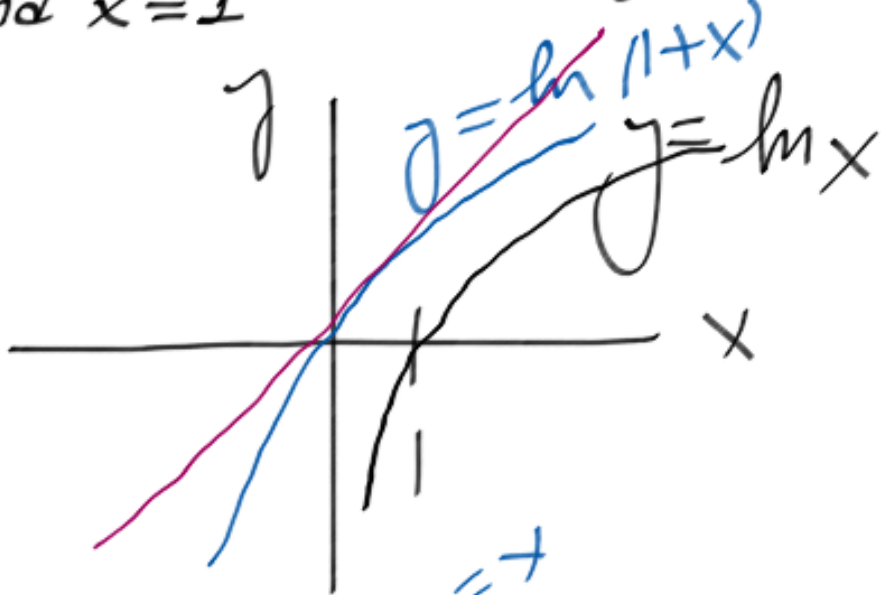
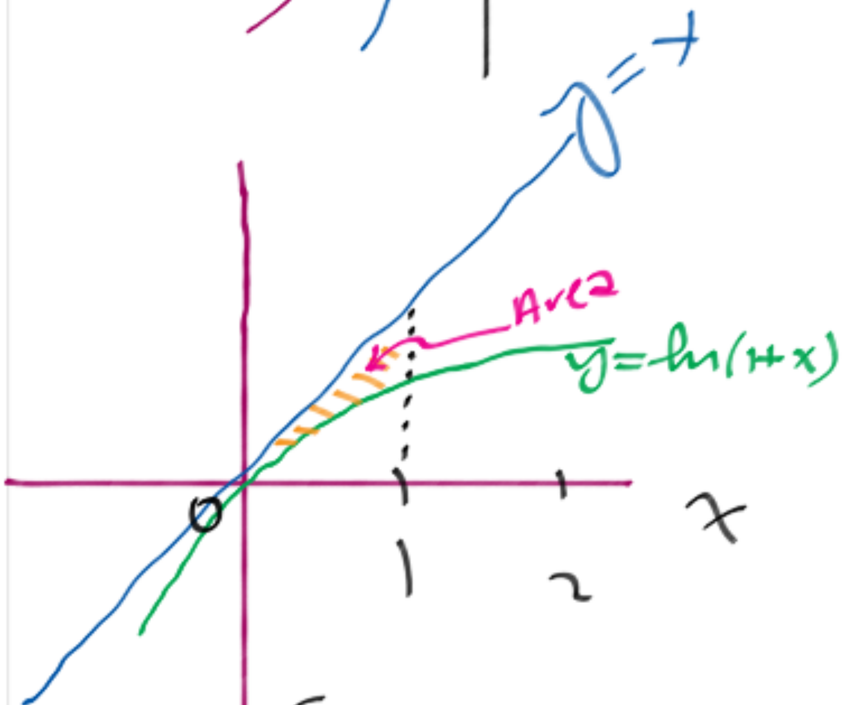
Ex. Find the area of region bounded by $y=\ln(1+x)$, $y=x$ and $x=1$



$$\ln(1+\underset{0}{x}) = \ln 1 = 0$$



$$\text{Area} = \int_0^1 (x - \ln(1+x))\,dx$$

$$= \int x\,dx - \int \underbrace{\ln(1+x)}_{u}\,\underbrace{dx}_{dv}$$

$$= \left.\frac{x^2}{2}\right|_0^1 \qquad du = \frac{dx}{1+x} \qquad v = x$$

$$\int \ln(1+x)\,dx = uv - \int v\,du = x\ln(1+x) - \int x\,\frac{dx}{1+x}$$

$$= \frac{x^2}{2}\Big|_0^1 - \left\{ x\ln(1+x) - \int_0^1 \frac{x}{1+x}\,dx \right\}$$

$$= \frac{x^2}{2} - \left\{ x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx \right\}$$

$$= \frac{x^2}{2} - x\ln(1+x) + x - \ln(1+x) \Big|_0^1$$

$$= \frac{3}{2} - 2\ln 2$$

$$\begin{array}{r|l} x & 1+x \\ \cancel{\mp x \mp 1} & 1 \\ \hline -1 & \end{array}$$

