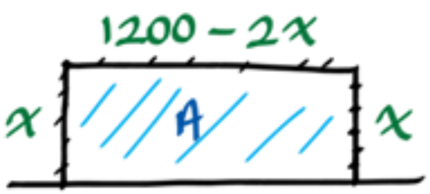
Max - Min Problems

Method:

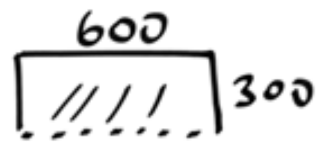
- Define your variables; draw a picture if possible
- Write the eqn of what is to be max/minimized
- Reduce to one variable, using the given information, if necessary
- Determine the interval where the solution is to be found
- Solve

Ex: A man has a stone wall outside a field. He has 1200 m of fencing material and he wishes to make a rectangular pen, using the wall as one side. What should the dimensions be in order to enclose the largest possible area?



$A(x) = x(1200 - 2x) = 1200x - 2x^2$

$A'(x) = 0 = 1200 - 4x \Rightarrow x = 1200/4 = 300$



$A = 300 \times 600 \ m^2$

Ex: Find the point on the graph of $y^2 = 4x$ which is nearest the point (2,3).

$$(dist)^2 = (x-2)^2 + (y-3)^2 \qquad x = y^2/4$$

$$f(y) = (y^2/4 - 2)^2 + (y-3)^2$$

$$f'(y) = 0 = \cancel{2}(y^2/4 - 2)\cdot\frac{\cancel{2}y}{\cancel{4}} + 2(y-3) \Rightarrow \frac{y^3}{4} - \cancel{2y} + \cancel{2y} - 6 = 0 \Rightarrow y = 2\sqrt[3]{3}$$

$$x = y^2/4 = \frac{1}{4}(2\sqrt[3]{3})^2 = \sqrt[3]{9}$$

Alternatively, you may use $y = 2\sqrt{x}$

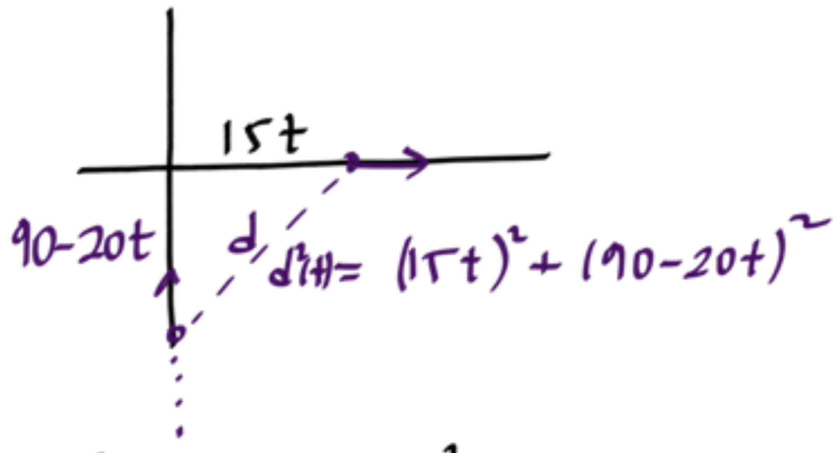
Ex: A business makes automobile transmissions selling for \$400 each. The total cost of marketing x units is $C(x) = 0.02x^2 + 160x + 400\,000$

How many transmissions should be sold for maximum profit?

$$P(x) = 400x - C(x)$$
$$= 400x - 0.02x^2 - 160x - 400\,000$$
$$P(x) = -0.02x^2 + 240x - 4\times10^5$$

$$P'(x) = 0 = -0.04x + 240 \Rightarrow x = \frac{240}{0.04} = 6000 \text{ units}$$

Ex: At midnight, ship B was 90 miles due south of ship A. Ship A sailed east 15 mph and ship B sailed north at 20 mph. At what time were they closest together? What was this closest distance?



$$f(t) = (15t)^2 + (90-20t)^2$$

$$f'(t) = 0 = 2(15t)\cdot 15 + 2(90-20t)\cdot(-20)$$

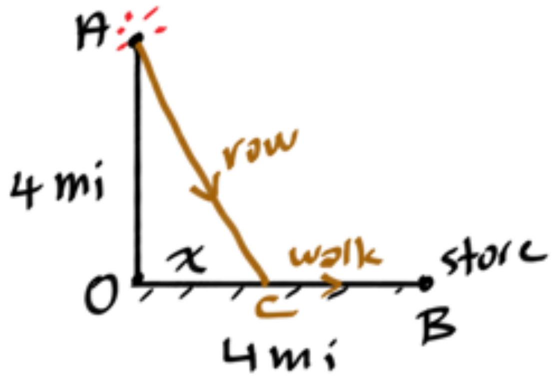
$$\cancel{2}\cdot\underset{3}{\cancel{15}}\cdot 15t = \cancel{2}\cdot\underset{4}{\cancel{20}}(90-20t)$$

