Chapter 7

Inverse functions and their derivatives

$$y(x) = f(x) = x^3$$

$$f^{-1}(x) = \sqrt[3]{x}$$

f^{-1} is inverse fn. of f

$$y = x^3$$

$$(y)^{1/3} = (x^3)^{1/3} \Rightarrow x = \sqrt[3]{\boxed{y}}$$

$$f^{-1}(x) = \sqrt[3]{x}$$

$f(x) = x^3$

$f^{-1}(x) = \sqrt[3]{x}$

$$\begin{tikzcd} 2 \arrow[r, bend left, "f"] & 8 \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

For a fn. to have its inverse, that is, to be able to invert it, the fn. must be one-to-one.

Def. A fn. $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

$$f(x) = y$$

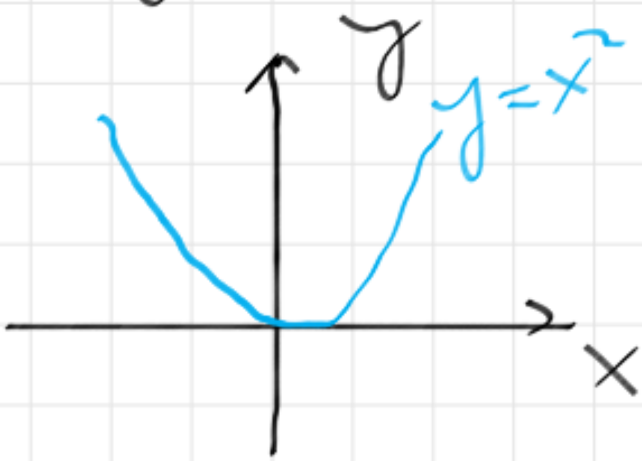
$$f^{-1}(y) = x$$

Ex

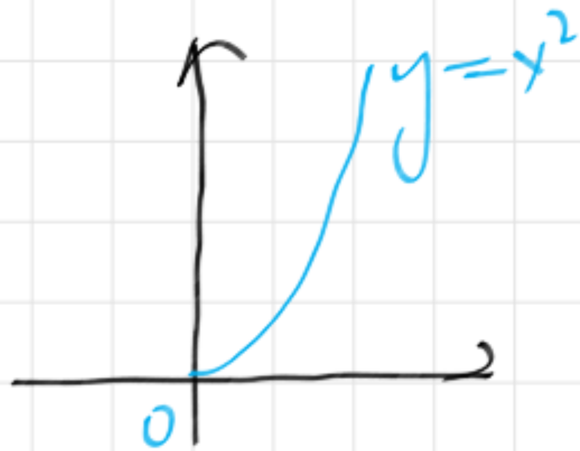
$y = x^2$ is <u>not</u> 1-1 fn.

