Chapter 3 APPLICATION of FIRST-ORDER EQUATIONS

1) Mechanics (Physics)
2) Electricity
3) Rate Problems (Chemistry)
4) Economics
5) Social problems
6) Geometrical Problems

Mechanics

Newton's 1st Law:

- If the body is in motion it stays in motion
- If the body is at rest it stays at rest

(Provided that No force acting on the object)

2nd Law:

The time rate of change of momentum of a body is proportional to the resultant force acting upon it, and is in the direction of this resultant force.

$$\frac{d}{dt}(mv) = \sum_{i=1}^{n} F_i$$

$$v\frac{dm}{dt} + m\frac{dv}{dt} = \sum_{i=1}^{n} F_i$$

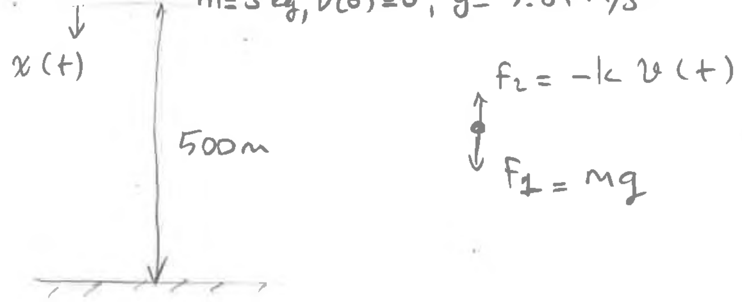
if m is constant then $dm/dt = 0$

$$m\frac{dv}{dt} = \sum_{i=1}^{n} F_i$$

$$\boxed{m \cdot a = \sum_{i=1}^{n} F_i}$$ vector equation $\vec{a}$, $\vec{F}$

ex: An object mass 3 kg is realeased from rest 500 m above the ground and allowed to fall under the influence of gravity. Assume that the gravitational force is constant with $g = 9.81\ m/s^2$, and force due to air resistance is proportional to the velocity of the object with proportionality constant $k = 3$ kg/s. Determine when the object will strike to the ground.

Soln: $m = 3\ kg$, $v(0) = 0$, $g = 9.81\ m/s^2$



Force due to gravity $\quad F_1 = mg$

force due to air resistance $\quad F_2 = -k v(t) \quad k > 0$

Total Force $\quad \Sigma f = f_1 + f_2 = mg - k v(t)$

Newtons 2nd Law

$$m \frac{dv}{dt} = mg - kv \qquad \text{I.C. } v(0) = 0$$

$$\left(\frac{dv}{dt} + \frac{k}{m} v = g \right) \quad \text{seperable}$$

$v(0) = 0$

$$\int \frac{dv}{mg - kv} = \int \frac{dt}{m}$$

$$-\frac{1}{k} \ln |mg - kv| = \frac{t}{m} + c$$

$$|mg - kv| = e^{-kc} \cdot e^{-kt/m}$$

or $mg - kv = A e^{-kt/m}$, $A = e^{-kc}$ (has same sign ($\pm$) as $(mg - kv)$)

solving for v

$$v = \frac{mg}{k} - \frac{A}{k} e^{-kt/m} \rightarrow \text{general sol}^{n}.$$

I.C $v(0) = v_0 \rightarrow v(0) = \frac{mg}{k} - \frac{A}{k} \rightarrow \frac{A}{k} = \frac{mg}{k} - v_0$

$$v = \frac{mg}{k} + \left(\cancel{v_0}^{0} - \frac{mg}{k}\right) e^{-kt/m} \quad \text{since } v(0) = v_0 = 0$$

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

$$x(t) = \int v(t)\,dt = \frac{mg}{k} t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right) . e^{-kt/m} + C$$

when $t = 0$, $x = 0$

$$0 = -\frac{m}{k}\left(v_0 - \frac{mg}{k}\right) + C \rightarrow C = \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)$$

Then $x(t) = \frac{mg}{k} t + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right)$

