CSE2023 Discrete Computational Structures

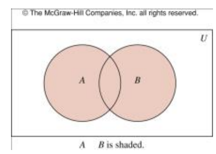
Lecture 9



## 2.2 Set operations

- **Union**: the set that contains those elements that are either in A or in B, or in both

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



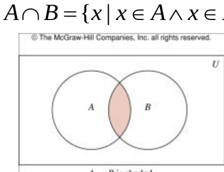

- A={1,3,5}, B={1,2,3}, A⋃B={1,2,3,5}



## Intersection

- **Intersection**: the set containing the elements in both A and B

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

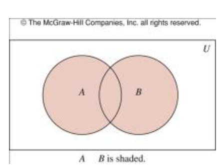


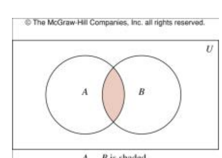

- A={1,3,5}, B={1,2,3}, A⋂B={1,3}



## Disjoint set

- Two sets are **disjoint** if their intersection is ∅
- A={1,3}, B={2,4}, A and B are disjoint
- Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$

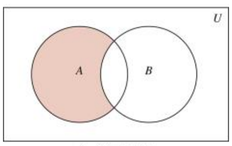


## Difference and complement

- A-B: the set containing those elements in A but not in B $A-B=\{x \mid x \in A \wedge x \notin B\}$






$A-B$ is shaded.

- A={1,3,5},B={1,2,3}, A-B={5}

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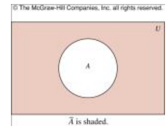
## Complement

- Once the universal set U is specified, the complement of a set can be defined
- **Complement** of A: $\overline{A}=\{x \mid x \notin A\}, \overline{A}=U-A$
- A-B is also called the complement of B with respect to A






$\overline{A}$ is shaded.

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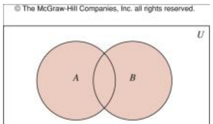
## Example

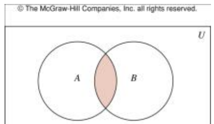
- A is the set of positive integers > 10 and the universal set is the set of all positive integers, then $\overline{A}=\{x \mid x \leq 10\}=\{1,2,3,4,5,6,7,8,9,10\}$
- A-B is also called the complement of B with respect to A






A B is shaded.






A B is shaded.

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## Set identities



**TABLE 1 Set Identities.**

| *Identity* | *Name* |
|---|---|
| $A \cup \emptyset = A$<br>$A \cap U = A$ | Identity laws |
| $A \cup U = U$<br>$A \cap \emptyset = \emptyset$ | Domination laws |
| $A \cup A = A$<br>$A \cap A = A$ | Idempotent laws |
| $\overline{(\overline{A})} = A$ | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Commutative laws |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ | Associative laws |
| $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | Distributive laws |
| $\overline{A \cup B} = \overline{A} \cap \overline{B}$<br>$\overline{A \cap B} = \overline{A} \cup \overline{B}$ | De Morgan's laws |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption laws |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \emptyset$ | Complement laws |

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## Example

- Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$
- (→): Suppose that $x \in \overline{A \cap B}$ , by definition of complement and use De Morgan's law

$$\neg(x \in A \wedge x \in B)$$
$$\equiv (\neg(x \in A)) \vee (\neg(x \in B))$$
$$\equiv (x \notin A) \vee (x \notin B)$$

- By definition of complement $x \in \overline{A}$ or $x \in \overline{B}$
- By definition of union $x \in \overline{A} \cup \in \overline{B}$



## Example

- (←): Suppose that $x \in \overline{A} \cup \overline{B}$
- By definition of union $x \in \overline{A} \vee x \in \overline{B}$
- By definition of complement $x \notin A \vee x \notin B$
- Thus $\neg(x \in A) \vee \neg(x \in B)$
- By De Morgan's law: $\neg(x \in A) \vee \neg(x \in B)$

$$\equiv \neg(x \in A \wedge x \in B)$$
$$\equiv \neg(x \in (A \cap B))$$

- By definition of complement, $x \in \overline{A \cap B}$



## Builder notation

- Prove it with builder notation

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \quad \text{(def of complement)} \\
&= \{x \mid \neg(x \in (A \cap B))\} \quad \text{(def of not belong to)} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \quad \text{(def of intersection)} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \quad \text{(De Morgan's law)} \\
&= \{x \mid x \notin A \vee x \notin B\} \quad \text{(def of not belong to)} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \quad \text{(def of complement)} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \quad \text{(def of union)} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$



