A Theoretical Model of Private Currency Dynamics

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Motivations

- There is an abundance of empirical papers documenting the nexus between privately-issued currencies and macroeconomic news.
  - Results are often contradictory (Benigno and Rosa, 2023 and Sören, 2023.)

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- Conclusions are highly data-dependent and preferences (deep parameters) for cryptos seem to have changed over time.

- We offer a macro model to study private currencies in a competitive economy. Our objective in this project is twofold:
  - How do cryptocurrencies fit in the standard monetary theory?
  - Is there a welfare-improving role for cryptocurrencies?

- Our framework relies on the seminal work of Lucas and Stokey (1983, 1987) on the introduction of *cash-in-advance* (CIA) constraints into the Real Business Cycle literature, namely the *cash-credit* model.

- A more fundamental inspiration for this project is the work of Marchiori (2021) on both cryptocurrencies and fiat currencies. However, he does not give too much weight on the monetary implications of cryptocurrency issuance.

- Our continuous-time model also follows the intuition developed in Holman and Neanidis (2005). Their version allows the modelling of two currencies as opposed to the standard CIA theory.

Preliminary Results

- Crypto-supply growth relative to money supply growth will determine appreciation or depreciation, but it will not affect anything apart from the nominal exchange rate.

- There are situations where cryptocurrencies provide superior *transactive* advantages than money.

- In a frinctionless economy, the mass of goods purchased using money increases with fees on cryptocurrencies and decreases with the consumption tax on money payments.

Household Problem

- Time is continuous and indexed by $t \in [0,\infty)$
- A single representative consumer
- The consumption good can be purchased with a combination of fiat money and cryptocurrency
- The household is infinitely-lived and maximizes intertemporal lifetime utility

$$U \equiv \int_0^{\infty} e^{-\rho t} ln \left[(c_t^m)^{\theta} (c_t^x)^{1-\theta} \right] dt, \tag{1}$$

where $\rho > 0$ is the utility discount rate and (c_t^m) and (c_t^x) indicate units of the consumption good purchased at time t with money and cryptos, respectively.

- Output is produced according to the production function $Y_t = F(K_t, a_t L)$, where L is a fixed amount of labor and a_t is labor productivity.

- Market clearing condition requires

$$F(K_t, a_t L) = c_t^m + c_t^x + \dot{K}_t = C_t + \dot{K}_t \tag{2}$$

- The CIA constraint thus reads

$$M_t = \phi P_t \dot{K}_t + P_t c_t^m + Q_t P_t^* c_t^x, \tag{3}$$

where M_t is nominal money, Q_t is the nominal exchange rate and ϕ allows us to distinguish between the model with a pure consumption-CIA constraint and the model with a comprehensive CIA constraint that includes K_t.

From the household point of view, the overall monetary budget constraint is

$$P_t\dot{K}_t + \dot{M}_t = P_tF(K_t, a_tL) + V_t - P_tc_t^m - Q_tP_t^*c_t^x, \tag{4}$$

where V_t is the lump-sum transfer from the government to the household.

To sum up, we write the current-value Hamiltonian for the household problem as

$$\mathcal{H} = ln[(c_t^m)^\theta(c_t^x)^{1-\theta}] + \lambda_t^K\dot{K}_t + \lambda_t^M\dot{m} + \lambda_t^S[m_t - \phi\dot{K}_t - c_t^m - q_tc_t^x] \tag{5}$$

Assuming our production function exhibits an AK-like setting, the first order conditions on control and co-state variables lead to two different sets of solutions:

Scenario 1 (complete neutrality): $\phi = 0$

$$F_{Kt} = \rho \tag{6}$$

$$\frac{\dot{C}_t}{C_t} = \alpha A - \rho \tag{7}$$

$$\pi_t = g^M - (\alpha A - \rho) \tag{8}$$

$$\pi_t^* = \frac{\dot{S}_t}{S_t} - (\alpha A - \rho) \tag{9}$$

$$\frac{\dot{Q}_t}{Q_t} = g^M - \frac{\dot{S}_t}{S_t} = \pi_t - \pi_t^*, \tag{10}$$

where $S_t = P_t^* c_t^x$.

Scenario 2 (crypto neutrality): $\phi = 1$

$$F_{Kt} = \rho \cdot (1 + \rho + \pi_t) \tag{11}$$

$$\frac{\dot{C}_t}{C_t} = \left[\frac{1}{1 + \rho + g^M} \cdot \alpha A - \rho\right] \tag{12}$$

$$\pi_t = g^M - \left[\frac{1}{1 + \rho + g^M} \cdot \alpha A - \rho\right] \tag{13}$$

$$\pi_t^* = \frac{\dot{S}_t}{S_t} - \left[\frac{1}{1 + \rho + g^M} \cdot \alpha A - \rho\right] \tag{14}$$

$$\frac{\dot{Q}_t}{Q_t} = g^M - \frac{\dot{S}_t}{S_t} = \pi_t - \pi_t^*, \tag{15}$$

Where are we now?

