Ex Find the critical points of $f(x) = \frac{x}{x^2+1}$ and identify the interval on which f is increasing and on which f is decreasing

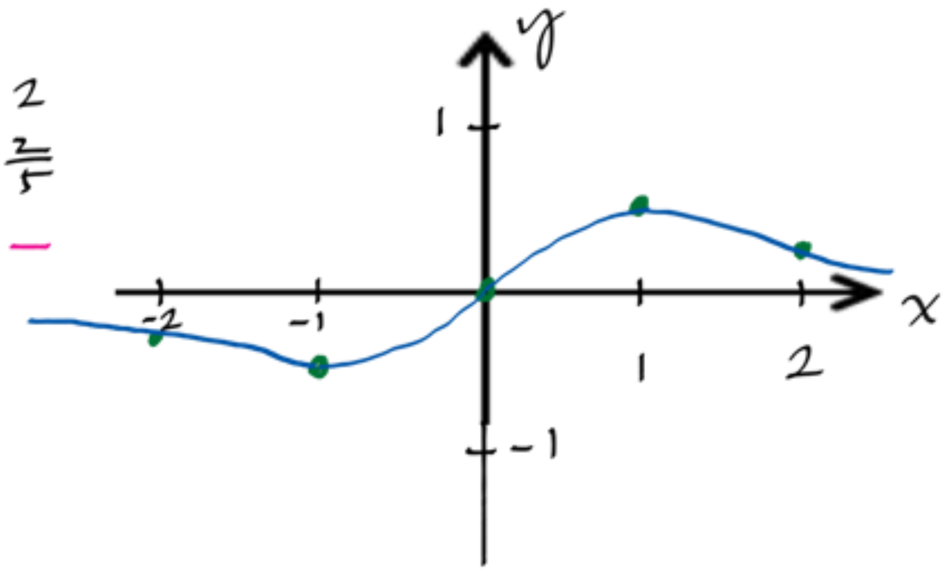
$$f(x) = \frac{x}{x^2+1} \qquad \text{odd fn.}$$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1/x}{1+1/x^2} = 0 , \quad y=0 \text{ is horizontal asymptote}$$

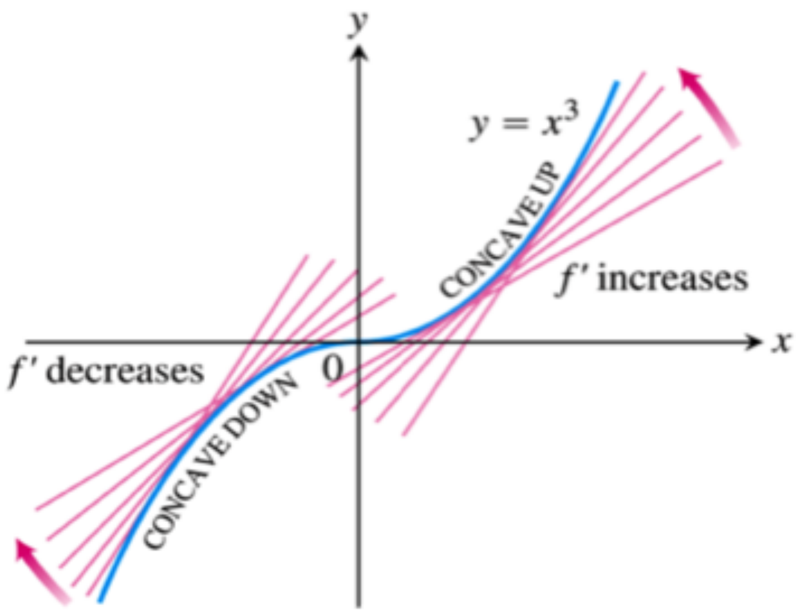
$$f(x) = x(x^2+1)^{-1} \Rightarrow f'(x) = (x^2+1)^{-1} - x(x^2+1)^{-2} \cdot 2x = \frac{1-x^2}{(x^2+1)^2}$$

$$f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1 , \text{ critical points}$$

	local min		local max		
x	-2	-1	0	1	2
f	$-\frac{2}{5}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{5}$
f'	$-$	0	$+$	0	$-$



Concavity and Curve Sketching



f is concave up on I if f' is increasing
concave down f' is decreasing

$\Rightarrow$ concave UP if $f'' > 0$
concave DOWN if $f'' < 0$

A point where the concavity changes is a POINT OF INFLECTION. At a point of inflection, either $f''(x_0)=0$ or $f''(x_0)$ does not exist.

