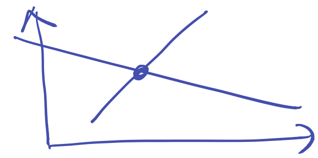
CH13 - Eigenvalues

Non homogeneous Eqⁿs:

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

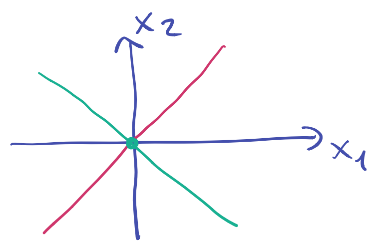


- If eqⁿs are independent $\Rightarrow$ a unique solution.
  (in 2D $\Rightarrow$ intersection of 2 straight lines)

Homogeneous system:

$$\underbrace{\underline{\underline{A}}\,\underline{x} = \underline{0}}$$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}\right\}$$



Two straight lines that intersect at zero.

Trivial solution : $\underline{x} = \underline{0}$

Eigenvalue Problems:

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x} \quad , \qquad \begin{aligned} &\lambda : \text{eigenvalue} \\ &\underline{x} : \text{eigenvector} \end{aligned}$$

$$\underline{\underline{A}}\,\underline{x} = \lambda \underbrace{\underline{\underline{I}}\,\underline{x}}_{\underline{x}}$$

identity matrix

$$\underline{\underline{A}}\,\underline{x} - \lambda\underline{\underline{I}}\,\underline{x} = \underline{0}$$

$$(\underline{\underline{A}} - \lambda\underline{\underline{I}})\,\underline{x} = \underline{0}$$

For nontrivial solution to exist, the determinant of the matrix must be zero:

$$|\underline{\underline{A}} - \lambda\underline{\underline{I}}| = 0$$

Expand the determinant $\Rightarrow$ get a polynomial in λ
(characteristic polynomial)

Two eq^n case:

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \begin{aligned} (a_{11}-\lambda)x_1 + x_2 &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 &= 0 \end{aligned}$$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21} = 0$$

$$\underbrace{\lambda_{1,2}}_{\text{2 solutions}} = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4a_{12}a_{21}}}{2}$$

__Ex__

$$(10-\lambda)x_1 - 5x_2 = 0 \qquad (1)$$
$$-5x_1 + (10-\lambda)x_2 = 0 \qquad (2)$$

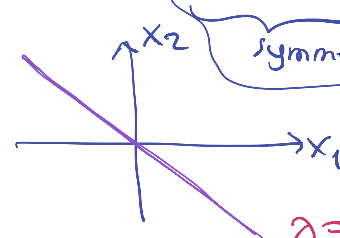
$$10x_1 - 5x_2 = 0$$
$$-5x_2 + 10x_2 = 0$$

$$A = \begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix}$$ symmetric

$$\begin{vmatrix} 10-\lambda & -5 \\ -5 & 10-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 20\lambda + 75 = 0$$

$\lambda_1 = 15$

$\lambda_2 = 5$



Put $\boxed{\lambda_1 = 15}$ into (1) and (2)

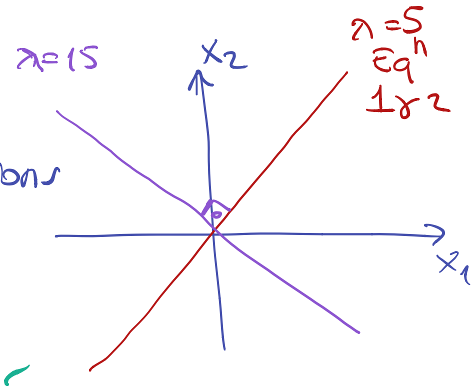
$$\left.\begin{aligned} -5x_1 - 5x_2 &= 0 \\ -5x_1 - 5x_2 &= 0. \end{aligned}\right\} \text{identical}$$

$\lambda = 15$
Eq^{n}s
1 and
2

infinite solutions: $x_2 = -x_1 \Rightarrow$ one solution: $\boxed{x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ eigenvector

Put $\lambda_2 = 5$ into (1) and (2):

$$\left.\begin{array}{l} 5x_1 - 5x_2 = 0 \\ -5x_1 + 5x_2 = 0 \end{array}\right\rangle \begin{array}{l}\text{identical} \\ \text{infinite solutions}\end{array}$$



$$x_2 = x_1$$

one solution : $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ↝ eigenvector

* Eigenvectors are orthogonal : a property of symmetric matrices with distinct eigenvalues.

* $Ax = \lambda x$
  information content of A is transformed into a scalar λ.