MA1200 Calculus and Basic Linear Algebra I

Lecture Note 9

More on Differentiation

Topic Covered

Differentiation Techniques (Finding $\frac{dy}{dx}$)

- Implicit Differentiation
- Differentiation of inverse function
- Logarithmic Differentiation
- Differentiation of Parametric Equation

High-Order derivatives -- Differentiate a given function for n-times ($n > 1$)

- Definition
- Some useful techniques in finding higher-order derivatives:
  - ➢ Leibnitz' Rule
  - ➢ Product-to-sum formula (for trigonometric functions)
  - ➢ Method of Partial fractions (for rational function)

Implicit Differentiation

- We would like to compute $\frac{dy}{dx}$ where x, y are governed by the following equation:

$$x^3 + \cos x + y^5 = 0,$$

  Intuitively, one can compute the derivative $\frac{dy}{dx}$ by first expressing y in terms of x and obtain $\frac{dy}{dx}$ using various differentiation techniques (chain rule, product rule, etc.)

$$y = \sqrt[5]{-x^3 - \cos x} \Rightarrow \frac{dy}{dx} = \cdots = \frac{1}{5}(-x^3 - \cos x)^{-\frac{4}{5}}(-3x^2 + \sin x).$$

- However in some cases, we may not able to write down the explicit expression of y (in terms of x). As an example, we consider the following equation:

$$x^3y + y^4 - \sin x = 3.$$

  It is clear that we are not able to express y in terms of x.

Question: How do we compute $\frac{dy}{dx}$ in this case?

One may try to obtain $\frac{dy}{dx}$ by differentiating the equation with respect to x:

$$\frac{d}{dx}(x^3y + y^4 - \sin x) = \frac{d}{dx}3$$

$$\Rightarrow y\frac{d(x^3)}{dx} + x^3\frac{dy}{dx} + \frac{dy^4}{dy}\frac{dy}{dx} - \cos x = 0$$

$$\Rightarrow 3x^2y + (x^3 + 4y^3)\frac{dy}{dx} - \cos x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x - 3x^2y}{x^3 + 4y^3}.$$

This technique is called *implicit differentiation*.

Remark:

Here, we observe that $\frac{dy}{dx}$ is now expressed in terms of x and y instead of x alone since we are not able to express y in terms of x.

Example 1

Find $\frac{dy}{dx}$ if x and y are governed by the equation:

$$xy^2 + 3x^3 = \frac{y}{x}.$$

☺ Solution:

We differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(xy^2 + 3x^3) = \frac{d}{dx}\left(\frac{y}{x}\right) \Rightarrow \underbrace{\left(y^2\frac{dx}{dx} + x\frac{dy^2}{dx}\right)}_{\frac{d}{dx}xy^2} + \frac{d}{dx}(3x^3) = \frac{x\frac{dy}{dx} - y\frac{dx}{dx}}{x^2}$$

$$\Rightarrow y^2 + x\left(\frac{dy^2}{dy}\frac{dy}{dx}\right) + 9x^2 = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

$$\Rightarrow y^2 + 2xy\frac{dy}{dx} + 9x^2 = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} \Rightarrow \frac{dy}{dx} = \frac{-\frac{y}{x^2} - y^2 - 9x^2}{2xy - \frac{1}{x}}.$$

Example 2

It is given that the equation of a conic section is $x^2 + 2y^2 - 6x + 4y - 16 = 0$.

(a) Identify this conic section.

(b) Find the tangent to the graph of this conic section at $(x, y) = (8, 0)$.

☺ Solution:

(a) Since there is no xy-term, one can identify the conic section using completing square technique:

$$x^2 + 2y^2 - 6x + 4y - 16 = 0 \Rightarrow (x-3)^2 + 2(y+1)^2 = 27$$

$$\Rightarrow \frac{(x-3)^2}{\left(\sqrt{27}\right)^2} + \frac{(y+1)^2}{\left(\sqrt{\frac{27}{2}}\right)^2} = 1.$$

which is an ellipse with center $(3, -1)$.

(b) We need to find the value of $\frac{dy}{dx}$ at $(x, y) = (8, 0)$.

By differentiating the equation with respect to x, we have

$$\frac{d}{dx}(x^2 + 2y^2 - 6x + 4y - 16) = \frac{d}{dx}0$$

$$\Rightarrow 2x + 4y\frac{dy}{dx} - 6 + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{6 - 2x}{4y + 4}.$$

Putting $(x, y) = (8, 0)$, then the slope of tangent is given by

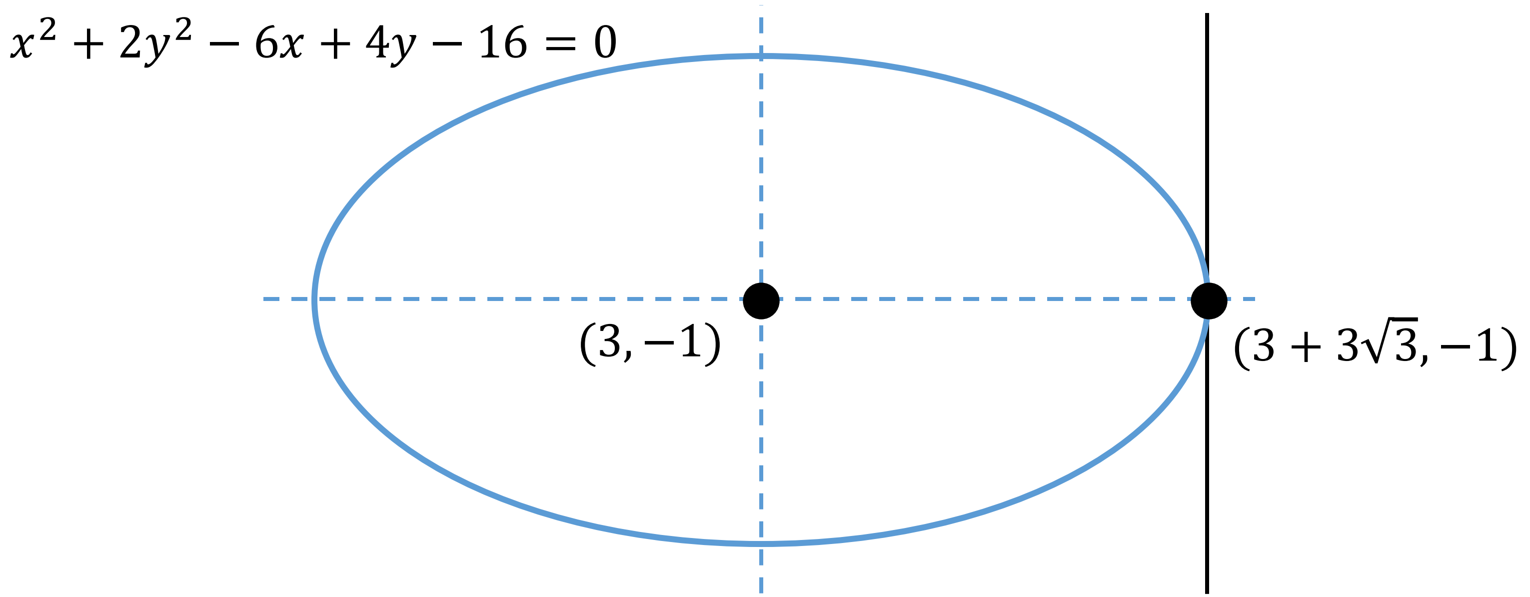
$$\frac{dy}{dx}\Big|_{(x,y)=(8,0)} = \frac{6 - 2(8)}{4(0) + 4} = -\frac{5}{2}.$$

The equation of tangent line at $(x, y) = (8, 0)$ is given by

$$\frac{y - 0}{x - 8} = -\frac{5}{2} \Rightarrow y = -\frac{5}{2}x + 20.$$

Remarks about implicit differentiation theorem

- When we carry out the implicit differentiation, we have assumed that y can be expressed as a differentiable function of x (i.e. $y = f(x)$ and $\frac{dy}{dx}$ exists). This assumption may not be true in some extreme cases.
- If we take another point $(x, y) = (3 + 3\sqrt{3}, -1)$ in Example 2, we find that the $\frac{dy}{dx}$ does not exist.

