Chapter 4

SOLVING HIGHER ORDER LINEAR DE.

Def^n of Linearly Dependent DE.:

A set of fns $f_1(x), f_2(x), \dots, f_n(x)$ is said to be linearly dependent (LD) in an interval $[a,b]$, if there exists a set of "n" constants, NOT ALL ZERO (at least one of them is not zero) such that in this interval

$$\sum_{i=1}^{n} c_i f_i(x) = c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \qquad (a \leq x \leq b)$$

otherwise the set is said to be linearly independent (LI)

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

$c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots + c_n f_n = 0$ implies if $c_1 = c_2 = c_3 = \dots = c_n = 0$?

ex: if $f_1 = e^x$, $f_2 = e^{-2x}$ $\qquad c_1 f_1 + c_2 f_2 = 0$, $\quad c_1 e^x + c_2 e^{-2x} = 0$ (LI) $\quad c_1 = c_2 = 0$

ex: $f_1(x) = \sin x$, $f_2(x) = \cos x$ $\qquad c_1 f_1 + c_2 f_2 = 0$, $\quad c_1 \sin x + c_2 \cos x = 0$ (LI) $\quad$ it can be zero only if ($c_1 = c_2 = 0$)

ex: $f_1(x) = \sin x$, $f_2(x) = 3\sin x$ $\qquad c_1 f_1(x) + c_2 f_2(x) = 0$, $\quad c_1 \sin x + c_2 3 \sin x = 0$ (LD) $\quad$ ($c_1 = -3$, $c_2 = 1$)

Def^n: If the set of fn's $f_1, f_2, \dots, f_n$ are Linearly Dependent we can express one of the functions, linearly interms of the others. This is called Linearly Dependency. If the converse happens (if we can not express any number of the sets linearly interms of the others) are called linearly independent.

<u>Ex:</u> $f_1 = x^2$, $f_2 = x$, $f_3 = 4x^2 - 3x$

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

$$c_1 x^2 + c_2 x + c_3 (4x^2 - 3x) = 0$$

$$x^2 (c_1 + 4c_3) + x (c_2 - 3c_3) = 0$$

$$\Rightarrow \quad c_1 + 4c_3 = 0 \quad , \quad c_2 - 3c_3 = 0$$

$$\left.\begin{aligned} c_3 &= -\frac{1}{4} c_1 \\ c_3 &= \frac{1}{3} c_2 \end{aligned}\right\} \quad c_3 = -\frac{c_1}{4} = \frac{c_2}{3}$$

$$-4 \cancel{c_3} f_1 + 3 \cancel{c_3} f_2 + \cancel{c_3} f_3 = 0$$

$$-4 f_1 + 3 f_2 + f_3 = 0$$

$$\left.\begin{aligned} f_3 &= 4 f_1 - 3 f_2 \\ f_2 &= \frac{4}{3} f_1 - \frac{1}{3} f_3 \\ f_1 &= \frac{3}{4} f_2 + \frac{1}{4} f_3 \end{aligned}\right\} \quad (LD)$$

Def^n of WRONSKIAN (Polish)

Let's generalize it !

Assume $f_1(x)$ and $f_2(x)$ are dependent

$$-f_1' / \quad c_1 f_1 + c_2 f_2 = 0 \qquad ; \text{ drive it wrt..}$$

$$f_1 / \quad c_1 f_1' + c_2 f_2' = 0$$

$$+ \overline{\qquad c_2 (f_1 f_2' - f_2 f_1') = 0 \qquad} ; \quad c_2 \neq 0 \text{ then}$$

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$; if this det. zero then $c_1, c_2 \neq 0$ then, linear dependency holds.

This theorem gives a simple criterion for determining whether or not n solns of $(*)$ are linearly independent.

$$\left[a_0(x)\, y^n + a_1(x)\, y^{n-1} + \dots + a_{n-1}(x)\, y' + a_n(x)\, y = 0 \right] \qquad (*)$$

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

(Only square matrices have determinants)

in which primes denote derivatives, is called the WRONSKIAN of these n fns.

if $W(f_1, f_2, \dots, f_n) = 0$ in $(a \leq x \leq b)$ (LD)

if $W(f_1, f_2, \dots, f_n) \neq 0$ in $(a \leq x \leq b)$ (LI)

ex: $f_1(x) = \sin x$, $f_2(x) = \cos x$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$\Rightarrow W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -(\sin^2 x + \cos^2 x)$$

$$= -1 \neq 0 \quad (\text{LI } \forall x)$$

ex: $f_1 = e^x$, $f_2 = e^{2x}$, $f_3 = e^{3x}$

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$\downarrow$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}$$

$$= (8-6)\, e^{6x} = 2e^{6x} \neq 0 \quad (\text{LI})$$

ex: $f_1(x) = x$
$f_2(x) = e^x$
$f_3(x) = xe^x$
$f_4(x) = (2-3x)e^x$
LD or LI (Notebook)

ex: $f_1(x) = 1$
$f_2(x) = \sin x$
$f_3(x) = \cos x$

$$W(1, \sin x, \cos x) = -\cos^2 x - \sin^2 x = -1 \neq 0 \quad (LI)$$

ex: $f_1(x) = \sin(x^2)$
$f_2(x) = \cos(x^2)$

$$W(\sin(x^2), \cos(x^2)) = \begin{vmatrix} \sin x^2 & \cos x^2 \\ 2x\cos x^2 & -2x\sin x^2 \end{vmatrix}$$

$$= -2x(\sin^2 x^2 + \cos^2 x^2) = -2x$$

$W(f_1, f_2) = 0$ if $x = 0$ (LD)
$W(f_1, f_2) \neq 0$ if $x \neq 0$ (LI)

Conclusion: f_1 & f_2 are linearly independent solutions on every finite closed interval which does not contain origin.

The General n^{th} order LDE

Defn: Generally a LDE of order "n" has the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}(y)}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \; ; \; a_0(x) \neq 0$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $F(x)$ depend only on x and NOT on y.

If one of them depend on y then non-linear DE.

Assume:
$a_0(x), a_1(x), \ldots, a_n(x), F(x)$ are continuous real-valued fns in $(-\infty, \infty)$

$n=1$

$$a_0(x)\frac{dy}{dx}+a_1(x)y=f(x),\quad a_0(x)\neq 0$$

$$\frac{dy}{dx}+\underbrace{\frac{a_1(x)}{a_0(x)}}_{P(x)}y=\underbrace{\frac{f(x)}{a_0(x)}}_{Q(x)}$$

$$y'+P(x)y=Q(x)$$ (1st Ord. LDE, Dep: y, Ind: x)

$n=2$:

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=f_2(x)\qquad a_0\neq 0$$

$$\frac{d^2y}{dx^2}+\frac{a_1(x)}{a_0(x)}\frac{dy}{dx}+\frac{a_2(x)}{a_0(x)}y=\frac{F(x)}{a_0(x)}$$

$$y''+P_1(x)y'+P_2(x)y=Q(x)$$ (2nd Ord. LDE, Dep: y, Ind: X)

$Q(x)=0 \rightarrow$ Homogeneous eqn

$Q(x)\neq 0 \rightarrow$ non-homogeneous eqn

$\downarrow$

not the meaning used for 1st and DE related with Linear

if $n\geq 2$ can not always be solved exactly.

