CSE2023 Discrete Computational Structures

Lecture 17

6.1 Basics of counting

- **Combinatorics**: they study of arrangements of objects
- **Enumeration**: the counting of objects with certain properties (an important part of combinatorics)
  - Enumerate the different telephone numbers possible in US
  - The allowable password on a computer
  - The different orders in which runners in a race can reach

Basic counting principles

- Two basic counting principles
  - Product rule
  - Sum rule
- **Product rule**: suppose that a procedure can be broken down into a sequence of two tasks
- If there are n_1 ways to do the 1st task, and each of these there are n_2 ways to do the 2nd task, then there are $n_1 \cdot n_2$ ways to do the procedure

Example

- The chairs of a room to be labeled with **a letter** and **a positive integer not exceeding 100**. What is the largest number of chairs that can be labeled differently?
- There are **26 letters** to assign for the 1st part and **100 possible integers** to assign for the 2nd part, so there are **$26 \cdot 100 = 2600$** different ways to label chairs

Product rule

- Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, i=1, 2, …, n can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure

Example

- How many different license plates are available if each plate contains a sequence of 3 letters followed by 3 digits (and non sequences of letters are prohibited, even if they are obscene)?
- License plate _ _ _ _ _ _: There are 26 choices for each letter and 10 choices for each digit. So, there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates

Counting functions

- How many **functions** are there from a set with **m** elements to a set with **n** elements?
- A function corresponds to one of the n elements in the codomain for each of the m elements in the domain
- Hence, by product rule there are $n \cdot n \ldots \cdot n = n^m$ functions from a set with m elements to one with n elements

Counting one-to-one functions

- How many one-to-one functions are there from a set with **m** elements to one with **n** elements?
- First note that when m>n there are no one-to-one functions from a set with m elements to one with n elements
- Let **m≤n.** Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value for the value at a_1
- As the function is one-to-one, the value of the function at a_2 can be picked in n-1 ways (the value used for a_1 cannot be used again)
- Using the product rule, there are **n(n-1)(n-2)…(n-m+1)** one-to-one functions from a set with m elements to one with n elements

Example

- How many one-to-one functions from a set with 3 elements to one with 5 elements?
- there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions

Example

- The format of telephone numbers in north America is specified by a numbering plan
- It consists of 10 digits, with 3-digit area code, 3-digit office code and 4-digit station code
- Each digit can take one form of
  - X: 0, 1, …, 9
  - N: 2, 3, …, 9
  - Y: 0, 1

Example

- In the old plan, the formats for area code, office code, and station code are NYX, NNX, and XXXX, respectively
- So the phone numbers had NYX-NNX-XXXX
- NYX: $8 \cdot 2 \cdot 10 = 160$ area codes
- NNX: $8 \cdot 8 \cdot 10 = 640$ office codes
- XXXX:$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes
- So, there are $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$ phone numbers

X: 0, 1, …, 9
N: 2, 3, …, 9
Y: 0, 1

Example

- In the new plan, the formats for area code, office code, and station code are NXX, NXX, and XXXX, respectively
- So the phone numbers had NXX-NXX-XXXX
- NXX: $8 \cdot 10 \cdot 10 = 800$ area codes
- NXX: $8 \cdot 10 \cdot 10 = 800$ office codes
- XXXX:$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes
- So, there are $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$ phone numbers

Product rule

- If A_1, A_2, …, A_m are **finite sets**, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set
- $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|$

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Sum rule

- If a task can be done either in **one of n_1 ways** or in **one of n_2 ways**,
- where **none of the set of n_1 ways is the same as any of the set of n_2 ways**,
- then there are n_1+n_2 ways to do the task

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Sum rule

- Example: suppose either a member of **faculty** or a **student** in CSE is chosen as a representative to a university committee.
- How many different choices are there for this representative if there are 8 members in faculty and 200 students?
- There are 8+200=208 ways to pick this representative

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Sum rule

- If A_1, A_2, …, A_m are **disjoint** finite sets, then the number of elements in the **union** of these sets is as follows

  $|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$

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More complex counting problems

- In a version of the BASIC programming language, the name of a variable is a string of **1 or 2 alphanumeric characters**, where uppercase and lowercase letters are not distinguished.
- Moreover, **a variable name must begin with a letter** and must be different from the **five strings of two characters that are reserved for programming use**
- How many different variables names are there?
- Let V_1 be the number of these variables of 1 character, and likewise V_2 for variables of 2 characters
- So, $V_1=26$, and $V_2=26\cdot 36-5=931$
- In total, there are $26+931=957$ different variables

