	FINAL EXAM	
Name, Surname:	Department:	GRADE
Student No:	Course: Linear Algebra	
Signature:	Exam Date: 11/06/2019	

Choose 5 out of 6 problems. Each problem is worth equal points. Duration is 70 minutes.

1. Let A be an ARBITRARY 3×3 matrix such that $A^T = -A$, that is A is skew-symmetric. Show that $\det(A) = 0$. (Bonus points (10pt): Show that the statement is true for $n \times n$ skew-symmetric matrices if n is an odd integer)

Solution: Skew symmetric means $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. $\det(A) = abc - abc = 0$ can be shown from this. Bonus part: $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$. If n is odd we have $\det(A) = -\det(A)$ and $\det(A) = 0$.

2. Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of a 2×2 matrix A corresponding to the eigenvalue 3 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2. (A) $A\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \boxed{}$ (B) Compute $A^2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \boxed{}$

Solution: $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$A\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\cdot 3\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$A^2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2\cdot 3\cdot A\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\cdot 3\cdot 3\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2)\cdot(-2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 26 \\ 22 \end{bmatrix}$

3. Let $A = \begin{bmatrix} a & -1 \\ 1 & 4 \end{bmatrix}$. Suppose that the matrix A has an eigenvalue 3. (A) $a = \boxed{}$. (B) If $\begin{bmatrix} 7 \\ b \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3, $b = \boxed{}$ so that .

Solution: Since 3 is an eigenvalue of the matrix A, we have $0 = \det(A - 3I) = a - 2$. $a = 2$. (B) $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. Then

$$\det(A - 3I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_2 = -v_1 \implies \mathbf{v} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$$

$b = -7$.

4. Let T be a linear transformation such that $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, $T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -4 \\ 8 \end{bmatrix}$ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \boxed{}$.

Solution: $\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ $a = y/2$, $b = (x-y)/4$. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{y}{2}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \frac{(x-y)}{4}\begin{bmatrix} 0 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} y \\ 2y-x \\ 2x \end{bmatrix}$.

5. Let M_2 be the vector space of all $n \times n$ real matrices. Let us fix a matrix $A \in M_2$. Define a map $T : M_2 \to M_2$ by $T(X) = AX + I$ where I is the identity matrix. Is $T : M_2 \to M_2$ a linear transformation?

Solution: $T(X_1 + X_2) = A(X_1 + X_2) + I = AX_1 + I + AX_2 + I - I = T(X_1) + T(X_2) - I \neq T(X_1) + T(X_2)$. So T is not linear.

6. Let $A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ where x, y, z are some real numbers. (A) Determine whether the matrix A is invertible or not. (B) If it is invertible, then find the inverse matrix A^{-1} .

Solution: (A) The matrix is invertible since $\det(A) = 1 \neq 0$. (B) By the method of row operations, $A^{-1} = \begin{bmatrix} 1 & -x & -y+xz \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$