CSE2023 Discrete Computational Structures

Lecture 11

2.3 Inverse function

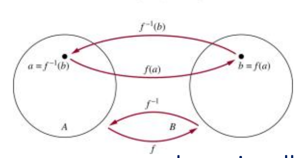
- Consider a one-to-one correspondence *f* from *A* to *B*
- Since *f* is onto, every element of B is the image of some element in A
- Since *f* is also one-to-one, every element of B is the image of a unique element of A
- Thus, we can define a new function from B to A that reverses the correspondence given by *f*

Inverse function

- Let *f* be a one-to-one correspondence from the set A to the set B
- The **inverse function** of *f* is the function that assigns an element b belonging to B the unique element a in A such that $f(a)=b$
- Denoted by f^{-1}, hence $f^{-1}(b)=a$ when $f(a)=b$
- Note f^{-1} is not the same as $1/f$

One-to-one correspondence and inverse function

- If a function *f* is not one-to-one correspondence, cannot define an inverse function of *f*



- A one-to-one correspondence is called **invertible**

Example

- *f* is a function from {a, b, c} to {1, 2, 3} with *f(a)=2, f(b)=3, f(c)=1*. Is it invertible? What is it its inverse?
- Let f: Z→Z such that *f(x)=x+1*, Is *f* invertible? If so, what is its inverse?

  $y=x+1$, $x=y-1$, $f^{-1}(y)=y-1$
- Let *f*: R→R with $f(x)=x^2$, Is it invertible?
  - Since f(2)=f(-2)=4, f is not one-to-one, and so not invertible

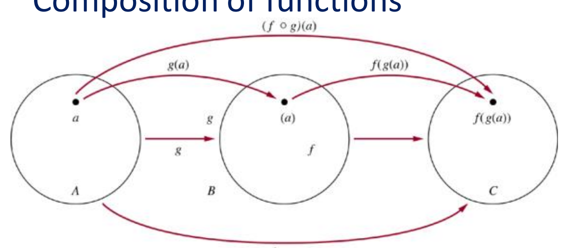
Example

- Sometimes we restrict the domain or the codomain of a function or both, to have an invertible function
- The function $f(x)=x^2$, from R^+ to R^+ is
  - one-to-one : If *f(x)=f(y),* then $x^2=y^2$, then x+y=0 or x-y=0, so x=-y or x=y
  - onto: $y= x^2$, every non-negative real number has a square root
  - inverse function: $f^{-1}(y)=\sqrt{y}$

Composition of functions

- Let *g* be a function from A to B and *f* be a function from B to C, the composition of the functions *f* and *g*, denoted by $f \circ g$, is defined by $(f \circ g)(a)=f(g(a))$
  - First apply *g* to a to obtain *g(a)*
  - Then apply *f* to *g(a)* to obtain $(f \circ g)(a)=f(g(a))$

Composition of functions



- Note $f \circ g$ cannot be defined unless the range of *g* is a subset of the domain of *f*

Example

- *g*: {a, b, c} → {a, b, c}, *g(a)=b, g(b)=c, g(c)=a,* and *f*:{a,b,c} →{1,2,3}, *f(a)=3, f(b)=2, f(c)=1.* What are $f \circ g$ and $g \circ f$?
  - $(f \circ g)(a) = f(g(a)) = f(b) = 2,$
  - $(f \circ g)(b) = f(g(b)) = f(c) = 1,$
  - $(f \circ g)(c) = f(a) = 3$
- $(g \circ f)(a)=g(f(a))=g(3)$ not defined. $g \circ f$ is not defined

