CSE2023 Discrete Computational Structures

Lecture 3

Logical equivalences

- S≡T: Two statements S and T involving predicates and quantifiers are **logically equivalent**
  - ***If*** and ***only if*** they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables.
  - Example: $\forall x(p(x) \wedge q(x)) \equiv \forall x p(x) \wedge \forall x q(x)$

  i.e., we can distribute a universal quantifier over a conjunction

$\forall x(p(x) \wedge q(x)) \equiv \forall x p(x) \wedge \forall x q(x)$

- Both statements must take the same truth value no matter the predicates p and q, and non matter which domain is used
- Show
  - If p is true, then q is true (p → q)
  - If q is true, then p is true (q → p)

$$\forall x(p(x) \wedge q(x)) \rightarrow \forall x p(x) \wedge \forall x q(x)$$
$$\forall x(p(x) \wedge q(x)) \leftarrow \forall x p(x) \wedge \forall x q(x)$$

$\forall x(p(x) \wedge q(x)) \equiv \forall x p(x) \wedge \forall x q(x)$

$\forall x(p(x) \wedge q(x)) \rightarrow \forall x p(x) \wedge \forall x q(x)$

- (→) If a is in the domain, then p(a)^q(a) is true. Hence, p(a) is true and q(a) is true. Because p(a) is true and q(a) is true for every element in the domain, so $\forall x p(x) \wedge \forall x q(x)$ is true
- (←) It follows that $\forall x p(x)$ and $\forall x q(x)$ are true. Hence, for a in the domain, p(a) is true and q(a) is true, hence p(a)^q(a) is true. If follows $\forall x(p(x) \wedge q(x))$ is true

Negating quantified expressions

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negations of the following statements
"There is an honest politician"
"Every politician is dishonest"
(Note "All politicians are not honest" is ambiguous)
"All Americans eat cheeseburgers"

Example

What are the negations of $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

$$\neg\forall x(x^2 > x) \qquad \neg\exists x(x^2 = 2)$$
$$\equiv \exists x\neg(x^2 > x) \qquad \equiv \forall x\neg(x^2 = 2)$$
$$\equiv \exists x(x^2 \le x) \qquad \equiv \forall x(x^2 \ne 2)$$

Show $\neg\forall x(p(x) \to q(x)) \equiv \exists x(p(x) \wedge \neg q(x))$

$$\neg\forall x(p(x) \to q(x)$$
$$\equiv \exists x(\neg(p(x) \to q(x)))$$
$$\equiv \exists x(\neg(\neg p(x) \vee q(x)))$$
$$\equiv \exists x(p(x) \wedge \neg q(x))$$

Translating English into logical expressions

- "Every student in this class has studied calculus"

  Let c(x) be the statement that "x has studied calculus". Let s(x) be the statement "x is in this class"

$\forall x\ c(x)$ if the domain consists of students of this class

$\forall x\ s(x) \to c(x)$ if the domain consists of all people

$\forall x\ s(x) \wedge c(x)$? if the domain consists of all people

Using quantifiers in system specifications

- "Every mail message larger than one megabyte will be compressed"

  Let s(m,y) be "mail message m is larger than y megabytes" where m has the domain of all mail messages and y is a positive real number. Let c(m) denote "message m will be compressed"

$$\forall m(s(m,1) \to c(m))$$

Example

- "If a user is active, at least one network link will be available"

  Let a(u) represent "user u is active" where u has the domain of all users, and let s(n, x) denote "network link n is in state x" where n has the domain of all network links, and x has the domain of all possible states, {available, unavailable}.

  $$\exists u\ a(u) \rightarrow \exists n\ s(n, \text{available})$$

1.5 Nested quantifiers

Let the variable domain be real numbers

$\forall x \exists y (x + y) = 0$

same as $\forall x q(x)$

where $q(x)$ is $\exists y p(x, y)$ and $p(x, y)$ is $x + y = 0$

$\forall x \forall y (x + y = y + x)$

$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$

$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

where the domain for these variables consists of real numbers

Quantification as loop

- For every x, for every y $\quad \forall x \forall y p(x, y)$
  - Loop through x and for each x loop through y
  - If we find p(x,y) is true for all x and y, then the statement is true
  - If we ever hit a value x for which we hit a value for which p(x,y) is false, the whole statement is false
- For every x, there exists y $\quad \forall x \exists y p(x, y)$
  - Loop through x until we find a y that p(x,y) is true
  - If for every x, we find such a y, then the statement is true

Quantification as loop

- $\exists x \forall y p(x, y)$: loop through the values for x until we find an x for which p(x,y) is always true when we loop through all values for y
  - Once found such one x, then it is true
- $\exists x \exists y p(x, y)$: loop though the values for x where for each x loop through the values of y until we find an x for which we find a y such that p(x,y) is true
  - False only if we never hit an x for which we never find y such that p(x,y) is true

Order of quantification

Let $p(x, y)$ be the statement $x + y = y + x$,
and the domain is real number
$\forall x \forall y p(x, y)$
$\forall x \forall y p(x, y) \equiv \forall y \forall x p(x, y)$?

Let $q(x, y)$ be the statement $x + y = 0$,
and the domain is real number
$\exists y \forall x q(x, y)$: There is a real number y s.t. for every real number x, $q(x,y)$
$\forall x \exists y q(x, y)$: For every real number x there is a real number y s.t. $q(x,y)$
$\exists y \forall x p(x, y) \equiv \forall x \exists y p(x, y)$?

Quantification of two variables

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Quantification with more variables

Let $q(x, y, z)$ be the statement $x + y = z$,
and the domain is real number
$\forall x \forall y \exists z q(x, y, z)$: What does it mean? Is it true?
$\exists z \forall x \forall y q(x, y, z)$: What does it mean? Is it true?

is false, because there is no value of z that satisfies the equation $x + y = z$ for all values of x and y.

Translating mathematical statements

- "The sum of two positive integers is always positive"

  $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$
  where the domain for both variables consists of all integers

  $\forall x \forall y(x + y > 0)$
  where the domain for both variables consists of all positive integers

Example

- “Every real number except zero has a multiplicative inverse”

$\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$

where the domain for both variables consists of real numbers

Express limit using quantifiers

Recall $\lim_{x \to a} f(x) = L$

For every real number ε>0, there exists a real number δ>0, such that |f(x)-L|<ε whenever 0<|x-a|<δ

$\forall \varepsilon \exists \delta \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$

where the domain for δ and ε is positive real number, and the domain for x is real number

$\forall \varepsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$

where the domain for $\varepsilon, \delta,$ and $x,$ is real number

Translating statements into English

- $\forall x(c(x) \vee \exists y(c(y) \wedge f(x, y)))$ where c(x) is “x has a computer”, f(x,y) is “x and y are friends”, and the domain for both x and y consists of all students in our school
- $\exists x \forall y \forall z((f(x, y) \wedge f(x, z) \wedge (y \neq z)) \rightarrow \neg f(y, z))$ where f(x,y) means x and y are friends, and the domain consists of all students in our school

Negating nested quantifiers

$$\neg \forall x \exists y (xy = 1)$$
$$\equiv \exists x \neg \exists y (xy = 1)$$
$$\equiv \exists x \forall y \neg (xy = 1)$$
$$\equiv \exists x \forall y (xy \neq 1)$$

- There does not exist a woman who has taken a flight on every airline in the world

$\neg \exists w \forall a \exists f (p(w, f) \wedge q(f, a))$

$\equiv \forall w \exists a \forall f (\neg p(w, f) \vee \neg q(f, a))$

where p(w,f) is “w has taken f”, and q(f,a) is “f is a flight on a