MATH 2055

Introduction to Differential Equations

Instructor

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Course Website

- http://akademik.marmara.edu.tr/mustafa.yilmaz/teaching for

  Syllabus, assignments, etc...

- https://www.wolframalpha.com for interactive answer check

Exams

- 2 Midterms:
  - Midterm 1: will be anounced
  - Midterm 2: will be anounced
- Final Exam: will be anounced

Grading Scheme & Grading Scale

- Scheme:
  - 9% HW, 24% Midterm 1, 27% Midterm 2, 40% Final Exam
- Scale:
  - Relative Evaluation System (Curve)

Mathematical Symbols And Abbreviations

Symbol	Say	Means
$\therefore$	therefore	therefore
$\because$	because	because
$\forall$	for all	for all
$\Leftarrow\Rightarrow$	if and only if or *iff*	if and only if or *iff*
$\propto$	proportional to	proportional to
$\Rightarrow$	implies	calculation on left of symbol imply those on the right
fn	function	function
wrt	with respect to	with respect to
LHS	Left-hand side	Left-hand side
RHS	right-hand side	right-hand side

Mathematical Symbols And Abbreviations

Symbol	Say	Means
$\frac{dy}{dx}$	Dee ***y*** dee ***x***	Differentiate fn ***y*** wrt ***x***
$\frac{d^2y}{dx^2}$	Dee 2 ***y*** dee ***x*** squared	Double differentiate fn ***y*** wrt ***x*** Second derivative of fn ***y***
$\frac{\partial y}{\partial x}, y_x$	Partial ***y*** wrt ***x***	Partial derivative of ***y*** wrt ***x***
$f'(x)$	f prime of ***x*** or f prime	Differentiate fn ***f(x)*** wrt ***x***, equivalent to $\frac{dy}{dx}$ if ***y=f(x)***
y'	***y*** prime	Differentiate fn ***y*** wrt ***x***, equivalent to $\frac{dy}{dx}$ if ***y=f(x)***
$\dot{x}$ **(dot above variable *x*)**	***x*** dot	Differentiate fn ***x*** wrt ***t***

Mathematical Symbols And Abbreviations

Symbol	Say	Means
$f''(x)$	*f* double prime of ***x*** or *f* double prime	Differentiate fn ***f(x)*** wrt ***x*** twice, Second derivative of fn ***f(x)***, equivalent to $\frac{d^2y}{dx^2}$, y'' if ***y=f(x)***
$\ddot{x}$	***x*** double dot	Differentiate fn ***x*** wrt ***t*** twice, Second derivative of fn ***x***, equivalent to $\frac{d^2x}{dt^2}$
$f'''(x)$	*f* triple prime of ***x*** or *f* double prime	Differentiate fn ***f(x)*** wrt ***x*** three times, Third derivative of fn ***f(x)***, equivalent to $\frac{d^3y}{dx^3}$, y''' if ***y=f(x)***

Mathematical Symbols And Abbreviations

Symbol	Say	Means
$\dddot{x}$	***x*** triple dot	Differentiate fn ***x*** wrt ***t*** three times, Third derivative of fn ***x,*** equivalent to $\frac{d^3x}{dt^3}$
$f^{(4)}(x)$	fourth derivative of $f(x)$ or f to four in parenthesis	Differentiate fn ***f(x)*** wrt ***x*** four times, Fourth derivative of fn ***f(x)***, equivalent to $\frac{d^4y}{dx^4}$, $y^{(4)}$ if ***y=f(x)***
$\mathcal{L}$	Laplace Transform	converts a time domain function to complex-domain function by integration from zero to infinity

Terminology

- This course deals with *ordinary differential equations*, which are equations that contain one or more derivatives of a function of a single variable. Such equations can be used to model a rich variety of phenomena of interest in the sciences, engineering, economics, ecological studies, and other areas.
- *Functions*: dependent variables and independent variables.

Concepts of differential equations

- In general, a differential equation is an equation that contains an unknown function and its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation.
- A function $\mathrm{y} = f(x)$ is called a solution of a differential equation if the equation is satisfied when $\mathrm{y} = f(x)$ and its derivatives are substituted into the equation.

$e.g.$

Show that $y = 2e^{2x}$ is a particular solution of the ordinary differential equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$$

Taking derivatives of fn

$$y' = 4e^{2x}$$
$$y'' = 8e^{2x}$$

Substituting into the DE, $8e^{2x} - 4e^{2x} - 2(2e^{2x}) = 0$

so $\boxed{y = 2e^{2x}}$ sol'n of the DE.

Why?

- One of the most important application of calculus is differential equations, which often arise in describing some phenomenon in engineering, physical science etc.
  - Population Dynamics
  - Circuit Design
  - Astrophysics
  - Geodesics (Pure Math)

Differential Equations

- The following are examples of physical phenomena involving rates of change:
  - Motion of fluids
  - Motion of mechanical systems
  - Flow of current in electrical circuits
  - Dissipation of heat in solid objects
  - Seismic waves
  - Population dynamics

Differential Equations (DE's)

- Equations that describe rates of change
- Equations that involve derivatives:

e.g., $p''(t) = -mg$ **(Falling object)**

$y'(t) = y(t)(1 - y(t))$ **(Logistic Equation)**

$u(t,x)_{tt} = cu(t,x)_{xx}$ **(Wave Equation)**

$L\dfrac{d^2Q(t)}{dt^2} + R\dfrac{dQ(t)}{dt} + \dfrac{1}{C}Q(t) = E(t)$ **(Current)**

$$y'' + y = 4x^2$$
$$y''(x) + y(x) = 4x^2$$
$$\frac{d^2y(x)}{dx^2} + y(x) = 4x^2$$

y : function of x

x : independent variable

Two Types

Ordinary (ODE)
(Only One Independent Variable)

$$p''(t) = -mg$$

$$y'(t) = y(t)(1 - y(t))$$

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

$$\frac{dv}{dt} = 9.8 - 0.2v, \quad \frac{dp}{dt} = 0.5p - 450$$

involves only **total derivatives** $\frac{df(x)}{dx}$

Partial (PDE)
(Multiple Independent Variables)

