Student: Huseyin Kerem Mican **Date:** 5/1/21	**Instructor:** Taylan Sengul **Course:** Linear Algebra	**Assignment:** Section 2.9 Homework

1. The vector x is in a subspace H with a basis B = {$\mathbf{b}_1$,$\mathbf{b}_2$}. Find the B-coordinate vector of **x**.

$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ -11 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

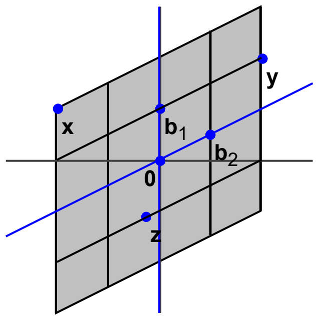
$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

2. The vector **x** is in a subspace H with a basis B = {$\mathbf{b}_1$,$\mathbf{b}_2$}. Find the B-coordinate vector of **x**.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -4 \\ -7 \\ 11 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -8 \\ -13 \\ 21 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

3. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -0.5 \\ -2.25 \end{bmatrix}$ and $B = \{\mathbf{b}_1,\mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_B$, $[\mathbf{y}]_B$, and $[\mathbf{z}]_B$. Confirm your estimates of $[\mathbf{y}]_B$ and $[\mathbf{z}]_B$ by using them and $\{\mathbf{b}_1,\mathbf{b}_2\}$ to compute **y** and **z**.



Use the figure to estimate $[\mathbf{x}]_B$. Choose the correct answer below.

- **A.** $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$
- **B.** $[\mathbf{x}]_B = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$
- **C.** $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
- **D.** $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

Use the figure to estimate $[\mathbf{y}]_B$. Choose the correct answer below.

- **A.** $[\mathbf{y}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- **B.** $[\mathbf{y}]_B = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$
- **C.** $[\mathbf{y}]_B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$
- **D.** $[\mathbf{y}]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Use the figure to estimate $[\mathbf{z}]_B$. Choose the correct answer below.

- **A.** $[\mathbf{z}]_B = \begin{bmatrix} -1 \\ -\frac{1}{4} \end{bmatrix}$
- **B.** $[\mathbf{z}]_B = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$
- **C.** $[\mathbf{z}]_B = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$
- **D.** $[\mathbf{z}]_B = \begin{bmatrix} -\frac{1}{4} \\ -1 \end{bmatrix}$

4. Given below is a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

$$A = \begin{bmatrix} 1 & 2 & 4 & -7 \\ 5 & 10 & 1 & 7 \\ 3 & 6 & -2 & 13 \\ 6 & 12 & 0 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A is given by $\left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 7 \\ 13 \\ 12 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of Col A is 3. (Type an integer.)

A basis for Nul A is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of Nul A is 1. (Type an integer.)

5. Find the bases for Col A and Nul A, and then state the dimension of these subspaces for the matrix A and an echelon form of A below.

$$A = \begin{bmatrix} 1 & 3 & 3 & 0 & -4 \\ 2 & 7 & 10 & 2 & -4 \\ -3 & -12 & -21 & -1 & 15 \\ 3 & 13 & 25 & 3 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 0 & -4 \\ 0 & 1 & 4 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -12 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

The dimension of Col A is 3.

A basis for Nul A is given by $\left\{ \begin{bmatrix} 9 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

The dimension of Nul A is 2.

6. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$\begin{bmatrix} 1 \\ -5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ -3 \\ 11 \end{bmatrix}$$

A basis for the subspace is given by $\left\{ \begin{bmatrix} 1 \\ -5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ -3 \\ 11 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of this subspace is 3 . (Type an integer.)

7. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \\ 13 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ -3 \\ 15 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 1 \\ -10 \end{bmatrix}$$

A basis for the subspace is given by $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 6 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

The dimension of this subspace is 2 .

8. Suppose a 4×6 matrix A has four pivot columns. Is Col A = $\mathbb{R}^4$? Is Nul A = $\mathbb{R}^2$? Explain your answers.

Is Col A = $\mathbb{R}^4$? Explain your answer. Choose the correct answer and reasoning below.

- **A.** Yes, because there are four pivot columns in A. These columns form a basis in four dimensions. Any 4-dimensional basis spans $\mathbb{R}^4$.
- **B.** Yes, because the column space of a 4×6 matrix is a subspace of $\mathbb{R}^4$. There is a pivot in each row, so the column space is 4-dimensional. Since any 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$, Col A = $\mathbb{R}^4$.
- **C.** No, Col A = $\mathbb{R}^2$. The number of pivot columns is equal to the dimension of the null space. Since the sum of the dimensions of the null space and column space equals the number of columns in the matrix, the dimension of the column space must be 2. Since any 2-dimensional basis is equal to $\mathbb{R}^2$, Col A = $\mathbb{R}^2$.
- **D.** No, because a 4×6 matrix exists in $\mathbb{R}^6$. If its pivot columns form a 4-dimensional basis, then Col A is isomorphic to $\mathbb{R}^4$ but is not strictly equal to $\mathbb{R}^4$.