Differential Operator:

D is a linear operator

$$Dy=\frac{dy}{dx},\quad Du=\frac{du}{dt},\quad Dx=\frac{dx}{dy},\quad Dt=\frac{dt}{dr}$$

ex: $D(\sin x)=\cos x$

$D^2(2t+\cos 3t)=D(2-3\sin 3t)=-9\cos 3t$

$$D[c_1 f_1(x) + c_2 f_2(x)] = c_1 D\{f_1(x)\} + c_2 D\{f_2(x)\} = \underbrace{c_1 f_1'(x) + c_2 f_2'(x)}_{\text{const.}}$$

Integral Operator: $\frac{1}{D}(\cdot) = \int (\cdot)\, dx$

linear operator

$$\frac{1}{D}(y) = \int y\, dx$$

$$\frac{1}{D}(\cos x) = \int \cos x\, dx$$

$$\frac{1}{D}\left\{c_1 f_1(x) + c_2 f_2(x)\right\} = c_1 \frac{1}{D}\{f_1(x)\} + c_2 \frac{1}{D}\{f_2(x)\}$$

what is L.O.

$$\gamma(f(x)) = 3 f(x)$$

If $\gamma(c_1 f(x)) = c_1 \gamma(f(x))$ (γ is linear)

$$\gamma(c_1 f(x)) = 3(c_1 f(x)) = 3c_1 f(x)$$

D is a linear operator; for which superposition holds. such that the response produced by simultaneous applications of different inputs is sum of individual responses.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x) y = F(x)$$

$$a_0(x) D^n y + a_1(x) D^{n-1} y + \dots + a_{n-1}(x) D y + a_n(x) y = F(x)$$

$$\underbrace{[a_0(x) D^n + a_1(x) D^{n-1} + \dots + a_{n-1}(x) D + a_n(x)]}\, y = F(x)$$

$$\phi(D)(y) = F(x)$$

$\phi(D) y = f(x)$ where $\phi(D) = a_0(x) D^n + a_1(x) D^{n-1} + \dots + a_{n-1}(x) D + a_n(x)$

n^{th} ord. LDE with <u>variable</u> coefficients.

If $a_0, a_1, a_2, \dots, a_n$ are constants

$L(D) y = F(x)$, where

$L(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$

n^{th} Order Linear DE with constant coefficients

Some properties of Operators : Let A, B, C be operators

Commutive $A + B = B + A$

Associativ $(A + B) + C = A + (B + C)$

$(A . B) . C = A . (B . C)$

Distributive $A(B + C) = AB + AC$

ex: $y'' + 3y' + 4y = \sin x$ (2^{nd} ord. LDE with const. coeff.)

$D^2 y + 3Dy + 4y = \sin x$

$\underbrace{(D^2 + 3D + 4)}_{L(D)} y = \sin x$ $L(D) = D^2 + 3D + 4$

$L(D)\, y = \sin x$

ex: $x^2 y'' + x y' + y = e^x$ (2^{nd} ord. LDE with variable coeff.)

$\phi(D)\, y = f(x) \longrightarrow f(x) \neq 0$

n^{th} Ord. LDE with variable coeff. (NON-HOMOGENOUS DE
RHS is not zero)

$\phi(D) y = 0$

n^{th} ord. LDE with variable coeff. (HOMOGENEOUS DE
RHS is zero)

$L(D) y = f(x)$

n^{th} ord. LDE with const. coeff. (NON-HOMOGENOUS EQN)

$L(D)\, y = 0$

n^{th} ord. LDE with const. coeff (HOMOGENEOUS EQN.)

ex: $9 \dfrac{d^2 y}{dx^2} - y = \sin x$

$(9D^2 - 1)\, y = \sin x$

$(3D - 1)(3D + 1) y = \sin x$

Existence theorem

Hypothesis: Consider

$\phi(D)y = f(x)$

$\phi(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$

1) or $a_0(x)\dfrac{d^n y}{dx^n} + a_1(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\dfrac{dy}{dx} + a_n(x)y = f(x)$ $(*)$

where $a_0, a_1, a_2, \ldots a_n$ and $f(x)$ are continuous real fns on a real interval $(a \leq x \leq b)$ and $a_0(x) \neq 0$ for any x in $a \leq x \leq b$.

2) let x_0 be any point of the interval $(a \leq x \leq b)$ and let $c_0, c_1, \ldots, c_{n-1}$ be "n" arbitrary real constants.

Conclusion: There exists a UNIQUE soln $f(x)$ of eqn $(*)$ such that $f(x_0) = c_0$, $f'(x_0) = c_1, \ldots, f^{n-1}(x_0) = c_{n-1}$ and this soln is defined over the entire interval $(a \leq x \leq b)$

ex: Consider the IVP

$\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + x^3 y = e^x$ $\qquad y(\overset{x_0}{1}) = 2 = c_0, \quad y'(\overset{x_0}{1}) = -5 = c_1$

1) $1, 3x, x^3$ and e^x are continuous $\forall x$ $(-\infty < x < \infty)$
2) The point $(x_0 = 1)$ belong to this interval
3) Real numbers $c_0 = 2$, $c_1 = -5$ belong to this interval

Conclusion: A soln of given IVP exists and is unique and defined $\forall x$ $(-\infty < x < \infty)$ trivial ↓

Corollary:

Hypothesis: Let $f(x)$ be a soln of the homogenous, linear, n^{th} order variable coeff. DE.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0$$

or $\phi(D)\,y = 0$ where $\phi(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$

such that $f(x)=0$ $(\Rightarrow y(x_0)=0)$, $f'(x_0)=0$, $f''(x_0)=0, \ldots, f^{n-1}(x_0)=0$

where x_0 is a point of the interval $(a \leq x \leq b)$ in which the coeff$\underline{s}$

$a_0, a_1, a_2, \ldots, a_n$ are all continuous and $a_0(x) \neq 0$.

Conclusion:

$f(x)=0 \quad \forall x$ on $(a \leq x \leq b)$

ex: $$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + x^2 y = 0$$

the sol$\underline{n}$ $y(x)$ or $f(x)$ of the 3rd ord. homogenous DE which is the trivial sol$\underline{n}$ of $f(x)$, such that

such that $f(2) = f'(2) = f''(2) = 0$

$f(x) = 0 \quad \forall x$ or $y(x) = 0 \quad \forall x$.

29 Nisan 2009

Basic Theorem on Lin. Homogenous DE

Theorem:

Hypothesis: Let $f_1(x), f_2(x), \ldots, f_n(x)$ be any (n) linearly independent set of sol$\underline{ns}$ of the homogereous Lin. DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0$$

same or $\phi(D)y = 0$ where $\phi(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$

Conclusion: Then $y = f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$ is also a sol$\underline{n}$ of eq$\underline{n}$ $\phi(D)y = 0$ where

$y = f(x)$ is also a sol$\underline{n}$ of eq$\underline{n}$ $\phi(D)y=0$ where $c_0, c_1, c_2, \ldots, c_n$ are essential arbitrary constants.

ex: $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$

$$\left.\begin{array}{l} y_1(x) = f_1(x) = e^x \\ y_2(x) = f_2(x) = e^{-x} \\ y_3(x) = f_3(x) = e^{2x} \end{array}\right\}$$ LI set of sol$\underline{ns}$ of the DE above

$\to e^x - 2e^x - e^x + 2e^x = 0$

$-e^x - 2e^{-x} + e^x + 2e^{-x}$

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} \neq 0 \quad \forall x$$

$\therefore$ LI

$$y_{general} = c_1y_1 + c_2y_2 + c_3y_3 = \underbrace{c_1e^x + c_2e^{-x} + c_3e^{2x}}_{\text{constant}}$$

for example $y = 2e^x - 3e^{-x} + \frac{2}{3}e^{2x}$ is a sol$\underline{n}$ of above DE.

ex: $y'' + y = 0$ $\quad y_1 = f_1(x) = \sin x$, $\quad y_2 = f_2(x) = \cos x$

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \neq 0 \text{ for } \forall x$$

$\therefore$ LI.