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{(heat equation)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave equation)}$$

$$u(t,x)_{tt} = cu(t,x)_{xx}$$

involves **partial derivatives** $\frac{\partial f(x.y)}{\partial x}, \frac{\partial f(x.y)}{\partial y}$

This Class

Ordinary
(Only One Independent Variable)

$$p''(t) = -mg$$

$$y'(t) = y(t)(1 - y(t))$$

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

$$\frac{dy}{dx} + 2 = 0,$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$$

Ordinary Differential Equations (ODE)

Definition: A differential equation is an equation containing an unknown function and its derivatives.

Examples:
(ex.'s)

1. $$\frac{dy}{dx} = 2x + 3$$
2. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$$
3. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

y is dependent variable (dep. var.) and ***x*** is independent variable (ind. var.)

Partial Differential Equations (PDE)

Definition: Partial Differential Equation: Differential equations that involve two or more independent variables are called partial differential equations.

ex.'s :

1. $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$$
2. $$u(t,x)_{tt} = cu(t,x)_{xx}$$
3. $$u_t + uu_x = 1 + u_{xx}$$

y & ***u*** are dependent variables and ***x*** and ***t*** are independent variables

Order (O)

The order of a differential equation is the *highest derivative* that appears in the differential equation.

Equation	**ind. var.**	**dep. var.**	**Order**
$\left(\frac{dy}{dx}\right)^2 + 2 = 0$	x	y	1
$xy' - y^2 = 3x$	x	y	1
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$	x	y	2
$u(t,x)_{tt} = cu(t,x)_{xx}$	x, t	u	2
$z^{(iv)} + z'' = -y$	y	z	4
$\frac{d^4w}{dt^4} - w^2 = e^{-2t}$	t	w	4

Degree (D)

The degree of a differential equation is the power of the *highest derivative* term.

Equation	ind. var.	dep. var.	Order	Degree
$\left(\frac{dy}{dx}\right)^2 + 2 = 0$	x	y	1	2
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$	x	y	2	1
$z^{(iv)} + z'' = -y$	y	z	4	1

Linear Differential Equations (LDE)

If a differential equation can be written as:

$$a_0(t)\mathrm{y}^{(n)} + a_1(t)\mathrm{y}^{(n-1)} + \cdots + a_n y = g(t)$$

A differential equation is called *linear* if there are

- no multiplications among dependent variables and
- their derivatives.

The term $g(t)$ is called a **forcing function**.

Linear Differential Equations (LDE)

In other words, all coefficients are functions of independent variables.

Equation	**ind. var.**	**dep. var.**	**Order**	**Degree**	**Linear**
$\left(\frac{dy}{dx}\right)^2 + 2 = 0$	x	y	1	2	-
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$	x	y	2	1	OK
$z^{(iv)} + z" = -\mathrm{y}$	y	z	4	1	OK

Non-linear Differential Equations

Differential equations that do not satisfy the definition of linear are non-linear.

Equation	ind. var.	dep. var.	Order	Degree	Linear	Non-Linear
$\left(\frac{dy}{dx}\right)^2 + 2 = 0$	x	y	1	2	-	OK
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$	x	y	2	1	OK	-
$z^{(iv)} + z'' = -\mathrm{y}$	y	z	4	1	OK	-

Constant Coefficient Linear ODE

Linear Differential equations with Constant Coefficient

$$b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \dots + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

$$3 \frac{d^4 y}{dt^4} + 5 \frac{d^3 y}{dt^3} + 7 \frac{d^2 y}{dt^2} + 8 \frac{dy}{dt} + 4y = \cos 5\, t + t^2$$

ex.'s :

Equation	ind. var.	dep. var.	Order	Degree	Linear	Non-Linear	Type
$\left(\frac{dy}{dx}\right)^2 + 2 = 0$	x	y	1	2	-	OK	ODE
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$	x	y	2	1	OK	-	ODE
$z^{(iv)} + z" = -y$	y	z	4	1	OK	-	ODE
$y' + ay = bt$	t	y	1	1	OK	-	ODE
$f" + f' + g(t)f = h(t)$	t	f	2	1	OK	-	ODE
$\frac{dy}{dx} = 2x + 3$	x	y	1	1	OK	-	ODE
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	x	y	3	1	-	OK	ODE
$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$	x, t	y	2	1	OK	-	PDE
$u(t,x)_{tt} = cu(t,x)_{xx}$ *or* $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$	x, t	u	2	1	OK	-	PDE

Solutions to Differential Equations

Three important questions in the study of differential equations:

- Is there a solution? (Existence)
- If there is a solution, is it unique? (Uniqueness)
- If there is a solution, how do we find it?
    - (Analytical Solution, Numerical Approximation, etc.)

General Solution (gen. sol'n)

- Solutions obtained from integrating the differential equations are called ***general solutions***.
- The general solution of an order ODE contains ***arbitrary constants*** resulting from integrating times.
- Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant *c*.

e.g., $y'' + 4y = 0$, sol'n: $y = \sin 2x$ for $x \in R$

$y' = \frac{y}{x} + 1$, sol'n: $y = x \ln x$ for $x > 0$

Particular Solution

- Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.
- A particular solution does not contain any arbitrary constants.

