The definite integral and the fundamental theorem of calculus

$f(x) \geq 0$, continuous on $[a,b]$.

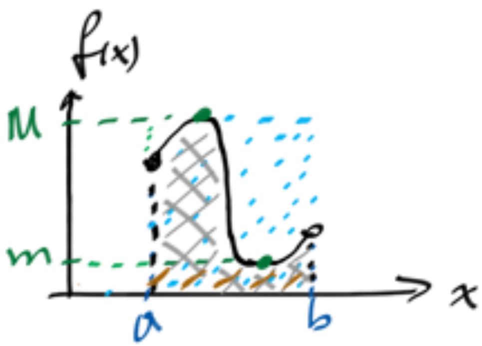
$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \int_a^b f(x)\,dx$$

upper limit

variable of integration

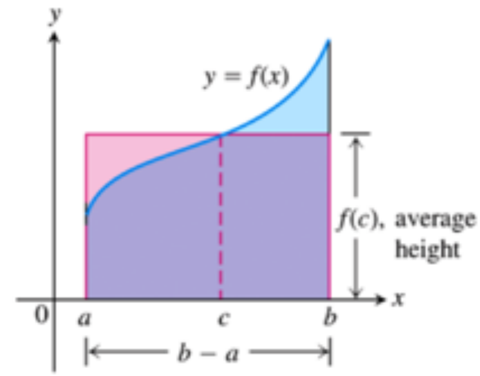
integrand

lower limit



Let $M = \max f$ on $[a,b]$
$m = \min f$ on $[a,b]$

$$m(b-a) \leq A = \int_a^b f(x)\,dx \leq M(b-a)$$



The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(c)$$

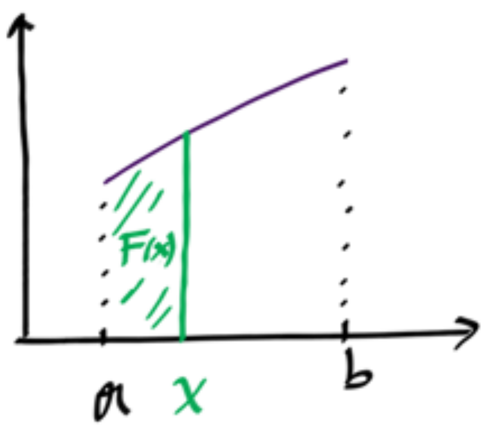
located between M and m

Def. The average (mean) value of f on $[a,b]$ is

$$\operatorname{avg} f = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\operatorname{avg} f$ is the height of a rectangle with base $[a,b]$ whose area equals A.

The Fundamental Theorem of Calculus



$$A = \int_a^b f(x)\,dx$$

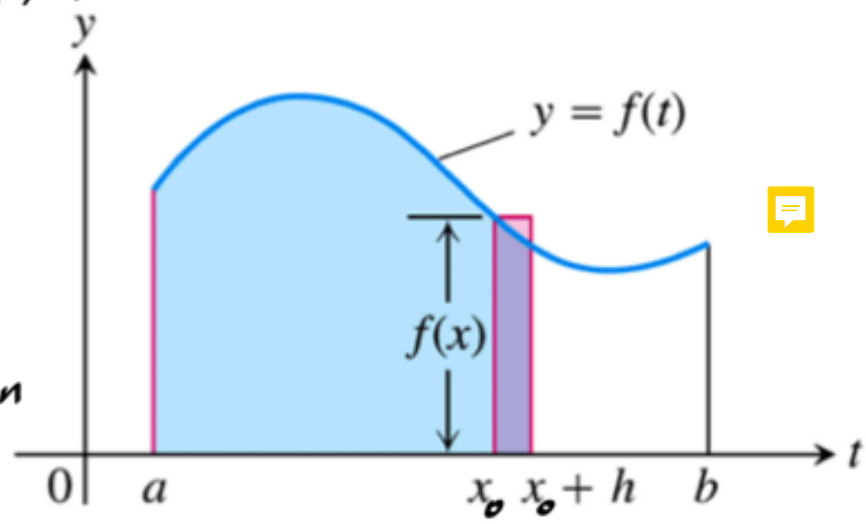
$$x \in (a,b)$$

$F(x) \equiv$ The area under f from a to x

$$F(x) = \int_a^x f(x)\,dx$$

Let $x_0 \in (a,b)$. Compute $F'(x_0)$:

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h}$$



$F(x_0)$ is the area of the blue region
$F(x_0+h)$ is the area of the blue region + the area of the little rectangle of $f(x) \cdot h$.

