#cse2046

Q1) For each of the following, indicate the class $\Theta(g(n))$ the function belongs to. Use the simplest g(n) possible.

(a) $\sum_{i=1}^{3n} \log_3 i$ (b) $\ln(\pi^{en}) + \ln(n^{\pi e})$ (c) $10^{n/2} + n^3 3^n$ (d) π^e

a) $\sum_{i=1}^{3n} \log_3 i = \log_3 1 + \log_3 2 + \log_3 3 + \dots + \log_3 3n$

$= \log_3 (1.2.3. \dots 3n) = \log_3 (3n!)$

$\rightarrow \boxed{\log(n!) \in \Theta(n \lg n)}$

$\underline{\log_3 (3n)!} = 3n \log_3 n \simeq n \log n \in \underline{\Theta(n \lg n)}$

b) $\ln(\pi^{en}) + \ln(n^{\pi e})$

$\underbrace{en \ln \pi}_{\text{constant}} + \cancel{\pi e} \ln n$

$\Theta(n) + \Theta(\ln n) = \Theta(n)$

c) $10^{n/2} + n^3 \cdot 3^n$

$\lim_{n \to \infty} \frac{10^{n/2}}{n^3 \cdot 3^n} = \infty \in \Theta(10^{n/2})$

d) $\pi^e \in \Theta(1)$

Q2) **For each of the following indicate whether it is true or false**

(i) $\log_2 n^2+1 \in O(n)$.

(ii) $\sqrt{n(n+1)} \in \Omega(n)$.

(iii) $(2n)! \in \Theta(n!)$.

(iv) $\log_2 \sqrt{n}$ and $(\log_2 n)^2$ are of the same asymptotical order.

i) $\log_2 n^2+1 \overset{?}{\in} O(n)$ True

$\log_2 n^2+1 \approx \log_2 n^2 = 2\log_2 n \rightarrow \Theta(\log n)$

$O(n)$

ii) $\sqrt{n(n+1)}$ True

$\sqrt{n^2+n} \approx \sqrt{n^2} = n \rightarrow \Theta(n)$

$\Omega(n)$

iii) $(2n)! \overset{?}{\in} \Theta(n!)$ False

$\lim_{n\to\infty} \frac{2n!}{n!} = \infty$

iv) $\log_2 \sqrt{n}$ $(\log_2 n)^2$ False

$\rightarrow \underline{\log_2 n \cdot \log_2 n}$

$\hookrightarrow \log_2 n^{1/2} = \underline{\frac{1}{2}\cdot\log_2 n}$

Q3) Solve the following recurrence relations using **backward substitution method**:

G(n) = 2G(n-1) if n is odd and n≥1;
G(n) = G(n-1) + G(n-2) if n is even and n>0;
G(0) = 1; *(Find the solution for both odd and even values of n.)*

for n is even

$$\underset{\text{even}}{G(n)} = \underset{\text{odd}}{G(n-1)} + \underset{\text{even}}{G(n-2)}$$

$$G(n-1) = 2G(n-1-1) = 2G(n-2)$$

$$= 2G(n-2) + G(n-2)$$

$$G(n) = 3G(n-2)$$

$$= 3(3G(n-4)) = 3^2 G(n-4)$$

$$= 3^2(3G(n-6)) = 3^3 G(n-6)$$

$$= 3^{n/2} \cdot G(\cancel{n}-n)$$

1

$$= 3^{n/2} \rightarrow G(n) = 3^{n/2}$$

$$G(n-1) = 3^{\frac{n-1}{2}}$$

for n is odd

$$\underset{\text{odd}}{G(n)} = 2\underset{\text{even}}{G(n-1)}$$

$$G(n) = 2 \cdot 3^{\frac{n-1}{3}}$$

Q4) How many ones does the following procedure print when run with input n? Compute the best bounds you can: the exact value if possible, a big-Θ expression if you can't find the exact value, or big-O and big-Ω bounds if you can't find a big-Θ expression.

```
Ones(n):
   if n = 0:
         print 1        ] Initial Condition
   else:
         for i = 1 to 2^n:     ] recursion relation
               Ones(n-1)
```

$P(n) = P(n-1) \cdot 2^n$ for $n > 0$, $P(0) = 1$

$= 2^{n-1} \cdot 2^n \cdot P(n-2)$

$= 2^{n-2} \cdot 2^{n-1} \cdot 2^n \cdot P(n-3)$

$\vdots$

$= 2^1 \cdot 2^2 \cdot 2^3 \cdots 2^n \cdot P(\underbrace{n-n}_{0}) \rightarrow 1$

$= 2^1 \cdot 2^3 \cdots 2^n = 2^{1+2+\cdots n} = 2^{\frac{n \cdot (n+1)}{2}}$

$\Theta\left(2^{n^2/2}\right)$

Q5) How many lines, as a function of n (in Θ(·) form), does the following programs print? Write a recurrence relation and solve it.