$m = 3$, $g = 9.81$, $k = 3$, $v_0 = 0$

$$x(t) = \frac{\cancel{3}(9.81)}{\cancel{3}} t - \frac{3^2 (9.81)}{\cancel{3^2}}\left(1 - e^{-t}\right)$$

$$500 = 9.81 t - (9.81)(1 - e^{-t})$$

$$t + e^{-t} = \frac{509.81}{9.81} \qquad \text{Neglect } e^{-t} \rightarrow \text{too small}$$

$$\underline{t = 51.97}$$

ORTHOGONAL & OBLIQUE TRAJECTORIES (ATLA) Go to 49

Orthogonal Trajectories:

Defn: Let $F(x,y;c)=0$ (A)

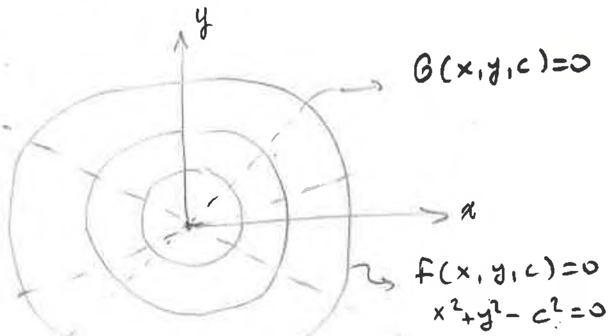
be a given one-parameter family of curves in the x-y plane. A curve that intersects the curves of the family (A) at right angels is called an orthogonal trajectory of the given family.

Ex: consider the family of circles

$x^2+y^2=c^2$ (center at origin & radius c)

Eeach straight line through the origin,

$y=kx$ is an orthogonal trajectory of the family of circles given by eqn (A). Conversely each circle is an orthogonal trajectory to the family of straight lines given by kx.



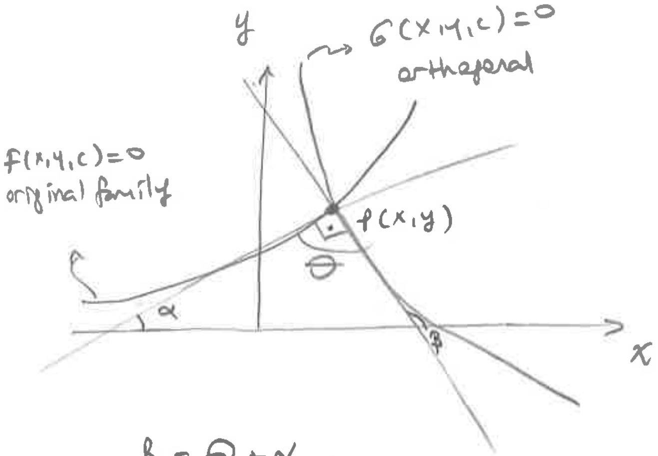
How to find orthogonal trajectories of curves

$F(x,y,c)=0$.

differentiate implicitly wrt. x.

$$\frac{dy}{dx}=f(x,y)$$

The curve c of the family (eqn. A) which passes through the the point (x,y) has the slope $f(x,y)$ there.



$\tan\alpha = y'_{original}$

$\tan\alpha = \frac{dy}{dx} = f(x,y)$

$\tan\beta = y'_{orthogonal} = -\frac{1}{f(x,y)}$

$\beta = \theta + \alpha$

$$\tan\theta = \tan(\beta - \theta)$$

$$= \frac{\tan\beta - \tan\alpha}{1 + \tan\beta * \tan\alpha}$$

$\theta = \frac{\pi}{2} \rightarrow \tan\theta \rightarrow \infty$

$1 + \tan\beta \cdot \tan\alpha = 0$

$1 + y'_{orth.} \cdot y'_{orig.} = 0 \rightarrow y'_{orth.} \cdot y'_{orig.} = -1$

$$\boxed{\tan\beta = y'_{orth} = \frac{-1}{y'_{org.}} = -\frac{1}{f(x,y)}} \quad \boxed{y'_{org.} = -\frac{1}{y'_{orth.}}}$$

Procedure for finding the Orthogonal trajectories of a given family of curves:

$$m_\alpha = y' = f(x,y) = \tan\alpha$$

$$\alpha = \tan^{-1}\{f(x,y)\}$$

$$\beta = \frac{\pi}{2} + \tan^{-1}\{f(x,y)\}$$

$$m_\beta = \tan\beta = \frac{\tan(\frac{\pi}{2}) + f(x,y)}{1 - \tan(\frac{\pi}{2}) \cdot f(x,y)} = -\frac{1}{f(x,y)} = y'$$

$$f(x,y) = -\frac{1}{y'}$$

<u>Step 1</u>: From the eqn

$$f(x,y,c)=0$$

find the DE

$\frac{dy}{dx} = f(x,y)$ of this family (Be sure to eliminate c)

<u>Step 2</u>: In the DE $\frac{dy}{dx} = f(x,y)$ so found in step 1, replace $f(x,y)$ by its negative reciprocal $-1/f(x,y)$, this gives the DE. $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ of the orthogonal trajectories.

