MA1200 Calculus and Basic Linear Algebra I

Lecture Note 10

Application of Differentiation

In this Chapter, we will discuss some applications of derivatives:

- Approximation of functions – Taylor Series

  (Application of Higher-order derivative)

- Optimization: Finding maximum value and minimum value of a function.

  (Application of first derivative and second derivative)

- Computing Limit: L’ Hopital’s Rule

- Root-finding algorithm: Newton’s method

Approximation of functions – Taylor Series

Question: How do we estimate the value of $e^{2.5}$ and $\cos 2.5$ when there is no calculator?

Let $f(x)$ be the "target" function. One may try to approximate the function using a polynomial since the calculation of polynomial can be done by simple addition, subtraction and multiplication:

$$f(x) \approx c_0 + c_1x + c_2x^2 + c_3x^3 \ldots + c_nx^n.$$

To obtain the coefficients, one can use method of substitution:

(Step 1) Put $x = 0$, then $f(0) = c_0$.

(Step 2) We consider

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1}.$$

Put $x = 0$ again, we then get $f'(0) = c_1$.

(Step 3) We consider

$$f''(x) = 2c_2 + 3(2)c_3x + 4(3)c_4x^2 + \cdots + n(n-1)c_nx^{n-2}.$$

Put $x = 0$, we get $f''(0) = 2c_2 \Rightarrow c_2 = \frac{f''(0)}{2!}$.

(Step 4) We consider

$$f^{(3)}(x) = 3(2)c_3 + 4(3)(2)c_4x + \cdots + n(n-1)(n-2)c_nx^{n-3}.$$

Put $x = 0$, we get $f^{(3)}(0) = 3!\,c_3 \Rightarrow c_3 = \frac{f^{(3)}(0)}{3!}$.

Repeating this process, one can get

$$c_4 = \frac{f^{(4)}(0)}{4!}, \ldots, c_n = \frac{f^{(n)}(0)}{n!}.$$

Hence, we get the following approximation

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \ldots + \frac{f^{(n)}(0)}{n!}x^n.$$

In general, one may consider the following general approximation (perhaps for better computational efficiency):

$$f(x) \approx c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots + c_n(x-a)^n.$$

One can obtain the coefficients c_i using the similar method (put $x = a$ instead of $x = 0$), we obtain

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Question: Is this true?

The following theorem, called Taylor theorem, shows that the above polynomial can give a good approximation to the function $f(x)$ provide that n is sufficiently large.

Taylor Theorem

Let f be n-times differentiable function at point $x = a$ (i.e. the derivatives $f'(a), f''(a), f^{(3)}(a), \dots, f^{(n)}(a)$ exists). Then the function $f(x)$ can be written as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n$$

where c is some number between x and a and the term $R_n = \frac{f^{(n)}(c)}{n!}(x-a)^n$ is called *residual term* or *error term*.

Some insights about Taylor Theorem

- The expression $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n$ is called *Taylor series* of $f(x)$ at $x = a$.

- Taylor theorem aims to approximate a function using a polynomial with suitable coefficients. One can approximate the value of $f(x)$ by computing the polynomial on the R.H.S. (provided that the $f^{(k)}(a)$ are known).

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}}_{polynomial\ (approximation\ of\ f(x))}$$

and the numerical error is given by $R_n = \frac{f^{(n)}(x_0)}{n!}(x-a)^n$.

This error R_n can be made to be small if n is sufficiently large (i.e. we pick more terms in approximating $f(x)$).

- Sometimes, the Taylor series of a function $f(x)$ is expressed using *infinite series* (provided that $f(x)$ can be differentiated as many times as we wish):

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- In general, a can be any real number at which $f(x)$ is differentiable.

  However, a must be chosen so that the EXACT VALUE of $f(a), f'(a), f''(a), \dots, f^{(n)}(a)$ in order to generate an useful formula for estimation.

- Usually, a is chosen to be 0 (you will see why in the next few examples) and the corresponding series is called Maclaurin Series.

Example 1 (Finding $e = e^1$)

Find the Taylor expansion of the function $f(x) = e^x$ at $x = 0$ and hence estimate the value of e.