Explicit Solution

Any solution that is given in the form *y* = *f*(*x*) (*independent variable*).

ex.'s: $$\begin{aligned} A(r) &= \pi r^2 + c \\ y(t) &= e^{-t} + c \end{aligned}$$ etc.

For most differential equations, it is *impossible* to find an explicit formula for the solution.

Implicit Solution

Any solution that isn't in explicit form *f*(*x*.*y*)=0.

ex.'s:
$$y^2x^2 = c$$
$$\sin(xy) = c$$
$$xe^y = c$$
etc.

Initial Value Problems

- An ODE together with an initial condition is called an ***initial value problem*** (**IVP**) .
  - $F(x, y, y') = 0, \qquad y(x_0) = y_0 \qquad \Rightarrow$ **initial condition**

Graphically, the particular integral curve passes through point (x_0, y_0)

Boundary Value Problems

- Problems with specified boundary conditions are called ***boundary value problems*** (**BVP'**s).
    - $F(x, y, y', y'') = 0, \quad y(x_0) = y_0, y(x_1) = y_1 \quad \Rightarrow$ **boundary conditions**

The objective is to obtain a ***unique*** solution

Initial Condition (IC)

In many physical problems we need to find the particular solution that satisfies a condition of the form $\boldsymbol{y(x_0)=y_0}$. Constrains that are specified at the initial point, generally time point, are called ***initial conditions***.

ex.'s:

y'+y=0 y(0)=1

z'=1 z(1)=-1

y'+y=2 y(1)=-5

General solution to (3): $y = 2 + ke^{-x}$

$$\because \quad y(1) = 2 + ke^{-1} = -5 \quad \therefore \quad k = -7e, \quad y = 2 - 7e^{1-x}$$

Boundary Condition (BC)

Constrains that are specified at the boundary points, generally space points, are called ***boundary conditions***.

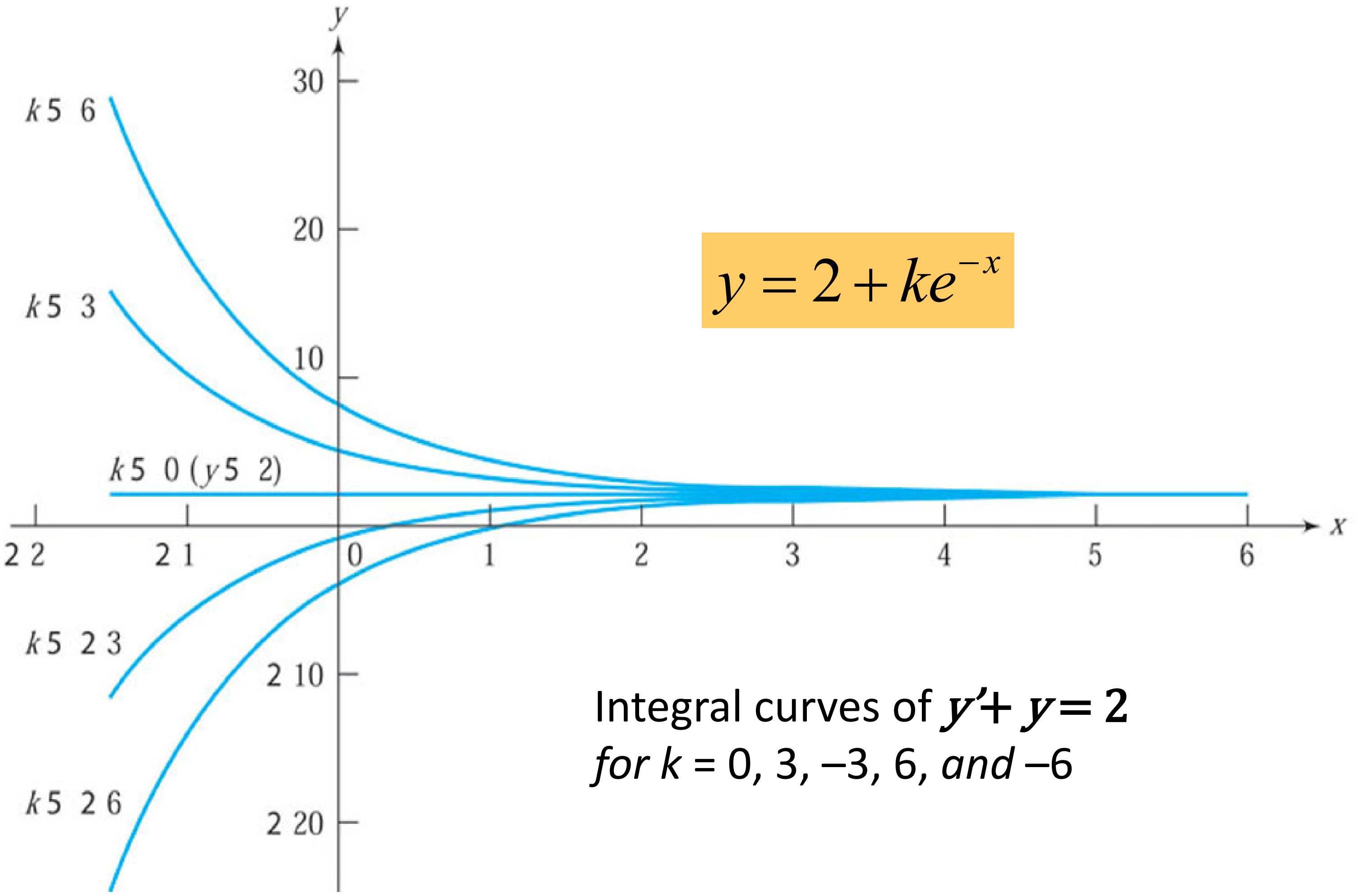
ex.'s:

$$z'' + z' + z = 1 \quad z(0) = 2 \quad z(2) = -1$$

$$f'' + f = 0 \quad f(0) = 0 \quad f(\pi/2) = 1$$

Integral Curves, Solution Curves

- A graph of the solution of an ODE is called a ***solution curve*, or an *integral curve*** of the equation.
- Help to comprehend the behavior of solution



$y = 2 + ke^{-x}$

Integral curves of $y' + y = 2$
for k = 0, 3, –3, 6, *and* –6

Families of Solutions

- A solution containing an arbitrary constant (parameter) represents a set $G(x, y, c) = 0$ of solutions to an ODE called a ***one-parameter family of solutions***.

