MA1200 – Calc & Basic Linear Algebra I

Mid-term Test for CE1, CF1, CG1 and CH1 Solution

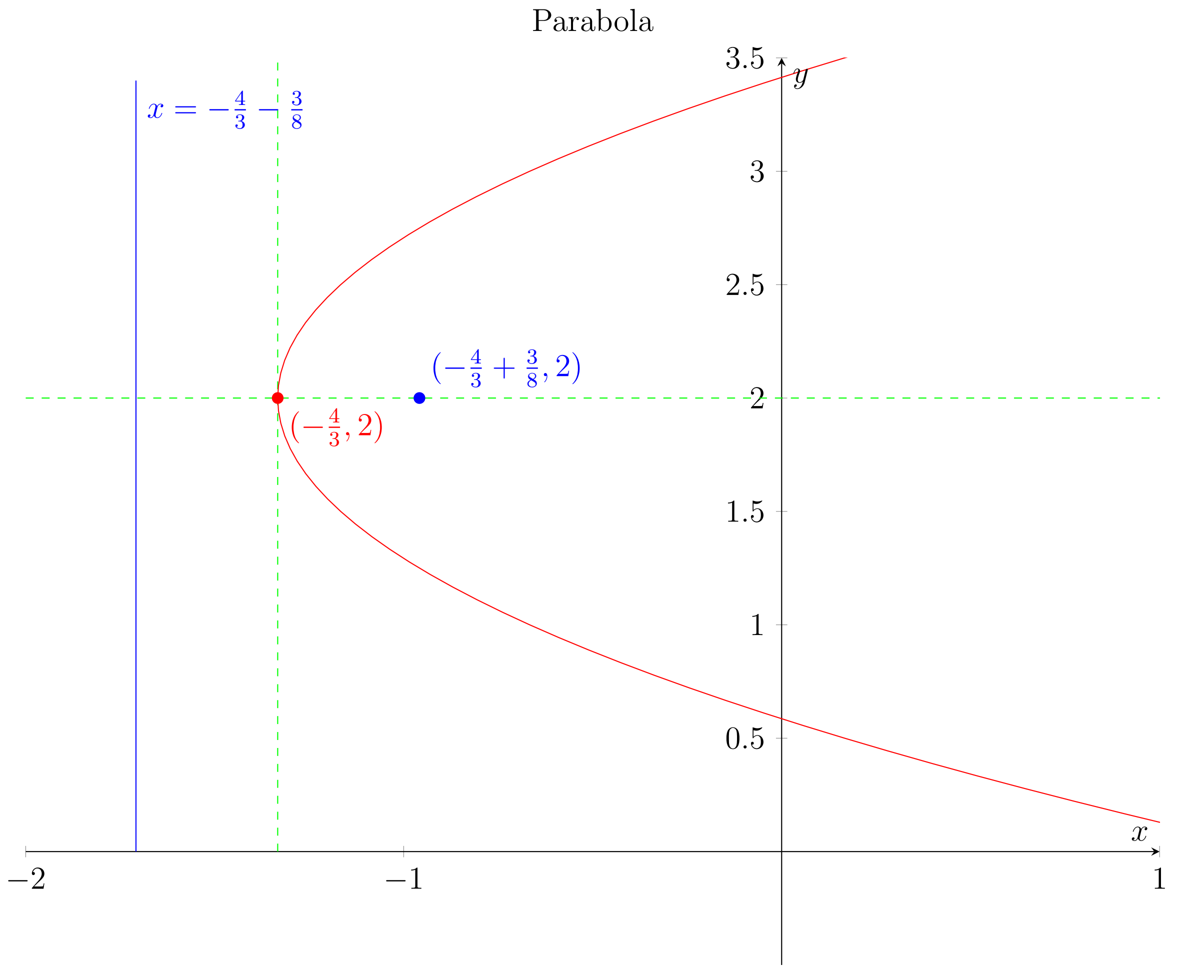
---

1. *Solution.*

   (a) $3x + 8y - 2y^2 - 4 = 0 \implies 4\left(\frac{3}{8}\right)\left(x + \frac{4}{3}\right) = (y-2)^2.$

   (b) Vertex:$(-\frac{4}{3}, 2)$.

   (c) Directrix:$x = -\frac{4}{3} - \frac{3}{8}$

   (d)

∎

2. Express the following rational functions in partial fraction.

Solution.

$$\frac{3x^3+5x^2+2x-10}{x^4-2x^2+1} = \frac{3x^3+5x^2+2x-10}{(x+1)^2(x-1)^2}$$

$$\text{Let } \frac{3x^3+5x^2+2x-10}{(x+1)^2(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

$$3x^3+5x^2+2x-10 = A(x+1)(x-1)^2 + B(x-1)^2 + C(x+1)^2(x-1) + D(x+1)^2 \qquad (*)$$

$$(*) \text{ with } x=1: \qquad 0 = 4D \implies D = 0$$

$$(*) \text{ with } x=-1: \qquad -10 = 4B \implies B = -\frac{5}{2}$$

$$(*) \text{ with } x=0: \qquad -10 = A+B-C \implies A-C = -\frac{15}{2}$$

$$(*) \text{ with } x=2: \qquad 38 = 3A+B+9C \implies 3A+9C = \frac{81}{2}$$

$$3(A-C)-(3A+9C) = -\frac{45}{2} - \frac{81}{2}$$

$$-12C = -\frac{126}{2} \implies C = \frac{21}{4}, A = -\frac{9}{4}$$

$$\text{Hence, } \frac{3x^3+5x^2+2x-10}{x^4-2x^2+1} = -\frac{9}{4(x+1)} - \frac{5}{2(x+1)^2} + \frac{21}{4(x-1)}$$

