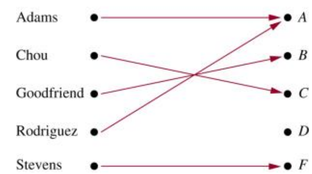
CSE2023 Discrete Computational Structures

Lecture 10

2.3 Functions

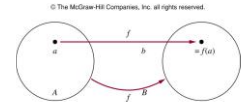
- Assign each element of a set to a particular element of a second set



Function

- A **function** f from A to B, f:A→B, is an assignment of **exactly one** element of B to each element of A
- f(a)=b if **b** is the unique element of B assigned by the function f to the element **a**
- Sometimes also called **mapping** or **transformation**

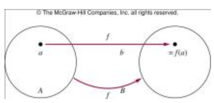
Function and relation



- f:A→B can be defined in terms of a relation from A to B
- Recall a **relation** from A to B is just a subset of A x B
- A relation from A to B that contains one, and only one, ordered pair (a,b) for every element a ∈ A, defines a function f from A to B
- f(a)=b where (a,b) is the <u>unique</u> <u>ordered pair</u> in the relation

Domain and range

- If f is a function from A to B
  – A is the **domain** of f
  – B is the **codomain** of f
  – f(a)=b, b is the **image** of a and a is **preimage** of b
  – **Range** of f: set of all images of element of A
  – f **maps** A to B



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Function

- Specify a function by
  – Domain
  – Codomain
  – Mapping of elements
- Two functions are equal if they have
  – Same domain, codomain, mapping of elements

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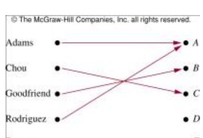
Example



- G: function that assigns a grade to a student, e.g., G(Adams)=A
- Domain of G: {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- Codomain of G: {A, B, C, D, F}
- Range of G is: {A, B, C, F}

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Example

- Let R be the relation consisting of (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24) and (Felicia, 22)
- f: f(Abdul)=22, f(Brenda)=24, f(Carla)=21, f(Desire)=22, f(Eddie)=24, and f(Felicia)=22
- Domain: {Abdul, Brenda, Carla, Desire, Eddie, Felicia}
- Codomain: set of positive integers
- Range: {21, 22, 24}

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Example

- f: assigns the last two bits of a bit string of length 2 or greater to that string, e.g., f(11010)=10
- Domain: all bit strings of length 2 or greater
- Codomain: {00, 01, 10, 11}
- Range: {00, 01, 10, 11}

Example

- f: Z → Z, assigns the square of an integer to its integer, $f(x)=x^2$
- Domain: the set of all integers
- Codomain: set of all integers
- Range: all integers that are perfect squares, i.e., {0, 1, 4, 9, …}

Example

- In programming languages
  - int floor(float x){…}
    - Domain: the set of real numbers
    - Codomain: the set of integers

Functions

- Two real-valued functions with the same domain can be added and multiplied
- Let f_1 and f_2 be functions from A to **R**, then f_1+f_2, and f_1f_2 are also functions from A to **R** defined by
  - $(f_1+f_2)(x)= f_1(x)+f_2(x)$
  - $(f_1f_2)(x)= f_1(x) f_2(x)$
- Note that the functions f_1+f_2 and f_1f_2 at x are defined in terms f_1 and f_2 at x

Example

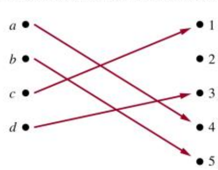
- $f_1(x) = x^2$ and $f_2(x) = x - x^2$
  - $(f_1+f_2)(x)=f_1(x)+f_2(x)=x^2+x-x^2=x$
  - $(f_1f_2)(x)=f_1(x)f_2(x)=x^2(x-x^2)=x^3-x^4$

One-to-one function

- A function f is said to be **one-to-one** or **injective**, <u>if and only if</u> f(a)=f(b) implies a=b for all a and b in the domain of f
$$\forall a \forall b (f(a) = f(b) \to a = b)$$
- A function f is one-to-one <u>if and only if</u> f(a)≠f(b) whenever a≠b $\forall a \forall b (a \neq b \to f(a) \neq f(b))$
- Using contrapositive of the implication in the definition (p→q ≡ q whenever p)
- Every element of B is the image of a unique element of A

Example

- f maps {a,b,c,d} to {1,2,3,4,5} with f(a)=4, f(b)=5, f(c)=1, f(d)=3
- Is f an one-to-one function?



Example

- Let $f(x)=x^2$, from the set of integers to the set of integers. Is it one-to-one?
- f(1)=1, f(-1)=1, f(1)=f(-1) but 1≠-1
- However, $f(x)=x^2$ is one-to-one for Z^+
- Determine f(x)=x+1 from real numbers to itself is one-to-one or not
- It is one-to-one. To show this, note that x+1 ≠ y+1 when x≠y

Increasing/decreasing functions

- Increasing (decreasing): if f(x)≤f(y) (f(x)≥f(y)), whenever x<y and x, y are in the domain of f

$$\forall x \forall y (x < y \rightarrow f(x) \le f(y))$$

- Strictly increasing (decreasing): if f(x)<f(y) (f(x) > f(y)) whenever x<y, and x, y are in the domain of f
- A function that is either strictly increasing or decreasing must be one-to-one

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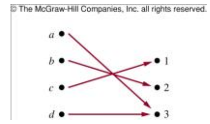
Onto functions

- **Onto**: A function from A to B is onto or **surjective**, if and only if for every element b ∈ B there is an element a ∈ A with f(a)=b

$\forall y \exists x (f(x) = y)$, where x is in the domain and y is the codomain

- Every element of B is the image of some element in A



f maps from {a, b, c, d} to {1, 2, 3}, is f onto?

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One-to-one correspondence

- The function f is a **one-and-one correspondence**, or **bijective**, if it is both one-to-one and onto
- Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f(a)=4, f(b)=2, f(c)=1, and f(d)=3, is f bijective?
  - It is one-to-one as no two values in the domain are assigned the same function value
  - It is onto as all four elements of the codomain are images of elements in the domain

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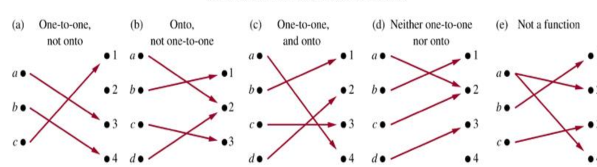
Example



- **Identity function**: $\iota_A : A \rightarrow A, \iota_A(x) = x, \forall x \in A$
  - It is one-to-one and onto

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