Linear Algebra: Lecture notes from Kolman and Hill 9th edition.

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Please let me know of any mistakes in these notes.

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Week 1-2

1.1 Systems of Linear Equations

1. Let $a_1, a_2, \ldots, a_n, b$ be constant numbers. The equation

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

is called a **linear equation** with n unknowns $x_1, x_2, \ldots, x_n$.

2. **A system of m linear equations in n unknowns** is

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

If the linear system has no solution, it is called **inconsistent**; if it has a solution it is called **consistent**. A linear system is called **homogeneous** if $b_1 = b_2 = \cdots = b_m = 0$; otherwise it is called **non-homogeneous**.
Example. $3x_1 + 2x_2 = 6$ *is a consistent, non-homogenous linear system since it has a solution* $x_1 = 2$, $x_2 = 0$.
Example. *Write a system which is inconsistent.*

3. A homogeneous system is always consistent because $x_1 = x_2 = \cdots = x_n = 0$ is a solution.

4. Two linear systems are called **equivalent** if they both exactly have the same solutions.

5. To find a solution to a linear system, we use the **method of elimination**.
   Example. *The linear system*

$$x - 3y = -7$$
$$2x - 6y = 7$$

   is inconsistent. To see, eliminate x from the second eq. to obtain

$$0 = 21$$

   Example.

$$x + 2y + 3z = 6$$
$$2x - 3y + 2z = 14$$
$$3x + y - z = -2$$

   Eliminate x from the second and third equations by using the first equation.

$$-7y - 4z = 2$$
$$-5y - 10z = -20$$

   Eliminate y from the second eq.

$$10z = 30 \implies z = 3$$

   Substitute to find $y = -2$ and $x = 1$.

By the elimination procedure we get the equivalent system

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3 \end{aligned}$$

This system has a unique solution.
Example.

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + y - 3z &= 4 \end{aligned}$$

Eliminate x,

$$-3y + 3z = 12$$

This system has solutions of the form

$$\begin{aligned} &x = z + 4 \\ &y = z - 4 \\ &z = \text{ any real number} \end{aligned}$$

This system has infinitely many solutions.

6. These examples suggest that a linear system may have a unique solution, no solution or infinitely many solutions.

7.

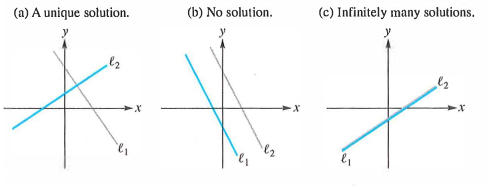
$$\begin{aligned} a_1x + a_2y = c_1 \\ b_1x + b_2y = c_2 \end{aligned}$$

The graph of each equation is a line. A solution must lie on both lines:

(a) If lines coincide: Infinitely many solutions.

(b) If lines intersect at exactly 1 point: unique solution.

(c) If lines are parallel and do not intersect: no solution.



8.

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
$$

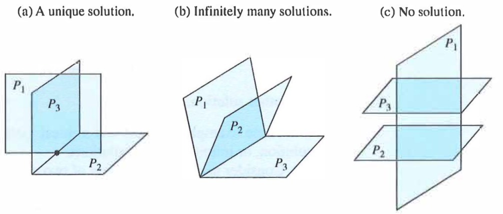
The graph of each equation is a plane.

(a) If all 3 planes coincide: infinitely many solution.

(b) If planes intersect on a line: infinitely many solutions.

(c) If planes intersect at exactly 1 point: unique solution.

(d) If two of the planes are parallel and do not intersect: no solution.



9. Unlike system of linear equations, system of nonlinear equations may have more than one but less then infinite solutions. Think about the intersection of two circles.

10. Exercises 1.1: 1-23

1.2 Matrices

- Define: an $m \times n$ matrix is a rectangular array of mn real (or complex) numbers in m rows and n columns. The ith row of a matrix, the jth column of a matrix. (i, j)- entry of a matrix.
- If $m = n$ the matrix is called a square matrix. Main diagonal of a square matrix: $a_{11}, a_{22}, \ldots, a_{mm}$.
- An $n \times 1$ matrix is called an n-vector.
- When are two matrices equal $A = [a_{ij}]$ and $B = [b_{ij}]$? If they have the same size and $a_{ij} = b_{ij}$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$.

- Matrix addition.
- Scalar multiplication of a matrix. Difference of two matrices.
- Linear combination of k matrices of size $m \times n$.
- Transpose of a matrix.
- Exercises 1.2. 5, 7, 9, 11, 13

1.3 Matrix Multiplication

- Dot product of two n-vectors.
  Example.

$$\textit{Let } \mathbf{a} = \begin{bmatrix} x \\ 2 \\ 3 \end{bmatrix} \textit{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}. \textit{ If } \mathbf{a} \cdot \mathbf{b} = -4, \textit{ find } x$$

  Answer: $x = -3$.

- Matrix multiplication. Which matrices can be multiplied? If $A_{m \times n}$ and $B_{p \times r}$ then AB is defined only if $n = p$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- If the matrix product AB is defined, BA may not be defined. If it is defined it may not be equal to AB.

- Matrix vector product.

$$A\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} (\mathrm{row}_1(A))^T \cdot \mathbf{c} \\ (\mathrm{row}_2(A))^T \cdot \mathbf{c} \\ \vdots \\ (\mathrm{row}_m(A))^T \cdot \mathbf{c} \end{bmatrix}$$

$$A\mathbf{c} = c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$A\mathbf{c} = c_1 \,\mathrm{col}_1(A) + c_2 \,\mathrm{col}_2(A) + \cdots + c_n \,\mathrm{col}_n(A)$$

- Express the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

  in the matrix form

$$A\mathbf{x} = \mathbf{b}$$

  with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called the coefficient matrix.

- Augmented matrix.
- Exercises 1.3: 1, 5, 9, 11, 15, 17, 19, 23, 30, 33, 37, 41, 51

1.4 Algebraic Properties of Matrix Operations

- Properties of matrix addition. If A, B, C are $m \times n$ matrices then (i) $A+B = B+A$, (ii) $A+(B+C) = (A+B)+C$, (iii) there is a unique zero matrix O, (iv) $A+(-A) = (-A)+A = O$ where $-A = -1A$.
- Properties of matrix multiplication. If A, B, C are matrices of appropriate sizes then (i)$(AB)C = A(BC)$, (ii) $A(B+C) = AB + AC$, (iii) $(A+B)C = AC + BC$.
- Properties of scalar multiplication. Theorem 1.3: If r, s are real numbers and A and B are matrices of appropriate sizes then (i) $r(sA) = (rs)A$, (ii) $(r+s)A = rA + sA$, (iii) $r(A+B) = rA + rB$, (iv) $A(rB) = r(AB) = (rA)B$.
- Properties of transpose. Theorem 1.4: If r is a scalar, A and B are matrices of appropriate sizes then (i) $\left(A^T\right)^T = A$, (ii) $(A+B)^T = A^T + B^T$, (iii) $(AB)^T = B^T A^T$, (iv) $(rA)^T = rA^T$. proof for $(AB)^T = B^T A^T$.

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k a_{jk} b_{ki} = \sum_k b^T_{ik} a^T_{kj} = (B^T A^T)_{ij}$$

Matrix multiplication has same properties unlike usual multiplication.