■

3. Solve the equation:

Solution.

$$3\sin^2 x + 2\cos x = 2$$

$$3(1-\cos^2 x) + 2\cos x = 2$$

$$0 = 3\cos^2 x - 2\cos x - 1$$

$$0 = 3y^2 - 2y - 1 \qquad \text{let } y = \cos x$$

$$0 = (y-1)(3y+1)$$

$$y = 1 \implies \cos x = 1 \implies x = 2n\pi \qquad n \in \mathbf{Z}$$

$$y = -\frac{1}{3} \implies \cos x = -\frac{1}{3} \implies x = \pm\cos^{-1}\left(-\frac{1}{3}\right) + 2n\pi \qquad n \in \mathbf{Z}$$

■

4. Solve the equation: $3^x = 9^{x^2-1}$.

Solution.

$$3^x = 9^{x^2-1} \Rightarrow 3^x = 3^{2x^2-2} \Rightarrow x = 2x^2 - 2$$

We need to solve quadratic function using formula

$$2x^2 - x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{17}}{4}.$$

■

5. *Solution.*

[a]

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-\sqrt{1-x}}{x^2+3x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-\sqrt{1-x}}{x^2+3x} \times \frac{\sqrt{x+1}+\sqrt{1-x}}{\sqrt{x+1}+\sqrt{1-x}} \\
&= \lim_{x\to 0} \frac{x+1-(1-x)}{x(x+3)(\sqrt{x+1}+\sqrt{1-x})} \\
&= \lim_{x\to 0} \frac{2}{(x+3)(\sqrt{x+1}+\sqrt{1-x})} \\
&= \frac{2}{3(1+1)} = \frac{1}{3}.
\end{aligned}$$

[b]

$$\begin{aligned}
\lim_{x\to 0}(\sin 5x)\cdot(\cot 3x) &= \lim_{x\to 0} 5x\frac{\sin 5x}{5x}\cdot\frac{\cos 3x}{\sin 3x} \\
&= \lim_{x\to 0} 5x\frac{\sin 5x}{5x}\cdot\frac{1}{3x}\frac{3x}{\sin 3x}\cdot\cos 3x = \frac{5}{3}\cdot 1\cdot 1\cdot 1 = \frac{5}{3}.
\end{aligned}$$

[c]

$$\begin{aligned}
\lim_{x\to+\infty}(1+\frac{1}{x}-\frac{1}{x-2})^{x^2} &= \lim_{x\to+\infty}\left(1+\frac{x-2-x}{x(x-2)}\right)^{x^2} = \lim_{x\to+\infty}\left(1+\frac{-2}{x(x-2)}\right)^{x(x-2)\cdot\frac{x^2}{x(x-2)}} \\
&= \lim_{x\to+\infty}\left[\left(1+\frac{-2}{x(x-2)}\right)^{x(x-2)}\right]^{\frac{x^2}{x(x-2)}}
\end{aligned}$$

By the fact (L6 Example 29) that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ for any $x \neq 0$, we know that

$$\lim_{x\to+\infty}\left[\left(1+\frac{-2}{x(x-2)}\right)^{x(x-2)}\right] = e^{-2}$$

Combing that $\lim_{x\to+\infty}\frac{x^2}{x(x-2)} = \lim_{x\to+\infty}\frac{1}{1-\frac{2}{x}} = 1$,

$$\lim_{x\to+\infty}(1+\frac{1}{x}-\frac{1}{x-2})^{x^2} = [e^{-2}]^{\lim_{x\to+\infty}\frac{x^2}{x(x-2)}} = e^{-2}.$$

■

6. Let $f(x) = \begin{cases} (\sin(4(x-3)))\cdot(\sin(\frac{3}{x-3})) & \text{if } x < 3; \\ x^3 + a & \text{if } x \geq 3. \end{cases}$ Here a is a real number. Find the value of a such that the function is continuous at $x = 3$. Justify your answer.

Solution.

In order for the function to be continuous at $x = 3$, we need to let $f(3) = 3^3 + a = \lim_{x\to3} f(x)$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} x^3 + a = 3^3 + a.$$

As $-1 \leq \sin(\frac{3}{x-3}) \leq 1$ and $\sin(4(x-3))$ is negative when $x \to 3^-$, so we have $(\sin(4(x-3))) \leq (\sin(4(x-3)))\cdot(\sin(\frac{3}{x-3})) \leq -(\sin(4(x-3)))$. We can show that $\lim_{x\to3^-} -(\sin(4(x-3))) = 0$ and $\lim_{x\to3^-}(\sin(4(x-3))) = 0$. By sandwich theorem,

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(\sin(4(x-3)))\cdot(\sin(\frac{3}{x-3})) = 0$$

Therefore, $3^3 + a = 0 \Rightarrow a = -27$. ■

— *End* —