$D^2y + y = 0 \Rightarrow (D^2+1)y = 0 \Rightarrow \cancel{y(D^2+1) = 0}$ Nonsense

$y_{gen} = c_1y_1 + c_2y_2 = c_1f_1 + c_2f_2 = c_1 \sin x + c_2 \cos x$

[The eq$\underline{n}$ we will solve must be LI]

$$\left.\begin{array}{l} y' = c_1\cos x - c_2 \sin x \\ y'' = -c_1 \sin x - c_2 \cos x \end{array}\right\} \Rightarrow \underbrace{-c_1\sin x - c_2\cos x}_{y''} + \underbrace{c_1 \sin x + c_2 \cos x}_{y} \overset{?}{=} 0$$

General sol$\underline{n}$ contains ess. arb. const$\underline{s}$. The number of ess. arb. const$\underline{s}$ give the order of DE.

Soln of n^{th} Order General Linear Homogeneous Const. Coeff. DE (RHS zero)

$L(D)y = 0$ where

$L(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$: all a's are consts.

or $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$

$y = f(x)$

$\frac{d^k f(x)}{dx^k} = C_0 f(x), \quad \forall x \quad f(x) = y = e^{mx}$

$\frac{d(e^{mx})}{dx} = m e^{mx}$

$\frac{d^2(e^{mx})}{dx^2} = m^2 e^{mx}, \quad \frac{d^n e^{mx}}{dx^n} = m^n e^{mx}$

$L(D)y = 0$, Assume $y = e^{mx}$

$a_0 \underbrace{\frac{d^n y}{dx^n}}_{m^n e^{mx}} + a_1 \underbrace{\frac{d^{n-1} y}{dx^{n-1}}}_{m^{n-1} e^{mx}} + \cdots + a_{n-1} \underbrace{\frac{dy}{dx}}_{m e^{mx}} + a_n \underbrace{y}_{e^{mx}} = 0$

$\Rightarrow a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$

$e^{mx} \underbrace{(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n)}_{L(m)} = 0$

$e^{mx} L(m) = 0, \quad e^{mx} \neq 0 \quad (-\infty < x < \infty) \quad L(m) = 0$

Characteristic eqn or auxilary eqn or m eqn.

$L(m) = \underbrace{a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n}_{\text{constants}} = 0$ (characteristic eqn)

n^{th} order polynomial has n roots.

$m_1, m_2, \ldots, m_n$ (roots).

Remark: Characteristic Eqn is useful only for const. coeff. Lin. DE.

- Reduction of Order Theorem: (Not good enough when the order of DE is greather than 2)

If we have non-trivial solution, say, $y = y_1$ of the n^{th} order DE $L(D)y=0$ then the substitutional $y = y_1 v(x)$ will transform the given DE into an equation of the $(n-1)^{th}$ order. And other particular solutions may be found from which $v(x)$ may be obtained.

$m\left(\frac{dv}{dt}\right)$ or $v'(x)$

Remark

This theorem is valid for constant or variable coefficient

and is also valid for the non-homogeneous Linear D.E,

$\phi(D)\,y = F(x)$

$\phi(n) = a_0(x)\,D^n + a_1(x)\,D^{n-1} + \dots + a_n$

ex: $y'' - 6y' + 9y = 0$ $\qquad L(m) = m^2 - 6m - 9 = 0$

$\underbrace{(D^2 - 6D + 9)}_{L(D)}\,y = 0$ $\qquad m_1 = m_2 = 3$ (equal roots)

$y = y_1\,v(x)$, $\quad y = e^{3x}\,v(x) \xrightarrow{} $ Transform

$v(x)$ reduction of order

$y = e^{3x}\,v(x)$

$y' = Dy = 3e^{3x}\,v(x) + e^{3x}\cdot v'(x)$

$y'' = D^2y = 9e^{3x}\,v(x) + 3e^{3x}\,v'(x) + 3e^{3x}\,v'(x) + e^{3x}\,v''(x)$

$y'' = 9e^{3x}\,v(x) + 6e^{3x}\,v'(x) + e^{3x}\,v''(x)$

$y'' - 6y' + 9y = 0$

$$\underbrace{9e^{3x}v(x) + 6e^{3x}v'(x) + e^{3x}v''(x)}_{y''} - \underbrace{18e^{3x}v(x) - 6e^{3x}v'(x)}_{y'} + \underbrace{9e^{3x}v(x)}_{9y} = 0$$

$e^{3x} v''(x) = 0, \quad e^{3x} \neq 0, \quad v''(x) = 0$

$\downarrow$

$v'(x) = c_2$

$v(x) = c_2 x + c_1$

$y = v e^{3x} = (c_1 + c_2 x) e^{3x} = \underset{y_1}{c_1 e^{3x}} + \underset{y_2}{c_2 x e^{3x}}$

$W(e^{3x}, xe^{3x}) \neq 0 \quad (L.I)$

ex: $(D-3)^3 y = 0$ $\quad$ multiplicity on root $m_1 = 3$

$L(m) = (m-3)^3 = 0$ $\quad m_1 = m_2 = m_3 = 3$

$y = c_1 e^{3x} + c_2 x e^{3x} + c_2 x^2 e^{3x} = (c_1 + c_2 x + c_3 x^2) e^{3x}$

ex: Given that $f(x) = e^{2x}$ is a soln of the eqn

$(2x+1) y'' - 4(x+1) y' + 4y = 0$

Find another linearly independent soln by reduction of order.

Soln: $y = e^{2x} v(x)$ $\quad y' = e^{2x} v'(x) + 2e^{2x} v(x)$

$y'' = e^{2x} v''(x) + 4e^{2x} v'(x) + 4e^{2x} v(x)$

Substituting into DE,

$(2x+1) e^{2x} \left(v''(x) + 4v'(x) + 4v(x) \right) - 4(x+1) \left(v'(x) + 2v(x) \right) + 4 e^{2x} v(x) = 0$

$(2x+1) e^{2x} v''(x) + (8x + \cancel{4} - 4x - \cancel{4}) e^{2x} v'(x) + (\cancel{4x} + \cancel{2} - \cancel{4x} - \cancel{4} + \cancel{2}) 4e^{2x} v(x) = 0$

Divide by e^{2x}

$(2x+1) v''(x) + 4x v'(x) = 0$

Letting $w(x) = v'(x)$, we obtain the eqn

$(2x+1) w'(x) + 4x w(x) = 0$ or $\frac{dw}{w} = -\frac{4x\,dx}{2x+1} = -2\left(\frac{2x}{2x+1}\right) = -2\left(\frac{2x+1-1}{2x+1}\right)$

$= -2\left(1 - \frac{1}{2x+1}\right)$

$\ln w = -2x + \ln(2x+1) + \ln c$

$w = e^{-2x}(2x+1) \cdot c$

choosing $c = 1$, and recalling $w = v'(x)$

$\frac{dv}{dx} = e^{-2x}(2x+1)$

$v = \int (2x e^{-2x} + e^{-2x})\, dx$

$= -e^{-2x}(x+1)$

So the 2^{nd} soln is $y_2 = e^{2x}(-e^{-2x})(x+1)$

$= (-x-1)$

$y_{gen} = c_1 e^{+2x} + c_2(-x-1)$

HW ex: 4.1- , P133

Case I :

Distinct (unequal) real roots

$m_1, m_2, \ldots, m_n$

$e^{m_1 x}, e^{m_2 x}, \ldots\ldots, e^{m_n x}$. L.I. set of sol'n $W(e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}) \neq 0$ L.I.

$$\boxed{y_{gen} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$$

ex: find the general sol'n

$y''' - 6y'' + 11y' - 6y = 0$ (3^{rd} Ord., Lin., Homogeneous, Const. coeff. DE)

$\underbrace{(D^3 - 6D + 11D - 6)}_{L(D)} y = 0$

$L(m) = m^3 - 6m^2 + 11m - 6 = 0$ $\begin{pmatrix} m_1 = 1 \\ m_2 = 2 \\ m_3 = 3 \end{pmatrix}$

$y_1 = e^{m_1 x} = e^x$

$y_2 = e^{m_2 x} = e^{2x}$

$y_3 = e^{m_3 x} = e^{3x}$

$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \neq 0$ L.I.

