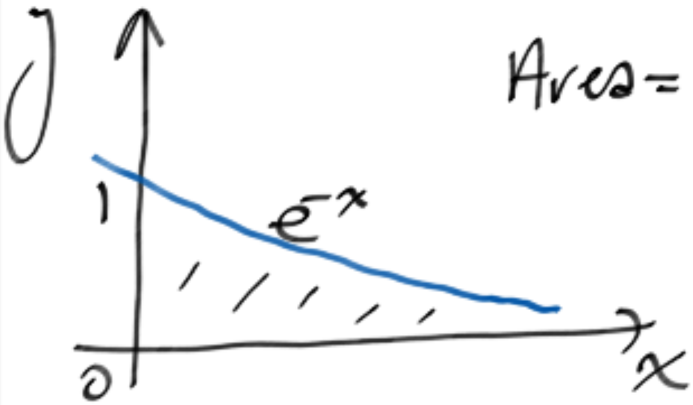
Improper Integrals

Ex Find the area between $y = e^{-x}$ and the x-axis, for $x \geq 0$.



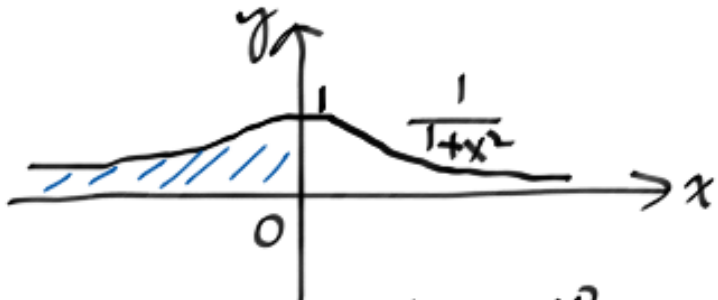
$$\text{Area} = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$u = -x, \; -du = dx$

$$= \lim_{b\to\infty} \int -e^{u}\,du = \lim_{b\to\infty} (-e^{-x}) \Big|_0^b$$

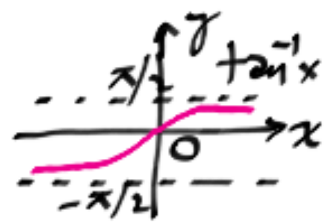
$$= \lim_{b\to\infty} [\underbrace{-\cancel{e^{-b}}}_{\to 0} + \underbrace{e^{0}}_{1}] = \underline{1}$$

Ex $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

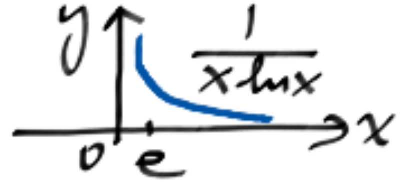


$$= \lim_{b\to-\infty} \int_b^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty} \text{Arctan}\, x \Big|_b^0$$

$$= \lim_{b\to-\infty} (\underbrace{\cancel{\text{Arctan}\, 0}}_{0} - \underbrace{\text{Arctan}\, b}_{-\pi/2}) = \pi/2$$



Ex $\int_e^\infty \frac{dx}{x\ln x}$

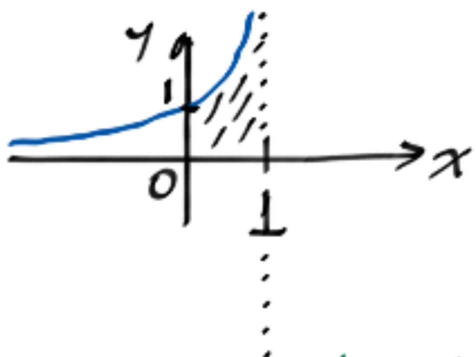


$= \lim_{b\to\infty}\int_e^b \frac{dx}{x\ln x}$ $u=\ln x,\ du=\frac{dx}{x}$

$= \lim_{b\to\infty}\int_e^b \frac{du}{u} = \lim_{b\to\infty} \ln(\ln x)\Big|_e^b = \lim_{b\to\infty}\left[\ln(\ln b) - \underbrace{\ln(\underbrace{\ln e}_{1})}_{0}\right]$

$= \infty$

Ex $\int_0^1 \frac{dx}{\sqrt{1-x}}$

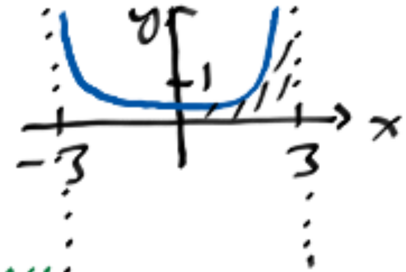


$x=1$ is the vertical asmp.

$= \lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x}}$ $u=1-x,\ -du=dx$

$= \lim_{b\to 1^-}\int -u^{-1/2}\,du = \lim_{b\to 1^-}\left(-2\sqrt{1-x}\right)\Big|_0^b = \lim_{b\to 1^-}\left(-2\underbrace{\sqrt{1-\underbrace{b}_{1}}}_{0}+2\right) = 2$

Ex $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

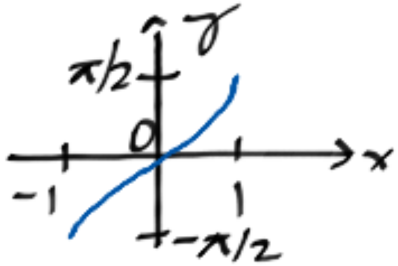


$$\lim_{b\to 3^-}\int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to 3^-} \frac{1}{3}\int \frac{dx}{\sqrt{1-(x/3)^2}}$$

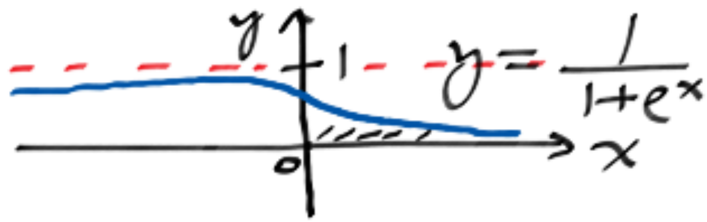
$u = x/3$
$3du = dx$

$$= \lim_{b\to 3^-} \int \frac{du}{\sqrt{1-u^2}} = \lim_{b\to 3^-} \operatorname{ArcSin}\left(\frac{x}{3}\right)\Big|_0^b$$

$$= \pi/2$$



Ex $\displaystyle\int_0^\infty \frac{dx}{1+e^x}$



$$= \lim_{b\to\infty}\int_0^b \frac{dx}{1+e^x}$$

$u = e^x$, $du = e^x dx$

$$= \lim_{b\to\infty}\int_1^b \frac{du}{u(1+u)} = \lim_{b\to\infty}\left(\ln e^x - \ln(e^x+1)\right)\Big|_0^b = 0 - \ln\frac{1}{2} = \ln 2$$

If there is a vertical asymptote at $x=c$, $a<c<b$, then

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx}_{\text{two improper integrals}}$$

Ex $\displaystyle\int_{-1}^{0} \frac{dx}{x+1} = \lim_{b\to -1^+} \ln(x+1)\Big|_0^b$

$$= \ln 1 - \lim_{b\to -1^+} \ln(b+1)$$

$$= -\infty$$