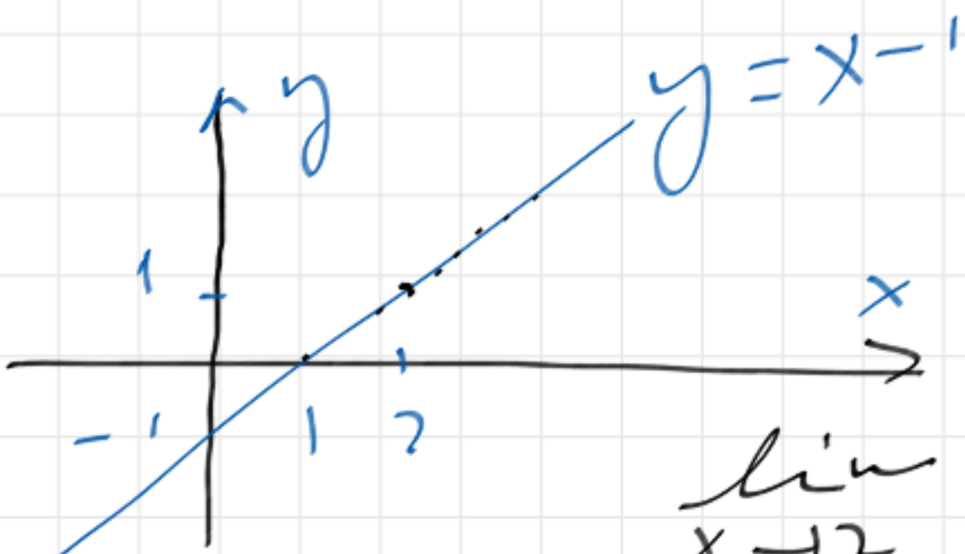
limits

Def. Let $y = f(x)$, $L \in \mathbb{R}$

The limit of $f(x)$ as x approaches x_0 is L.