- A solution to an *n−th* order ODE is a ***n-parameter family of solutions*** $F(x, y, y', \cdots, y^n) = 0$.

- Since the parameter can be assigned an infinite number of values, an ODE can have an infinite number of solutions.

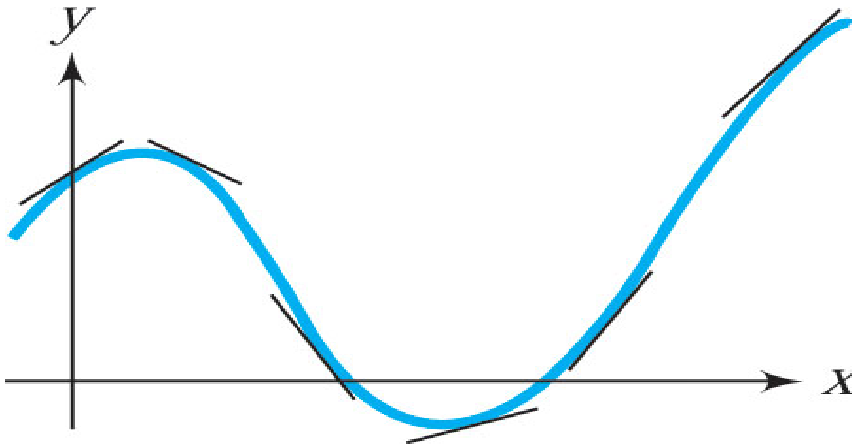
Direction Field

- Suppose we are asked to sketch the graph of the solution of the initial-value problem $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y}(\boldsymbol{x_0}) = \boldsymbol{y_0}$. The equation tells us that the slope at any point $(\boldsymbol{x}, \boldsymbol{y})$ on the graph is $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$.
- A plot of short line segments drawn at various points in the $\boldsymbol{x}, \boldsymbol{y}$ -plane showing the slope of the solution curve this is called a "*direction field*" for the differential equation.
- The direction field gives us the " ***flow of solutions*** ".

Direction Field

- We plot small lines representing slopes at each coordinate (x,y) in the xy-plane. From this, we can infer solution curves. Short tangent segments suggest the shape of the curve
  - Giving $F(x, y, y') = 0$, instead of solving for y,

    solving for $y' = \frac{dy}{dx} = f(x, y)$ $\begin{cases} \text{slope} \\ \text{direction field} \end{cases}$



Direction Field

***ex*. 1:** Sketch a direction field for $y' = x + y$.

At each (x,y) coordinate, we determine y'. Some examples are:

At (1,1) we have $y' = 2$.

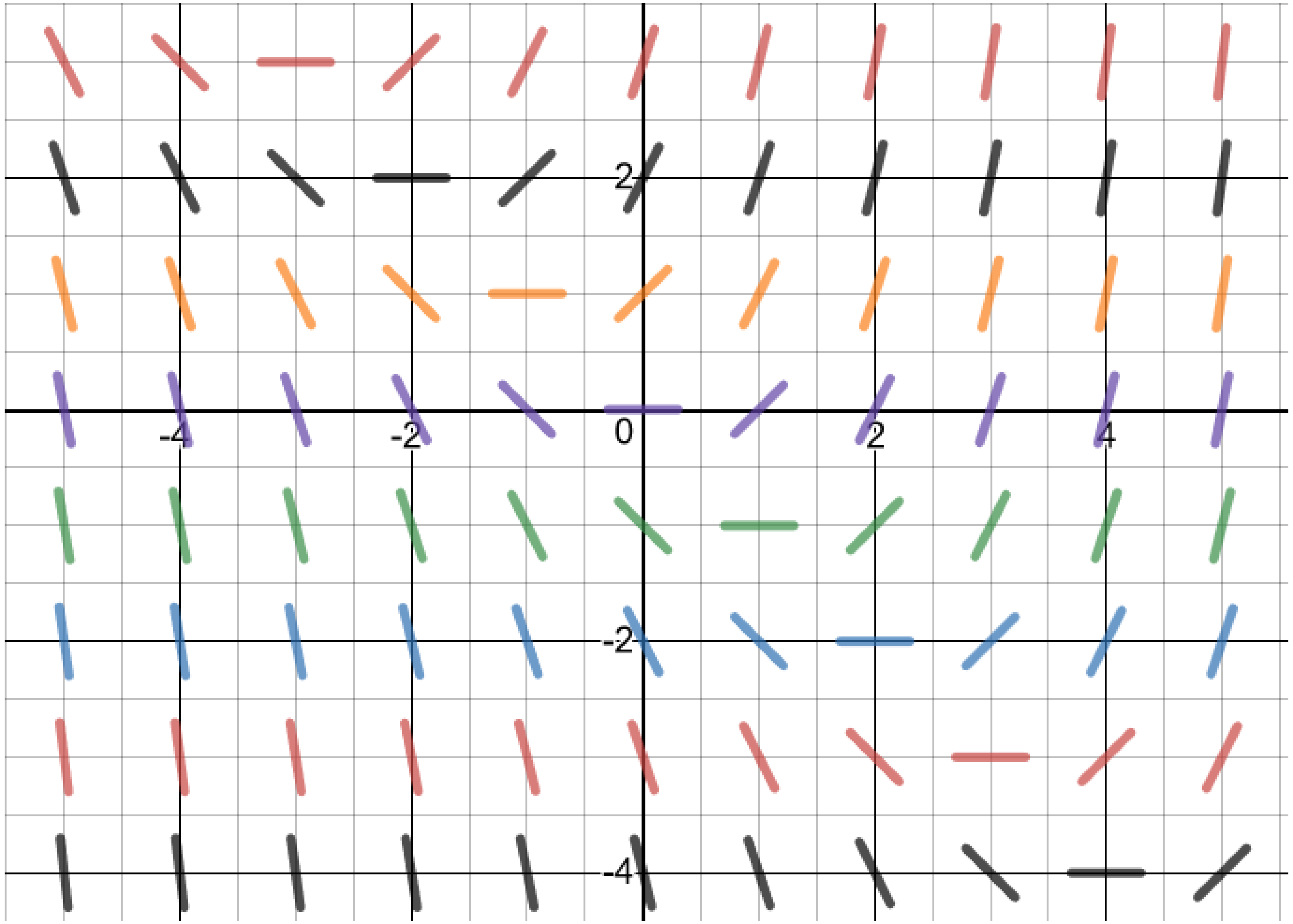
At (2,-3), we have $y' = -1$. And so on.