$y_{gen} = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

ex: $y'' - y = 0$ $y(0) = 1$, $y'(0) = -2$

$D^2 y - y = 0$. $y_1 = e^{m_1 x} = e^x$ $W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$

$(D^2 - 1) y = 0$ $y_2 = e^{m_2 x} = e^{-x}$ $\neq 0$

$\forall (x)$ L.I.

$L(m) = m^2 - 1$

$\begin{pmatrix} m_1 = 1 \\ m_2 = -1 \end{pmatrix}$

$y_{gen} = c_1 y_1 + c_2 y_2$

$= \underbrace{c_1 e^x + c_2 e^{-x}}_{\text{2 ess. arb. const.}}$

$\left.\begin{array}{l} y(0) = 1 = c_1 + c_2 \\ y'(0) = -2 = c_1 - c_2 \end{array}\right\}$ $c_1 = -1/2$

$c_2 = 3/2$

$y = -\frac{1}{2} e^x + \frac{3}{2} e^{-x}$

Methods for Finding Roots :

I. Synthetic Method :

$L(m) = m^3 - 6m^2 + 11m - 6 = 0$ $\qquad$ $-6 = \pm 2 \cdot \mp 3$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $-6 = \pm 1 \cdot \mp 6$

$L(1) = 1 - 6 + 11 - 6 = 0 \rightarrow m_1 = 1$

$$
\begin{array}{l|l}
m^3 - 6m^2 + 11m - 6 & m-1 \\
\cline{2-2}
\underline{m^3 - m^2} & m^2 - 5m + 6 \longrightarrow (m-2)(m-3) \\
\quad -5m^2 + 11m - 6 & \\
\quad \underline{-5m^2 + 5m} & m_2 = 2 \\
\qquad\quad 6m - 6 & m_3 = 3
\end{array}
$$

II. Newton's Approx. Method :

$L(m) = m^3 + m + 1 = 0$ $\qquad$ $m_{i+1} = m_i - \dfrac{f(m_i)}{f'(m_i)}$

i	m_i
0	$-1/2$
1	$3/8$
2	$\vdots$
$\vdots$	

repeat until $L(m_i) = 0$

Case II Characteristic Eqn. $L(m) = 0$, has equal roots

ex: $y'' - 4y' + 4y = 0$ $\qquad$ $L(m) = m^2 - 4m + 4 = 0$

$D^2 y - 4Dy + 4y = 0$ $\qquad$ $L(m) = (m-2)^2 = 0$

$\underbrace{(D^2 - 4D + 4)}_{L(D)} y = 0$ $\qquad$ $\begin{pmatrix} m_1 = 2 \\ m_2 = 2 \end{pmatrix}$ Equal

$$\left.\begin{aligned} y_1 &= e^{m_1 x} = e^{2x} \\ y_2 &= e^{m_2 x} = e^{2x} \end{aligned}\right\} (L.D) \qquad W(e^{2x}, e^{2x}) = \begin{vmatrix} e^{2x} & e^{2x} \\ 2e^{2x} & 2e^{2x} \end{vmatrix} = 0 \quad (L.D)$$

$y_{gen} = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{2x} = e^{2x}(c_1 + c_2)$

↳ Not a general sol^n. Because we have only one essential arbitrary const. we must have 2 ess. arb. const. since D.E is 2^{nd} Ord.

Rule for Equal Roots

Consider the n^{th} ord. linear, const. coeff. DE $L(D)y=0$

If auxilary or characteristic equation $L(m)=0$ has the real root (m), multiplicity "k" times then the general soln of $L(D)y=0$ is

$$y=(c_1+c_2x+c_3x^2+\dots+c_kx^{k-1})e^{mx}$$

Ex: $\underbrace{(D^6-6D^5+12D^4-6D^3-9D^2+12D-4)}_{L(D)}.y=0$ ord: 6

$L(m)=m^6-6m^5+12m^4-6m^3-9m^2+12m+4=0$

$\underbrace{m_1=m_2=m_3=1}_{\text{multiplicity 3 times}}$ $\quad \underbrace{m_4=m_5=2}_{\text{multiplicity 2 times}}$, $m_6=-1$

* $y=\underbrace{(c_1+c_2x+c_3x^2)e^x+(c_4+c_5x)e^{2x}+c_6e^{-x}}_{\text{6 ess. arb. const., because}}$ ord: 6

Case III

$L(m)=0$ has complex conjugate roots:

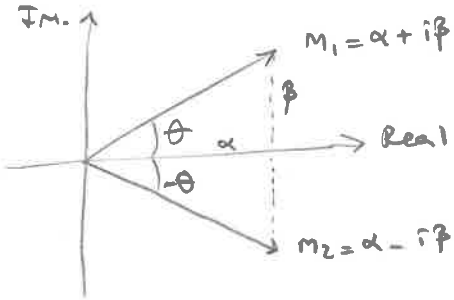
$L(m)=0 \quad m_1=\alpha+i\beta, \quad m_2=\alpha-i\beta$

$m_1=\alpha+i\beta \rightarrow$ complex root

$\overline{m_1}=\overline{\alpha+i\beta}=m_2=\alpha-i\beta \rightarrow$ conjugate root

$i^2=-1$

Any complex number is a vector



$\alpha = Re(m_1)$

$\beta = Im(m_1)$

$|m_1| = \sqrt{\alpha^2 + \beta^2}$

$\theta = \arctan(\beta/\alpha)$

$z = \alpha + i\beta \quad , \quad \bar{z} = \alpha - i\beta$

$z \cdot \bar{z} = |z|^2 = \alpha^2 + \beta^2 \qquad (i^2 = 1)$

$(\underbrace{\cos\theta}_{Real} + i \underbrace{\sin\theta}_{Im.}) = e^{i\theta}$ (Euler's Formula)

$e^{i\pi/2} = (\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}) = 0 + i \; , \quad i = e^{i\pi/2}$

$-i = e^{-i\pi/2}$

$-1 = e^{i\pi}$

$1 = e^{i2\pi}$

$\pm\sqrt{i} = \pm\sqrt{e^{i\pi/2}} = \pm e^{i\pi/4} = \pm(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}) = \pm(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}})$

$\underbrace{m_1 = \alpha + i\beta \quad , \quad m_2 = \alpha - i\beta}_{\text{Distinct, unequal}}$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$= e^{\alpha x}[c_1(\cos(\beta x) + i\sin(\beta x)) + c_2(\cos(\beta x) - i\sin(\beta x))]$

$= e^{\alpha x}[\underbrace{(c_1 + c_2)}_{C_1}\cos\beta x + \underbrace{(ic_1 - ic_2)}_{C_2}\sin\beta x]$

