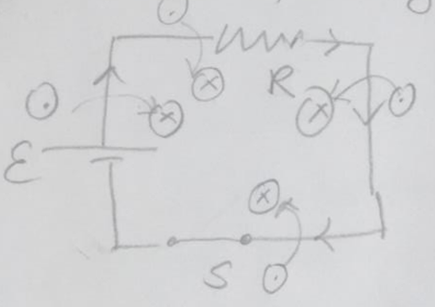
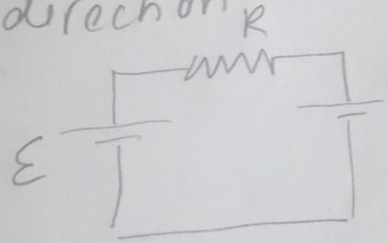
INDUCTANCE

Consider the following circuit:



When S is closed, a current starts to flow in the circuit. This current creates B (a magnetic field) in the loop, inwards ⊗. This B, which depends on I, increases as I increases & reaches its steady state value: I = ε/R. So, I starts from 0 and reaches ε/R.

During this transitory period, we have a time dependent B $\left(\frac{dB}{dt} > 0\right)$ hence the flux through the loop increases. We know from Faraday's law that a time dependent flux induces an ε (EMF) in the opposite direction. So we have a ε in the opposite direction.



$\leftarrow \varepsilon_{induced} = -\frac{d\phi(t)}{dt}$. This is called BACK ε.

This back ε exists only until I = ε/R and then it disappears. The effect of the ε_{ind} is to delay the increase in the current. This effect is called SELF-INDUCTION

We know that $\varepsilon_{ind} = -\frac{d\phi(t)}{dt} = -\frac{d}{dt}(\vec{B}\cdot\vec{A})$

$B \propto I \qquad \varepsilon_{ind} \propto \frac{dI}{dt} \qquad \boxed{\varepsilon_L = -L\frac{dI}{dt}}$

where L is a constant.

2

$\mathcal{E}_L = -N \frac{d\phi}{dt}$ (for N loops), $N\frac{d\phi}{dt} = L\frac{dI}{dt}$

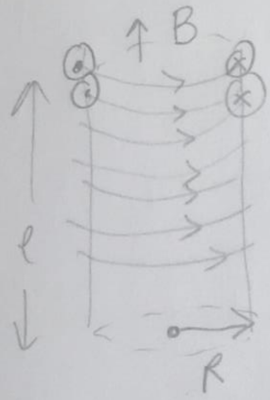
$$\boxed{L = \frac{N\phi}{I}}$$ = inductance measured in "Henry" (H)

Now if we introduce a circuit element like a solenoid in the circuit:

The increased flux will increase the inductance hence the self induced back $\mathcal{E}$.

Inductance (L) of course depends on the geometry.

Let's calculate L for a Solenoid of radius R.



$B = \mu_0 n I$ or $B = \mu_0 \frac{NI}{\ell}$ $\quad n = \frac{N}{\ell}$

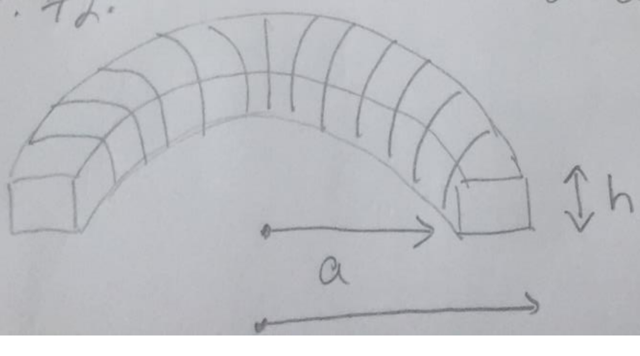
where ℓ is the length of the solenoid and N is the # of windings.

$\phi = BA = \mu_0 \frac{NI}{\ell} A$ where $A = \pi R^2$

$$L = \frac{N\phi}{I} = \boxed{\frac{\mu_0 N^2 A}{\ell}}$$

Inductance of a toroid (only half of the toroid is shown in the figure but we have a complete toroid.

P. 72.



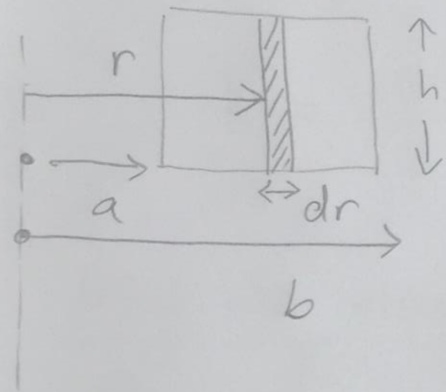
What is the inductance of the toroid?

