Sec. 7.5 Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty$$

$\lim_{x\to 0^+} x^2/x$, x/x, x/x^2 have different answers.

THEOREM 5— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Ex $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0/0$

$\lim_{\theta\to 0} \frac{\sin\theta}{1} = 0$

$\sec x = \frac{1}{\cos x}$

Ex $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\sec^2\theta}{1} = 1$

Ex

$$\lim_{x \to 0^+} \frac{\tan x}{1 - \cos x} = \lim_{x \to 0^+} \frac{\sec^2 x}{\sin x}$$

$$= \frac{1}{0} = \infty$$

Ex

$$\lim_{\theta \to \infty} \frac{\sin \theta}{\theta} = 0$$

(by Sandwich Thm.)

$$\frac{-1}{\theta} \leq \frac{\sin\theta}{\theta} \leq \frac{1}{\theta}$$

$\lim_{\theta \to \infty}$: $\frac{-1}{\theta} \to 0$, $\frac{\sin\theta}{\theta}$ has to be zero, $\frac{1}{\theta} \to 0$

Ex $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right) = \infty - \infty$

$$\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)\frac{\left(\sqrt{9x^2+x} + 3x\right)}{\sqrt{9x^2+x} + 3x} =$$

$$\lim_{x\to\infty} \frac{\cancel{9x^2}+x-\cancel{9x^2}}{\sqrt{9x^2+x}+3x} =$$

$$\lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{9+\underset{\substack{\uparrow\\0}}{1/\infty}}+3\right)} = \frac{1}{6} \text{ //}$$

Ex $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$\theta = \frac{1}{x}$, As $x\to\infty$, $\theta\to 0$: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Indeterminate Powers

$1^\infty, \ 0^0, \ \infty^0$

Ex $\lim_{x \to 0} (1+x)^{1/x} = 1^\infty$

$$y = (1+x)^{1/x}$$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} \quad \text{L'Hosp.}$$

$$= 1$$

$$\lim_{x \to 0} \ln y = 1$$

$$\lim_{x \to 0} y = e^1 \quad ; \quad y = (1+x)^{1/x}$$

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

Ex $$\lim_{x \to 0} (1+\sin x)^{1/x} = e$$

$$y = (1+\sin x)^{1/x}$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+\sin x)}{x}$$

l'Hop. Rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\frac{\cos x}{1+\sin x}}{1} = 1$$

$$\lim_{x \to 0} \ln y = 1 \Rightarrow \lim_{x \to 0} y = e^1$$

Ex $\lim_{x\to 0^+} x^{\frac{1}{\ln x}} = e$

Ex $\lim_{x\to\infty} x^{1/x} = 1$

Ex $\lim_{x\to\infty} (\ln x)^{1/x} = 1$

Ex $\lim_{x\to 0^+} (\cot x)^{x} = 1$

Ex $\lim_{x\to 0^+} (\cos\sqrt{x})^{1/x} = e^{-\frac{1}{2}}$

Ex $\lim_{x \to 0} \left(e^{-\cot x}\right)^x$

$$y = \left(e^{-\cot x}\right)^x$$

$$\ln y = x(-\cot x)\underbrace{\ln e}_{1}$$

$$\lim_{x \to 0} \ln y = -\lim_{x \to 0} \frac{\overset{1}{\cos x}}{\underset{1}{\frac{\sin x}{x}}} = -1$$

$$\lim_{x \to 0} y = e^{-1} = 1/e$$

$$\lim_{x \to 0} \left(e^{-\cot x}\right)^x = 1/e$$