$$\lim_{x \to x_0} f(x) = L$$

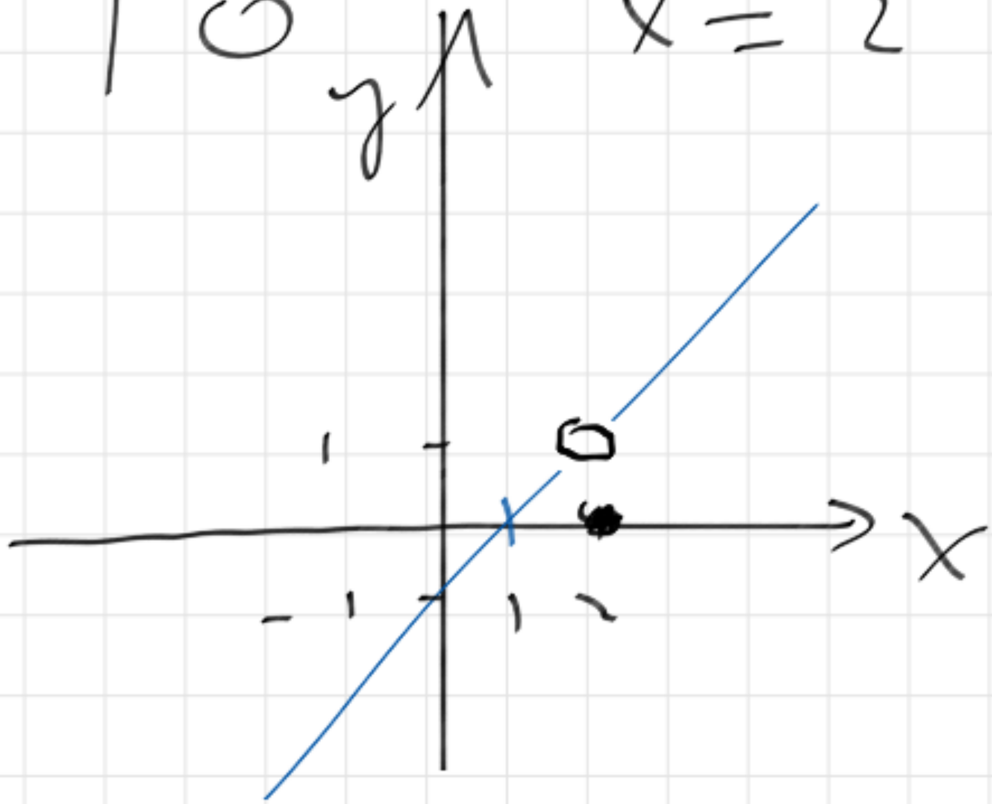
Ex



$$\lim_{x \to 2} x - 1 = 1$$

EX

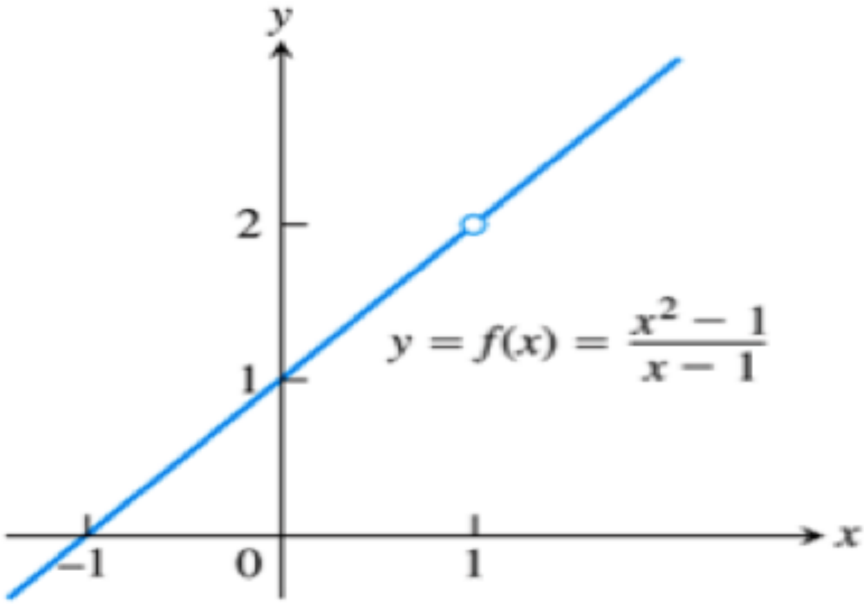
$$y = \begin{cases} x-1 & x \neq 2 \\ 0 & x = 2 \end{cases}$$



$f(2) = 0$

However

$$\lim_{x \to 2} x-1 = 1$$



$f(1) = ?$ not defined!

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \quad 0/0$$

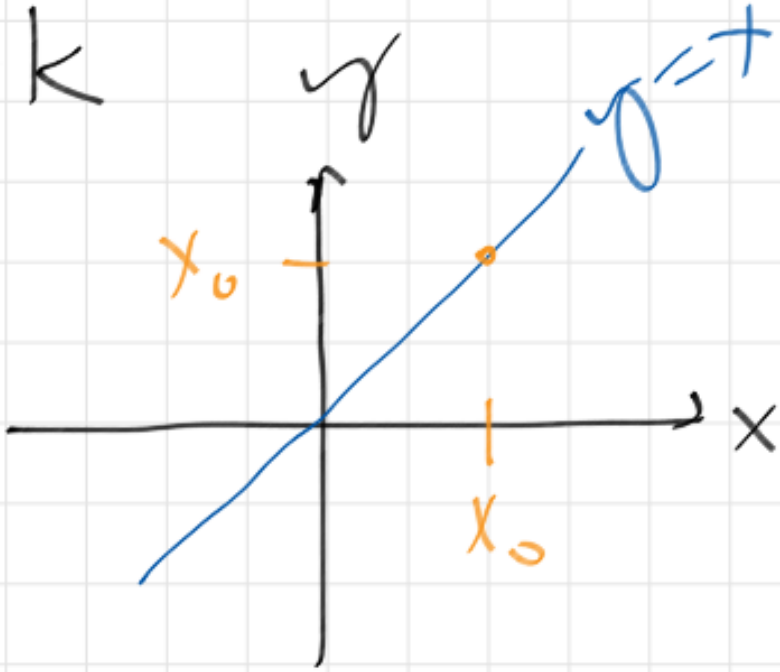
$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} =$$