Example. *If $AB = 0$ then it may happen that $A \neq 0$ and $B \neq 0$.*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example. *If $AB = AC$ then it may happen that $B \neq C$.*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$

Then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}$$

Thus for matrices of appropriate sizes

1. AB need not equal BA
2. AB may be the zero matrix with $A \neq O$ and $B \neq O$
3. AB may equal AC with $B \neq C$.

- Exercises 1.4: 22, 30, 31, 32.

1.5 Special Types of Matrices and Partitioned Matrices

- A square matrix is called a **diagonal matrix** if its entries satisfy $a_{ij} = 0$ if $i \neq j$. A **scalar matrix** is a diagonal matrix whose diagonal elements are equal. **Identity matrix** I_n is a scalar matrix with diagonal elements equal to 1.

- An $n \times n$ square matrix is called the identity matrix I_n if its entries satisfy

$$a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

If A is $m \times n$:

$$AI_n = A, \quad \text{and} \quad I_m A = A$$

- For a square matrix, define powers of A

$$A^0 = I_n, \qquad A^p = \underbrace{AA \cdots A}_{\text{p times}}, \quad p \geq 1$$

Example. *Let A be a diagonal matrix with diagonal entries $\{2, -3, 5\}$. Compute A^4.*

By definition, we set $A^0 = I_n$. For any non-negative integers p, q

$$A^p A^q = A^{p+q}, \qquad (A^p)^q = A^{pq}$$

Note that in general,

$$(AB)^p \neq A^p B^p$$

However

$$AB = BA \implies (AB)^p = A^p B^p = (BA)^p$$

- An $n \times n$ matrix is called **upper triangular** if its entries satisfy $a_{ij} = 0$ if $i > j$, that is all its entries below the main diagonal are zero. Similarly we can define **lower triangular** matrices.

- A square matrix is **symmetric** if $A^T = A$ and **skew-symmetric** if $A^T = -A$. Write the general form of 3×3 symmetric and skew symmetric matrices.

  Example. *Any square matrix A can be written as a sum of symmetric and skew-symmetric matrices. This decomposition is unique.*

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = S + K$$

  Write $S = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} 0 & d \\ -d & 0 \end{smallmatrix}\right)$ and solve for a, b, c, d to find $a = 1$, $b = 5/2$, $c = 4$, $d = -1/2$.

- Partitioned matrices. We skip this topic for now.

- If for an $n \times n$ matrix A there exists an $n \times n$ matrix B such that $AB = BA = I_n$, then A called a **nonsingular** or **invertible** matrix and B is called an **inverse** of A. If A is not invertible then it is called **singular** or **non-invertible**.

- The inverse matrix is unique. proof. Suppose

$$AB = BA = I_n \quad \text{and} \quad AC = CA = I_n$$

  Then

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

  Because of uniqueness, we write the inverse of a matrix as A^{-1}.

- Example. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solve

$$a + 2c = 1$$
$$2a + 4c = 0$$

$$b + 2d = 0$$
$$2b + 4d = 1$$

to get

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

- Theorem 1.6. If A and B have inverses, then so does AB. And its inverse is $(AB)^{-1} = B^{-1}A^{-1}$. proof Check the inverse identity from left and right.
- Corollary 1.1. In general if $A_1, \ldots, A_n$ are invertible then so is the product matrix $A_1 \cdots A_n$ and $(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}$.
- Theorem 1.7. $(A^{-1})^{-1} = A$ proof. This is evident from the identity $AA^{-1} = A^{-1}A = I$.
- Theorem 1.8. $(A^{-1})^T = (A^T)^{-1}$. proof. $\left(A^{-1}\right)^T A^T = I_n^T = I_n$ and $\left(A^T\right)\left(A^{-1}\right)^T = I_n$

- Linear Systems and Inverses. Suppose A is a nonsingular matrix and $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ \left(A^{-1}A\right)\mathbf{x} &= A^{-1}\mathbf{b} \\ I_n\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

  We showed that if A is an $n \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. Moreover. if $\mathbf{b} = \mathbf{0}$, then the unique solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

- <u>Example</u>. Solve the systems

$$\begin{aligned} x + 2y &= 8 \\ 3x + 4y &= 6 \end{aligned}, \qquad \text{and} \qquad \begin{aligned} x + 2y &= 10 \\ 3x + 4y &= 2 \end{aligned}$$

  <u>Solution</u>. Write the system as $A\mathbf{x} = \mathbf{b}$. Then pre-multiply with A^{-1}.

- <u>Example</u>. Suppose $A^2\mathbf{x} = \mathbf{b}$, A is nonsingular with

$$A^{-1} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

  Find the solution $\mathbf{x}$.

  <u>Solution</u>.

$$A^{-1}A^{-1}A^2\mathbf{x} = A^{-1}A^{-1}\mathbf{b} \implies \mathbf{x} = (A^{-1})^2\mathbf{b} = \begin{bmatrix} 9 & 0 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ -6 \end{bmatrix}$$

- <u>Exercises 1.5</u>: 17, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40

Week 3-4

2.1 Echelon Form of a Matrix

- An $m \times n$ matrix A is said to be in **reduced row echelon form (RREF)** if it satisfies the following properties:

  a) All zero rows, if there are any, appear at the bottom of the matrix.

  b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.

  c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

  d) If a column contains a leading one, then all other entries in that column are zero.

  An ,$m \times n$ matrix satisfying properties (a), (b), and (c) is said to be in **row echelon form (REF)**.

  example. Give examples where a matrix breaks one of each rule.

- There are three types of **elementary row operations**

  1. Interchange row i and row j: $r_i \leftrightarrow r_j$.

  2. Replace row i by $k \neq 0$ times row i: $kr_i \rightarrow r_i$.

  3. Replace row j by k time row i + row j: $kr_i + r_j \rightarrow r_j$.

- If the matrix $B_{m\times n}$ can be obtained from the matrix $A_{m\times n}$ by elementary row operations then B is called **row equivalent** to A.
- If A is row equivalent to B then B is row equivalent to A. proof. Since each elementary row operation has an inverse which is again an elementary row operation. For example the inverse of the operation $r_i \leftrightarrow r_j$ is itself, since applying it twice gives the original matrix.
- Theorem 2.1. Every non-zero matrix is row-equivalent to a matrix in REF. The REF of a matrix may not be unique. Moreover, every matrix is equivalent to a unique matrix in RREF. proof. The proof for bringing to REF is the below algorithm. Let A be an $m \times n$ matrix with $m \geq 2$.
    1. Let $i = 1$.
    2. Let the first non-zero entry in the column i of A below row i be in the row j. If there are none then go to last step.
    3. $r_1 \leftrightarrow r_j$.
    4. By choosing suitable numbers k, make $a_{ji} = 0$ for $j \geq i+1$ by the row operation $kr_i + r_j \to r_j$.
    5. If $i = m-1$ then stop. If not increase i by 1.
- In the next section, we will solve examples of finding a matrix in REF which is row equivalent to a given matrix.
- Exercises 2.1: 1, 3, 5

2.2 Solving Linear Systems

- Take a system of m equations and n unknowns. Then write it in matrix form $A\mathbf{x} = \mathbf{b}$. Then write it in the **augmented matrix form** $[A \mid \mathbf{b}]$.

- Theorem. Let $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ be two linear systems. If the augmented matrices $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are row equivalent then the linear systems are equivalent, i.e. they have the same solutions.

- **To solve a linear system:**

  1. Write the linear system in the augmented form: $[A \mid \mathbf{b}]$
  2. **Gaussian Elimination**: Reduce $[A \mid \mathbf{b}]$ into row echelon form and solve by back substitution.
  3. **Gauss-Jordan reduction**: Reduce $[A \mid \mathbf{b}]$ into reduced row echelon form.

