Mathematical Symbols And Abbreviations		
Symbol	Say	Means
$\therefore$	therefore	therefore
$\because$	because	because
$\forall$	for all	for all
$\Longleftrightarrow$	if and only if or ***iff***	if and only if or ***iff***
$\propto$	proportional to	proportional to
$\Rightarrow$	implies	calculation on left of symbol imply those on the right
fn	function	function
wrt	with respect to	with respect to
LHS	Left-hand side	Left-hand side
RHS	right-hand side	right-hand side
$\frac{dy}{dx}$	Dee ***y*** dee ***x***	Differentiate fn ***y*** wrt ***x***
$\frac{d^2y}{dx^2}$	Dee 2 ***y*** dee ***x*** squared	Double differentiate fn ***y*** wrt ***x*** Second derivative of fn ***y***
f′(***x***)	f prime of ***x*** or f prime	Differentiate fn ***f(x)*** wrt ***x***, equivalent to $\frac{dy}{dx}$ if ***y=f(x)***
y′	***y*** prime	Differentiate fn ***y*** wrt ***x***, equivalent to $\frac{dy}{dx}$ if ***y=f(x)***
$\dot{x}$ (dot above variable ***x***)	***x*** dot	Differentiate fn ***x*** wrt t
f″(x)	f double prime of ***x*** or f double prime	Differentiate fn ***f(x)*** wrt ***x*** twice, Second derivative of fn ***f(x)***, equivalent to $\frac{d^2y}{dx^2}$ if ***y=f(x)***
f‴(x)	f triple prime of ***x*** or f triple prime	Differentiate fn ***f(x)*** wrt ***x*** three times, Third derivative of fn ***f(x)***, equivalent to $\frac{d^3y}{dx^3}$ if ***y=f(x)***

DIFFERENTIAL EQUATIONS

Natural World

eg: dissipation of heat
- Fluid flow (liquid / gas)
- Vibration of mass

Understanding and modeling the behaviour of natural world like motion of fluid, flow of electrical current, dissipation of heat, propagation of waves etc. usually requires assumptions and simplifications.

↓ Assumptions & Simplifications

Assumptions are usually made to simplify the model of the behaviour of natural world. This is a critical step in modeling because,

under simplified models – lead to over complication to solve

over simplified models – result does not represent the actual system

Physical Model

Simplification of natural world leads to a physical model of the actual system.

↓ Physical Laws

eg: Newton's 2^{nd} law
- Kirchoffs law
- rates of population growth

There exist many principles and laws to be applied to the physical model such as ⇒

Mathematical Model

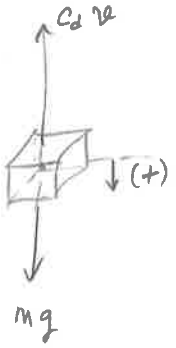
usually involve derivative terms

Application of physical laws to physical model leads to a mathematical model of the system. Mathematical models usually involves derivative terms. The equation involving derivatives are called differential equations (DE).

↓ Solution Techniques (Scope of our lectures)

Solution

ex: Consider a problem of modeling an object falling in the atmosphere near sea level.



C_d is proportionality constant (drag coefficient)

(1) In assumption and simplification step,

- we determine the main quantities of interest playing essential role during the actual event. Let the main parameters be
  - time (t)
  - velocity of the object (v)
- And ignoring the other parameters. The other possible effects like
  - gravitation of moon & sun
  - effect of air flow
  - variation of gravity
  - wind
  - area of object, etc are neglected.

(2) In order to form a proper physical model, next step is to assign a letter to the selected parameters and determine on the proper units. Let

- t denote time in second $[s]$
- v represents the velocity of the falling object in meter per second $[m/s]$

- Presumably, the velocity v will change with time, $v(t)$. In other word
  - t is independent variable
  - v is dependent variable
- Physical law that governs the motion of object is Newton's 2nd Law

$F = m.a$ where m is the mass of the object $[kg]$
F is the net force exerted on the object $[N]$
a is the acceleration of the object $[m/s^2]$

(3) In order to obtain a mathematical model, a positive direction of motion must be selected. Let's select

- downwards to be positive direction.