$$\lim_{x \to 1} x + 1 = 2 \quad //$$

$$f(x) = k = \text{const}$$

$$\lim_{x \to x_0} f(x) = k$$

$$f(x) = x$$



$$\lim_{x \to x_0} f(x) =$$

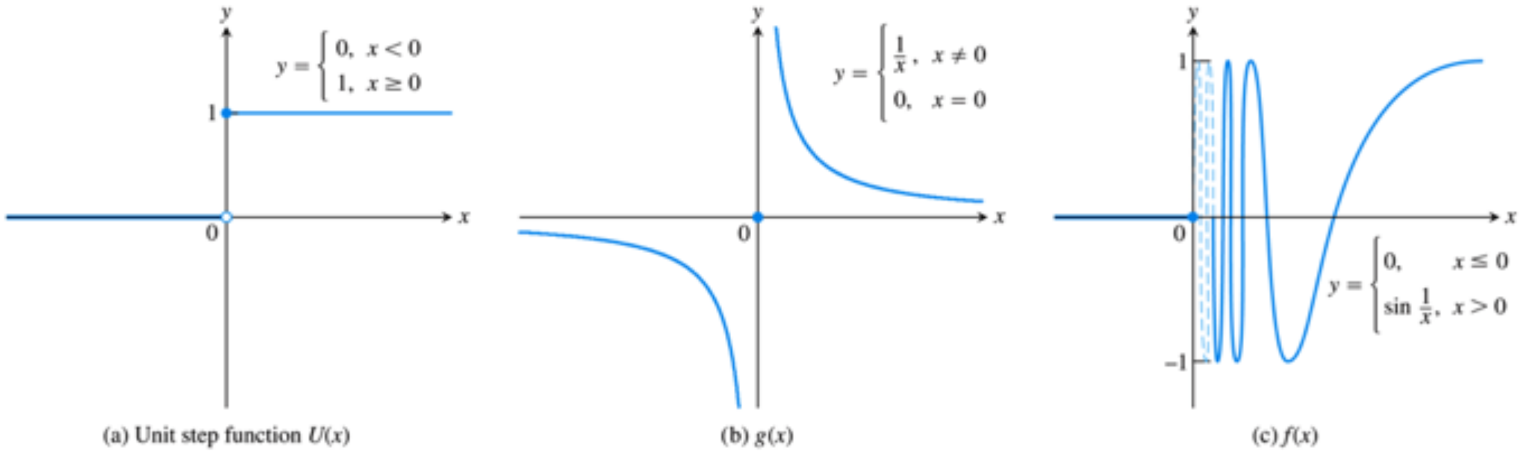
$$\lim_{x \to x_0} x = x_0$$

Ex $f(x) = x^3 + 3x - 2$

$$\lim_{x \to 2} f(x) = \left(\lim_{x \to 2} x\right)^3 + 3\left(\lim_{x \to 2} x\right) - 2$$

$$= 2^3 + 3 \times 2 - 2$$

Some way limits can fail



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

$\lim_{x \to 0} f(x)$ does NOT exist!

