10.7 Power Series

DEFINITIONS A **power series about $x = 0$** is a series of the form

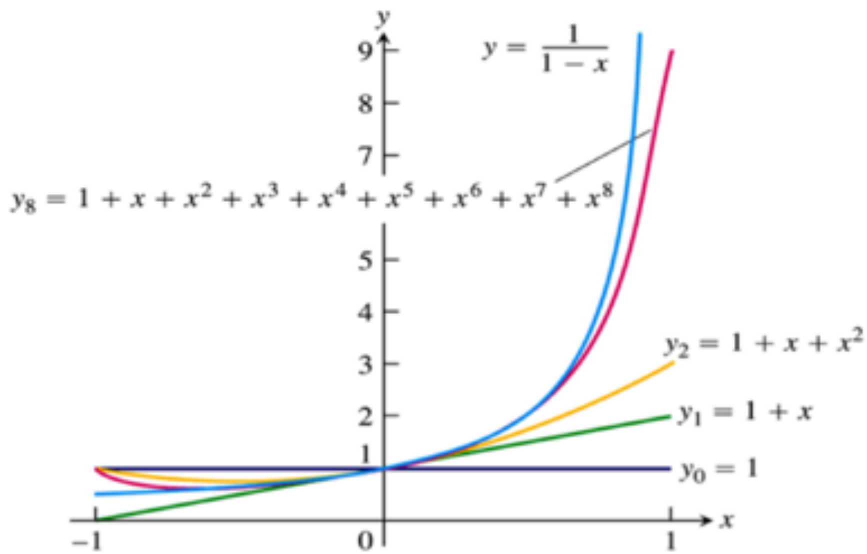
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Ex $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \dfrac{1}{1-x}, \quad |x| < 1 \Rightarrow -1 < x < 1$

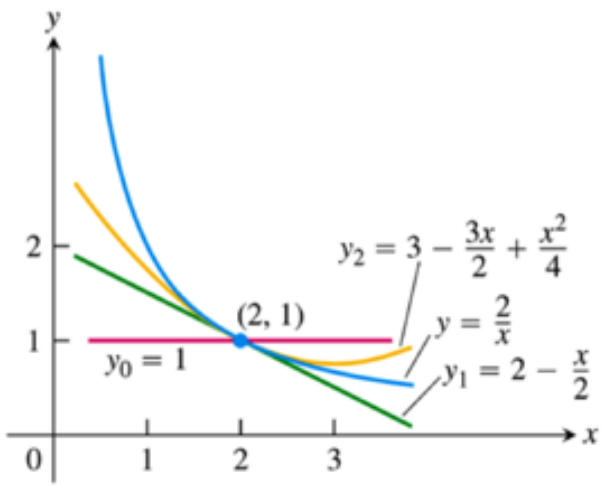


The graphs of $f(x) = 1/(1-x)$ in Example and four of its polynomial approximations.

Ex $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots = \frac{1}{1+\frac{x-2}{2}}$

$= 2/x$

$\left| -\frac{x-2}{2} \right| < 1 \Rightarrow x > 0,\ x < 4$



The graphs of $f(x) = 2/x$ and its first three polynomial approximations

For $x=0$, $\sum_{n=0}^{\infty} 2^n$, diverges

For $x=4$, $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\cdots$, diverges

$\therefore \quad 0 < x < 4$

Generalized Ratio Test

Let $\sum a_n$ be any series, let $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(i) the series converges absolutely if $\rho < 1$.
(ii) the series diverges if $\rho > 1$.
(iii) the test gives no information if $\rho = 1$.

Ex Find all the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n\, n^2} \quad \text{is convergent.}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(x+1)^{n+1}}{2^{n+1}(n+1)^2}}{\dfrac{(x+1)^n}{2^n\, n^2}} \right| = \lim_{n\to\infty} \frac{1}{2}|x+1| \left(\frac{n}{n+1}\right)^2$$

$$\rho = \frac{1}{2}|x+1| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \frac{1}{2}|x+1| \lim_{n\to\infty} \left(\frac{1}{1+\underbrace{1/n}_{0}}\right)^2 = \frac{1}{2}|x+1|$$

The series converges absolutely if

$$\frac{1}{2}|x+1| < 1 \Rightarrow -3 < x < 1$$

Now, we must test the end pts $x=1$ and $x=-3$:

For $x=1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$ converges absolutely, since it is a p-series with $p=2$.

For $x=-3$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n\, n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$ converges absolutely. $\therefore -3 \le x \le 1$

The set of all x for which a power series is convergent is called the interval of covergence. Notice that for a power series of the type $\sum c_n (x-a)^n$ the ratio of two consecutive terms will always contain a term like $|x-a|$, and $\rho = |x-a| \cdot$ something.

If the something is positive real number, then the series converges on an interval. If the something is zero, the series has ratio $\rho = 0$ for all x, so converges everywhere. If the something is ∞ , the series diverges everywhere except at x=a, where the series collapses to a_0. This consideration gives the following theorem:

THEOREM : The convergence of the series $\Sigma c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

R is called the radius of convergence of the power series.

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!\, x^n}{10^n}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!\, x^{n+1}}{10^{n+1}}}{\frac{n!\, x^n}{10^n}}\right| = \frac{|x|}{10} \lim_{n\to\infty} (n+1) = \infty,$$

unless $x=0$

The series converges only at $x=0$.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$\rho = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 \quad \text{for all } x.$$

The series converges for all x.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1|$$

The series converges if $|x-1| < 1 \Rightarrow 0 < x < 2$

$$\left.\begin{array}{l} \text{For } x=0: \ \sum -\frac{1}{n} = -\sum \frac{1}{n} \text{, diverges} \\ \text{For } x=2: \ \sum \frac{(-1)^n}{n} \text{, known to be convergent} \end{array}\right\} 0 < x \leq 2$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{|\arctan n|}{n}$, we shall use comparison test

$$\lim_{n\to\infty} \frac{\frac{|\arctan n|}{\cancel{n}}}{\frac{1}{\cancel{n}}} = \lim_{n\to\infty} |\arctan n| = \pi/2$$

$\sum \frac{|\arctan n|}{n}$ and $\sum \frac{1}{n}$ are comparable. Since $\sum \frac{1}{n}$ is divergent, $\sum \frac{|\arctan n|}{n}$ is also divergent.

Ex $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s}$ we shall use the integral test

$$\int_2^{\infty} \frac{dx}{x(\ln x)^s} = \left.\frac{(\ln x)^{1-s}}{1-s}\right|_2^{\infty} = \begin{cases} \text{Div.} & 1-s>0 \\ \text{conv.} & 1-s<0 \end{cases}$$

Thus the series diverges for $s<1$, converges for $s>1$; this is for positive s.

Hw: study if $s \leq 0$ (it diverges)