

$F = \frac{s_1^2}{s_2^2}$ where s_1^2 is the larger of the two sample variances.

with a rejection region based on an F distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$.

Confidence interval:

$$\frac{s_1^2}{s_2^2}\frac{1}{F_{df_1,df_2}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}F_{df_2,df_1}$$

Example

An experimenter has performed a lab experiment using two groups of rats. He wants to test H_0: $\mu_1 = \mu_2$, but first he wants to make sure that the population variances are equal.

	Standard (2)	Experimental (1)
Sample size	10	11
Sample mean	13.64	12.42
Sample Std Dev	2.3	5.8

Preliminary test:

$H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_a : \sigma_1^2 \neq \sigma_2^2$

Example

	Standard (2)	Experimental (1)
Sample size	10	11
Sample Std Dev	2.3	5.8

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_a : \sigma_1^2 \neq \sigma_2^2$

Test statistic:

$$F = \frac{s_1^2}{s_2^2} = \frac{5.8^2}{2.3^2} = 6.36$$

We designate the sample with the larger standard deviation as sample 1, to force the test statistic into the upper tail of the F distribution.

Example

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_a : \sigma_1^2 \neq \sigma_2^2$

Test statistic:

$$F = \frac{s_1^2}{s_2^2} = \frac{5.8^2}{2.3^2} = 6.36$$

The rejection region is two-tailed, with α = 0.05, but we only need to find the upper critical value, which has $\alpha/2$ = 0.025 to its right.

From Table 6, with df_1=10 and df_2 = 9, we reject H_0 if F > 3.96.

CONCLUSION: Reject H_0. There is sufficient evidence to indicate that the variances are **unequal.** Do not rely on the assumption of equal variances for your *t test!*

Key Concepts

I. Experimental Designs for Small Samples

1. Single random sample: The sampled population must be normal.
2. Two independent random samples: Both sampled populations must be normal.
   a. Populations have a common variance σ^2.
   b. Populations have different variances
3. Paired-difference or matched-pairs design: The samples are not independent.

Key Concepts

II. Statistical Tests of Significance

1. Based on the t, F, and χ^2 distributions
2. Use the same procedure as in Chapter 9
3. Rejection region — critical values and significance levels: based on the t, F, and χ^2 distributions with the appropriate degrees of freedom
4. Tests of population parameters: a single mean, the difference between two means, a single variance, and the ratio of two variances

III. Small Sample Test Statistics

To test one of the population parameters when the sample sizes are small, use the following test statistics:

Key Concepts

Parameter	Test Statistic	Degrees of Freedom
μ	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$n - 1$
$\mu_1 - \mu_2$ (equal variances)	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$	$n_1 + n_2 - 2$
$\mu_1 - \mu_2$ (unequal variances)	$t \approx \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$	Satterthwaite's approximation
$\mu_1 - \mu_2$ (paired samples)	$t = \dfrac{\bar{d} - \mu_d}{s_d/\sqrt{n}}$	$n - 1$
σ^2	$\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	$n - 1$
σ_1^2/σ_2^2	$F = s_1^2/s_2^2$	$n_1 - 1$ and $n_2 - 1$