A Universal Turing Machine

A limitation of Turing Machines:

Turing Machines are "hardwired"

they execute
only one program

Real Computers are re-programmable

Solution: Universal Turing Machine

Attributes:

- Reprogrammable machine
- Simulates any other Turing Machine

Universal Turing Machine

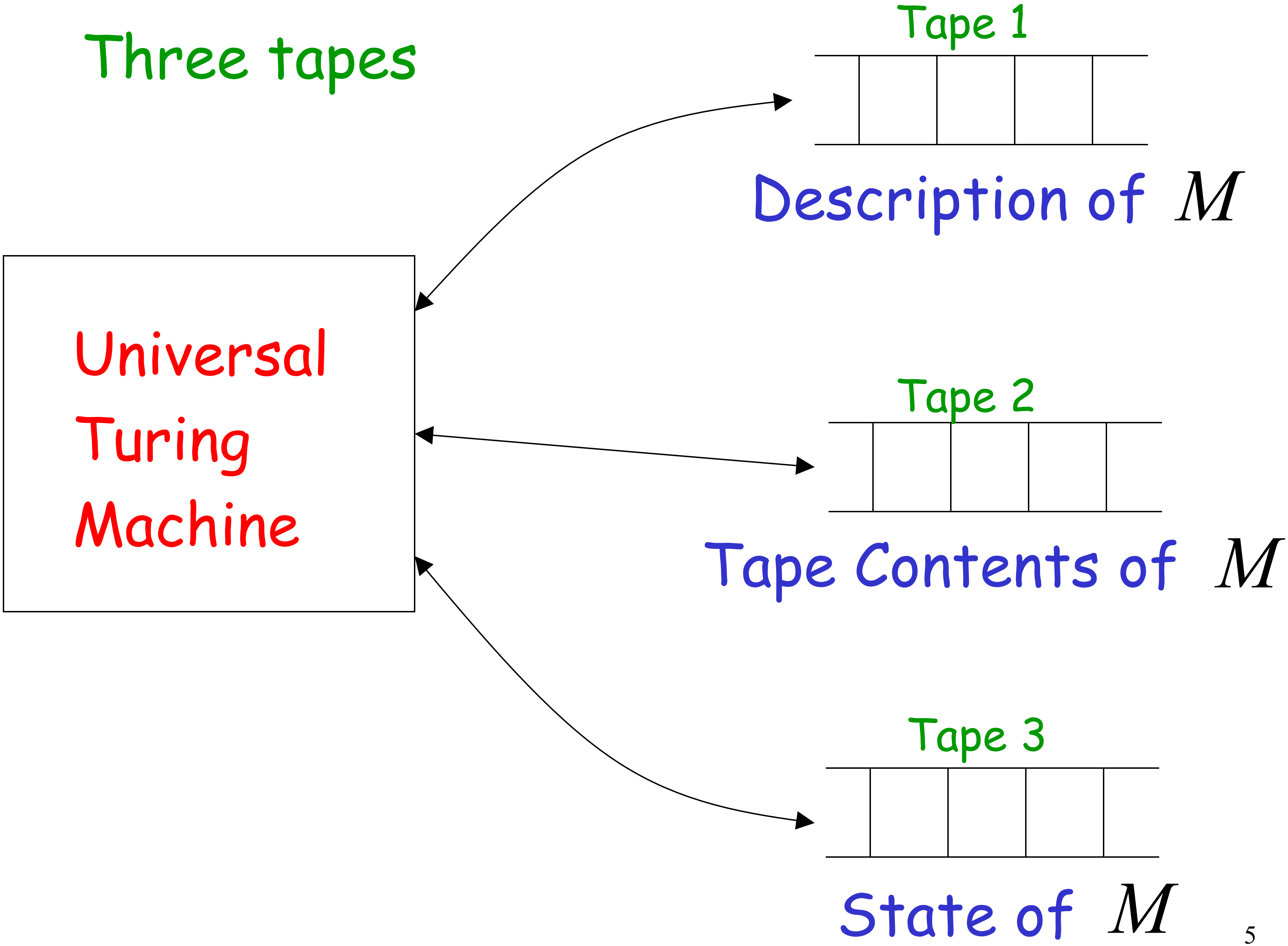
simulates any Turing Machine M

Input of Universal Turing Machine:

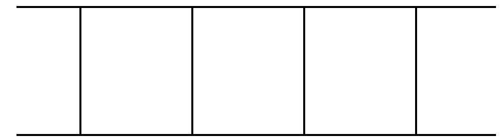
Description of transitions of M

Input string of M

Three tapes


Universal
Turing
Machine
Tape 1
Description of M
Tape 2
Tape Contents of M
Tape 3
State of M