<u>Step 3</u>: Obtain a one-parameter family

$G(x,y,c)=0$ or $y=f(x,c)$ ← orthogonal trajectory.

<u>ex</u>: Find the orthogonal trajectories of circle $x^2+y^2=c_1^2$

1) $x^2+y^2=c_1^2$ $\quad f(x,y,c_1)=0 \quad$ (A)

$x^2+y^2-c_1^2=0$ (the eqn of the original family)

2) Find DE of the original family

$$\left.\begin{aligned} x^2+y^2-c_1^2 &= 0 \\ 2x+2yy' &= 0 \end{aligned}\right\} \; y'_{orig} = -\frac{x}{y} \rightarrow \left.\frac{dy}{dx}\right|_{orig} = -\frac{x}{y} \quad (B)$$

3) Find the DE of the Orthogonal family

Replace y' by $\frac{-1}{y'}$ in (B) to get the DE of the Orth. family

$$-\frac{1}{y'_{orth.}} = -\frac{x}{y} \Rightarrow \boxed{y'_{orth.} = \frac{y}{x}} \quad \text{DE of ORTH. FAMILY (C)}$$

4) Solve (c) for $y = y(x)$ to get the eq$\underline{n}$ of orth. family

$$\frac{dy}{dx} = y'_{orth.} = \frac{y}{x} \quad \text{(seperable)}$$

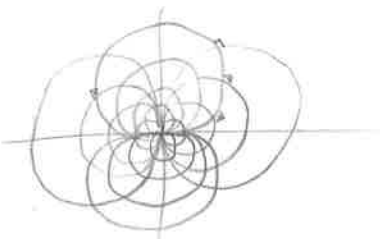
$$\int \frac{dy}{y} = \int \frac{dx}{x} \rightarrow \ln y = \ln x + \ln c_2 = \ln (x c_2)$$

$$y = C_2 x$$

67.6 + HW

ex: Find the orthogonal trajectories of $r = c_1 \cos \theta$

1) Find the DE of the original family



$$r = c_1 \cos \theta \quad \text{(orig. family)}$$

$$\frac{dr}{d\theta} = - c_1 \sin\theta$$

$$\frac{r}{dr/d\theta} = - \frac{\cancel{c_1}\cos\theta}{\cancel{c_1} \sin\theta} \rightarrow \frac{r}{r'} = - \frac{\cos\theta}{\sin\theta} \quad \text{(eq. 1)}$$

DE of orig. family.

2) Find the DE of ORTH. FAMILY

Replace $\left(\frac{r}{r'}\right)$ by $\left(-\frac{r'}{r}\right)$ in Eq. (1)

$$\left.\begin{aligned} -\frac{r'}{r} &= -\frac{\cos\theta}{\sin\theta} \\ \frac{r'}{r} &= \frac{\cos\theta}{\sin\theta} \end{aligned}\right\} \text{Eq}\underline{n}\text{ (2) the DE of the ORTH. FAMILY}$$

Step 3) Solve Eq$\underline{n}$ (2) for $r = r(\theta)$ to get the eq$\underline{n}$. of ORTH. FAMILY

$$\frac{r'}{r} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{\frac{dr}{d\theta}}{r} = \frac{\cos\theta}{\sin\theta} \quad \text{(seperable DE)}$$

$$\int \frac{dr}{r} = \int \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$\ln|r| = \ln|\sin\theta| + \ln C_2 = \ln|\sin\theta \cdot C_2|$$

$$\boxed{r = C_2 \sin\theta}$$ This is ORTH. TRAJECTORIES

RATE PROBLEMS (ANLAT)

a) Rate of growth and rate of decay

b) Mixture Problems

c) Population Problems

Rate Problems:

In certain problems the rate at which a quantity changes is a known fn of the amount present and/or the time, and it is desired to find the quantity itself. If x denotes the amount of the quantity present at time t, then dx/dt denotes the rate at which the quantity changes and we are at once led to a differential eqn.

