More Applications

of

the Pumping Lemma

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The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer p (critical length)
- for any string $w \in L$ with length $|w| \geq p$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq p$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \quad i = 0, 1, 2, \ldots$

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \quad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let p be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq p$

We pick $w = a^p b^p b^p a^p$

From the Pumping Lemma:

We can write: $w = a^p b^p b^p a^p = x\, y\, z$

with lengths: $|\, x\, y\, | \leq p, \quad |\, y\, | \geq 1$

$$w = xyz = \underbrace{a \ldots a}_{x}\underbrace{\overbrace{a \ldots a}^{p}}_{y}\ldots a\overbrace{b \ldots b}^{p}\overbrace{b \ldots b}^{p}\overbrace{a \ldots a}^{p}$$

$$z$$

Thus: $y = a^k, \;\; 1 \leq k \leq p$

$$x\ y\ z = a^p b^p b^p a^p \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$i = 0, 1, 2, \ldots$

Thus: $x\ y^2\ z \ \in\ L$

$$x\,y\,z = a^p b^p b^p a^p \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{a \ldots a a \ldots a}^{p+k}\ldots$$

$$xy^2z = \underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}\ldots a\underbrace{b\ldots b b\ldots b a\ldots a}_{z} \in L$$

Thus: $a^{p+k} b^p b^p a^p \in L$

$$a^{p+k} b^p b^p a^p \quad \in L \quad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{p+k} b^p b^p a^p \quad \notin L$$

CONTRADICTION!

Therefore: Our assumption that L is a regular language is not true.

Conclusion: L is not a regular language.

END OF PROOF

Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

is not regular.

Proof: Use the Pumping Lemma.

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Assume for contradiction

that L is a regular language.

Since L is infinite

we can apply the Pumping Lemma.

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Let p be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq p$

We pick $w = a^p b^p c^{2p}$

From the Pumping Lemma:

We can write $w = a^p b^p c^{2p} = x\ y\ z$

with lengths $|x\ y| \le p, \quad |y| \ge 1$

$$\boldsymbol{w} = xyz = \underbrace{a\ldots a}_{x}\overbrace{\underbrace{a\ldots a}_{y}a\ldots a}^{p}\underbrace{\overbrace{b\ldots b}^{p}\overbrace{c\ldots c}^{2p}}_{z}$$

Thus: $y = a^k, \quad 1 \le k \le p$

$$x\,y\,z = a^p b^p c^{2p} \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\,y^i\,z \;\in\; L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^0\,z = xz \;\in L$

$$x\,y\,z = a^p b^p c^{2p} \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $xz \in L$

$$xz = \underbrace{a...a}_{x}\overbrace{a...a}^{p-k}\underbrace{\overbrace{b...b}^{p}\overbrace{c...c}^{2p}}_{z} \in L$$

Thus: $a^{p-k} b^p c^{2p} \in L$

$$a^{p-k} b^p c^{2p} \quad \in L \quad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$

$\Downarrow$

$$a^{p-k} b^p c^{2p} \quad \notin L$$

CONTRADICTION!

Therefore: Our assumption that L is a regular language is not true.

Conclusion: L is not a regular language.

END OF PROOF

Theorem: The language $L = \{a^{n!} : \ n \geq 0\}$

is not regular.

Proof: Use the Pumping Lemma.

$$L = \{a^{n!} : \ n \geq 0\}$$

Assume for contradiction
that L is a regular language.

Since L is infinite
we can apply the Pumping Lemma.

$$L = \{a^{n!} : n \geq 0\}$$

Let p be the critical length of L.

Pick a string w such that: $w \in L$

length $|w| \geq p$

$$\boxed{\text{We pick } \quad w = a^{p!}}$$

From the Pumping Lemma:

We can write $w = a^{p!} = x\,y\,z$

with lengths $|x\,y| \leq p, \quad |y| \geq 1$

$$w = xyz = a^{p!} = \underbrace{a\ldots a}_{x}\overbrace{\underbrace{a\ldots a}_{y}a\ldots a}^{p}\underbrace{a\ldots a\overbrace{a\ldots a}^{p!-p}}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq p$

$$x\,y\,z = a^{p!} \qquad y = a^k,\;\; 1 \le k \le p$$

From the Pumping Lemma: $x\,y^i\,z \;\in\; L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \;\in\; L$

$$x\ y\ z = a^{p!} \qquad y = a^k,\ \ 1 \le k \le p$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\overbrace{\underbrace{a...a}_{y}}^{p+k}a...a\,\underbrace{a...a\overbrace{a...a}^{p!-p}}_{z}a...a \ \in\ L$$

Thus: $a^{p!+k} \ \in L$

$$a^{p!+k} \in L \qquad 1 \le k \le p$$

Since: $L = \{a^{n!} : \ n \ge 0\}$

$\Downarrow$

There must exist z such that:

$$p!+k = z!$$

However: $p!+k \leq p!+p$ for $p > 1$

$$\leq p!+p!$$

$$< p!\,p + p!$$

$$= p!(p+1)$$

$$= (p+1)!$$

$$p!+k < (p+1)!$$

$p!+k \neq z!$ for any z

for $p = 1$

we could pick string $w = a^{p'!}$

where $p' > p$

and we would obtain the same conclusion:

$$p'! + k \neq z! \quad \text{for any} \quad z$$

$$a^{p!+k} \in L \quad 1 \le k \le p$$

BUT: $L = \{a^{n!} : n \ge 0\}$

$\Downarrow$

$$a^{p!+k} \notin L$$

CONTRADICTION!

Therefore: Our assumption that L is a regular language is not true.

Conclusion: L is not a regular language

END OF PROOF