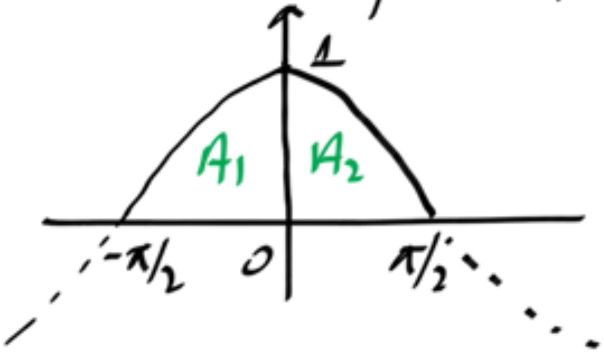
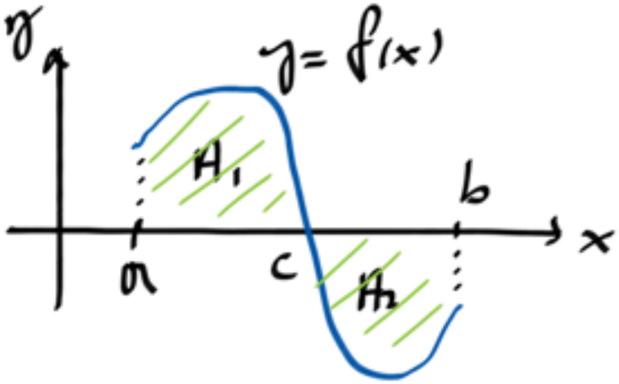
Ex Same question, for $y = \cos x$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$$A = A_1 + A_2 = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x \Big|_{-\pi/2}^{\pi/2} = 1+1 = 2$$

Alternatively, by symmetry,

$$A = 2\int_0^{\pi/2} \cos x \, dx = 2 \sin x \Big|_0^{\pi/2} = 2 \cdot 1 - 0 = 2 \,//$$



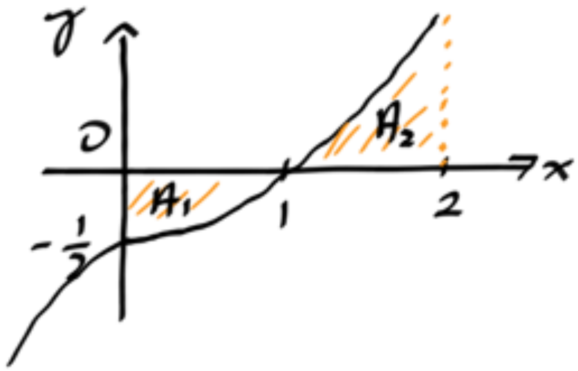
watch out!

$$A_1 = \int_a^c f(x)\,dx, \quad A_2 = -\int_c^b f(x)\,dx$$

In other words, if $f > 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ = area under f

if $f < 0$ on $[a,b]$, then $\int_a^b f(x)\,dx = -$(are under f)

Ex Find the area of the region between the x-axis and the graph of
$y = \frac{1}{4}(x^3 + x - 2)$ on $[0, 2]$.



$y' = \frac{3}{4}x^2 + 1 > 0 \Rightarrow y$ is increasing

$y(1) = 0 \Rightarrow x = 1$ is a root; no other roots

$$A_1 = -\int_0^1 \frac{1}{4}(x^3 + x - 2)\,dx = -\frac{1}{4}\left(\frac{x^4}{4} + \frac{x^2}{2} - 2x\right)\Big|_0^1 = 5/16$$

$$A_2 = \int_1^2 \frac{1}{4}(x^3 + x - 2)\,dx = \frac{1}{4}\left(\frac{x^4}{4} + \frac{x^2}{2} - 2x\right)\Big|_1^2 = 13/16$$

$$A = A_1 + A_2 = \frac{5}{16} + \frac{13}{16} = \frac{18}{16} = \frac{9}{8}$$

Ex Remember that the FTC states

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

FTC (part 1)

$$\frac{d}{dx}\int_1^x x^2\,dx = \int_1^x \frac{d}{dx}x^2\,dx = \int_1^2 2x\,dx = \cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big|_1^x = x^2-1$$

