Ch10 sequences & Infinite Series

A sequence is an unending succession of numbers, called terms. It is understood that the terms have a definite order.

$$a_1, a_2, a_3, \ldots$$

Ex

(a) $1, 2, 3, 4, \ldots$ (b) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (c) $2, 4, 6, 8, \ldots$ (d) $1, -1, 1, -1, \ldots$

It is better to have a rule or formula for generation of terms. One way of doing this is to look for a function that relates each term in the sequence to its term number.

Ex

(a) $2, 4, 6, 8, \ldots, 2n, \ldots$ $\qquad f(n) = 2n$

(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$ $\qquad f(n) = \frac{n}{n+1}$

(c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$ $\qquad f(n) = \frac{1}{2^n}$

(d) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$ $\qquad f(n) = (-1)^{n+1}\frac{n}{n+1}$

When the general term of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is known, we only write the general term in braces:

$$\{a_n\}_{n=1}^{+\infty}$$

Ex $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$, $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$, $\{2n-1\}_{n=1}^{+\infty}$, $\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{+\infty}$ or $\left\{\frac{1}{2^n}\right\}_{n=0}^{+\infty}$

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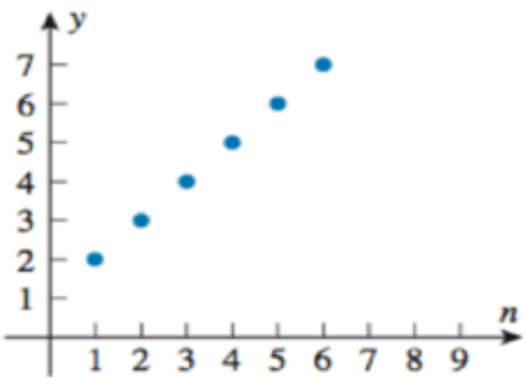
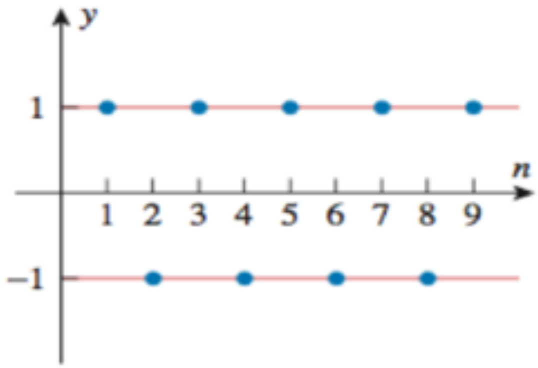
Definition: A sequence is a function whose domain is a set of integers.

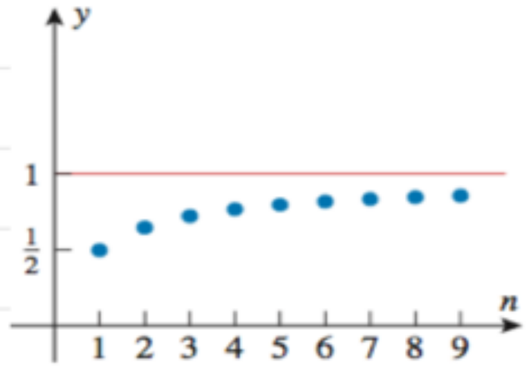
Limit of a sequences



$\{n+1\}_{n=1}^{+\infty}$



$\{(-1)^{n+1}\}_{n=1}^{+\infty}$

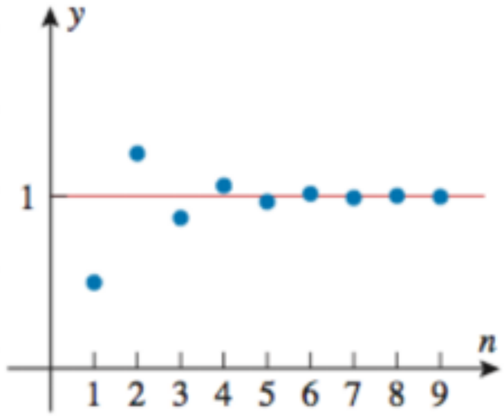


$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$

increases without bound

oscillates between 1 and -1

increases toward a "limiting value" of 1.