We do this for "all" possible points in the plane. We get…

example 1: Continuation

Direction field for

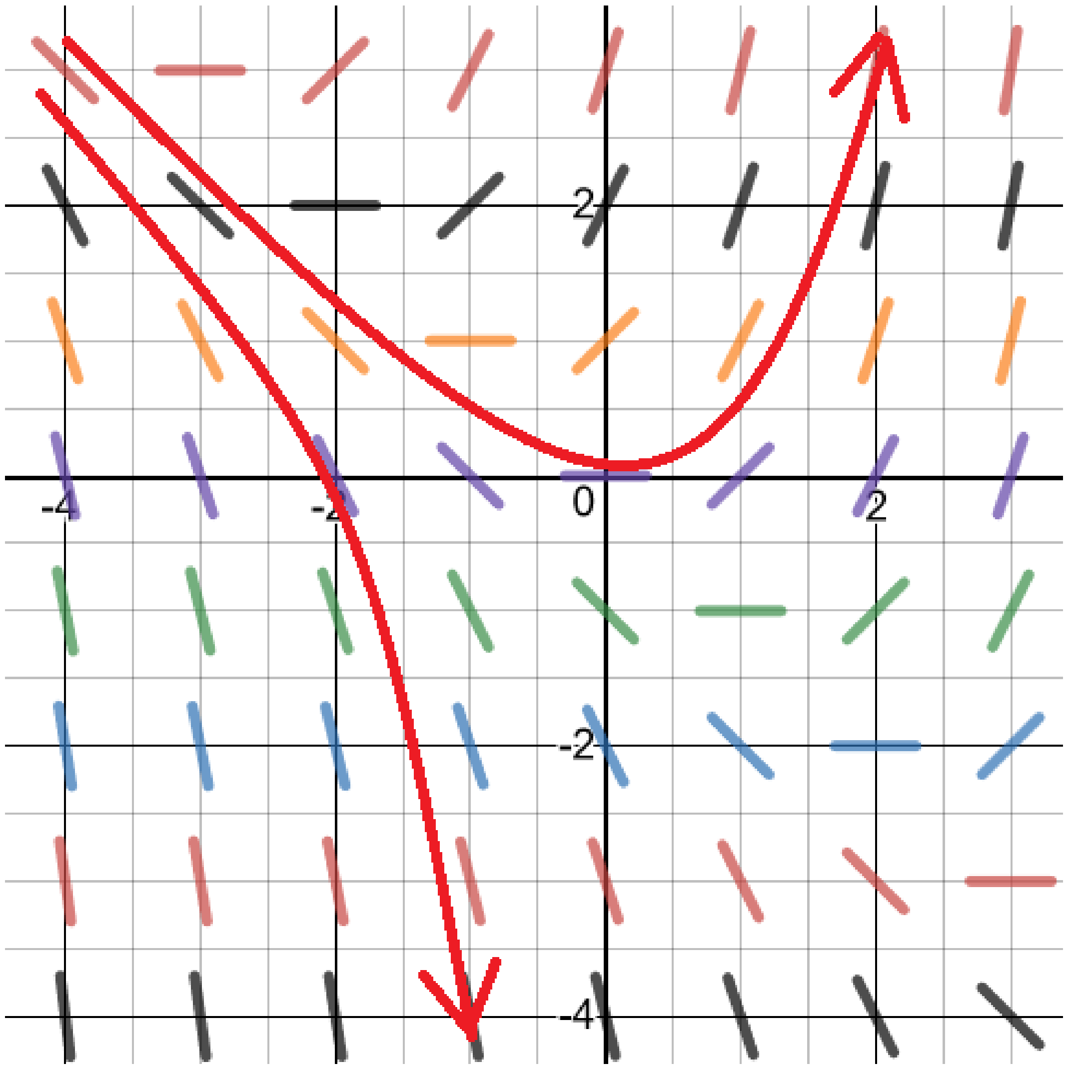
$$y' = x + y$$



ex. 1: Continuation

We can infer possible solution curves.

