

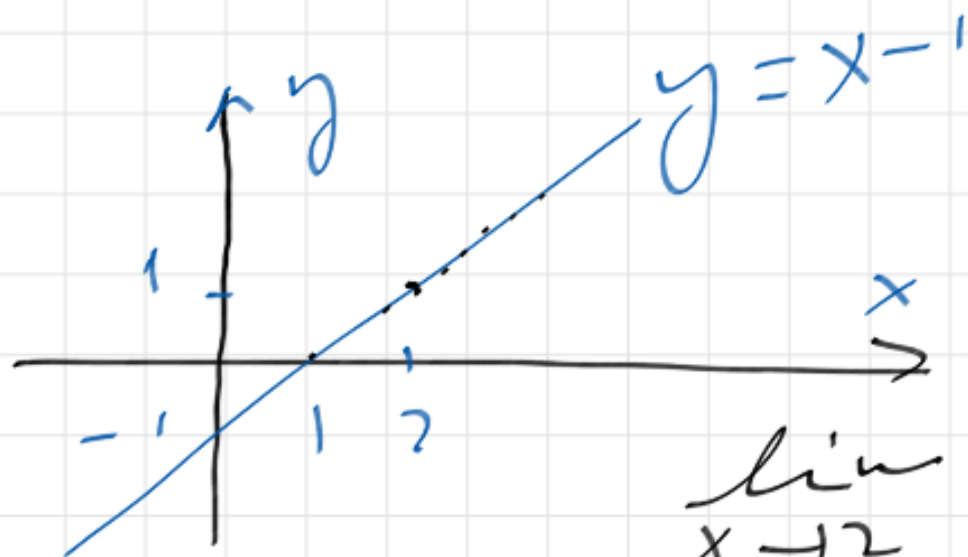
limits

Def. Let $y = f(x)$, $L \in \mathbb{R}$

The limit of $f(x)$ as x approaches x_0 is L .