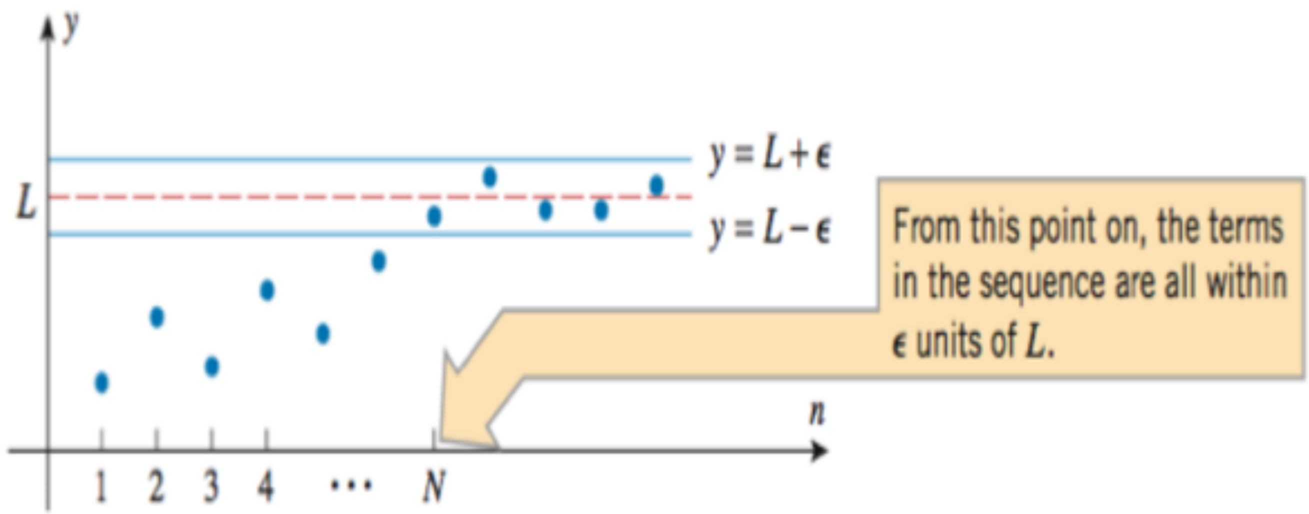
$\left\{1+\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{+\infty}$

tends toward a "limiting value" of 1, but do so in an oscillation fashion.

DEFINITION A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.



Ex $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1$ L' Hospital rule

Ex $\lim_{n\to\infty} \left[1 + (1-\frac{1}{2})^n\right] = 1$

THEOREM *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then:*

(a) $\lim_{n\to+\infty} c = c$

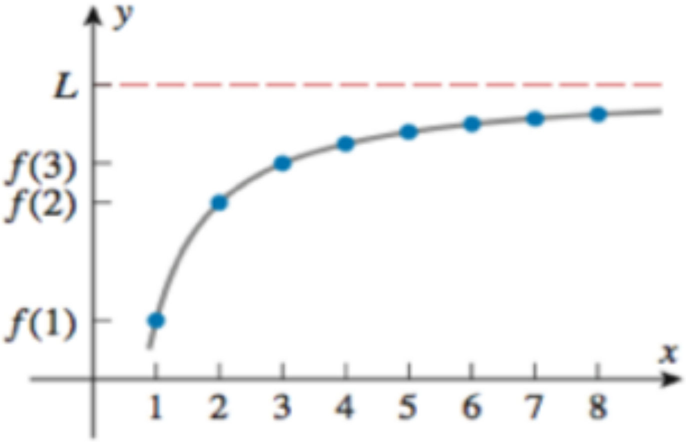
(b) $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n = cL_1$

(c) $\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

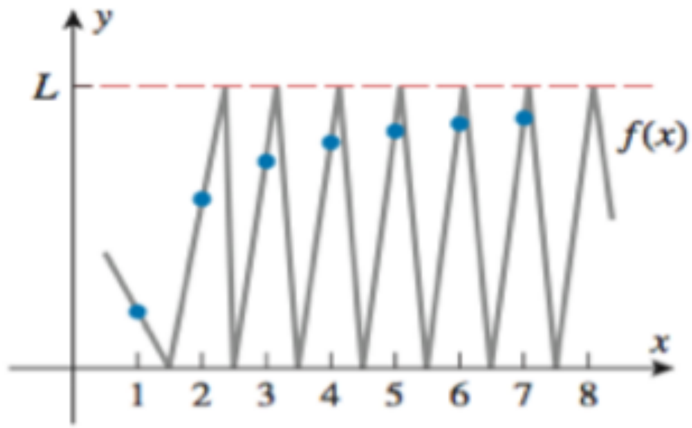
(d) $\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(e) $\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(f) $\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2}$ (if $L_2 \neq 0$)



If $f(x) \to L$ as $x \to +\infty$,
then $f(n) \to L$ as $n \to +\infty$.