- Considering the relation between velocity and acceleration, that is the later be the time derivative of the former, mathematical model of the system is obtained as,

$$\text{net Force} = mg - C_d v = m.a = F_{net} \quad \text{and} \quad a = \frac{dv}{dt}$$

$$mg - C_d v = m.\frac{dv}{dt}$$

mathematical model of an object falling

$$\boxed{\frac{dv}{dt} = g - \frac{C_d}{m} v} \rightarrow DE$$

$m = 10$ kg $\rightarrow$ $\dfrac{dv}{dt} = 10 - \dfrac{1}{5} v$ (rate)

$c_d = 2$

$g \cong 10\ m/s^2$

$0 = 10 - \dfrac{1}{5} v \Rightarrow v = 50\ m/s$

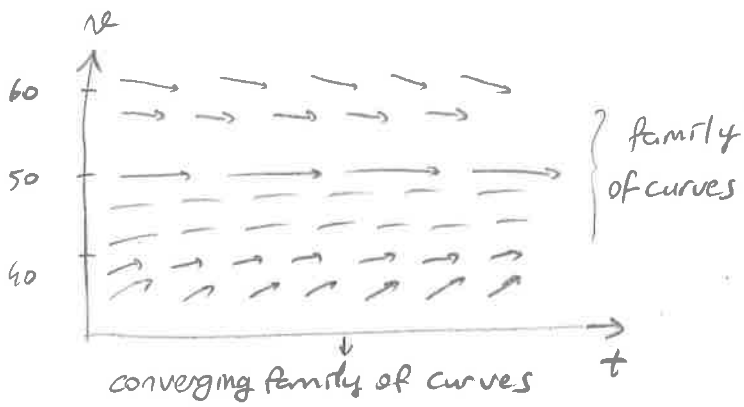
if $v = 40$ $\quad \dfrac{dv}{dt} = 10 - \dfrac{1}{5} \cdot 40 = 2$

if $v = 60$ $\quad \dfrac{dv}{dt} = 10 - \dfrac{1}{5} \cdot 60 = -2$

The force of gravity will be in balance with the air resistance after a sufficiently long time. The net force is zero and acceleration has ceased. Then that constant velocity is called "terminal velocity".

if $v < v_T$ all line segments have positive slopes

if $v < v_T$ all line segments have negative slopes



converging family of curves

<u>ex</u>: Population of mice : Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population.

population of mice (p), time (t), rate constant (r) $= 0.5$ /month (growth rate)

then the assumption about population growth can be expresed by the eqn.

$$\frac{dp}{dt} = rp$$

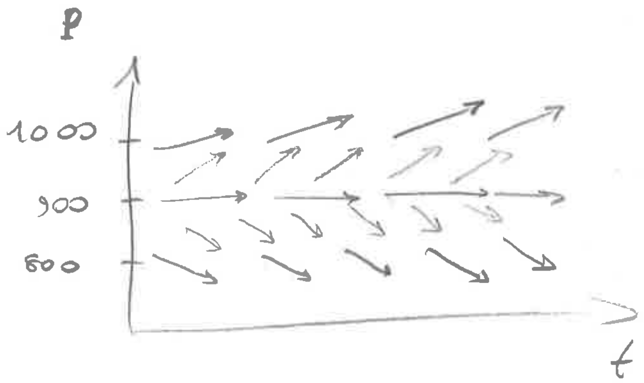
Now let us add the problem by supposing that several owls live in the same neighbourhood and they kill 15 field mice per day. To incorporate this information into model, so that it becomes

$$\frac{dp}{dt} = 0.5p - 450$$

$$\frac{dp}{dt} = 0 = 0.5p = 450 \Rightarrow p = 900$$

if $p = 800 \rightarrow \dfrac{dp}{dt} = 0.5 * 800 - 450 = -50$

$p = 1000 \rightarrow \dfrac{dp}{dt} = 0.5 * 1000 - 450 = 50$



diverging family of curves

Boyce

ex 1. find solution of this equation

$$\frac{dp}{dt} = \frac{p-900}{2} \longrightarrow \frac{dp/dt}{p-900} = \frac{1}{2}$$