OKU

ex : The rate at which radiactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radio-active nuclei have undergone disintegration in a period of 1500 years.

a) What percentage of the original radiactive nuclei remain after 4500 years.

b) In howmany years will only one-tenth of the original number remains?

1) Mathematical formulation:

Let $x(t)$ be the amount of the radiactive nuclei present after (t) years, $\frac{dx}{dt}$ = represents the rate at which the nuclei decay

$$\frac{dx}{dt} \propto x(t)$$

$$\frac{dx}{dt} = kx \quad , \quad \frac{dx}{dt} = \pm kx$$

growth

decay

$$\frac{dx}{dt} = -kx \quad \text{(DECAYING)}$$

$x(0) = x_0$ (initially the amount of the rad. act. nuclei)

$$x(1500) = \frac{x_0}{2}$$

Soln : solve $x(t) = ?$

$$\frac{dx}{dt} = -kx \quad \text{(seperable)}$$

$$\frac{dx}{x} = -\int k\,dt \rightarrow \ln|x| + \ln|c_1| = -kt$$

$$\ln|c_1 x| = -kt$$

$$xc_1 = e^{-kt}$$

$$x(t) = \frac{1}{c_1} e^{-kt} = c_2 e^{-kt}$$

$$x(0) = x_0 = C_2 e^0 = C_2 \rightarrow C_2 = x_0$$

$$x(t) = x_0 e^{-kt}$$

$$x(1500) = \frac{x_0}{2} = x_0 e^{-k.1500}$$

$$\left(\frac{1}{2}\right)^{1/1500} = \left(e^{-k1500}\right)^{1/1500}$$

$$\left[\left(\frac{1}{2}\right)^{1/1500}\right]^t = \left[e^{-k}\right]^t$$

$$\left(\frac{1}{2}\right)^{t/1500} = e^{-kt} \rightarrow x(t) = x_0 . \left(\frac{1}{2}\right)^{t/1500}$$

$$x(4500) = x_0 \left(\frac{1}{2}\right)^{4500/1500} = \frac{x_0}{2^3} \Rightarrow x(4500) = 0.125\, x_0$$

$$12.5\ \%$$

b) $$x(t) = \frac{x_0}{10} = x_0 \left(\frac{1}{2}\right)^{t/1500}$$

$$\frac{1}{10} = \left(\frac{1}{2}\right)^{t/1500} \rightarrow \ln\left(\frac{1}{10}\right) = \frac{t}{1500} \ln\left(\frac{1}{2}\right)$$

$$-\ln 10 = \frac{t}{1500}(-\ln 2)$$

$$t = 1500 \frac{\ln 10}{\ln 2} \rightarrow t \cong 4985 \text{ years}$$

Mixture Problems

A substance is allowed to flow into a certain mixture in a container at a certain rate, and the mixture is kept uniform by stirring.

Further in such situation, this uniform mixture simultaneously flows out of the container at another (generally different) rate another situation this may not be the case. In either case we seek to determine the quantity of the substance S present in the mixture at time t.

Let $x(t)$ denote amount of S present at time "t" the derivative $\left(\frac{dx}{dt}\right)$ denotes the rate of change of $x(t)$ with respect to "t".

$$\left(\frac{dx}{dt}\right) = \text{IN} - \text{OUT}$$

from which we determine the amount $x(t)$ of S at time "t".

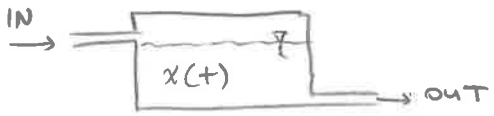
ex: A tank initially contains 50 lt of pure water. Starting at $t=0$ a brine containing 2 kg of dissolved salt per/lt flows into the tank at the rate of 3 lt/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the same rate.

a) How much salt is in the tank any time $t > 0$

b) How much salt is present at the end of 25 min.

c) How much salt is present after a long time.