$f(n) \to L$ as $n \to +\infty$, but $f(x)$
diverges by oscillation as $x \to +\infty$.

Ex $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ L'Hosp.

Ex $\lim_{n\to\infty} (-1)^{n+1} \frac{n}{2n+1}$ oscillates between 1 and -1, diverges

Ex $\lim_{n\to\infty} (8-2n) = -\infty$, diverges

Ex Does the following sequences converge or not?

(a) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$ $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$ $\lim_{n\to\infty} \frac{1}{2^n} = 0$

(b) $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$ $\quad \lim_{n\to\infty} 2^n = \infty$, seq. $\{2^n\}$ diverges.

Ex Find the limit of the sequence $\left\{ \frac{n}{e^n} \right\}_{n=1}^{+\infty}$

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 \qquad \text{L' Hospt.}$$

Ex Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$y = n^{1/n} \Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n} \ln n = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1/n}{1} = 0 \qquad \text{L' Hosp.}$$

$$\lim_{n\to\infty} \ln y = 0$$

$$\lim_{n\to\infty} y = e^0 = 1$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Ex $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{+\infty}$, does this sequence converge or not?

$$y = \left(1+\frac{1}{n}\right)^n \Rightarrow \ln y = n \ln(1+1/n) = \frac{\ln(1+1/n)}{1/n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{\cancel{-1/n^2}}{1+1/n}}{\cancel{-1/n^2}} = 1$$

$$\lim_{n\to\infty} \ln y = 1$$

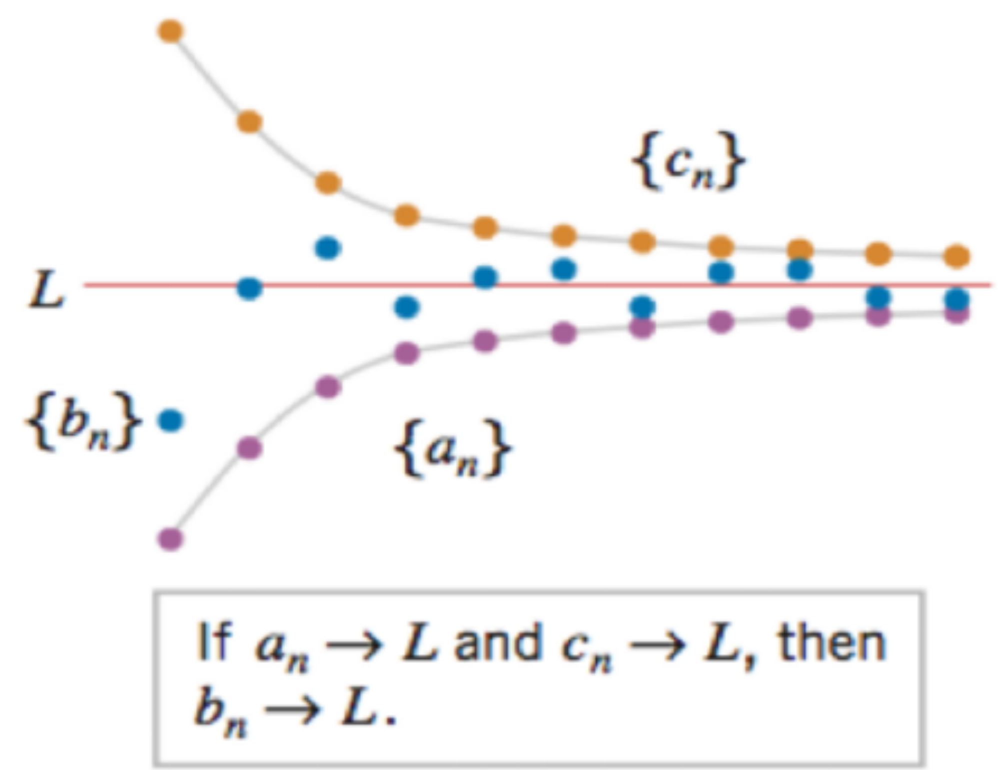
$$\lim_{n\to\infty} y = e^1$$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, the sequence converges to e.