$$\longrightarrow \frac{d}{dt} \ln|p-900| = \frac{1}{2}$$

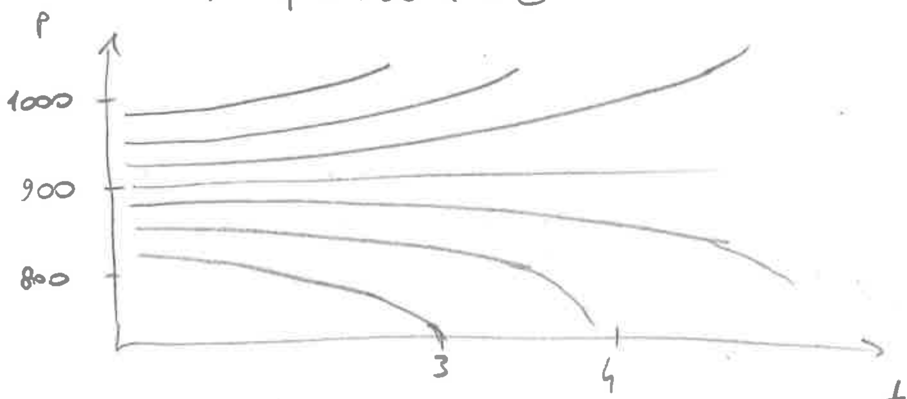
$$\longrightarrow \ln|p-900| = \frac{t}{2} + C$$ → arbitrary constant

by taking the exponential of both sides

$$\rightarrow |p-900| = e^{(t/2)+C} = e^{C} e^{t/2}$$

$$\rightarrow p-900 = \pm e^{C} e^{t/2}$$

$$\rightarrow \quad p = 900 + c\, e^{t/2}$$



for several of c

CH 1 DE's and Their Solutions :

Definition : An eqn. involving derivatives or Differentials of one or more dependent variables with respect to (wrt) one or more independent variables is called differential equations (DE).

examples :

① $\dfrac{dy}{dt} = ky$ (Newton's Law of Cooling)

y: dep. var.
x: indep. var.

Mechanical vibration ② $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = f(t)$

③ $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$

④ $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$

⑤ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

⑥ $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + p(p+1)y = 0$ Legendre's eqn.

⑦ $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-p^2)y = 0$ Bessel's eqn.

⑧ $\dfrac{d^2y}{dx^2} + xy = 0$ Airy's eqn.

⑨ $y'' - 2xy' + 2py = 0$ Chebyshev's eqn.

Hermite's eqn.

⑩ $\dfrac{dy}{dx} = k \cdot y$
is the eqn. of exponential decay (or of biological growth)

$\dfrac{d(uv)}{dx} = \dfrac{du}{dx}v + \dfrac{dv}{dx}u$ → differential identity → NOT a DE like $(x+1)^2 = x^2+2x+1$

$\dfrac{d}{dx}(e^{ax}) = ae^{ax}$ → "

$\dfrac{d}{dx}(\sin x) = \cos x$ → "

y: dep. var.
x: indep. var.
dx: an infinetely small change in x

Diff. Eqn.'s
- According to number of independent parameters
  - ordinary DE (ODE)
  - Partial DE (PDE)
- According to structure
  - Linear
  - Non-Linear

Ordinary DE (ODE)

Defn: A DE involving ordinary derivatives of one or more dependent variables w.r.t. a single independent variable is called an ordinary DE (Total derivatives only) (1 & 2). They have a single independent variable.

examples:

① $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$

② $\frac{d^3y}{dt^3} + t\frac{d^2y}{dt^2} + \frac{dy}{dt} + y^2 = \tan t$

③ $\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0 \rightarrow$ not ODE (Transport Eqn)

Partial DE (PDE)

Def'n: A DE involving partial derivatives of one or more dependent variables w.r.t two or more independent variables is called PDE (3,4,5).

examples:

① $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (Heat eqn)

② $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ (Wave eqn in 2-D)

③ $\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = \nu\frac{\partial^2 y}{\partial x^2}$ (Burger's eqn)

Order of a DE

Defn: The order of highest ordered derivative involve in a DE is called order of the DE.

ex: ① $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \rightarrow O = 2$, dep. var. $= y$, ind. var. $= x$

② $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \rightarrow O = 4$, $= x$, $= t$

③ $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \rightarrow O = 1$, $= v$, $= t, s$

Degree of a DE

Def'n: If a DE can be rationalized and cleared from fractions with regard to all derivatives present, the exponent of the highest order derivative is called the degree of DE.