Example

- $f(x)=2x+3$, $g(x)=3x+2$. What are $f \circ g$ and $g \circ f$?
- $(f \circ g)(x)=f(g(x))=f(3x+2)=2(3x+2)+3=6x+7$
- $(g \circ f)(x)=g(f(x))=g(2x+3)=3(2x+3)+2=6x+11$
- Note that $f \circ g$ and $g \circ f$ are defined in this example, but they are not equal
- The commutative law does not hold for composition of functions

f and f^{-1}

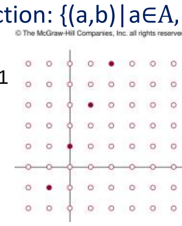
- f and f^{-1} form an identity function in any order
- Let f: A →B with f(a)=b
- Suppose f is one-to-one correspondence from A to B
- Then f^{-1} is one-to-one correspondence from B to A
- The inverse function reverses the correspondence of f, so $f^{-1}(b)=a$ when $f(a)=b$, and $f(a)=b$ when $f^{-1}(b)=a$
- $(f^{-1} \circ f)(a)=f^{-1}(f(a))=f^{-1}(b)=a$, and
- $(f \circ f^{-1})(b)=f(f^{-1}(b))=f(a)=b$

$$f^{-1} \circ f = \iota_A,\ f \circ f^{-1} = \iota_B,\ \iota_A, \iota_B \text{ are identity functions for A and B}$$
$$(f^{-1})^{-1} = f$$

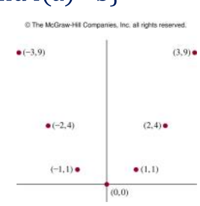
Graphs of functions

- Associate a set of pairs in A x B to each function from A to B
- The set of pairs is called the graph of the function: {(a,b)|a∈A, b ∈ B, and f(a)=b}

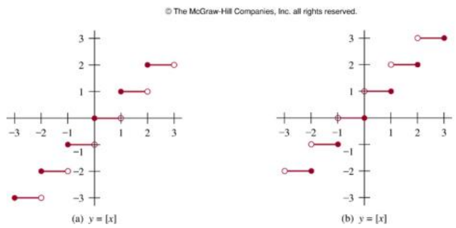
f(x)=2x+1



f(x)=x^2



Example



$$\text{floor}: y = \lfloor x \rfloor \qquad \text{ceiling}: y = \lceil x \rceil$$

2.4 Sequences

- Ordered list of elements
  - e.g., 1, 2, 3, 5, 8 is a sequence with 5 elements
  - 1, 3, 9, 27, 81, …, 30, …, is an infinite sequence
- **Sequence** $\{a_n\}$: a function from a subset of the set of integers (usually either the set of {0, 1, 2, …} or the set {1, 2, 3, …}) to a set S
- Use a_n to denote the image of the integer n
- Call a_n a **term** of the sequence

Sequences

- Example: $\{a_n\}$ where $a_n=1/n$
  - $a_1, a_2, a_3, a_4, \ldots$
  - 1, ½, 1/3, ¼,…
- When the elements of an infinite set can be listed, the set is called **countable**
- Will show that the set of positive rational numbers is countable, but the set of real numbers is not

Geometric progression

- **Geometric progression**: a sequence of the form
  $a, ar, ar^2, ar^3, \ldots, ar^n$
  where the initial term a and common ratio r are real numbers
- Can be written as $f(x)=a \cdot r^x$
- The sequences $\{b_n\}$ with $b_n=(-1)^n$, $\{c_n\}$ with $c_n=2\cdot 5^n$, $\{d_n\}$ with $d_n=6 \cdot (1/3)^n$ are geometric progression
  - b_n : 1, -1, 1, -1, 1, …
  - c_n: 2, 10, 50, 250, 1250, …
  - d_n: 6, 2, 2/3, 2/9, 2/27, …

Arithmetic progression

- **Arithmetic progression**: a sequence of the form
  a, a+d, a+2d, …, a+nd
  where the initial term a and the common difference d are real numbers
- Can be written as $f(x)=a+dx$
- $\{s_n\}$ with $s_n=-1+4n$, $\{t_n\}$ with $t_n=7-3n$
  - $\{s_n\}$: -1, 3, 7, 11, …
  - $\{t_n\}$: 7, 4, 1, 02, …