THEOREM ***(The Sandwich Theorem for Sequences)*** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \le b_n \le c_n \quad (\textit{for all values of } n \textit{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.



Ex Find the limit of $\left\{ \frac{\sin(n!)}{n^2}(n+1) \right\}_{n=1}^{+\infty}$

$$\frac{(n+1)}{n^2}(-1) \leq \frac{\sin n!\,(n+1)}{n^2} \leq 1 \cdot \frac{(n+1)}{n^2}$$

$$-\frac{n+1}{n^2} \leq \frac{\sin n!}{n^2}(n+1) \leq \frac{n+1}{n^2}$$

$$\lim_{n\to\infty} -\left(\frac{1}{n}+\frac{1}{n^2}\right) \leq \frac{\sin n!}{n^2}(n+1) \leq \left(\frac{1}{n}+\frac{1}{n^2}\right)$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$\lim_{n\to\infty} \frac{\sin n!}{n^2}(n+1) = 0$$

Hw $\{\cos n/n\}$, $\{1/2^n\}$, $\{(-1)^n \frac{1}{n}\}$ find their limits.

Sequences defined recursively

Ex $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, $x_1 = 1$

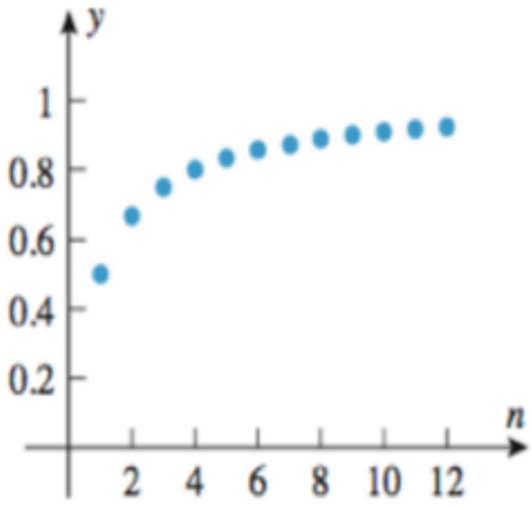
$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 3/2$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right)$$

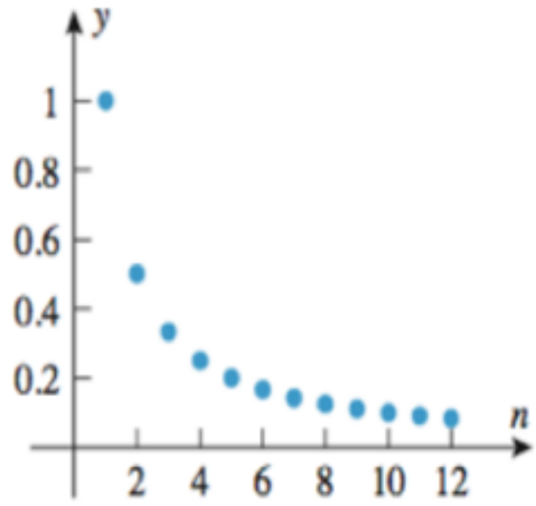
$\vdots$

Monotone Sequences

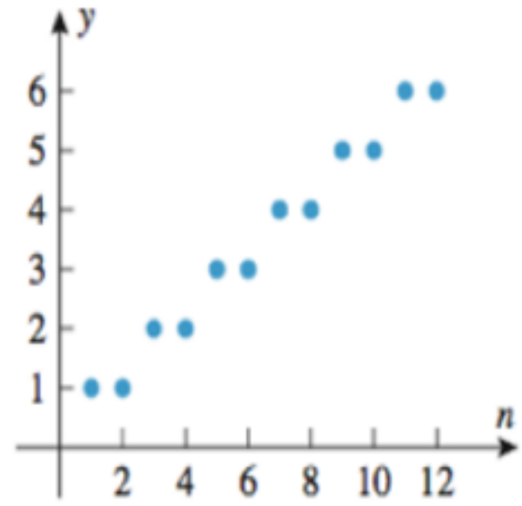
SEQUENCE	DESCRIPTION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	Strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	Strictly decreasing
$1, 1, 2, 2, 3, 3, \ldots$	Increasing; not strictly increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	Decreasing; not strictly decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing nor decreasing