We had determined the B for a toroid using Ampere's Law in chapter 30 (Please review) as,

$$B = \frac{\mu_0 N I}{2\pi r}$$

So B inside the toroid is not constant but depends on r which is the distance from the center.

We find the flux through one loop (rectangular)



$$d\phi = B\,dA = \frac{\mu_0 N I}{2\pi r} h\,dr$$

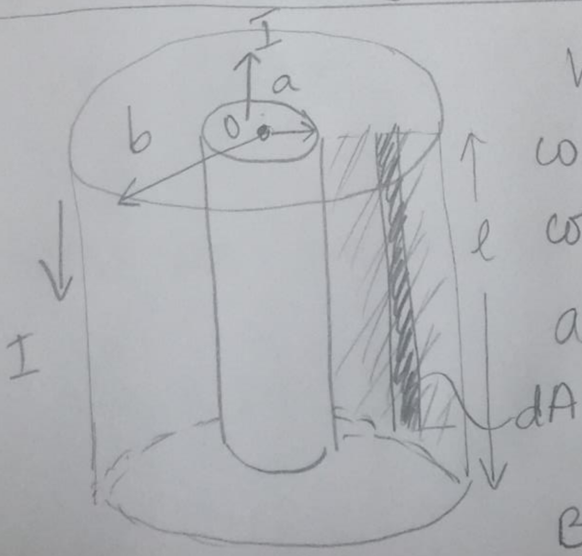
$$\phi = \int B\,dA = \int_a^b \frac{\mu_0 N I h}{2\pi r}\,dr$$

$$\phi = \frac{\mu_0 N I h}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 N I h}{2\pi} \ln\frac{b}{a}$$

$$L = \frac{N\phi}{I}$$ where N is the total # of turns.

$$\boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a}}$$

Inductance of a Coaxial Wire:



We had determined B in the coaxial region between the conductors (i.e. at $a < r < b$) as:

$$B = \frac{\mu_0 I}{2\pi r}$$

wher r is The distance from O.

B is tangent to the ccw circle.

We consider the flux through the shaded area.
But over that area B is not constant. So we consider
$d\phi$ over the small shaded area dA.

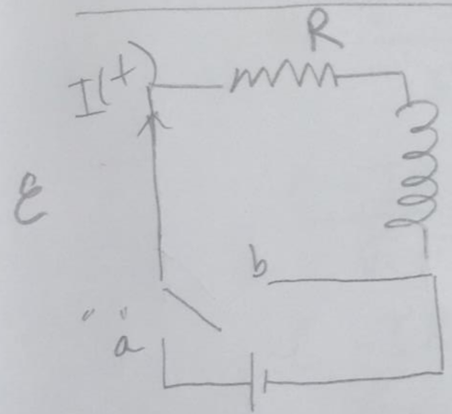
$$d\phi = B\,dA = \frac{\mu_0 I}{2\pi r}\,\ell\,dr$$

(Notice that B and dA are parallel)
(Both ⊗ in)

$$\phi = \int_a^b \frac{\mu_0 I}{2\pi r}\,\ell\,dr = \frac{\mu_0 I \ell}{2\pi}\ln\frac{b}{a}$$

$$L = \frac{N\phi}{I} = \frac{\mu_0 \ell}{2\pi}\ln\frac{b}{a}$$ where $N = 1$

Now we consider a circuit with an inductance.



S is brought to "a" at $t=0$. We write the loop equation in the transitory period before the current reaches its final value $I = \mathcal{E}/R$. Also note that the inductor has no resistence like the connecting wires.

$$\mathcal{E} - I(t)R - L\frac{dI(t)}{dt} = 0$$

Now we want to solve for $I(t)$:

$$\frac{\mathcal{E}}{L} - I(t)\frac{R}{L} - \frac{dI(t)}{dt} = 0$$

5

$$\frac{\varepsilon - I(t)R}{L} = \frac{dI(t)}{dt}$$

$$\int_0^t \frac{dt}{L} = \int_0^I \frac{dI(t)}{\varepsilon - I(t)R}$$

Let $\varepsilon - I(t)R = u$

$-R\,dI(t) = du$

$dI(t) = -\frac{1}{R}\,du.$

$$\int_0^t \frac{dt}{L} = -\frac{1}{R}\int_{\varepsilon}^{\varepsilon - IR} \frac{du}{u}$$