String

- Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science
- These finite sequences are also called **strings**
- The length of the string S is the number of terms
- The empty string, denoted by **λ**, is the string has no terms

Recurrence relation

- Express a_n in terms of one or more of the previous terms of the sequence
- Example: $a_n=a_{n-1}+3$ for n=1,2,3,… and $a_1=2$
  - $a_2=a_1+3=2+3=5$, $a_3=a_2+3=(2+3)+3=2+3x2=8$, $a_4=a_3+3=(2+3+3)+3=2+3+3+3=2+3x3=11$
  - $a_n=2+3(n-1)$
  - $a_n=a_{n-1}+3=(a_{n-2}+3)+3=a_{n-2}+3x2$
    $=(a_{n-3}+3)+3x2=a_{n-3}+3x3$
    $=a_2+3(n-2)=(a_1+3)+3(n-2)=2+3(n-1)$

Fibonacci sequence

- $f_0=0$, $f_1=1$, $\mathbf{f_n=f_{n-1}+f_{n-2}}$, for n=2, 3, 4
  - $f_2=f_1+f_0=1+0=1$
  - $f_3=f_2+f_1=1+1=2$
  - $f_4=f_3+f_2=2+1=3$
  - $f_5=f_4+f_3=3+2=5$
  - $f_6=f_5+f_4=5+3=8$

Closed formula

- Determine whether the sequence $\{a_n\}$, $a_n=3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n=2a_{n-1}-a_{n-2}$ for n=2,3,4,
  - For n>=2, $a_n=2a_{n-1}-a_{n-2}=2(3(n-1))-3(n-2)=3n=a_n$
- Suppose $a_n=2^n$, Note that $a_0=1$, $a_1=2$, $a_2=4$, but $2a_1-a_0=2\times2-1=3 \neq a_2$, thus $a_n=2^n$ is not a solution of the recurrence relation

Special integer sequences

- Finding some patterns among the terms
- Are terms obtained from previous terms
  - by adding the same amount or an amount depends on the position in the sequence?
  - by multiplying a particular amount?
  - By combining previous terms in a certain way?
  - In some cycle?

Summations

- The sum of terms: a_m, a_{m+1}, …, a_n from $\{a_n\}$

$$\sum_{j=m}^{n} a_j,\ \sum\nolimits_{j=m}^{n} a_j,\ \text{or } \sum\nolimits_{m\le j\le n} a_j$$

that represents $a_m + a_{m+1} + \ldots + a_n$

  - Here j is the index of summation (can be replaced arbitrarily by i or k)
  - The index runs from the lower limit m to upper limit n
  - The usual laws for arithmetic applies

$$\sum\nolimits_{j=1}^{n}(ax_j+by_j)=a\sum\nolimits_{j=1}^{n}x_j+b\sum\nolimits_{j=1}^{n}y_j \text{ where } a,b \text{ are real numbers}$$

Example

- Express the sum of the first 100 terms of the sequence $\{a_n\}$ where $a_n=1/n$, n=1, 2, 3, …

$$\sum\nolimits_{j=1}^{100}\frac{1}{j}$$

- What is the value of $\sum\nolimits_{k=1}^{5}k^2$

$$\sum\nolimits_{k=1}^{5}k^2=1^2+2^2+3^2+4^2+5^2=1+4+9+16+25=55$$

- What is the value of $\sum\nolimits_{k=4}^{8}(-1)^k$

$$\sum\nolimits_{k=4}^{8}(-1)^k=(-1)^4+(-1)^5+(-1)^6+(-1)^7+(-1)^8=1+(-1)+1+(-1)+1=1$$

- Shift index:

$$\sum_{j=1}^{5}j^2=\sum_{k=0}^{4}(k+1)^2 \text{ by setting } j=k+1, \text{or } k=j-1$$

Geometric series

- **Geometric series**: sums of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

$$\begin{aligned} S &= \sum_{j=0}^{n} ar^j \\ rS &= \sum_{j=0}^{n} ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a) \\ &= S + (ar^{n+1} - a) \\ S &= \frac{ar^{n+1} - a}{r-1} \end{aligned}$$