If you know a point on a particular curve, the rest of the curve can be inferred by "following" the direction field. (*We always read left to right, i.e. x is increasing*).



Direction Field

ex. 2: Sketch the direction field for $y' = y^3 - 4y$.

Hint: note that this only depends on y. Thus, if we find one slope-line for a particular y-value, we can extend it left and right.

Another hint: Note that $y' = 0$ represents a horizontal slope-line and this occurs when $y^3 - 4y = 0$. Factoring, we have

$$y(y^2 - 4) = y(y + 2)(y - 2) = 0.$$

Thus, when $y = 2, -2 \text{ or } 0$, then $y' = 0$.

ex. 2: Continuation

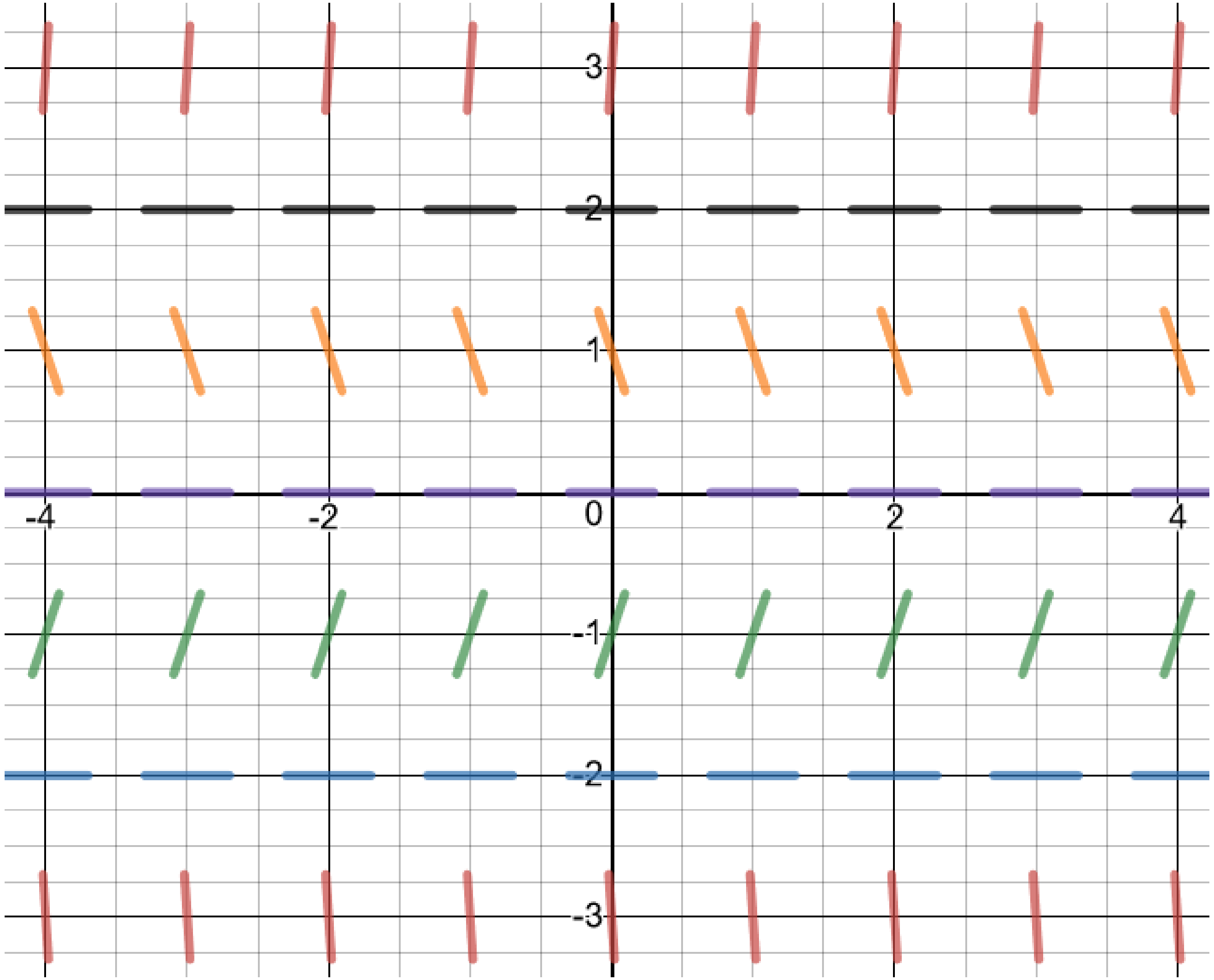
The direction field for

$$y' = y^3 - 4y$$

Note the horizontal slopes when

y = -2, y = 0 or y = 2. These are

equilibrium solutions.



ex. 2: Continuation

Note how the flow lines (representing possible solution curves) veer towards the equilibrium solution $y = 0$.

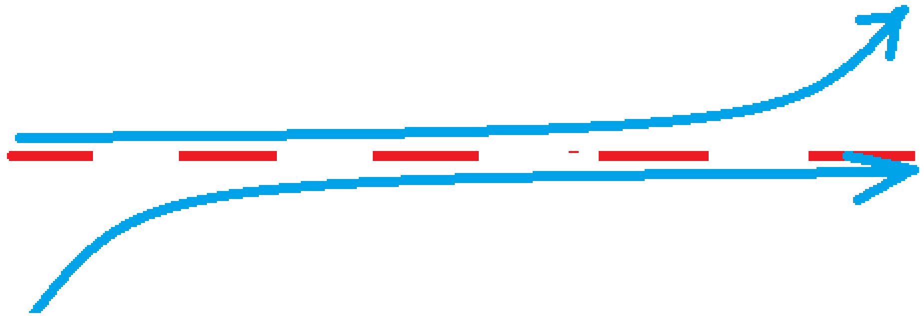
Thus, $y = 0$ is a stable equilibrium.

ex. 2: Continuation

The flow lines veer away from the equilibriums $y = 2$ and $y = -2$. These are unstable equilibriums.

