Chapter 2

1^{st} Order Equations which exact solutions are obtainable.

A) Standart Forms of First Order DEs.

In this chapter we will study DE in either of the following forms;

a) $\frac{dy}{dx} = f(x,y)$ → derivative form

b) $M(x,y)\,dx + N(x,y)\,dy = 0$ → differential form

One can be written in the other form.

ex: $\frac{dy}{dx} = \frac{x^2+y^2}{x-y}$ → derivative form → $(x-y)\,dy - (x^2+y^2)\,dx = 0$ → diff. form

ex: $(\sin x + y)\,dx + (x+3y)\,dy = 0$ → differential form

$\frac{dy}{dx} = -\frac{\sin x + y}{x+3y}$ → derivative form

y is dependent variable
x is independent variable

B) Exact DEs

Defn: The expression

$M(x,y)\,dx + N(x,y)\,dy$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential. $dF(x,y)$ for all $(x,y) \in D$.

This expression is an exact differential in D if there exist a function F such that,

$\frac{\partial F(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial F(x,y)}{\partial y} = N(x,y)$ for all $(x,y) \in D$.

If $dF(x,y) = M(x,y)\,dx + N(x,y)\,dy$ is an exact differential form, then the DE,

$M(x,y)\,dx + N(x,y)\,dy = 0$ is called an exact DE.

ex: $y^2 dx + 2xy\,dy = 0$ is an exact DE

it is the total differential of the function F defined for all (x,y) by

- $F(x,y) = xy^2$ $\quad \frac{\partial F}{\partial x} = y^2 = M$

$\frac{\partial F}{\partial y} = 2xy = N$

- $\underbrace{(2xy^2+1)}_{M} dx + \underbrace{(2x^2y)}_{N} dy = 0$ $\quad \frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x} \rightarrow$ exact

- $y\,dx + 2x\,dy = 0$ is not exact

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Theorem: The necessary and sufficient condition for exactness of the DE

$M\,dx + N\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof: By definition of an exact differential, there exist a function F such that

$\frac{\partial F(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial F(x,y)}{\partial y} = N(x,y)$ for all $(x,y) \in D$ is rectangular domain

Then,

$\frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$

But, using the continuity of the first partial derivatives of M & N, we have

$$\frac{\partial^2 F(x,y)}{\partial y\,\partial x} = \frac{\partial^2 F(x,y)}{\partial x\,\partial y}$$

$\Rightarrow$ Hence $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$

2) Conversely, if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$ for all $(x,y) \in D$, then

$M\,dx + N\,dy$ is exact.

ex: $\underbrace{(2xy+3x^2)}_{M}\,dx + \underbrace{x^2}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$ for $\forall x$ $\therefore$ exact

ex: $\underbrace{y}_{M}\,dx + \underbrace{2x}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2$$ $\rightarrow$ $\therefore$ not exact

ex: $(2x\sin y + y^3 e^x)\,dx + (x^2\cos y + 3y^2 e^x)\,dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2x\cos y + 3y^2 e^x \\ \frac{\partial N}{\partial x} &= 2x\cos y + 3y^2 e^x \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \therefore \text{exact}$$

C. The Soln of the exact DE s:

Theorem: Suppose the DE $M(x,y)\,dx + N(x,y)\,dy = 0$ satisfies differentiability requirements of exactness, then a one-parameter family of solns of this DE is given by $F(x,y) = c$, where F is a function such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y),$$ for all $(x,y) \in D$ and c is arbitrary constant

$F(x,y) = \int M(x,y)\,dx + \phi(y)$ if we derive it w.r.t. "y"

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + \frac{d\phi}{dy} = N(x,y)$$

from here, $\frac{d\phi}{dy} = \underbrace{\psi}_{(psi)}(y)$ obtained ex.: $(2xy+3x^2)\,dx + x^2\,dy = 0$ Find the soln of this DE.

Our first duty is to determine whether or not the eqn is exact.

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$M = 2xy + 3x^2 = \frac{\partial F(x,y)}{\partial x}$$

$$N = x^2 = \frac{\partial F(x,y)}{\partial y}$$

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = d(c_1) = 0$$

$$\int \partial F(x,y) = \int x^2\,\partial y$$

$$F(x,y) = x^2 y + \underbrace{\phi(x)}_{\text{arbitrary fn.}}$$

$$\frac{\partial F(x,y)}{\partial x} = 2xy + \phi'(x) = M(x,y) = 2xy + 3x^2$$

$\phi'(x) = 3x^2$ → There must not be any "y" in the expression fn of x only

$$\frac{d\phi(x)}{dx} = 3x^2 \rightarrow \int d\phi(x) = \int 3x^2\,dx$$

$$\phi(x) = x^3 + C_2$$

Hence, $F(x,y) = x^2 y + x^3 + C_2 = C_1$ → $\boxed{F(x,y) = x^2 y + x^3 = C_0}$

$= x^2 y + x^3 = C_1 - C_2 = C_0$

Method of Grouping: Quick but, need experience and inequity

$(2xy+3x^2)\,dx + x^2\,dy = 0$ we now recognize this as,

$(3x^2)\,dx + [(2xy)\,dx + (x^2)\,dy] = d(C_0)$

$d(x^3) + d(x^2 y) = d(C_0)$ from this we have

$$\boxed{x^3 + x^2 y = C}$$

ex: Find the soln of DE given below.

$$\frac{dy}{dx} = -\frac{3x^2 \sin y + y^2}{x^3 \cos y + 2yx}$$

Soln: $M(x,y) = 3x^2 \sin y + y^2 \rightarrow \frac{\partial M}{\partial y} = 3x^2 \cos y + 2y$

$N(x,y) = x^3 \cos y + 2xy \rightarrow \frac{\partial N}{\partial x} = 3x^2 \cos y + 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \therefore$ exact

$$f_x(x,y) = \frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x^2 \sin y + y^2$$

$$f_y(x,y) = \frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 \cos y + 2yx$$

$$F(x,y) = \int N(x,y)\, \partial y + \phi(x) = \int (x^3 \cos y + 2yx)\, dy + \phi(x)$$

$$= x^3 \sin y + y^2 x + \phi(x)$$

From this, $f_x(x,y) = 3x^2 \sin y + y^2 + \phi'(x)$

$\therefore \; 3x^2 \cancel{\sin y} + \cancel{y^2} = 3x^2 \cancel{\sin y} + \cancel{y^2} + \phi'(x)$

$\phi(x) = C_1$

$$\boxed{f(x,y) = x^3 \sin y + y^2 x + C = 0}$$

by grouping

$(3x^2 \sin y + y^2)\, dx + (x^3 \cos y + 2yx)\, dy = 0$

$(3x^2 \sin y\, dx + x^3 \cos y\, dy) + (y^2\, dx + 2yx\, dy) = 0$

$d(x^3 \sin y + y^2 x) = d(C_0)$

$$\boxed{x^3 \sin y + y^2 x = C}$$

ex: (Sect. 2.1, p.36)

(3) Show that $(2xy+1)\,dx+(x^2+4y)\,dy=0$ is exact. Find sol$\underline{n}$:

Sol$\underline{n}$: $M(x,y)=2xy+1$ $\quad M_x = 2x$
$N(x,y)=x^2+4y$ $\quad N_y = 2x$ $\Big\}$ $M_x = N_y \rightarrow \therefore$ DE is exact

$$F_x(x,y)=\frac{\partial F(x,y)}{\partial x}=M(x,y)=2xy+1$$

$$F_y(x,y)=\frac{\partial F(x,y)}{\partial y}=N(x,y)=x^2+4y$$

$$f(x,y)=\int M(x,y)\,dx+\varnothing(y)$$
$$=\int(2xy+1)\,dx+\varnothing(y)$$
$$=x^2y+x+\varnothing(y)$$

From this, $f_y(x,y)=x^2+\varnothing'(y)$

$$\therefore\ x^2+4y=x^2+\varnothing'(y) \Rightarrow \frac{d\varnothing(y)}{dy}=4y$$
$$\varnothing(y)=2y^2+C_0$$

Thus, $F(x,y)=x^2y+x+2y^2+C_0$

The one-parameter family of sol$\underline{ns}$ $F(x,y)=C_1$ is

$\boxed{x^2y+x+2y^2=C}$ where $C=C_1-C_0$

by grouping, $(2xy+1)\,dx+(x^2+4y)\,dy=0$

$$(2xy\,dx+x^2\,dy)+dx+4y\,dy=0$$
$$d(x^2y)+d(x)+d(2y^2)=d(C)$$
or $$d(x^2y+x+2y^2)=d(C)$$

Hence sol$\underline{n}$ is $\boxed{x^2y+x+2y^2=C}$

HW: (1) $(6xy+2y^2-5)\,dx+(3x^2+4xy-6)\,dy=0$

Sol1: $M=6xy+2y^2-5 \rightarrow M_y=6x+4y$
$N=3x^2+4y-6 \rightarrow N_x=6x+4y$ $\Big\}$ $M_y=N_x \rightarrow \therefore$ EXACT

$$(6xy\,dx+3x^2\,dy)+(2y^2dx+4xy\,dy)-5\,dx-6\,dy=0$$
$$d(3x^2y)+d(2y^2x)-d(5x)-d(6y)=d(C)$$
$$\boxed{3x^2y+2y^2x-5x-6y=C_1}$$

(12) Solve the following initial value problem (IVP)

$(3x^2y^2 - y^3 + 2x)\,dx + (2x^3y - 3xy^2 + 1)\,dy = 0 \qquad y(-2) = 1$

Soln: $M(x,y) = 3x^2y^2 - y^3 + 2x \rightarrow M_y(x,y) = 6x^2y - 3y^2$

$N(x,y) = 2x^3y - 3xy^2 + 1 \rightarrow N_x(x,y) = 6x^2y - 3y^2$ $\Big\}\, M_y = N_x \rightarrow$ exact

Find $F(x,y)$, such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$

$$F(x,y) = \int M(x,y)\,dx + \phi(y)$$
$$= \int (3x^2y^2 - y^3 + 2x)\,dx + \phi(y)$$
$$= x^3y^2 - xy^3 + x^2 + \phi(y)$$

From this, $F_y(x,y) = \cancel{2x^3y} - \cancel{3xy^2} + \phi'(y) = N(x,y) = \cancel{2x^3y} - \cancel{3xy^2} + 1$

$\phi'(y) = 1$

$\phi(y) = y + C_0$ & $F(x,y) = C_1$

Thus, $F(x,y) = x^3y^2 - xy^3 + x^2 + y + C_0$ where $C = C_1 - C_0$

$$\boxed{x^3y^2 - xy^3 + x^2 + y = C}$$

I.C. $y(-2) = 1 \rightarrow x = -2,\ y = 1$

$\underbrace{(-2)^3(1)^2}_{-8} - \underbrace{(-2)(1)^3}_{+2} + \underbrace{(-2)^2}_{+4} + 1 = C \rightarrow \underline{C = -1}$

Thus the particular soln is

$$\boxed{\begin{aligned} &x^3y^2 - xy^3 + x^2 + y = -1 \quad \text{or} \\ &x^3y^2 - xy^3 + x^2 + y + 1 = 0 \end{aligned}}$$

by grouping, $(3x^2y^2\,dx + 2x^3y\,dy) - (y^3\,dx + 3xy^2\,dy) + 2x\,dx + dy = 0$

$d(x^3y^2) - d(xy^3) + d(x^2) + d(y) = d(c)$

$$\boxed{x^3y^2 - xy^3 + x^2 + y = C}$$

HW's
1. $4xy + (2x^2 - 6y)\,y' = 0, \quad y(1) = 1$
2. $y + 3y^{-1} + 2xy' = 0, \quad y(1) = 1$
3. $1 + 2x^2 - (x + 4xy)y' = 0, \quad \mu = \mu(x)$
4. $(e^x + e^y + x^2y)\,y' = -(e^xy + xy^2)$
5. $(1 + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$

from Lowenders's book

Roney

<u>ex:</u> Solve the DE

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\,y' = 0$$
$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

$M(x,y) = y\cos x + 2xe^y \rightarrow M_y = \cos x + 2xe^y$
$N(x,y) = \sin x + x^2e^y - 1 \rightarrow N_x = \cos x + 2xe^y$
$\Big\}\; M_y = N_x \rightarrow$ exact

$$F(x,y) = \int M(x,y)\,\partial x + \phi(y)$$
$$= \int (y\cos x + 2xe^y)\,dy + \phi(y)$$
$$= y\sin x + x^2e^y + \phi(y)$$

From this, $f_y(x,y) = \cancel{\sin x} + \cancel{x^2e^y} + \phi'(y) = N(x,y) = \cancel{\sin x} + \cancel{x^2e^y} - 1$

$$\phi'(y) = -1$$
$$\phi(y) = -y + C_0$$

Thus, $F(x,y) = y\sin x + x^2e^y - y + C_0$

$$\Rightarrow \boxed{y\sin x + x^2e^y - y = C}$$

by grouping, $(y\cos x\,dx + \sin x\,dy) + (2xe^y dx + x^2e^y\,dy) - dy = 0$

$$d(y\sin x) + d(x^2e^y) - d(y) = 0$$
$$\boxed{y\sin x + x^2e^y - y = C}$$

INTEGRATING FACTORS

If $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$ then the eqn is not exact.

Then we multiply the eqn $F(x,y)$ by a factor $\mu(x,y)$ to make the eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an exact eqn. $\mu(x,y)$ is called "integrating factor".

In simple cases we may find integrating factors by inspection or
perhaps after some trials. Therefore; while in principle IF are powerfull
tools for solving DEs, in practice they can be found only in special
cases. The most important situations which simple integrating factors
can be found occur when μ is a fn of only one of the variables x or y,
instead of both.

Let us determine necessary conditions on M and N so that

$M(x,y)\,dx + N(x,y)\,dy = 0$

has an IF μ that depends on x only. Assuming that μ is a fn of x only, we have,

$$(\mu M)_y = \mu M_y \quad , \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

Thus, if $(\mu M)_y = (\mu N)_x$ it is necessary that

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

If μ depends on y only, in similar manner

$$\boxed{\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu}$$

HW: If $M + Ny' = 0$ has an IF of $\mu(x,y)$. Find a general formula for this integrating factor.