Double summations

- Often used in programs

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i = 6 + 12 + 18 + 24 = 60$$

- Can also write summation to add values of a function of a set

$$\sum_{s \in S} f(s)$$

$$\sum_{s \in \{0,2,4\}} s = \sum 0 + 2 + 4 = 6$$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Example

- Find $\sum_{k=50}^{100} k^2$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338350 - 40425 = 297925$$

- Let x be a real number with |x|<1, Find $\sum_{n=0}^{\infty} x^n$

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}, \quad \sum_{n=0}^{k} x^n = \frac{x^{k+1}-1}{x-1}, \quad \sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \frac{x^{k+1}-1}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$$

- Differentiating both sides of $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$

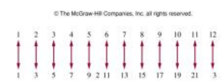
$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

2.5 Cardinality

- The sets A and B have the same cardinality, |A|=|B|, if and only if there is a one-to-one correspondence from A to B
- **Countable**: A set that is *either* **finite** *or* has the same cardinality as the set of **positive integers**
- A set that is not countable is called **uncountable**
- When an infinite set S is countable, we denote the cardinality of S by $\mathbb{N}_0$, i.e., $|S| = \mathbb{N}_0$

Example

- Is the set of odd positive integers countable?
  - f(n)=2n-1 from Z^+ to the set of odd positive integers
  - One-to-one: suppose that f(n)=f(m) then 2n-1=2m-1, so n=m
  - Onto: suppose t is an odd positive integer, then t is 1 less than an even integer 2k where k is a natural number. Hence t=2k-1=f(k)



Infinite set

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence
- The reason being that a one-to-one correspondence f from the set of **positive integers** to a set S can be expressed by

  $a_1, a_2, \ldots, a_n, \ldots$ where $a_1=f(1), a_2=f(2), \ldots a_n=f(n)$
- For instance, the set of odd integers, $a_n=2n-1$

Example

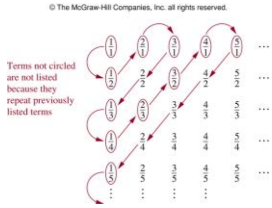
- Show the set of all integers is countable
- We can list all integers in a sequence by 0, 1, -1, 2, -2, …
- Or f(n)=n/2 when n is even and f(n)=-(n-1)/2 when n is odd (n=1, 2, 3, …)

Example

- Is the set of positive rational numbers countable?
- Every positive rational number is p/q
- First consider p+q=2, then p+q=3, p+q=4, …

1, ½, 2,3,1/3,
¼, 2/3, 3/2, 4, 5,…

Because all positive rational numbers are listed once, the set is countable



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Example

- Is the set of real numbers uncountable?
- Proof by contradiction
- Suppose the set is <u>countable</u>, then <u>the subset of all real numbers that fall between 0 and 1 would be countable</u> (as any subset of a countable set is also countable)
- The real numbers can then be listed in some order, say, r_1, r_2, r_3, …

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Example

- So

$$\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\cdots, \text{ where } d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\} \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\cdots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\cdots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\cdots \\
r &= 0.23794101\text{...(for example)}
\end{aligned}$$

- Form a new real number with

$$r = 0.d_1d_2d_3d_4\cdots$$

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

$$\begin{aligned}
r_1 &= 0.23794102\cdots \\
r_2 &= 0.44590138\cdots \\
r_3 &= 0.09118764\cdots \\
r_4 &= 0.80553900\cdots \\
r &= 0.4544...
\end{aligned}$$

- Every real number has a unique decimal expansion
- The real number r is not equal to r_1, r_2, … as its decimal expansion of r_i in the i-th place differs from others
- So there is a real number between 0 and 1 that is not in the list
- So the assumption that all real numbers can between 0 and 1 can be listed must be false
- So all the real numbers between 0 and 1 cannot be listed
- The set of real numbers between 0 and 1 is uncountable

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