But

$y = x^2$ is 1-1 on $D: [0, \infty)$

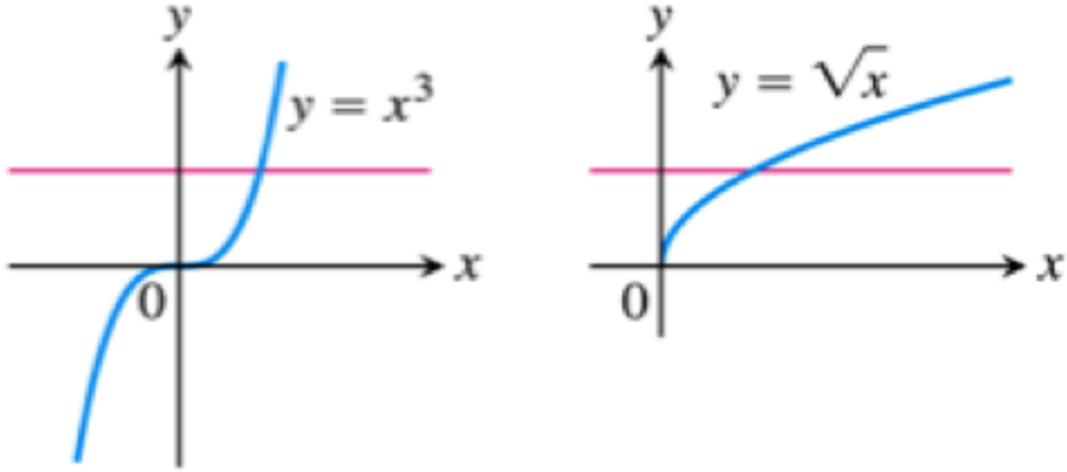


$D: (-\infty, \infty)$

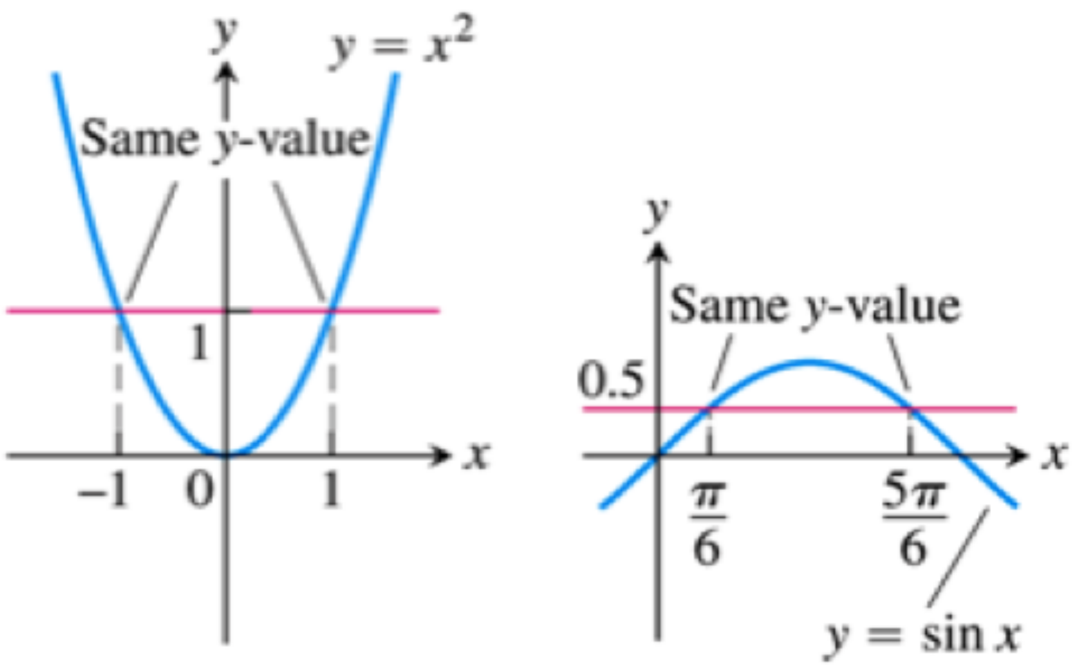


$D: [0, \infty)$

Horizontal line test



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

Ex

$$f(x) = 3x - 2 \quad , \quad f^{-1}(x) = ?$$

$$y = 3x - 2$$

$$y + 2 = 3x$$

$$\frac{1}{3}(y+2) = x$$

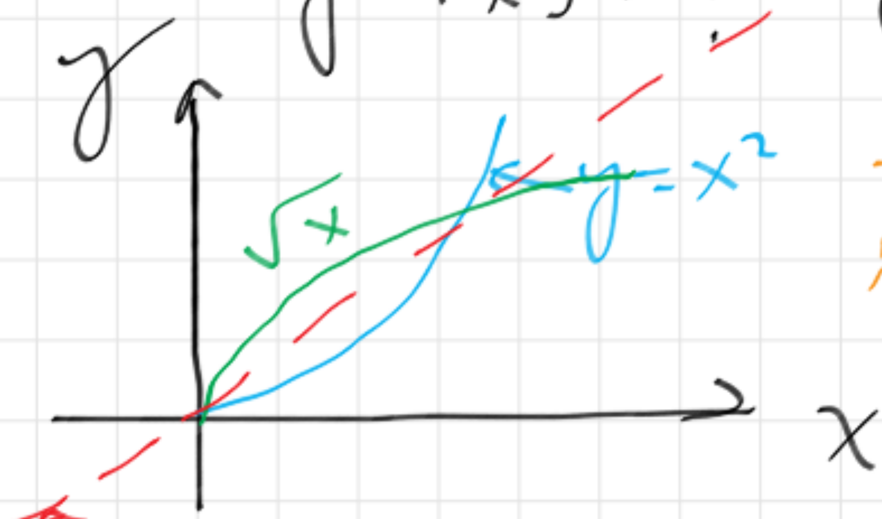
$$\frac{1}{3}(\underset{x}{\underset{\uparrow}{y}}+2) = \underset{f^{-1}(x)}{\underset{\uparrow}{x}}$$