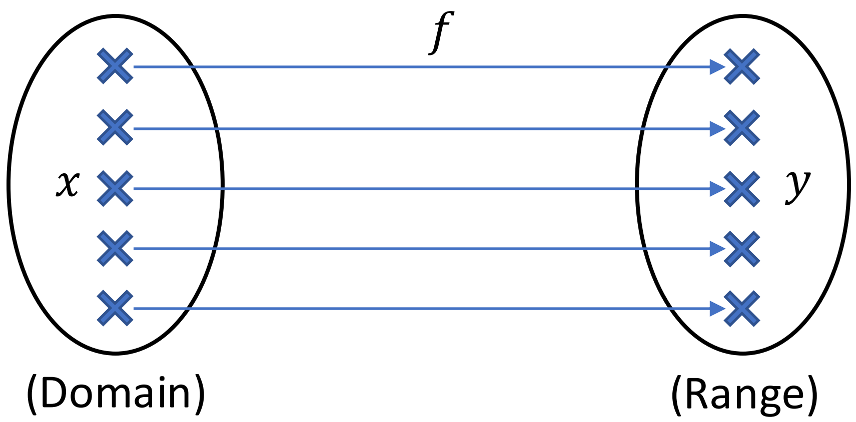


- In advanced level mathematics, a theorem called *implicit function theorem* is developed to address this issue. Roughly speaking, the implicit function theorem states that given an equation $F(x, y) = 0$, if $F(x, y)$ is "nice" enough (i.e. continuous differentiable), then y can be expressed as a function of x and $\frac{dy}{dx}$ exists as a number so that one can carry out the implicit differentiation to obtain $\frac{dy}{dx}$.

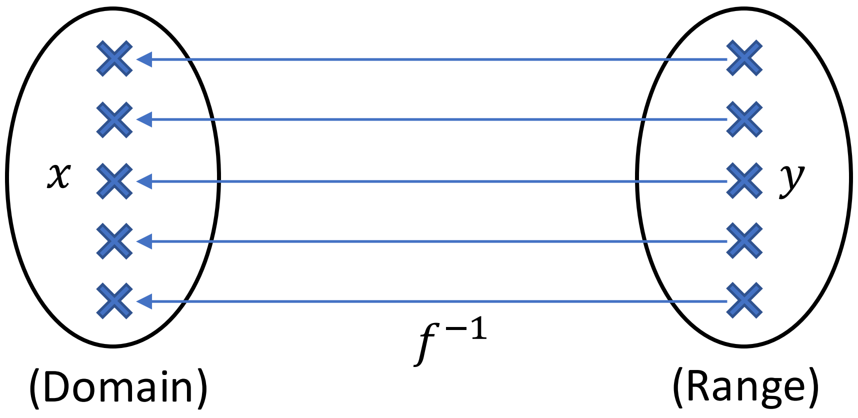
- In this course, we only handle the cases when $F(x, y)$ is nice enough and you can do the implicit differentiation freely.

Differentiation of inverse function ($\frac{d}{dx}f^{-1}(x)$)

Recall that a function $f(x)$ takes a number x from its domain and assigns it to a single value y from its range.



The inverse function of $f(x)$, denoted by $f^{-1}(x)$, is a function which takes a number y from the range of $f(x)$ and "bring back" to x.



Recall that the inverse of a function $f(x)$ exists if and only if $f(x)$ is one-to-one.

Some other common inverse functions used in Mathematics

$f(x)$	Inverse of $f(x)$
$f_1: \mathbb{R} \to (0, \infty),\ f_1(x) = 10^x$	$f_1^{-1}(x) = \log_{10} x$
$f_2: \mathbb{R} \to (0, \infty),\ f_2(x) = e^x$	$f_2^{-1}(x) = \ln x\ (= \log_e x)$
$f_3: [0, \infty) \to [0, \infty),\ f_3(x) = x^2$	$f_3^{-1}(x) = \sqrt{x}$
$f_4: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1],\ f_4(x) = \sin x$	$f_4^{-1}(x) = \sin^{-1} x\ (= \arcsin x)$
$f_5: [0, \pi] \to [-1,1],\ f_5(x) = \cos x$	$f_5^{-1}(x) = \cos^{-1} x\ (= \arccos x)$
$f_6: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R},\ f_6(x) = \tan x$	$f_6^{-1}(x) = \tan^{-1} x\ (= \arctan x)$

Example 3 (Inverse of trigonometric function)

What is the derivative of $\sin^{-1} x$, $-1 < x < 1$? (Here, $-\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}$)

☺ Solution:

Let $y = \sin^{-1} x$, then

$y = \sin^{-1} x \Rightarrow \sin y = \sin(\sin^{-1} x) \Rightarrow \sin y = x.$

Differentiate the equation both sides with respect to x, we have

$$\frac{d}{dx}(\sin y) = \frac{d}{dx} x \Rightarrow \frac{d(\sin y)}{dy}\frac{dy}{dx} = 1 \Rightarrow \cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}.$$

Substitute $y = \sin^{-1} x$ and note that

$$\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} \overset{since\ -\frac{\pi}{2}<y=\sin^{-1} x<\frac{\pi}{2}}{\geq} 0,$$ we finally get

$$\frac{d}{dx}\sin^{-1} x = \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - (\sin(\sin^{-1} x))^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

Example 4

Find the derivative of $y = \sec^{-1} x$ where $|x| > 1$ and $0 < \sec^{-1} x < \pi$.

☺ Solution:

Let $y = \sec^{-1} x$, then

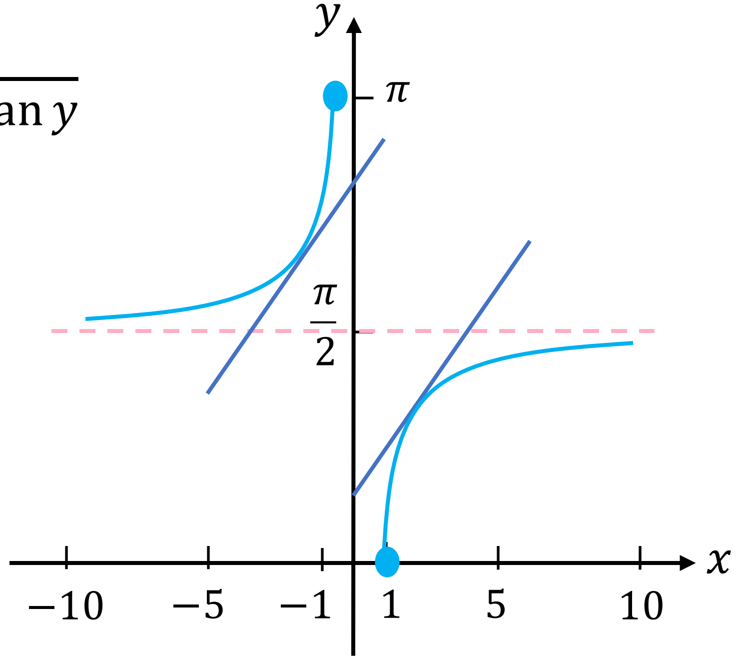
$y = \sec^{-1} x \Rightarrow \sec y = x.$

Differentiate the equation both sides with respect to x, we have

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x \Rightarrow \frac{d(\sec y)}{dy}\frac{dy}{dx} = 1 \Rightarrow \sec y \tan y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\overset{\tan^2\theta+1=\sec^2\theta}{\Rightarrow} \frac{dy}{dx} = \frac{1}{\sec y\left(\pm\sqrt{\sec^2 y - 1}\right)} = \frac{1}{\pm x\sqrt{x^2 - 1}}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \dfrac{1}{+x\sqrt{x^2-1}} & if\ x > 1 \\ \dfrac{1}{-x\sqrt{x^2-1}} & if\ x < -1 \end{cases} = \frac{1}{|x|\sqrt{x^2-1}}.$$