- **Example 6 from Section 2.2** The system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

in augmented matrix form is

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

Use the row operations

1. $-2r_1 + r_2 \to r_2$, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$,

2. $-3r_1 + r_3 \to r_3$, $\left[\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$,

3. $\frac{-1}{5} r_3 \to r_3$, $r_2 \leftrightarrow r_3$, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right]$,

4. $7r_2 + r_3 \to r_3$, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right]$

5. $\frac{1}{10}\mathrm{r}_3 \to \mathrm{r}_3$, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Then solve by back substitution. That is

$$\begin{aligned} z &= 3, \\ y &= 4 - 2z = -2 \\ x &= 6 - 2y - 3z = 1 \end{aligned}$$

- Examples of systems with exactly one solution, no solution, infinitely many solutions with 1, 2 free parameters. If a system has no solutions we say that the system is **inconsistent**, otherwise we say that the system is **consistent**.

Example. *If*

$$\left[\begin{array}{c|c} C & \mathbf{d} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

then $C\mathbf{x} = \mathbf{d}$ has no solutions since the last equation is

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1.$$

Example. *The solutions of the system*

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & -1 & 7 \\ 0 & 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right]$$

are in the form

$$\begin{aligned} x_1 &= -1 - 10r \\ x_2 &= 2 + 5r \\ x_3 &= -11 + r \\ x_4 &= 9 - 2r \\ x_5 &= r, \ \textit{any real number.} \end{aligned}$$

Example. *The solutions of the system*

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & -\frac{5}{2} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

are of the form

$$\begin{aligned}
x_1 &= \frac{2}{3} - r - 2s + \frac{5}{2}t \\
x_2 &= r \\
x_3 &= s \\
x_4 &= \frac{1}{2} - \frac{1}{2}t \\
x_5 &= t
\end{aligned}$$

with r, s, t being real numbers.

- Application: quadratic interpolation. Find the quadratic polynomial passing from the points (1, -5), (-1, 1), (2, 7).

- Consider a system of the form $A\mathbf{x} = \mathbf{b}$, where $A_{m\times n}$ is a matrix. The system is called a **homogeneous system** if $\mathbf{b} = \mathbf{0}$ where $\mathbf{0}$ is the $m \times 1$ zero vector. If $\mathbf{b} \neq \mathbf{0}$ then the system is called a **non-homogeneous system**.

- A homogeneous system $A\mathbf{x} = \mathbf{0}$, where $A_{m\times n}$ is a matrix, always has a solution. Namely the solution $\mathbf{x} = \mathbf{0}$ which is the $n \times 1$ zero vector. This solution is called the **trivial solution**. Hence, **a homogeneous system is always consistent**. We discussed that for linear systems, there are either 0, 1 or ∞ many solutions. Hence for homogeneous systems the question is whether the trivial solution is the only solution or there are infinitely many solutions.

  <u>Theorem</u>. A homogeneous system $A\mathbf{x} = \mathbf{0}$, where $A_{m\times n}$ is a matrix, has infinitely many solutions if $m < n$, that is there are fewer equations than unknowns. <u>proof</u>. Consider the RREF form B of A. Then each column of

B which does not have a leading one will be a free parameter. Since each row has at most 1 leading one, if there are more columns than there are rows, each column can not have a leading one.

Example.

$$\begin{aligned} x + y + z + w &= 0 \\ x + w &= 0 \\ x + 2y + z &= 0 \end{aligned}$$

whose augmented matrix in REF is

$$\left[\begin{array}{cccr|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Thus the solution is

$$\begin{aligned} x &= -r \\ y &= r \\ z &= -r \\ w &= r, \text{ any real number.} \end{aligned}$$

- **Example.** *Balance the chemical reaction*

$$x\mathrm{NaOH} + y\mathrm{H_2SO_4} \to z\mathrm{Na_2SO_4} + w\mathrm{H_2O}.$$

That is, find x, y, z.

Solution.

$$\begin{aligned}&\textit{Na: } x = 2z\\&\mathrm{O} : x + 4y = 4z + w\\&\mathrm{H} : x + 2y = 2w\\&\mathrm{S} : y = z\end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A non-trivial solution is

$$\mathrm{2NaOH + H_2SO_4 \rightarrow Na_2SO_4 + 2H_2O}$$

Example. *Section 2.2, exercise 14. Determine all values of a such that the resulting linear system has (a) no solution; (b) a unique solution; (c) infinitely many solutions:*

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + \left(a^2 - 5\right) z &= a \end{aligned}$$

Example. *Section 2.2, exercise 27. Find an equation relating a, b, and c so that the linear system*

$$\begin{aligned} 2x + 2y + 3z &= a \\ 3x - y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

- The relationship between homogeneous and non-homogeneous linear systems. If $A\mathbf{x} = \mathbf{b}$ has a particular solution $\mathbf{x}_p$ and $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x}_h$, then $\mathbf{x}_h + \mathbf{x}_p$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

- Exercises 2.2: 1, 3, 5, 7, 10, 15, 20, 26, 30, 39.

2.3 Elementary Matrices; Finding A^{-1}

- An $m \times m$ matrix is called an **elementary matrix** if it can be obtained from the identity matrix I_m by means of a single elementary row operation.

  example. The following are elementary matrices

$$E_1 = (I_3)_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = (I_3)_{-2r_2 \to r_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = (I_3)_{2r_2 + r_1 \to r_1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = (I_3)_{3r_3 + r_1 \to r_3} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Theorem. Let A be an $m \times n$ matrix. Then

$$(A)_{\text{some row operation}} = (I_m)_{\text{same row operation}} A$$

  example.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{r_1 \leftrightarrow r_2} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (I_2)_{r_1 \leftrightarrow r_2} A$$

- Theorem. If A and B are $m \times n$ matrices, then A is row equivalent to B if and only if there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

- Theorem. An elementary matrix E is invertible, and its inverse is an elementary matrix of the same type. proof. This follows from the fact that every row operation can be inverted.

- Theorem. $A_{n\times n}$ is nonsingular if and only if A is a product of elementary matrices. proof. If $A = E_1 \cdots E_k$ then since E_i is invertible, $A^{-1} = E_k^{-1} \cdots E_1^{-1}$. Conversely, if A is nonsingular, then $A\mathbf{x} = \mathbf{0}$ has only zero solution since $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$. Then each column of RREF(A) must have a leading one, otherwise the system would have a non-trivial solution. Thus $RREF(A) = I_n$ and A is row equivalent to I_n. By the above theorem $A = E_k \cdots E_1 I_n$.

- Summary. For $A_{n\times n}$ the following are equivalent.

  1. A is nonsingular.

  2. $Ax = 0$ has only the trivial solution.

  3. A is row equivalent to I_n. (RREF of A is I_n)

  4. The linear system $Ax = b$ has a unique solution for every $b_{n\times 1}$. proof $x = A^{-1}b$.

  5. A is a product of elementary matrices.

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Show that A is singular. Solution Show that RREF of A

is not I_2.