$$f^{-1}(x) = \frac{1}{3}(x+2) \quad //$$

Ex

$$y(x) = f(x) = x^2$$

$$f^{-1}(x) = ? \quad \text{for } D: [0, \infty)$$



the graph of f^{-1} is reflected across $y = x$

$$y = x^2 \Longrightarrow x = \sqrt{y}$$

$$f^{-1}(x) = \sqrt{x}$$

Theorem: The Derivative Rule for inverse functions

$$\boxed{f^{-1\,\prime} = \frac{1}{f'(x)}}$$

Ex

$$f(x) = x^3 - 3x^2 - 1 \ , \quad x \geq 2$$

Find the der. of $f^{-1}(x)$ when $x = -1 = f(3)$

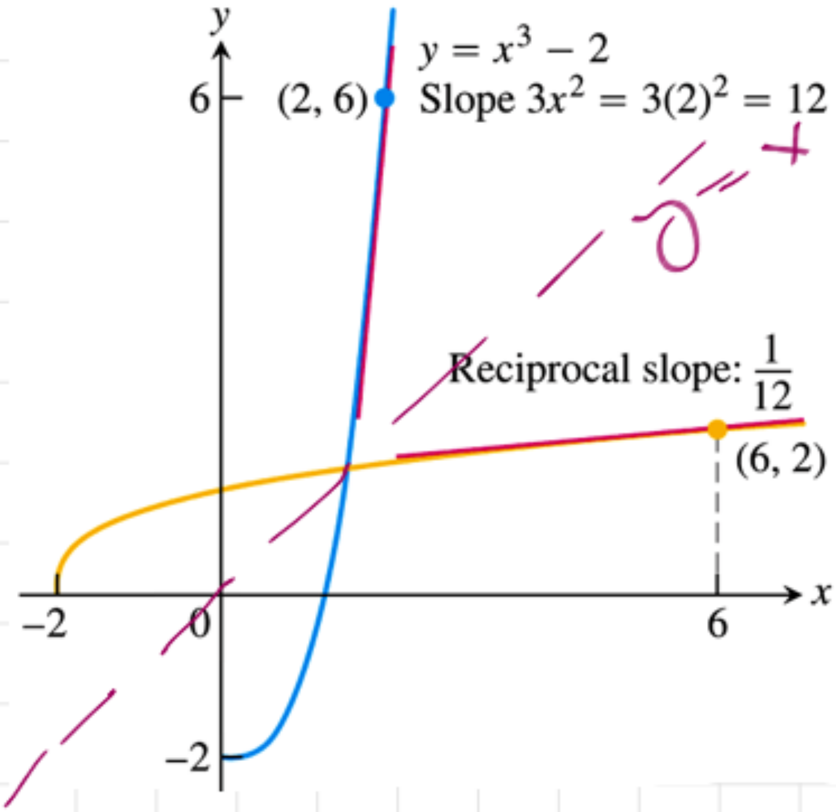
$$f'(x) = 3x^2 - 6x = 3x(x-2) > 0$$

for $x > 2$.

$$\left(f^{-1}\right)' = \frac{1}{f'(x)} = \left.\frac{1}{3x^2 - 6x}\right|_{x=3}$$

$$\left(f^{-1}\right)' = \frac{1}{3 \times 3^2 - 6 \times 3} = \frac{1}{9} \ //$$