Using similar method, one can obtain the derivatives of the inverse of other trigonometric functions. We summarize the result in the following table:

$y = f(x)$	The derivative $f'(x)$
$y = \sin^{-1} x$ $\left(-1 < x < 1,\ -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}\right)$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
$y = \cos^{-1} x$ $(-1 < x < 1,\ 0 < \cos^{-1} x < \pi)$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$
$y = \tan^{-1} x$ $\left(-\infty < x < \infty, -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}\right)$	$\frac{dy}{dx} = \frac{1}{1+x^2}$
$y = \cot^{-1} x$ $(-\infty < x < \infty,\ 0 < \cot^{-1} x < \pi)$	$\frac{dy}{dx} = -\frac{1}{1+x^2}$
$y = \sec^{-1} x$ $(\lvert x\rvert > 1,\ 0 < \sec^{-1} x < \pi)$	$\frac{dy}{dx} = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$y = \csc^{-1} x\ (or\ \mathrm{cosec}^{-1} x)$ $\left(\lvert x\rvert > 1,\ -\frac{\pi}{2} < \csc^{-1} x < \frac{\pi}{2}\right)$	$\frac{dy}{dx} = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Example 5 (Inverse of exponential function)

The natural logarithm, denoted by $\ln x$, is defined as the inverse of e^x (i.e. $e^{\ln x} = x$, $\ln(e^x) = x$). What is the derivative of $\ln x$, $x > 0$? (i.e. $\frac{d}{dx}\ln x$)

☺ Solution:

We let $y = \ln x$, using similar method (take $f(x) = e^x$, $f^{-1}(x) = \ln x$), we get $y = \ln x \Rightarrow e^y = e^{\ln x} \Rightarrow e^y = x$.

We differentiate the equation both sides with respect to x, we have

$$\frac{d}{dx}e^y = \frac{d}{dx}x \Rightarrow \frac{de^y}{dy}\frac{dy}{dx} = 1 \Rightarrow e^y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y}.$$

Substitute $y = \ln x$, we finally get

$$\frac{d}{dx}\ln x = \frac{dy}{dx} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Example 6

We consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{x^3-3x^2+9x}$.

(a) Is the function increasing or decreasing? Explain your answer.

(b) Show that its inverse function f^{-1} exists.

(c) Find $\frac{d}{dx} f^{-1}(x)$.

☺ Solution:

(a) One can check its monotonicity by considering its derivative

$$\frac{d}{dx} f(x) = \frac{d\left(e^{x^3-3x^2+9x}\right)}{dx} \Rightarrow \frac{de^{x^3-3x^2+9x}}{d(x^3-3x^2+9x)} \frac{d(x^3-3x^2+9x)}{dx}$$
$$= e^{x^3-3x^2+9x}(3x^2-6x+9) \Rightarrow \underbrace{3}_{>0} \underbrace{(x^2-2x+3)}_{\substack{=(x-1)^2+2\\ \geq 2>0}} \underbrace{e^{x^3-3x^2+9x}}_{>0}.$$

One can observe that $\frac{d}{dx} f(x) > 0$, this implies that $f(x)$ is strictly increasing.

(b) Since the function is strictly increasing, then for any $x_1 \neq x_2$, we must have $f(x_1) \neq f(x_2)$. This implies that $f(x)$ is one-to-one and its inverse exists.

(c) *IDEA: Let $y = e^{x^3-3x^2+9x}$, it appears that it is difficult for us to express x in terms of y so that the expression f^{-1} is not available.*

Let $y = f^{-1}(x)$, then $f(y) = x \Rightarrow e^{y^3-3y^2+9y} = x$.

Differentiate the equation both sides with respect to x, we get

$$\frac{d\left(e^{y^3-3y^2+9y}\right)}{dx} = \frac{d}{dx}x \Rightarrow \frac{d\left(e^{y^3-3y^2+9y}\right)}{d(y^3-3y^2+9y)} \frac{d(y^3-3y^2+9y)}{dy} \frac{dy}{dx} = 1$$

$$= e^{y^3-3y^2+9y}(3y^2-6y+9)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{(3y^2-6y+9)e^{y^3-3y^2+9y}}.$$

Here, we cannot express $\frac{dy}{dx}$ in terms of x since the explicit formula for $y = f^{-1}(x)$ is not known.

Logarithmic Differentiation

Sometimes, we may need to calculate the derivatives of the following function:

$$\frac{d}{dx} f(x)^{g(x)}.$$

To compute these derivatives, one can compute the derivatives by first letting $y = f(x)^{g(x)}$ (so we need to find $\frac{dy}{dx}$). Then we take the (natural) logarithm on both sides, then the expression becomes

$$\ln y = \ln f(x)^{g(x)} = \underbrace{g(x)\ln f(x)}_{product\ of\ functions}.$$

Then we differentiate both sides with respect to x and get

$$\frac{d}{dx}\ln y = \frac{d}{dx} g(x)\ln f(x) \Rightarrow \frac{d(\ln y)}{dy}\frac{dy}{dx} = \frac{d}{dx} g(x)\ln f(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx} g(x)\ln f(x) \Rightarrow \frac{dy}{dx} = y\frac{d}{dx} g(x)\ln f(x). \quad \left(\frac{d}{dx}\ln x = \frac{1}{x}\right)$$

The derivative $\frac{d}{dx} g(x)\ln f(x)$ can be computed easily.

Example 7

Compute the derivative

$$\frac{d}{dx}x^x.$$

☺ Solution:

Let $y = x^x$, we then take natural logarithm on both sides:

$$\ln y = \ln x^x = x \ln x\,.$$

We differentiate both sides of the equation with respect to x, we have

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln x \Rightarrow \frac{d(\ln y)}{dy}\frac{dy}{dx} = \ln x\frac{d}{dx}x + x\frac{d}{dx}\ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x(1) + x\left(\frac{1}{x}\right) \Rightarrow \frac{dy}{dx} = x^x(\ln x + 1).$$

Example 8

Compute the derivative

$$\frac{d}{dx}(x^2 - 2x + 3)^{\cos x}.$$

☺ Solution:

We let $y = (x^2 - 2x + 3)^{\cos x}$ and apply natural logarithm on both sides, we have

$$\ln y = \ln(x^2 - 2x + 3)^{\cos x} \Rightarrow \ln y = \cos x \ln(x^2 - 2x + 3)\,.$$

We differentiate both sides with respect to x, we get

$$\frac{d}{dx}\ln y = \frac{d}{dx}\cos x \ln(x^2 - 2x + 3)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(x^2 - 2x + 3)\frac{d}{dx}\cos x + \cos x \frac{d}{dx}\ln(x^2 - 2x + 3)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(x^2 - 2x + 3)\,(-\sin x) + \cos x\,\frac{d[\ln(x^2 - 2x + 3)]}{d(x^2 - 2x + 3)}\frac{d(x^2 - 2x + 3)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\sin x \ln(x^2 - 2x + 3) + \cos x \left(\frac{1}{x^2 - 2x + 3}\right)(2x - 2)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2x - 2}{x^2 - 2x + 3}\cos x - \sin x \ln(x^2 - 2x + 3)\right].$$

Substitute $y = (x^2 - 2x + 3)^{\cos x}$, we finally get

$$\frac{dy}{dx} = (x^2 - 2x + 3)^{\cos x}\left[\frac{2x - 2}{x^2 - 2x + 3}\cos x - \sin x \ln(x^2 - 2x + 3)\right].$$

Example 9

Compute

$$\frac{d}{dx}(a^x), \qquad a > 0.$$

☺ Solution:

Let $y = a^x$, we apply the natural logarithm on both sides:

$$\ln y = \ln(a^x) \Rightarrow \ln y = x \ln a\,.$$

We differentiate both sides with respect to x, we have

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln a \Rightarrow \frac{d(\ln y)}{dy}\frac{dy}{dx} = \ln a\frac{dx}{dx} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y\ln a\,.$$

Putting $y = a^x$, we finally get $\frac{d}{dx}a^x = \frac{dy}{dx} = a^x \ln a\,.$

Again, $\frac{d}{dx}a^x = a^x \ln a \neq a^x$ in general. We only have $\frac{d}{dx}e^x = e^x$!!