$$\frac{t}{L} = -\frac{1}{R}\ln\frac{\varepsilon - IR}{\varepsilon}$$

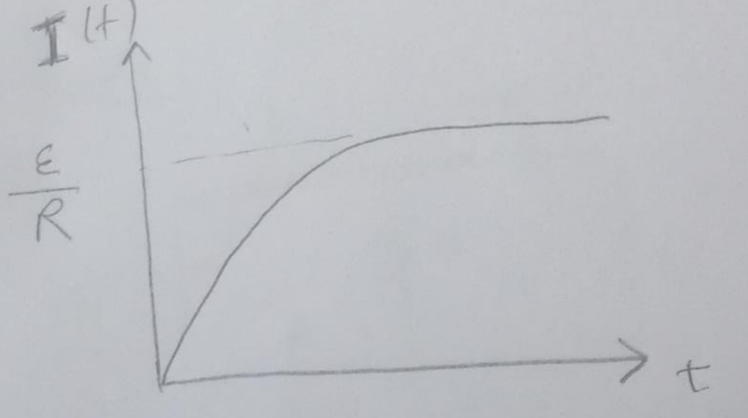
$$\ln\frac{\varepsilon - IR}{\varepsilon} = -\frac{R}{L}t$$

$$\frac{\varepsilon - IR}{\varepsilon} = e^{-\frac{R}{L}t}$$

$$\varepsilon - IR = \varepsilon e^{-\frac{R}{L}t}$$

$$I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$\tau = \frac{L}{R}$ (time constant)



when $t = 0$

$e^{-\frac{R}{L}t} = 1$

$I = 0$

when $t \to \infty$

$e^{-\frac{R}{L}t} \to 0$

$I = \frac{\varepsilon}{R}$

Let us now assume that after the current reaches its steady state value $\mathcal{E}/R$, we bring the switch S to "b". Now the loop equation:

$$-I(t)R - L\frac{dI(t)}{dt} = 0$$

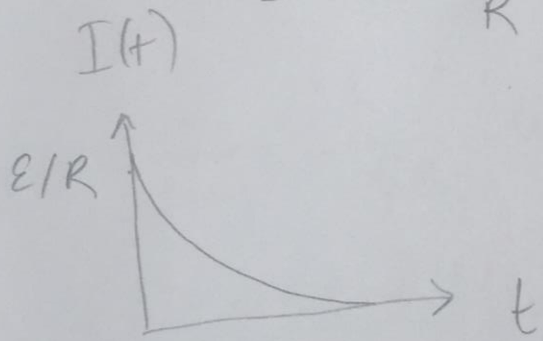
$$I(t)\frac{R}{L} = -\frac{dI(t)}{dt} \qquad \frac{dI(t)}{I(t)} = -(dt)\frac{R}{L}$$

$$\int_{\mathcal{E}/R}^{I(t)} \frac{dI(t)}{I(t)} = -\int_{0}^{t} \frac{R}{L}\,dt$$

$$\ln\frac{I(t)}{\mathcal{E}/R} = -\frac{R}{L}t$$

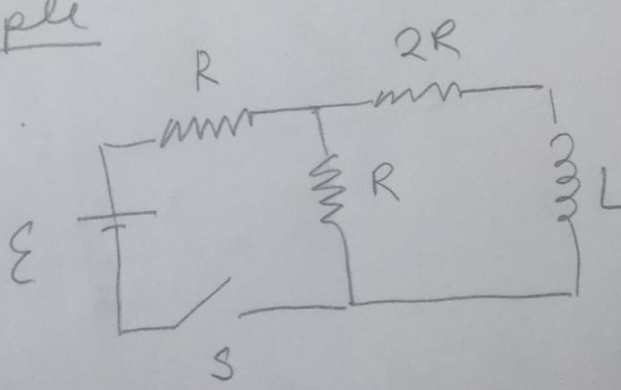
$$I(t)/\mathcal{E}/R = e^{-R/L\,t}$$

$$I(t) = \frac{\mathcal{E}}{R}e^{-(R/L)t} = \boxed{\frac{\mathcal{E}}{R}e^{-t/\tau}}$$



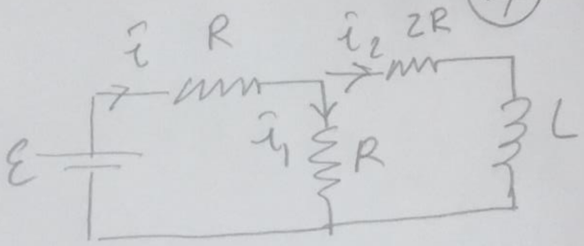
Example

P. 26



S is closed at $t = 0$. Find the current in the inductor L and the current in the switch S as functions