Ex



$$y = f(x) = x^3 - 2$$

$$y + 2 = x^3$$

$$(y+2)^{1/3} = x$$

$$f^{-1}(x) = (x+2)^{1/3}$$

$$f'(x) = 3x^2, \quad f^{-1\,'} = \frac{1}{3}\,\frac{1}{(x+2)^{2/3}}$$

$$f'(x) = 3x^2, \quad f^{-1\,'} = \frac{1}{3}\frac{1}{(x+2)^{2/3}}$$

$$f'(2) = 3\cdot 2^2 = 12$$

$$f^{-1\,'}(6) = \frac{1}{3}\frac{1}{(6+2)^{2/3}}$$

$$= \frac{1}{12}$$

$$f^{-1\,'}(6) = \frac{1}{f'(2)} = \frac{1}{12}$$

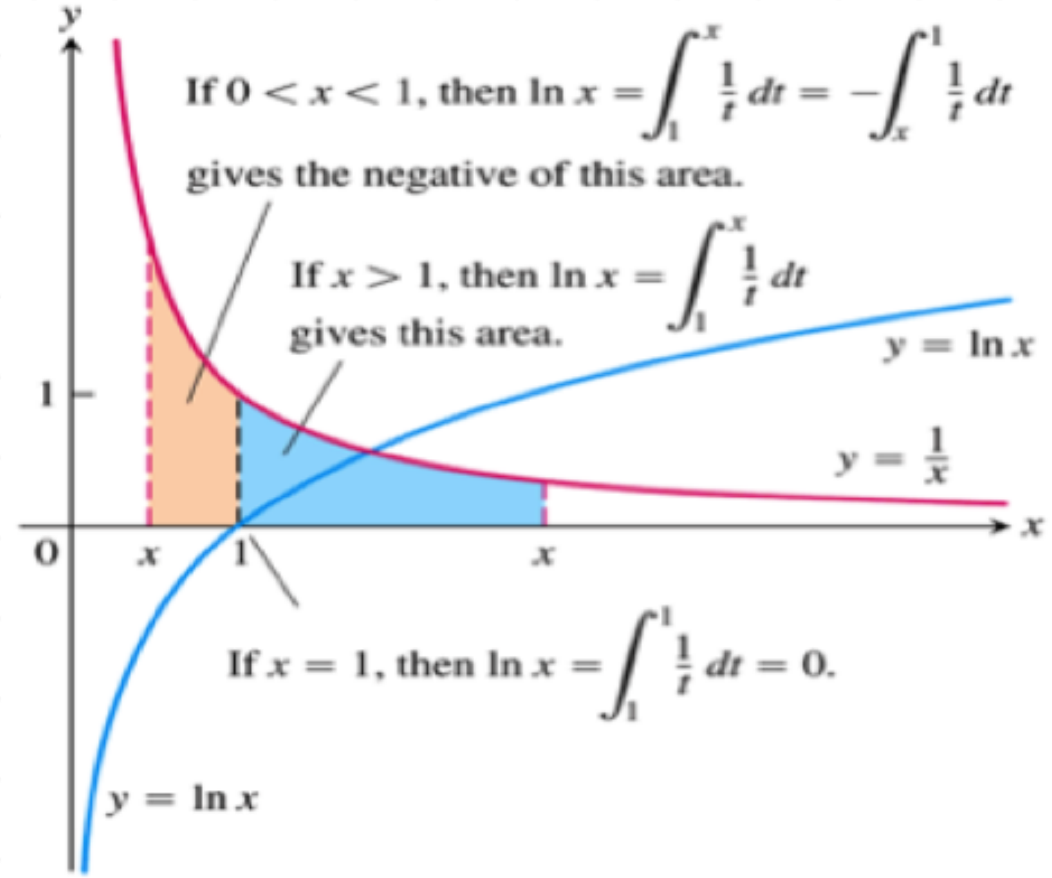
Natural Logarithm

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0. \tag{1}$$

$\ln x$ is the area fn. of $f(t) = \frac{1}{t}$, if $x \geq 1$.