Example 10

Compute

$$\frac{d}{dx}\left[\frac{(x+1)^2}{(x-2)^3(3x+4)^5}\right].$$

☺ Solution:

Let $y = \frac{(x+1)^2}{(x-2)^3(3x+4)^5}$, we take natural logarithm on both sides:

$$\ln y = \ln \frac{(x+1)^2}{(x-2)^3(3x+4)^5} \Rightarrow \ln(x+1)^2 - \ln(x-2)^3 - \ln(3x+4)^5$$

$$\Rightarrow \ln y = 2\ln(x+1) - 3\ln(x-2) - 5\ln(3x+4).$$

Next, we differentiate the equation with respect to x,

$$\frac{d}{dx}\ln y = 2\frac{d}{dx}\ln(x+1) - 3\frac{d}{dx}\ln(x-2) - 5\frac{d}{dx}\ln(3x+4)$$

Note:

$$\ln x^a = a\ln x$$

$$\ln xy = \ln x + \ln y$$

$$\ln\frac{x}{y} = \ln x - \ln y$$

$$\Rightarrow \overbrace{\frac{1}{y}\frac{dy}{dx}}^{\frac{d(\ln y)}{dy}\frac{dy}{dx}} = 2\overbrace{\left(\frac{1}{x+1}\right)}^{\frac{d\ln(x+1)}{d(x+1)}\frac{d(x+1)}{dx}} - 3\overbrace{\left(\frac{1}{x-2}\right)}^{\frac{d\ln(x-2)}{d(x-2)}\frac{d(x-2)}{dx}} - 5\overbrace{\left(\frac{3}{3x+4}\right)}^{\frac{d\ln(3x+4)}{d(3x+4)}\frac{d(3x+4)}{dx}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1)^2}{(x-2)^3(3x+4)^5}\left(\frac{2}{x+1} - \frac{3}{x-2} - \frac{15}{3x+4}\right).$$

Remark:

Of course, one can find the derivative of this function using quotient rule and product rule, i.e.

$$\frac{d}{dx}\left[\frac{(x+1)^2}{(x-2)^3(3x+4)^5}\right] = \frac{(x-2)^3(3x+4)^5\frac{d}{dx}(x+1)^2 - (x+1)^2\frac{d}{dx}(x-2)^3(3x+4)^5}{[(x-2)^3(3x+4)^5]^2}.$$

The calculation is very tedious.

Example 11

Compute the derivative

$$\frac{d}{dx}\sqrt[3]{\frac{(a+x)(b+x)}{(a-x)(b-x)}}.$$

☺ Solution:

Again, we can let $y = \sqrt[3]{\frac{(a+x)(b+x)}{(a-x)(b-x)}}$ and take the natural logarithm on both sides:

$$\ln y = \ln \sqrt[3]{\frac{(a+x)(b+x)}{(a-x)(b-x)}} \Rightarrow \ln y = \ln \left[\frac{(a+x)(b+x)}{(a-x)(b-x)}\right]^{\frac{1}{3}} \Rightarrow \ln y = \frac{1}{3}\ln\left[\frac{(a+x)(b+x)}{(a-x)(b-x)}\right]$$

$$\Rightarrow \ln y = \frac{1}{3}\ln(a+x) + \frac{1}{3}\ln(b+x) - \frac{1}{3}\ln(a-x) - \frac{1}{3}\ln(b-x).$$

Next, one can differentiate both sides of the equation with respect to x and we get

$$\overbrace{\frac{1}{y}\frac{dy}{dx}}^{\frac{d(\ln y)}{dy}\frac{dy}{dx}} = \frac{1}{3}\overbrace{\left(\frac{1}{a+x}\right)}^{\frac{d\ln(a+x)}{d(a+x)}\frac{d(a+x)}{dx}} + \frac{1}{3}\overbrace{\left(\frac{1}{b+x}\right)}^{\frac{d\ln(b+x)}{d(b+x)}\frac{d(b+x)}{dx}} + \frac{1}{3}\overbrace{\left(\frac{1}{a-x}\right)}^{\frac{d\ln(a-x)}{d(a-x)}\frac{d(a-x)}{dx}} + \frac{1}{3}\overbrace{\left(\frac{1}{b-x}\right)}^{\frac{d\ln(b-x)}{d(b-x)}\frac{d(b-x)}{dx}}$$

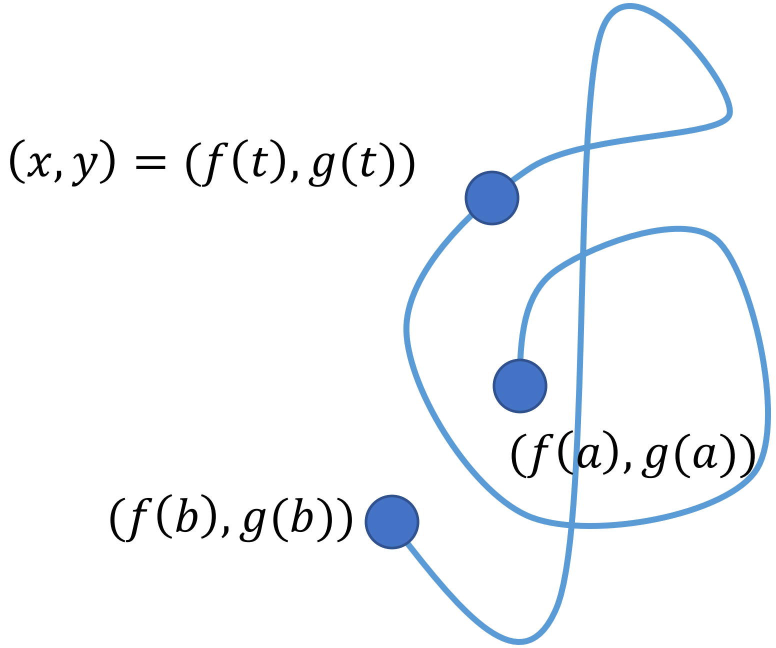
$$\Rightarrow \frac{dy}{dx} = \frac{1}{3}\sqrt[3]{\frac{(a+x)(b+x)}{(a-x)(b-x)}}\left(\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{a-x}+\frac{1}{b-x}\right).$$

Parametric Equations

In Physics, we usually describe the motion of a moving particle by the following pair of equations:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in [a, b]$$

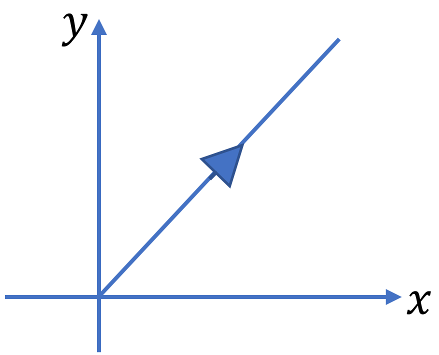
where t is a parameter (which may be interpreted as time) and $f(t)$, $g(t)$ are some functions of t. This pair of equation is called parametric equations.