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

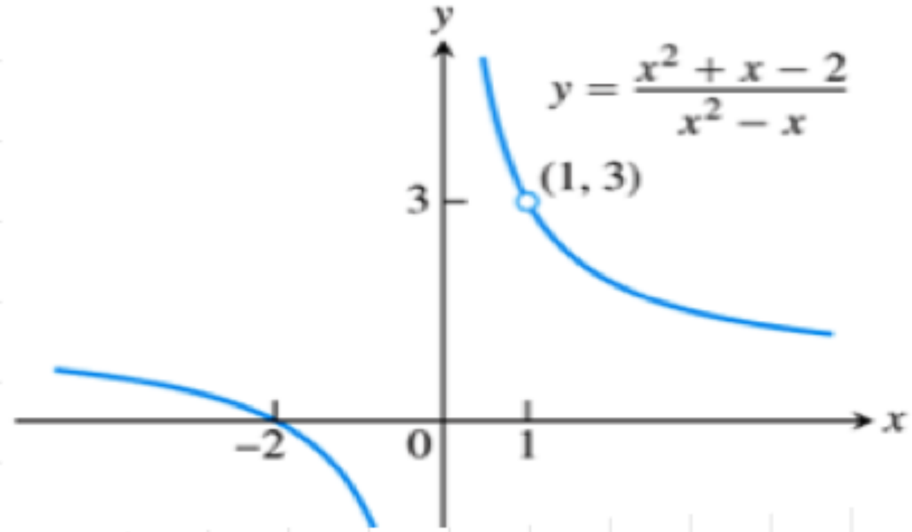
(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Ex $\lim_{x \to 3} \dfrac{x^2 - 2x - 3}{x - 3} = 0/0$

$\lim_{x \to 3} \dfrac{(x+1)\cancel{(x-3)}}{\cancel{x-3}} =$

$\lim_{x \to 3} (x+1) = 4$

Ex



$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} =$$

$$\lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} =$$

$$\lim_{x \to 1} (x+2) = 3$$

Ex

$$\lim_{x\to 3} \frac{x-3}{x^2-9} =$$

$$\lim_{x\to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)} =$$

$$\lim_{x\to 3} \frac{1}{x+3} = \frac{1}{6}$$

Examples on Rationalization

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} =$$

$$\lim_{h\to 0} \frac{\cancel{4}+\cancel{h}-\cancel{2^2}}{\cancel{h}(\sqrt{4+h}+2)} =$$

$$\lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

Ex

$$\lim_{h\to 0} \frac{(2+h)^2-4}{h} =$$

2^2

$$\lim_{h\to 0} \frac{(\cancel{2}+\cancel{h}-\cancel{2})(2+h+2)}{\cancel{h}} =$$

$$\lim_{h\to 0} (4+h) = 4 \quad //$$

Ex

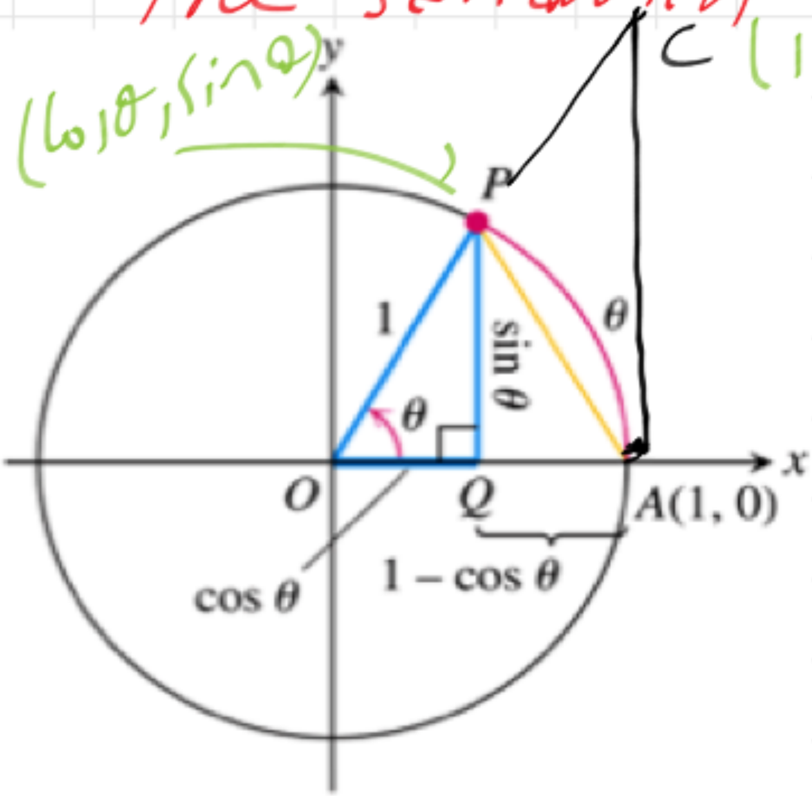
$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=$$

