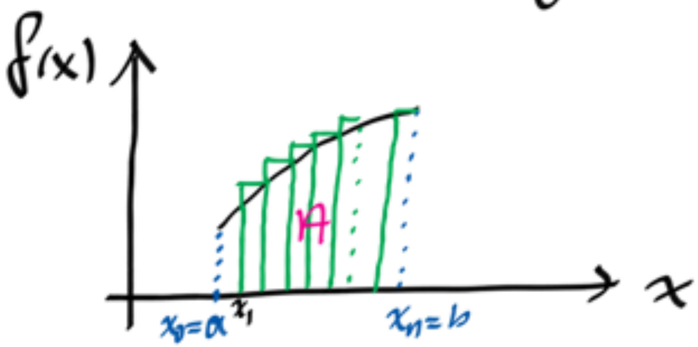
Ch 5 The Area Problems

$f(x) \geq 0$ and continuous on $[a,b]$

$A \equiv$ Area under graph of f, above the x-axis, from $x=a$ to $x=b$.



We can approximate area A by an upper (lower) sum.

Take a partition $a = x_0 < x_1 < x_2 \cdots < x_n = b$

Let $\Delta_i x = x_i - x_{i-1}$

Let $M_i = \max f$ on $[x_{i-1}, x_i]$

$$\left.\begin{array}{l} \Delta_i x = x_i - x_{i-1} \\ M_i = \max f \text{ on } [x_{i-1}, x_i] \end{array}\right\} S_n = \sum_{i=1}^{n} M_i \Delta_i x$$

, is an upper sum

Similarly, let $m_i = \min f$ on $[x_{i-1}, x_i] \Rightarrow s_n = \sum_{i=1}^{n} m_i \Delta_i x$, is a lower sum

$s_n \leq A \leq S_n$; $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = A$, by Sandwitch Thm.

Sigma Notation

Ex

$$\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2$$

$$\sum_{k=1}^{6} (-1)^k = -1+1-1+1-1+1 = 0$$

$$1+3+5+7 = \sum_{k=0}^{3}(2k+1)$$

Ex

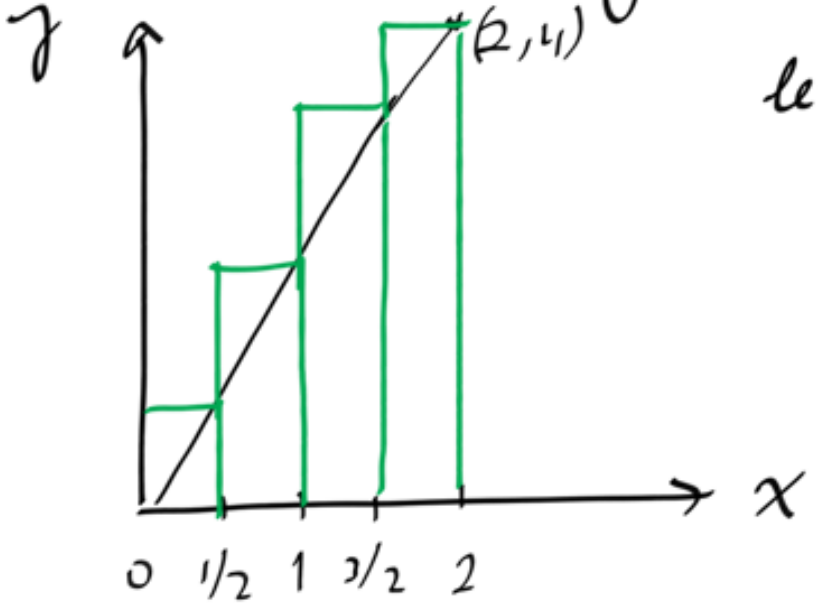
Show that $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

$$
\begin{array}{rl}
 & S = 1+2+3+\cdots+n \\
+ & S = n+(n-1)+(n-2)+\cdots+1 \\
\hline
 & 2S = n.(n+1)
\end{array}
$$

$$S_n = \frac{n(n+1)}{2}$$

Ex

Find the upper and lower sums for A, where A is the area under $y=2x$ on $[0,2]$.



let $n=4$, $\Delta x = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$

<u>lower sum</u>: $\sum_{i=1}^{n} m_i \Delta_i x$

$$S_4 = 2\cdot 0\cdot\frac{1}{2} + 2\cdot\frac{1}{2}\cdot\frac{1}{2} + 2\cdot 1\cdot\frac{1}{2} + 2\cdot\frac{3}{2}\cdot\frac{1}{2}$$

$$S_n = \frac{1}{2}(0+1+2+3) = 3$$

Upper sum:

$$\overline{S}_4 = 2\cdot\frac{1}{2}\cdot\frac{1}{2} + 2\cdot 1\cdot\frac{1}{2} + 2\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 2\cdot\frac{1}{2}$$

$$= \frac{1}{2}(1+2+3+4) = 5$$

Actually, the real area is $A = \frac{1}{2}\cdot 2\cdot 4 = 4$ //

Ex Find A by using an upper sum. Subdivide into n pieces, $y=2x$ again on $[0,2]$.

$$\Delta_i x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_0 = 0,\ x_1 = \underbrace{\Delta_i x}_{2/n},\ x_2 = 2\Delta_i x,\ \dots,\ x_n = \cancel{n}\underbrace{\Delta_i x}_{2/\cancel{n}} = 2$$

$$S_n = \Delta_i x\,[2x_1 + 2x_2 + \cdots + 2x_n]$$

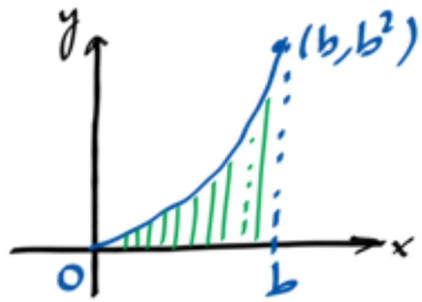
$$= 2\Delta_i x\,[\underbrace{x_1}_{\Delta_i x} + \underbrace{x_2}_{2\Delta_i x} + \cdots + \underbrace{x_n}_{n\Delta_i x}]$$

$$= 2(\Delta_i x)^2\,[1+2+3+\cdots+n]$$

$$= \cancel{2}\cdot\left(\frac{2}{n}\right)^2\left[\frac{n(n+1)}{\cancel{2}}\right] = 4\left(\frac{n+1}{n}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 4\left(\frac{n+1}{n}\right) = 4\underbrace{\lim_{n\to\infty}\frac{n+1}{n}}_{1} = 4$$

Ex Find A by using an upper sum, subdividing into n pieces, for $y = x^2$ on $[0, b]$.



$$\Delta_i x = \frac{b-0}{n} = \frac{b}{n}$$

$$x_0 = 0$$
$$x_1 = \Delta x$$
$$x_2 = 2\Delta x$$
$$\vdots$$
$$x_n = n\overset{b/n}{\Delta x} = b$$

$$S_n = \Delta x\left[(\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \dots + (n\Delta x)^2\right]$$

$$= (\Delta x)^3\left[1^2 + 2^2 + 3^2 + \dots + n^2\right]$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \left(\frac{b}{n}\right)^3\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} S_n = b^3 \underbrace{\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}}_{1/3} = \frac{b^3}{3} = A$$