Some Example of Parametric Equation

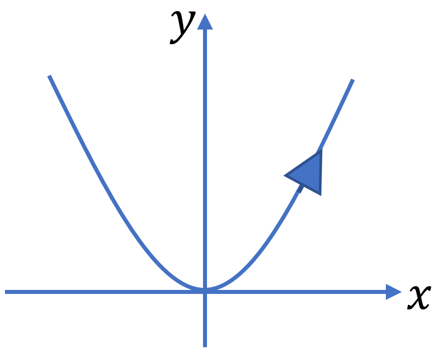
1. (Straight Line: $y = x$)

$$\begin{cases} x(t) = t \\ y(t) = t \end{cases}, \qquad t \in [0, \infty)$$



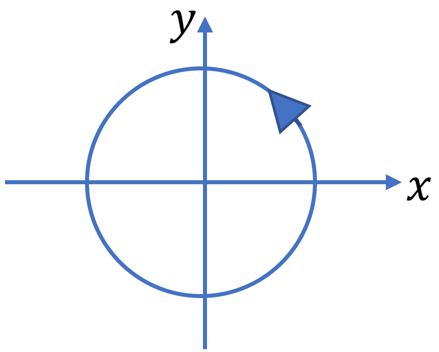
2. (Parabola: $y = x^2$)

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}, \qquad t \in (-\infty, \infty)$$



3. (Circle: $x^2 + y^2 = 1$)

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}, \qquad t \in [0, \infty)$$



One may be interested in knowing the direction of the trace of the curve. To investigate this, we need to find its derivative $\frac{dy}{dx}$.

The following theorem demonstrates how we obtain this derivative.

Theorem

We let $x = f(t)$ and $y = g(t)$ where $f(t)$ and $g(t)$ are two differentiable functions on some interval. Provided that $f'(t) = \frac{dx}{dt} \neq 0$, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$

Example 12

Find $\frac{dy}{dx}$ for each of the following parametric equation

(a) $\begin{cases} x(t) = t^4 + 3 \\ y(t) = 2t^5 + 4t - 2 \end{cases}$ $for\ t \geq 0.$

(b) $\begin{cases} x(t) = e^t - t^2 \\ y(t) = \sqrt{t} + e^{t^2} \end{cases}.$

☺ Solution:

$$(a)\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \overset{\frac{d}{dx}x^a = ax^{a-1}}{\cong} \frac{10t^4 + 4}{4t^3} = \frac{10}{4}t + \frac{4}{4t^3} = \frac{5}{2}t + \frac{1}{t^3}.$$

$$(b)\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2}t^{-\frac{1}{2}} + \overbrace{2te^{t^2}}^{\frac{de^{t^2}}{dt^2}\frac{dt^2}{dt}}}{e^t - 2t}.$$

Higher order derivatives

Given a differentiable function $f(x)$, we can obtain its derivative $f'(x)$.

Suppose that the function $f'(x)$ is also differentiable function, then one can differentiate this function and obtain another derivative $f'(x)$. This derivative is called second-order derivative of $f(x)$ since it is obtained by differentiating the function $f(x)$ twice.

$$\boxed{f(x) = x^3} \xrightarrow{\dfrac{d}{dx}} \underset{\text{First derivative}}{\boxed{f'(x) = 3x^2}} \xrightarrow{\dfrac{d}{dx}} \underset{\text{Second derivative}}{\boxed{f''(x) = (f'(x))' = 6x}}$$

In general, when a function is being differentiated for n times (provided that it is feasible), the function obtained is called nth-order derivative of $f(x)$.

Definition (Higher order derivatives)

The nth order derivative of a function $f(x)$, denoted by $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$, is defined as the function which is obtained by differentiating $f(x)$ with respect to x for n times:

$$f^{(n)}(x) = \overbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\cdots\left(\frac{d}{dx}f(x)\right)\right)\right)}^{n\ times}.$$

Remark:

- We say $f(x)$ is n-time differentiable if nth order derivative $f^{(n)}(x)$ exists.
- If $f^{(n)}(x)$ exists, then the lower order derivatives $f^{(n-1)}, f^{(n-2)}, \dots, f^{(2)}, f'$ also exist.
- Here, we take $f^{(0)}(x) = f(x)$.

Example 13

Compute $\frac{d^2}{dx^2}\cos(x^2+1)$.

☺ Solution:

$$\frac{d}{dx}\cos(x^2+1) = \frac{d\cos(x^2+1)}{d(x^2+1)}\frac{d(x^2+1)}{dx} = \cdots = -2x\sin(x^2+1)\,,$$

$$\frac{d^2}{dx^2}\cos(x^2+1) = \frac{d}{dx}(-2x\sin(x^2+1))$$

$$= -2\left[\sin(x^2+1)\frac{dx}{dx} + x\frac{d[\sin(x^2+1)]}{dx}\right]$$

$$= -2\left[\sin(x^2+1) + x\frac{d[\sin(x^2+1)]}{d(x^2+1)}\frac{d(x^2+1)}{dx}\right]$$

$$= -2\sin(x^2+1) - 4x^2\cos(x^2+1)\,.$$

Example 14

Compute $\frac{d^4}{dx^4}(x^3 \cos x)$.

☺ Solution:

$$\frac{d}{dx} x^3 \cos x = \cos x \frac{d}{dx} x^3 + x^3 \frac{d}{dx} \cos x = \cdots = 3x^2 \cos x - x^3 \sin x\,.$$

$$\frac{d^2}{dx^2} x^3 \cos x = \frac{d}{dx}(3x^2 \cos x - x^3 \sin x)$$
$$= 3\left(\cos x \frac{d}{dx} x^2 + x^2 \frac{d}{dx} \cos x\right) - \left(\sin x \frac{d}{dx} x^3 + x^3 \frac{d}{dx} \sin x\right)$$
$$= 6x \cos x - 3x^2 \sin x - 3x^2 \sin x - x^3 \cos x$$
$$= 6x \cos x - 6x^2 \sin x - x^3 \cos x\,.$$

$$\frac{d^3}{dx^3}x^3\cos x = \frac{d}{dx}(6x\cos x - 6x^2\sin x - x^3\cos x)$$

$$= 6\left(\cos x\frac{d}{dx}x + x\frac{d}{dx}\cos x\right) - 6\left(\sin x\frac{d}{dx}x^2 + x^2\frac{d}{dx}\sin x\right) - \left(\cos x\frac{d}{dx}x^3 + x^3\frac{d}{dx}\cos x\right)$$

$$= 6\cos x - 6x\sin x - 12x\sin x - 6x^2\cos x - 3x^2\cos x + x^3\sin x$$

$$= 6\cos x - 18x\sin x - 9x^2\cos x + x^3\sin x\,.$$

$$\frac{d^4}{dx^4}x^3\cos x = \frac{d}{dx}(6\cos x - 18x\sin x - 9x^2\cos x + x^3\sin x)$$

$$= \cdots = -24\sin x - 36x\cos x + 12x^2\sin x + x^3\cos x\,.$$

Example 15

(a) Let $f(x) = e^{ax}$, find $\frac{df}{dx}, \frac{d^2 f}{dx^2}, \frac{d^3 f}{dx^3}$.

(b) Using (a), try to guess the general form of $\frac{d^n f}{dx^n}$, where n is non-negative integer.

☺ Solution:

(a) Using Chain Rule, we have

$$\frac{df}{dx} = \frac{d(e^{ax})}{dx} = \frac{d(e^{ax})}{d(ax)}\frac{d(ax)}{dx} \overset{\frac{d}{dy}e^y = e^y}{\triangleq} e^{ax}(a) = ae^{ax}.$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(ae^{ax}) = a\frac{d}{dx}e^{ax} = a(ae^{ax}) = a^2 e^{ax}.$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) = \frac{d}{dx}(a^2 e^{ax}) = a^2\frac{d}{dx}e^{ax} = a^2(ae^{ax}) = a^3 e^{ax}.$$

(b) By observing the pattern in (a), we guess that $\frac{d^n f}{dx^n} = a^n e^{ax}$.

Example 16

Let $f(x) = \ln x$, find $\frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \frac{d^4f}{dx^4}$.

Hence, guess the general form of $\frac{d^nf}{dx^n}$, where $n \geq 1$.