- To find A^{-1} . Suppose A is nonsingular. Then there exists elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A = I_n$. This means $A^{-1} = E_k \cdots E_1$. From this observation, we get

$$\begin{aligned}(E_k E_{k-1} \cdots E_2 E_1)\,[A \mid I_n] &= [E_k E_{k-1} \cdots E_2 E_1 A \mid E_k E_{k-1} \cdots E_2 E_1 I_n] \\ &= \left[I_n \mid A^{-1}\right]\end{aligned}$$

Example. Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

if it exists. Solution. Write

$$\left[\; A \;\middle|\; I_3 \;\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

Use elementary row operations $-5\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3$, $\frac{1}{2}\mathbf{r}_2 \to \mathbf{r}_2$, $-\frac{1}{4}\mathbf{r}_3 \to \mathbf{r}_3$, $-\frac{3}{2}\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2$, $-\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1$, $-1\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1$ to bring it into the form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] = \left[\; I_3 \;\middle|\; A^{-1} \;\right]$$

- Theorem. $A_{n\times n}$ is singular, if and only if it is row equivalent to a matrix which has a row of zeros. proof. In this case $Ax = 0$ has a nontrivial solution and hence can not be invertible. <u>example</u>. Show that the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$$

is singular.

- <u>Theorem</u> If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$. Thus $B = A^{-1}$.

  <u>proof</u>. First, we will show that A is invertible. If not then $RREF(A) = C$ which has a row of zeros and $C = E_k \cdots E_1 A$. Hence $CB = E_k \cdots E_1 AB = E_k \cdot E_1$. Hence CB is invertible since it is a product of elementary matrices. But this is not possible since CB also has a row of zeros. Hence A^{-1} must exist. Then $B = A^{-1}AB = A^{-1}I = A^{-1}$. So $B = A^{-1}$.

  <u>Remark</u> By definition A and B are inverses of each other if $AB = I_n$ and $BA = I_n$. This theorem says that to be inverses of each other, it suffices check only one equation $AB = I_n$.

- <u>Exercises 2.3</u>: 2, 7, 8, 9, 11, 17, 19, 21

Week 5-6

3.1 Definition of Determinant

- Define $S_n = \{1, 2, \ldots, n\}$. A rearrangement (with no repetation of elements) $j_1 j_2 \ldots j_n$ of the elements of S_n is called a **permutation** of S_n. The set S_n has a total A permutation is said to have an **inversion** if a larger integer comes before than a smaller integer. If the total number of inversions is even, then the permutation is called **even** otherwise it is called **odd**.

  Example. *The permutation* 4132 *of* S_4 *has 4 inversions:* $4 > 1$, $4 > 3$, $4 > 2$, $3 > 2$ *so it is even.*

  Example. *The permutations of the set* $S_2 = \{1, 2\}$ *are* 12 *and* 21. *The permutation* 12 *is even (zero inversions), and the permutation* 21 *is odd (1 inversion).*

  Example. *The permutations of the set* $S_3 = \{1, 2, 3\}$ *are* 123, 231, 312 *which are even and* 132, 321, 213 *which are odd.*

  Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** of A is

  $$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

  where the summation is over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \ldots, n\}$. The sign is taken as $+$ or $-$ according to whether the permutation $j_1 j_2 \cdots j_n$ is even or odd.

Thus determinant of an $n \times n$ matrix is a sum of $n!$ terms each of which is a product of n terms. Each term in a product comes from a distinct row and column. That is no product contains terms from the same row or column.

- For a 2×2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example. *For*

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$$

$\det(A) = (2)(5) - (-3)(4) = 22.$

- For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we have

$$\begin{aligned} \det(A) =& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

Example. *The determinant of*

$$\begin{vmatrix} 0 & 3 & 7 \\ 5 & 6 & 5 \\ -1 & 5 & 5 \end{vmatrix}$$

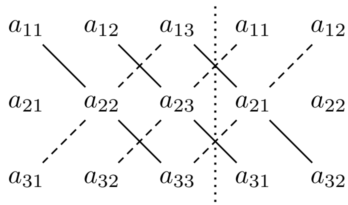


Figure 1: Sarrus rule to compute 3×3 determinants. This method does not work for 4×4 matrices.

is 127.

- The definition of determinant is not practical when n is as large as 10. It involves the sum of $10!$ terms each of which is a product of 10 terms. Around 3.7×10^7 operations.
- Exercises 3.1: 8, 11, 13

3.2 Properties of Determinant

$$\det(A) = \det\left(A^T\right).$$

Example. $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \left|A^T\right| = ad - bc$

> $\det(A_{r_i \leftrightarrow r_j}) = -\det(A)$ if $i \neq j$.

Example. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$

> If two rows (or columns) of a are equal then $\det(a) = 0$.

proof suppose $row_i = row_j$, $i \neq j$ then $a = a_{r_i \leftrightarrow r_j}$ so that $\det a = \det a_{r_i \leftrightarrow r_j} = -\det a$.

Example. $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$ *and* $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0$

> If a row (or column) of a consists entirely of zeros then $\det(a) = 0$.

proof. This follows from the fact that each summand in the determinant sum formula contains exactly one element from each row (or column).

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

> $\det(A_{kr_i \to r_i}) = k\det(A)$.

Example. $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\det(A_{r_i+kr_j\to r_i}) = \det(A)$, $i \neq j$.

Example. $\begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} = (a(kb+d) - b(ka+c) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 4$*, obtained by adding twice the second row to the first row.*

Example. *If* $\det(A) = 4$ *and* $B = A_{r_1+2r_2\to r_1}$ *then* $\det(B) = 4$*.*

If a row (or column) of a matrix is a multiple of another row then its determinant is zero.

proof. Since the row which is a multiple of the other can be made zero by the operation $r_i + kr_j \to r_i$ which gives a zero determinant.

Let A be an upper (or lower) triangular matrix. Then $\det(A)$ is equal to the product of the diagonal entries of A.

proof. Recall that determinant is a sum of product of n terms each coming from a different row and column. Thus for an upper diagonal matrix, only the terms including a_{11} can be non-zero. But for any product containing a_{11}, no other elements can be chosen from the first column. Thus the only products including $a_{11}a_{22}$ can be non-zero. Going this way, we obtain the conclusion.

Since a diagonal matrix is both lower and upper triangular, The determinant of a diagonal matrix is the product of elements on its diagonal.

Example.

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 3 \times 8 \times 1 = 24.$$

Example. *Compute*

$$\begin{vmatrix} 3 & 4 & 5 \\ 5 & 2 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

After $r_1 \leftrightarrow r_3$ *determinant changes by* -1 *and the matrix is in triangular form. The answer is* $-1 \times -1 \times 2 \times 5 = 10$.

Since $\det(A) = \det(A^T)$, column operations can be used too instead of row operations. That is

1. $\det(A) = -\det(A_{c_i \leftrightarrow c_j})$, $i \neq j$.
2. $\det(A_{kc_i \to c_i}) = k \det(A)$.
3. $\det(A_{c_i + kc_j \to c_i}) = \det(A)$, $i \neq j$.

Example. *Compute*

$$\begin{vmatrix} 3 & 6 & 5 \\ 4 & 8 & 0 \\ -1 & -2 & 1 \end{vmatrix}$$

Second column can be made zero by $c_2 - 2c_1 \to c_2$. *So the determinant is zero.*

Example. *Compute* $\det(A)$ *for*

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix}$$

After the operations

1. $1/2r_1 \to r_1$
2. $-2r_1 + r_3 \to r_3$,
3. $7r_2 + r_3 \to r_3$

$$\det(A) = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{vmatrix} = 2(1)(1)9 = 18$$

This method is called as the ***computation of determinant via reduction to triangular form****.*

Determinant of three types of elementary matrices:

- $\det(I_{r_i \leftrightarrow r_j}) = -1$,
- $\det\left(I_{kr_i \to r_i}\right) = k$,
- $\det\left(I_{kr_j + r_i \to r_i}\right) = 1$.

