Ex Find the Maclaurin series of $f(x) = x^3 + x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x^3+x	0
1	$3x^2+1$	1
2	$6x$	0
3	6	6
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f^{(0)}(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

$$= 0 + x + 0 + \frac{6}{3!} x^3 + 0 + 0 + 0 + \cdots$$

$$= x^3 + x$$

Maclaurin (or Taylor) series of a polynomial will not be an infinite series but themselves.

Hw: Find Taylor series of $f(x) = x^3 + x$ about $x = 1$.

The Binomial Series

We will derive the Maclaurin series for a function of the form

$(1+x)^m$, where m is any real number.

Suppose m is a positive integer; then $(1+x)^m$ is a polynomial of degree m:

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n \quad \text{where} \quad \binom{m}{n} = \frac{m!}{n!\,(m-n)!}$$

is a binomial coeff.

For other values of m, such as $\frac{1}{2}$, -1, π or $-3/2$, the series does not terminate.

Let us determine the coefficients:

$$f(x) = (1+x)^m$$
$$f'(x) = m(1+x)^{m-1}$$
$$f''(x) = m(m-1)(1+x)^{m-2}$$
$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)x^n$$

$$f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)$$

Thus

$$(1+x)^m \sim 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n$$

where $1 = f(0)$, is the leading term.

This series converges absolutely for $|x| < 1$.

If m is a positive integer,

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m(m-1)(m-2)\cdots(m-n+1)\cancel{(m-n)!}}{n!\cancel{(m-n)!}}$$

$$= \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$$

So formally, the coefficients of the series are like that of the polynomial.

As a special case, take $m=\frac{1}{2}$. Then,

$$(1+x)^m \sim 1+\sum_{k=1}^{\infty}\binom{m}{k}x^k$$

$$\sqrt{1+x}\sim 1+\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}x^k$$

$$\sim 1+\binom{1/2}{1}x+\binom{1/2}{2}x^2+\binom{1/2}{3}x^3+\cdots$$

$$\sim 1+\frac{\frac{1}{2}!}{1!\,(1-\frac{1}{2})!}x+\frac{\frac{1}{2}!}{2!\,(\frac{1}{2}-2)!}x^2+\frac{\frac{1}{2}!}{3!\,(\frac{1}{2}-3)!}x^3+\cdots$$

$$\sim 1+\frac{\frac{1}{2}\cdot\cancel{(\frac{1}{2}-1)!}}{1!\,\cancel{(\frac{1}{2}-1)!}}x+\frac{\frac{1}{2}(\frac{1}{2}-1)\cancel{(\frac{1}{2}-2)!}}{2!\,\cancel{(\frac{1}{2}-2)!}}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cancel{(\frac{1}{2}-3)!}}{3!\,\cancel{(\frac{1}{2}-3)!}}x^3+\cdots$$

$$\sim 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\cdots$$

For example, we may use the Binomial to estimate $\sqrt{1.75}$, where $m=1/2$

$$\sqrt{1.75}=(1+1/4)^{1/2}=1+\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)+\frac{\frac{1}{2}(-\frac{1}{2})}{2}(1/4)^2+\cdots=1+\frac{1}{8}-\frac{1}{128}+\cdots$$

Another special case is $m=-1$. Then $(1+x)^{-1}=\frac{1}{1+x}$ is a geometric series.

$$\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n=\frac{1}{1+x}=(1+x)^{-1}$$