Tape 1



Description of M

We describe Turing machine M
as a string of symbols:

We encode M as a string of symbols

Alphabet Encoding

Symbols:	a	b	c	d	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Encoding:	1	11	111	1111	

State Encoding

$$\begin{array}{lccccc} \text{States:} & q_1 & q_2 & q_3 & q_4 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{Encoding:} & 1 & 11 & 111 & 1111 & \end{array}$$

Head Move Encoding

$$\begin{array}{lcc} \text{Move:} & L & R \\ & \downarrow & \downarrow \\ \text{Encoding:} & 1 & 11 \end{array}$$

Transition Encoding

Transition: $\delta(q_1, a) = (q_2, b, L)$

Encoding: $1\,0\,1\,0\,11\,0\,11\,0\,1$

separator

Turing Machine Encoding

Transitions:

$$\delta(q_1, a) = (q_2, b, L) \qquad \delta(q_2, b) = (q_3, c, R)$$

Encoding:

1 0 1 0 11 0 11 0 1 00 11 0 11 0 111 0 111 0 11

separator

Tape 1 contents of Universal Turing Machine:

binary encoding
of the simulated machine M

Tape 1

1 0 1 0 11 0 11 0 10011 0 1 10 111 0 111 0 1100…

A Turing Machine is described with a binary string of 0's and 1's

Therefore:

The set of Turing machines forms a language:

each string of this language is the binary encoding of a Turing Machine

Language of Turing Machines

L = { 1010110101, (Turing Machine 1)

101011101011, (Turing Machine 2)

11101011110101111, ……

…… }

Countable Sets

Infinite sets are either:

Countable

or

Uncountable

Countable set:

There is a one to one correspondence (injection)

of

elements of the set

to

Positive integers (1,2,3,...)

Every element of the set is mapped to a positive number such that no two elements are mapped to same number

Example: The set of even integers is countable

Even integers: $0, \ 2, \ 4, \ 6, \ \ldots$
(positive)

Correspondence: $\downarrow \ \downarrow \ \downarrow \ \downarrow$

Positive integers: $1, \ 2, \ 3, \ 4, \ \ldots$

$2n$ corresponds to $n+1$

Example: The set of rational numbers is countable

Rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Naïve Approach

Nominator 1

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Correspondence:

$$\downarrow \quad \downarrow \quad \downarrow$$

Positive integers: $1, \ 2, \ 3, \ \ldots$

Doesn't work:

we will never count
numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach

$$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \cdots$$

$$\frac{2}{1} \quad \frac{2}{2} \quad \frac{2}{3} \quad \cdots$$

$$\frac{3}{1} \quad \frac{3}{2} \quad \cdots$$

$$\frac{4}{1} \quad \cdots$$

$$
\begin{array}{ccccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \cdots \\
\\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \cdots & & \\
\\
\frac{3}{1} & & \frac{3}{2} & & \cdots & & & & \\
\\
\frac{4}{1} & & \cdots & & & & & &
\end{array}
$$

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}$$

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

Correspondence:

Positive Integers: $1, \; 2, \; 3, \; 4, \; 5, \ldots$

We proved:

the set of rational numbers is countable
by describing an enumeration procedure
(enumerator)
for the correspondence to natural numbers

Definition

Let S be a set of strings (Language)

An **enumerator** for S is a Turing Machine that generates (prints on tape) all the strings of S one by one

and

each string is generated in finite time

strings $s_1, s_2, s_3, \ldots \in S$

Enumerator Machine for S

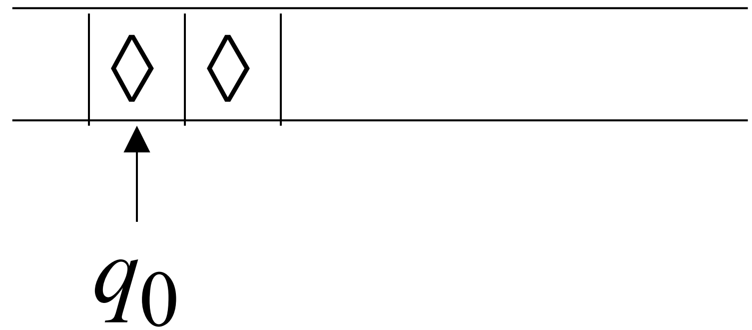
output (on tape) $s_1, s_2, s_3, \ldots$