If E is an elementary matrix then $\det(EA) = \det(E)\det(A)$.

proof. Take $E = I_{r_i \leftrightarrow r_j}$ then EA is the matrix A with row i and j swapped. Thus $\det(EA) = -\det(A)$. On the other hand $\det(E)\det(A) = -\det(A)$ since $\det(E) = -1$. Thus $\det(EA) = \det(E)\det(A)$. The same is true for other types of elementary matrices.

If B is row equivalent to A then $\det(A)$ is non-zero if and only if $\det(B)$ is non-zero.

proof. Suppose B is row equivalent to A. We know $B = E_k E_{k-1} \cdots E_1 A$.

$$\begin{aligned}\det(B) &= \det(E_k E_{k-1} \cdots E_1 A) = \det(E_k)\det(E_{k-1} \cdots E_1 A) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)\end{aligned}$$

Theorem

$A_{n\times n}$ is nonsingular if and only if $\det(A) \neq 0$.

proof. If A is nonsingular then A is a product of elementary matrices and hence its determinant is non zero. If A is singular, then A is row equivalent to a matrix B that has a row of zeros. Hence its determinant is zero.

If A is $n \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det(A) = 0$.

Example. *Does the system*

$$\begin{aligned} x + 3y + 2z &= 0 \\ 2x - 2y &= 0 \\ 3x + 9y + 6z &= 0 \end{aligned}$$

have any non-trivial solution?

solution. Yes because

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -2 & 0 \\ 3 & 9 & 6 \end{pmatrix}$$

its determinant is zero since the 3rd row is a multiple of the first row.

Example.

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

have a non-trivial solution. solution. No because the determinant of the matrix is 10. This system has only the solution $(x_1, x_2) = (0, 0)$.

Example. *Let A be a 4×4 matrix with det $(A) = -2$*

1. *Describe the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. answer. since det $(A) \neq 0$, the homogeneous system has only the trivial solution.*

2. *If A is transformed to reduced row echelon form B, what is B? answer. I_n.*

3. *Can the linear system $A\mathbf{x} = \mathbf{b}$ have more than one solution? Explain. answer. no the system has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

4. *Does A^{-1} exist? answer. Yes.*

Theorem

If A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.

proof. If A is nonsingular then $A = E_k \cdots E_1$, product of elementary matrices.

$$\det(AB) = \det(E_k \cdots E_1 B) = \det(E_k) \cdots \det(E_1)\det(B) = \det(A)\det(B)$$

If A is singular then A is row equivalent to C which has a row of zeros. Thus

$$\det(AB) = \det(E_k \cdots E_1 CB) = \det(E_k) \cdots \det(E_1)\det(CB)$$

Notice that CB has a row of zeros and $\det(CB) = 0$.
Example. *If* $\det(A) = 5$, $\det(B) = -3$ *then* $\det(AB) = -15$ *and* $\det(A^2) = 25$.

Theorem

If A is nonsingular then $\det(A^{-1}) = \frac{1}{\det(A)}$.

proof. $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$.

In general (most usually) $\det(A + B) \neq \det(A) + \det(B)$.

Exercises 3.2: 1-5, 8, 9, 10, 13, 14, 15, 17, 22, 26, 30, 31.

3.3 Cofactor Expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column. $A_{ij} = (-1)^{i+j} \det(M_{ij})$ is called the **cofactor** A_{ij} of a_{ij}.

Example

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -2 & 0 \\ 3 & 9 & 6 \end{pmatrix}$$

Find the cofactor A_{22}.
answer.

$$A_{22} = (-1)^{2+2} \times \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$$

Determinant as cofactor expansion theorem

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then **expansion of** $\det(A)$ **along the ith row** is

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

and the expansion of $\det(A)$ along the jth column is

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Let us verify that the statement for the 3×3 case and for expansion along the first row.

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}
\end{aligned}$$

Best way to compute a matrix determinant is to expand it along a row/column with most zeros.

Example

Use cofactor expansion method to compute the determinant.

$$\begin{vmatrix} 4 & 3 & 1 \\ 0 & -2 & 0 \\ 5 & -3 & 6 \end{vmatrix}$$

Solution. We expand along the second row.

$$(-1)^{2+1} \times 0 + (-1)^{2+2}(-2) \begin{vmatrix} 4 & 1 \\ 5 & 6 \end{vmatrix} + (-1)^{2+3} \times 0 = -38$$

Check that expansion along a different row or column gives the same result.

$$\begin{vmatrix} 4 & 8 & -9 & 10 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & 0 & 0 \end{vmatrix} = 4 \times (-1)^{1+1} \begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 5 & 0 & 0 \end{vmatrix} = 4 \times 5 \times (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -40$$

In the first determinant, we expand along the first column. In the second determinant we expand along third row.

We can combine row operations with cofactor method to evaluate determinants.

Example. *Find all values of t for which*

$$\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0$$

Solution. Expansion along the third row gives

$$(-1)^{3+3}(t+1)((t-1)(t+2)-0) = 0$$

gives $t = -2, -1, 1$.

Exercises 3.3. 3, 11, 12

3.4 Inverse of a Matrix

If $A = [a_{ij}]$is an $n \times n$ matrix then

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \begin{cases} \det(A) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where A_{kj} is the cofactor of a_{kj}.

proof. If B is a matrix obtained from A by replacing the kth row of A by its ith row then B has two identical rows and $\det(B) = 0$. On the other hand $\det(B) = a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn}$.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & 5 & -2 \end{bmatrix}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = 19$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -14$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 3$$

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = (1)(19) + (2)(-14) + (3)(3) = 0$$

$$a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = (-2)(19) + (3)(-14) + (1)(3) = -77 = \det A$$

$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = (4)(19) + (5)(-14) + (-2)(3) = 0$$

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **adjoint** of A is the $n \times n$ matrix

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T,$$

where A_{ij} is the cofactor of a_{ji}. (Be careful about the transpose)

Theorem

If A is $n \times n$ matrix

$$A(\text{adj}A) = (\text{adj}A)A = \det(A)I_n.$$

Thus if $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)}(\text{adj}A)$$

proof. This is a consequence of

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \begin{cases} \det(A) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where A_{kj} is the cofactor of a_{kj}. The left hand side is the product of kth row of $\mathrm{adj}(A)$ with ith row of A.

Example. *Let*

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{pmatrix}$$

Compute adj$A$. Verify $A(\text{adj}A) = (\text{adj}A)A = \det(A)I_n$.

answer.

$$\mathrm{adj}\, A = \begin{pmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{pmatrix}$$

Example. *Use the adjoint method to find the inverse of*

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution. The answer is

$$\frac{1}{24}\begin{pmatrix} 6 & 2 & -9 \\ 0 & -8 & 12 \\ 0 & 0 & 12 \end{pmatrix}$$

This method of inverting is much less efficient than the method given in Chapter 2: $[A \mid I_n] \to [I_n \mid A^{-1}]$.

Exercises 3.4: 2, 9, 10, 12

3.5 Other Applications of Determinants

Theorem. Cramer's Rule

Let

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n\end{aligned}$$

If $\det(A) \neq 0$, then the system has the unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_i is the matrix A with ith column replaced by $\mathbf{b}$,

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

proof. Since $\det A \neq 0$, A^{-1} exists. $A\mathbf{x} = \mathbf{b}$ implies

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A}\operatorname{adj}(A)\mathbf{b}$$

For example, suppose A is 3×3. Then

$$\begin{aligned}
x_1 &= \frac{1}{\det A}(A_{11}b_1 + A_{21}b_2 + A_{31}b_n) \\
&= \frac{1}{\det A}\left((-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} b_1 + \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} b_2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} b_3\right) \\
&= \frac{1}{\det A}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

This is exactly the determinant $\det(A_1)$ defined in the theorem.

