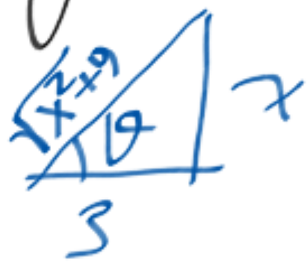
Sec 8.3 Trigonometric Substitutions

Integrals involving $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 - a^2}$

Ex

$$\int \frac{dx}{\sqrt{x^2+9}}$$



$\tan\theta = x/3$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+9}/3$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\underbrace{\sqrt{\tan^2\theta+1}}_{\sec^2\theta}}$$

$$= \int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)\, d\theta}{\sec\theta+\tan\theta}$$

$u = \sec\theta + \tan\theta$

$$= \int \frac{du}{u}$$

$u = \sec\theta + \tan\theta$

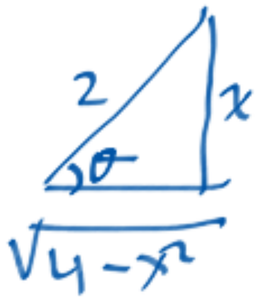
$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$

$$= \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

Ex $\displaystyle\int \frac{x^2\,dx}{\sqrt{4-x^2}}$



$\sin\theta = \frac{x}{2}$ $\to \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$\cos\theta = \sqrt{4-x^2}/2$

$$= \int \frac{4\sin^2\theta \cdot 2\cos\theta\,d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{4\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}} = \int 4\sin^2\theta\,d\theta$$

$$\cos(2\theta) = \cos(\theta+\theta)$$
$$= \cos^2\theta - \sin^2\theta$$
$$= 1 - \sin^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= 4\int \sin^2 d\theta = 4\cdot\frac{1}{2}\int (1-\cos 2\theta)\,d\theta$$
$$= 2\Big(\theta - \frac{1}{2}\underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\Big) + C$$

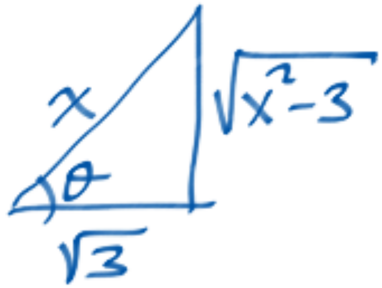
$$\sin\theta = x/2$$
$$\cos\theta = \frac{\sqrt{4-x^2}}{2}$$

$$= 2\left(\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\cdot\frac{x}{2}\sqrt{4-x^2}\right) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C$$

Ex

$$\int \frac{x^3\,dx}{\sqrt{x^2-3}}$$



$$\sec\theta = \frac{x}{\sqrt{3}}$$

$$\tan\theta = \frac{\sqrt{x^2-3}}{\sqrt{3}}$$

$$x = \sqrt{3}\sec\theta$$

$$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$1+\tan^2\theta = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$= \int \frac{3\sqrt{3}\,\cancel{\sqrt{3}}\,\sec^4\theta\tan\theta\,d\theta}{\cancel{\sqrt{3}}\,\sqrt{\underbrace{\sec^2\theta-1}_{\tan^2\theta}}}$$

$$= \int \frac{3\sqrt{3}\,\sec^4\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}}$$

$$= 3\sqrt{3}\int \sec^2\theta\sec^2\theta\,d\theta$$

$$= 3\sqrt{3}\int (1+\tan^2\theta)\sec^2\theta\,d\theta$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= 3\sqrt{3}\int (1+u^2)\,du$$

$$= 3\sqrt{3}\left(u + \frac{1}{3}u^3\right) + C$$

$$= 3\sqrt{3}\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= 3\sqrt{3}\left[\frac{\sqrt{x^2-3}}{\sqrt{3}} + \frac{1}{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}}\right)^3\right] + C$$

$$= \frac{1}{3}\sqrt{x^2-3}\,(x^2-6) + C$$

Ex $\displaystyle\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x}\,dx$

$\sin\theta = x$

$\cos\theta\, d\theta = dx$

$\cos\theta = \sqrt{1-x^2}$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin\theta} \cos\theta \, d\theta$$

$$= \int \frac{\cos^2\theta \, d\theta}{\sin\theta} = \int \frac{1-\sin^2\theta}{\sin\theta} \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (\underbrace{\csc\theta}_{1/\sin\theta} - \sin\theta) \, d\theta$$

$$= \int \csc\theta \, d\theta - \underbrace{\int \sin\theta \, d\theta}_{-\cos\theta}$$

$$= \int \csc\theta \cdot \left(\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}\right) d\theta + \cos\theta \Big|_{\pi/6}^{\pi/2}$$

$$u = \csc\theta - \cot\theta$$

$$du = (\csc^2\theta - \csc\theta\cot\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{du}{u} + \cos\theta\Big|_{\pi/6}^{\pi/2}$$

$$= \ln|u| + \cos\theta\Big|_{\pi/6}^{\pi/2}$$

$$= \ln|\csc\theta - \cot\theta\Big|_{\pi/6}^{\pi/2} + \cos\theta\Big|_{\pi/6}^{\pi/2}$$

Homework:

$$\int \frac{x\,dx}{\sqrt{16-x^4}} \qquad u=x^2$$

$$u=\sin\theta$$

$$du=\cos\theta\,d\theta$$