Mathematical Induction (December 16, 2020)

Principle of Mathematical Induction:

Let $p(m), p(m+1), p(m+2), \ldots$ be a sequence of propositions.

If

✓ (B) $p(m)$ is true, and

✓ (I) $p(k+1)$ is true whenever $p(k)$ is true and $m \le k$,

then all propositions are true.

Condition (B) in the principle of induction is called basis, and (I) is the inductive step. Given a list of propositions, these principles help us to organize a proof that all the propositions are true.

Ex: Let $p(n)$ be $\boxed{n! > 2^n}$ for $n \geq 4$. ✓

Prove by induction.

$p(n) : n! > 2^n, n \geq 4$

$p(4) : 24 > 16$

$p(5) : 120 > 32$

$p(6) : 720 > 64$

$\vdots$

(B) $\boxed{p(4) \text{ is true}}$ since $\boxed{4! = 24 > 16 = 2^4}$

(I) Suppose $p(k)$ is true, that is $\boxed{k! > 2^k}$. $(k \geq 4)$

Then let us see that $p(k+1)$ is also true, that is $(k+1)! > 2^{k+1}$

$\rightarrow (k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$

$\boxed{p(k) \rightarrow p(k+1)}$

So, $(k+1)! > 2^{k+1}$ is true.

$p(k)$ ✓

Ex: Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all n in $\mathbb{Z}^+$ by induction.

Let $p(n)$ be $\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$. Then

(B): $p(1)$: $\sum_{i=1}^{1} i = \frac{1 \cdot (1+1)}{2} = 1$ is true.

(I): Assume that $p(k)$ is true for some $k \in \mathbb{Z}^+$ i.e. $\sum_{i=1}^{k} i = \frac{k \cdot (k+1)}{2}$

We want to show that this assumption implies that $p(k+1)$ is true,

i.e. $\sum_{i=1}^{k+1} = \boxed{\frac{(k+1)\cdot(k+2)}{2}}$

$$\sum_{i=1}^{k+1} i = \underline{\sum_{i=1}^{k} i + k+1} = \frac{k\cdot(k+1)}{2} + k+1 = \left(\frac{k}{2}+1\right)\cdot(k+1)$$
$$= \left(\frac{k+2}{2}\right)\cdot(k+1)$$
$$= \frac{(k+2)\cdot(k+1)}{2} \checkmark$$

Ex: Show that $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$ is true for $r\neq 0$, $r\neq 1$ and $n\in\mathbb{N}$.

Let $p(n)$ be $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$. Then

(B) $p(0)$: $\sum_{i=0}^{0} r^i = \frac{r^1-1}{r-1} = r^0$ is true.

(I) Assume $p(k)$ is true. Let us see that $p(k+1)$ is true.

$$\boxed{\sum_{i=0}^{k+1} r^i} = \sum_{i=0}^{k} r^i + r^{k+1} = \frac{r^{k+1}-1}{r-1} + r^{k+1} = \frac{\cancel{r^{k+1}}-1+r^{k+2}-\cancel{r^{k+1}}}{r-1}$$
$$= \boxed{\frac{r^{k+2}-1}{r-1}} \checkmark$$

Ex: Show that $11^n - 4^n$ is divisible by 7 for all $n\in\mathbb{Z}^+$

(B) : For $n=1$: $11^1 - 4^1 = 7$ is divisible by 7, obviously.

(I) : Assume $\underline{11^k - 4^k = 7\cdot \ell}$ for some $\ell\in\mathbb{Z}$.

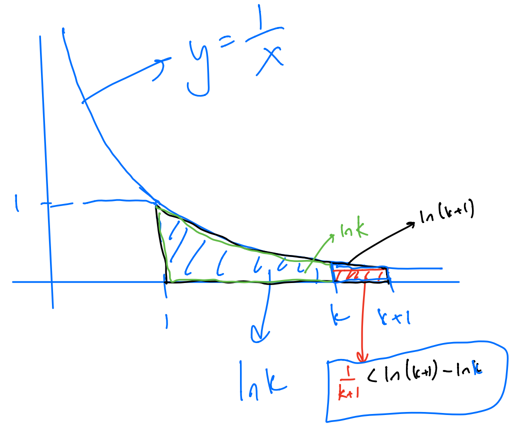
Let us see that $11^{k+1} - 4^{k+1}$ is divisible by 7.

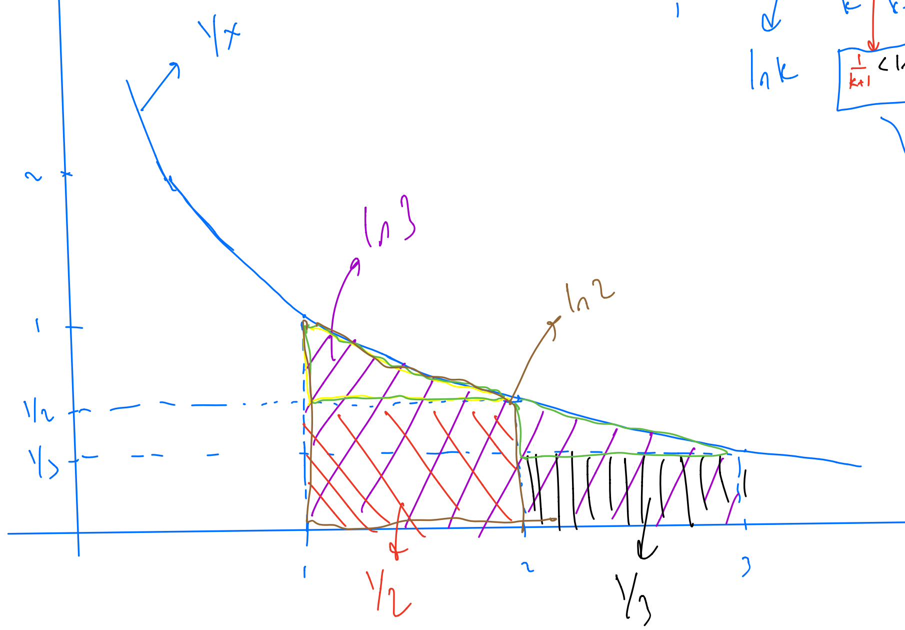
$$11^{k+1} - 4^{k+1} = 11\cdot 11^k - 4\cdot 4^k$$
$$= 4\cdot 11^k + 7\cdot 11^k - 4\cdot 4^k$$
$$= 4\cdot\underbrace{(11^k - 4^k)}_{7\ell} + 7\cdot 11^k = 4\cdot 7\cdot \ell + 7\cdot 11^k$$
$$= 7\cdot(4\cdot\ell + 11^k)$$

is divisible by 7.

Ex: Show that $\boxed{\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} < \ln n}$ for $n \geqslant 2$.

Hint: $\ln x = \int_1^x \frac{1}{x}\,dx$





see that $1/2 + 1/3 < \ln 3$

(B) $\rightarrow$ $1/2 < \ln 2$

(I) Assume $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} < \ln k$

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1} < \ln k + \boxed{\frac{1}{k+1}} < \cancel{\ln k} + \boxed{\ln(k+1) - \cancel{\ln k}}$$

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k+1} < \ln(k+1) \quad \checkmark$$