Finite time: $t_1, t_2, t_3, \ldots$

Enumerator Machine

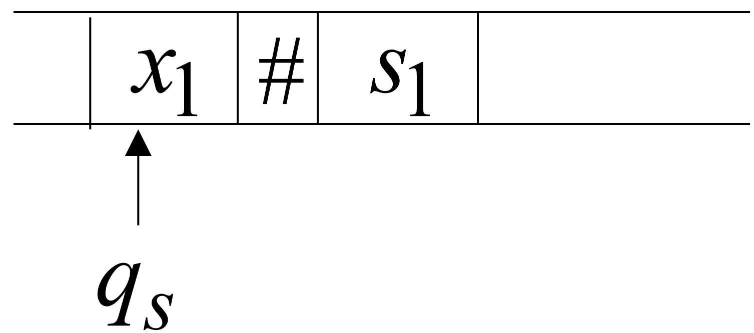
Configuration

Time 0



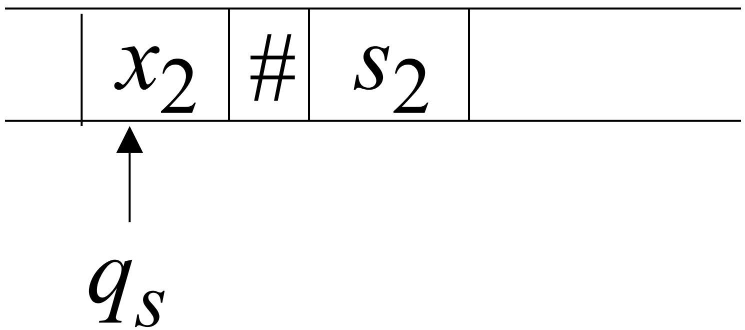
prints s_1

Time t_1



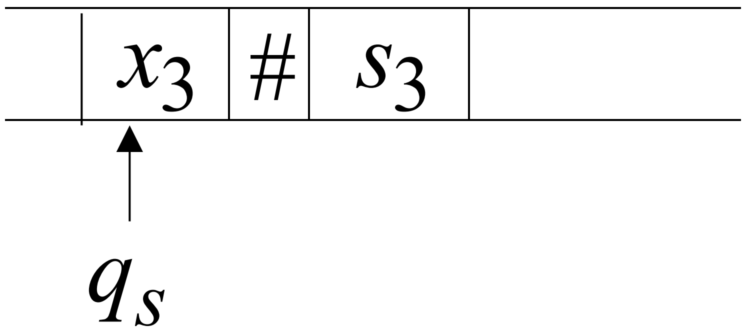
prints s_2

Time t_2



prints s_3

Time t_3



Observation:

If for a set S there is an enumerator, then the set is countable

The enumerator describes the correspondence of S to natural numbers

Example: The set of strings $S = \{a,b,c\}^+$ is countable

Approach:

We will describe an enumerator for S

Naive enumerator:

Produce the strings in lexicographic order:

$$s_1 = a$$

$$s_2 = aa$$

$$\vdots \quad aaa$$

$$aaaa$$

$$\ldots\ldots$$

Doesn't work:

strings starting with b

will never be produced

Better procedure: **Proper Order**

(Canonical Order)

1. Produce all strings of length 1

2. Produce all strings of length 2

3. Produce all strings of length 3

4. Produce all strings of length 4

⋮

Produce strings in

Proper Order:

$$\left.\begin{array}{l} s_1 = a \\ s_2 = b \\ \vdots \quad c \end{array}\right\} \text{length 1}$$

$$\left.\begin{array}{l} aa \\ ab \\ ac \\ ba \\ bb \\ bc \\ ca \\ cb \\ cc \end{array}\right\} \text{length 2}$$

$$\left.\begin{array}{l} aaa \\ aab \\ aac \\ \ldots\ldots \end{array}\right\} \text{length 3}$$

Theorem: The set of all Turing Machines is countable

Proof: Any Turing Machine can be encoded with a binary string of 0's and 1's

Find an enumeration procedure for the set of Turing Machine strings

Enumerator:

Repeat

1. Generate the next binary string of 0's and 1's in proper order

2. Check if the string describes a Turing Machine

   if **YES:** print string on output tape

   if **NO:** ignore string

Binary strings | Turing Machines

0 ignore

1 ignore

00 ignore

01

⋮

1 0 1 0 11 0 11 0 0

1 0 1 0 11 0 11 0 1 $\xrightarrow{s_1}$ 1 0 1 0 11 0 11 0 1

⋮

1 0 11 0 1010010101101 $\xrightarrow{s_2}$ 1 0 11 0 1010010101101

⋮

End of Proof

Simpler Proof:

Each Turing machine binary string is mapped to the number representing its value

Uncountable Sets

We will prove that there is a language L which is not accepted by any Turing machine

Technique:

Turing machines are countable

Languages are uncountable

(there are more languages than Turing Machines)

Theorem:

If S is an infinite countable set, then

the powerset 2^S of S is uncountable.