Example. Cramer's Rule

Use Cramer's rule to solve

$$\begin{aligned}
-2x_1 + 3x_2 - x_3 &= 1 \\
x_1 + 2x_2 - x_3 &= 4 \\
-2x_1 - x_2 + x_3 &= -3
\end{aligned}$$

solution.

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$$

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2, \qquad x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3$$

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \frac{-8}{-2} = 4$$

Cramer's rule is applicable only when there are n equations in n unknowns and the coefficient matrix A is nonsingular. Otherwise we must use the Gaussian elimination or Gauss-Jordan reduction methods. Cramer's rule becomes computationally inefficient for $n \geq 4$.

Exercises from the book

- Exercises 3.5. 1, 3, 5.
- Chapter 3 Supplementary Exercises. 1, 2, 3, 4, 5, 8.
- Chapter 3 Quiz. 1, 2, 3, 4, 5.
- This is optional. Read section 3.6 for computational complexity of computing determinants with various methods, inverse matrix, etc.

Week 7

Example

Prove that for any a_i, b_i, c_1, d_1

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & 0 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{vmatrix} = 0$$

Example

Find

$$\begin{vmatrix} 0 & 1 & 2 & 0 \\ 1 & 3 & 4 & 5 \\ -2 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \end{vmatrix}$$

Example

Let B be the matrix obtained from A after the row operations $2\mathbf{r}_3 \to \mathbf{r}_3, \mathbf{r}_1 \leftrightarrow \mathbf{r}_2, 4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3$, and $-2\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4$ have been performed. If $\det(B) = 2$ find $\det(A)$.

Example

Let A, B, and C be 2×2 matrices with $\det(A) = 3$ $\det(B) = -2$, and $\det(C) = 4$. Compute $\det\left(6A^TBC^{-1}\right)$.

Example

Show that if $A^n = O$, the zero matrix, for some positive integer n then $\det(A) = 0$.

True or False

1. $\det(A + B) = \det(A) + \det(B)$.
2. $\det(A^{-1}B) = \frac{\det(B)}{\det(A)}$.
3. If $\det(A) = 0$ then A has at least two equal rows.
4. A is singular if and only if $\det(A) = 0$.
5. If B is the reduced row echelon form of A then $\det(A) = \det(B)$.
6. $\frac{1}{c}\det(cA) = \det(A)$.
7. $\det(AB^TA^{-1}) = \det(B)$.
8. $\det(AA^T) \geq 0$.

Example

Verify

$$\begin{vmatrix} a-b & 1 & a \\ b-c & 1 & b \\ c-a & 1 & c \end{vmatrix} = \begin{vmatrix} a & 1 & b \\ b & 1 & c \\ c & 1 & a \end{vmatrix}$$

Week 8

4.1 Vectors in the plane and in 3-space

1. 2-Vectors can be added and multiplied by scalars. There is a zero vector and can take the difference of vectors. Same holds for 3-vectors.

2. If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and c and d are real scalars, then the following properties are valid:

   (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

   (b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

   (c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

   (d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

   (e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

   (f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

   (g) $c(d\mathbf{u}) = (cd)\mathbf{u}$

   (h) $1\mathbf{u} = \mathbf{u}$

4.2 Vector Spaces

There are many structures satisfying the above conditions other then $\mathbb{R}^2$ and $\mathbb{R}^3$ equipped with vector addition and scalar multiplication.

1. Definition. A real vector space is a set V of elements on which we have two operations $\oplus$ and $\odot$ defined with the following properties:

   (a) If $\mathbf{u}$ and $\mathbf{v}$ are any elements in V, then $\mathbf{u} \oplus \mathbf{v}$ is in V . (We say that V is closed under the operation $\oplus$.)

      (1) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v}$ in V

(2) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V

(3) There exists an element 0 in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

(4) For each $\mathbf{u}$ in V there exists an element $-\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = -\mathbf{u} \oplus \mathbf{u} = \mathbf{0}$.

(b) If u is any element in V and c is any real number, then $c \odot u$ is in V (i.e. V is closed under the operation $\odot$)

(5) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$ in V and any real number c.

(6) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

(7) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d

(8) $1 \odot \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V

The elements of V are called **vectors**; the elements of the set of real numbers $\mathbb{R}$ are called **scalars**. The operation $\oplus$ is called **vector addition**; the operation $\odot$ is called **scalar multiplication**. The vector $\mathbf{0}$ in property (3) is called a **zero vector**. The vector $-\mathbf{u}$ in property (4) is called a **negative of u**. It can be shown that $\mathbf{0}$ and $-\mathbf{u}$ are unique. If we allow the scalars to be complex numbers, we obtain a **complex vector space**.

2. example. $\mathbb{R}^n$, the set of n-vectors with usual vector addition and scalar product is a real vector space for any integer $n \geq 1$.

3. example. More generally, the set of all $m \times n$ matrices with usual matrix addition and scalar product is a real vector space.

4. example. The set of all 2×2 matrices with trace equal to zero is a real vector space with usual matrix addition and scalar product.

5. example. The set P_n of all polynomials with degree $\leq n$ with polynomial addition and multiplication by scalar is a real vector space.

6. example. More generally, the set of all real-valued functions defined on $\mathbb{R}^1$ form a real vector space with usual function addition and product of functions with scalars.

7. example. Let V be the set of all real numbers with the operations $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} - \mathbf{v}$($\oplus$ is ordinary subtraction) and $c \odot \mathbf{u} = c\mathbf{u}$($\odot$ is ordinary multiplication). Is V a vector space? If it is not, which properties in definition fail to hold? answer. (2), (3), (4), (6). Thus V is not a vector space.

8. example. Let V be the set of all ordered triples of real numbers (x, y, z) with the operations $(x, y, z) \oplus (x', y', z') = (x', y + y', z + z')$; $c \odot (x, y, z) = (cx, cy, cz)$. We can readily verify that properties $(1), (3), (4)$, and (6) of Definition fail to hold.

9. example. Let V be the set of all integers; define $\oplus$ as ordinary addition and $\odot$ as ordinary multiplication. Here V is not a vector space, because if u is any nonzero vector in V and $c = \sqrt{3}$, then $c \odot \mathbf{u}$ is not in V. Thus (b) fails to hold.

10. The following are not in axioms of a vector space but they hold for any vector space.

Theorem 4.2 If V is a vector space, then

(a) $0 \odot \mathbf{u} = \mathbf{0}$ for any vector $\mathbf{u}$ in V

(b) $c \odot 0 = 0$ for any scalar c

(c) If $c \odot \mathbf{u} = \mathbf{0}$, then either $c = 0$ or $\mathbf{u} = \mathbf{0}$

(d) $(-1) \odot \mathbf{u} = -\mathbf{u}$ for any vector $\mathbf{u}$ in V

proof (a) $0 \odot \mathbf{u} = (0 + 0) \odot \mathbf{u} = 0 \odot \mathbf{u} + 0 \odot \mathbf{u}$ then subtract $0 \odot \mathbf{u}$ from both sides.

