CH 12 : Iterative Methods

For solving simultaneous (linear and nonlinear) eq^{n}s.

Linear Systems: Gauss Siedel

$$Ax = b$$

Let the system be 3x3

(If diagonal elements are non-zero)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

First equation ⟹ solve for x_1 :
$$x_1^j = \frac{b_1 - a_{12}x_2^{j-1} - a_{13}x_3^{j-1}}{a_{11}} \quad (1)$$

Second equation ⟹ solve for x_2:
$$x_2^j = \frac{b_2 - a_{21}x_1^j - a_{23}x_3^{j-1}}{a_{22}} \quad (2)$$

Third equation ⟹ solve for x_3:
$$x_3^j = \frac{b_3 - a_{31}x_1^j - a_{32}x_2^j}{a_{33}} \quad (3)$$

$j, j-1$: present and past iterations

Initial guess: assume they're all zero.

visit equations (1) – (2) – (3)

Convergence criterion:
$$\varepsilon_{a,i} = \left| \frac{x_i^j - x_i^{j-1}}{x_i^j} \right| \times 100\% \leq \varepsilon_s$$

Ex Use Gauss-Siedel method to solve:

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \end{cases}$$

(Note: sol^n:
$x_1 = 3$
$x_2 = -2.5$
$x_3 = 7$)

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \quad (1)$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \quad (2)$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \quad (3)$$

Start with $x_2 = 0,\ x_3 = 0 \Rightarrow$

$$\left.\begin{aligned} x_1 &= 2.61 \quad \text{from } (1) \\ x_2 &= -2.79 \quad \text{from } (2) \\ x_3 &= 7.005 \quad \text{from } (3) \end{aligned}\right\} \text{iteration } 1$$

Second iteration:

$$x_1 = 2.99$$
$$x_2 = -2.49$$
$$x_3 = 7.00$$

$$\varepsilon_{a,1} = \left| \frac{2.99 - 2.61}{2.99} \right| \times 100\,\% = 12.5\% \quad (\text{for } x_1)$$

$$\varepsilon_{a,2} = 11.8\% \quad (\text{for } x_2) \qquad \varepsilon_{a,3} = 0.076\,\% \quad (\text{for } x_3)$$

Jacobi iterations:

→ Compute new x's on the basis of old x's.

→ New values of x's are not immediately used.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$$x_1^j = \frac{b_1 - a_{12}x_2^{j-1} - a_{13}x_3^{j-1}}{a_{11}} \quad (1)$$

$$x_2^j = \frac{b_2 - a_{21}x_1^{j-1} - a_{23}x_3^{j-1}}{a_{22}} \quad (2)$$

$$x_3^j = \frac{b_3 - a_{31}x_1^{j-1} - a_{32}x_2^{j-1}}{a_{33}} \quad (3)$$

Convergence of Gauss-seidel

If the following condition holds, Gauss-Seidel will convergence:

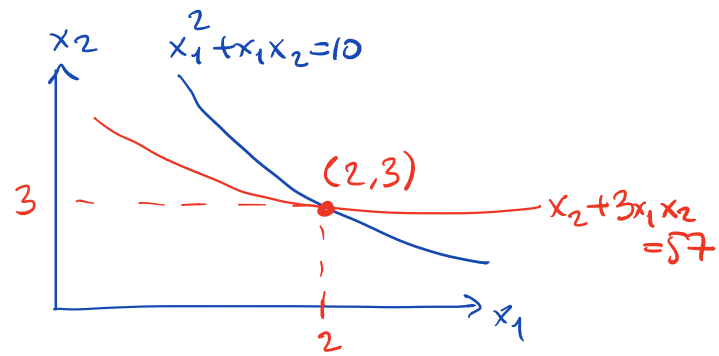
$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \longrightarrow \text{diagonally dominant system}$$

→ Sufficient but not necessary condition.

→ may sometimes converge even if cond^n is not met.

Nonlinear Systems

$$x_1^2 + x_1 x_2 = 10$$
$$x_2 + 3x_1 x_2 = 57$$



Similar to the roots for single nonlinear eq^{n}s:

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots x_n) &= 0 \\ f_2(x_1, \ldots x_n) &= 0 \\ &\vdots \\ f_n(x_1, \ldots x_n) &= 0 \end{aligned}\right\}$$ solution : x's that make the eq^n zero.

Successive Substitution ($\sim$ fixed-pt iteration)

→ Each of the nonlinear eq^{n}s can be solved for one of the unknowns.

→ Implement iteratively ($\sim$ Gauss-siedel iteration)

Ex

$$\left.\begin{aligned} x_1^2 + x_1 x_2 &= 10 \quad (1) \\ x_2 + 3x_1 x_2^2 &= 57 \quad (2) \end{aligned}\right\}$$ sol^n : $x_1 = 2$ and $x_2 = 3$

initial guesses $x_1 = 1.5$, $x_2 = 3.5$

Solve (1) for x_1 ⇒ $x_1 = \dfrac{10 - x_1^2}{x_2}$ (3)

Solve (2) for x_2 ⇒ $x_2 = 57 - 3x_1 x_2^2$ (4)

iteration 1 $\begin{cases} \text{from (3)} \Rightarrow x_1 = \dfrac{10 - (1.5)^2}{3.5} = 2.21 \\ \text{from (4)} \Rightarrow x_2 = 57 - 3(2.21)(3.5)^2 = -24.37 \end{cases}$

(seems to be diverging)

converging)

iteration 2 $\begin{cases} x_1 = -0.209 \\ x_2 = 429.70 \end{cases}$ (diverging!!)

Sol^n: Repeat with original eq^n's setup in a different way:

$$\begin{cases} x_1 = \sqrt{10 - x_1 x_2} \\ x_2 = \sqrt{\dfrac{57 - x_2}{3x_1}} \end{cases}$$

iteration 1 : $\begin{cases} x_1 = 2.17 \\ x_2 = 2.86 \end{cases}$

iteration 2 : $\underbrace{\begin{cases} x_1 = 1.94 \\ x_2 = 3.04 \end{cases}}_{\text{converging}}$

Newton-Raphson

Taylor expansion:

single equation form

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) \quad , x_i = \text{initial expansion}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \Rightarrow \text{for zero of a single eq}^n.$$

$$\left.\begin{matrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{matrix}\right\} \text{system of nonlinear eq}^n.$$

$f_1(x_{1,i}, x_{2,i})$

$$\begin{cases} f_{1,i+1} = f_{1,i} + (x_{1,i+1} - x_{1,i}) \dfrac{\partial f_{1,i}}{\partial x_1} + (x_{2,i+1} - x_{2,i}) \dfrac{\partial f_{1,i}}{\partial x_2} \\ f_{2,i+1} = f_{2,i} + (x_{1,i+1} - x_{1,i}) \dfrac{\partial f_{2,i}}{\partial x_1} + (\quad " \quad) \dfrac{\partial f_{2,i}}{\partial x_2} \end{cases}$$

arrange

update eq^{n}s.

$$\begin{cases} x_{1,i+1} = x_{1,i} + \ldots \\ x_{2,i+1} = x_{2,i} + \ldots \end{cases}$$