$$\lim_{t\to 0}\left(\frac{t^2+\cancel{t}-\cancel{t}}{t(t^2+t)}\right)=$$

$$\lim_{t\to 0}\frac{\cancel{t^2}}{\cancel{t^2}(t+1)}=$$

$$\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{1}=1$$

The Sandwich Thm



$$\frac{\text{Segment Area}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Segment Area} = \frac{1}{2}\theta r^2$$

Area $\triangle$ OAC $\geq$ Seg. Area $\geq$ Area $\triangle$ OAP

$$\frac{1}{\cancel{2}} \cdot 1 \cdot \tan\theta \geq \frac{1}{\cancel{2}}\theta \cdot 1^2 \geq \frac{1}{\cancel{2}} \cdot 1 \cdot 1 \cdot \sin\theta$$

$1/\sin\theta$ $$\frac{\sin\theta}{\cos\theta} \geq \theta \geq \sin\theta$$

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \qquad \leq 1 \qquad \lim_{\theta \to 0}$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Now, we shall consider the sandwich thm.

$$g(x) \leq f(x) \leq h(x)$$

$$L \qquad\qquad L \qquad \lim_{x \to a}$$

$$\Longrightarrow \lim_{x \to a} f(x) = L$$

Ey

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 4x} =$$

$$\lim_{x \to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 4x}{4x}} \cdot \frac{5}{4} = \frac{5}{4}$$

Ex

$$\lim_{x \to 0^-} \frac{h}{\sin 3h} =$$

$$\lim_{x \to 0^-} \frac{\underbrace{3h}_{\theta}}{\sin \underbrace{3h}_{\theta}} \cdot \frac{1}{3} =$$ change of variable

$$\frac{1}{3} \lim_{\theta \to 0^-} \frac{\theta}{\sin \theta} = \frac{1}{3} \lim_{\theta \to 0^-} \frac{1}{\sin\theta / \theta}$$

$$= 1/3$$

Ex

$$\lim_{t\to 0} \frac{t\sin t}{\sin 2t^2} =$$

$$\lim_{t\to 0} \frac{\frac{\sin t}{t}}{\frac{\sin 2t^2}{2t^2}} \cdot \frac{1}{2} = \frac{1}{2}$$

1

1

Ex

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} =$$

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} =$$

$$\lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} =$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{\theta}{1+\cos\theta} =$$

$$\frac{0}{1+1} = \frac{0}{2} = 0 \quad //$$

Ex

$$\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) =$$

$$\cos\left[\lim_{\theta\to 0} \frac{\pi}{\frac{\sin\theta}{\theta}}\right] =$$

$$\cos(\pi) = -1 \quad //$$

EX

$$\lim_{x \to \frac{\pi}{2}} \tan x (\sin x - 1) =$$

$$\lim_{x \to \pi/2} \tan x (\sin x - 1) \frac{(\sin x + 1)}{\sin x + 1} =$$

$$\lim_{x \to \pi/2} \underbrace{\tan x}_{\frac{\sin x}{\cos x}} \frac{\sin^2 x - 1}{1 + \sin x} \quad -\cos^2 x$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cancel{\cos x}} \cdot \frac{-\cos^{\cancel{2}} x}{1 + \sin x} =$$

$$-\lim_{x \to \pi/2} \frac{\sin x \cos x}{1 + \sin x} = 0/2 = 0 \,//$$