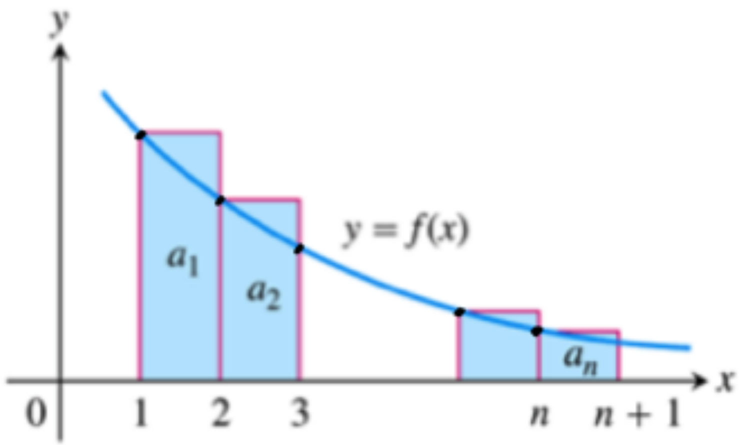
Integral Test



(a)

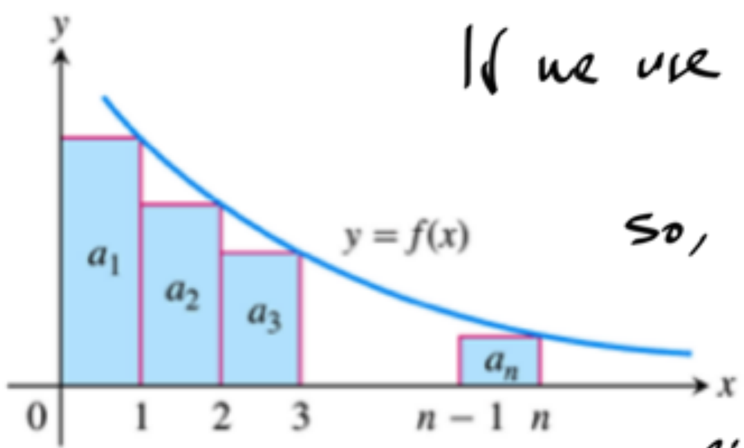
The area under curve is $y=f(x)$, which is an improper integral and the series may approximate this area.

$$\text{Area} = \int_1^\infty f(x)\,dx = \lim_{b\to\infty}\int_1^b f(x)\,dx$$

In terms of rectangles, if we use n rect:

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \quad \text{(for Fig. a)}$$

Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.



(b)

If we use Fig. b: $a_1 + a_2 + \cdots \le \int_1^{n+1} f(x)\,dx$

So, $$\underset{A}{\sum_{k=2}^{n+1} a_k} \le \underset{B}{\int_1^{n+1} f(x)\,dx} \le \underset{C}{\sum_{k=1}^{n} a_k}$$

$\lim_{n\to\infty}$ B $<\infty$, implies that $\lim_{n\to\infty}$ A $<\infty$, so that series must converge.

Ex $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ $\quad f(x) = \frac{1}{x\ln x}$, is decreasing, since

Now, integral test: $\quad f'(x) = -\frac{1}{x^2\ln^2 x}(1+\ln x) < 0$

$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b = \infty$, so, by integral test, it diverges.

The p-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$, for what values of p, does it converge?

$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \frac{x^{-p+1}}{1-p}\Big|_1^b$, if $p>1$, it converges; diverges otherwise.

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$, since $p=1$, $\sum \frac{1}{n}$ diverges

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad \int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \operatorname{Arctan} x\Big|_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$

it converges

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$

$$\int_1^{\infty} \frac{dx}{\sqrt[3]{2x-1}} = \frac{1}{2}\int_1^{\infty} \frac{2\,dx}{\sqrt[3]{2x-1}} = \frac{1}{2}\int u^{1/3}\,du = \frac{1}{2}\cdot\frac{3}{2}u^{2/3} = \frac{3}{4}\lim_{n\to\infty}(2x-1)^{2/3}\Big|_1^n$$

$$= \infty$$

$u = 2x-1$

$du = 2x\,dx$

The series diverges by integral test!

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1}k}{1+k^2}$

$$\int_1^{\infty} \frac{\tan^{-1}x\,dx}{1+x^2} = \int u\,du = \frac{1}{2}u^2 = \lim_{n\to\infty} \frac{1}{2}(\tan^{-1}x)^2\Big|_1^n = \frac{1}{2}\left[(\tan^{-1}(\infty))^2 - (\tan^{-1}(1))^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$$

$$= \frac{3}{32}\pi^2$$

$u = \tan^{-1}x$

$x = \tan u$

$\frac{dx}{du} = 1+\tan^2 u$

$\frac{dx}{1+\tan^2 u} = du$

$\frac{dx}{1+x^2} = du$

Comparison Test

We have to have a nice collection of known convergent series to compare an unknown series to.

Ex $\sum_{n=1}^{\infty} \frac{1}{n!}$, $\frac{1}{n!} \leq \frac{1}{2^n}$ for all n.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to zero, so does $\sum_{n=1}^{\infty} \frac{1}{n!}$

THEOREM —The Comparison Test Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \qquad \text{for all} \qquad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Ex $\sum_{n=2}^{\infty} \frac{1}{\ln n}$, we know that $\sum \frac{1}{n}$ is divergent

$$\frac{1}{\ln n} \geq \frac{1}{n}$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

Ex

$\sum \frac{1}{n(n+1)}$, is it convergent? It is similar to $\sum \frac{1}{n^2}$, is conv.

However, $\frac{1}{n^2} > \frac{1}{n(n+1)}$, therefore we need to apply the following second test.

Limit form (Asymptotic form) Comparison test

THEOREM —Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

$$\lim_{n\to\infty} \frac{1/n^2}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right) = 1$$

Ex $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}+3n^{1/3}-5}$ is it convergent or not?

In $\sum \frac{1}{n^{1/2}}$, since $p = 1/2 < 1$, it is divergent

we can compare with $\sum \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/n^{1/2}}{n^{1/2}+3n^{1/3}-5} = \lim_{n\to\infty} \left(1+\frac{3}{n^{1/6}}-\frac{5}{n^{1/2}}\right) = 1$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{1/2}+3n^{1/3}-5}$ is divergent!