Alternating Series, Absolute and Conditional Convergence

THEOREM —The Alternating Series Test (Leibniz's Test) The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Ex $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converges (conditionally)

Ex $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots$

Actually, this is a geometric series with $a=1$, $r=-\frac{1}{2}$; $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

$S_1 = 1$

$S_2 = 1 - \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{4}$

$\vdots$

$S_8 = 0.6640625$

$S_9 = 0.66796875$

$S_8 < 2/3 < S_9$

$0.664065 < 2/3 < 0.66796875$

$\frac{2}{3} - S_8 = 0.002604166$ is (+) and less than $\frac{1}{256} = 0.00390625$

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

DEFINITION A series that converges but does not converge absolutely **converges conditionally**.

THEOREM —The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$, since $\sum \frac{1}{n^2}$ is a p-series with $p=2$, this series converges absolutely.

Ex $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \dots$, which contains both + and − terms

the corresponding series of absolute value is

$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{2} + \dots$ converges by comparision with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $|\sin n| \le 1$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$, $p>0$; which converges.

If $p>1$, the series converges absolutely. If $0<p<1$, the series converges conditionally.

Conditional convergence: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$; $p = 1/2$

Absolute convergence: $1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$; $p = 3/2$

THEOREM —The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

The above theorem guarantees that the terms of an abolutely convergent series can be summed in any order without affecting the result.

Summary of the Tests

1. **The nth-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Ex

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \qquad \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \quad \text{(L' Hosp.)}$$

by the nth-term test, it converges.

Ex

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}, \quad \rho = \lim_{n\to\infty} \sqrt[n]{2^n/n^2} = \lim_{n\to\infty} \frac{2}{n^{2/n}} = 2 \lim_{n\to\infty} \overset{y}{n^{-2/n}} = 2 \times 1 = 2 > 1$$

diverges by nth-root test.

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} -\frac{2}{x} \ln x = -2 \lim_{x\to\infty} \frac{\ln x}{x} = -2 \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{x\to\infty} y = e^0 = 1$$

Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$
3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n3^n}$
4. $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$
5. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$
6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+5}{n^2+4}$
7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$
8. $\sum_{n=1}^{\infty}(-1)^{n}\frac{10^n}{(n+1)!}$
9. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$
10. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$
11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$
12. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$
14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute and Conditional Convergence

Which of the series in Exercises 15–26 converge absolutely, which converge, and which diverge? Give reasons for your answers.

15. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$
16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$
17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$
19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$
20. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$
21. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$
22. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$
23. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$
24. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$
25. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$
26. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt[n]{10}\right)$