- We are replacing the Cobb-Doulas bundle with a Constant Elasticity of Substitution (CES) bundle.

- Technically, we are forcing cryptocurrencies to interact with fiat currencies. Our aim is to have *equilibria* where the dynamic of capital is influenced by the crypto-money interaction.

- The CES instantaneous utility function is written as

$$U \equiv \int_0^\infty e^{-\rho t} ln \left[\theta \cdot (c_t^m)^{\frac{\delta-1}{\delta}} + (1-\theta) \cdot (c_t^x)^{\frac{\delta}{\delta-1}} \right] dt \tag{16}$$

Where are we now?

- We are putting more structure (micro-foundation) on the functional form we use to study the preference and production side of the economy.
- We are "endogenizing" some of the preference parameters that would normally be taken as given in a standard macro model.
  - For instance, our CIA constraint tells us that people use cryptos but it does not say why nor make inference on welfare considerations.
  - we build a static model to address that and with the hope of improving the preference and production sides of the dynamic model.

An overview of the Static Model

- The economy is populated by L consumers.
- Each consuming (a continuous finite mass of) N different goods indexed by $n \in [0, N]$.
- Only a fraction ϵ of the L consumers has access to cryptocurrencies.
- The remaining $(1-\epsilon)L$ consumers purchase all the N goods using money.
- In a nutshell, consumers can be grouped as *crypto-less consumers* and *crypto-using consumers*
- Crypto transactions come with fees denoted $\varphi' > 0$ (conversion fee of money to crypto by consumers) , $\varphi'' > 0$ (fee charged by merchants) and φ''' (paid by firms).

The expenditure problem of the *crypto-less consumer* is

$$\max_{\{\tilde{c}(n)\}} \tilde{U} = \int_0^N \ln\left[\tilde{c}(n) \cdot (1 - \delta(n))\right] dn$$

subject to

$$\tilde{x} = \int_0^N p(n)\, \tilde{c}(n) \cdot (1 + \tau)\, dn \qquad (17)$$

where $\tilde{c}(n)$ is the consumed quantity of the n-th good, $p(n)$ is the market price of the good in terms of money, $\tilde{x}$ is individual spending on consumption goods, $\delta(n)$ is a good-specific disutility parameter, τ is a money-transaction fee (sunk monetary cost).

An overview of the Static Model

The expenditure problem solved by the *crypto-using consumer* is

$$\max_{\{c(j),c(i)\}} \int_0^{\bar{n}} \ln\left[c\left(j\right)\left(1-\delta\left(j\right)\right)\right] dj + \int_{\bar{n}}^{N} \ln c\left(i\right) di$$

subject to

$$x = \int_0^{\bar{n}} p\left(j\right) c\left(j\right)\left(1+\tau\right) dj + \int_{\bar{n}}^{N} sq\left(i\right) c\left(i\right)\left(1+\varphi'+\varphi''\right) di, \tag{17}$$

The set of goods is now broken down into a subset indexed by $j \in [0, \bar{n})$ is purchased using *money* and another subset indexed by $i \in [\bar{n}, N]$ is purchased using the *crypto-currency*. **Implicitly the expression above define a threshold good, indexed by $n = \bar{n}$, which splits the set $[0, N]$ by payment characteristics.**

In order to choose the best payment option for a given good $n \in [0, N]$,the consumer compares opportunity costs in terms of utility.

$$u_n\left(c^{money}\right) - u_n\left(c^{crypto}\right) = \ln\left[\frac{sq\left(n\right)}{p\left(n\right)} \cdot \frac{\left(1+\varphi'+\varphi''\right)\left(1-\delta\left(n\right)\right)}{\left(1+\tau\right)}\right]. \quad (18)$$

Therefore, the condition for using the crypto-currency, $u_n\left(\tilde{c}'\right) \leqslant u_n\left(\tilde{c}''\right)$, is

$$sq\left(n\right) \cdot \left(1+\varphi'+\varphi''\right) \leqslant p\left(n\right) \cdot \frac{1+\tau}{1-\delta\left(n\right)}. \quad (19)$$

Inequality (19) describes the situation in which *crypto-payments are superior to money-payments*.

