CSE2023 Discrete Computational Structures

Lecture 8

2. 1 Basic structures

- Sets
- Functions
- Sequences
- Sums

Sets

- Used to group objects together
- Objects of a set often have similar properties
  – all students enrolled at MU
  – all students currently taking discrete mathematics
- A **set** is an unordered collection of objects
- The objects in a set are called the **elements** or **members** of the set
- A set is said to **contain** its elements

Notation

- $a \in A$: a is an elemnet of the set A. $a \notin A$: otherwise
- The set of all vowels in the English alphabet can be written as V={a, e, i, o, u}
- The set of odd positive integers less than 10 can be expressed by O={1, 3, 5, 7, 9}
- Nothing prevents a set from having seemingly unrelated elements, {a, 2, Fred, New Jersey}
- The set of positive integers<100: {1,2,3,…, 99}

Notation

- **Set builder**: characterize the elements by stating the property or properties
- The set O of all odd positive integers < 10:
  O={x|x is an odd positive integer < 10}
  or specify as
  $$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$
- The set of positive rational numbers

$$Q^+ = \{x \in R \mid x = p/q \text{ for some positive integers } p \text{ and } q\}$$

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Notation

$N = \{1,2,3,...\}$ the set of natural numbers

$Z = \{...,-2, -1, 0, 1,...\}$ the set of integers

$Z^+ = \{1,2,3,...\}$ the set of positive integers

$\mathrm{Q} = \{p/q \mid p \in Z, q \in Z, \text{and } q \neq 0\}$ the set of rational numbers

R, the set of real numbers

- The set {N, Z, Q, R} is a set containing four elements, each of which is a set

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Sets and operations

- A **datatype** or **type** is the name of a set,
- Together with a set of operations that can be performed on objects from that set
- **Boolean**: the name of the set {0,1} together with operations on one or more elements of this set such as AND, OR, and NOT

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Sets

- Two sets are equal **if and only if** they have the same elements
- That is if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write A=B if A and B are equal sets
- The sets {1, 3, 5} and {3, 5, 1} are equal
- The sets {1, 3, 3, 3, 5, 5, 5, 5} is the same as {1, 3, 5} because the have the same elements

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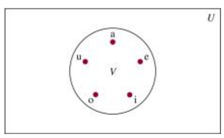
Venn diagram

- Rectangle: **Universal** set that contains all the objects
- Circle: sets
  - U: 26 letters of English alphabet
  - V: a set of vowels in the English alphabet



Empty set and singleton

- **Empty** (**null**) set: denoted by {} or Ø
- The set of positive integers that are greater than their squares is the null set
- **Singleton**: A set with one element
- A common mistake is to confuse Ø with {Ø}

Subset

- The set A is a subset of B **if and only if** every element of A is also an element of B
- Denote by $A \subseteq B$
- We see $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$

Empty set and the set S itself

- Theorem: or every set S
  - (i) $\emptyset \subseteq S$, and
  - (ii) $S \subseteq S$
- Let S be a set, to show $\emptyset \subseteq S$, we need to show $\forall x(x \in \emptyset \rightarrow x \in S)$ is true.
- But $x \in \emptyset$ is always false, and thus the conditional statement is always true
- An example of vacuous proof
- (ii) is left as an exercise

Proper subset

- A is a **proper subset** of B: Emphasize that A is a subset of B but that A≠B, and write it as A⊂B

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

- One way to show that two sets have the same elements is to show that each set is a subset of the other, i.e., if A⊆B and B⊆A, then A=B

$$\forall x(x \in A \leftrightarrow x \in B)$$

Sets have other sets as members

- A={∅, {a}, {b}, {a,b}}
- B={x|x is a subset of the set {a, b}}
- Note that A=B and {a}∈A but a∉A
- Sets are used extensively in computing problem