11. Exercises 4.2.: 1-4, 7-12, 16-18

4.3 Subspaces

1. Let V be a vector space and W a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a **subspace** of V .

2. Let V be a vector space with operations $\oplus$ and $\odot$ and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold: (a) If u and v are any vectors in W, then $u \oplus v$ is in W (b) If c is any real number and $\mathbf{u}$ is any vector in W, then $c \odot \mathbf{u}$ is in W

3. Every vector space has at least two subspaces, itself and the subspace $\{\mathbf{0}\}$ consisting only of the zero-vector, called the **zero subspace**.

4. P_2 the set of all polynomials of degree ≤ 2 is a subspace of the vector space of all polynomials P.

5. Is the set of all polynomials of degree exactly 2 a vector space? No! The sum of polynomials $x^2 + 1$ and $-x^2 + x$ is a polynomial of degree 1.

6. Which of the following subsets of R^2 with the usual operations of vector addition and scalar multiplication are subspaces?

   (a) The set of all vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x \geq 0$.

   (b) The set of all vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 0$

7. Is the set $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$, a subspace of $\mathbb{R}^3$ with usual operations?

8. Given two vectors v_1 and v_2 in a vector space V, the set $\{a_1v_1 + a_2v_2 : a_1, a_2 \in \mathbb{R}\}$ is a subspace of V. <u>Prove</u> that this set is closed with respect to vector addition and scalar multiplication.

9. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. A vector $\mathbf{v}$ in V is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \sum_{j=1}^{k} a_j\mathbf{v}_j$ for some real numbers $a_1, a_2, \ldots, a_k$.

10. In a previous example we showed that W, the set of all vectors in R^3 of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ where a and b are any real numbers, is a subspace of

R^3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.Then every vector in W is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since $a\mathbf{v}_1 + b\mathbf{v}_2 = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$

11. example Is the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

solution. Yes if we can find real numbers a_1, a_2, and a_3 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{v}$ which gives

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

which is equivalent to

$$\begin{aligned} a_1 + a_2 + a_3 &= 2 \\ 2a_1 + a_3 &= 1 \\ a_1 + 2a_2 &= 5 \end{aligned}$$

We can solve this system to find $a_1 = 1$, $a_2 = 2$ and $a_3 = -1$ so that $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$.

12. Let $A_{m\times n}$ be a matrix. Then the solution set $A\mathbf{x} = \mathbf{0}$ is a subset of $\mathbb{R}^n$. In fact this is a subspace of $\mathbb{R}^n$.

13. Exercises 4.3: 1-19, 23-35

Week 9

1 class hour was canceled due to national holiday.

4.4 Span

1. We have seen that the set of all linear combination of two vectors of a vector space V is a subspace of V. More generally we have:

   Theorem 4.4 The set of all possible linear combinations of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V is a subspace of V. This subspace is known as the **span** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

2. What is the span of vectors

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

   in the vector space $M_{2\times 3}$ of 2×3 matrices.

answer. It consists of all matrices that can be written as

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where a, b, c, and d are real numbers, that is $\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$

3. example. Let $S = \left\{t^2, t\right\}$ be a subset of the vector space P_2. Then span S is the subspace of all polynomials of the form $at^2 + bt$, where a and b are any real numbers.

4. example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Is

$$\mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$?

answer. The problem asks if we can find scalars a_1, a_2 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{v}$, that is

$$a_1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

The augmented form is

$$\left[\begin{array}{rr|r} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 3 & -7 \end{array}\right]$$

which has RREF

$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent with $a_1 = 2$, $a_2 = 3$. The answer is YES.

5. <u>example</u>. Do vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

span R^3?

<u>answer</u> Pick any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and determine whether there are constants a_1, a_2, a_3 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{v}$. This leads to the linear system

$$\begin{aligned} a_1 + a_2 + a_3 &= a \\ 2a_1 + a_3 &= b \\ a_1 + 2a_2 &= c \end{aligned}$$

which has solution

$$a_1 = \frac{-2a + 2b + c}{3}, \quad a_2 = \frac{a - b + c}{3}, \quad a_3 = \frac{4a - b - 2c}{3}$$

The answer is YES!

6. example. Let V be the vector space P_2. Let $\mathbf{v}_1 = t^2 + 2t + 1$ and $\mathbf{v}_2 = t^2 + 2$. Does $\{\mathbf{v}_1, \mathbf{v}_2\}$ span V?

   answer. No, see book example 9.

7. example. Suppose the RREF of augmented matrix of the equation $A\mathbf{x} = \mathbf{0}$ is

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}$$

   Find a set which spans the solution space of $A\mathbf{x} = \mathbf{0}$.

   answer. Note that the solution is

$$x_1 = -r - 2s, \quad x_2 = r, \quad x_3 = s, \quad x_4 = s$$

   so that the general solution is

$$\mathbf{x} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

   Hence the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the solution space.

8. 4.4 Exercises. 2-11

4.5 Linear Independence

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1. We have seen that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$$

is a subspace of R^3. Each of the following sets span W:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

S_2 is a more efficient spanning set since it has fewer elements.

2. We want to find a spanning set for a given vector space which contains the least number of elements. For this we make the following definition.

<u>definition</u>. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, \ldots, a_k$, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if, whenever $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ we have $a_1 = a_2 = \cdots = a_k = 0$

3. <u>example</u>. Are the following vectors linearly independent? $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

   <u>answer</u>. Write $a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, that is $\left[\begin{array}{ccr|c} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$
   The RREF is $\left[\begin{array}{ccr|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, which has a non-trivial solution $\begin{bmatrix} k \\ -2k \\ k \end{bmatrix}, \quad k \neq 0$ so the vectors are linearly dependent.

4. <u>example</u>. Are the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ linearly independent? <u>answer</u>. YES!

5. <u>example</u>. Are the vectors $\mathbf{v}_1 = t^2 + t + 2, \mathbf{v}_2 = 2t^2 + t$, and $\mathbf{v}_3 = 3t^2 + 2t + 2$ in P_2 linearly independent? <u>answer</u>. No since $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$

6. <u>theorem 4.5</u> Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors in R^n (R_n). Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $\det(A) \neq 0$

   <u>proof</u>. If S is linearly independent then RREF(A)=I_n and $\det(A) \neq 0$. Con-

verse also holds.

7. example. Is $S = \left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -1 \end{bmatrix} \right\}$ a linearly independent set of vec- tors in R^3?

   solution. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$ Since $\det(A) = 2$, S is linearly independent.

8. Let $S_1 \subset S_2$ be two subsets of a vector space. If S_1 is linearly dependent then so is S_2. If S_2 is linearly independent then so is S_1.

9. Geometric meaning of linear independence. Two non-zero vectors in R^2 are linearly dependent if they are parallel (one is a scalar multiple of the other) and independent if not. In R^3, three vectors are linearly dependent if one of the vectors is a linear combination of the other two. That means these three vectors lie in the same plane. And if they do no lie in the same plane then they are independent.

10. example. Let $V = R_3$ and also $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 & 2 & -1 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$. We find (verify) that $\mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$ so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3,$ and $\mathbf{v}_4$ are linearly dependent. We then have $\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3$.

11. Section 4.5 exercises 1-4, 11-17, 20-23

Week 10

1 class hour was canceled due to April 23rd holiday. 2 class hours was canceled due to May 1st holiday.

