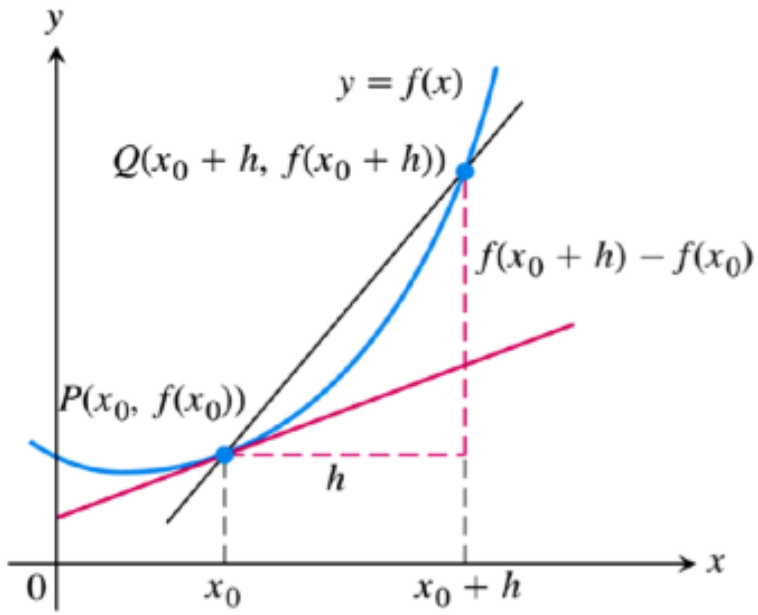
Chapter 3: Differentiation



The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Ex $y = f(x) = \frac{1}{x}$, the slope at $x_0 = 2$

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f(2) = \frac{1}{2}, \quad f(2+h) = \frac{1}{2+h}$$

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2} - \cancel{2} - \cancel{h}}{2\cancel{h}(2+h)}$$

$$= \lim_{h \to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}$$

The eqn of the tangent line at (2, 1/2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$-\frac{1}{4} = \frac{y - 1/2}{x - 2}$$

$$4y - 2 = -x + 2$$

$$y = -\frac{1}{4}x + 1 \quad //$$

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

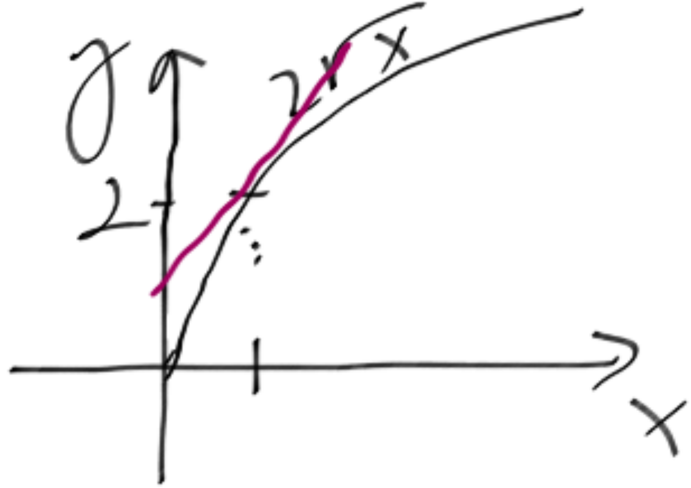
provided this limit exists.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

Ex $y=f(x)=2\sqrt{x}$, $f'(1)=?$



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x+h)=2\sqrt{x+h}$$

$$f'(x)=\lim_{h\to 0}\frac{2\sqrt{x+h}-2\sqrt{x}}{h}$$

$$=2\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x)=2\lim_{h\to 0}\frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{2}{2\sqrt{x}}$$

$f'(1)=1$ //

to find eqn of the tangent line at (1,2) is

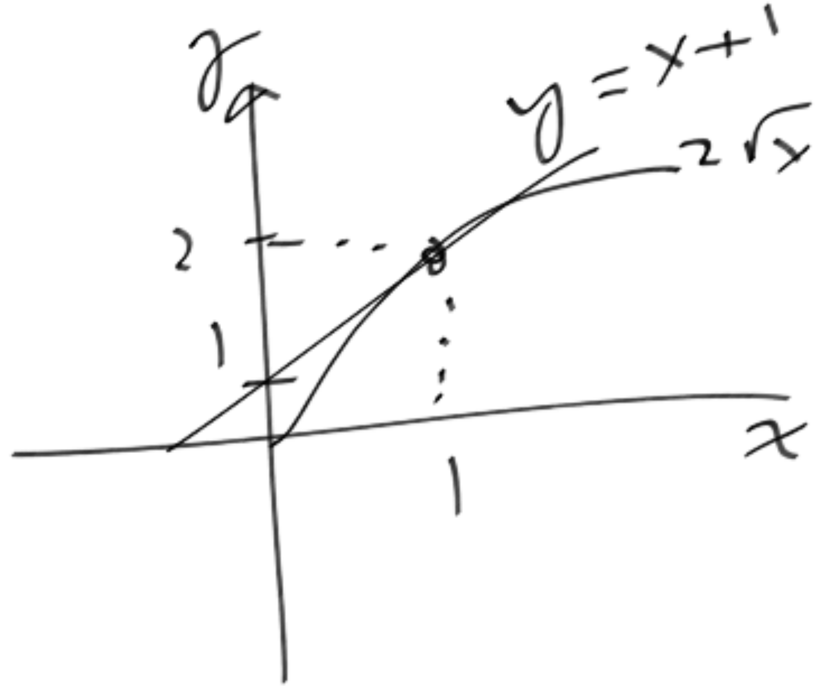
$$f'(1) = m = \text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$x_1 = 1, \quad y_1 = 2$$