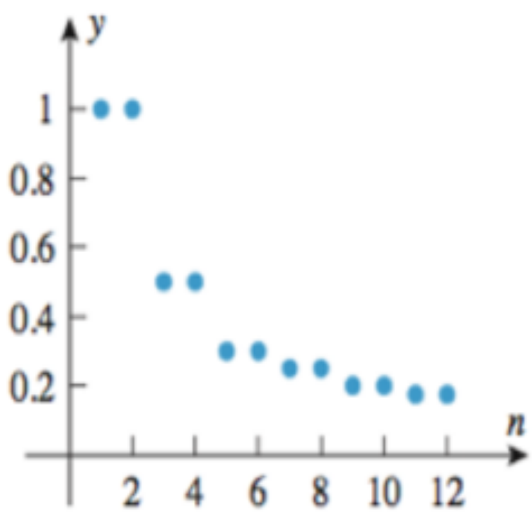
$\left\{\dfrac{n}{n+1}\right\}_{n=1}^{+\infty}$



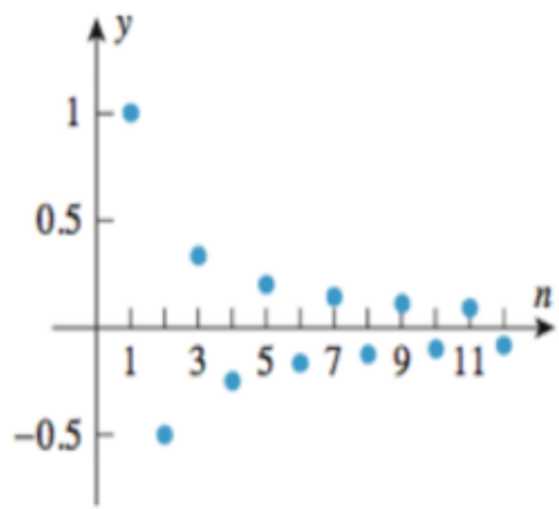
$\left\{\dfrac{1}{n}\right\}_{n=1}^{+\infty}$



$1, 1, 2, 2, 3, 3, \ldots$



$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$



$\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{+\infty}$

DIFFERENCE BETWEEN SUCCESSIVE TERMS	RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1} - a_n > 0$	$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1} - a_n < 0$	$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1} - a_n \leq 0$	$a_{n+1}/a_n \leq 1$	Decreasing

DERIVATIVE OF f FOR $x \geq 1$	CONCLUSION FOR THE SEQUENCE WITH $a_n = f(n)$
$f'(x) > 0$	Strictly increasing
$f'(x) < 0$	Strictly decreasing
$f'(x) \geq 0$	Increasing
$f'(x) \leq 0$	Decreasing

Convergence of monotone sequences

THEOREM *If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:*

(a) *There is a constant M, called an* **upper bound** *for the sequence, such that $a_n \le M$ for all n, in which case the sequence converges to a limit L satisfying $L \le M$.*

(b) *No upper bound exists, in which case $\lim_{n\to+\infty} a_n = +\infty$.*

THEOREM *If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:*

(a) *There is a constant M, called a* **lower bound** *for the sequence, such that $a_n \ge M$ for all n, in which case the sequence converges to a limit L satisfying $L \ge M$.*

(b) *No lower bound exists, in which case $\lim_{n\to+\infty} a_n = -\infty$.*

Ex Show that $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ converges and find its limit.

$a_n = \frac{10^n}{n!}$ $\quad a_{n+1} = \frac{10^{n+1}}{(n+1)!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n} = \frac{10}{n+1} < 1$

$a_{n+1} = \frac{10^{n+1}}{(n+1)!} = \frac{10}{n+1}\cdot\underbrace{\frac{10^n}{n!}}_{a_n} = \frac{10}{n+1}a_n$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{10}{n+1} a_n$$

$$= \underbrace{\lim_{n\to\infty} \frac{10}{n+1}}_{0} \cdot \underbrace{\lim_{n\to\infty} a_n}_{L} = 0$$

$$L = \lim_{n\to\infty} \frac{10^n}{n!} = 0$$

$$\boxed{\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L}$$ Thm.

For any real value of x, $\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$

Ex Does $\left\{\frac{n-1}{n}\right\}$ converge?
limit?

let us use calculus: $f(x) = (1 - \frac{1}{x}) \Rightarrow f'(x) = \frac{1}{x^2} > 0$, $x \geq 1$

$f(x)$ monotonically increases, so does the sequence. $\lim_{x\to\infty}(1 - \frac{1}{x}) = 1$