$$\lim_{x \rightarrow x_0} f(x) = L$$

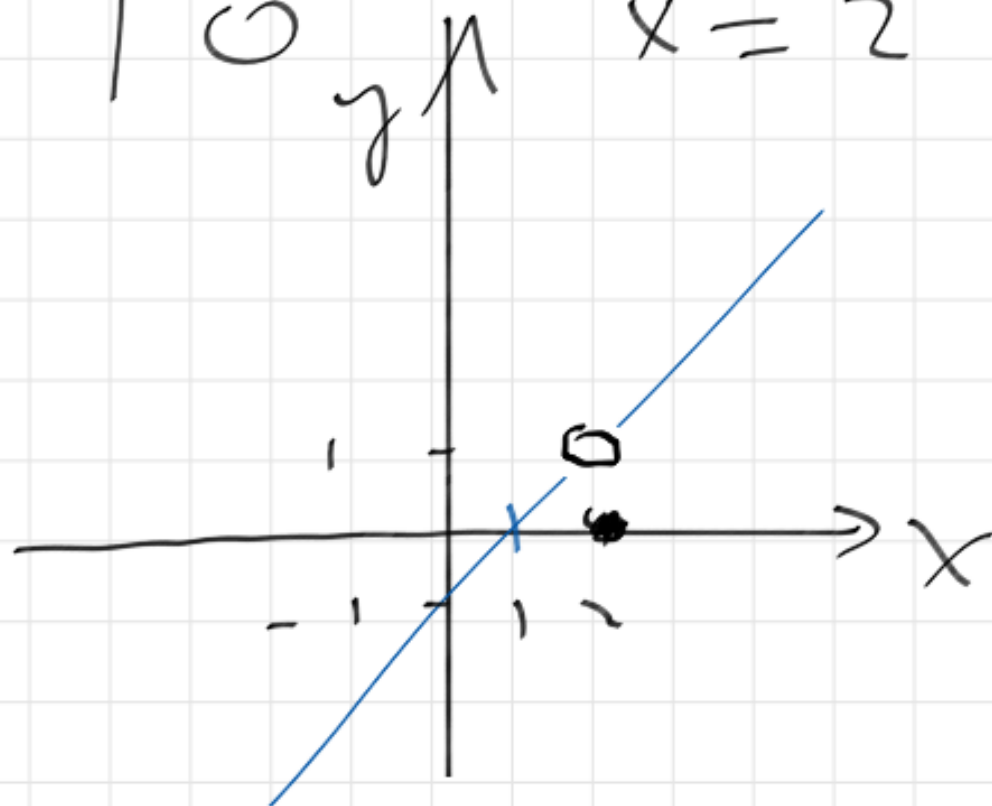
Ex



$$\lim_{x \rightarrow 2} x - 1 = 1$$

Ex

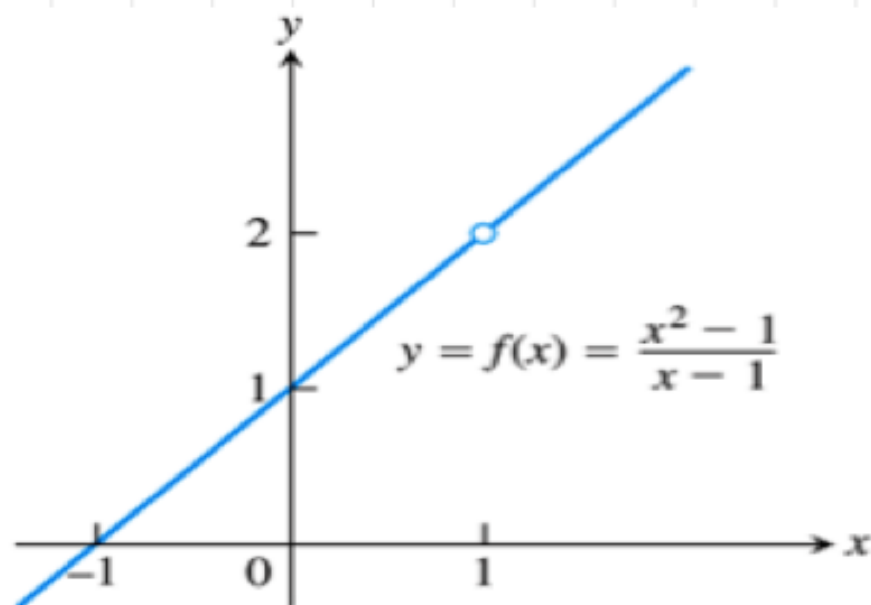
$$y = \begin{cases} x-1 & x \neq 2 \\ 0 & x = 2 \end{cases}$$



$$f(2) = 0$$

However

$$\lim_{x \rightarrow 2} (x-1) = 1$$



$f(1) = ?$ not defined!

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

(Note: In the original image, there are orange annotations: a downward arrow from x^2 to x , a percentage sign $\%$ next to the fraction, and a squiggly arrow pointing from the denominator $x-1$ to the numerator x^2-1 .)

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} =$$

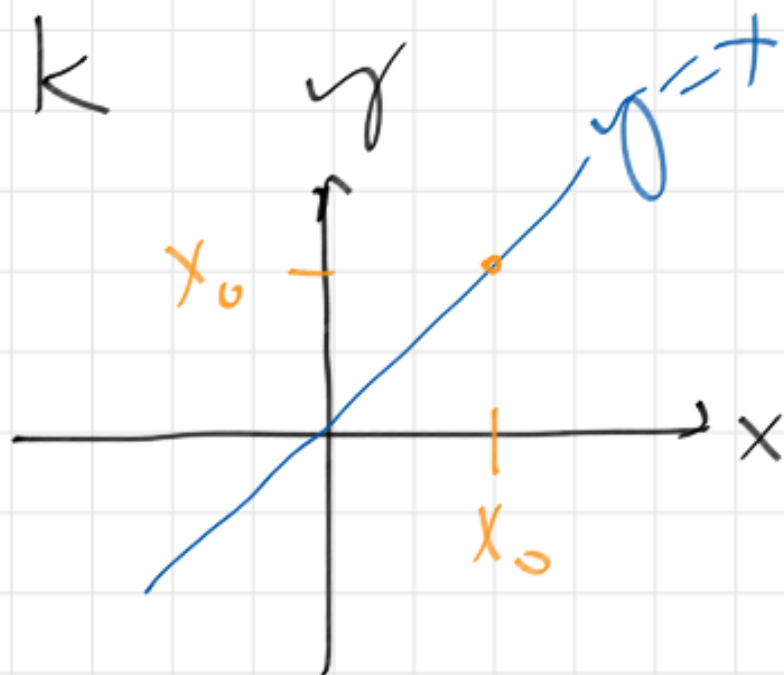
(Note: In the original image, the $(x-1)$ terms in the numerator and denominator are crossed out with orange lines, and an orange arrow points from the remaining $(x+1)$ in the numerator to the next line.)

