3.2 The Derivative as a Function

$$g(x) = f'(x)$$

$$f'(x) = y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex $\quad y = f(x) = \sqrt{x}$

$$f'(x) = y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

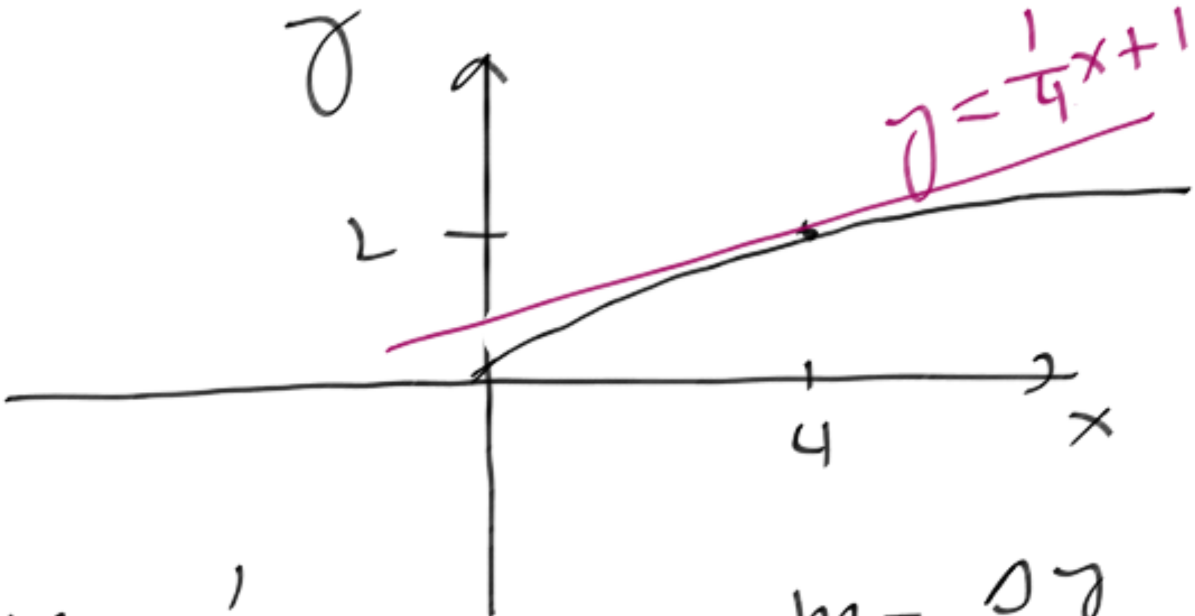
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}\left[\sqrt{x+h} + \sqrt{x}\right]} = \frac{1}{2\sqrt{x}}$$

Find the tangent line at $x=4$, i.e, at $(4,2)$.