Is Nul A = $\mathbb{R}^2$? Explain your answer. Choose the correct answer and reasoning below.

- **A.** No, because although the null space is 2-dimensional, its basis consists of four vectors and not two. Therefore, it cannot be equal to $\mathbb{R}^2$.
- **B.** Yes, because the linearly dependent vectors in A form a basis in two dimensions. Any basis in two dimensions is also a basis for $\mathbb{R}^2$. Therefore, Nul A = $\mathbb{R}^2$.
- **C.** Yes, because a 4×6 matrix exists in $\mathbb{R}^4$. Therefore, if its null space is 2-dimensional and contained within $\mathbb{R}^4$, it must be equal to $\mathbb{R}^2$.
- **D.** No, because the null space of a 4×6 matrix is a subspace of $\mathbb{R}^6$. Although dim Nul A = 2, it is not strictly equal to $\mathbb{R}^2$ because each vector in Nul A has six components. Each vector in $\mathbb{R}^2$ has two components. Therefore, Nul A is isomorphic to $\mathbb{R}^2$, but not equal.

9. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×8 matrix, what is the rank of A?

rank A = 5 (Type a whole number.)

10. If the rank of a 4×8 matrix A is 4, what is the dimension of the solution space $A\mathbf{x} = \mathbf{0}$?

The dimension of the solution space is 4 .

11. If possible, construct a 3×5 matrix A such that dim Nul A = 3 and dim Col A = 2.

Which 3×5 matrix has a null space with a dimension of 3 and a column space with a dimension of 2? Choose the correct answer below.

- **A.** $A = \begin{bmatrix} 1 & 2 & 7 & 8 & 9 \\ 2 & 3 & 5 & 7 & 8 \\ 3 & 4 & 7 & 10 & 11 \end{bmatrix}$
- ✔ **B.** $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 5 & 6 & 1 \\ 3 & 4 & 7 & 8 & 1 \end{bmatrix}$
- **C.** $A = \begin{bmatrix} 3 & 1 & 0 & 1 & 3 \\ 0 & 5 & 0 & 1 & 1 \\ 0 & 0 & 7 & 1 & 1 \end{bmatrix}$
- **D.** There is no 3×5 matrix A such that dim Nul A = 3 and dim Col A = 2.

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12. Let A be an $n \times p$ matrix whose column space, denoted Col A, is p-dimensional. Explain why the columns of A must be linearly independent.

Choose the correct answer below.

- **A.** Since A is an $n \times p$ matrix and Col A is p-dimensional, the basis for Col A has n vectors and therefore the columns of A are linearly independent.
- **B.** Since A is an $n \times p$ matrix and Col A is p-dimensional, rank A = n and this implies that the columns of A are linearly independent.
- ✔ **C.** Since Col A is p-dimensional, then the p columns of A span Col A and therefore, the spanning set of p columns is automatically a basis for Col A. This implies that the p columns of A are linearly independent.
- **D.** Since the Col A is p-dimensional this implies that the dimension of the null space is n and the columns of an $n \times p$ matrix must be linearly independent for this to be true.

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13. Let H = Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that **x** is in H, and find the B-coordinate vector of **x**, when $\mathbf{v}_1$, $\mathbf{v}_2$, and **x** are as below.

$$\mathbf{v}_1 = \begin{bmatrix} 13 \\ -4 \\ 12 \\ 9 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 16 \\ -7 \\ 15 \\ 12 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 21 \\ -12 \\ 20 \\ 17 \end{bmatrix}$$

Reduce the augmented matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x} \end{bmatrix}$ to reduced echelon form.

$$\begin{bmatrix} 13 & 16 & 21 \\ -4 & -7 & -12 \\ 12 & 15 & 20 \\ 9 & 12 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

How can it be shown that **x** is in H?

- **A.** The last two rows of the augmented matrix has zero for all entries and this implies that **x** must be in H.
- **B.** The first two columns of the augmented matrix are pivot columns and therefore **x** is in H.
- **C.** The augmented matrix is upper triangular and row equivalent to $\begin{bmatrix} B & \mathbf{x} \end{bmatrix}$, therefore **x** is in H because H is the Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$.
- **D.** The augmented matrix shows that the system of equations is consistent and therefore **x** is in H.

This implies that the B-coordinate vector is $[\mathbf{x}]_B = \begin{bmatrix} -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix}$.