$2/3 \to$ irrational

ex: $\left(\dfrac{d^2y}{dt^2}\right)^{2/3} = 1 + \dfrac{dy}{dt}$

$\left(\dfrac{d^2y}{dt^2}\right)^2 = \left(1+\dfrac{dy}{dt}\right)^3 \to$ ODE, $O=2$, $D=2$, dep. var. $=y$; ind. var. $= t$

$\dfrac{\partial^4 z}{\partial x^4} + \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^6 = x \to$ PDE, $O=4$, $D=1$, dep. var. $= z$, ind. var. $= x, y$

$y'' + (y')^2 = \ln y'' \to$ ODE, $O=2$, $D=1$, dep. var. $= y$, ind. var. $= t$

$\dot{y} = e^t e^y \to$ ODE, $O=1$, $D=1$, dep. var. $= y$

LINEARITY & NON-LINEARITY

Def'n: A DE is called linear if

a) every dependent variable and every derivative involved occurs to the 1^{st} degree only, and

b) no products of dependent variables and/or derivatives occur

If a DE which is not linear is called non-linear DE.

These are true for ODE and PDE as well.

exp's:

① $\dfrac{d^4y}{dx^4} + x^2\dfrac{d^3y}{dx^3} + x^3\dfrac{dy}{dx} = xe^x \to$ Lin., ODE, $O=4$, $D=1$, dep $= y$, ind. $= x$

② $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0 \to$ Lin., ODE, $O=2$, $D=1$, dep $= y$, ind. $= x$

③ $x\dfrac{d^2y}{dx^2} + \overbrace{y\dfrac{dy}{dx}}^{\text{product}} = e^x \to$ non-Lin., ODE, $O=2$, $D=1$, dep $= y$, ind. $= x$

④ $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^{\overset{\text{power}}{3}} + 6y = 0 \to$ non-Lin., ODE, $O=2$, $D=1$, dep $= y$, ind. $= x$

⑤ $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0$ → non-Lin., ODE, O = 2, D = 1, dep = y, ind. = x

(function)

⑥ $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ → Lin., PDE, O = 2, D = 1, dep = y, ind. = x, t

⑦ $\left(\frac{\partial y}{\partial t}\right)^2 = \frac{\partial^2 y}{\partial x^2}$ → non-Lin., PDE, O = 2, D = 1, dep. = y, ind. = x, t

Constant Coefficients, Variable Coefficients:

Defn: Linear DE are further classified according to the nature of the coefficients of the dependent variables and their derivatives:

ex: (2) is said to be Lin. with constant coefficient

ex: (1) is said to be Lin. with variable coefficient

HOMEWORK 1

Exercises: Classify each of the following DE as: PDE, ODE, Lin, non-Lin., Determine the order and degree of them.

① $\frac{dy}{dx} + x^2y = xe^x$ → Lin., ODE, O = 1, D = 1, dep. = y, ind = x

② $\frac{d^3y}{dt^3} = \sqrt{x+y}$ → non-Lin., ODE, O = 3, D = 1, dep = y, ind = t

③ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ → Lin., PDE, O = 2, D = 1, dep = u, ind = x, y

④ $\frac{d^2y}{dx^2} + x\sin y = 0$ → non-Lin., ODE, O = 2, D = 1, dep = y, ind = x

⑤ $\frac{d^2y}{dx^2} + y\sin x = 0$ → Lin., ODE, O = 2, D = 1, dep = y, ind = x

⑥ $\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0$ → Lin., PDE, O = 4, D = 1, dep = u, ind = x, y

⑦ $\frac{d^6x}{dt^6} + \left(\frac{d^4x}{dt^4}\right)\left(\frac{d^3x}{dt^3}\right) + x = t$ → non-Lin., ODE, O = 6, D = 1, dep = x, ind = t

⑧ $\left(\frac{dr}{ds}\right)^3 = \sqrt{\frac{d^2r}{ds^2} + 1}$ → non-Lin., ODE, O = 2, D = 1, dep = r, ind = s

