Ex

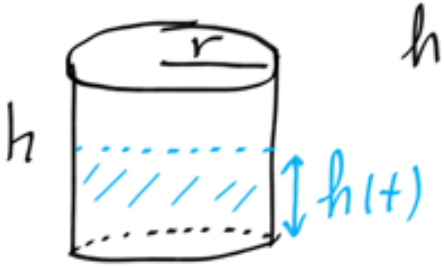
3.8 Related Rates

Water is poured into a cylindrical jar of radius 3 cm so that at time t (minute) the height of the water is t^0.5 cm. (i.e., $\sqrt{t}$)

(a) What is the average rate of change of the volume of water over $0 \le t \le 1$?
(a) What is the instantaneous rate of change of the volume when t=1 min ?

$r = 3$ cm = const.

$h = \sqrt{t}$



$V = \pi r^2 h$

(a) $V(t) = \pi r^2 h(t)$

$V(t) = \pi \cdot 3^2 \cdot \sqrt{t} = 9\pi\sqrt{t}$

$$\text{Average rate} = \frac{\Delta V}{\Delta t} = \frac{V(1) - V(0)}{1 - 0}$$

$V(0) = 0$, $V(1) = 9\pi\sqrt{1} = 9\pi$

$$\text{Av. rate} = \frac{9\pi - 0}{1 - 0} = 9\pi \text{ cm}^3/\text{m}$$

(b) $V = \pi r^2 h$

$h = \sqrt{t} = t^{1/2}$

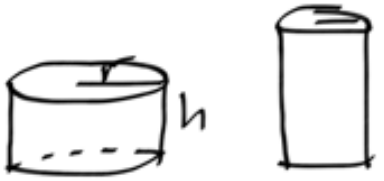
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$$

$$\frac{dV}{dt} = \pi r^2 \frac{1}{2\sqrt{t}} = \frac{\pi r^2}{2\sqrt{t}} = \frac{\pi \cdot 3^2}{2\sqrt{1}} = \frac{9}{2}\pi \text{ cm}^3/\text{m}$$

Ex

A cylinder is being compressed from the side and streched vertically so that the radius decreases at the rate of 2 cm/s and the height increases at the rate of 5 cm/s. Find the rate at which the volume is changing when r=6 cm and h=8 cm. Is the volume increasing or decreasing?



$\frac{dr}{dt} = -2$ cm/s $\quad \frac{dh}{dt} = +5$ cm/s

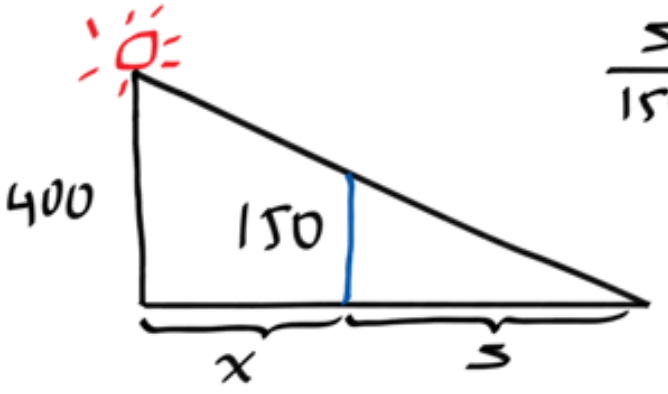
$V = \pi r^2 h$

$$\frac{dV}{dt} = \pi h 2r \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$= \pi \cdot 8 \cdot 2 \cdot 6(-2) + \pi (6)^2 \cdot 5 = -12\pi \text{ cm}^3/\text{s}, \text{ decreasing.}$$

Ex

A light shines on top of a lamppost 4 m above the ground. A woman 150 cm tall walks away from the light. (a) Find the rate at which her shadow is increasing if she walks away at 120 cm/s. (b) Find the rate at which her shadow is increasing if she walks toward the lamppost at 120 cm/s.



$\frac{s}{150} = \frac{x+s}{400}$ (from similar triangles)

$\frac{400}{150} = \frac{x+s}{s} \Rightarrow \frac{8}{3} = \frac{x+s}{s}$

$\frac{8}{3}s - s = x \Rightarrow \frac{5}{3}s = x \Rightarrow s = \frac{3}{5}x$

$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt}$, $\frac{dx}{dt} = +120$ cm/s

(a)