Sol'n: Mathematical Model: Let x be amount of salt at time t.



concentration of salt

$$IN = 2\,\frac{kg}{\cancel{lt}}\;3\,\frac{\cancel{lt}}{min} = 6\ kg/min \quad , \quad OUT: \frac{x\,(kg)}{50\,(lt)}\;3\,\frac{lt}{min} = \frac{3x}{50}\ (kg/min)$$

$$\begin{cases} \dfrac{dx}{dt} = IN - OUT = 6 - \dfrac{3x}{50} \\ x(0) = 0 \rightarrow \text{pure water} \end{cases}$$

$$\frac{dx}{dt} = 6 - \frac{3x}{50} \rightarrow \frac{dx}{dt} + \underbrace{\frac{3}{50}}_{p(t)}x = \underbrace{6}_{Q(t)}$$

$$\mu = e^{\int p(t)\,dt} = e^{\int \frac{3}{50}dt} = e^{(3/50)t}$$

$$\int d\left[x e^{(3/50)t}\right] = 6\int e^{(3/50)t}\,dt$$

$$x e^{(3/50)t} = 6 \cdot \frac{50}{3} e^{(3/50)t} + c$$

$$x = 100 + c\,e^{(-3/50)t}$$

a) $x(0) = 0 \Rightarrow 0 = 100 + c \rightarrow c = -100$

$$x(t) = 100\,(1 - e^{(-3/50)t})$$

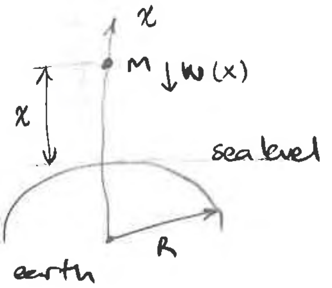
b) $x(25) = 100\,(1 - e^{(-3/50)\,25})$

$\cong 78$ kg

c) $x(\infty) \longrightarrow e^{(-3/50)\infty} \longrightarrow 0 \;\rightarrow\; \underline{x(\infty) = 100 \text{ kg}}$

Escape Velocity: (ATLA)

A body of mass (m) is projected upward from the earth's surface with an initial velocity of v_0. Assuming there is no air resistence but taking into consideration the variation of earth's gravitational force with altitude, let us find the slowest initial velocity for which the body will not return to earth. This is called escape velocity.



$$w(x) = \frac{\overset{\to\,\text{const.}}{k}}{(R+x)^2}$$ (from Newton's inverse square law)

$$f = \frac{\overbrace{M_1 M_2}^{\to\, k}}{R^2 \longrightarrow (R+x)^2}$$

$t = 0,\; v(0) = v_0,\; x = 0,\; v = v_0,\; w = mg$

At sea level $x = 0$ $\quad w = mg$

$$w(0) = mg = \frac{k}{(R+0)^2} = \frac{k}{R^2} \rightarrow k = mgR^2$$

$$w(x) = \frac{k}{(R+x)^2} = \frac{mgR^2}{(R+x)^2}$$

Newton's 2^{nd} law (m) is constant

$$-\frac{mgR^2}{(R+x)^2} = -w(x) = \sum f_i = ma = m\frac{dV}{dt} = m\frac{dV}{dx}\frac{dx}{dt} \to V$$

$$= mV\frac{dV}{dx}$$

$$\cancel{m}V\frac{dV}{dx} = -\frac{\cancel{m}gR^2}{(R+x)^2} \quad \text{(seperable)}$$

$(R+x) = u$

$dx = du$

$$\int V\,dV = -\int \frac{gR^2}{\underbrace{(R+x)^2}_{u}}\,dx$$

$$\frac{V^2}{2} = -\int gR^2\frac{du}{u^2}$$

$$\boxed{\frac{V^2}{2} = \frac{1}{(R+x)}gR^2 + C_1}$$

$x = 0, \quad V = V_0$

$$\frac{1}{2}V_0^2 = \frac{gR^2}{R} + C_1 \to C_1 = \frac{1}{2}V_0^2 - gR$$

$$\boxed{V^2 = \frac{2gR^2}{(R+x)} + V_0^2 - 2gR}$$

$V^2 > 0$ then $\frac{2gR^2}{(R+x)}$ already >0

so let $V_0^2 - 2gR \geq 0$

$$V_0^2 - 2gR = 0$$

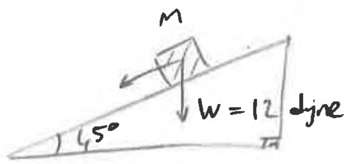
$$V_e = \sqrt{2gR} \cong 11500\ m/s$$

$\downarrow$
escape velocity

Remark: In calculating the escape velocity, the air resistance has been neglected. The actual escape velocity can be reduced if the body is transported some distance above sea level, before being fired.

ex: An object weighing 12 dyne is released from rest at top of a plane metal slide that is inclined 45° to the horizontal. Air resistance is numerically equal to $\frac{1}{3}$ of velocity and coefficient of friction is one-quarter.

what is the velocity of the object at any time



μ : friction coeff.