$$\lim_{x \rightarrow 1} x + 1 = 2 //$$

$$f(x) = k = \text{const}$$

$$\lim_{x \rightarrow x_0} f(x) = k$$

$$f(x) = x$$



$$\lim_{x \rightarrow x_0} f(x) =$$

$$\lim_{x \rightarrow x_0} x = x_0$$

Ex $f(x) = x^3 + 3x - 2$

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \left(\lim_{x \rightarrow 2} x \right)^3 + 3 \lim_{x \rightarrow 2} x - 2 \\ &= 2^3 + 3 \times 2 - 2 \end{aligned}$$

Some way limits can fail

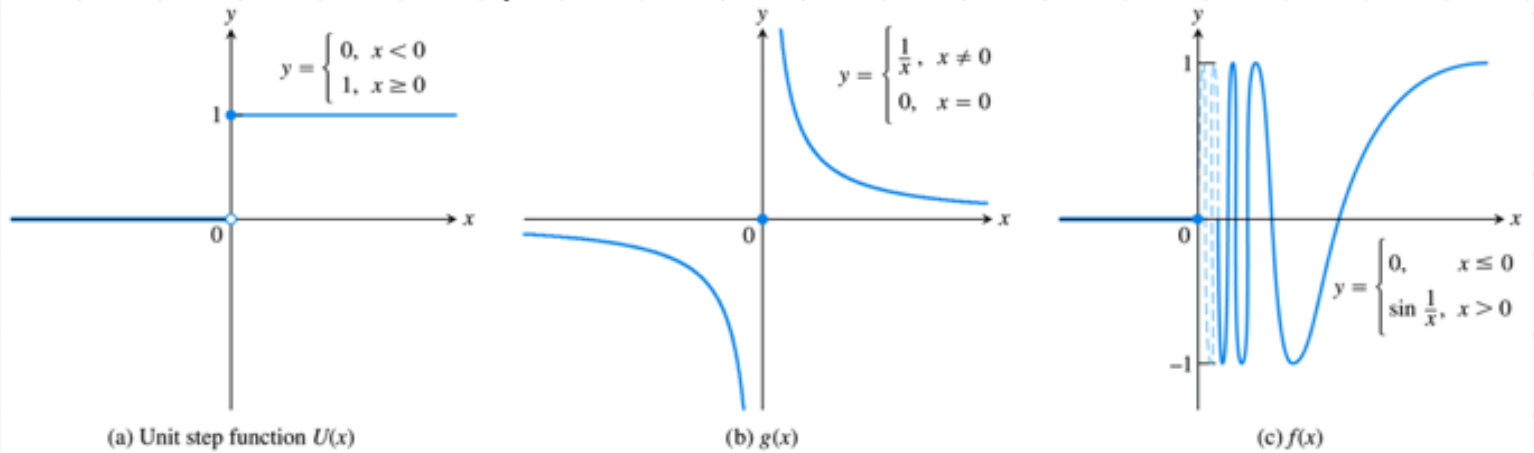


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

$\lim_{x \rightarrow 0} f(x)$ does NOT exist!