The powerset 2^S contains all possible subsets of S

Example: $S = \{a, b\}$ $\quad 2^S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Proof:

Since S is countable, we can list its elements in some order

$$S = \{s_1, s_2, s_3, \ldots\}$$

Elements of S

Elements of the powerset 2^S have the form:

$$\emptyset$$

$$\{s_1, s_3\}$$

$$\{s_5, s_7, s_9, s_{10}\}$$

$$\vdots$$

They are subsets of S

We encode each subset of S
with a binary string of 0's and 1's

Subset of S	Binary encoding				
	s_1	s_2	s_3	s_4	$\cdots$
$\{s_1\}$	1	0	0	0	$\cdots$
$\{s_2, s_3\}$	0	1	1	0	$\cdots$
$\{s_1, s_3, s_4\}$	1	0	1	1	$\cdots$

Every infinite binary string corresponds to a subset of S:

Example: $$1\,0\,0\,1\,1\,1\,0 \;\cdots$$

Corresponds to: $$\{s_1, s_4, s_5, s_6, \ldots\} \in 2^S$$

Let's assume (for contradiction) that the powerset 2^S is countable

Then: we can list the elements of the powerset in some order

$$2^S = \{t_1, t_2, t_3, \ldots\}$$

Subsets of S

Powerset element	Binary encoding example					
t_1	1	0	0	0	0	$\cdots$
t_2	1	1	0	0	0	$\cdots$
t_3	1	1	0	1	0	$\cdots$
t_4	1	1	0	0	1	$\cdots$
$\cdots$			$\cdots$			

$t =$ the binary string whose bits are the complement of the diagonal

t_1	1	0	0	0	0	$\cdots$
t_2	1	1	0	0	0	$\cdots$
t_3	1	1	0	1	0	$\cdots$
t_4	1	1	0	0	1	$\cdots$

Binary string: $t = 0011\cdots$

(birary complement of diagonal)

The binary string $t = 0011\ldots$

corresponds
to a subset of S: $t = \{s_3, s_4, \ldots\} \in 2^S$

t = the binary string whose bits are the complement of the diagonal

$$\begin{array}{llllllll} t_1 & & 1 & 0 & 0 & 0 & 0 & \cdots \\ t_2 & & 1 & 1 & 0 & 0 & 0 & \cdots \\ t_3 & & 1 & 1 & 0 & 1 & 0 & \cdots \\ t_4 & & 1 & 1 & 0 & 0 & 1 & \cdots \end{array}$$

$$t = 0011\cdots$$

Question: $t = t_1$? **NO:** differ in 1st bit

t = the binary string whose bits are the complement of the diagonal

$$
\begin{array}{llllll}
t_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
t_2 & 1 & 1 & 0 & 0 & 0 & \cdots \\
t_3 & 1 & 1 & 0 & 1 & 0 & \cdots \\
t_4 & 1 & 1 & 0 & 0 & 1 & \cdots
\end{array}
$$

$$t = 0011\cdots$$

Question: $t = t_2$? NO: differ in 2nd bit

t = the binary string whose bits are the complement of the diagonal

$$\begin{array}{llllll} t_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ t_2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ t_3 & 1 & 1 & 0 & 1 & 0 & \cdots \\ t_4 & 1 & 1 & 0 & 0 & 1 & \cdots \end{array}$$

$$t = 0011\cdots$$

Question: $t = t_3$? NO: differ in 3rd bit

Thus: $t \neq t_i$ for every i

since they differ in the ith bit

However, $t \in 2^S \Rightarrow t = t_i$ for some i

Contradiction!!!

Therefore the powerset 2^S is uncountable

End of proof

An Application: Languages

Consider Alphabet : $A = \{a, b\}$

The set of all strings:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

because we can enumerate
the strings in proper order

Consider Alphabet : $A = \{a, b\}$

The set of all strings:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

Any language is a subset of S :

$$L = \{aa, ab, aab\}$$

Consider Alphabet : $A = \{a, b\}$

The set of all Strings:

$$S = A^* = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

The powerset of S contains all languages:

$$2^S = \{\varnothing, \{\varepsilon\}, \{a\}, \{a, b\}, \{aa, b\}, \ldots, \{aa, ab, aab\}, \ldots\}$$