Cardinality

- Let S be a set. If there are exactly n distinct elements in S where n is a non-negative integer
- S is a **finite** set
- |S|=n, n is the **cardinality** of S
  - Let A be the set of odd positive integers < 10, |A|=5
  - Let S be the set of letters in English alphabet, |S|=26
  - The null set has no elements, thus |∅|=0

Infinite set and power set

- A set is said to be infinite if it is not finite
  - The set of positive integers is infinite
- Given a set S, the power set of S is the set of all subsets of the set S. The **power set** of S is denoted by P(S)
- The power set of {0,1,2}
  - P({0,1,2})={∅,{0},{1},{2},{0,1},{1,2},{0,2},{0,1,2}}
  - Note the empty set and set itself are members of this set of subsets

Example

- What is the power set of the empty set?
  - P(∅)={∅}
- The set {∅} has exactly two subsets, i.e., ∅, and the set {∅}. Thus P({∅})={∅,{∅}}
- If a set has n elements, then its power set has 2^n elements

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Cartesian product

- Sets are unordered
- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, and an as its nth element
- $(a_1, a_2, \ldots, a_n)= (b_1, b_2, \ldots, b_n)$ if and only if $a_i=b_i$ for i=1, 2, .., n

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Ordered pairs

- 2-tupels are called ordered pairs
- (a, b) and (c, d) are equal if and only if a=c and b=d
- Note that (a, b) and (b, a) are not equal unless a=b

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Cartesian product

- The **Cartesian product** of sets A and B, denoted by A x B, is the set of all ordered pairs (a,b), where a ∈ A and b ∈ B

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- A: students of MU, B: all courses offered at MU
- A x B consists of all ordered pairs of (a, b), i.e., all possible enrollments of students at MU

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Example

- A={1, 2}, B={a, b, c}, What is A x B?
  - A x B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
- A subset R of the Cartesian product A x B is called a **relation**
- A={a, b, c} and B={0, 1, 2, 3}, R={(a, 0), (a, 1), (a,3), (b, 1), (b, 2), (c, 0), (c,3)} is a relation from A to B
- A x B $\neq$ B x A
  - B x A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

Cartesian product: general case

- Cartesian product of A_1, A_2, …, A_n, is denoted by A_1 x A_2 x … x A_n is the set of ordered n-tuples (a_1, a_2, …, a_n) where ai belongs to A_i for i=1, 2, …, n

  $A_1 \times A_2 \times \cdots A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{for } i = 1,2,\ldots,n$
- A={0,1}, B={1,2}, C={0,1,2}

  A x B x C={{0, 1, 0},{0, 1, 1}, {0, 1, 2}, {0, 2, 0}, {0, 2, 1}, {0, 2, 2}, {1, 1, 0}, {1, 1, 1}, {1, 1, 2}, {1, 2, 0}, {1, 2, 1}, {1, 2, 2}}

Using set notation with quantifiers

- $\forall x \in S(P(x))$ denotes the universal quantification P(x) over all elements in the set S
- Shorthand for $\forall x(x \in S \to (P(x))$
- $\exists x S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$
- What do they man? $\forall x \in R(x^2 \geq 0), \exists x \in Z(x^2 = 1)$
  - The square of every real number is non-negative
  - There is an integer whose square is 1

Truth sets of quantifiers

- Predicate P, and a domain D, the truth set of P is the set of elements x in D for which P(x) is true, denote by {x ∈ D|P(x)}
- P(x) is |x|=1, Q(x) is x^2=2, and R(x) is |x|=x and the domain is the set of integers
  - Truth set of P, {x∈Z||x|=1}, i.e., the truth set of P is {-1,1}
  - Truth set of Q, {x∈Z|x^2=2}, i.e., the truth set is ∅
  - Truth set of R, {x∈Z||x|=x}, i.e., the truth set is N

Example

- $\forall x P(x)$ is true over the domain U if and only if P is the set U
- $\exists x P(x)$ is true over the domain U if and only if P is non-empty

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