Soln:

$$\mu M + \mu N y' = 0$$

$$(\mu M)_y = (\mu N)_x \Rightarrow \mu_y M - \mu_x N = \mu N_x - \mu M_y$$

Suppose that $\Rightarrow N_x - M_y = R\,(xM - yN)$ in which $R(z)$ where $z = xy$

modified form of equation is exact $\Rightarrow \mu_y M - \mu_x N = \mu\, R(xM - yN)$

$$= R(\mu x M - \mu y N)$$

this relation is satisfied if

$$\mu_y = (\mu x)R \quad \text{and} \quad \mu_x = (\mu y)R$$

Now consider $\mu = \mu(xy) \rightarrow \begin{array}{l} \mu_x = \mu' y \\ \mu_y = \mu' x \end{array}$ Note that $\mu' = \dfrac{d\mu}{dz}$

Thus, $\mu'(z) = R(z)$

Then, $\mu'(z) = R(z)$

$d\mu = R(z)\,dz$ seperable

$\mu(z) = \int R(z)\,dz \quad \longrightarrow \therefore \; R = R(xy)$

$\mu = \mu(xy)$ is possible to determine.

ex: $\underbrace{(3xy+y^2)}_{M} + \underbrace{(x^2+xy)}_{N} y' = 0$

Soln: $\frac{\partial M}{\partial y} = 3x+2y$, $\frac{\partial N}{\partial x} = 2x+y$ $\Big\}$ $M_x \neq N_y \rightarrow$ not exact

Let us DE has μ depends on x only

$$\frac{d\mu}{dx} = \frac{3x+2y-(2x+y)}{x^2+xy}\mu = \frac{\mu}{x} \rightarrow \mu(x) = x$$

Then, $(3x^2y + xy^2) + (x^3 + x^2y)\, y' = 0$

$\frac{\partial M}{\partial y} = 3x^2+2xy$, $\frac{\partial N}{\partial x} = x^3+x^2y$ $\Big\}$ $M_y = N_x \rightarrow$ exact

by grouping, $(3x^2y\,dx + x^3dy) + (xy^2dx + x^2y\,dy) = 0$

$$d(x^3y) + d\left(\tfrac{1}{2}x^2y^2\right) = d(c)$$

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c}$$

ex: $\underbrace{(3y+4xy^2)}_{M}dx + \underbrace{(2x+3x^2y)}_{N}dy = 0$

Soln: $\frac{\partial M}{\partial y} = 3+8xy$, $\frac{\partial N}{\partial x} = 2+6xy$ $\Big\}$ $M_y \neq N_x \rightarrow$ not exact

Let $\mu(x,y) = x^2y$

Then, $(3x^2y^2+4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0 \rightarrow$ EXACT since,

$$\frac{\partial M}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial N}{\partial x} = 6x^2y + 12x^3y^2$$

by grouping, $(3x^2y^2dx + 2x^3y\,dy) + (4x^3y^3dx + 3x^4y^2dy) = 0$

$$d(x^3y^2) + d(x^4y^3) = d(c)$$

$$\boxed{x^3y^2 + x^4y^3 = C_1}$$

ex: (sec 2.1, Q 21, P37)
Consider the DE

$(4x+3y^2)\,dx + 2xy\,dy = 0$

a) check exactness

b) Find an IF in the form of x^n (n is positive integer)

c) Multiply the IF with DE and solve it

Soln: $M = 4x+3y^2 \rightarrow \partial M/\partial y = 6y$; $M_y \neq N_x \rightarrow$ not EXACT

(a) $N = 2xy \rightarrow \partial N/\partial x = 2y$

(b) $x^n(4x+3y^2)\,dx + 2x^{n+1}y\,dy = 0$

$\frac{\partial M}{\partial y} = 6yx^n$

$\frac{\partial N}{\partial x} = 2(n+1)x^n y$

$6yx^n = 2(n+1)x^n y \rightarrow \underline{n=2}$

(c) $(4x^3+3y^2x^2)\,dx + 2x^3y\,dy = 0$

$(4x^3\,dx) + (3y^2x^2\,dx + 2x^3y\,dy) = 0$

$d(x^4) + d(x^3y^2) = d(c)$

$\boxed{x^4 + x^3y^2 = C}$

Qiz: Solve the DE given below. If it has an IF $\mu(x)$ depends x only.

$(4xy+3y^2-x)\,dx + x(x+2y)\,dy = 0$

Soln: $\frac{\partial M}{\partial y} = 4x+6y$; $M_y \neq N_x \rightarrow$ not EXACT

$\frac{\partial N}{\partial x} = 2(x+y)$

$x^n(4xy+3y^2-x)\,dx + x^{n+1}(x+2y)\,dy = 0$

$\frac{\partial M}{\partial y} = x^n(4x+6y)$

$\frac{\partial N}{\partial x} = (n+1)x^n(x+2y) + x^{n+1}$

$\cancel{x^n}(4x+6y) = (n+1)\cancel{x^n}(x+2y) + x^{n+1}$

$3(x+\cancel{2y}) = (n+1)(x+\cancel{2y}) \rightarrow \underline{n=2}$

Multiply the org. DE by IF (x^2)

$(4x^3y+3x^2y^2-x^3)\,dx + (x^4+2x^3y)\,dy = 0$

by grouping we get, $(4x^3y\,dx + x^4\,dy) + (3x^2y^2\,dx + 2x^3y\,dy) - x^3\,dx = 0$

$d(x^4y) + d(x^3y^2) - d(\frac{1}{4}x^4) = d(C)$

$\boxed{x^4y + x^3y^2 - \frac{1}{4}x^4 = C}$

2.2 SEPERABLE EQUATIONS :

An eqn of the form

$$F(x)\,G(y)\,dx + f(x)\,g(y)\,dy = 0 \quad \text{---- (A)}$$

is called seperable eqn.

In general SEPERABLE eqns are not exact, but they possess an integrating factor

$$\mu(x,y) = \frac{1}{f(x)\,G(y)}$$

Multiply (A) by μ yields,

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0$$ This eqn is exact, since,

$$\frac{\partial}{\partial y}\Bigg[\underbrace{\frac{F(x)}{f(x)}}_{M}\Bigg] = 0 = \frac{\partial}{\partial x}\Bigg[\underbrace{\frac{g(y)}{G(y)}}_{N}\Bigg] = 0$$

$$M\,dx + N\,dy = 0$$

$$\boxed{\int M(x)\,dx + \int N(y)\,dy = C}$$

ex1: $(xy + x^2)\,dx + yx^2\,dy = 0$

$x(y+x)\,dx + yx^2\,dy = 0 \rightarrow$ not seperable

ex2: $\underbrace{x^2 y}_{F(x)\,G(y)}\,dx + \underbrace{xy^2}_{f(x)\,g(y)}\,dy = 0 \rightarrow$ seperable

$\mu(x,y) = \dfrac{1}{f(x)\,G(y)} = \dfrac{1}{xy} \rightarrow (x \neq 0,\ y \neq 0)$

$x\,dx + y\,dy = 0 \rightarrow \int x\,dx + \int y\,dy = c_1 \rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} = C_1 \rightarrow \boxed{x^2 + y^2 = C}$ where $C = 2C_1$

ex3: $(x-4)\,y^4\,dx - x^3(y^2-3)\,dy = 0 \rightarrow$ seperable, not exact

$\mu = \dfrac{1}{x^3 y^4} \rightarrow (x^3 \neq 0,\ y^4 \neq 0)$

$\dfrac{x-4}{x^3}\,dx - \dfrac{y^2-3}{y^4}\,dy = 0 \rightarrow (x^{-2} - 4x^{-3})\,dx - (y^{-2} - 3y^{-4})\,dy = 0$

$-\dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{1}{y} - \dfrac{1}{y^3} = c$ ($x=0$, $y=0$ not member)

But in case we use

$\dfrac{dy}{dx} = \dfrac{(x-4)\,y^4}{x^3(y^2-3)} \rightarrow$ ($y=0$ is also member) $\rightarrow$ The soln was lost in seperation process.