## Example

- Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- (→): Suppose that $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$. By definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$. Consequently, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$
- By definition of intersection, it follows $x \in A \cap B$ or $x \in A \cap C$
- By definition of union, $x \in (A \cap B) \cup (A \cap C)$

## Membership table



**TABLE 2** A Membership Table for the Distributive Property.

| $A$ | $B$ | $C$ | $B \cup C$ | $A \cap (B \cup C)$ | $A \cap B$ | $A \cap C$ | $(A \cap B) \cup (A \cap C)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |



## Example

- Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$



## Example

- Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{B \cap C} \quad \text{(De Morgan' s law)}$$



## Example

- Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{B \cap C} \quad \text{(De Morgan' s law)}$$
$$= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{(De Morgan' s law)}$$

## Example

- Show that $\overline{A\cup(B\cap C)}=(\overline{C}\cup\overline{B})\cap\overline{A}$

$$\begin{aligned}\overline{A\cup(B\cap C)}&=\overline{A}\cap\overline{B\cap C}\quad\text{(De Morgan's law)}\\&=\overline{A}\cap(\overline{B}\cup\overline{C})\quad\text{(De Morgan's law)}\\&=(\overline{B}\cup\overline{C})\cap\overline{A}\quad\text{(commutative law)}\end{aligned}$$



## Example

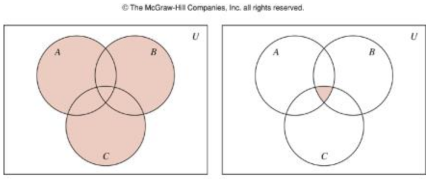
- Show that $\overline{A\cup(B\cap C)}=(\overline{C}\cup\overline{B})\cap\overline{A}$

$$\begin{aligned}\overline{A\cup(B\cap C)}&=\overline{A}\cap\overline{B\cap C}\quad\text{(De Morgan's law)}\\&=\overline{A}\cap(\overline{B}\cup\overline{C})\quad\text{(De Morgan's law)}\\&=(\overline{B}\cup\overline{C})\cap\overline{A}\quad\text{(commutative law)}\\&=(\overline{C}\cup\overline{B})\cap\overline{A}\quad\text{(commutative law)}\end{aligned}$$



## Generalized union and intersection






(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

- A={0,2,4,6,8}, B={0,1,2,3,4}, C={0,3,6,9}
- A∪B∪C={0,1,2,3,4,6,8,9}
- A∩B∩C={0}



## General case

- Union: $A_1\cup A_2\cup\cdots\cup A_n=\bigcup_{i=1}^{n}A_i$
- Intersection $A_1\cap A_2\cap\cdots\cap A_n=\bigcap_{i=1}^{n}A_i$
- Union: $A_1\cup A_2\cup\cdots\cup A_n\cup\cdots=\bigcup_{i=1}^{\infty}A_i$
- Intersection: $A_1\cap A_2\cap\cdots\cap A_n\cap\cdots=\bigcap_{i=1}^{\infty}A_i$
- Suppose $A_i$={1, 2, 3,…, i} for i=1,2,3,…

$$\bigcup_{i=1}^{\infty}A_i=\bigcup_{i=1}^{\infty}\{1,2,3,\ldots,i\}=\{1,2,3,\ldots\}=Z^{+}$$
$$\bigcap_{i=1}^{\infty}A_i=\bigcap_{i=1}^{\infty}\{1,2,3,\ldots,i\}=\{1\}$$

## Computer representation of sets

- U={1,2,3,4,5,6,7,8,9,10}
- A={1,3,5,7,9} (odd integer ≤10),B={1,2,3,4,5} (integer ≤5)
- Represent A and B as 1010101010, and 1111100000
- Complement of A: 0101010101
- A⋂B: 1010101010^1111100000=1010100000

  which corresponds to {1,3,5}