3.4 The Derivative as a Rate of Change

Ex. A bacteria grows according to eq $b(t) = 10^6 + 10^4 t - 10^3 t^2$

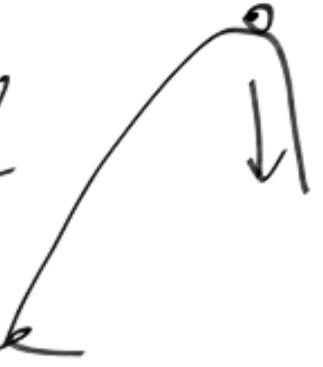
Find the growth rate at t=0, 5, and 10 hours

$b'(t) = 10^4 - 2\times 10^3 t$

$b'(0) = 10^4$ increase

$b'(5) = 10^4 - 10^4 = 0$ stopped

$b'(10) = 10^4 - 2\times 10^4 = -10^4$ decrease



indicates decrease.

Ex Find the tangent line to the "Witch of Agnesi"

$$y(x) = \frac{8}{x^2+4} \quad \text{at} \quad (2,1)$$

$$y' = -\frac{2x \cdot 8}{(x^2+4)^2}$$

$$y'(2) = -\frac{2 \cdot 2 \cdot 8}{(4+4)^2} = -\frac{1}{2}$$

$$-\frac{1}{2} = \frac{y-1}{x-2} \Rightarrow \underline{y = -\frac{1}{2}x + 2}$$

Ex. Find the values of a and b that makes the following fn differentiable for all x values

$$f(x) = \begin{cases} ax+b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

$$\frac{d}{dx}(ax+b)\Big|_{x=-1} = \frac{d}{dx}(bx^2-3)\Big|_{x=-1}$$

$$a\Big|_{x=-1} = 2bx\Big|_{x=-1}$$

$$a = -2b \Rightarrow -a = 2b$$

$$\lim_{x \to -1^-} bx^2 - 3 = \lim_{x \to -1^+} ax + b$$

$$b - 3 = -a + b \Rightarrow$$

$$-a = 7b$$

$$b - 3 = -a + b$$

$$b - 3 = 2b + b$$

$$-3/2 = b$$

$$-a = 7b$$

$$-a = 7 \times \left(-\frac{3}{2}\right)$$

$$a = 3$$

Derivatives of Trigonometric Functions

Trigonometric fns.

$\sin x, \cos x, \tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x},$

$\sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}$

$\frac{d}{dx} \sin x = ?$ $\quad f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$\cos h = \cos\left(\frac{h}{2} + \frac{h}{2}\right) = \cos^2\frac{h}{2} - \sin^2\frac{h}{2}$$

$$= 1 - \sin^2\frac{h}{2} - \sin^2\frac{h}{2}$$

$$= 1 - 2\sin^2\frac{h}{2}$$

$$= \sin x \lim_{h\to 0} \frac{1 - 2\sin^2\frac{h}{2} - 1}{h} + \cos x$$

$$= -\sin x \lim_{h\to 0} \sin\frac{h}{2} \cdot \frac{\sin\frac{h}{2}}{h/2} + \cos x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x = 1+\tan^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Ex $y = \dfrac{\sin x}{x}$

$$y' = \frac{x\cos x - \sin x}{x^2}$$

Ex $y = \dfrac{\cos x}{1+\sin x}$, $y' = \dfrac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2}$

$$y' = -\frac{1}{1+\sin x}$$

Ex $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

$$y' = \frac{-x\sin x - \cos x}{x^2} + \frac{\cos x + x\sin x}{\cos^2 x}$$

Ex $y = \sqrt{x}\sec x + 3 = x^{\frac{1}{2}}\sec x + 3$

$y' = \frac{1}{2}x^{-\frac{1}{2}}\sec x + \sqrt{x}\sec x\tan x$

$y' = \left(\frac{1}{2\sqrt{x}} + \sqrt{x}\tan x\right)\sec x$

Ex

$y = c\csc x - 4\sqrt{x} + 7$

$y' = -\csc x\cot x - 4\cdot\frac{1}{2}x^{-\frac{1}{2}}$

$= -\csc x\cot x - 2/\sqrt{x}$

Ex $y = \sin x\tan x$

$y' = \cos x\tan x + \sec^2 x\cdot\sin x$

$y' = \cancel{\cos x}\frac{\sin x}{\cancel{\cos x}} + \sec^2 x\sin x = \sin x(1+\sec^2 x)$

Ex $y = (\sec x + \tan x)(\sec x - \tan x)$

$y' = (\sec x \tan x + \sec^2 x)(\sec x - \tan x) +$

$(\sec x + \tan x)(\sec x \tan x - \sec^2 x)$

$= 2\sec^2 x \tan x - 2\tan x \sec^2 x$

$= 0$

Ex $y = (\sin x + \cos x)\sec x$

$y' = (\cos x - \sin x)\sec x + (\sin x + \cos x)\sec x \tan x$

$y' = \sec x\left(\cos x - \sin x + \sin x \tan x + \sin x\right)$

$y' = \sec x\left(\cos x + \frac{\sin^2 x}{\cos x}\right)$

$= \sec x\left(\frac{\cos^2 x + \sin^2 x}{\cos x}\right)$

$= \frac{1}{\cos^2 x}$ //

Ex $y(x) = \csc x \cot x$

$y' = -\csc x \cot x \cdot \cot x - \csc^2 x \cdot \csc x$

$y' = -\csc x \cot^2 x - \csc^3 x$

Ex $y = x^2 \cot x - \left(\frac{1}{x^2}\right) x^{-2}$

$y' = 2x \cot x - x^2 \csc x + \frac{2}{x^3}$