HOMOGENEOUS EQNS :

Defn : The First-order DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be homogeneous, if, when written in the derivative form $(dy/dx) = f(x,y)$, there exists a fn g such that $f(x,y)$ can be expressed in the form $g(y/x)$

I.e: if a DE can be put in the form

$y' = f(y/x)$ or $y' = g(x/y)$ it is called HOMOGENEOUS DE.

ex : DE $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$. HOMOGENEOUS ?

derivative form

$$\frac{dy}{dx} = \frac{-x^2+3y^2}{2xy} \rightarrow \frac{3y^2-x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

Then, $\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right) \rightarrow f(x,y) = g(y/x) \rightarrow$ HOMOGENEOUS !

ex : $(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0 \rightarrow$ HOMOGENEOUS ?

derivative form $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} \mp \frac{\sqrt{x^2+y^2}}{\sqrt{x^2}} = \frac{y}{x} \mp \sqrt{1+\left(\frac{y}{x}\right)^2}$

Then, $f(x,y) = g(y/x) \rightarrow$ HOMOGENEOUS !

Degree of HOMOGENEITY :

if $F(tx,ty) = t^{n} F(x,y)$ then, F is called homogeneous of degree n

n → degree of homogeneity

Exs : ① $f(x,y) = x^2 - xy + y^2$

$f(tx,ty) = (tx)^2 - (tx)(ty) + (ty)^2 = t^2(x^2 - xy + y^2)$ — t^2 → D. of H.

② $f(x,y) = xy \sin\left(\frac{x^2+y^2}{xy}\right)$

$f(tx,ty) = t^2 xy \sin\left(\frac{x^2+y^2}{xy}\right) \rightarrow t^2\left[xy \sin\left(\frac{x^2+y^2}{xy}\right)\right]$ — t^2 → D. of H.

③ $f(x,y) = \sqrt{x^3 + x^2y}$

$f(tx,ty) = \sqrt{t^3x^3 + t^3x^2y} \rightarrow t^{3/2}\left(\sqrt{x^3 + x^2y}\right)$ — $t^{3/2}$ → D. of H.

Theorem: If $M(x,y)\,dx + N(x,y)\,dy = 0$ .... (A)

is a homogeneous eqn, then the change of variables $y = vx$, transforms the DE (A) into a seperable eqn in the variables v and x.

Proof: since $M(x,y)\,dx + N(x,y)\,dy = 0$, is homogeneous, it may be written as,

$$\frac{dy}{dx} = g(y/x)$$

Let $y = vx$, Then, $\frac{dy}{dx} = \frac{d}{dx}(vx)$

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ and DE (A) becomes

$$v + x\frac{dv}{dx} = g(v)$$

$[v - g(v)]\,dx + x\,dv = 0 \rightarrow$ seperable

$\int \frac{dv}{v-g(v)} + \int \frac{dx}{x} = c_1$, Let $F(v) = \int \frac{dv}{v-g(v)}$

$F(v) + \ln|x| = c_1 \leftarrow v = y/x$

$F(y/x) + \ln|x| = c_1$

ex: Solve the IVP $(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0$, $y(1) = 0$

1^{st} is it homogeneous?

$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \rightarrow \frac{dy}{dx} = \frac{y}{x} \mp \sqrt{1+\left(\frac{y}{x}\right)^2} \rightarrow$ HOMOGENEOUS

since $x>0$ take positive value

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

Let $y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = v + x\frac{dv}{dx}$$

$$\cancel{v} + \sqrt{1+v^2} = \cancel{v} + x\frac{dv}{dx}$$

$x\frac{dv}{dx} = \sqrt{1+v^2} \longrightarrow$ seperate the variables

$\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}} \longrightarrow \ln|x| + \ln|c| = \ln|v + \sqrt{v^2+1}|$

$\ln|cx| = \ln|v + \sqrt{v^2+1}|$

$v + \sqrt{v^2+1} = cx$, then replace v by y/x

$$x\left(\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx\right) \rightarrow y+\sqrt{y^2+x^2} = cx^2$$

I.C. when $y=0$, $x=1$

$$0+\sqrt{0+1} = 1.c \rightarrow c=1$$

$$y+\sqrt{y^2+x^2} = x^2 \rightarrow \boxed{y=\frac{1}{2}(x^2-1)}$$

$$\cancel{y^2}+\cancel{x^2} = x^4-2x^2y+\cancel{y^2}$$

$$y = \frac{1}{2}(x^2-1)$$

HW

ex: Solve the eqn $(x^2-3y^2)\,dx+2xy\,dy=0$

Soln: $x^{-2}[(x^2-3y^2)\,dx+2xy\,dy]=0$

$$\left[1-3\left(\frac{y}{x}\right)^2\right]dx+2\left(\frac{y}{x}\right)dy=0 \qquad \frac{y}{x}=v \rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$(1-3v^2)\,dx+2v\,dy=0 \rightarrow \frac{dy}{dx}=\frac{3v^2-1}{2v}$$

$$\rightarrow v+x\frac{dv}{dx}=\frac{3v^2-1}{2v}-v$$

$$\rightarrow \frac{dx}{x}=\frac{2v}{v^2-1}\,dv$$

$$\int\frac{2v}{v^2-1}\,dv=\frac{a}{v-1}+\frac{b}{v+1}$$

$av+a+bv-b$

$v(a+b)+(a-b)$

$a+b=2$

$a-b=0$

$a=1$

$b=1$

$$\rightarrow \ln|x| = \int\frac{1}{v-1}\,dv+\int\frac{1}{v+1}\,dv$$

$$\rightarrow \ln|x| = \ln|v-1|+\ln|v+1|$$

$$\rightarrow x = (v^2-1) \leftarrow v=\frac{y}{x}$$

$$\rightarrow x = \left(\frac{y}{x}\right)^2-1 \rightarrow \boxed{y=\sqrt{x^3+x^2}}$$

HW ex: $(x^2+3y^2)\,dx-2xy\,dy=0$ $\quad y(2)=6$

Sol'n: $x^{-2}\left[(x^2+3y^2)\,dx-2xy\,dy\right]=0$ $\qquad y=vx \rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\left[1+3\left(\frac{y}{x}\right)^2\right]dx-2\left(\frac{y}{x}\right)dy=0$$

$$(1+3v^2)\,dx-2v\,dy = \frac{dy}{dx}=\frac{1+3v^2}{2v}$$

$v^2+1=u$

$2v\,dv=du$

$$v+x\frac{dv}{dx}=\frac{1+3v^2}{2v}$$

$$\frac{dx}{x}=\frac{2v}{1+v^2}\,dv$$

$$\ln|x| = \frac{du}{u} = \ln u+\ln c$$

$$\ln|x| = \ln(u \cdot c)$$

$$|x| = u.c = (v^2+1)\,c$$

$$x = \left[\left(\frac{y}{x}\right)^2+1\right]c \rightarrow c=\frac{1}{5}$$

$$y=\sqrt{(5x-1)\,x^2}$$

HW

1. $(3x+8)(y^2+4)\,dx-4y(x^2+5x+6)\,dy=0$

   $y(0)=0$

Ans: $4(x+3)(x+2)^2=3(y^2+4)^2$

Linear Equations and Bernoulli Equations

Defn: A first-order O.D.E. is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \text{------} \quad (A)$$

for ex, $x\frac{dy}{dx} + (x+1)\,y = x^3$ $\quad 1^{st}$ Order L.D.E

$$\frac{dy}{dx} + \left(1+\frac{1}{x}\right)y = x^2 \longrightarrow P(x) = 1+\frac{1}{x}, \quad Q(x) = x^2$$

if we assume $Q(x) = 0$

$$\frac{dy}{dx} + P(x)\,y = 0 \Rightarrow \frac{dy}{y} = -P(x)\,dx$$

$$\Rightarrow y = c\,e^{-\int P(x)\,dx}$$

called homogeneous soln.