Week 11

4.6 Basis and Dimension

1. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis** for V if $(\mathrm{a})\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \operatorname{span} V$ and $(\mathrm{b})\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.
2. Remark. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for a vector space V, then they must be distinct and nonzero.
3. Standard basis of R^3: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Standard basis for R_3:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

   . Generalize this to R^n and R_n.
4. <u>example</u>. Show that the set $S = \left\{t^2+1, t-1, 2t+2\right\}$ is a basis for the vector space P_2.

solution. 1st step. Show that for each a, b, c we can find a_1, a_2, a_3 such that

$$at^2 + bt + c = a_1\left(t^2+1\right) + a_2(t-1) + a_3(2t+2)$$

This reduces to

$$\begin{aligned} a_1 &= a \\ a_2 + 2a_3 &= b \\ a_1 - a_2 + 2a_3 &= c \end{aligned}$$

Show that this system has a solution. 2nd step. Show that the vectors given are linearly independent. To show this, write $a_1\left(t^2+1\right)+a_2(t-1)+a_3(2t+2)=0$ and show that only solution is the trivial solution $a_1 = a_2 = a_3 = 0$. This reduces to

$$\begin{aligned} a_1 &= 0 \\ a_2 + 2a_3 &= 0 \\ a_1 - a_2 + 2a_3 &= 0 \end{aligned}$$

5. example. Find a basis for the subspace V of P_2, consisting of all vectors of the form $at^2 + bt + c$, where $c = a - b$.

   solution. Every vector in P_2 is of the form $at^2+bt+a-b$, or $a\left(t^2+1\right)+b(t-1)$. The vectors t^2+1 and $t-1$ span V. These vectors are linearly independent since neither one is a multiple of the other.

6. A vector space V is called finite-dimensional if there is a finite subset of V that is a basis for V. If there is no such finite subset of V, then V is called infinite-dimensional.

7. Theorem 4.8 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

8. Theorem 4.9 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in a vector space V and let $W = \operatorname{span} S$. Then some subset of S is a basis for W.

9. example. Find a basis for the span(S) where $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$, and $\mathbf{v}_5 = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}$. Find a subset of S that is a basis for R_3.

   solution. Step 1. Write $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 = \mathbf{0}$.

   Step 2.
   $$\begin{aligned} a_1 + a_3 + a_4 - a_5 &= 0 \\ a_2 + a_3 + 2a_4 + a_5 &= 0 \\ a_1 + a_2 + 2a_3 + a_4 - 2a_5 &= 0 \end{aligned}$$

   Step 3. RREF is $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$

   This has infinitely many solutions so the vectors are not linearly independent.

   Step 4. If $\mathbf{v}_3$ and $\mathbf{v}_5$ were not present the RREF would be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ and the system would be linearly independent.

10. We can have different basis for the same vector space. For example $B_1 =$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ are both basis for R^2. But it turns out that (see Corollary 4.1 in the book) the number of vectors in a basis set is always same for a given vector space. This number is called the **dimension** of the vector space.

11. The set $S = \left\{t^2, t, 1\right\}$ is a basis for P_2, so dim $P_2 = 3$.

12. Let V be the subspace of R_3 spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$. We find that S is linearly dependent, and $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ (verify). Thus $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ also spans V. Since S_1 is linearly independent (verify), we conclude that it is a basis for V. Hence $\dim V = 2$.

13. If vector space V has dimension n, then any subset of $m > n$ vectors must be linearly dependent.

14. If vector space V has dimension n, then any subset of $m < n$ vectors cannot span V.

15. 4.6 Exercises. 1-15, 19-24.

4.7 Homogeneous Systems

1. Let A be $m \times n$ matrix. Then the solution set of $Ax = 0$ is a subspace of R^n. This solution space is also called nullspace of A and its dimension is called nullity of A.

2. <u>example</u>. Suppose the RREF of $Ax = 0$ is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Then the solution space is $\mathbf{x} = \begin{bmatrix} -2s - t \\ -2s + t \\ s \\ -2t \\ t \end{bmatrix}$ where s, t are any real numbers. Let $\mathbf{x}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for the solution space of A with dimension 2. The nullity of A is 2.

3. <u>example</u>. Find a basis for the solution space of the homogeneous system $(\lambda I_3 - A)\,\mathbf{x} = \mathbf{0}$ for $\lambda = -2$ and $A = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

<u>solution</u>. We form $-2I_3 - A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. RREF is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. $\begin{bmatrix} -1 \\ \frac{2}{3} \\ 1 \end{bmatrix}$ is a basis for the solution space.

4. Warning The solution set of a non-homogeneous equation $Ax = b$, $b \neq 0$ is not a vector space. Any solution of a non-homogeneous equation can be written as a sum of a particular solution and the solution of the associated homogeneous equation.

5. 4.7 Exercises. 1-20.

6.1 Linear Transformations and Matrices. Definition and Examples

1. Let V and W be vector spaces. A function $L : V \to W$ is called a **linear transformation** of V into W if
(a) $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for every $\mathbf{u}$ and $\mathbf{v}$ in V
(b) $L(c\mathbf{u}) = cL(\mathbf{u})$ for any $\mathbf{u}$ in V, and c any real number.

2. Let A be an $m \times n$ matrix. Then $L : R^n \to R^m$ defined by $L(\mathbf{u}) = A\mathbf{u}$ is a linear transformation.

3. example. Let $L : R^3 \to R^3$ be defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \end{bmatrix}$. Is L a linear transformation?

   answer. No because in general $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$.

4. example. Let $L : R_2 \to R_2$, $L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1^2 & 2u_2 \end{bmatrix}$. Is L a linear transformation?

   answer. No! Show that $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$.

5. example. Let $L : P_1 \to P_2$ be defined by $L[p(t)] = tp(t)$. Show that L is a linear transformation.

6. Let $L : V \to W$ be a linear transformation. Then
(a) $L(\mathbf{0}_V) = \mathbf{0}_{\mathrm{W}}$ where $\mathbf{0}_V$ is the zero vector in V and $\mathbf{0}_W$ is the zero vector in W.
(b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$, for $\mathbf{u}, \mathbf{v}$ in V

7. $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix}$ is not a linear transformation because
$L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

8. example. Let $L : R_2 \to R_2$ be a linear transformation for which we know that $L\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \end{bmatrix}$ and $L([-1 \quad 1]) = \begin{bmatrix} 2 & 3 \end{bmatrix}$. What is $L\left(\begin{bmatrix} -1 & 5 \end{bmatrix}\right)$? What is $L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right)$?

9. Let $L : R^n \to R^m$ be a linear transformation and consider the natural basis $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$. Let A be the $m \times n$ matrix whose j th column is $L(\mathbf{e}_j)$. A has the property $L(\mathbf{x}) = A\mathbf{x}$ for every $x \in R^n$. This matrix is called the **standard matrix representation** of L.

10. example. Find the standar matrix representation of L where $L\left(\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$. answer. $A = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & L(\mathbf{e}_3] \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$

11. example. Let $L : P_1 \to P_1$ be a linear transformation for which we know that $L(t+1) = 2t+3$ and $L(t-1) = 3t-2$. Find $L(6t-4)$. Find $L(at+b)$.

12. 6.1 Exercises: 1-4, 7-16, 20, 23

Week 12

7.1 Eigenvalues and Eigenvectors

1. Let A be an $n \times n$ matrix. If λ is a scalar (real or complex) and $\mathbf{x} \neq 0$ is a vector in R^n or (C^n) such that $A\mathbf{x} = \lambda\mathbf{x}$ then we say that λ is an **eigenvalue** of A and x is an **eigenvector** of A associated with A.

2. example 7.1.10.

3. definition 7.2

4. example 7.1.11.

5. example 7.1.12

6. example 7.1.13

7. exercises 7.1: 5-11