Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2 hx - \sin^2 hx = 1$$

$$\sin 2hx = 2\sinh x \cosh x$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

Chapter 8. Techniques of Integration

Integration by parts

Product rule:

$$d(uv) = uv' + u'v$$

$$\cancel{\int}\cancel{d}(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex

(1) $\int x e^x dx \longrightarrow \int x^n e^{ax} dx$

$u = x \qquad \int dv = \int e^x dx$

$du = dx \qquad v = e^x$

$$\int x e^x dx = \int u\, dv = uv - \int v\, du$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + C$$

$$= e^x (x-1) + C$$

(2) $\int x^2 \sin x\, dx \longrightarrow \int x^n \sin ax\, dx$

$\longrightarrow \int x^n \cos ax\, dx$

$n = 1, 2, \ldots$

$u = x^2 \qquad dv = \sin x\, dx$

$du = 2x\, dx \qquad v = -\cos x$

$$\int \underbrace{x^2}_{u} \underbrace{\sin x\, dx}_{dv} = uv - \int v\, du$$

$$= -x^2 \cos x + \int \cos x \cdot 2x\, dx$$

$$= -x^2 \cos x + 2 \int \underbrace{x}_{u} \underbrace{\cos x\, dx}_{dv}$$

$u = x$ $\quad dv = \cos x\, dx$

$du = dx$ $\quad v = \sin x$

$$= -x^2 \cos x + 2 \left\{ x \sin x - \int \sin x\, dx \right\}$$

$$= -x^2 \cos x + 2x \sin x + \cos x + C$$

(3) $\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int \cancel{x} \cdot \frac{dx}{\cancel{x}}$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

(4) $\int x \ln x \, dx$

$u = \ln x \qquad dv = x \, dx$

$du = \frac{dx}{x} \qquad v = \frac{1}{2} x^2$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{1}{2} x^{\cancel{2}} \frac{dx}{\cancel{x}} = \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$$

(5)

$$\int \underbrace{\text{Arctan}\,x}_{u}\, \underbrace{dx}_{dv}$$

$$du = \frac{dx}{1+x^2} \qquad v = x$$

$$\int \text{Arctan}\,x\,dx = x\,\text{Arctan}\,x - \int \frac{x}{1+x^2}\,dx$$

$$s = 1+x^2$$
$$ds = 2x\,dx$$
$$\frac{ds}{2} = x\,dx$$

$$= x\,\text{Arctan}\,x - \frac{1}{2}\underbrace{\int \frac{ds}{s}}_{\ln s}$$

$$= x\,\text{Arctan}\,x - \frac{1}{2}\ln|1+x^2| + C$$

16) $\int x^3\sqrt{9-x^2}\,dx = \int u\,dv$

$u = x^2$

$dv = \frac{1}{2} 2x\sqrt{9-x^2}\,dx$

$du = 2x\,dx$

$s = 9 - x^2$

$-ds = +2x\,dx$

$\int dv = \frac{-1}{2}\int \sqrt{s}\,ds$

$v = -\cancel{\frac{1}{2}}\,\cancel{\frac{2}{3}}\,s^{3/2}$

$v = -\frac{1}{3}(9-x^2)^{3/2}$

$\int x^3\sqrt{9-x^2}\,dx = -\frac{1}{3}x^2(9-x^2)^{3/2} +$

$\int \frac{1}{3}(\underbrace{\boxed{9-x^2}}_{z})^{3/2}\cdot \underbrace{2x\,dx}_{-dz}$

$$\int x^3\sqrt{9-x^2}\,dx = -\frac{1}{7}x^2(9-x^2)^{7/2} - \frac{2}{5}\cdot\frac{1}{7}(9-x^2)^{5/2} + C$$

(7)

$$I = \int e^x \underbrace{\sin x}_{u}\,dx$$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x\,dx \qquad v = e^x$

$$I = e^x \sin x - \int e^x \underbrace{\cos x}_{u}\,dx$$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x\,dx \qquad v = e^x$

$$I = e^x \sin x - \Big[e^x \cos x + \underbrace{\int e^x \sin x\,dx}_{I}\Big]$$

$$2I = e^x(\sin x - \cos x) + C$$

(8) $I = \int \sec^3 x\,dx = \int \underbrace{\sec x}_{u}\ \underbrace{\sec^2 x\,dx}_{dv}$

$du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$

$v = \tan x$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan x \sec x \tan x\,dx$$

$$= \sec x \tan x - \int \underset{\substack{\uparrow \\ (\sec^2 x - 1)}}{\tan^2 x} \sec x\,dx$$

$$= \sec x \tan x - \underline{\underline{\int \sec^3 x\,dx}} + \int \sec x\,dx$$

$$2I = \sec x \tan x + \int \sec x\,dx$$

$$2I = \sec x\tan x + \int \sec x \cdot \frac{\overset{du}{\sec x\tan x}}{\sec x + \tan x}\,dx$$

$$2I = \sec x \tan x + \ln|u| + C \qquad u$$

$$= \sec x \tan x + \ln|\sec x + \tan x| + C$$

(4) $\int \sin(\ln x)\,dx = \int u\,dv = uv - \int v\,du$

$u = \sin(\ln x) \qquad dv = dx$

$du = \cos(\ln x)\cdot\frac{1}{x}\,dx \qquad v = x$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cancel{x}\,\frac{\cos(\ln x)\,dx}{\cancel{x}}$$

$$= x\sin(\ln x) - \int \underbrace{\cos(\ln x)}_{u}\,\underbrace{dx}_{dv}$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$$

$$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C$$

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$$

(10) $\int (\ln x)^2\,dx$

$u = (\ln x)^2$ $\quad dv = dx$

$v = x$

$du = 2(\ln x)\frac{dx}{x}$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int \cancel{x}\,\frac{2\ln x}{\cancel{x}}\,dx$$

$$= x(\ln x)^2 - 2\int \ln x\,dx$$

In conclusion, integration by parts is good for

$$\int x^n \ln x \, dx, \int x^n e^{ax} \, dx,$$

$$\int x^n \sin ax \, dx, \int \operatorname{Arctan} x \, dx,$$

$$\int e^{ax} \cos hx \, dx, \text{ etc.}$$