Let's write in the form,

$$[P(x)\,y - Q(x)]\,dx + dy = 0 \quad \text{----} \quad (B)$$

This is of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where $M(x,y) = P(x)\,y - Q(x)$, $N(x,y) = 1$

$$\frac{\partial M(x,y)}{\partial y} = P(x), \quad \frac{\partial N(x,y)}{\partial x} = 0$$

Eqn (A) is not exact unless $P(x)=0$. Then if $P(x)=0$ (A) become seperable. However, (B) possesses an integrating factor that only depends on x.

$$\left[\mu(x)P(x)y - \mu(x)Q(x)\right]dx + \mu(x)\,dy = 0 \quad \text{- - - - (C)}$$

$\mu(x)$ is an integrating factor only if (C) is exact. That is, iff

$$\frac{\partial}{\partial y}\left[\mu(x)P(x)y - \mu(x)Q(x)\right] = \frac{d}{dx}\left[\mu(x)\right]$$

this condition reduces to

$$\mu(x)P(x) = \frac{d}{dx}\left(\mu(x)\right]$$

P is a known function of ind. var. x but. μ is unknown function of x. Determine μ

$$\mu P(x) = \frac{d\mu}{dx} \Rightarrow \frac{d\mu}{\mu} = P(x)dx \Rightarrow \ln|\mu| = \int P(x)dx$$

$$\Rightarrow \ln|\mu| = \int P(x)dx$$

$$\Rightarrow \mu = e^{\int P(x)dx} \quad \text{where } \mu > 0$$

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}Q(x)$$

Integrating this, we obtain the solution to (A)

$$\boxed{e^{\int P(x)\,dx} \cdot y = \int Q(x)\, e^{\int P(x)\,dx}\, dx + C}$$

$\int d\,[\text{dep. var.} * \text{integ. factor}]\,dx$

$\int d\left[y\, e^{\int P(x)\,dx}\right] = \int e^{\int P(x)\,dx}\, Q(x)\,dx$

$$y = \underbrace{e^{-\int P(x)\,dx} \cdot \int Q(x)\, e^{\int P(x)\,dx}\, dx}_{\substack{\text{Particular sol}^{n} \\ \text{does not contain constant}}} + \underbrace{C \cdot e^{-\int P(x)\,dx}}_{\substack{\text{homogeneous sol}^{n} \\ \text{contains constant}}}$$

ex: $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right) y = e^{-2x}$

$P(x) = \dfrac{2x+1}{x}$, $Q(x) = e^{-2x}$

$$\mu = \exp\left[\int P(x)\,dx\right] = \exp\left[\int\left(\frac{2x+1}{x}\right) dx\right] = \exp\left(2x + \ln|x|\right)$$

$$= \exp(2x) \cdot \exp(\ln|x|) = x e^{2x}$$

$$\left(x e^{2x} \frac{dy}{dx} + x e^{2x} \left(\frac{2x+1}{x}\right) y\right) = x \cancel{e^{2x}}\, \cancel{e^{-2x}}$$

$\rightarrow x e^{2x} \dfrac{dy}{dx} + \left[e^{2x} + 2x e^{2x}\right] y$

$e^{2x}(1+2x)\, y$

$$\frac{d}{dx}\left(x e^{2x} y\right) = x \Rightarrow x e^{2x} y = \frac{x^2}{2} + C$$

$$\boxed{y = \frac{1}{2} x e^{-2x} + \frac{C}{x} e^{-2x}}$$

ex: Find the general sol^n of the following DE.

$x y' + 2y = x^3$ $\qquad y(1) = -2$

$y' + 2\dfrac{y}{x} = x^2$ $\qquad (x \neq 0)$

$\dfrac{dy}{dx} + \dfrac{2}{x} y = x^2 \rightarrow P(x) = \dfrac{2}{x}$, $Q(x) = x^2$

$$dy + \left(\frac{2}{x}y - x^2\right)dx = 0 \rightarrow \frac{\partial M}{\partial y} = \frac{2}{x} \neq \frac{\partial N}{\partial x} = 0 \quad \therefore \text{not exact}$$

$$\mu = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Multiply by μ

$$x^2 \frac{dy}{dx} + x^2 \frac{2y}{\cancel{x}} = x^2 x^2$$

$$x^2\,dy + x\,(2y - x^3)\,dx = 0 \rightarrow \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x \quad \text{exact}$$

$$d\,(y x^2) - (x^4)\,dx = 0$$

$$\boxed{y x^2 - \frac{x^5}{5} = c_1}$$

ex: $y^2\,dx + (3xy - 1)\,dy = 0$

Solving for dy/dx, this becomes

$$\frac{dy}{dx} = \frac{y^2}{1-3xy} \rightarrow \text{not Lin. in } y$$

$\downarrow$

Not exact
Not separable
Not homogeneous

interchange the roles of x with y

Then, $\dfrac{dx}{dy} = \dfrac{1-3xy}{y^2}$

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2} \Rightarrow \frac{dx}{dy} + P(y)\,x = Q(y) \rightarrow \text{Lin. in } x$$

$$\mu = e^{\int P(y)\,dy} = e^{\int \frac{3}{y}dy} = e^{3\ln|y|} = e^{\ln|y^3|} = y^3$$

$$y^3 \frac{dx}{dy} + 3y^2 x = y$$

HW ① $\frac{dy}{dx} + 2xy = 4x$ → QUIZ

② $2(y - 4x^2)\,dx + x\,dy = 0$

③ $y \ln y\,dx + (x - \ln y)\,dy = 0$

$$\frac{d}{dy}\left[y^3 x\right] = y$$

$$y^3 x = \frac{y^2}{2} + c \rightarrow x = \frac{1}{2y} + \frac{c}{y^3} \qquad c: \text{const}$$

Bernoulli Equations:

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \quad , n \neq 0 \, , n \neq 1 \, , \text{Dep}: y \, , \text{Ind.}: x$$

$n = 0 \rightarrow 1^{st}$ Ord. L.D.E

$n = 1 \rightarrow 1^{st}$ Ord. seperable

$n \neq 0$ or $n \neq 1 \rightarrow$ Bernoulli's non-Linear DE

→ if the transformation $v = y^{1-n}$ is used the Bernoulli eq$\underline{n}$ becomes a linear equation in v.