☺ Solution:

Step 1: Write the Taylor Series of this function

From the Taylor expansion, we have

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

Step 2: Compute $f(0), f'(0), f''(0), \ldots, f^{(n)}(0)$

Since $f(x) = e^x$, it is easy to see that

$$f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x, \ldots . \Rightarrow f^{(n)}(x) = e^x.$$

$$\Rightarrow f(0) = e^0 = 1, f^{(n)}(0) = e^0 = 1 \ \ for\, n = 1, 2, 3, \ldots .$$

Step 3: Substitute

Hence, the Taylor expansion is then given by

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To estimate the value of e, we simply substitute $x = 1$ in the above series:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

One can estimate e by compute the first n-term of the R.H.S.:

n	5	10	20	50
Estimate of e	2.71666666666	2.718281801146385	2.718281828459046	2.718281828459046

(*Note: The exact value of e is 2.718281828459046).

Example 2

Find the Taylor series of the function $f(x) = \cos x$ at $x = 0$. Compute $\cos(0.5)$.

☺ Solution:

Step 1: Write the Taylor Series of this function

From the Taylor expansion, we have

$$\cos x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

Step 2: Compute $f(0), f'(0), f''(0), \dots, f^{(n)}(0)$

From Example ?? of Lecture Note 9, we have

$$f^{(n)}(x) = \frac{d^n}{dx^n}\cos x = \cos\left(\frac{n\pi}{2} + x\right).$$

Hence, one can find that

$$f(0) = \cos 0 = 1, \qquad f'(0) = \cos\left(\frac{\pi}{2}\right) = 0, \qquad f^{(2)}(0) = \cos(\pi) = -1,$$

$$f^{(3)}(0) = \cos\left(\frac{3\pi}{2}\right) = 0, \qquad f^{(4)}(0) = \cos(2\pi) = 1,$$

$$f^{(5)}(0) = \cos\frac{5\pi}{2} = 0, \qquad f^{(6)}(0) = \cos 3\pi = -1.$$

Step 3: Substitute

Hence, the Taylor expansion of $f(x) = \cos x$ is then given by

$$\cos x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

To compute $\cos 0.5$, we substitute $x = 0.5$ and compute the first four terms, we have

$$\cos 0.5 \approx 1 - \frac{1}{2}(0.5)^2 + \frac{1}{4!}(0.5)^4 - \frac{1}{6!}(0.5)^6 \approx 0.877582$$ (True value: 0.8775825618904)

Example 3 (Generalized Binomial Theorem)

The "classical" Binomial theorem states that for any positive integer n, we have

$$(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n.$$

In this example, we will extend this theorem to general case.

Find the Maclaurin series of $f(x) = (1+x)^\alpha$, **where $\boldsymbol{\alpha}$ is real**, as far as the term x^4.

☺ Solution:

The Maclaurin series of $f(x) = (1+x)^\alpha$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots.$$

One can find that

$$f(0) = (1+0)^\alpha = 1, \qquad f'(0) = \frac{d}{dx}(1+x)^\alpha\Big|_{x=0} = \alpha(1+x)^{\alpha-1}\Big|_{x=0} = \alpha.$$

$$f''(0) = \frac{d^2}{dx^2}(1+x)^\alpha|_{x=0} = \frac{d}{dx}\alpha(1+x)^{\alpha-1}|_{x=0} = \alpha(\alpha-1)(1+x)^{\alpha-2}|_{x=0} = \alpha(\alpha-1).$$

$$f^{(3)}(0) = \frac{d^3}{dx^3}(1+x)^\alpha|_{x=0} = \frac{d}{dx}\alpha(\alpha-1)(1+x)^{\alpha-2}|_{x=0}$$
$$= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}|_{x=0} = \alpha(\alpha-1)(\alpha-2).$$

$$f^{(4)}(0) = \frac{d^4}{dx^4}(1+x)^\alpha|_{x=0} = \frac{d}{dx}\alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}|_{x=0}$$
$$= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-4}|_{x=0}$$
$$= \alpha(\alpha-1)(\alpha-2)(\alpha-3).$$

Thus the Maclaurin series of $f(x) = (1+x)^\alpha$ is then given by

$$(1+x)^\alpha \approx \overbrace{1}^{f(0)} + \overbrace{\alpha}^{f'(0)}\,x + \frac{\overbrace{\alpha(\alpha-1)}^{f''(0)}}{2!}x^2 + \frac{\overbrace{\alpha(\alpha-1)(\alpha-2)}^{f^{(3)}(0)}}{3!}x^3 + \frac{\overbrace{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}^{f^{(4)}(0)}}{4!}x^4 + \cdots.$$

Some Maclaurin series of some elementary functions

Using similar technique, one can derive the Maclaurin series (Taylor series at $x = 0$) of the following elementary functions:

1. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad for \ -1 < x < 1$

2. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots \quad for\ all\ real\ x$

3. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \quad for\ all\ real\ x$

4. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots \quad for\ all\ real\ x$

5. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \cdots \quad for \ -1 < x < 1.$

Example 4

Find the Maclaurin series for $\ln(\cos x)$ as far as the term in x^4.