(a)

```
function func1(n)

if n = 1:
        print_line("Ayinesi iştir kişinin lafa bakılmaz.")
else:
        func1(⌊n/3⌋)
        for i=1: ⌊n/3⌋
            print_line("Ayinesi iştir kişinin lafa bakılmaz.")
        end for
```

$$P(n) = P(n/3) + \frac{n}{3} \quad \text{for } n>1, \quad P(1) = 1$$

$$n = 3^k$$

$$P(3^k) = P(3^{k-1}) + 3^{k-1}$$

$$= P(3^{k-2}) + 3^{k-2} + 3^{k-1}$$

$$= P(3^{k-3}) + 3^{k-3} + 3^{k-2} + 3^{k-1}$$

$$\vdots$$

$$= P(\cancel{3^{k-k}})^{1} + 3^0 + 3^1 + \dots + 3^{k-1}$$

$$= 1 + \underbrace{3^0 + 3^1 + \dots + 3^{k-1}}_{\frac{3^{k-1+1} - 1}{3-1}} = 1 + \frac{3^k - 1}{2}$$

$$= \frac{2 + n - 1}{2} = \frac{n+1}{2} \in \Theta(n)$$

(b)

```
function func2(n)

if n > 1:
    func2(⌊n/3⌋)
    print_line("Görünür kişinin rütbe-i aklı eserinde.")
    func2(⌊n/3⌋)
    print_line("Görünür kişinin rütbe-i aklı eserinde.")
    func2(⌊n/3⌋)
```

$P(n) = 3P(n/3) + 2n^0 \rightarrow$ Master Theorem

$T(n) = aT(n/b) + f(n)$

where $f(n) \in n^d$, $d \geq 0$

$a = 3$ $\quad a > b^d$
$b = 3$ $\quad a < b^d$
$d = 0$ $\quad a = b^d$

$3 \quad 3^0$

$3 > 1 \rightarrow n^{\log_b a}$

$n^{\log_3 3} = n^1 = n$

$\rightarrow P(n) = 3.P(n/3) + 2$, $P(1) = 0$

$n = 3^k$

$P(3^k) = 3.P(3^{k-1}) + 2$

$= 3(3P(3^{k-2}) + 2) + 2 = 3^2P(3^{k-2}) + 3.2 + 2$

$= 3^2P(3^{k-2}) + 3.2 + 2$

$= 3^2(3P(3^{k-3}) + 2) + 3.2 + 2$

$= 3^3P(3^{k-3}) + 3^2.2 + 3.2 + 2$

$\vdots$

$= 3^k.\cancel{P(3^{k-k})}^{0} + 3^{k-1}.2 + 3^{k-2}.2 + \dots + 3.2 + 2$

$= 2(\underline{3^{k-1} + 3^{k-2} + \dots + 3 + 3^0})$

$= 2.\dfrac{3^{k-1+1} - 1}{3-2} = \cancel{2}.\dfrac{3^k - 1}{\cancel{2}} = 3^k - 1$

$= n - 1$

$\in \Theta(n)$

Q6) Consider the following program.

```
void func1(int n, int x) {
if (x <= 0)
      foo1 (n);
else
      foo2 (n);
```

It is known that x can get both negative and positive values (but not certainly with the same probability). Time complexities of **foo1(n)** and **foo2(n)** are given in the following table.

	Worst case	Best case	Average case
foo1 (n)	$\Theta(n^2)$	$\Theta(n)$	$\Theta(n^2)$
foo2 (n)	$\Theta(n \log n)$	$\Theta(1)$	$\Theta(n \log n)$

For each of the following, indicate whether it is "true", "false", or "there is no enough information". Give a short reasoning. Answers without any comments will not be graded.