· Multiply the Bernoulli eq$\underline{n}$ by y^{-n}

$$y^{-n} \frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

$v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)\, y^{-n} \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1-n} y^{n} \frac{dv}{dx}$$

$$\frac{1}{1-n} \frac{dv}{dx} + P(x)\, v = Q(x)$$

or, equivalently,

$$\frac{dv}{dx} + (1-n)\, P(x)\, v = (1-n)\, Q(x)$$

Let $\quad P_1(x) = (1-n)\, P(x)$

$\qquad\; Q_1(x) = (1-n)\, Q(x)$

then $\frac{dv}{dx} + P_1(x)\, v = Q_1(x)$ which is linear in v

$$\mu = e^{\int P_1(x)\, dx}$$

ex: $(12 e^{2x} y^2 - y)\, dx = dy \quad y(0) = 1$

$$\rightarrow \frac{dy}{dx} = 12 e^{2x} y^2 - y$$

$$\rightarrow \frac{dy}{dx} + y = 12 e^{2x} y^2$$ (Non-Lin. Bernouilli)

$P(x) = 1 \quad , \quad Q(x) = 12 e^{2x}$

Let $y^{-1} = v$

$$-y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-2} \frac{dy}{dx} + y^{-1} = 12 e^{2x}$$

$$y^{-2} \frac{dy}{dx} = -\frac{dv}{dx}$$

Substituting into the org. eqn.

$$-\frac{dv}{dx} + v = 12 e^{2x}$$

$$\frac{dv}{dx} - v = -12 e^{2x} \qquad P(x) = -1 \;,\; Q(x) = -12 . e^{2x}$$

1st Ord. Lin. Dep: v, Ind: x

$$\mu = e^{\int p(x) dx} = e^{\int -dx} = e^{-x}$$

$dv - (v - 12 e^{2x})\, dx = 0 \longrightarrow$ not exact

$e^{-x} dv - v e^{-x} dx = -12 e^{x} dx$ exact

$$\int d(v e^{-x}) = \int -12 e^{x} dx$$

$$v e^{-x} = -12 e^{x} + c_1$$

$$\frac{1}{y} = v = -12 e^{2x} + c_1 e^{x} \longrightarrow \boxed{\frac{1}{y} = -12 e^{2x} + c_1 e^{x}}$$ general Soln

$y(0) = 1 \longrightarrow 1 = -12 \cancel{e^{2.0}}^{1} + c_1 \cancel{e^{0}}^{1} \Rightarrow \underline{c_1 = 13}$

Thus, $\boxed{\frac{1}{y} = -12 e^{2x} + 13 e^{x}}$

HW: ① $\frac{dy}{dx} + 2xy + xy^4 = 0$

② $x\,dy - [y + xy^3(1 + \ln x)]\, dx = 0$

for M1

ex: Solve $\frac{dy}{dx} - 5y = -\frac{5}{2} xy^3$ $\rightarrow$ Bernoulli Eqn $n=3$

$P(x) = -5$, $Q(x) = -\frac{5}{2}x$

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x$$

Let $v = y^{-2}$, Since $dv/dx = -2y^{-3}\frac{dy}{dx}$

$$-\frac{1}{2}\cdot\frac{dv}{dx} - 5v = -\frac{5}{2}x$$

$$\frac{dv}{dx} + 10v = 5x \longrightarrow \text{Linear, } P(x)=10,\ Q(x)=5x$$

$$\mu = e^{\int 10\,dx} = e^{10x}$$

$u = 10x \rightarrow \frac{du}{dx} = 10 \rightarrow dx = \frac{du}{10}$

$\frac{1}{20}\int u\,e^{u}\,du = \frac{1}{20}(u-1)e^{u}$

$\frac{1}{2}\int 10x\,e^{10x}\,dx = \frac{1}{20}(10x-1)e^{10x}$

$$\int d\left[v\cdot e^{10x}\right] = \int 5x\,e^{10x}\,dx$$

$$v\cdot e^{10x} = \frac{1}{20}(10x-1)e^{10x} + c_1$$

$$\downarrow$$

$$\frac{1}{y^2} = \frac{1}{20}(10x-1) + c_1 e^{-10x}$$

$$\boxed{y^{-2} = \frac{x}{2} - \frac{1}{20} + c_1 e^{-10x}}$$

Finding Integrating Factors:

The seperable eqns always possess integrating factors.

Theorem: Consider the DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\mu M\,dx + \mu N\,dy = 0$$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \quad \text{.........} \quad (A)$$

i) if $\mu = \mu(x)$ only

$$\mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{d\mu}{dx}$$

$$\frac{d\mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

If $\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ depends upon x only, then

$\mu(x) = \exp\left\{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx\right\}$ is an integrating factor of the DE.

If $\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$ depends upon y only, then

$\mu(y) = \exp\left\{\int \frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] dy\right\}$ is an integrating factor of the DE.

If $M\,dx + N\,dy = 0$ is neither seperable nor linear, compute

$\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the eqn is exact. If it is not exact, consider

(*) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$, if this is fn of x only then

$\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$ is the integrating factor.

If not, consider

(**) $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$, if this is fn of y only, then

$$\mu(y) = e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy}$$ is the integrating factor

ex: consider the DE

$(2x^2+y)\,dx + (x^2y-x)\,dy = 0$ → not seperable, not linear

$\frac{\partial M}{\partial y} = 1 \not\equiv \frac{\partial N}{\partial x} = 2xy-1$ → not exact

Compute,

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1-(2xy-1)}{x^2y-1} = \frac{-2xy+2}{x^2y-x} = \frac{2(\cancel{1-xy})}{-x(\cancel{1-xy})} = -\frac{2}{x}$$ fn of x only.

$$\mu(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = x^{-2}$$

$$x^{-2}(2x^2+y)\,dx + x^{-2}(x^2y-x)\,dy = 0$$

$$(2+yx^{-2})\,dx + (y-x^{-1})\,dy = 0$$

implicit sol'n

$$2x - \frac{y}{x} + \frac{y^2}{2} = C$$

ex. $y(x+y+1)\,dx + x(x+3y+2)\,dy = 0$

$M_y = x+2y+1 \neq N_x = 2x+3y+2$

$M_y - N_x = x+2y+1-(2x+3y+2) = -(x+y+1)$

$\frac{1}{N}(M_y - N_x) = \frac{-(x+y+1)}{x(x+3y+2)}$ depends on x & y

$\frac{1}{M}(N_x - M_y) = \frac{\cancel{(x+y+1)}}{y\cancel{(x+y+1)}} = \frac{1}{y}$ depends only on y.

$\mu(y) = e^{\int \frac{dy}{y}} = e^{\ln y} = y$

$y^2(x+y+1)\,dx + xy(x+3y+2)\,dy = 0$

$(xy^2+y^3+y^2)\,dx + (x^2y+3y^2x+2xy)\,dy = 0$

$(xy^2\,dx + x^2y\,dy) + (y^3dx + 3y^2x\,dy) + (y^2dx + 2xy\,dy) = 0$

$d\left(\frac{1}{2}x^2y^2 + y^3x + y^2x\right) = d(C)$

$\boxed{\frac{1}{2}x^2y^2 + y^3x + y^2x = C}$

HW: Solve the DE

$$(2x+\tan y)\,dx + (x - x^2\tan y)\,dy = 0$$

QUIZ: $y(x+y)\,dx + (x+2y-1)\,dy = 0$

Special Transformation (Eqns with Linear Coefficients)

$(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2)\,dy = 0$ (D.E) if $c_1 = c_2 = 0$ (HDE)

where a_1, b_1, c_1, a_2, b_2 and c_2 are constants.