$$\int_a^x \frac{d}{dx}f(x)\,dx \neq f(x) \quad \text{always}$$

Properties of the Definite Integral

$$\boxed{\int_a^b f(x)\,dx = F(b)-F(a)}$$

where F is antiderivative

FTC (part 2)

$$\int_a^b (f\pm g)\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \therefore \qquad \int_a^a f(x)\,dx = 0$$

$$\int_a^b \underbrace{c}_{\text{a const.}} f(x) = c\int_a^b f(x)\,dx$$

Techniques of Integration

<u>Chain rule</u>: $(f\circ g)'(x) = f'(g(x))\,g'(x)$

$$\int f'(g(x))\,g'(x)\,dx = \int f(u)\,du = (f\circ g)(x) + C$$

Ex We know that

$$\int \cos x\,dx = \sin x + C$$

However, $\int \cos 3x\,dx = ?$

$u = 3x$
$du = 3dx$
$\frac{1}{3}du = dx$

$$\int \cos u \frac{du}{3} = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$$

Ex

$$\int (3x+4)^{10}\,dx =$$

$u = 3x+4$
$du = 3dx$
$\frac{1}{3}du = dx$

$$\int u^{10}\frac{du}{3} = \frac{1}{3}\int u^{10}\,du = \frac{1}{3}\,\frac{u^{11}}{11} + C$$

$$= \frac{1}{33}(3x+4)^{11} + C$$

Ex

$$\int \Big(\underbrace{\tfrac{1}{2}x^2-4}_{u}\Big)^{2/5} \underbrace{x\,dx}_{du} = \int u^{2/5}\,du = \frac{u^{\frac{2}{5}+1}}{\frac{2}{5}+1} + C = \frac{5}{7}u^{7/5} + C$$

$$= \frac{5}{7}\left(\tfrac{1}{2}x^2-4\right)^{7/5} + C$$

$u = \frac{1}{2}x^2 - 4$

$du = x\,dx$

Ex

$$\int \underbrace{\sin^{5/2} x}_{u^{5/2}}\, \underbrace{\cos x\,dx}_{du} = \int u^{5/2}\,du = \frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1} + C = \frac{2}{7}u^{7/2} + C$$

$$= \frac{2}{7}\sin^{7/2} x + C$$

$u = \sin x$

$du = \cos x\,dx$

Ex

$$\int x^5\sqrt{1-x^3}\,dx = \int x^3\sqrt{1-x^3}\,x^2\,dx = \int (1-u)\,u^{1/2}\left(-\frac{du}{3}\right)$$

$u = 1 - x^3 \Longrightarrow x^3 = 1 - u$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{3}\left\{ \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right\} + C$$

$$= \frac{1}{3}\left[\frac{2}{5}(1-x^3)^{5/2} - \frac{2}{3}(1-x^3)^{3/2} \right] + C$$

Substitution in definite integrals

Ex $$\int_0^{\pi/4} \sin x \sqrt{\cos x}\, dx = -\int u^{1/2} du = -\frac{2}{3} \cos^{3/2} x \Big|_0^{\pi/4} = -\frac{2}{3}\left[\cos^{3/2} \pi/4 - \cos^{3/2} 0\right]$$

$$= -\frac{2}{3}\left(\left(\frac{1}{\sqrt{2}}\right)^{3/2} - 1 \right)$$

$u = \cos x$

$-du = \sin x dx$

or

$$\int_0^{\pi/2} \sin x \sqrt{\cos x}\, dx = -\int_{u(0)}^{u(\pi/4)} u^{1/2} du = -\frac{2}{3} u^{3/2} \Big|_{u(0)}^{u(\pi/4)}$$

$$= \frac{2}{3} u^{3/2} \Big|_{u(\pi/4)}^{u(0)}$$

$$= \frac{2}{3}\left(u^{3/2}(0) - u^{3/2}(\pi/4) \right)$$

$$= \frac{2}{3}\left(1 - \left(1/\sqrt{2}\right)^{3/2} \right)$$

$u(x) = \cos x$

$u(0) = 1$

$u(\pi/4) = 1/\sqrt{2}$