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}$$

Sec 8.2. Trigonometric Integrals

Odd powers of sines and cosines

Ex $\int \cos^5 x \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$$= -\int u^5\, du = -\frac{1}{6}u^6 + c$$

$$= -\frac{1}{6}\cos^6 x + c'$$

Ex $\int \sin^3 x \cos^2 x \, dx =$

$$\int \sin^2 x \sin x \cos^2 x \, dx =$$

$$\int (1-\cos^2 x) \cos^2 x \sin x \, dx =$$

$u = \cos x$, $du = -\sin x \, dx$

$$-\int (1-u^2) u^2 \, du = -\int (u^2 - u^4) \, du$$

$$= -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C$$

$$= \frac{-1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C$$

More generally, $\int \sin^n x \cos^m x \, dx$,

where one exponent is odd (say n)

use $u = \cos x$ etc.

Even Powers of sines and cosines

Use half angle formulas.

Ex $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx =$

$$\int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1+2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}}) dx$$

$$= \frac{x}{4} + \frac{\sin 2x}{2} + \frac{x}{8} + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C$$

Ex

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \int \frac{(1-\sin^2 x)\cos x\,dx}{\sqrt{\sin x}}$$

$u = \sin x \,, \quad du = \cos x\,dx$

$$= \int \frac{(1-u^2)}{\sqrt{u}}\,du = \int \left(u^{-\frac{1}{2}} - u^{-\frac{1}{2}+2}\right) du$$

$$= \int \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$

$$= 2\sqrt{\sin x} - \frac{2}{5}\sin^{5/2} x + C$$

Ex $$\int \frac{\tan^3\theta}{\sec\theta}\,d\theta = \int \frac{\sin^3\theta}{\cos^{\cancel{3}\,2}\theta}\cdot \cancel{\cos\theta}\,d\theta$$

$1/\cos\theta$

$$= \int \frac{\sin^2\theta\,\sin\theta}{\cos^2\theta}\,d\theta = \int \frac{(1-\cos^2\theta)\sin\theta\,d\theta}{\cos^2\theta}$$

$u=\cos\theta,\ du=-\sin\theta\,d\theta$

$$= -\int \frac{(1-u^2)}{u^2}\,du = \int (1-u^{-2})\,du$$

$$= u + \frac{1}{u} + C$$

$u=\cos\theta$

$\frac{1}{u}=\sec\theta$

$$= \cos\theta + \sec\theta + C$$

Products of Sines and Cosines

$$\int \sin mx \sin nx\, dx, \quad \int \sin mx \cos nx\, dx,$$

$$\int \cos mx \sin nx\, dx$$

$$\sin(A+B) = \sin A \cos B + \cancel{\cos A \sin B}$$

$$+ \; \sin(A-B) = \sin A \cos B - \cancel{\cos A \sin B}$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Ex

$$\int \sin 2x \cos 3x \, dx \quad ; \quad \begin{array}{l} A = 2x \\ B = 3x \end{array}$$

$$= \frac{1}{2} \int \left(\sin 5x + \sin(-x) \right) dx$$

$$= \frac{1}{2} \int \sin 5x \, dx - \frac{1}{2} \int \sin x \, dx$$

$$= -\frac{1}{2} \cdot \frac{1}{5} \cos 5x + \frac{1}{2} \cos x + C$$

Integrals involving tan and sec

$$\int \sec^n x \tan^m x\,dx$$

n even: keep one $\sec^2 x$

Ex $\int \sec^4 x \tan x\,dx = \int \underbrace{\sec^2 x}_{1+\tan^2 x} \sec^2 x \tan x\,dx$

$$= \int \tan x \sec^2 x\,dx + \int \tan^3 x \sec^2 x\,dx$$

$$= \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$$

$u = \tan x$

$du = \sec^2 x\,dx$

m odd: keep one $\sec x \tan x$

Ex

$$\int \sec x \tan^3 x \, dx$$

$$= \int \sec x \tan x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx$$

$$= \int (\sec^2 x - 1) \sec x \tan x \, dx$$

$u = \sec x$, $du = \sec x \tan x$

$$= \int (u^2 - 1) \, du = \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\sec^3 x - \sec x + C$$