$= e^{\alpha x}[C_1\cos(\beta x) + C_2\sin(\beta x)]$

ex: $y'' + 2y' + 5y = 0 \qquad y(0)=1,\ y'(0)=0$

$$D^2 y + 2Dy + 5y = 0$$

$$(D^2 + 2D + 5)y = 0$$

$$L(m) = m^2 + 2m + 5 = 0 \rightarrow m_1 = -1 \mp 2i \quad \begin{cases} \alpha = -1 \\ \beta = 2 \end{cases}$$

$$y = e^{-x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

– if it was $(D^2 + 2D + 5)^2 y = 0$

$$L(m) = (m^2 + 2m + 5)^2 = 0$$

$m_1 = -1 + 2i \qquad m_3 = -1 + 2i$

$m_2 = -1 - 2i \qquad m_4 = -1 - 2i$

$$y = e^{-x}\left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$$

$e^{-x}\cos 2x \qquad e^{-x}\sin 2x$

$xe^{-x}\cos 2x \qquad xe^{-x}\sin 2x$

– if $(D^2 + 2D + 5)^3 = 0$

$$L(m) = (m^2 + 2m + 5)^3 = 0$$

$m_1 = -1 + 2i \qquad m_3 = -1 + 2i \qquad m_5 = -1 + 2i$

$m_2 = -1 - 2i \qquad m_4 = -1 - 2i \qquad m_6 = -1 - 2i$

$$y = e^{-x}\left[(c_1 + c_2 x + c_3 x^2)\cos 2x + (c_4 + c_5 x + c_6 x^2)\sin 2x\right]$$

<u>RULE</u>: If $\alpha + i\beta$ and $\alpha - i\beta$ are each has multiplicity "k" times then the general soln of $L(D)y = 0$ is

$$\boxed{y = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1})\cos \beta x + (d_1 + d_2 x + d_3 x^2 + \dots + d_k x^{k-1})\sin \beta x\right]}$$

ex: $y'' - x\,y' + 5y = 0$
$\hookrightarrow$ NOT const.

ex: Solve $\underbrace{(D^4 - 5D^2 + 12D + 28)}_{L(D)}\, y = 0$

$L(m) = m^4 - 5m^2 + 12m + 28 = 0$

$28 = \mp 1 * \mp 28$
$\mp 4 * \mp 7$
$\mp 2 * \mp 14$

$m_1 = -2$

$L(-2) = 16 - 20 - 24 + 28 = 0$

$$\begin{array}{l|l} m^4 - 5m^2 + 12m + 28 & m+2 \\ \hline m^4 + 2m^3 & m^3 - 2m^2 - m + 14 \\ \hline -2m^3 - 5m^2 & \\ -2m^3 - 4m^2 & \\ \hline -m^2 + 12m & \\ -m^2 - 2m & \\ \hline 14m + 28 & \\ 14m + 28 & \\ \hline \end{array}$$

$\hookrightarrow m_2 = -2 \rightarrow -8 - 8 + 2 + 14 = 0.$

$m_3 = +2 + i\sqrt{3}$

$m_4 = +2 - i\sqrt{3}$

$L(m) = (m+2)^2 \, (m^2 - 4m + 7) = 0$

$m_1 = m_2 = -2$ $\quad$ $m_3 = +2 + i\sqrt{3}$
$m_4 = +2 - i\sqrt{3}$

$$y = (c_1 + c_2 x)\, e^{-2x} + e^{+2x}\left(c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x) \right)$$

$L(D)\,y = 0$, $L(m) = 0$

ex: $m \rightarrow 2, -1, 0, 0, 0, 3 \mp 5i, 2, 0, 3 \mp 5i$ $\rightarrow$ write the sol^{n}

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)\, e^{0x} + (c_5 + c_6 x)\, e^{2x} + c_7 e^{-x} + e^{3x}\Big[(c_8 + c_9 x)\cos(5x) + (c_{10} + c_{11} x)\sin(5x) \Big]$$

Find the char. eqn.

$$L(m) = (m-2)^2 . m^4 . (m+1)\,(m - (3+5i))^2\,(m - (3-5i))^2$$
$$= m^4\, (m-2)^2\, \left[(m-3-5i)(m-3+5i) \right]^2 . (m+1)$$

Sol$\underline{ns}$ of n^{th} order Linear, Const. Coeff. Non-Homogeneous DE:

$$L(D)\,y = F(x)$$

$L(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$, all a's are constant

$L(D)y = 0 \Rightarrow y_c \rightarrow$ complementary sol$\underline{n}$ $\rightarrow$ soln. of homogenous part

$+ \quad L(D)y = f(x) \Rightarrow y_p \rightarrow$ Particular sol$\underline{n}$ (Integral Sol$\underline{n}$) (Steady-state sol$\underline{n}$)

- U.C (Undetermined Coeff. meth.)
- Operator Meth. (shortcut meth.)
- V.P (Variation of parameter meth. or Lagrange meth.)

$$L(D)y_c + L(D)y_p = 0 + f(x)$$

Since $L(D)$ is a Linear operator

$$L(D)\,(\underbrace{y_c + y_p}_{y_{gen}}) = f(x)$$

$\boxed{y_{gen} = y_c + y_p}$ is a general sol$\underline{n}$ of $L(D)y = f(x)$

$\rightarrow$ contains no. arb. const.

HW = from Ross, page 143 25, 32, 36, 51, 55

The Method of Undetermined Coeff. (Simple but restricted)

Def$\underline{n}$: A fn is called UC fn if it is either

(1) a fn defined by one of the following:

a) x^n, where n is a positive integer or zero

b) e^{ax}, where a is a const. $\neq 0$

c) $\sin(bx+c)$, where b & c are const. $b \neq 0$

d) $\cos(bx+c)$, " " " " "

or

(2) a fn defined as a finite product of two or more fns of this four types.

Rules for Solving LDE with const. coeff.

1) write the complementary sol$\underline{n}$ y_c
2) Assume a particular soln corresponding to the terms on the RHS of the eqn. (y_p)
   a) for a polynomial of degree n, assume a polynomial of degree n
   b) for terms $\sin\beta x$, $\cos\beta x$ or sums and differences of such terms assume
   $$a \sin\beta x + b\cos\beta x$$
   c) for terms $e^{\alpha x}$ assume $ae^{\alpha x}$
3) If any of these assumed terms in 2a, 2b and 2c occurs in the complementary sol$\underline{n}$, multiply these assumed terms by a power of x which is sufficiently high (but not higher) so that none of these assumed will occurs in the complementary sol$\underline{n}$.
4) Write the assumed form for the particular sol$\underline{n}$ and evaluate these coefficients.
5) And y_p to y_c to obtain the required general sol$\underline{n}$.
$$y_{gen} = y_c + y_p$$

* whenever UC is aplicable the operator method is also aclicable

$$L(D)y = f(x) = e^{\alpha x}\left[(A_0 + A_1 x + A_2 x^2 + \dots)\cos(\beta x) + (B_0 + B_1 x + B_2 x^2 + \dots)\sin(\beta x)\right]$$

ex: $y'' + 4y = 4e^{2x}$

$(D^2+4)y = 4e^{2x} \rightarrow$ complementary sol$\underline{n}$. $(D^2+4)y_c = 0$ & particular sol$\underline{n}$. $(D^2+4)y_p = 4e^{2x}$

$m = \mp 2i$

$y_c = c_1\cos 2x + c_2 \sin 2x$

Assume (Trial)

$y_p = a e^{2x}$

$y'_p = Dy_p = 2a e^{2x}$

$y''_p = D^2 y_p = 4a e^{2x}$

$(D^2+4)y_p = 4a(e^{2x} + e^{2x}) = 4e^{2x}$

$8a e^{2x} = 4e^{2x}$

$a = 1/2 \rightarrow y_p = \frac{1}{2}e^{2x}$

$$\boxed{y_{gen} = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}e^{2x}}$$

ex: $y'' + 3y' + 2y = 4e^{-2x}$

$D^2 + 3Dy + 2y = 4e^{-2x}$

$(\underbrace{D^2 + 3D + 2}_{L(D)})y = 4e^{-2x}$

$L(m) = m^2 + 3m + 2 = 0$

$m_1 = -1, \; m_2 = -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

repetation of e^{-2x} (one time)

$$\boxed{\begin{aligned} y_{gen} &= y_c + y_p \\ &= c_1 e^{-x} + (c_2 - 4x)e^{-2x} \end{aligned}}$$