$\left(\frac{dr}{ds}\right)^6 = \frac{d^2r}{ds^2} + 1$

⑨ $y'' + 3y + 5x = 0$

⑩ $x'' + tx^2 = t$

Solutions of DE

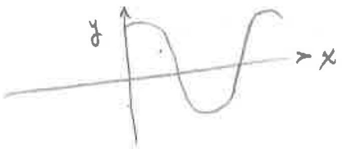
Any non-derivative relation between the variables of a DE which satisfies the given DE is called a sol'n of it. The graphs of the solns of a DE are called its INTEGRAL CURVES (soln curves)

Method of soln :
- Closed form soln
  - Series method
  - Numerical methods
  - Graphical methods

Solns of a DE
1) Explicit soln } soln
2) Implicit soln }
3) formal soln → non-real values

ex 1 : $f(x) = 2\sin x + 3\cos x$ is an explicit soln of

$$\frac{d^2y}{dx^2} + y = 0$$



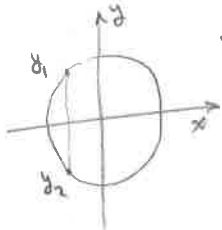
for each "x" there is only one "y"

ex 2 : $x^2 + y^2 - 25 = 0$ is an implicit soln of

$$x + y\frac{dy}{dx} = 0$$

$f_1(x) = \sqrt{25-x^2}$ → 1st explicit sol'n.

$f_2(x) = -\sqrt{25-x^2}$ → 2nd explicit sol'n.



for each "x" there are two "y" values

ex 3 : $x^2 + y^2 + 25 = 0$ is a formal soln of

$$2x + 2y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y} \rightarrow \therefore y = \mp\sqrt{-25-x^2}$$ non real

ex : $y = cx^2$ is a soln of

$xy' = 2y \quad (y' = \frac{dy}{dx})$ Lin., ODE, O=1, D=1

Let's Check!

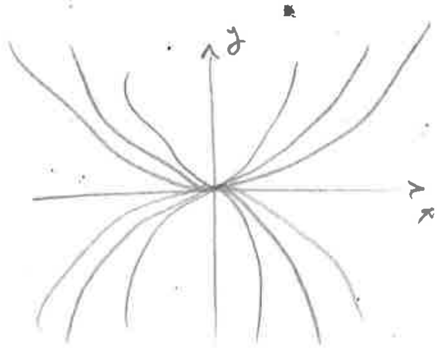
$$x(2cx) = 2(cx^2)$$
$$2cx^2 = 2cx^2$$

$c=-2$, $y = -2x^2$

$c=-1$, $y = -x^2$

$c=1$, $y = x^2$

$c=2$, $y = 2x^2$



} family of parabolas

Initial - Value Problems, Boundary Value Problems
Existence of Sol^{n}:

Initial (boundary, side) conditions (IC)

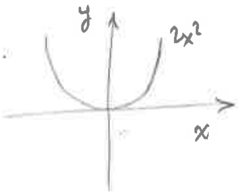
ex: $xy' = 2y$ & $\underset{x}{y(1)} = \underset{y}{2}$

$y = cx^2$

$2 = c \cdot 1^2$

$c = 2$

$y = 2x^2$

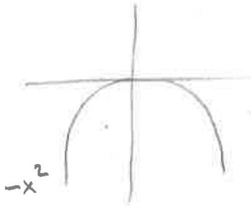


if $\underset{x}{y(1)} = \underset{y}{-1}$

$-1 = c(1)^2$

$c = -1$

$y = -x^2$



if $y(0) = 1 \Rightarrow y = cx^2$
$1 = c(0)$ → no sol^{n}.

Initial Value Problem (IVP):

If the boundary conditions (cond$\underline{s}$) relate to one independent value (x) the problem is called IVP.

ex: $\dfrac{d^2y}{dx^2} + y = 0$ $\quad y(1) = 3$, $\; y'(1) = -4$

Boundary Value Problem (BVP):

If the cond$\underline{s}$ relate to two different x values, the problem is called BVP.

ex: $\dfrac{d^2y}{dx^2} + y = 0$ $\quad y(0) = 1$, $\; y(\pi/2) = 5$

Finding DE from the General Sol'^{n}:

Essential Arbitrary Constants: If the number of constants can not be replaced by a smaller number of constants such constants are called ESS. ARB. CONST.$\underline{S}$.