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An overview of the Static Model

The threshold good is determined by

$$\bar{n} = \frac{1 + \varphi' + \varphi'' - (1 + \tau)(1 - \varphi''')}{1 + \varphi' + \varphi''} \cdot \frac{N}{\beta} \tag{20}$$

Expression (20) confirms the most intuitive properties of the threshold index, namely,

$$\frac{\partial \bar{n}}{\partial \varphi'} > 0, \; \frac{\partial \bar{n}}{\partial \varphi''} > 0, \; \frac{\partial \bar{n}}{\partial \varphi'''} > 0 \text{ and } \frac{\partial \bar{n}}{\partial \tau} < 0,$$

that is, the mass of goods exclusively purchased using money $\bar{n}$ is higher the higher the crypto-fee rates and the lower the consumption tax on money payments, τ.

Conclusion

- We propose a model to explain interaction between cryptocurrencies and fiat currencies.
- The theoretical literature tends to model cryptocurrencies as a "monetary innovation" on its own. We provide insights on possible interactions between the two worlds.
- We study equilibrium conditions for using crypto versus money.
- In equilibrium, cryptocurrencies do not affect real resource allocations.
- Our aim is to combine the static model with the dynamic framework to present a streamlined explanation of the crypto market. A possible extension is a model that also accounts for central bank digital currencies.

Appendix

Figure 1: Regression of crypto returns on implied volatility (256-day rolling window)

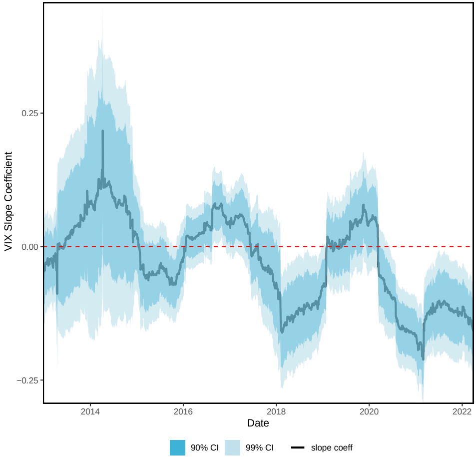

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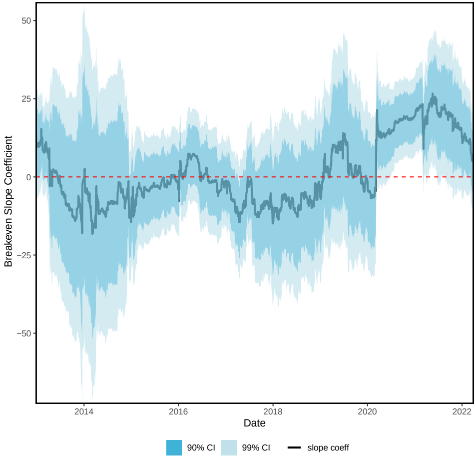
Figure 2: Regression of crypto returns on breakeven inflation (256-day rolling window)



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Appendix

Proposition 1: *The optimal payment method for any good $n \in [0, N]$ is determined by*

$$\delta(n) \quad < \quad 1 - \frac{(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''} \Longrightarrow \textit{Money} \tag{21}$$

$$\delta(n) \quad \geqslant \quad 1 - \frac{(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''} \Longrightarrow \textit{Cryptocurrency} \tag{22}$$

Given $\delta(n) = \beta \cdot (n/N)$, there are three possible scenarios. Scenario I: if $0 < \beta < 1 - \frac{(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''}$, all the N goods are purchased using money. Scenario II: if $1 - \frac{(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''} < 0 < \beta$, all the N goods are purchased using the crypto-currency. Scenario III: if

$$0 < 1 - \frac{(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''} < \beta \tag{23}$$

there exists a unique threshold good $\bar{n} \in (0, N)$ such that all goods $n \in [0, \bar{n})$ are purchased using money, whereas all goods $n \in [\bar{n}, N]$ are purchased using the crypto-currency. The threshold is determined by

$$\bar{n} = \frac{1+\varphi'+\varphi''-(1+\tau)(1-\varphi''')}{1+\varphi'+\varphi''} \cdot \frac{N}{\beta} \tag{24}$$

Appendix

Proof. *Inequalities (21)-(22) follow directly by substituting* $s \cdot q(n) \cdot (1 - \varphi''') = p(n)$ *for each* $n \in [0, N]$ *into (19) and solving for* $\delta(n)$*. Using* $\delta(n) = \beta \cdot (n/N)$ *with* $0 < \beta < 1$*. to substitute* $\delta(n)$ *in the equilibrium condition for overall no-arbitrage between money-payments and crypto-payments for goods. Scenarios I-III follow from the parameter restrictions that would respectively imply* $\bar{n} > N$, $\bar{n} < 0$ *and* $0 < \bar{n} < N$ *in (24).* ■

The equilibrium no-arbitrage condition between money and crypto is given by

$$\frac{1 + \varphi' + \varphi''}{1 - \varphi'''} = \frac{1 + \tau}{1 - \delta(\bar{n})}. \tag{25}$$

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