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} = m$$

slope



$m=\frac{1}{4}, \ b=?$

$m=\frac{1}{4}$

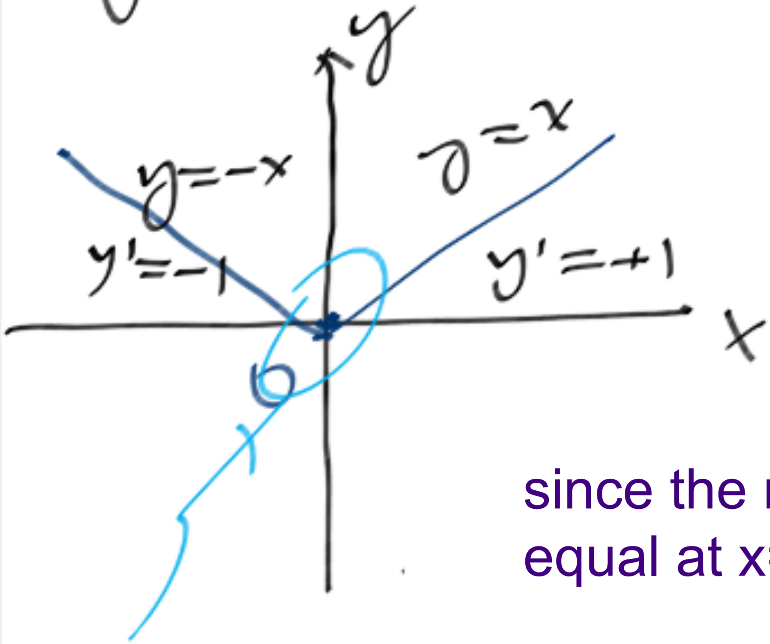
$b=1$

$$m = \frac{\Delta y}{\Delta x} = \frac{y-2}{x-4}$$

$\frac{1}{4}$

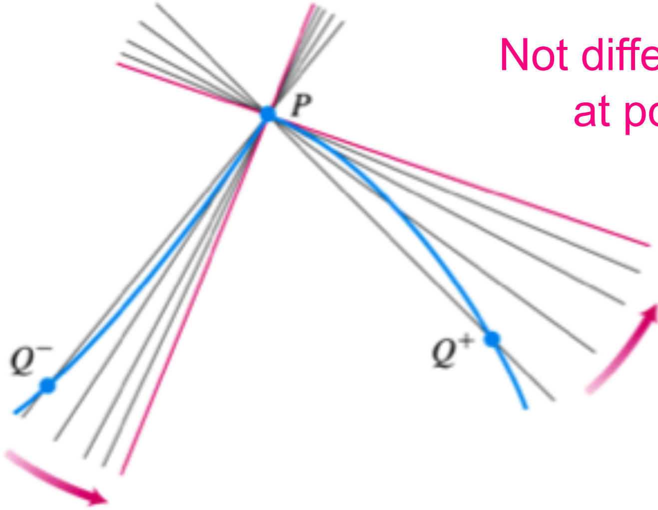
$y=\frac{1}{4}x+1$

$y = f(x) = |x|$ is NOT differentiable at $x = 0$

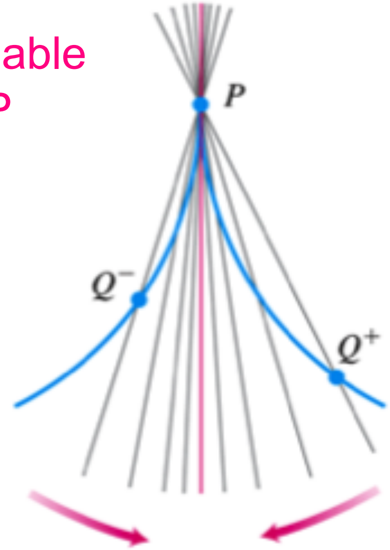


Sharp Corner

since the right and left derivatives are NOT equal at x=0, y=|x| is not differentiable at x=0.

