3.6 The Chain Rule

THEOREM -The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

where dy/du is evaluated at $u = g(x)$.

Ex $y = 3u - 9$, $u = \frac{1}{2}x^4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3 \cdot \frac{4}{2}x^3 = 6x^3$$

Ex $y = \cos u$, $u = \sin x \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\sin(\sin x)\cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\frac{dy}{du} = -\sin u$, $\frac{du}{dx} = \cos x$

Ex $y = 2u^3, \quad u = 8x-1$ $\qquad \frac{dy}{du} = 6u^2; \quad \frac{du}{dx} = 8$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 6u^2 \cdot 8 = 48(8x-1)^2$$

Ex $y = \tan(\underbrace{10x-5}_{u}), \quad u = 10x-5$

$$y = \tan u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot 10 = 10\sec^2(10x-5)$$

Ex $y = \cos(-\frac{x}{3}), \quad u = -x/3$

$$y = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \cdot (-1/3)$$

$$= \frac{1}{3}\sin(-x/3)$$

Ex $y = -\sec(x^2+7x)$, $u = x^2+7x$

$y = -\sec u$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= -\sec u \tan u \cdot (2x+7)$$

$$= -(2x+7)\sec(x^2+7x)\tan(x^2+7x)$$

Ex $y = \sqrt{3x^2-4x+6}$ $\quad u = 3x^2-4x+6$

$y = u^{1/2}$ $\quad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ $\quad \frac{du}{dx} = 6x-4$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = -\frac{1}{2\sqrt{u}}\cdot(6x-4)$$

$$= \frac{3x-2}{\sqrt{3x^2-4x+6}}$$

Ex $y = \sin^3 x = (\sin x)^3$ $\qquad u = \sin x$

$y = u^3$, $\frac{dy}{du} = 3u^2$ $\qquad \frac{du}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cdot \cos x = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x$$

Ex $y = \sec(\tan x)$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$y = \sec u$ $\qquad \frac{dy}{du} = \sec u \tan u = \sec(\tan x)\tan(\tan x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec(\tan x) \cdot \tan(\tan x) \cdot \sec^2 x$$

Ex $y = 5\cos^{-4} x^3 = 5(\cos x^3)^{-4}$ $\qquad u = \cos x^3$

$\frac{du}{dx} = -\sin x^3 \cdot 3x^2$

$y = 5u^{-4}$, $\frac{dy}{du} = -4 \cdot 5 \cdot u^{-5}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 60x^2 \cos^{-5} x^3 \cdot \sin x^3$$

Ex $y = \sin^3 3x + 3\sin x^3$

$$y' = 3\sin^2 3x \cdot 3\cos 3x + 3\cos x^3 \cdot 3x^2$$
$$= 9\sin^2 3x \cos 3x + 9x^2 \cos x^3$$

Ex $y = \sin(\cos x) \rightarrow y' = \cos(\cos x).(-\sin x)$

$$y' = -\sin x \cos(\cos x)$$

Ex $y = \left(\dfrac{x^2+1}{x^3+2x}\right)^4 \qquad u = \dfrac{x^2+1}{x^3+2x}$

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = 4\left(\frac{x^2+1}{x^3+2x}\right)^3 \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^3+2x}\right)$$

Ex $y = \sqrt[m]{x}$ $\quad$ m is an integer

$$y = x^{\frac{1}{m}} \Longrightarrow y' = \frac{1}{m} x^{\frac{1}{m}-1}$$

Ex $\frac{d}{dx}\sqrt[3]{x^3-2x+1} = \frac{1}{3}(x^3-2x+1)^{\frac{1}{3}-1}\cdot\frac{d}{dx}(x^3-2x+1)$

$= \frac{1}{3}(x^3-2x+1)^{-\frac{2}{3}}(3x^2-2)$

Ex $y = 4\sin(\sqrt{1+\sqrt{t}})$

$\frac{dy}{dt} = \cancel{4}\cos(\sqrt{1+\sqrt{t}})\cdot\frac{1}{\cancel{2}}(1+\sqrt{t})^{-\frac{1}{2}}\cdot\frac{1}{\cancel{2}}t^{-\frac{1}{2}}$

$= \frac{\cos\sqrt{1+\sqrt{t}}}{\sqrt{t+t\sqrt{t}}}$

Ex if $f'(x)=\sqrt{3x^2-1}$ and $y=f(x^2)$, $u=x^2$

Find $\frac{dy}{dx}$ $\qquad x\to u\to y$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= f'(x^2)\cdot 2x = \sqrt{3x^4-1}\cdot 2x$

Ex $f(x) = \sqrt{7 + x\sec x}$

$u = 7 + x\sec x$

$\frac{du}{dx} = \sec x + x\sec x\tan x$

$y = u^{1/2}$, $\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{7 + x\sec x}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{\sec x + x\sec x\tan x}{2\sqrt{7 + x\sec x}}$$