Taylor and Maclaurin Series

If a function f(x) has derivatives of all orders on an interval I, can it be expressed as a power series on I? And if it can, what will its coefficients be?

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\
&= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots, \\
f''(x) &= 1\cdot 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots, \\
f'''(x) &= 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-a) + 3\cdot 4\cdot 5a_5(x-a)^2 + \cdots,
\end{aligned}
$$

with the nth derivative, for all n, being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \qquad f''(a) = 1\cdot 2a_2, \qquad f'''(a) = 1\cdot 2\cdot 3a_3,$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

$$
\begin{aligned}
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.
\end{aligned}
$$

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f** is $(a=0)$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Ex Find the Taylor series and the Taylor Polynomials generated by $f(x) = e^x$ at $x=0$ (i.e., $a=0$)

$f(x) = e^x$

$f'(x) = e^x, \; f'' = e^x, \; f''' = e^x, \ldots$

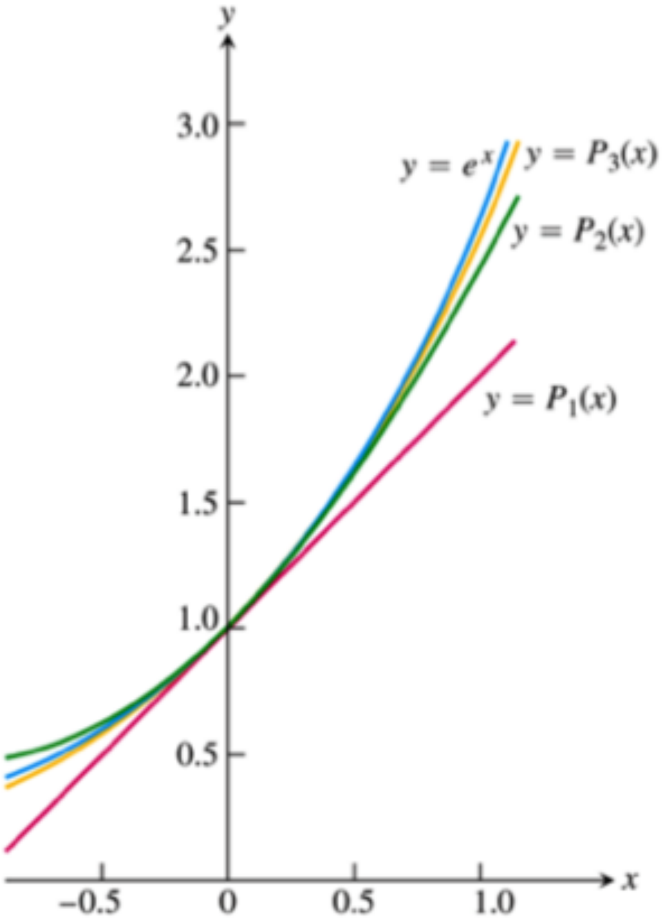
$f^{(k)}(x) = e^x$

$f^{(k)}(0) = 1$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f^{(0)}(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$



$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{x^2}{2!}$

$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Ex Find the Maclaurin series for $\sin x$, i.e., Taylor series for $a=0$

$$\sin x \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$ which is Maclaurin series

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1
$\vdots$	$\vdots$	

repeats itself

$$\sin x \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$ odd terms left!

This is not sufficient, we need to write down the n^{th} term of this series.

$$\sin x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Now, we shall find the interval of convergence, using generalized ratio thm:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} = x^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)}$$

$$= 0, \text{ for all } x$$

Thus, the series converges everywhere.

HW: Study $\cos x \sim \sum_{n=0}^{\infty} \frac{(-1)\,x^{2n}}{(2n)!}$, using Taylor series at $x=0$.

Ex Find the Taylor series of $\ln x$ near $x=1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	x^{-1}	1
2	$-x^{-2}$	-1
3	$2\cdot x^{-3}$	$2!$
4	$-2\cdot 3\,x^{-4}$	$-3!$
$\vdots$		
n	$(-1)^{n-1}\cdot(n-1)!\,x^{-n}$	

$$\Longrightarrow f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

$$\ln x \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n, \quad a=1$$

$$\ln x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{(n-1)!}}{n!}(x-1)^n$$

$$n! = n\cancel{(n-1)!}$$

$$\ln x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$