Example

- Each user on a computer system has a password, which is **6 to 8 characters long**, where each character is an **uppercase letter or a digit**. Each password **must contain at least one digit**. How many possible passwords are there?
- Let P be the number of all possible passwords and $P=P_6+P_7+P_8$ where P_i is a password of i characters
- $P_6=36^6-26^6=1{,}867{,}866{,}560$
- $P_7=36^7-26^7=70{,}332{,}353{,}920$
- $P_8=36^8-26^8=208{,}827{,}064{,}576$
- $P=P_6+P_7+P_8=2{,}684{,}483{,}063{,}360$

Example: Internet address

Bit Number	0	1	2	3	4	8 … 16	24 … 31
Class A	0	netid					hostid
Class B	1	0	netid				hostid
Class C	1	1	0	netid			hostid
Class D	1	1	1	0	Multicast Address		
Class E	1	1	1	1	0	Address	

- Internet protocol (IPv4)
  - Class A: largest network
  - Class B: medium-sized networks
  - Class C : smallest networks
  - Class D: multicast (not assigned for IP address)
  - Class E: future use
  - Some are reserved: netid 1111111, hostid all 1's and 0's
- Neither class D or E addresses are assigned as the IPv4 addresses
- How may different IPv4 addresses are available?

Example: I

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Bit Number	0	1	2	3	4	8 … 16	24 … 31
Class A	0	netid					hostid
Class B	1	0	netid				hostid
Class C	1	1	0	netid			hostid
Class D	1	1	1	0	Multicast Address		
Class E	1	1	1	1	0	Address	

- Let the total number of address be x, and $x=x_A+x_B+x_C$
- Class A: there are $2^7-1=127$ netids (1111111 is reserved). For each netid, there are $2^{24}-2=16{,}777{,}214$ hostids (as hostids of all 0s and 1s are reserved), so there are $x_A=127\cdot 16{,}777{,}214=2{,}130{,}706{,}178$ addresses
- Class B, C: $2^{14}=16{,}384$ Class B netids and $2^{21}=2{,}097{,}152$ Class C netids. $2^{16}-2=65{,}534$ Class B hostids, and $2^8-2=254$ Class C hostids. So, $x_B=1{,}073{,}709{,}056$, and $x_C=532{,}676{,}608$
- So, $x=x_A+x_B+x_C=3{,}737{,}091{,}842$

Inclusion-exclusion principle

- Suppose that a task can be done in n_1 or in n_2 ways, but **some of the set of n_1 ways to do the task are the same as some of the n_2 ways** to do the task
- Cannot simply add n_1 and n_2, but need to subtract the number of ways to the task that is common in both sets
- This technique is called **principle of inclusion-exclusion** or **subtraction principle**

Example

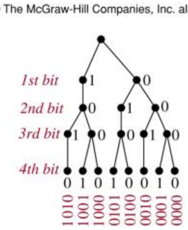
- How many bit strings of length 8 either start with a 1 or end with two bits 00?
- 1 _ _ _ _ _ _ _: 2^7=128 ways
- _ _ _ _ _ _ 00: 2^6=64 ways
- 1 _ _ _ _ _ 00: 2^5=32 ways
- Total number of possible bit strings is 128+64-32=160

Inclusion-exclusion principle

- Using sets to explain
  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Tree diagrams

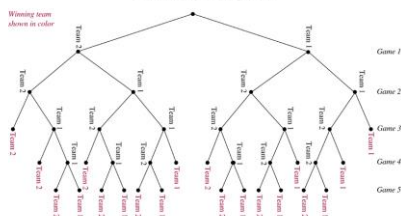
- How many bit strings of length 4 do not have two consecutive 1s?
- In some cases, we can use tree diagrams for counting



8 without two consecutive 1s

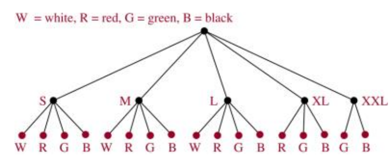
Example

- A playoff between 2 teams consists of at most 5 games. The 1st team that wins 3 games wins the playoff. How many different ways are there?



Example

- Suppose a T-shirt comes in 5 different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in 4 colors, white, green, red, and black except for XL which comes only in red, green and black, and XXL which comes only in green and black. How many possible size and color of the T-shirt?