$$1 = \frac{y-2}{x-1}$$

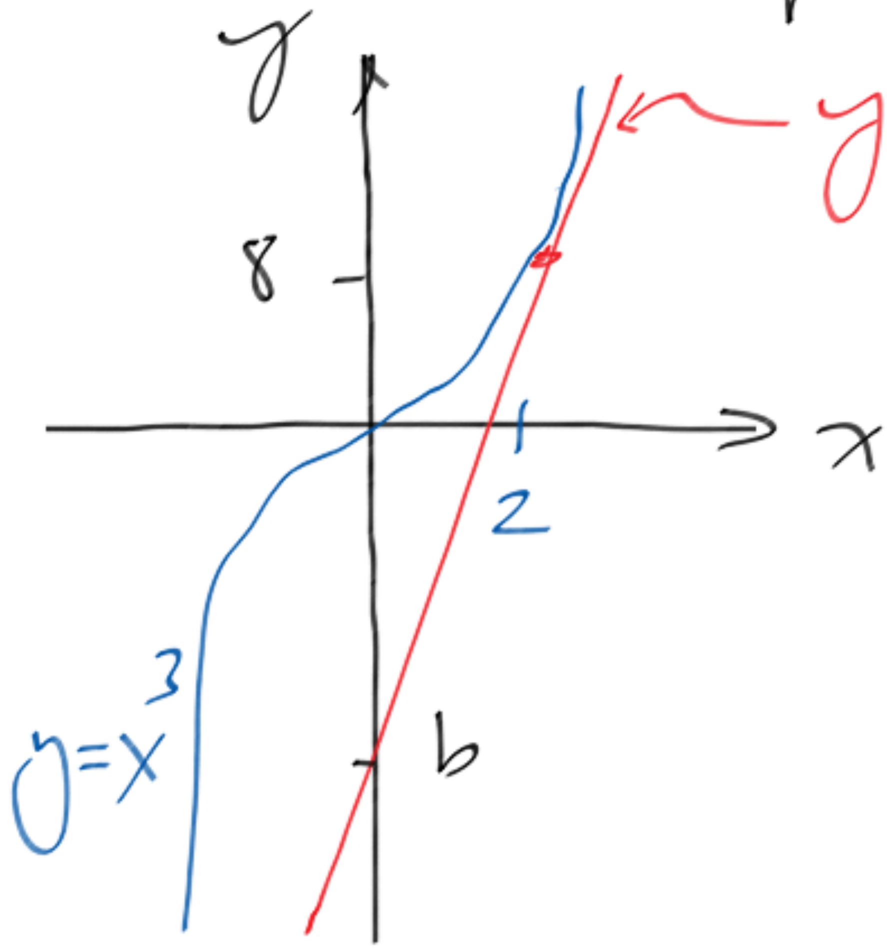
$$x - 1 = y - 2$$

$$y = x + 1$$



Ex

$y = f(x) = x^3$, find the eqn. of tangent line at $(2, 8)$.



$y = mx + b$

$m = ?$

$b = ?$

$$f'(x_0) = m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f(x) = x^3, \quad f(2) = 8, \quad f(2+h) = (2+h)^3$$

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{(\cancel{2}+\cancel{h}-\cancel{2})\left[(2+h)^2 + 2(2+h) + 4\right]}{\cancel{h}}$$

$$f'(2) = 4 + 4 + 4 = 12 \qquad P(2, 8)$$

$$f'(2) = m = 12 = \frac{y - 8}{x - 2}$$

$$y = 12x - 16 \quad // \quad \begin{array}{l} m = 12 \\ b = -16 \end{array}$$

Ex $f(x)=\frac{1}{x^2}$, $f'(-1)=?$

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$f(-1)=1, \quad f(-1+h)=\frac{1}{(-1+h)^2}$$

$$f'(-1)=\lim_{h\to 0}\frac{\frac{1}{(h-1)^2}-1}{h}=\frac{\cancel{1}-h^2+2h-\cancel{1}}{h(h-1)^2}$$

$$f'(-1)=\lim_{h\to 0}\frac{\cancel{h}(-h+2)}{\cancel{h}(h-1)^2}=2$$