☺ Solution:

The Maclaurin series of $f(x) = \ln(\cos x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots.$$

One can find that

$f(0) = \ln(\cos 0) = \ln 1 = 0$

$$f'(0) = \frac{d}{dx}\ln(\cos x)\,|_{x=0} = \frac{d(\ln(\cos x))}{d(\cos x)}\frac{d(\cos x)}{dx}|_{x=0} = \frac{1}{\cos x}(-\sin x)|_{x=0} = -\tan x\,|_{x=0} = 0.$$

$$f''(0) = \frac{d^2}{dx^2}\ln(\cos x)\,|_{x=0} = \frac{d}{dx}(-\tan x)|_{x=0} = -\sec^2 x\,|_{x=0} = -1.$$

$$f^{(3)}(0) = \frac{d^3}{dx^3}\ln(\cos x)\,|_{x=0} = \frac{d}{dx}(-\sec^2 x)|_{x=0}$$

$$= -\frac{d(\sec^2 x)}{d(\sec x)}\frac{d(\sec x)}{dx}|_{x=0} = -2\sec x\,(\sec x\tan x)|_{x=0}$$

$$= -2\sec^2 x\tan x\,|_{x=0} = 0.$$

$$f^{(4)}(0) = \frac{d^4}{dx^4}\ln(\cos x)\,|_{x=0} = \frac{d}{dx}(-2\sec^2 x\tan x)|_{x=0} = \cdots$$

$$= -4\sec^2 x\tan^2 x - 2\sec^4 x\,|_{x=0} = -2.$$

So the Maclaurin series for $f(x) = \ln(\cos x)$ is given by

$$\ln(\cos x) \approx \overbrace{0}^{f(0)} + \overbrace{0}^{f'(0)}\,x + \frac{\overbrace{-1}^{f''(0)}}{2!}x^2 + \frac{\overbrace{0}^{f^{(3)}(0)}}{3!}x^3 + \frac{\overbrace{-2}^{f^{(4)}(0)}}{4!}x^4 = -\frac{x^2}{2} - \frac{x^4}{12}.$$

Example 5

Let $f(x) = \sin(\ln(1+x))$

(a) Show that

$$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0.$$

(b) By differentiating the whole equation with respect to x for n times and using Leibnitz's Rule, show that

$$(1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+1)f^{(n)}(x) = 0.$$

(c) Hence, find the Maclaurin series of $f(x) = \sin(\ln(1+x))$ as far as the term x^5.

☺ Solution:

(a) It can be found that

$$f'(x) = \frac{d}{dx}\sin(\ln(1+x)) = \cos(\ln(1+x))\frac{1}{1+x}.$$

$$f''(x) = \frac{d}{dx}\cos(\ln(1+x))\frac{1}{1+x} = -\sin(\ln(1+x))\frac{1}{(1+x)^2} - \cos(\ln(1+x))\frac{1}{(1+x)^2}.$$

Substitute the result into L.H.S. of the equation, we get

$$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0.$$

(b) To obtain the general formula, we differentiate the equation in (a) with respect to x for n times

$$\frac{d^n}{dx^n}[(1+x)^2 f''(x) + (1+x)f'(x) + f(x)] = 0$$

$$\frac{d^n}{dx^n}(1+x)^2 f''(x) + \frac{d^n}{dx^n}(1+x)f'(x) + f^{(n)}(x) = 0 \ldots (*)$$

To compute $\frac{d^n}{dx^n}(1+x)^2 f''(x)$, we can use Leibnitz's rule

$$\frac{d^n}{dx^n}(1+x)^2 f''(x) \sum_{r=0}^{n} C_r^n \frac{d^r}{dx^r}(1+x)^2 \frac{d^{n-r}}{dx^{n-r}} f''(x)$$

$$= C_0^n \overbrace{\frac{d^0}{dx^0}(1+x)^2}^{(1+x)^2} \overbrace{\frac{d^n}{dx^n} f''(x)}^{f^{(n+2)}(x)} + C_1^n \overbrace{\frac{d^1}{dx^1}(1+x)^2}^{2(1+x)} \overbrace{\frac{d^{n-1}}{dx^{n-1}} f''(x)}^{f^{(n+1)}(x)}$$