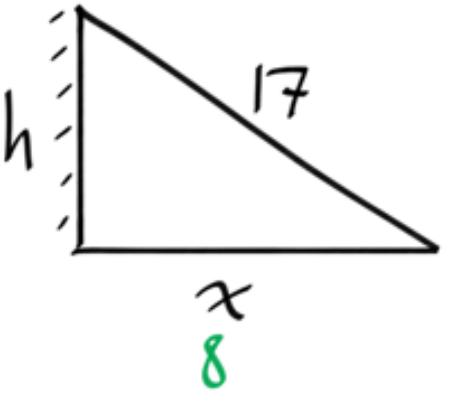
$\frac{ds}{dt} = \frac{3}{5} \cdot 120 = 3 \times 24 = 72$ cm/s

(b) $\frac{dx}{dt} = -120$ cm/s

$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt} = \frac{3}{5}(-120) = -72$ cm/s

Ex

A ladder 17 m long leans against a vertical wall. The lower end of the ladder is pulled away from the foot of the wall at 3 m/s. How fast is the top descending when the lower end is 8 m from the wall?



$x^2 + h^2 = 17^2$ $\quad \frac{dx}{dt} = +3 \text{ m/s}, \quad \frac{dh}{dt} = ?$

$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

$x\frac{dx}{dt} + h\frac{dh}{dt} = 0$

$8 \cdot 3 + 15\frac{dh}{dt} = 0$

$\frac{dh}{dt} = -\frac{8}{5} \text{ m/s}$

$x^2 + h^2 = 17^2$

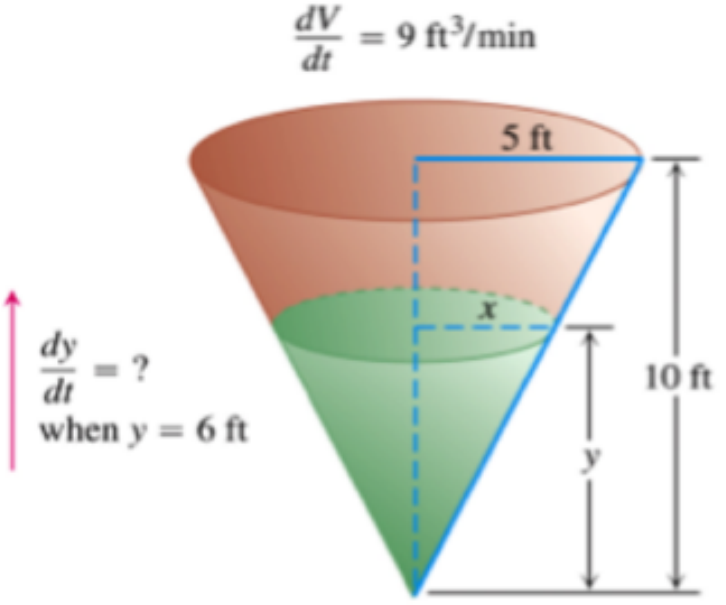
$8^2 + h^2 = 17^2$

$h^2 = 17^2 - 8^2$

$h^2 = (17-8)(17+8)$

$h = 15$

EXAMPLE Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



$$V = \frac{1}{3}\pi x^2 y$$

Again from similar triangles:

$$\frac{10}{5} = \frac{y}{x} \Rightarrow x = y/2$$

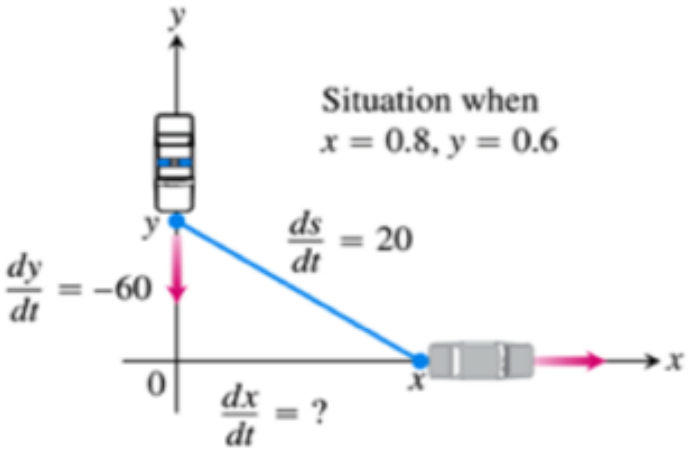
$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y$$

$$V = \frac{1}{12}\pi y^3$$

$$\frac{dV}{dt} = \frac{1}{12}\pi 3y^2 \frac{dy}{dt}$$

$$9 = \frac{1}{4}\pi (6)^2 \frac{dy}{dt} \rightarrow \frac{dy}{dt} = \quad ft/m$$

EXAMPLE A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



$$s^2 = x^2 + y^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\frac{ds}{dt} = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\frac{ds}{dt} = 20,\ \frac{dy}{dt} = -60,\ x = 0.8,\ y = 0.6,\ \frac{dx}{dt} = ?$$