$(D^2 + 3D + 2)y_p = 4e^{-2x}$

Assumed trial $y_p = ae^{-2x} \cdot x$

$Dy_p = (-2ae^{-2x})x + ae^{-2x}$

$D^2y_p = 4ae^{-2x}x - 2ae^{-2x} - 2ae^{-2x}$

$\underbrace{\cancel{4ae^{-2x}x} - 4ae^{-2x}}_{D^2y_p} \underbrace{- \cancel{6axe^{-2x}} + 3ae^{-2x}}_{3Dy_p} \underbrace{+ \cancel{2axe^{-2x}}}_{2y_p} = 4e^{-2x}$

$-a\cancel{e^{-2x}} = 4\cancel{e^{-2x}} \rightarrow \underline{a = -4}$

$y_p = -4xe^{-2x}$ (in y_p there is no ess. arb. const)

ex: $(D^2 + 4D + 4)y = 6\sin 3x$

$(\underbrace{D^2 + 4D + 4}_{L(D)})y_c = 0$

$L(m) = m^2 + 4m + 4 = 0$

$m_{1,2} = -2$

$y_c = (c_1 + c_2x)e^{-2x}$

$(D^2 + 4D + 4)y_p = 6\sin 3x$

Assumed $y_p = a\cos 3x + b\sin 3x$

$Dy_p = -3a\sin 3x + 3b\cos 3x$

$D^2y_p = -9a\cos 3x - 9b\sin 3x$

$\underbrace{-9a\cos 3x - 9b\sin 3x}_{D^2y_p} \underbrace{- 12a\sin 3x + 12b\cos 3x}_{Dy_p} + \underbrace{4a\cos 3x + 4b\sin 3x}_{4y_p} = 6\sin 3x$

$\left.\begin{aligned} -9a + 12b + 4a &= 0 \\ -9b - 12a + 4b &= 6 \end{aligned}\right\} \quad a = -\frac{72}{169}, \quad b = -\frac{30}{169}$

$$\boxed{\begin{aligned} y_{gen} &= y_c + y_p \\ &= (c_1 + c_2x)e^{-2x} - \frac{72}{169}\cos 3x - \frac{30}{169}\sin 3x \end{aligned}}$$

2017

ex: $(D^2+4D+9)\,y = x^2+3x$

$\underbrace{(D^2+4D+9)}_{L(D)}\, y_c = 0$

m^2+4m+9

$\alpha = m_1 = -2\,,\ \beta = \sqrt{5}$

$y_c = e^{-2x}\left[C_1 \cos(\sqrt{5}\,x) + C_2 \sin(\sqrt{5}\,x)\right]$

$y_p = ax^2+bx+c$

$(D^2+4D+9)\,y_p = x^2+3x$

$a = \frac{1}{9}\,,\ b = \frac{19}{81}\,,\ c = -\frac{94}{729}$

$y_p = \frac{1}{9}x^2 + \frac{19}{81}x - \frac{94}{729}$

ex: $y''+2y'+y = 2\cos 2x + 7x + 2 + 3e^x$

Complementary sol$\underline{n}$

$D^2+2Dy+y = 2\cos 2x + 7x + 2 + 3e^x$

$(D^2+2D+1)\,y_c = 0$

$(m+1)^2 = 0$

$m_{1,2} = -1$

$\boxed{y_c = (C_1 + C_2 x)\,e^{-x}}$

$y_{gen} = y_c + y_p$

for finding particuler sol$\underline{n}$, y_p,

$(D^2+2D+1)\,y_{p_1} = 2\cos 2x \rightarrow y_{p_1} = a\cos 2x + b\sin 2x$

$(D^2+2D+1)\,y_{p_2} = 7x+2 \rightarrow y_{p_2} = cx+d$

$(D^2+2D+1)\,y_{p_3} = 3e^x \rightarrow y_{p_3} = fe^x$

$a = -6/25\,,\ b = 8/25\,,\ c = 7\ \ d = -12\ \ f = 3/4$

$$\boxed{\begin{aligned} y_p &= y_{p_1}+y_{p_2}+y_{p_3} \\ &= -\frac{6}{25}\cos 2x + \frac{8}{25}\sin 2x + 3x - 4 + \frac{3}{4}e^x \end{aligned}}$$

ex: $y''+4y = 6\sin 2x + 3x^2$

$D^2y+4y = 6\sin 2x + 3x^2$

$(D^2+4)\,y = 6\sin 2x + 3x^2$

$L(m) = m^2+4 = 0$

$m^2 = -4$

$m_{1,2} = \mp 2i$

$\alpha = 0\,,\ \beta = 2$

$\boxed{y_c = c_1\cos 2x + C_2 \sin 2x}$

$y_{gen} = y_c + y_p$

$(D^2+4)\,y_p = 6\sin 2x + 3x^2$

$(D^2+4)\,y_{p_1} = 6\sin 2x \rightarrow y_{p_1} = (a\cos 2x + b\sin 2x).\,x$

$(D^2+4)\,y_{p_2} = 3x^2 \rightarrow y_{p_2} = (cx^2+dx+f)\,x^0$

$a = -3/2\,,\ b = 0\,,\ c = 3/4\,,\ d = 0\,,\ f = -3/8$

$$\boxed{\begin{aligned} y_p &= y_{p_1}+y_{p_2} \\ &= \left(-\frac{3}{2}\cos 2x\right)x + \frac{3}{4}x^2 - \frac{3}{8} \end{aligned}}$$

Append (The Method of UC)

examples of UC functions:

$x^3,\ e^{-5x},\ \sin\frac{5}{2}x,\ \cos(2x+\frac{\pi}{4}),\ x^3e^{4x},\ e^{2x}\sin 3x,\ x^3e^{4x}\sin 5x$

UC set of fn's

fn's itself and its linearly independent succesive derivatives that are UC fn's themselves form a UC set of fn's.

ex: $f(x)=x^3$ UC set?

$y'=3x^2$
$y''=6x$
$y'''=x$

UC set $=\{x^3, x^2, x, 1\}$

ex: $f(x)=\sin 2x$ UC set $=\{\sin 2x, \cos 2x\}$

$y'=2\cos 2x$
$y''=-4\sin 2x \rightarrow$ LD with $f(x)$
$\vdots$

If $h=fg$ (f & g are UC fn's), then UC set of
$h=$ UC set of f $*$ UC set of g

ex: $h(x)=x^2\sin x$ $f(x)=x^2$ UC set of $f(x)=\{x^2, x, 1\}$
$g(x)=\sin x$ " " " $g(x)=\{\sin x, \cos x\}$

UC set of $h(x)=\{x^2\sin x, x^2\cos x, x\sin x, x\cos x, \sin x, \cos x\}$

UC fn's	UC set
x^n	$\{x^n, x^{n-1}, x^{n-2}, ----, x, 1\}$
e^{ax}	$\{e^{ax}\}$
$\sin(bx+c)$ or $\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
x^ne^{ax}	$\{x^ne^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \ldots\ldots, xe^{ax}, e^{ax}\}$
$x^n\sin(bx+c)$ or $x^n\cos(bx+c)$	$\{x^n\sin(bx+c), x^n\cos(bx+c), x^{n-1}\sin(bx+c), x^{n-1}\cos(bx+c)\ldots\ldots x\sin(bx+c), x\cos(bx+c), \sin(bx+c), \cos(bx+c)\}$
$e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
$e^{ax}x^n\sin(bx+c)$ or $e^{ax}\cos(bx+c)$	$\{x^ne^{ax}\sin(bx+c), x^ne^{ax}\cos(bx+c), x^{n-1}e^{ax}\sin(bx+c), x^{n-1}e^{ax}\sin(bx+c), x^{n-1}e^{ax}\cos(bx+c), \ldots x^{n-1}e^{ax}\cos(bx+c), \ldots xe^{ax}\sin(bx+c), xe^{ax}\cos(bx+c), e^{ax}\sin(bx+c), e^{ax}\cos(bx+c).$