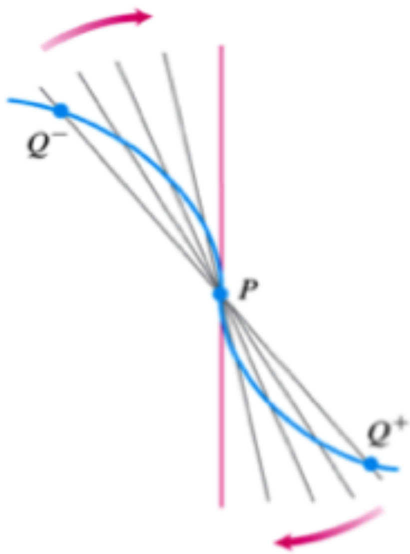


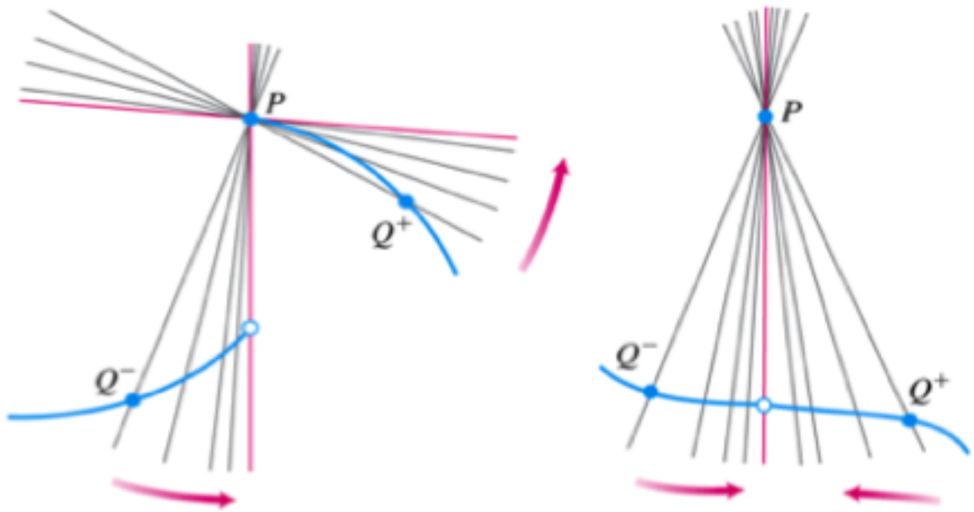
Not differentiable
at point P



1. a *corner*, where the one-sided derivatives differ.

2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.





3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).

4. a *discontinuity* (two examples shown).

Not differentiable
at point P

> **THEOREM —Differentiability Implies Continuity** If f has a derivative at $x = c$, then f is continuous at $x = c$.

Ex $y = f(x) = x^3 - 2x^2 + 3$ $\qquad f'(2) = ?$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h)^2 + 3 - x^3 + 2x^2 - 3}{h}$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2\cancel{h} + 3x\cancel{h^2} + \cancel{h^3} - \cancel{2x^2} - 4xh - 2\cancel{h^2} + \cancel{3} \cancel{- x^3} \cancel{+ 2x^2} \cancel{- 3}}{\cancel{h}}$$

$$f'(x) = 3x^2 - 4x \longrightarrow f'(2) = 3 \cdot 2^2 - 4 \cdot 2 = 4$$

3.3 Rules of Differentiation

(1) $f = \text{constant}$, $f' = 0$

(2) $(f \pm g)' = f' \pm g'$

(3) $(cf)' = cf'$, $c = \text{const.}$

(4) $(fg)' = f'g + fg'$

(5) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

(6) $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

(7) The general power rule:

$$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

n is any real number

Ex

$p(x) = \alpha x^3 + 2x - 5$

find α so that $p'(1) = 5$

$p'(x) = 3\alpha x^2 + 2$

$p'(1) = 3\alpha + 2$

$5 = 3\alpha + 2 \Rightarrow \underline{\alpha = 1}$

Ex

$$y = \frac{x+3}{x^2+7} \qquad y'(0) = ?$$

$$y' = \frac{x^2+7 - 2x(x+3)}{(x^2+7)^2}$$

$$y' = -\frac{x^2+6x\overset{+7\;\;-1}{-7}}{(x^2+7)^2} = -\frac{(x-1)(x+7)}{(x^2+7)^2}$$

$$y'(0) = -\frac{(-1)(7)}{7^2} = \frac{1}{7}$$

Higher Order Derivatives

$$f'' = (f')' \quad \text{or} \quad \frac{d^2y}{dx^2}$$

Ex

$$y = x^3$$

$$y' = 3x^2$$

$$y'' = 6x$$

$$y''' = 6$$

$$y^{(4)} = 0$$

Ex

$$y = \sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$y' = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

Ex

$$y = \sqrt{x^{2+\pi}} = x^{\frac{2+\pi}{2}} = x^{1+\pi/2}$$

$$y' = \left(1+\frac{\pi}{2}\right) x^{\cancel{1}+\frac{\pi}{2}\cancel{-1}}$$

$$y' = \left(1+\frac{\pi}{2}\right) x^{\pi/2}$$

Ex

$$y = x^{\sqrt{2}}$$

$$y' = \sqrt{2}\, x^{\sqrt{2}-1}$$

Ex

$$y = \frac{1}{x^4} = x^{-4}$$

$$y' = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

Ex

$$y = x^{-4/3}$$

$$y' = -\frac{4}{3} x^{-\frac{4}{3}-1} = -\frac{4}{3} x^{-7/3}$$

Ex

Find all pts on the graph of $g(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+1$ where the tangent line is parallel to $8x-2y=1$

solution:

$$8x-2y=1$$

$$y=4x-\frac{1}{2}$$

4

$g' = m \equiv$ slope

$$g'(x)=x^2-3x$$

$$4 = x^2 - 3x$$

$$x^2 - 3x - 4 = 0$$

$+1 \quad -4$

$$(x+1)(x-4) = 0$$

$$x_1 = -1$$

$$x_2 = +4$$

//