7



$$i = i_1 + i_2$$

$$\varepsilon - iR - i_1 R = 0$$

$$\varepsilon - (i_1 + i_2)R - i_1 R = 0$$

$$\varepsilon - 2 i_1 R - i_2 R = 0$$

$$\frac{\varepsilon - i_2 R}{2} = i_1 R$$

$$-i_2 2R - L\frac{di_2}{dt} + i_1 R = 0$$

$$-i_2 2R - L\frac{di_2}{dt} + \frac{\varepsilon - i_2 R}{2} = 0$$

$$-4 i_2 R - 2L\frac{di_2}{dt} + \varepsilon - i_2 R = 0$$

$$\varepsilon - 5 i_2 R - 2L\frac{di_2}{dt} = 0$$

$$\frac{\varepsilon - 5 i_2 R}{2L} = \frac{di_2}{dt}$$

$$\int_0^t \frac{dt}{2L} = \int_0^{i_2(t)} \frac{di_2}{\varepsilon - 5R i_2}$$

Let $\varepsilon - 5R i_2 = u$

$$-5R\, di_2 = du.$$

$$di_2 = -\frac{du}{5R}$$

$$\frac{t}{2L} = -\frac{1}{5R}\int_{\varepsilon}^{\varepsilon - 5R i_2(t)} \frac{du}{u}$$

$$\frac{t}{2L} = -\frac{1}{5R}\ln\frac{\varepsilon - 5R i_2(t)}{\varepsilon}$$

$$\frac{\varepsilon - 5R\, i_2(t)}{\varepsilon} = e^{-t\,5R/2L}$$

$$\boxed{i_2(t) = \frac{\varepsilon}{5R}\left(1 - e^{-\frac{5R}{2L}t}\right)}$$

$$\frac{\varepsilon - i_2 R}{2R} = i_1$$

$$i_1 = \frac{\varepsilon}{2R} - \frac{1}{2} i_2$$

$$i_1(t) = \frac{\varepsilon}{2R} - \frac{\varepsilon}{10R}\left(1 - e^{-5Rt/2L}\right)$$

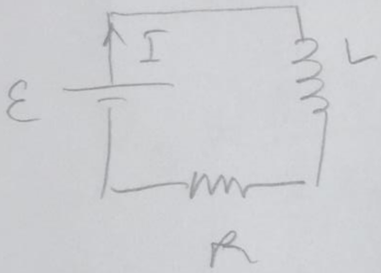
$$i = i_1 + i_2 = \frac{\mathcal{E}}{2R} - \frac{\mathcal{E}}{10R}\left(1-e^{-5Rt/2L}\right) + \frac{\mathcal{E}}{5R}\left(1-e^{-5Rt/2L}\right)$$

$$i = \frac{\mathcal{E}}{2R} + \left(1-e^{-5Rt/2L}\right)\left(\frac{\mathcal{E}}{5R} - \frac{\mathcal{E}}{10R}\right)$$

$$i = \frac{\mathcal{E}}{2R} + \frac{\mathcal{E}}{10R}\left(1-e^{-t/\tau}\right) \quad \text{where } \tau = \frac{2L}{5R}$$

Energy Stored in an Inductor

In the circuit, we write the loop eqn:



$$\mathcal{E} - L\frac{dI}{dt} - IR = 0$$

Multiply by I

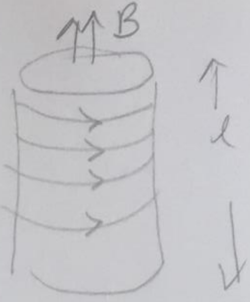
$$\mathcal{E} - LI\frac{dI}{dt} - I^2R = 0 \Rightarrow \text{Power eqn.}$$

The power in the inductor must be:

$$LI\frac{dI}{dt} = \frac{du_B}{dt} \Rightarrow$$ the rate at which energy is stored in the inductor

$$\int_0^{u_B} du = \int_0^{I} LI\,dI$$

$$u_B = L\frac{I^2}{2}$$



Consider a solenoid of length l and N windings

$B = \mu_0 n I = \mu_0 \frac{NI}{l}$ $\quad \phi = \mu_0 \frac{NIA}{l}$

$$L = \frac{N\phi}{I} = \mu_0 \frac{N^2 A}{l} = \frac{\mu_0 N^2 A l}{l^2} = \mu_0 n^2 V$$

where $\frac{N}{l} = n$ and $Al = \text{Volume}$.

$$U_B = \frac{LI^2}{2} = \boxed{\frac{\mu_0 n^2 V I^2}{2}} \quad \text{or} = \frac{\mu_0^2 n^2 I^2 V}{2\mu_0}$$

if we write this in terms of B

$$U_B = \frac{B^2 V}{2\mu_0}$$

and divide by V: energy per volume:

$$u_B = \frac{B^2}{2\mu_0}$$