Chapter 13 – Eigenvalues – Power Method

Goal: Solve the following eigenvalue problem:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Solution. The system is first written in the form of Eq. (13.12):

$$\begin{aligned} 40X_1 - 20X_2 \qquad &= \lambda X_1 \\ -20X_1 + 40X_2 - 20X_3 &= \lambda X_2 \\ -20X_2 + 40X_3 &= \lambda X_3 \end{aligned}$$

At this point, we can specify initial values of the X's and use the left-hand side to compute an eigenvalue and eigenvector. A good first choice is to assume that all the X's on the left-hand side of the equation are equal to one:

$$\begin{aligned} 40(1) - 20(1) \qquad &= 20 \\ -20(1) + 40(1) - 20(1) &= 0 \\ -20(1) + 40(1) &= 20 \end{aligned}$$

Next, the right-hand side is normalized by 20 to make the largest element equal to one:

$$\begin{Bmatrix} 20 \\ 0 \\ 20 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

Thus, the normalization factor is our first estimate of the eigenvalue (20) and the corresponding eigenvector is $\{1\ 0\ 1\}^T$. This iteration can be expressed concisely in matrix form as

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 20 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

The next iteration consists of multiplying the matrix by the eigenvector from the last iteration, $\{1\ 0\ 1\}^T$ to give

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -40 \\ 40 \end{Bmatrix} = 40 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

eigenvector estimate

Therefore, the eigenvalue estimate for the second iteration is 40, which can be employed to determine an error estimate:

$$|\varepsilon_a| = \left|\frac{40 - 20}{40}\right| \times 100\% = 50\%$$

The process can then be repeated.

Third iteration:

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$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -80 \\ 60 \end{Bmatrix} = -80 \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix}$$

where $|\varepsilon_a| = 150\%$ (which is high because of the sign change).

Fourth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 70 \\ -50 \end{Bmatrix} = 70 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

where $|\varepsilon_a| = 214\%$ (another sign change).

Fifth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix} = \begin{Bmatrix} -48.51714 \\ 68.51714 \\ -48.51714 \end{Bmatrix} = 68.51714 \begin{Bmatrix} -0.70833 \\ 1 \\ -0.70833 \end{Bmatrix}$$

where $|\varepsilon_a| = 2.08\%$.

Thus, the eigenvalue is converging. After several more iterations, it stabilizes on a value of 68.28427 with a corresponding eigenvector of $\{-0.707107 \; 1 \; -0.707107\}^T$.

MATLAB FUNCTION: eig

As might be expected, MATLAB has powerful and robust capabilities for evaluating eigenvalues and eigenvectors. The function `eig`, which is used for this purpose, can be employed to generate a vector of the eigenvalues as in

```
>> e = eig(A)
```

where `e` is a vector containing the eigenvalues of a square matrix `A`. Alternatively, it can be invoked as

```
>> [V,D] = eig(A)
```

where `D` is a diagonal matrix of the eigenvalues and `V` is a full matrix whose columns are the corresponding eigenvectors.

It should be noted that MATLAB scales the eigenvectors by dividing them by their Euclidean distance. Thus, as shown in the following example, although their magnitude may be different from values computed with say the polynomial method, the ratio of their elements will be identical.

Eigenvalues and Eigenvectors with MATLAB

Problem Statement. Use MATLAB to determine all the eigenvalues and eigenvectors for the system described in Example 13.3.

Solution. Recall that the matrix to be analyzed is

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix}$$

The matrix can be entered as

```
>> A = [40 -20 0;-20 40 -20;0 -20 40];
```

If we just desire the eigenvalues, we can enter

```
>> e = eig(A)

e =
   11.7157
   40.0000
   68.2843
```

Notice that the highest eigenvalue (68.2843) is consistent with the value previously determined with the power method in Example 13.3.

If we want both the eigenvalues and eigenvectors, we can enter

```
>> [v,d] = eig(A)

v =
    0.5000    -0.7071   -0.5000
    0.7071    -0.0000    0.7071
    0.5000     0.7071   -0.5000

d =
   11.7157          0          0
         0    40.0000          0
         0          0     8.2843
```

Although the results are scaled differently, the eigenvector corresponding to the highest eigenvalue $\{-0.5\ 0.7071\ -0.5\}^T$ is consistent with the value previously determined with the power method in Example 13.3: $\{-0.707107\ 1\ -0.707107\}^T$. The can be demonstrated by dividing the eigenvector from the power method by its Euclidean norm:

```
>> vpower = [-0.7071 1 -0.7071]';
>> vMATLAB = vpower/norm(vpower)

vMATLAB =
   -0.5000
    0.7071
   -0.5000
```

Thus, although the magnitudes of the elements differ, their ratios are identical.