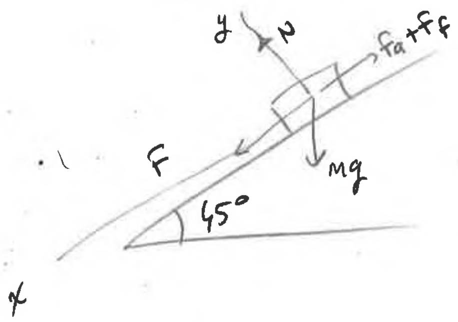
$\mu = \frac{1}{4}$

$(1 \text{ dyn} = \text{gr cm}/s^2)$

air resistance force : $\frac{1}{3} v$

$v(0) = 0, \; v(t) = ?$

$mg = 12$ dyne $\longrightarrow m = \frac{12}{g} = \frac{12}{980}$ gr



$F = mg \sin 45 = 12 \cdot \sin 45 = 8.46$ dyne

$N = mg \cos 45 = 12 \cdot \cos 45 = 8.46$ dyne

$f_f = N\mu = 8.46 * \frac{1}{4} = 2.12$ dyne

$f_a = \frac{1}{3} v$

$F_{net} = F - f_f - f_a = ma$

$$ma = m\frac{dv}{dt} = 8.46 - 2.12 - \frac{1}{3} v$$

$$\frac{12}{980} \cdot \frac{dv}{dt} = 6.34 - \frac{1}{3} v$$

$$\begin{cases} \frac{dv}{dt} + 27.2 v = 522.7 \\ v(0) = 0 \end{cases}$$

(1^{st} Ord. LDE)

$$\mu = e^{\int 27.2 dt} = e^{27.2t}$$

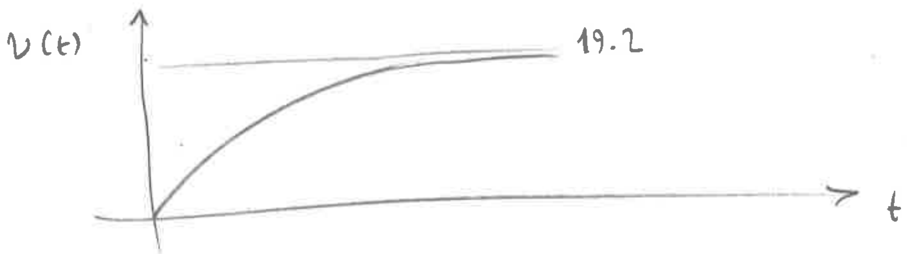
$$\int d\left[v\, e^{27.2t}\right] = \int 522.7\, e^{27.2t}\, dt$$

$$v\, e^{27.2t} = \frac{522.7}{27.2}\, e^{27.2t} + c$$

$$\boxed{v = 19.2 + c\, e^{-27t}}$$

$$v(0) = 0 = 19.2 + c\, e^{-27.2 * 0} \longrightarrow c = -19.2$$

Then $v(t) = 19.2\,(1 - e^{-27.2t})$



HW

Ex: According to population countings, population of Turkey in 1975 is 35 million, and in 1980 40 million; then determine that when the population of Turkey will become 100 million.

Soln: Population of Turkey

1975 $\longrightarrow$ 35 mil.

1980 $\longrightarrow$ 46 mil.

? $\longrightarrow$ 100 mil.

$N(t)$: the population of Turkey at any time (t)

$$\frac{dN}{dt} \propto N(t) \longrightarrow \frac{dN}{dt} = kN$$

$$\int \frac{dN}{N} = \int k\, dt$$

$N(0) = 35$ mil.

$N(5) = 40$ mil.