(a) Time complexity of func1 is in $\Theta(n^2)$ False , $O(n^2)$
(b) Worst case time complexity of func1 is in $\Omega(n^2)$ True $\Theta(n^2) \rightarrow \Omega(n^2)$
$\rightarrow O(n^2)$
(c) Average case time complexity of func1 is in $O(n^2)$ True
(d) Average case time complexity of func1 is in $\Theta(n^2)$ False ,

9)

Q7) ~~6.~~ What is the time complexity of the following function? Indicate your answer in **Θ(·)** form.

```
void func2(int n) {
  int i = n;
      int x = 0;
      int count = 0;

      while (i > 1){
            x = x + 2;
            i = i/3;
      }
      for(int j = 1; j <= x; j++)
            for(int k = 1; k <= x; k++)
                  count = count + 1;
}
```

$\log_3 n \rightarrow 2 \cdot \log_3 n = x$

$$\sum_{j=1}^{x} \sum_{k=1}^{x} 1 = \sum_{j=1}^{x} x = x^2 = (2 \log_3 n)^2 = 2^2 \cdot \log_3^2 n$$

$$\Rightarrow \Theta(\log^2 n)$$

1. Solve the following recurrence relations using **backward substitution method**:

(a) T(n)=T(n-2)+n for n>1, T(1)=1, T(2)=2. for n>1, T(1)=1, T(2)=2. *(Solve for both odd and even values of n.)*

$T(n) = T(n-2) + n \quad , \quad T(1) = 1 \quad T(2) = 2$

$= T(n-4) + n-2 + n$

$= T(n-6) + n-4 + n-2 + n$

$\vdots$

$= T(n-(n-2)) + 4 + 6 + \cdots + n$

$= \underbrace{T(2)}_{2} + 4 + 6 \cdots + n$

$= 2(1 + 2 + \cdots + n/2) = 2 \cdot \frac{\left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}+1\right)}{2}$

$= \frac{n^2}{4} + \frac{n}{2} \in \Theta(n^2)$

(b) T(n)=4T($\lceil n/2 \rceil$)+n for n>1, T(1)=1. *(solve for $n = 2^k$)*.

$T(n) = 4T(n/2) + n \quad , \quad T(1) = 1$

$T(2^k) = 4T(2^{k-1}) + 2^k$

$= 4(4T(2^{k-2}) + 2^{k-1}) + 2^k$

$= 4^2 T(2^{k-2}) + 4 . 2^{k-1} + 2^k$

$\vdots$

$= 4^k T(\cancel{2^{k-k}}_{1}) + 4^{k-1} . 2 + 4^{k-2} . 2^2 + \dots + 4^1 . 2^{k-1} + 2^k$

$= 4^k + 4^{k-1} . 2 + 4^{k-2} . 2^2 + \dots + 4^1 . 2^{k-1} + 2^k$

$= 2^{2k} + 2^{2k-2} . 2 + 2^{2k-4} . 2^2 + \dots + 2^2 . 2^{k-1} + 2^k$

$= 2^{2k} + 2^{2k-1} + 2^{2k-2} + \dots + 2^k$

$= 2^k (\underbrace{2^k + 2^{k-1} + 2^{k-2} + \dots + 1})$ $\quad \frac{2^{k+1} - 1}{2-1} = 2^{k+1} - 1$

$= 2^k (2^{k+1} - 1)$

$= 2^{2k+1} - 1 = 2^{2k} . 2 - 1 = n^2 . 2 - 1$

$= 2 . n^2 - 1$

$\Theta(n^2)$

(c) T($\sqrt{n}$)+1, T(2) = 1. *(solve for* $n = 2^{2^k}$*)*.

$T(n) = T(\sqrt{n}) + 1 \quad , \quad T(2) = 1$

$T(2^{2^k}) = T\left(\left(2^{2^k}\right)^{1/2}\right) + 1$

$= T\left(2^{2^{(k-1)}}\right) + 1$

$= T\left(2^{2^{k-2}}\right) + 1 + 1$

$= T\left(2^{2^{k-3}}\right) + 1 + 1 + 1$

$\vdots$

$= T\left(2^{2^0}\right) + \underbrace{1 + \cdots + 1}_{k} = T(2^1) + k$

$= 1 + k = 1 + \log\log n$

$n = 2^{2^k} \Rightarrow \log n = 2^k \cdot \frac{\log 2}{1}$

$\log\log n = k \frac{\log 2}{1} \Rightarrow k = \log\log n$