$$+C_2^n \overbrace{\frac{d^2}{dx^2}(1+x)^2}^{2} \overbrace{\frac{d^{n-2}}{dx^{n-2}} f''(x)}^{f^{(n)}(x)} + C_3^n \overbrace{\frac{d^3}{dx^3}(1+x)^2}^{0} \overbrace{\frac{d^{n-3}}{dx^{n-3}} f''(x)}^{f^{(n-1)}(x)} + \cdots + C_n^n \overbrace{\frac{d^n}{dx^n}(1+x)^2}^{=0} \overbrace{\frac{d^0}{dx^0} f''(x)}^{f^{(2)}(x)}$$

$$= (1+x)^2 f^{(n+2)}(x) + 2n(1+x)f^{(n+1)}(x) + n(n-1)f^{(n)}(x).$$

Similarly, one can find that

$$\frac{d^n}{dx^n}[(1+x)f'(x)] = (1+x)f^{(n+1)}(x) + nf^{(n)}(x).$$

Substitute all result into $(*)$, we finally get

$$(1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+1)f^{(n)}(x) = 0.$$

(c) The Maclaurin Series of the function $f(x) = \sin(\ln(1+x))$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5.$$

By direct substitution, one can find that

$$f(0) = \sin(\ln 1) = 0, f'(0) = 1 \text{ and } f''(0) = -1.$$

To obtain $f^{(3)}(0)$, $f^{(4)}(0)$ and $f^{(5)}(0)$, one can use the equation obtained in (b). To do this, we first substitute $x = 0$ into the equation and get

$$f^{(n+2)}(0) + (2n+1)f^{(n+1)}(0) + (n^2+1)f^{(n)}(0) = 0.$$

Put $n = 1$, we get

$$f^{(3)}(0) + (2(1)+1)\overbrace{f^{(2)}(0)}^{=-1} + (1^2+1)\overbrace{f^{(1)}(0)}^{1} = 0 \Rightarrow f^{(3)}(0) = 1.$$

Put $n = 2$, we get

$$f^{(4)}(0) + (2(2) + 1)\overbrace{f^{(3)}(0)}^{=1} + (2^2 + 1)\overbrace{f^{(1)}(0)}^{-1} = 0 \Rightarrow f^{(4)}(0) = 0.$$

Put $n = 3$, we get

$$f^{(5)}(0) + (2(3) + 1)\overbrace{f^{(4)}(0)}^{=0} + (3^2 + 1)\overbrace{f^{(3)}(0)}^{1} = 0 \Rightarrow f^{(5)}(0) = -10.$$

Thus the Maclaurin series of $f(x)$ is given by

$$f(x) \approx x + \frac{-1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{-10}{5!}x^5 = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{12}.$$

Application of Taylor Series – An overview

1. Computation of Limits

Taylor theorem allows to approximate some "ugly" terms by simple polynomial so that we can compute the limits in an easier way.

Example 6

Compute the limits

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2}$$

☺ Solution:

Using the Taylor series of $\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2} = \lim_{x\to 0}\frac{x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \cdots}{x^2} = \lim_{x\to 0}\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \cdots\right) = 1$$

Example 7

We consider the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}.$$

It is clear that the function is continuous at $x = 0$ since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = f(0)$.

Determine whether the function $f(x)$ is differentiable at $x = 0$.

☺ Solution: Using the first principle, we consider

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\dfrac{\sin h}{h} - 1}{h} = \lim_{h\to 0} \frac{\sin h - h}{h^2}$$

$$= \lim_{h\to 0} \frac{\left(h - \dfrac{h^3}{3!} + \dfrac{h^5}{5!} - \cdots\right) - h}{h^2} = \lim_{h\to 0} \left(-\frac{h}{3!} + \frac{h^3}{5!} - \cdots\right) = 0.$$

So the function is differentiable at $x = 0$ and $f'(0) = 0$.

2. Computation of π

Consider the Taylor Series of $\tan^{-1} x$:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots.$$

This series can be derived using the result obtained in Example 24 of Lecture Note 9.

Substitute $x = 1$ into the above equation and note that $\tan^{-1} 1 = \frac{\pi}{4}$, we can obtain the formula for calculating π

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \Rightarrow \pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

However, this formula is not useful at all. If we compute the first 500000 terms to approximate π, we get

$\pi \approx 3.141590653$ (True value: 3.1415926).