Direction Field

A third type of equilibrium is *semi-stable*, in which the solution curves approach the equilibrium asymptotically from one side, yet veer away from the other.

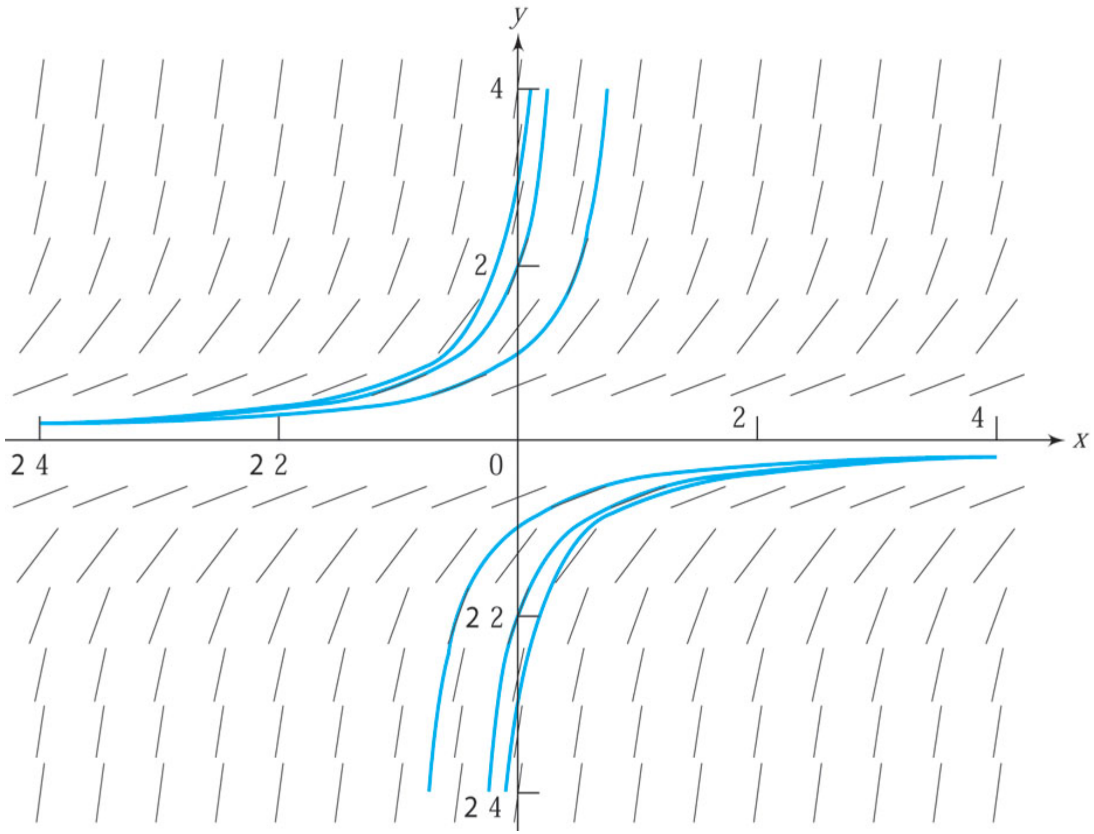


Red: equilibrium solution.
Blue: flow lines.

Direction Field

ex. 3:

$y' = y^2$



Direction field for $y' = y^2$ and integral curves through (0,1), (0,2), (0,3), - 1, (0, - 2), and (0, - 3).

Slope: y^2

General Solution: $y = -\dfrac{1}{x+k}$

$$y' = \frac{dy}{dx} = y^2$$

$$y^{-2}dy = dx$$

$$-y^{-1} = x + c$$

$$y = \frac{-1}{x+c}$$

Worksheet 1

Equation	ind. var.	dep. var.	Order	Degree	Linearity	Type
$\left(\frac{dy}{dz}\right)^3 + 2\mathrm{y} = 0$						
$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$						
$z^{(iv)} + z" = 2$						
$p' + ap = bt$						
$f" + f + g(t) = h(t)$						
$\frac{d^4y}{d\mathrm{t}^4} - \mathrm{t}\frac{d^2y}{d\mathrm{t}^2} + 1 = \mathrm{t}^2$						
$u_{xx} + uu_{yy} = \sin t$						
$u_{xx} + \sin u\, u_{yy} = \cos t$						
$y'' + 3e^{\mathrm{y}}y' - 2t = 0$						
$y'' + 3y' - 2\mathrm{t}^2 = 0$						

Worksheet 2

Equation	ind. var.	dep. var.	Order	Degree	Linearity	Type
$\frac{dy}{dx} + x^2 y = xe^x$						
$\frac{d^3y}{dt^3} = \sqrt{x+y}$						
$\frac{\partial^2 \mathrm{u}}{\partial x^2} + \frac{\partial^2 \mathrm{u}}{\partial y^2} = 0$						
$\frac{d^2y}{dx^2} + x \sin y = 0$						
$\frac{d^2y}{dx^2} + y \sin x = 0$						
$\left(\frac{d\mathrm{r}}{ds}\right)^3 = \sqrt{\frac{d^2\mathrm{r}}{ds^2} + 1}$						
$\frac{\partial^4 \mathrm{u}}{\partial x^2 \partial y^2} + \frac{\partial^2 \mathrm{u}}{\partial x^2} + \frac{\partial^2 \mathrm{u}}{\partial y^2} + \mathrm{u} = 0$						
$u_{xx} + \sin u \, u_{yy} = \cos t$						
$x'' + tx^2 = t$						
$y'' + 3y' + 5x = 0$						

Verification of a solution by substitution

ex. 4: $$y'' - 2y' + y = 0; \quad y = xe^x$$

$$y' = (x+1)e^x, \qquad y'' = (x+2)e^x$$

left hand side:

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$$

right-hand side: 0

The DE possesses the constant *y*=0 ⇨ **trivial solution**

Worksheet 2

Check whether the given expression a solution of the corresponding equation?