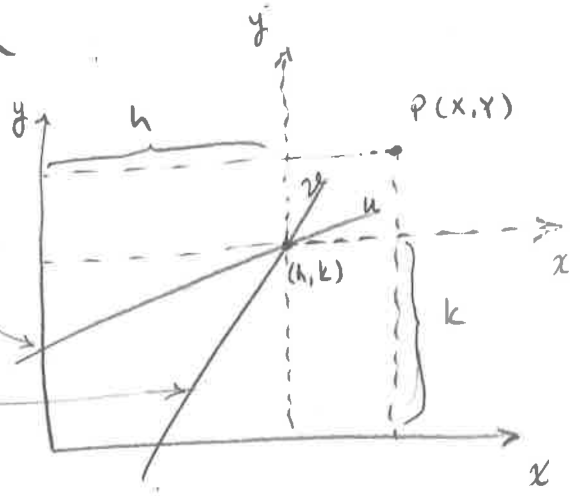
case 1:

If $a_2/a_1 \neq b_2/b_1$ then the transformation

$x = X + h \rightarrow dx = dX$

$y = Y + k \rightarrow dy = dY$

where (h,k) is the soln of system



$$\left.\begin{array}{ll} "L_1" & a_1h+b_1k+c_1=0, \\ "L_2" & a_2h+b_2k+c_2=0, \end{array}\right\}$$ find h & k by setting $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$

reduces the DE to the homogenous eqn $\frac{dx}{dy}=f(\;)=g(y/x)$

$$[a_1(X+h)+b_1(Y+k)+c_1]\,dX+[a_2(X+h)+b_2(Y+h)+c_2]\,dY=0$$

$$[a_1X+b_1Y+\underbrace{a_1h+b_1k+c_1}_{0}]\,dX+[a_2X+b_2Y+\underbrace{a_2h+b_2k+c_2}_{0}]\,dY=0$$

$(a_1X+b_1Y)\,dX+(a_2X+b_2Y)\,dY=0$ in the variables X and Y

Case II :

$$\frac{a_2}{a_1}=\frac{b_2}{b_1}=m$$

L_1 L_2

$$(a_1x+b_1y+c_1)\,dx+(\underset{ma_1}{a_2}x+\underset{mb_1}{b_2}y+c_2)\,dy=0 \qquad (DE)$$

$$(a_1x+b_1y+c_1)\,dx+(m(a_1x+b_1y)+c_2)\,dy=0$$

$z=a_1x+b_1y$ | Transformation

$$dz=a_1dx+b_1\,dy \longrightarrow dy=\frac{dz-a_1dx}{b_1}$$

$$\frac{dz}{dx}=a_1+b_1\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{b_1}\frac{dz}{dx}-\frac{a_1}{b_1}$$

(DE) is reduced to a seperable one in variable x and z

$$(z+c_1)\,dx+(mz+c_2)\left(\frac{dz-a_1dx}{b_1}\right)=0$$

ex: $(x+2y-4)\,dx-(2x+y-5)\,dy=0$

$$\frac{a_2}{a_1}\overset{?}{=}\frac{b_2}{b_1}\Rightarrow\frac{-2}{1}\neq\frac{-1}{2}$$

$$2/\ h+2k-4=0$$
$$2h+k-5=0$$
$$\overline{3k-3=0}\Rightarrow\begin{matrix}k=1\\h=2\end{matrix}\longrightarrow\begin{matrix}x=X+h\to x=X+2\to dx=dX\\y=Y+k\to y=Y+1\to dy=dY\end{matrix}$$

$[(X+\cancel{2})+2(Y+\cancel{1})-\cancel{4}]\,dX-[\cancel{2}(X+\cancel{2})+(Y+\cancel{1})-\cancel{5}]\,dY=0$ (always the constants must cancel eachother)

$(X+2Y)\,dX-(2X+Y)dY=0$ (HDE) in terms of (X,Y)

$$\frac{dX}{dY}=\frac{2X+Y}{X+2Y}=f(X,Y)=\frac{(2\frac{X}{Y}+1)}{(\frac{X}{Y}+2)}=f\left(\frac{X}{Y}\right)\Rightarrow \text{Transf}:\ \frac{X}{Y}=z$$

$$dX=dz\,Y+z\,dY\to\frac{dX}{dY}=Y\frac{dz}{dY}+z$$

$$z+Y\frac{dz}{dY}=\frac{2z+1}{z+2}\Rightarrow Y\frac{dz}{dY}=\frac{2z+1}{z+2}-z$$

$$Y\frac{dz}{dY}=\frac{\cancel{2z}+1-z^2-\cancel{2z}}{z+2}=\frac{1-z^2}{z+2}$$ (seperation of variable meth.)

$$\frac{z+2}{1-z^2}\,dz=\frac{dY}{Y}$$

$$-\int\frac{z+2}{z^2-1}\,dz=\int\frac{dY}{Y}=\ln Y+\ln C_1$$

$$\frac{z+2}{z^2-1}=\frac{z+2}{(z-1)(z+1)}=\frac{a}{z-1}+\frac{b}{z+1}\to a=\frac{3}{2},\ b=-1/2$$

Then,

$$-\frac{3}{2}\int\frac{1}{z-1}\,dz+\frac{1}{2}\int\frac{1}{z+1}\,dz=\ln(YC_1)$$

$$-\frac{3}{2}\ln|z-1|+\frac{1}{2}\ln|z+1| = \ln(yC_1)$$

$$\cancel{\ln}\left([z-1]^{-3/2}\cdot[z+1]^{1/2}\right) = \cancel{\ln}(yC_1)$$

$$(z-1)^{-3/2}\cdot(z+1)^{1/2} = y\cdot C_1$$

$$(z-1)^{-3}\cdot(z+1) = y^2C_1^2 = y^2C_3$$

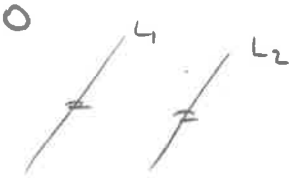
$$\boxed{\frac{z+1}{(z-1)^3} = y^2c^3.}$$ convert to x, y plane

$$\frac{y^2\cdot y\left(\frac{x}{y}+1\right)}{y^3\left(\frac{x}{y}-1\right)^3} = y^2C_3 \rightarrow \frac{\cancel{y^2}(x+y)}{(x-y)^3} = \cancel{y^2}C_3$$

$$\frac{x-2+y-1}{(x-2-y+1)^3} = C_3$$

$$\Rightarrow \boxed{(x+y-3) = C_3(x-y-1)^3}$$

<u>ex</u>: $(x+2y+3)\,dx+(2x+4y-1)\,dy = 0$



$a_1=1,\ a_2=2$
$b_1=2,\ b_2=4$

$$\frac{a_1}{a_2} = \frac{1}{2} = \frac{b_1}{b_2}$$

$$z = x+2y \rightarrow dz = dx+2dy \rightarrow dy = \frac{dz-dx}{2}$$

$$(z+3)\,dx+(2z-1)\left(\frac{dz-dx}{2}\right)=0$$

$$2(z+3)\,dx+(2z-1)\,dz-(2z-1)\,dx=0$$

$$7\,dx+(2z-1)\,dz=0$$

$$7x+z^2-z=C \rightarrow z=x+2y$$

$$7x+(x+2y)^2-(x+2y)=C$$

$$\boxed{x^2+4xy+4y^2+6x-2y=C}$$

HW

ex: Solve $(-3x+y+6)\,dx + (x+y+2)\,dy = 0$ (NEEGLE & SAFF, P 75)

$$\frac{a_2}{a_1} = \frac{1}{-3} \neq \frac{b_2}{b_1} = \frac{1}{1} \rightarrow \text{case I} \qquad x = X + h, \quad y = Y + k$$

$$\left.\begin{array}{l} -3h + k + 6 = 0 \\ h + k + 2 = 0 \end{array}\right\} \Rightarrow h = 1,\ k = -3 \rightarrow \begin{array}{l} x = X + 1 \rightarrow dx = dX \\ y = Y - 3 \rightarrow dy = dY \end{array}$$