6.2 Pigeonhole principle

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost
- Thus, **at least** 1 of these 19 pigeonholes must have **at least** 2 pigeons
- Why? If each pigeonhole had at most one pigeon in it, at most 19 pigeons, 1 per hole, could be accommodated
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole with at least 2 pigeons in it

Example

13 pigeons and 12 pigeonholes

Pigeonhole principle

- Theorem 1: If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- Proof: suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction as there are at least k+1 objects
- Also known as **Dirichlet drawer principle**

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Pigeonhole principle

- Corollary 1: A function f from a set with k+1 or more elements to a set with k elements is not one-to-one
- Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain f s.t. f(x)=y
- As the domain contains k+1 or more elements and the codomain contain only k elements, the pigeonhole principle tells us that one of these boxes contains 2 or more elements x of the domain
- This means that f cannot be one-to-one

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Example

- Among any group of 367 people, there must be at least 2 with the same birthday
- How many students must be in a class to guarantee that at least 2 students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points

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Generalized pigeonhole principle

- Theorem 2: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects
- Proof: Proof by contradiction. Suppose that none of the boxes contains more than $\lceil N/k \rceil$ -1 objects. Then the total number of objects is at most $k(\lceil N/k \rceil -1)<k((N/k+1)-1)=N$
  where the inequality $\lceil N/k \rceil <N/k+1$ is used
- This is a contradiction as there are a total of N objects

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Generalized pigeonhole principle

- A common type of problem asks for the minimum number of objects s.t. at least r of these objects must be in one of k boxes when these objects are distributed among boxes
- When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as $\lceil N/k \rceil \geq r$. Recall $N/k+1 > \lceil N/k \rceil$. The smallest integer N with $N/k > r-1$, i.e., $\mathbf{N=k(r-1)+1}$ is the smallest integer satisfying the inequality $\lceil N/k \rceil \geq r$

Example

- Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month
- What is the minimum number of students required in a discrete mathematics class to be sure that at least 6 will receive the same grade, if there are 5 possible grades,?
- The minimum number of studeA, B, C, D, and Fnts needed to ensure at least 6 students receive the same grade is the smallest integer N s.t. $\lceil N/5 \rceil = 6$. Thus, the smallest $N=5 \cdot 5+1=26$

Example

- How many cards must be selected from a standard deck of 52 cards to guarantee that a least 3 cards of the same suit are chosen?
- Suppose there are 4 boxes, one for each suit. If N cards are selected, using the generalized pigeonhole principle, there is at lest one box containing at least $\lceil N/4 \rceil$ cards
- Thus to have $\lceil N/4 \rceil \geq 3$, the smallest N is $2 \cdot 4+1=9$. So at least 9 cards need to be selected

Example

- How many cards must be selected to guarantee that at least 3 hearts are selected?
- We do not use the generalized pigeonhole principle to answer this as we want to make sure that there are 3 hearts, not just 3 cards of one suit
- Note in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all before selecting a single heart
- The next 3 cards will be all hearts, so we may need to select 42 cars to guarantee 3 hearts are selected

Applications of Pigeonhole principle

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games
- Let a_j be the number of games played on or before jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinctive positive integers with $1 \le a_j \le 45$. Moreover $a_1+14, a_2+14, \ldots, a_{30}+14$ is also an increasing sequence of distinct positive integers with $15 \le a_j+14 \le 59$
- The 60 positive integers, $a_1, a_2, \ldots, a_{30}, a_1+14, a_2+14, \ldots, a_{30}+14$ are all less than or equal to 59. Hence, by the pigeonhole principle, two of these integers must be equal, i.e., there must be some I and j with $a_i=a_j+14$. This means exactly 14 games were played from day j+1 to day i

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Ramsey theory

- Example: Assume that in a group of 6 people, each pair of individuals consists of two friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group
- Let A be one of the 6 people. Of the 5 other people in the group, there are either 3 or more who are friends of A, or 3 or more are enemies of A
- This follows from the generalized pigeonholes principles, as 5 objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements

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Ramsey number

- **Ramsey number** R(m, n) where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party s.t. there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies
- In the previous example, $R(3,3) \le 6$
- We conclude that $R(3,3)=6$ as in a group of 5 people where every two people are friends or enemies, there may not be 3 mutual friends or 3 mutual enemies

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