$F(x_0)$ is the area to the left of x. Also, $F(x_0 + h)$ is the area to the left of $x_0 + h$. The difference quotient $[F(x_0 + h) - F(x_0)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

Suppose $h > 0$. Then

$$mh \leq F(x_0+h) - F(x_0) \leq Mh$$

$$\frac{m\cancel{h}}{\cancel{h}} \leq \underbrace{\frac{F(x_0+h) - F(x_0)}{h}}_{F'(x_0)} \leq \frac{M\cancel{h}}{\cancel{h}}$$

$M = \max f$ on $[a,b]$
$m = \min f$ on $[a,b]$

$\lim_{h \to 0}$ By the Sandwich Theorem

$$F'(x_0) = f(x_0) = M = m \Rightarrow F' = f. \therefore F \text{ is an antiderivative of } f.$$

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Fundamental Theorem of Calculus

$$F(a) = 0\,, \quad F(b) = A$$

Let G be an antiderivative of f. Then

$$(G-F)' = G' - F' = f - f = 0 \Rightarrow G - F = \underset{\text{constant}}{C}$$

$$G(a) = \underbrace{F(a)}_{0} + C$$

$$F(x) = G(x) - G(a)$$

$$A = F(b) = G(b) - G(a)$$

To find the area, take any antiderivative of f, say G, and

$$A = \int_a^b f(t)\,dt = G(b) - G(a) = G(x)\Big|_a^b$$

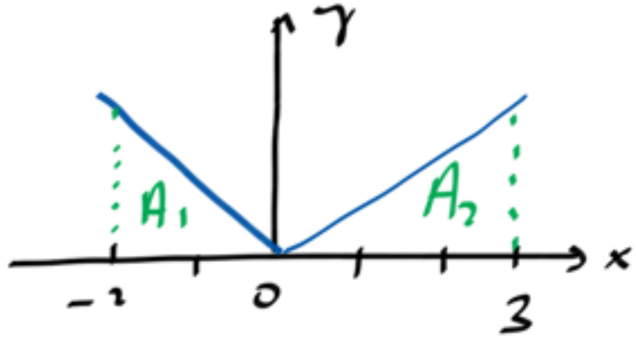
Ex Find the area under $y=x^3$ on $[0,2]$

$$A=\int_a^b f(x)\,dx$$

$$A=\int_0^2 x^3\,dx = \frac{x^{3+1}}{3+1}\Big|_0^2 = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(2)^4 - \underbrace{\frac{1}{4}(0)^4}_{0} = 4$$

$$\underbrace{F(b)-F(a)}_{\text{net change in } F \text{ from } a \text{ to } b} = \underbrace{\int_a^b F'(x)\,dx}_{\text{the integral of rate of change of } F}$$

Ex Same question, for $y=|x|$, $[-2,3]$



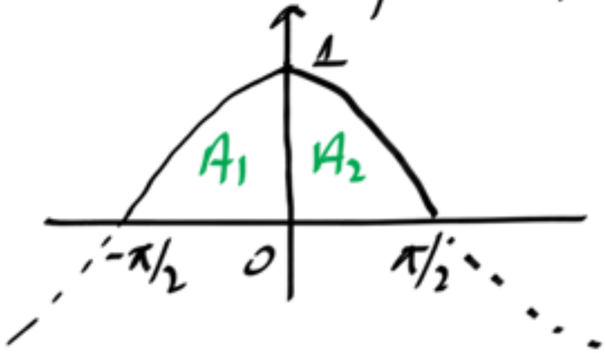
$$A = A_1 + A_2$$

$$= \int_{-2}^{0}(-x)\,dx + \int_0^3 x\,dx = -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^3$$

watch out sign

$$= 2 + 9/2 = 13/2$$

Ex Same question, for $y = \cos x$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$$A = A_1 + A_2 = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x \Big|_{-\pi/2}^{\pi/2} = 1+1 = 2$$

Alternatively, by symmetry,

$$A = 2\int_0^{\pi/2} \cos x \, dx = 2 \sin x \Big|_0^{\pi/2} = 2.1 - 0 = 2 \,/\!/$$