☺ Solution:

$$\frac{df}{dx} = \frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) \overset{\frac{d}{dx}x^a = ax^{a-1}}{=} -x^{-2}.$$

$$\frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) = -\frac{d}{dx}(x^{-2}) = -(-2)x^{-3} = 2x^{-3}.$$

$$\frac{d^4f}{dx^4} = \frac{d}{dx}\left(\frac{d^3f}{dx^3}\right) = \frac{d}{dx}(2x^{-3}) = 2(-3)x^{-4} = -(2)(3)x^{-4} = -6x^{-4}.$$

By observing the pattern above, we guess that $\frac{d^n}{dx^n}\ln x = (-1)^{n-1}(n-1)!\,x^{-n}$.

Example 17

Find $\frac{d^n}{dx^n}\cos(ax+b)$ where a, b are constants.

☺ Solution:

Note that

$$\frac{d}{dx}\cos(ax+b) = \frac{d(\cos(ax+b))}{d(ax+b)}\frac{d(ax+b)}{dx} \overset{\frac{d}{dy}\cos y = -\sin y}{=} -\sin(ax+b)\,(a) = -a\sin(ax+b)$$

$$= a\cos\left(\frac{\pi}{2}+ax+b\right).$$

$$\frac{d^2}{dx^2}\cos(ax+b) = \frac{d}{dx}\left(\frac{d}{dx}\cos(ax+b)\right) = \frac{d}{dx}(-a\sin(ax+b)) = -a\frac{d(\sin(ax+b))}{d(ax+b)}\frac{d(ax+b)}{dx}$$

$$= -a\cos(ax+b)\,(a) = -a^2\cos(ax+b) = a^2\cos(\pi+ax+b).$$

$$\frac{d^3}{dx^3}\cos(ax+b) = \frac{d}{dx}\left(\frac{d^2}{dx^2}\cos(ax+b)\right) = \frac{d}{dx}\left(-a^2\cos(ax+b)\right) = -a^2\frac{d(\cos(ax+b))}{d(ax+b)}\frac{d(ax+b)}{dx}$$

$$= a^2\sin(ax+b)\,(a) = a^3\sin(ax+b) = a^3\cos\left(\frac{3\pi}{2}+ax+b\right).$$

$$\frac{d^4}{dx^4}\cos(ax+b) = \frac{d}{dx}\left(\frac{d^3}{dx^3}\cos(ax+b)\right) = \frac{d}{dx}\left(a^3\sin(ax+b)\right) = a^3\frac{d(\sin(ax+b))}{d(ax+b)}\frac{d(ax+b)}{dx}$$

$$= a^3\cos(ax+b)\,(a) = a^4\cos(ax+b) = a^4\cos(2\pi+ax+b).$$

By observing the above pattern, we guess that

$$\frac{d^n}{dx^n}\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2}+ax+b\right).$$

Example 18 (A harder example)

(a) Show that

$$\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)=\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}.$$

(b) Hence, compute

$$\frac{d^n}{dx^n}\frac{x^2+9x+10}{(x+1)(x-1)(2x+3)}.$$

(Hint: Using method partial fraction).

☺ Solution:

(a) Note that

$$\frac{d}{dx}\left(\frac{1}{ax+b}\right)=\frac{d}{dx}(ax+b)^{-1}=\frac{d(ax+b)^{-1}}{d(ax+b)}\frac{d(ax+b)}{dx}=-(ax+b)^{-2}(a)=-a(ax+b)^{-2}.$$

$$\frac{d^2}{dx^2}\left(\frac{1}{ax+b}\right)=\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{1}{ax+b}\right)\right)=\frac{d}{dx}(-a(ax+b)^{-2})=-a\frac{d(ax+b)^{-2}}{d(ax+b)}\frac{d(ax+b)}{dx}$$
$$=-a(-2)(ax+b)^{-3}(a)=2a^2(ax+b)^{-3}.$$

$$\frac{d^3}{dx^3}\left(\frac{1}{ax+b}\right)=\frac{d}{dx}\left(\frac{d^2}{dx^2}\left(\frac{1}{ax+b}\right)\right)=\frac{d}{dx}[2a^2(ax+b)^{-3}]=2a^2\frac{d(ax+b)^{-3}}{d(ax+b)}\frac{d(ax+b)}{dx}$$
$$=2a^2(-3)(ax+b)^{-4}(a)=-(3)2a^3(ax+b)^{-4}.$$

By observing the pattern, we conclude that

$$\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)=(-1)^n(n\times(n-1)\times\cdots\times 2\times 1)a^n(ax+b)^{-(n+1)}=\frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}.$$

(b) According to the hint, one can rewrite the expression as

$$\frac{x^2+9x+10}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3}$$

$$\Rightarrow A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1) = x^2+9x+10.$$

To obtain the unknowns A, B and C, we use substitution method

- Put $x = 1$ (so that $x - 1 = 0$), then

$$A(0)(3) + B(2)(5) + C(2)(0) = 20 \Rightarrow B = 2.$$

- Put $x = -1$ (so that $x + 1 = 0$), then

$$A(-2)(1) + B(0)(1) + C(0)(-2) = 2 \Rightarrow A = -1.$$

- Put $x = -\frac{3}{2}$ (so that $2x + 3 = 0$), then

$$A\left(-\frac{5}{2}\right)(0) + B\left(-\frac{1}{2}\right)(0) + C\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right) = -\frac{5}{4} \Rightarrow C = -1$$

$$\Rightarrow \frac{x^2+9x+10}{(x+1)(x-1)(2x+3)} = -\frac{1}{x+1} + \frac{2}{x-1} - \frac{1}{2x+3}.$$

Using the result of (a), we finally get

$$\frac{d^n}{dx^n}\left[\frac{x^2+9x+10}{(x+1)(x-1)(2x+3)}\right]$$

$$= -\frac{d^n}{dx^n}\left(\frac{1}{x+1}\right) + 2\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) - \frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right)$$

$$= -\frac{(-1)^n n!}{(x+1)^{n+1}} + 2\frac{(-1)^n n!}{(x+2)^{n+1}} - \frac{(-1)^n n!\,2^n}{(2x+3)^{n+1}}.$$

Here, we use the method of partial fraction (see Chapter 4 for details) to decompose a rational fraction (involving quotient of polynomial into the sum with simpler denominator which can be differentiated easily).

Example 19 (Use of product-to-sum formula)

Compute

$$\frac{d^n}{dx^n}\sin^2 3x \cos 5x\,.$$

☺ Solution:

Using product-to-sum formula, one can rewrite the function as

$$\sin^2 3x \cos 5x = (\sin 3x \sin 3x)\cos 5x = -\frac{1}{2}[\cos(3x+3x) - \cos(3x-3x)]\cos 5x$$

$$= -\frac{1}{2}\cos 6x \cos 5x + \frac{1}{2}\cos 5x = -\frac{1}{4}[\cos(6x+5x) + \cos(6x-5x)] + \frac{1}{2}\cos 5x$$

$$= -\frac{1}{4}\cos 11x - \frac{1}{4}\cos x + \frac{1}{2}\cos 5x\,.$$

Note that

$$\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right),$$

we then have

$$\frac{d^n}{dx^n}\sin^2 3x \cos 5x = -\frac{1}{4}\frac{d^n}{dx^n}\cos 11x - \frac{1}{4}\frac{d^n}{dx^n}\cos x + \frac{1}{2}\frac{d^n}{dx^n}\cos 5x$$

$$= -\frac{11^n}{4}\cos\left(\frac{n\pi}{2} + 11x\right) - \frac{1}{4}\cos\left(\frac{n\pi}{2} + x\right) + \frac{5^n}{2}\cos\left(\frac{n\pi}{2} + 5x\right).$$

Remark of this Example:

Although one can obtain the derivative using product rule, the derivation is tedious, the product-to-sum formula allows us to separate the product of trigonometric functions into the sum of trigonometric functions so that we can avoid to use product rule. In differentiation, "doing sum" is always better than "doing product".