THEOREM 1—Limit LawsIf L, M, c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \rightarrow c} [f(x)]^n = L^n, n \text{ a positive integer}$
7. *Root Rule:* $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n \text{ a positive integer}$

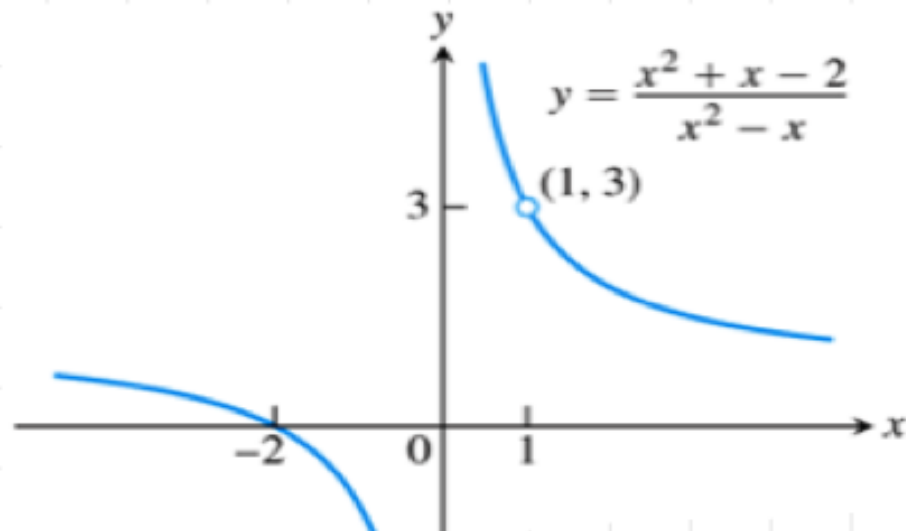
(If n is even, we assume that $\lim_{x \rightarrow c} f(x) = L > 0$.)

Ex $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3} = \frac{0}{0}$

$$\lim_{x \rightarrow 3} \frac{(x+1)(x-3)}{x-3} =$$

$$\lim_{x \rightarrow 3} (x+1) = 4$$

Ex



$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} =$$

$$\lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{x(x-1)} =$$

$$\lim_{x \rightarrow 1} (x+2) = 3 //$$

Ex

$$\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} =$$

$$\lim_{x \rightarrow 3} \frac{x-3}{(x-3)(x+3)} =$$

$$\lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{6}$$

Examples on Rationalization

$$\lim_{h \rightarrow 0} \frac{\sqrt{4+h}-2}{h} =$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} =$$

$$\lim_{h \rightarrow 0} \frac{\cancel{4} + \cancel{h} - \cancel{2^2}}{\cancel{h} (\sqrt{\cancel{4} + \cancel{h}} + 2)} =$$

$$\lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}$$

Ex

$$\lim_{h \rightarrow 0} \frac{(2+h)^2 - \textcircled{4}^2}{h} =$$

$$\lim_{h \rightarrow 0} \frac{(\cancel{2} + \cancel{h} - \cancel{2})(\cancel{2} + \cancel{h} + \cancel{2})}{\cancel{h}} =$$

$$\lim_{h \rightarrow 0} (4+h) = 4 //$$

Ex

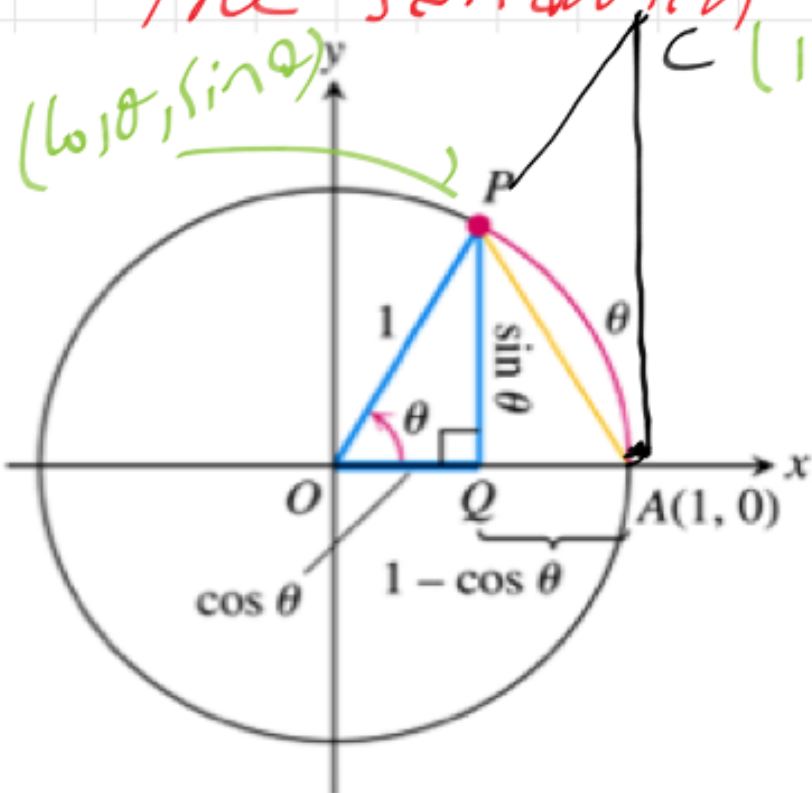
$$\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right) =$$