uncountable

Consider Alphabet : $A = \{a, b\}$

countable

Turing machines: $M_1 \quad M_2 \quad M_3 \quad \cdots$

accepts

Languages accepted
By Turing Machines: $L_1 \quad L_2 \quad L_3 \quad \cdots$

countable

Denote: $X = \{L_1, L_2, L_3, \ldots\}$
countable

Note: $X \subseteq 2^S$

$(S = \{a, b\}^*)$

Languages accepted by Turing machines: X countable

All possible languages: 2^S uncountable

Therefore: $X \neq 2^S$

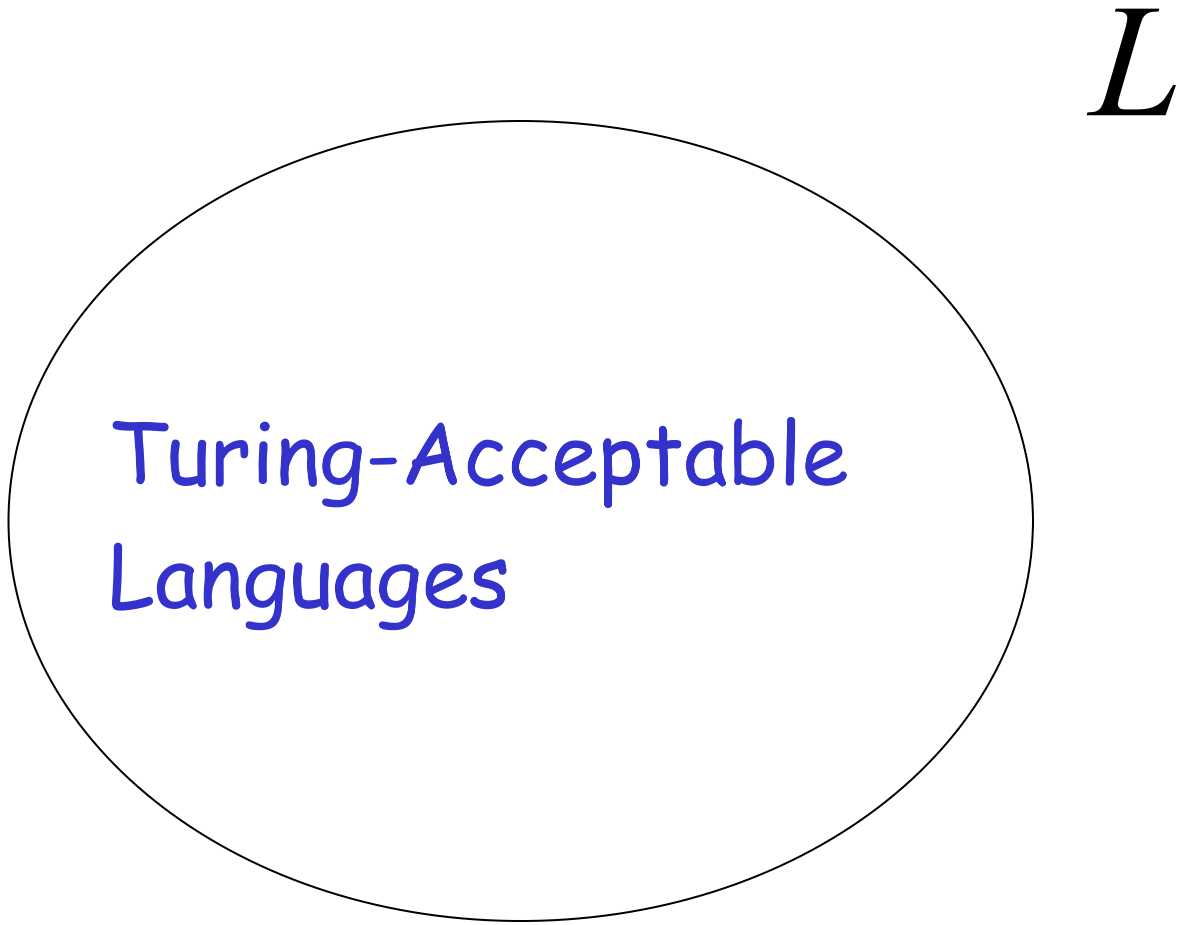
$$\left(\text{since } X \subseteq 2^S, \text{ we get } X \subset 2^S\right)$$

Conclusion:

There is a language L not accepted by any Turing Machine:

$$X \subset 2^S \quad \Longrightarrow \quad \exists L \in 2^S \text{ and } L \notin X$$

Non Turing-Acceptable Languages



Note that: $X = \{L_1, L_2, L_3, \ldots\}$

is a *multi-set* (elements may repeat) since a language may be accepted by more than one Turing machine

However, if we remove the repeated elements, the resulting set is again countable since every element still corresponds to a positive integer