Higher-order derivative of parametric equation

Suppose that x and y are given by the following parametric equation:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}.$$

One can find the first derivative using the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}.\ \left(let\ z = h(t) = \frac{g'(t)}{f'(t)}\right)$$

To obtain higher-order derivate $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, we can repeat the above process:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}z = \frac{dz}{dx} \underbrace{\overset{\substack{z=h(t)\\x=x(t)}}{=}} \frac{dz/dt}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} = w(t).$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}w(t) = \frac{dw/dt}{dx/dt}.$$

Example 20

It is given that $x(t) = 2t - t^2$ and $y(t) = t^3$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$.

☺ Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{d}{dx}t^3}{\frac{d}{dx}(2t - t^2)} = \frac{3t^2}{2 - 2t}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t^2}{2 - 2t}\right) \overset{\substack{z=\frac{3t^2}{2-2t} \\ x=x(t)}}{\underset{\smile}{=}} \frac{\frac{d}{dt}\left(\frac{3t^2}{2 - 2t}\right)}{dx/dt} = \frac{\frac{12t - 6t^2}{(2 - 2t)^2}}{2 - 2t} = \frac{12t - 6t^2}{(2 - 2t)^3}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{12t - 6t^2}{(2 - 2t)^3}\right) \overset{\substack{z=\frac{12t-6t^2}{(2-2t)^3} \\ x=x(t)}}{\underset{\smile}{=}} \frac{\frac{d}{dt}\left(\frac{12t - 6t^2}{(2 - 2t)^3}\right)}{dx/dt} = \cdots = \frac{6(4 + 4t - 2t^2)}{(2 - 2t)^5}.$$

Leibnitz' Rule – A shortcut for calculating the higher order derivative

- In Example 19, we calculate the higher order derivative of the form $\frac{d^n}{dx^n} f(x)g(x)$. Although we can easily calculate the derivatives by using product rule repeatedly, the calculation is lengthy.
- One would like to ask whether there is any shortcut of doing this. Luckily, there is a theorem which provides a general formula of the $\frac{d^n}{dx^n} f(x)g(x)$. This is called Leibnitz' Rule.

Leibnitz' Rule

Let $f(x)$ and $g(x)$ be two n-times differentiable functions. Then the nth derivative of the product $f(x)g(x)$ is given by

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{r=0}^{n} C_r^n \underbrace{\frac{d^r}{dx^r} f(x)}_{\substack{differentiate \\ r\ times}} \underbrace{\frac{d^{n-r}}{dx^{n-r}} g(x)}_{\substack{differentiate \\ (n-r)\ times}} = \sum_{r=0}^{n} C_r^n f^{(r)}(x) g^{(n-r)}(x).$$

Example 21

Compute

$$\frac{d^4}{dx^4}(1-x^2)e^{2x}.$$

☺ Solution:

Step 1: Write down the "general" formula using Leibnitz's rule

We let $f(x) = 1 - x^2$ and $g(x) = e^{2x}$. From Leibnitz's rule, we have

$$\frac{d^4}{dx^4}(1-x^2)e^{2x} = \frac{d^4}{dx^4}f(x)g(x) \overset{n=4}{\cong} \sum_{r=0}^{4} C\,{}_{r}^{4} f^{(r)}g^{(4-r)}$$

$$= C\,{}_0^4 f^{(0)}g^{(4)} + C\,{}_1^4 f^{(1)}g^{(3)} + C\,{}_2^4 f^{(2)}g^{(2)} + C\,{}_3^4 f^{(3)}g^{(1)} + C\,{}_4^4 f^{(4)}g^{(0)}.$$

Step 2: Compute all numbers required in the formula

By direct calculation, we get

r	0	1	2	3	4
C^4_r	$C^4_0 = 1$	$C^4_1 = 4$	$C^4_2 = 6$	$C^4_3 = 4$	$C^4_4 = 1$
$f^{(r)}(x)$	$1 - x^2$	$-2x$	-2	0	0
$g^{(r)}(x)$	e^{2x}	$2e^{2x}$	$2(2e^{2x}) = 4e^{2x}$	$4(2e^{2x}) = 8e^{2x}$	$8(2e^{2x}) = 16e^{2x}$

Step 3: Substitute everything into the formula

$$\frac{d^4}{dx^4}(1 - x^2)e^{2x}$$

$$= C^4_0 f^{(0)} g^{(4)} + C^4_1 f^{(1)} g^{(3)} + C^4_2 f^{(2)} g^{(2)} + C^4_3 f^{(3)} g^{(1)} + C^4_4 f^{(4)} g^{(0)}$$

$$= (1 - x^2)(16e^{2x}) + 4(-2x)(8e^{2x}) + 6(-2)(4e^{2x}) + 4(0)(2e^{2x}) + 1(0)(e^{2x})$$

$$= (-16x^2 - 64x - 32)e^{2x}.$$

Example 22

Compute

$$\frac{d^3}{dx^3}\cos x \ln x\,.$$

☺ Solution:

Step 1: Write down the "general" formula using Leibnitz's rule

We let $f(x) = \cos x$ and $g(x) = \ln x$. From Leibnitz's rule, we have

$$\frac{d^3}{dx^3}\cos x \ln x = \frac{d^3}{dx^3} f(x)g(x) \overset{n=3}{\cong} \sum_{r=0}^{3} C\,{}_{r}^{3}\, f^{(r)} g^{(3-r)}$$

$$= C\,{}_{0}^{3} f^{(0)} g^{(3)} + C\,{}_{1}^{3} f^{(1)} g^{(2)} + C\,{}_{2}^{3} f^{(2)} g^{(1)} + C\,{}_{3}^{3} f^{(3)} g^{(0)}.$$

Step 2: Compute all numbers required in the formula

By direct calculation, we get

r	0	1	2	3
C^3_r	$C^3_0 = 1$	$C^3_1 = 3$	$C^3_2 = 3$	$C^3_3 = 1$
$f^{(r)}(x)$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$g^{(r)}(x)$	$\ln x$	$\dfrac{1}{x} = x^{-1}$	$-x^{-2}$	$-(-2x^{-3}) = 2x^{-3}$

Step 3: Substitute everything into the formula

$$\frac{d^3}{dx^3}\cos x \ln x = C^3_0 f^{(0)} g^{(3)} + C^3_1 f^{(1)} g^{(2)} + C^3_2 f^{(2)} g^{(1)} + C^3_3 f^{(3)} g^{(0)}$$

$$= \cos x\,(2x^{-3}) + 3(-\sin x)(-x^{-2}) + 3(-\cos x)(x^{-1}) + \sin x\,(\ln x)$$

$$= (2x^{-3} - 3x^{-1})\cos x + (\ln x - 3x^{-2})\sin x.$$

Example 23

For any positive integer n, compute

$$\frac{d^n}{dx^n} x^2 \cos 3x\,.$$

☺ Solution:

Step 1: Write down the "general" formula using Leibnitz's rule

We let $f(x) = x^2$ and $g(x) = \cos 3x$. From Leibnitz's rule, we have

$$\frac{d^n}{dx^n} x^2 \cos 3x = \frac{d^n}{dx^n} f(x)g(x) = \sum_{r=0}^{n} C_r^n f^{(r)} g^{(n-r)}$$

$$= C_0^n f^{(0)} g^{(n)} + C_1^n f^{(1)} g^{(n-1)} + C_2^n f^{(2)} g^{(n-2)} + C_3^n f^{(3)} g^{(n-3)} + +C_4^n f^{(4)} g^{(n-4)} + \cdots + C_n^n f^{(n)} g^{(0)}.$$

