Infinite Series

Let $\{a_n\}$ be a given sequence. We construct a new sequence from this as follows:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$ the sequence $\{S_n\}$ is called the sequence of partial sums of the infinite series

$$a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k$$

We say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges to sum L if its sequence of partial sums has limit L. Otherwise, we say that the series diverges.

Ex Repeating decimals. Say $0.222\ldots$, really means

$$\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots = \frac{2}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) = \frac{2}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n \text{ or } \frac{2}{10}\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^{k-1}$$

Geometric Series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

The n^{th} term of the sequence of its partial sums is

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ -\ rS_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline \end{aligned}$$

$$(1-r)S_n = a - ar^n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

If $|r|<1$, $\quad \lim_{n\to\infty} r^n = 0$

$$\boxed{\lim_{n\to\infty} S_n = \frac{a}{1-r}} \quad \text{that is} \quad \boxed{\lim_{n\to\infty} \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}, \ |r|<1$$

If $|r| \geq 1$, it diverges!

Ex Again the decimal example 0.222...

$$0.222\ldots = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots = \frac{2}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right)$$

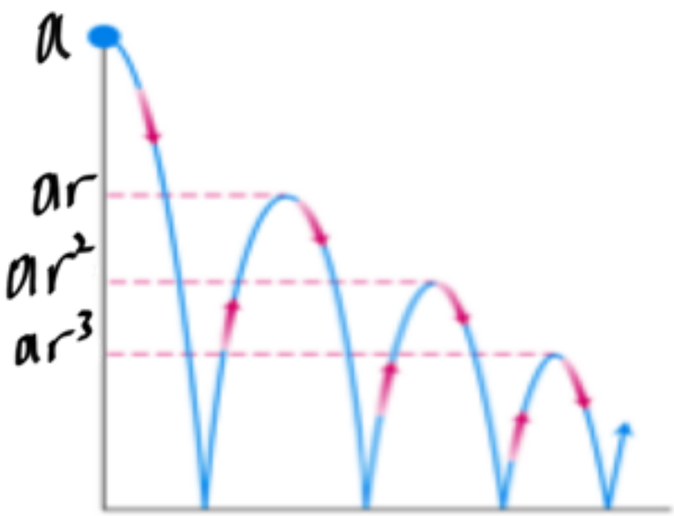
$$S_n = \frac{2}{10}\sum_{n=0}^{n}\left(\frac{1}{10}\right)^n \qquad a = 1,\ r = 1/10$$

$$= \frac{2}{10}\cdot\frac{1}{1-1/10} = \frac{2}{\cancel{10}}\cdot\frac{\cancel{10}}{9} = \frac{2}{9}$$

Ex

$$\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n = \frac{5}{1+1/4} = 4$$

EX. You drop a ball from *a* meters above a flat surface. Each time the ball hits the surface after falling a distance *h*, it rebounds a distance *rh*, where *r* is positive but less than 1. Find the total distance the ball travels up and down



(a)

$$S = a + 2ar + 2ar^2 + 2ar^3 + \cdots$$

$$= a + 2ar(1 + r + r^2 + \cdots)$$

$$= a + 2ar\underbrace{\sum_{n=0}^{\infty} r^n}_{\frac{1}{1-r}} = a + 2ar\frac{1}{1-r} = a\left(\frac{1+r}{1-r}\right)$$

Telescoping Series

Ex-1: Determine if $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or not.

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$$

The trick to use here is partial fractions.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow A=1, \; B=-1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n = 1$, which is the sum of the series.

Ex-2: $S_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1} + \cdots$

The n^{th} term of the sequence of partial sums is

$S_n = \ln\frac{1.2.3\ldots}{2.3.4\ldots} = \ln\frac{1}{n+1} = -\ln(n+1)$, $\lim_{n\to\infty} -\ln(n+1) = -\infty$, is divergent!

We shall not confuse infinite series with infinite sequences.

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Caution: $\lim_{n\to\infty} a_n = 0$ does not necessarily mean that $\sum a_n$ is a convergent series.

Ex The series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent since

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0, \text{ divergent} \quad (\text{L'Hsp.})$$

Ex Remember that $\lim_{n\to\infty} a_n = 0$ does not necessarily mean that the series is convergent.

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$, but $\left\{\frac{1}{n}\right\}$ sequence is divergent (in Integral test) see p series

Sometimes we can decompose a series into the sum or difference of two known convergent ones

THEOREM If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

Ex

$$\sum_{n=1}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n - \sum_{n=0} \frac{1}{3}\left(\frac{1}{3}\right)^n$$

$$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} - \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$