$$\lim_{t \rightarrow 0} \left(\frac{\cancel{t^2} + \cancel{t} - \cancel{t}}{t(t^2 + t)} \right) =$$

$$\lim_{t \rightarrow 0} \frac{\cancel{t^2}}{\cancel{t^2}(t+1)} =$$

$$\lim_{t \rightarrow 0} \frac{1}{t+1} = \frac{1}{1} = 1$$

The Sandwich Thm



Segment

$$\frac{\text{Area}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Segment Area} = \frac{1}{2} \theta r^2$$

$$\text{Area } \triangle OAC \geq \text{Seg. Area} \geq \text{Area } \triangle OAP$$

$$\frac{1}{2} \cdot 1 \cdot \tan \theta \geq \frac{1}{2} \theta \cdot 1^2 \geq \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

$$\frac{1}{\cos \theta} \geq \frac{\theta}{\sin \theta} \geq 1$$

$$\cos \theta \geq \frac{\sin \theta}{\theta} \geq 1$$

\downarrow
 \perp

\downarrow
 \perp

$\lim_{\theta \rightarrow 0}$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Now, we shall consider the sandwich theorem.

$$g(x) \leq f(x) \leq h(x)$$

\downarrow
 L

$\Rightarrow \text{Ligad}(x) = L$

Ex $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x} =$

$$\lim_{x \rightarrow 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 4x}{4x}} \cdot \frac{5}{4} = \frac{5}{4}$$

Ex $\lim_{x \rightarrow 0^-} \frac{h}{\sin 3h} =$

$$\lim_{x \rightarrow 0^-} \frac{3h}{\sin 3h} \cdot \frac{1}{3} =$$

change of variable

$$\frac{1}{3} \lim_{\theta \rightarrow 0^-} \frac{\theta}{\sin \theta} = \frac{1}{3} \lim_{\theta \rightarrow 0^-} \frac{1}{\sin \theta / \theta} = 1/3$$

Ex

$$\lim_{t \rightarrow 0} \frac{t \sin t}{\sin 2t^2} =$$

$$\lim_{t \rightarrow 0} \frac{\frac{\sin t}{t}}{\frac{\sin 2t^2}{2t^2}} \cdot \frac{1}{2} = \frac{1}{2}$$

Ex

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} =$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} =$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} =$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\theta} \cdot \frac{\theta}{1 + \cos \theta} =$$

$$\frac{0}{1+1} = \frac{0}{2} = 0 //$$

Ex

$$\lim_{\theta \rightarrow 0} \cos\left(\frac{\pi \theta}{\sin \theta}\right) =$$

$$\hookrightarrow \cos\left[\lim_{\theta \rightarrow 0} \frac{\pi}{\frac{\sin \theta}{\theta}}\right] =$$

$$\cos(\pi) = -1 //$$

Ex

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x (\sin x - 1) =$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x (\sin x - 1) \frac{(\sin x + 1)}{(\sin x + 1)} =$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x \frac{\sin^2 x - 1}{1 + \sin x} = \lim_{x \rightarrow \frac{\pi}{2}} \tan x \frac{(\sin x - 1)(\sin x + 1)}{1 + \sin x}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{-\cos^2 x}{1 + \sin x} =$$
$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x \cos x}{1 + \sin x} = 0/2 = 0 //$$