Step 2: Compute all numbers required in the formula

Note that for any positive integer k, we have

$$g^{(k)}(x) = \frac{d^k}{dx^k}\cos 3x = 3^k \cos\left(\frac{k\pi}{2} + 3x\right).$$

r	0	1	2	3	…	n
$f^{(r)}(x)$	x^2	$2x$	2	0	…	0

Step 3: Substitute everything into the formula

$$\frac{d^n}{dx^n} x^2 \cos 3x$$

$$= C^n_0 f^{(0)} g^{(n)} + C^n_1 f^{(1)} g^{(n-1)} + C^n_2 f^{(2)} g^{(n-2)} + \overbrace{C^n_3 f^{(3)} g^{(n-3)} + C^n_4 f^{(4)} g^{(n-4)} + \cdots + C^n_n f^{(n)} g^{(0)}}^{=0\ since\ f^{(k)}(x)=0\ for\ k\geq 3}$$

$$= C^n_0 x^2 3^n \cos\left(\frac{n\pi}{2} + 3x\right) + C^n_1 (2x)\left[3^{n-1}\cos\left(\frac{(n-1)\pi}{2} + 3x\right)\right] + C^n_2 (2)\left[3^{n-2}\cos\left(\frac{(n-2)\pi}{2} + 3x\right)\right].$$

Recall that $C^n_r = \frac{n!}{r!(n-r)!}$, so we have

$$C^n_0 = 1, \quad C^n_1 = n, \quad C^n_2 = \frac{n(n-1)}{2}.$$

$$\frac{d^n}{dx^n} x^2 \cos 3x$$

$$= x^2 3^n \cos\left(\frac{n\pi}{2} + 3x\right) + n(2x)\left[3^{n-1}\cos\left(\frac{(n-1)\pi}{2} + 3x\right)\right] + n(n-1)\left[3^{n-2}\cos\left(\frac{(n-2)\pi}{2} + 3x\right)\right].$$

Example 24 (Harder Example)

Let $f(x) = \tan^{-1} x$

(a) Show that $(1+x^2)f''(x) + 2xf'(x) = 0$.

(b) Let n be a positive integer

(i) Using Leibnitz's rule, show that

$$(1+x^2)f^{(n+2)}(x) + 2(n+1)xf^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0.$$

(ii) Hence, find $f^{(3)}(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$.

☺ Solution:

(a) Using direct differentiation, we get

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2}.$$

Then

$$(1+x^2)f''(x) + 2xf'(x) = (1+x^2)\left(-\frac{2x}{(1+x^2)^2}\right) + 2x\left(\frac{1}{1+x^2}\right) = 0.$$

(b) To obtain the equation (i), one has to differentiate the equation in (a) with respect to x for n times:

$$\frac{d^n}{dx^n}(1+x^2)f''(x)+\frac{d^n}{dx^n}[2xf'(x)]=\frac{d^n}{dx^n}0=0\ \ldots\ldots(*)$$

We proceed to compute the two derivatives on L.H.S.

Using Leibnitz's Rule, we get

$$\frac{d^n}{dx^n}(1+x^2)f''(x)=\sum_{r=0}^{n} C^n_r\frac{d^r}{dx^r}(1+x^2)\frac{d^{n-r}}{dx^{n-r}}f''(x)$$

$$=C^n_0\overbrace{\frac{d^0}{dx^0}(1+x^2)}^{1+x^2}\overbrace{\frac{d^n}{dx^n}f''(x)}^{f^{(n+2)}(x)}+C^n_1\overbrace{\frac{d^1}{dx^1}(1+x^2)}^{2x}\overbrace{\frac{d^{n-1}}{dx^{n-1}}f''(x)}^{f^{(n+1)}(x)}+C^n_2\overbrace{\frac{d^2}{dx^2}(1+x^2)}^{2}\overbrace{\frac{d^{n-2}}{dx^{n-2}}f''(x)}^{f^{(n)}(x)}$$

$$+C^n_3\overbrace{\frac{d^3}{dx^3}(1+x^2)}^{0}\overbrace{\frac{d^{n-3}}{dx^{n-3}}f''(x)}^{f^{(n-1)}(x)}+\cdots+C^n_n\overbrace{\frac{d^n}{dx^n}(1+x^2)}^{=0}\overbrace{\frac{d^0}{dx^0}f''(x)}^{f^{(2)}(x)}$$

$$= (1+x^2)f^{(n+2)}(x) + n(2x)f^{(n+1)}(x) + \frac{n(n-1)}{2}2f^{(n)}(x)$$

$$= (1+x^2)f^{(n+2)}(x) + 2nxf^{(n+1)}(x) + n(n-1)f^{(n)}(x).$$

Similarly, one can find that

$$\frac{d^n}{dx^n}[2xf'(x)] = 2xf^{(n+1)}(x) + 2nf^{(n)}(x).$$

Substitute the formula obtained into the equation $(*)$, we get

$$(1+x^2)f^{(n+2)}(x) + 2nxf^{(n+1)}(x) + n(n-1)f^{(n)}(x) + 2xf^{(n+1)}(x) + 2nf^{(n)}(x) = 0$$

$$\Rightarrow (1+x^2)f^{(n+2)}(x) + (2n+2)xf^{(n+1)}(x) + (n^2+n)f^{(n)}(x) = 0.$$

(b) (ii)

To obtain the derivatives $f^{(3)}(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$, one can use the equation derived in (b)(i).

We first put $n = 1$ into the equation, we get

$$(1+x^2)f^{(3)}(x) + 4x\,\overbrace{f^{(2)}(x)}^{=\frac{-2x}{(1+x^2)^2}} + 2\,\overbrace{f^{(1)}(x)}^{=\frac{1}{(1+x^2)}} = 0 \Rightarrow f^{(3)}(x) = \frac{6x^2-2}{(1+x^2)^3}.$$

Put $n = 2$ into the equation, we get

$$(1+x^2)f^{(4)}(x) + 6x\,\overbrace{f^{(3)}(x)}^{=\frac{6x^2-2}{(1+x^2)^3}} + 6\,\overbrace{f^{(2)}(x)}^{=\frac{-2x}{(1+x^2)^2}} = 0 \Rightarrow f^{(4)}(x) = \frac{24x-24x^3}{(1+x^2)^4}.$$

Finally, we put $n = 3$ into the equation, we get

$$(1+x^2)f^{(5)}(x) + 8x\,\overbrace{f^{(4)}(x)}^{=\frac{24x-24x^3}{(1+x^2)^4}} + 12\,\overbrace{f^{(3)}(x)}^{=\frac{6x^2-2}{(1+x^2)^3}} = 0 \Rightarrow f^{(5)}(x) = \frac{120x^4-240x^2+24}{(1+x^2)^5}.$$

In general, to find the higher-order derivatives of a complicated function, say $f(x) = \sin(\ln(x+1))$, one can do this by following procedure:

Step 1: Compute the first derivative and second derivative $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ and use them to derive a differential equation

$$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0.$$

Step 2: We obtain the *general equation* by differentiating the equation in Step 1 with respect to x for n-times (by Leibnitz's Rule), i.e.

$$\frac{d^n}{dx^n}[(1+x)^2 f''(x) + (1+x)f'(x) + f(x)] = \frac{d^n}{dx^n}0$$

$$\Rightarrow (1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+1)f^{(n)}(x) = 0.$$

Step 3: Find $f^{(3)}(x), f^{(4)}(x), f^{(5)}(x)$, ... by putting $n = 1, 2, 3, \ldots$ into the general equation.

Remark about using Leibnitz's Rule

- Although the Leibnitz's rule provide a useful formula in finding the higher order derivative, it is only efficient in the case when the function is a product of some elementary functions and the general formula of nth derivative of these elementary functions are available, e.g.

$$f(x) = e^x \sin x , \qquad g(x) = x^2 \sin x .$$

- In some cases, method of partial fractions or product-to-sum formula may be more useful than Leibnitz's rule when finding the derivatives such as

$$\frac{d}{dx} \sin 3x \cos 4x , \qquad \frac{d}{dx} \frac{2}{(x-1)(x+3)(2x-1)} .$$