$$45t = 360 - 80t$$

$$125t = 360$$

$$t = \frac{360}{125}\ h \text{ later.}$$

Ex: A lighthouse is at point A, 4 mi offshore from the nearest point O of a straight beach. A store is at point B, 4 mi down the beach from O. If the lighthouse keeper can row 4 mph and walks 5 mph, how should he proceed in order to get from the lighthouse to the store in the least possible time?



$$t = \frac{|AC|}{4} + \frac{|CB|}{5}$$

$$|AC| = \sqrt{4^2+x^2}$$
$$|CB| = 4 - x$$

$$t(x) = \frac{\sqrt{4^2+x^2}}{4} + \frac{4-x}{5}$$

$$t'(x) = 0 = \frac{1}{4}\cdot\frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{4^2+x^2}} - \frac{1}{5}$$

$$\frac{x}{\sqrt{4^2+x^2}} = \frac{4}{5}$$

$$\frac{x^2}{16+x^2} = \frac{16}{25}$$

$0 \leqslant x \leqslant 4$

$$9x^2 = 16\times 16$$

$x = \pm \frac{16}{3} > 4$, outside of interval

Now, try end pts:

$x=0$: $t(0) = 1 + \frac{4}{5} = 9/5$ | for $x=4$: $t(4) = \sqrt{2}$
$\sqrt{2} < 9/5$

He must row from A to B directly!

Ex: A silo is to be built in the farm of a right circular cylinder surmounted by a hemisphere. If the cost/m2 of the material for the floor costs twice as much as the sides, and the hemispheric part costs 3 times as much as the sides, find the most economic proportions for a given capacity V

$$V = \pi r^2 h + \frac{2}{3}\pi r^3$$

Cost function:

$$C = 2\pi r h k + \pi r^2 (2k) + 2\pi r^2 (3k)$$

$$h = \frac{V - \frac{2}{3}\pi r^2}{\pi r^2}$$

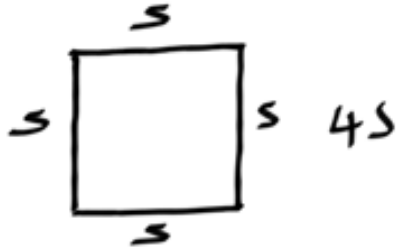
$$C(r) = 2\pi k \left\{ \frac{1}{\pi r}\left(V - \frac{2}{3}\pi r^3\right) + r^2 + 3r^2 \right\}$$

$$C(r) = 2\pi k \left[\frac{V}{\pi} \cdot \frac{1}{r} + \frac{10}{3} r^2 \right]$$

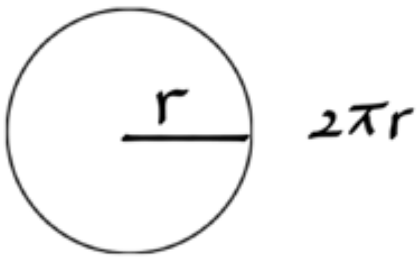
$$C'(r) = 0 = 2\pi k \Big[\underbrace{\frac{V}{\pi}\left(-\frac{1}{r^2}\right) + 2 \times \frac{10}{3} r}_{0} \Big] \Rightarrow r = \sqrt[3]{\frac{3V}{20\pi}}$$

$$h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{9V}{10\pi}\left(\frac{20\pi}{3V}\right)^{2/3}$$

Ex: A piece of wire of length L is cut into two parts, one of which is bent into a square and the other into a circle. How should the wire be cut so that the sum of the enclosed areas is (a) minimum, (b) maximum.



$$4s + 2\pi r = L \quad \Rightarrow s = \frac{L - 2\pi r}{4}$$



$$A = s^2 + \pi r^2$$

$$A(r) = \left(\frac{L-2\pi r}{4}\right)^2 + \pi r^2$$

$$A'(r) = 0 = \frac{2}{16}(L - 2\pi r).(-\cancel{2}\pi) + \cancel{2}\pi r \Rightarrow r = \frac{L}{8+2\pi}, \text{ local min.}$$

For the max, check the end pts.

only square: $s = L/4$, $A = L^2/16$

only circle: $2\pi r = L$, $A = \pi\left(\frac{L}{2\pi}\right)^2 = L^2/4\pi$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \frac{L^2}{4\pi} > \frac{L^2}{16}$$