In above graph, $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, $f''(x) = 6x \Rightarrow x=0$, inf. pt

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up. +
2. If $f'' < 0$ on I, the graph of f over I is concave down. −

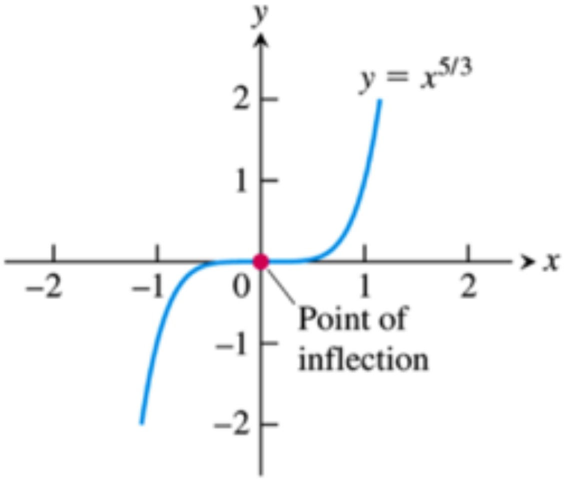


FIGURE The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$

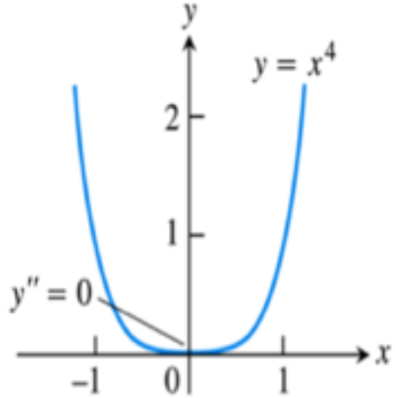


FIGURE The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there

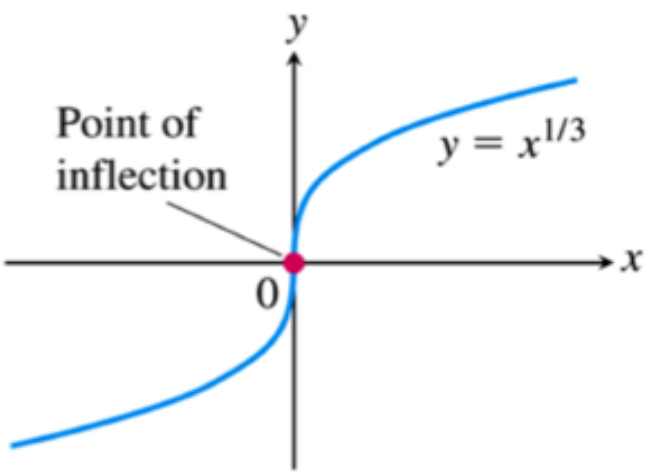


FIGURE A point of inflection where y' and y'' fail to exist

$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Ex $y = \frac{1}{4}x^4 - \frac{3}{2}x^2$ even fn. $\Rightarrow y = 0 = x^2(x^2/4 - 3/2)$

$\Rightarrow x = 0, \pm\sqrt{6}$

$y' = x^3 - 3x = x(x^2-3) \Rightarrow x = 0, \pm\sqrt{3}$, critical pts.

$y'' = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$

$x = -1, +1$, pts of inflection

x	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
y	-2	$-9/4$	$-5/4$	0	$-5/4$	$-9/4$	-2
y'	$-$	0	$+$	0	$-$	0	$+$
y''	$+$	$+$	0	$-$	0	$+$	$+$