DE: $(D^3-3D^2+3D-1)y = 2e^x$

$(D^3-3D^2+3D-1)y_c = 0$

$L(m)=m^3-3m^2+3-1 = 0$

$m_1=m_2=m_3=1$

$$\boxed{y_c = (c_1+c_2x+c_3x^2)e^x}$$

$y_{gen} = y_c + y_p$

$(D^3-3D^2+3D-1)y_p = 2e^x$

$y_p = (ae^x)\cdot x^3$

$Dy_p = 3ax^2e^x + ax^3e^x$

$D^2y_p = 6axe^x + (ax^3+3ax^2)e^x + 3ax^2e^x + ax^3e^x$

$= (2ax^3+6ax^2+6ax)e^x$

$D^3y_p = (6ax^2+12ax+6a)e^x + (2ax^3+6ax^2+6ax)e^x$

$(\cancel{2ax^3}+\cancel{12ax^2}+18\cancel{a}x+6a)e^x = 2e^x$

$6ae^x = 2e^x$

$a = 1/3$

$y_p = \frac{1}{3}x^3e^x$

HW

ex: $y^{(4)}+8y''+16y = x^3\sin 2x + x^2\cos 2x$

$(D^4+8D^2+16)y_c = 0$

$L(m) = m^4+8m^2+16$

$(r^2+4)^2 = 0$

$r_{1,2,3,4} = \pm 2i$

$y_c = c_1\cos 2x + c_2x\cos 2x + c_3\sin 2x + c_4x\sin 2x$

$(D^4+8D^2+16)y_{p_1} = x^3\sin 2x$

$(D^4+8D^2+16)y_{p_2} = x^2\cos 2x$

$y_{p_1} = \{x^3\sin 2x, x^3\cos 2x, x^2\sin 2x, x^2\cos 2x, x\sin x, x\cos x, \sin x, \cos x\}$

$y_{p_2} = \{x^2\sin 2x, x^2\cos 2x, x\sin x, x\cos x, \sin x, \cos x\}$

UC set of y_{p_1} includes y_{p_2}, then we disregard y_{p_2} and y_{p_2} also contains elements of y_c, therefore we multiply the elements in y_c by x^2, and we obtain $y^*_{p_1}$ that does not contains any elements of y_c

So $y_p = Ax^5\sin 2x + Bx^5\cos 2x + Cx^4\sin 2x + Dx^4\cos 2x + Ex^3\sin 2x + Fx^3\cos 2x + Gx^2\sin 2x + Hx^2\cos 2x$

$y_{gen} = y_c + y_p$

The Method of Variation of Parameter :

V.P. Method or LAGRANGE METHOD for finding y_p

The method of U.C. is effective only when the RHS of the DE

$L(D)\, y = f(x)$ is of special type

$$f(x) = e^{\alpha x}\left[(a_0+a_1 x+a_2 x^2+\dots)\cos\beta x + (b_0+b_1 x+b_2 x^2+\dots)\sin\beta x\right]$$

LAGRANGE's method is applicable where the method of U.C. cannot work, as well as where it does.

This is another method we can apply to find a particular soln of a non-homogeneous LDE. We will develop the method in connection with a 2nd ord. linear ODE with variable coeffs.

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = f(x) \qquad \text{---}(1)$$

Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent soln of the corresponding homogeneous eqn.

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0$$

then the complementary soln of eqn (1) is:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 & c_2 are arb. consts.

Replace the arb. consts c_1 & c_2 in the complementary soln by two fns $v_1(x)$ & $v_2(x)$ to be determined such that

$$y_p(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x)$$

is a particular soln of eqn (1).

This is the first condition we impose in order to determine $v_1(x)$ & $v_2(x)$.

Since there are two fns, we still have a 2nd condition to impose, provided that this second condition does not violate the first condition.

Let's differentiate the $f\underline{n}$ y_p :

$$y_p' = v_1(x)\, y_1'(x) + v_1'(x)\, y_1(x) + v_2(x)\, y_2'(x) + v_2'(x)\, y_2(x)$$

At this point we impose the second condition. We simplify y_p' by demanding

Assume $\quad v_1'(x) y_1(x) + v_2'(x) y_2(x) = 0 \quad \text{-----} \ (*)$

So, y_p' becomes

$$y_p' = v_1(x)\, y_1'(x) + v_2(x)\, y_2'(x)$$

And the 2nd derivative is

$$y_p'' = v_1(x)\, y_1''(x) + v_1'(x) y_1'(x) + v_2(x)\, y_2''(x) + v_2'(x)\, y_2'(x)$$

Substituting in $eq\underline{n}$ (1)

$$a_0(x)\left[v_1(x) y''(x) + v_1'(x) y_1'(x) + v_2(x) y_2''(x) + v_2'(x) y_2'(x)\right] + a_2(x)\left[v_1(x)\, y_1(x) + v_2(x) y_2(x)\right] = f(x)$$
$$+ a_1(x)\left[v_1(x) y_1'(x) + v_2(x) y_2'(x)\right]$$

or

$$v_1(x)\left[a_0(x) y_1''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x)\right]$$
$$+ v_2(x)\left[a_0(x) y_2''(x) + a_1(x)\, y_2'(x) + a_2(x) y_2(x)\right]$$
$$+ a_0(x)\left[v_1'(x) y_1'(x) + v_2'(x)\, y_2'(x)\right] = f(x)$$

Since y_1 & y_2 are $sol\underline{ns}$ of the corresponding homogeneous $eq\underline{n}$, the first two terms on the left side of the eqn are zero. Then,

$$a_0(x)\left[v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x)\right] = f(x)$$

or

$$v_1'(x)\, y_1'(x) + v_2'(x) y_2'(x) = f(x)/a_0(x) \quad \text{-----}(**)$$

So, the two imposed condition have created a system of two $eq\underline{ns}$ that the derivatives of the two $f\underline{ns}$ v_1 & v_2 are satisfying.

Solve (*) and (**)

$$v_1'(x) = -\frac{y_2(x)\, f(x)}{a_0\left[y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)\right]}$$

$$v_2'(x) = \frac{y_1(x)\, f(x)}{a_0\left[y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)\right]}$$

Take integrals

$$v_1(x) = -\int \frac{y_2(x)\, f(x)}{a_0\, W(y_1, y_2)}\, dx$$

$$v_2(x) = \int \frac{y_1(x)\, f(x)}{a_0\, W(y_1, y_2)}\, dx$$

where

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$\therefore$ L.I.

ex: $D(D+2)y = 2x+1$

U.C. method

$D(D+2)\,y_c = 0$

$D^2 + 2D \rightarrow m^2 + 2m = 0$

$m_1 = 0,\ m_2 = -2$

$y_c = C_1 + C_2 e^{-2x}$

$y_p = (ax+b)\,x$

$y_p = ax^2 + bx$

$y_p' = 2ax + b$

$y_p'' = 2a$

$D^2 y_p + 2D y_p = 2x+1$

$2a + 2(2ax+b) = 2x+1 \rightarrow a = \frac{1}{2}$

$2(a+b) = 1 \rightarrow b = 0$

$y_p = \frac{1}{2}x^2$

$y_{gen} = y_c + y_p$

$= \frac{1}{2}x + C_1 + C_2 e^{-2x}$

by using Lagrange method:

$y_c = C_1 + C_2 e^{-2x}$

$y_p = A(x) + B(x)\, e^{-2x}$

$D y_p = A'(x) + B'(x)\, e^{-2x} - 2B(x)\, e^{-2x}$

Assume, $\quad A'(x) + B'(x)\, e^{-2x} = 0$

$D y_p = -2 B(x) e^{-2x}$

$D^2 y_p = -2 B'(x) e^{-2x} + 4 B(x) e^{-2x}$

$(D^2 + 2D)\, y_p = 2x+1$

$-2 B'(x) e^{-2x} + \cancel{4 B(x) e^{-2x}} - \cancel{4 B(x) e^{-2x}} = 2x+1$

$-2 B'(x) e^{-2x} = 2x+1$ $\qquad x = u \quad dv = e^{2x}$

$\qquad dx = du \quad v = \frac{1}{2} e^{2x}$

$B'(x) = -x e^{2x} - \frac{1}{2} e^{2x}$

$B(x) = -\int x e^{2x}\, dx - \frac{1}{2}\int e^{2x}\, dx$

$= -\left[uv - \int v\, du\right] - \frac{1}{4} e^{2x}$

$= -\left[\frac{x}{2} e^{2x} - \frac{1}{2}\int e^{2x}\, dx\right] - \frac{1}{4} e^{2x}$

$B(x) = -\frac{x}{2} e^{2x} + \cancel{\frac{1}{4} e^{2x}} - \cancel{\frac{1}{4} e^{2x}}$

$B(x) = -\frac{x}{2} e^{2x} + \cancel{c_1}$

$A'(x) + B'(x) e^{-2x} = 0$

$A(x) = \int \left(x e^{2x} + \frac{1}{2} e^{2x}\right) e^{-2x}\, dx$

$A(x) = \int \left(x + \frac{1}{2}\right) dx = \frac{x^2}{2} + \frac{1}{2} x$

$y_p = A(x) + B(x) e^{-2x} = \frac{x^2}{2} + \cancel{\frac{1}{2} x} + -\cancel{\frac{x}{2} e^{2x}} \cdot \cancel{e^{-2x}} = \frac{1}{2} x^2$

$y_{gen} = \frac{x^2}{2} + c_1 + c_2 e^{-2x}$

ex: $y'' + y = \sec x \tan x$

$(D^2+1)\, y_c = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

Assume, $y_p = A(x) \cos x + B(x) \sin x$

$Dy_p = -A(x)\sin x + A'(x)\cos x + B(x)\cos x + B'(x)\sin x$

Assume, $A'(x)\cos x + B'(x)\sin x = 0$

$Dy_p = -A(x)\sin x + B(x)\cos x$

$D^2 y_p = -A(x)\cos x - A'(x)\sin x - B(x)\sin x + B'(x)\cos x$

$(D^2+1)\, y_p = \sec x \tan x$

$\cancel{-A(x)\cos x} - A'(x)\sin x \cancel{- B(x)\sin x} + B'(x)\cos x \cancel{+ A(x)\cos x} \cancel{+ B(x)\sin x}$

$= \sec x \tan x$

$-A'(x)\sin x + B'(x)\cos x = \sec x \tan x$

$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \neq 0$

$= \cos^2 x + \sin^2 x = 1$

$A'(x) = -\dfrac{y_2(x) f(x)}{a_0 W(y_1(x), y_2(x))}$

$= -\dfrac{\sin x \sec x \tan x}{1} = -\tan^2 x$

$B'(x) = \dfrac{y_1(x) f(x)}{a_0 W(y_1(x), y_2(x))}$

$= \dfrac{\cos x \sec x \tan x}{1} = \tan x$

Then,

$$A(x) = -\int \tan^2 x \, dx = -\int (\sec^2 x - 1)\, dx = -\tan x + x = x - \tan x$$

$$B(x) = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x|$$

$$y_p = (x - \tan x)\cos x + \ln|\sec x| \sin x$$

$$y_{gen} = y_c + y_p$$

ex: $(x^2+2x)y'' - 2(x+1)y' + 2y = (x+2)^2$

Find the general sol$\underline{n}$. Given that $y = x+1$ & $y = x^2$ are LI sol$\underline{n}$ of the DE above.

Sol'$\underline{n}$: $y_c = c_1(x+1) + c_2 x^2$

$$y_p = A(x)(x+1) + B(x)x^2$$

$$y_p' = A(x) + A'(x)(x+1) + 2x\,B(x) + B'(x)\,x^2$$

$$A'(x)(x+1) + B'(x)x^2 = 0$$

Then, $y_p' = A(x) + 2x\,B(x)$

$$y_p'' = A'(x) + 2B(x) + 2x\,B'(x)$$

Substituting in DE,

$$(x^2+2x)\left[A'(x) + 2B(x) + 2x\,B'(x)\right] - 2(x+1)\left[A(x) + 2x\,B(x)\right]$$

$$+ 2\left[A(x)(x+1) + B(x)x^2\right] = (x+2)^2$$

Finally, $A'(x) + 2x\,B'(x) = \dfrac{x+2}{x}$

$$W(x+1, x^2) = \begin{vmatrix} x+1 & x^2 \\ 1 & 2x \end{vmatrix} \neq 0$$

$$= 2x(x+1) - x^2$$

$$= x(x+2)$$

$$A'(x) = -\frac{x^2(x+2)^2}{[\underbrace{x(x+2)}_{a(x)} . \underbrace{x(x+2)}_{W}]} = -1$$

$$B'(x) = \frac{(x+1)\,\cancel{(x+2)^2}}{[x\,\cancel{(x+2)}]^2} = \frac{1}{x} + \frac{1}{x^2}$$

$$A(x) = -\int dx = -x$$

$$B(x) = \int\left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \ln|x| - x^{-1}$$

$$y_p = -x(x+1) + \left(\ln|x| - \frac{1}{x}\right) x^2$$

$$= -x^2 - 2x + x^2 \ln|x|$$

$$y_{gen} = y_c + y_p$$

HW P.171, 24, ch.4.4

ex: $(2x+1)(x+1)y'' + 2xy' - 2y = (2x+1)^2$

$y = x$ & $y = (x+1)^{-1}$ are LI sol^n of DE. Find y_{gen}.

HW p.170, 13, ch.4h

ex: $y'' + 3y' + 2y = \frac{1}{1+e^x}$

$(D^2+3D+2)\,y_c = 0$

$m^2+3m+2 \rightarrow m_1 = -1,\ m_2 = -2$

$\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$

$W(e^{-x}, e^{-2x}) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$

Assume $y_p = A(x)\,e^{-x} + B(x)\,e^{-2x}$

$Dy_p = A'(x)e^{-x} - A(x)e^{-x} + B'(x)e^{-2x} - 2B(x)e^{-2x}$

Assume $A'(x)e^{-x} + B'(x)e^{-2x} = 0$

$-A'(x)e^{-x} - 2B'(x)e^{-2x} = \frac{1}{1+e^x}$

$A'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix}}{-e^{-3x}} = \frac{\cancel{-}e^{-2x}}{\cancel{-}(1+e^x)\,e^{-3x}} = \frac{e^x}{1+e^x}$

$1+e^x = u$

$e^x dx = du$

$A(x) = \int \frac{e^x}{1+e^x}dx = \int \frac{du}{u} = \ln u = \boxed{\ln(1+e^x)}$

$B'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix}}{-e^{-3x}} = \frac{e^{-x}}{-(1+e^x)\,e^{-3x}} = -\frac{e^{2x}}{1+e^x}$

$B(x) = -\int \frac{e^{2x}}{1+e^x}dx = -\int \frac{(u-1)}{u}du = -\int du + \int \frac{du}{u} = -u + \ln u$

$\boxed{B(x) = -(1+e^x) + \ln(1+e^x)}$