ex: $y = \underbrace{c_1 e^{2c_2 + x}}_{2\text{ const.}} = \underbrace{c_1 e^{2c_2}}_{1\text{ ess. const.}} e^{x}$ → 1^{st} Ord. ODE

$y = c_1 \cos x + c_2 \sin x$ (2 Ess. Arb. Const.s) → 2^{nd} Ord. ODE

$x = c_1 + c_2 \sin t + c_3 \cos t$ (3 Ess. Arb. Const.s) → 3^{rd} Ord. ODE

* The number of ESS. ARB. CONST.$\underline{S}$ in the sol^{n} corresponds the order of the given DE.

ex: Obtain the DE corresponding the following soln

$y = c_1 e^{x} + c_2 e^{-x}$ [given the soln find the corr. DE]

2 ess. arb. const. $\Rightarrow$ O=2

1^{st} step: Elimination of Arb. Const.

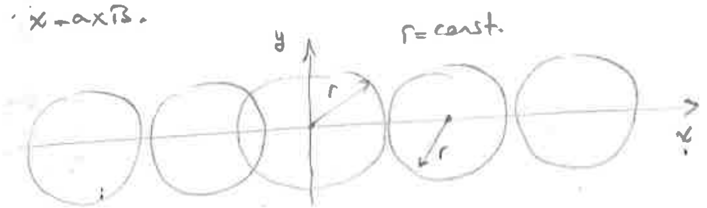
$$\left.\begin{aligned} y &= c_1 e^{x} + c_2 e^{-x} \\ y' &= c_1 e^{x} - c_2 e^{-x} \\ y'' &= c_1 e^{x} + c_2 e^{-x} \end{aligned}\right\} \text{eliminate } c_1 \,\&\, c_2$$

2^{nd} step: by applying mathematical manupulation

$$\left.\begin{aligned} y + y' &= 2 c_1 e^{x} \\ y' + y'' &= 2 c_1 e^{x} \end{aligned}\right\} \begin{aligned} & y + y' = y' + y'' \\ & \boxed{y'' - y = 0} \end{aligned}$$ ANS

(Lin., ODE, O=2, D=1, dep. = y, ind. = x)

ex: Find the DE of the family of circles of fixed radius (r) with centers on the x-axis.



integral curves (soln curves)

$(x - c_1)^2 + (y - 0)^2 = r^2 = \text{const.} \rightarrow$ soln is given

only 1 ess. arb. const. $\rightarrow \therefore$ 1^{st} ord. DE

eliminate c_1

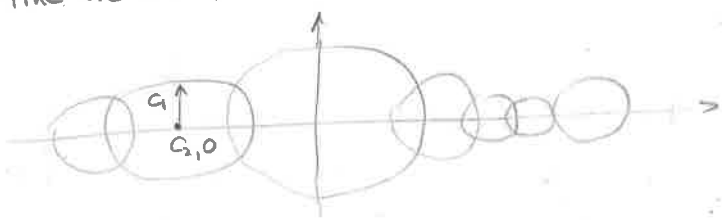
$$2(x - c_1) + 2y\,y' = 0$$

$$x = c_1 - y y'$$

$$(c_1 - yy' - c_1)^2 + (y - 0)^2 = r^2$$

$$(yy')^2 + y^2 = r^2$$ (Non-Lin., ODE, O=1, D=2, dep. = y, ind. = x)

ex: Find the DE of the circles of various radius c_1 with centers on x-axis.



Sol$\underline{n}$: $\underbrace{(x-c_2)^2 + (y-0)^2 = c_1^2}_{\text{2 Ess. Arb. const.s}} \Rightarrow 2^{nd}$ ord. ODE