Equation	**Solution to Check**
$y'' + 4y = 0$	$y(x) = c_1 \sin 2x + c_2 \cos 2x$
$(y')^4 + y^2 = -1$	$y = x^2 - 1$
$y' = ay, \quad y(0) = 1$	$y(t) = e^{at}$
$y' + y = 10$	$y(t) = 10 - ce - t$
$w_t + 3w_x = 0$	$w(x,t) = 1/(1 + (x - 3t)^2)$
$x'' + 4x = 0$	$x = \cos(2t) + \sin(2t) + c$
$y'' + y = 0$	$y_1(t) = \text{sin}t, y_2(t) = -\text{cos}t, y_3(t) = 2\text{sin}t$
$\frac{dy}{dx} = -\frac{(1 + ye^{xy})}{(1 + xe^{xy})}$	$x + y + e^{xy} = 0$

Simple Integrable Forms

$$b_k \frac{d^k y}{dt^k} = f(t)$$

In theory, this equation may be solved by integrating both sides k times. It may be convenient to introduce new variables so that only first derivative forms need be integrated at each step.

ex. 5:

An object is initially at rest and dropped from a height *h* at *t* = 0. Determine velocity *v* and displacement *y*.

$$\frac{dv}{dt} = g$$

$$dv = gdt$$

$$v = gt + C_1 \quad C_1 = 0 \quad v = gt$$

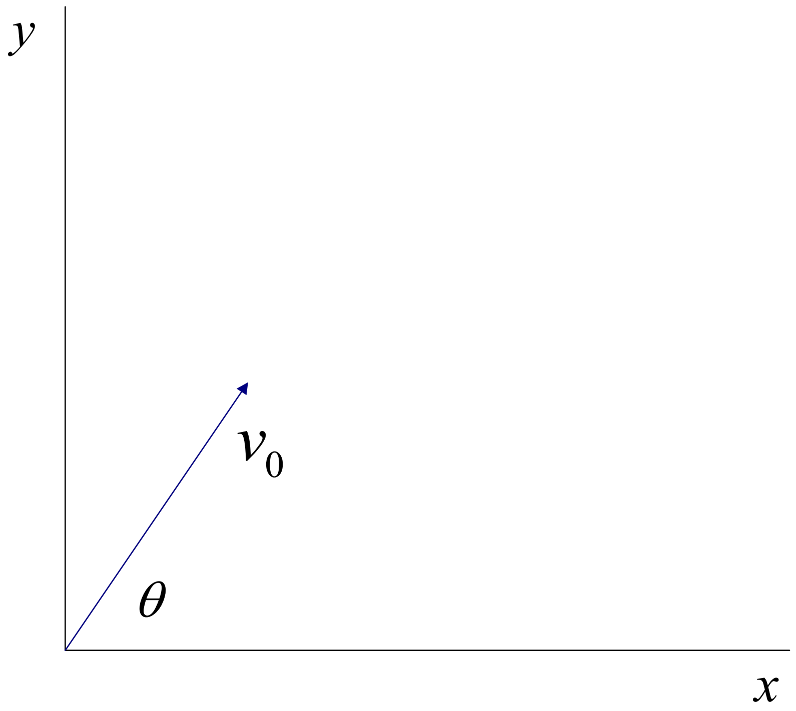
$$\frac{dy}{dt} = gt$$

$$dy = gtdt$$

$$y = \frac{1}{2}gt^2 + C_2 \quad C_2 = 0 \quad y = \frac{1}{2}gt^2$$

ex. 6

Consider situation below and solve for velocity and displacement in both x and y directions.



ex. 6: Continuation

$$\frac{dv_y}{dt} = -g$$

$$v_y = -gt + C_1$$

$$v_y(0) = v_0 \sin\theta \quad C_1 = v_0 \sin\theta \qquad v_y = -gt + v_0 \sin\theta$$

$$\frac{dy}{dt} = v_y = -gt + v_0 \sin\theta$$

$$y = -\frac{1}{2} g t^2 + (v_0 \sin\theta) t + C_2$$

$$y(0) = 0 \qquad C_2 = 0 \quad y = -\frac{1}{2} g t^2 + (v_0 \sin\theta) t$$

ex. 6: Continuation

$$\frac{dv_x}{dt} = 0$$

$$v_x = C_3$$

$$v_x(0) = v_0 \cos\theta \quad C_3 = v_0 \cos\theta \quad v_x = v_0 \cos\theta$$

$$\frac{dx}{dt} = v_x = v_0 \cos\theta$$

$$x = (v_0 \cos\theta)t + C_4$$

$$x(0) = 0 \qquad C_4 = 0 \quad x = (v_0 \cos\theta)t$$

Standard Integrals

$f\left(x\right)$	$\int f(x)dx$	$f\left(x\right)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$\left[g\left(x\right)\right]^n g'\left(x\right)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g\left(x\right)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

Standard Integrals

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}[\sin^{-1}(\frac{x}{a})$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$+\frac{x\sqrt{a^2-x^2}}{a^2}]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$