$N(t) = 100$ mil.

$$\ln N + \ln c_1 = kt$$

$$\ln(N c_1) = kt$$

$$N(t) = c_2 e^{kt}$$

$$N(0) = 35 = c_2 e^0 \Rightarrow c_2 = 35$$

$$N(t) = 35 e^{kt} \rightarrow N(5) = 35 e^{kt} = 40 \text{ mil.}$$

$$e^{5k} = \frac{40}{35} = \frac{8}{7}$$

$$e^{k} = \left(\frac{8}{7}\right)^{1/5} \Rightarrow e^{kt} = \left(\frac{8}{7}\right)^{t/5}$$

$$N(t) = 35 e^{kt} = 35 \left(8/7\right)^{t/5}$$

$$100 = 35 \left(8/7\right)^{t/5}$$

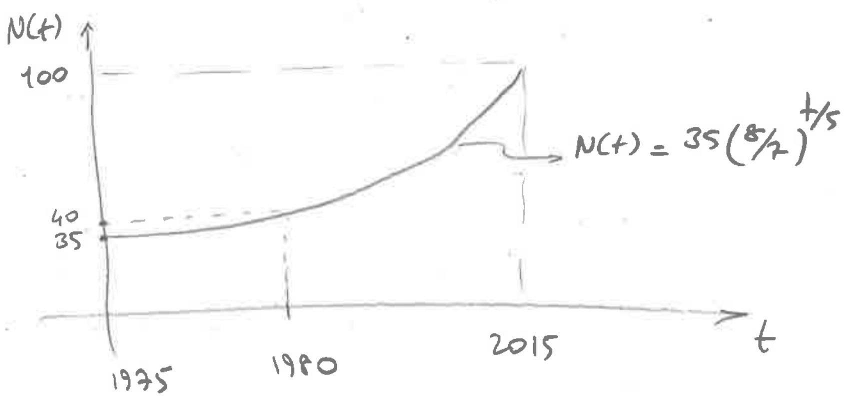
$$\frac{20}{7} = \left(\frac{8}{7}\right)^{t/5} \Rightarrow \frac{t}{5} \ln\left(\frac{8}{7}\right) = \ln\left(\frac{20}{7}\right)$$

$$\Rightarrow t = 5 \frac{\ln(20/7)}{\ln(8/7)} = 5 \cdot \frac{\ln 20 - \ln 7}{\ln 8 - \ln 7}$$

$$= 5 \cdot \frac{1.049}{0.134} \cong 39.24$$

$$1975 + 39.24 = 2014.30$$

100 million $\xrightarrow{\sim}$ 2015



Chapter 4

SOLVING HIGHER ORDER LINEAR DE.

Defn of Linearly Dependent DE.:

A set of fns $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be linearly dependent (LD) in an interval $[a,b]$, if there exists a set of "n" constants, NOT ALL ZERO (at least one of them is not zero) such that in this interval

$$\sum_{i=1}^{n} c_i f_i(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad (a \leq x \leq b)$$

otherwise the set is said to be linearly independent (LI)

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

$c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots + c_n f_n = 0$ implies if $c_1 = c_2 = c_3 = \cdots = c_n = 0$?

ex: if $f_1 = e^x$, $f_2 = e^{-2x}$ $\qquad c_1 f_1 + c_2 f_2 = 0$, $c_1 e^x + c_2 e^{-2x} = 0$ (LI) $c_1 = c_2 = 0$

ex: $f_1(x) = \sin x$, $f_2(x) = \cos x$ $\qquad c_1 f_1 + c_2 f_2 = 0$, $c_1 \sin x + c_2 \cos x = 0$ (LI) $\qquad$ it can be zero only if $(c_1 = c_2 = 0)$

ex: $f_1(x) = \sin x$, $f_2(x) = 3\sin x$ $\qquad c_1 f_1(x) + c_2 f_2(x) = 0$, $c_1 \sin x + c_2 3\sin x = 0$ (LD) $\qquad (c_1 = -3,\ c_2 = 1)$

Defn : If the set of fn's $f_1, f_2, \ldots, f_n$ are Linearly Dependent we can express one of the functions, linearly interms of the others. This is called Linearly Dependency. If the converse happens (if we can not express any number of the sets linearly interms of the others) are called linearly independent.