Since $f(t) = \frac{1}{t}$ is a positive fn on $t > 0$, the area fn. is monotone decreasing, hence it is 1–1 $\therefore$ invertable.



Considering area of $y = \ln x$ curve in above graph,

$\ln 2 \sim 0{,}69$

$\ln 3 \sim 1.10$

$\ln e = 1$, $2 < e < 3$

$\ln e = \int_1^e \frac{1}{t}\,dt = 1$ $e = 2.71828\ldots$

By the fundamental theorem of Calculus,

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Ex $\quad \frac{d}{dx}\ln ax = \frac{1}{ax}a = \frac{1}{x}$

if $x<0$ and $b=-1$, then

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}|x| = \frac{1}{x}}, \quad x \neq 0$$

Properties of $\ln x$

$\ln bx = \ln b + \ln x$ prod. rule

$\ln \frac{b}{x} = \ln b - \ln x$ quotient rule

$\ln \frac{1}{x} = -\ln x$ reciprocal rule

$\ln x^r = r \ln x$ power rule

From graph of $\ln x$

$\ln 1 = 0$, domain $x > 0$, invertable.

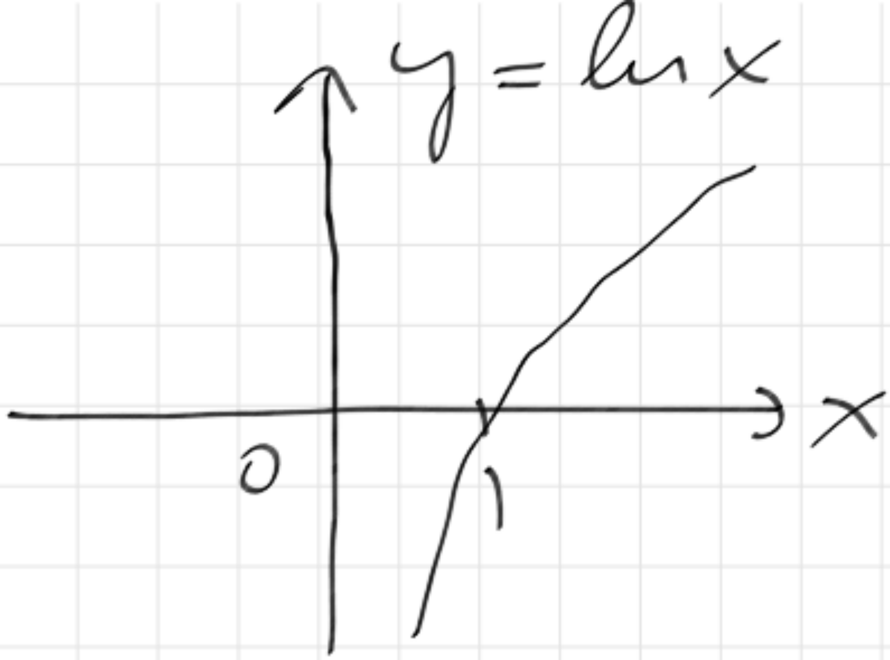
$$\frac{d^2}{dx^2}\ln x = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\ln 2^n = n\ln 2 > n/2$$

$$\lim_{n\to\infty} \ln 2^n = \infty$$

$$\ln 2^{-n} = -n\ln 2 < -n/2$$

$$\lim_{n\to\infty} \ln 2^{-n} = -\infty$$



$$\lim_{x \to 0^+} \ln x = -\infty$$

Ex

$$\int \frac{f(x)\,dx}{f'(x)} = \int \frac{du}{u}$$

$$= \ln u + c$$

$$= \ln|f(x)| + c$$

Ex

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x}$$

$$u = \cos x, \quad du = -\sin x \, dx$$

$$\int \tan x \, dx = -\int \frac{du}{u} = -\ln u + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

Similarly

$$\int \cot x \, dx = \int \frac{\cos x \, dx}{\sin x} = \ln|\sin x| + C$$