$(-3X+Y)\,dX + (X+Y)\,dY = 0$ or

$$\frac{dY}{dX} = \frac{3-(Y/X)}{1+(Y/X)} \longrightarrow \text{homogenous} \rightarrow z = \frac{Y}{X} \rightarrow \frac{dY}{dX} = z + X\frac{dz}{dX}$$

$$z + X\frac{dz}{dX} = \frac{3-z}{1+z} \rightarrow \int \frac{z+1}{z^2+2z-3} = \int -\frac{1}{X}\,dX = -\ln(Xc)$$

$$\frac{z+1}{(z+3)(z-1)} = \frac{a}{(z+3)} + \frac{b}{(z-1)} \rightarrow a = b = \frac{1}{2}$$

$$\frac{1}{2}\left(\int \frac{dz}{z+3} + \int \frac{dz}{z-1}\right) = \ln(Xc_1)^{-1}$$

$$\frac{1}{2}\left(\ln|z+3| + \ln|z-1|\right) = \ln(Xc_1)^{-1}$$

$$\cancel{\ln}\left[(z+3)(z-1)\right]^{1/2} = \cancel{\ln}(Xc_1)^{-1}$$

$$(z+3)(z-1) = (Xc_1)^{-2}$$

$$X^2\left(\frac{Y}{X}+3\right)\left(\frac{Y}{X}-1\right) = (\cancel{X}c_1)^{-2} \quad \cancel{X^2}$$

$(Y+3X)(Y-X) = c_2$ convert to x,y plane

$$((y+3)+3(x-1))(y+3-x+1) = c_2$$

$$\boxed{(y+3x)(y-x+4) = c}$$

Reduction of Order

Some 2^{nd} ord eqn.'s can be reduced to 1^{st} ord. eqn.'s
The following are three particular types of such 2^{nd} ord. eqn.'s.

Type 1: 2^{nd} ord. eqn.'s with the dependent variable missing.

ex: $y''+y'=x$

Let $w=y'$
$w'=y''$

Substituting into the DE, it becomes

$w'+w=x$ (dep. var. y is missing)

$\Downarrow$

LDE

$u=x \quad dv=e^x dx$
$du=dx \quad v=e^x$

$$\int xe^x dx = xe^x-\int e^x dx$$
$$= xe^x - e^x + C_1$$

$$\mu(x)=e^{\int dx}=e^x$$

$$e^x \frac{dw}{dx}+e^x w = xe^x$$

$$\cancel{\int d}(e^x w) = \int x e^x dx$$

$$e^x w = xe^x - e^x + C_1$$

$$w = x-1+C_1e^{-x}$$

Replace w by y'

$$y'=x-1+C_1e^{-x}$$

$$\boxed{y=\frac{x^2}{2}-x+C_1e^{x}+C_2}$$

ex: $xy''-2y'=10x^4$

Let $w=y'$
$w'=y''$

$$xw'-2w=10x^4$$
$$w'-\frac{2}{x}w=10x^3 \rightarrow \text{LDE}$$

$$\mu(x)=e^{-\int\frac{2}{x}dx}=x^{-2}$$

$$\int d(x^{-2}w)=\int 10x\,dx$$

$$w=5x^4+Cx^2$$

$$y'=5x^4+Cx^2$$

$$\boxed{y=x^5+C_1x^3+C_2}$$

where $C_1=\frac{C}{3}$

ex: $y'' + (y')^2 = 0 \quad y(0) = y'(0) = 1$ Let $w = y'$, $w' = y''$

$w' + w^2 = 0 \rightarrow$ sep.

$$\int -\frac{dw}{w^2} = \int dx$$

$$\frac{1}{w} = x + C_1 \Rightarrow w = \frac{1}{x+C_1}$$

$$y = \int \frac{1}{x+C_1} dx = \ln|x+C_1| + C_2$$

Applying I.C. $y(0) = 1 = \ln C_1 + C_2$

$y'(0) = 1 = \frac{1}{C_1} \Rightarrow \boxed{\begin{array}{c} C_1 = 1 \\ C_2 = 1 \end{array}}$

Type 2 = 2^{nd} ord non-linear eqn's with the independent variable missing.

ex: $y^2 y'' - (y')^3 = 0$ the ind. var. x, does not explicitly appear in the eqn.

Let $w = y'$, $w' = y''$ again evaluate w'

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx} = \frac{dw}{dy} \cdot w$$

$$y^2 w \frac{dw}{dy} - w^3 = 0$$

$$\int \frac{dw}{w^2} = \int \frac{dy}{y^2} \rightarrow \text{sep.}$$

$$\cancel{-}\frac{1}{w} = \cancel{-}\frac{1}{y} \cancel{+} C_1$$

$$\frac{dx}{dy} = (y^{-1} + C_1) \rightarrow \text{sep.}$$

$$(y^{-1} + C_1)\, dy = dx$$

$$\ln y + C_1 y = x + C_2$$

Type 3: 2^{nd} ord homogeneous LDE where one (non zero) sol'n is known.

ex: $x^2y'' - xy' + y = 0$. it is known that $y = x$ satisfies the eqn.

Let $y = y_1 \cdot v$ where $y_1 = x$ & $v = v(x)$

$= x \cdot v$

$y' = x v' + v$

$y'' = x v'' + 2v'$

Substituting into the eqn

$$x^2(xv'' + \cancel{2v'}) - x(xv' \cancel{+v}) + \cancel{xv} = 0$$

$$x^3 v'' + x^2 v' = 0$$

$$v'' + \frac{1}{x} v' = 0 \qquad \text{Let } w = v', \; w' = v''$$

$$w' + \frac{1}{x} w = 0 \rightarrow \text{LDE} \qquad \mu(x) = e^{\int \frac{dx}{x}} = x$$

$$\int d(xw) = \int 0 \, dx$$

$$xw = c_1$$

$$w = c_1 x^{-1}$$

$$v' = c_1 x^{-1} \Rightarrow v = \ln x$$

$$y_2 = y_1 \cdot v = \boxed{x \ln x}$$

$$y = y_1 + y_2 = \boxed{c_1 x + c_2 \ln x}$$

<u>ex</u>: $xy'' - (x+1)y' + y = 0$. $y_1 = e^{x}$ is a known sol'n of the DE.

<u>ex</u>: $x^2y'' - 3xy' + 4y = 0$, $x > 0$ and $y_1 = x^2$ is a known sol'n of the DE.