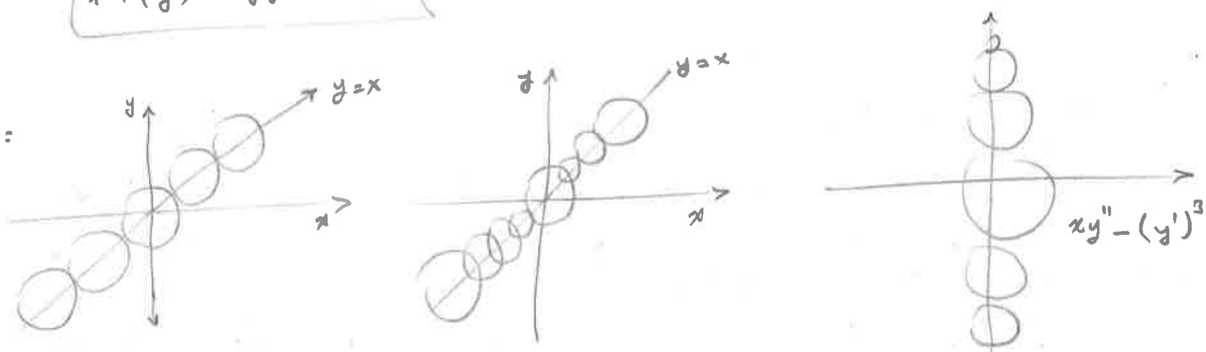
$$2(x-c_2) + 2yy' = 0$$

$$2 + 2[yy'' + y'y'] = 0$$

$$\boxed{1 + (y')^2 + yy'' = 0}$$

ANS (Non-Lin., ODE, O = 2, D = 1, dep. = y, ind. = x)

HW:



$xy'' - (y')^3 - y' = 0$

HW: According to Newton's law of cooling, the temperature $u(t)$ of an object satisfies the DE $\quad \frac{du}{dt} = -k(u-T)$

where T is const. ambient temperature and k is positive const. Suppose that the initial temperature of the object is

$$u(0) = u_0$$

Find the temperature of the object at any time

a) Find the temperature of the object at any time

b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k & τ.

Sol$\underline{n}$:

$$\frac{du}{u-T} = -k\,dt$$

$$d.[\ln(u-T)] = -kt + c_1$$

$$u - T = ce^{-kt}$$

$$u = ce^{-kt} + T$$

$u = u_0$ @ $t = 0$

$u_0 = c + T \Rightarrow c_1 = u_0 - T$

(a) $u = (u_0 - T)e^{-kt} + T$

$u = \frac{u_0 - T}{2}$ @ $t = \tau$

$$\frac{u_0 - T}{2} = (u_0 - T)e^{-k\tau} + T$$

$$\ln \frac{u_0 - 3T}{2} = \ln\left[(u_0 - T) \cdot e^{-k\tau}\right]$$

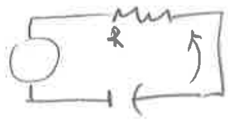
$$\ln \frac{u_0 - 3T}{2} - \ln(u_0 - T) = -k\tau$$

$$\tau = -\frac{1}{k} \ln\left[\frac{u_0 - 3T}{2(u_0 - T)}\right]$$

D.P

1.17 Consider an electric circuit containing a capacitor, resistor & battery shown as below. The charge $Q(t)$ on the capacitor satisfies the eqn.

$$R\frac{dQ}{dt}+\frac{Q}{C}=V$$



where R is the resistance, C is the capacitance and V is the constant voltage supplied by the battery.

a) If $Q(0)=0$, Find $Q(t)$ at any time.

b) Find the limiting value Q_L that $Q(t)$ approachs a long time

c) Suppose that $Q(t_1)=Q_L$ and that at time $t=t_1$ the battery is removed and circuit closed again. Find $Q(t)$ for $t>t_1$

Soln:
$$\frac{dQ}{dt}=\frac{V}{R}-\frac{Q}{CR}$$

$$\frac{dQ}{dt}=\frac{CV-Q}{CR}$$

(a) $$\frac{dQ/dt}{Q-CV}=\frac{-1}{CR}$$, thus integrating & simplifying we get,

$$Q=De^{-t/CR}+CV$$

$Q(0)=0 \rightarrow D=-CV$ and thus

$$Q(t)=CV\left(1-e^{-t/CR}\right)$$

(b) $\lim_{t\to\infty} Q(t)=CV$ since $\lim_{t\to\infty} e^{-t/CV}=0$

(c) $$R\frac{dQ}{dt}+\frac{Q}{C}=0$$

$$Q(t_1)=CV$$

DE is $Q(t)=Ee^{-t/CR}$

$$Q(t_1)=Ee^{-t_1/CR}=CV$$

$$E=CVe^{t_1/CR}$$ thus

$$Q(t)=CVe^{t_1/CR}\cdot e^{-t/CR}=CVe^{-(t-t_1)/CR}$$