In order to improve the efficiency, one tries to decompose the $\tan^{-1} 1$ into the sum of $\tan^{-1} x$ with smaller value of x. To do this, let's recall the compound angle formula for $\tan x$:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Put $A = \tan^{-1} a$ and $B = \tan^{-1} b$, we then have

$$\tan(\tan^{-1} a + \tan^{-1} b) = \frac{\overbrace{\tan(\tan^{-1} a)}^{=a} + \overbrace{\tan(\tan^{-1} b)}^{b}}{1 - \underbrace{\tan(\tan^{-1} a)}_{a}\underbrace{\tan(\tan^{-1} b)}_{b}}$$

$$\Rightarrow \tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}$$

To decompose $\tan^{-1} 1$, one can set

$$\frac{a+b}{1-ab} = 1, a = \frac{1}{2} \Rightarrow b = \frac{1}{3}.$$

So that $\tan^{-1} 1 = \frac{\pi}{4}$ can be expressed as

$$\frac{\pi}{4} = \tan^{-1} 1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \Rightarrow \pi = 4\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right).$$

The newer formula is more efficient since both series

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \frac{\left(\frac{1}{2}\right)^9}{9} + \cdots = \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} - \cdots$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \frac{\left(\frac{1}{3}\right)^9}{9} + \cdots = \frac{1}{3} - \frac{1}{81} + \frac{1}{1215} - \frac{1}{15309} \cdots$$

have better convergence speed than $\tan^{-1} 1$. For instance, one can compute the first 500 terms of each of the series and obtain the following estimation of π: (True value: $\pi = 3.141592653589793$)

$$\pi \approx 3.141592653589794$$

One can continue the process in order to obtain more efficient formula. Some examples of such formulas are

- Machin's formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

- Euler

$$\pi = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79}.$$

- Yasumasa Kanada (Dec, 2002)

$$\pi = 4\left(12\tan^{-1}\frac{1}{49} + 32\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239} + 12\tan^{-1}\frac{1}{110443}\right).$$

(*Note: At the time, he used this formula to calculate π up to 1.24 trillion $= 1.24 \times 10^{12}$ digits)

Optimization – Finding maximum and minimum value of a function

The subject of optimization is important in many decision-making problems such as portfolio selection (finance), inventory management (logistic), production problem (economic) etc.

These problems can be formulated as an optimization problem: Given a function (called objective function) $f(x)$, where x is the decision variable, we would like to find x such that the function $f(x)$ is maximized (or minimized).

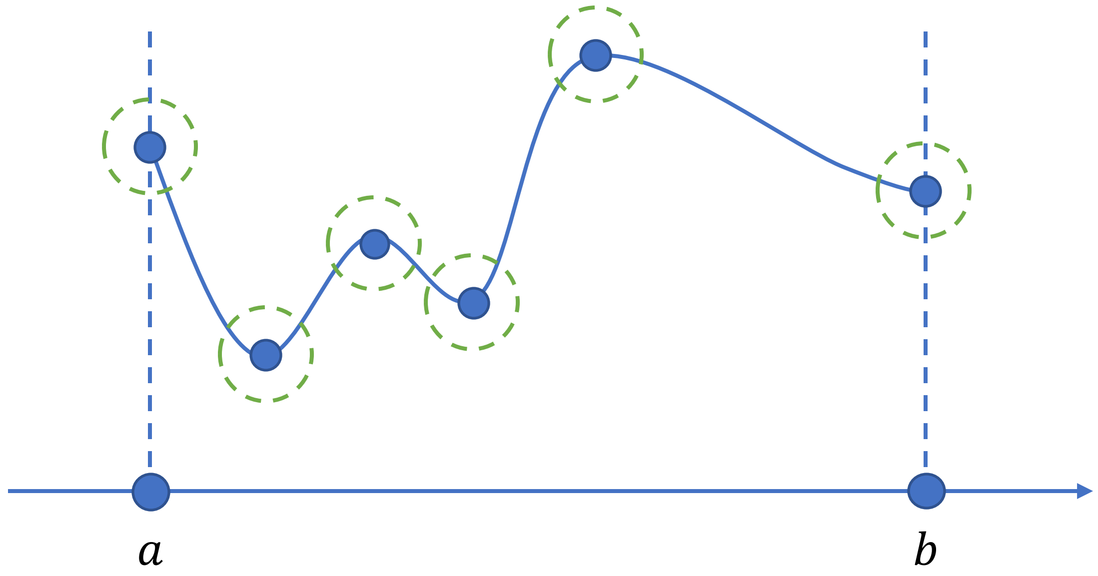
Problem: Find x^* such that

$$f(x^*) = \max_x f(x) \quad \left(or\ f(x^*) = \min_x f(x)\right)$$

In this section, we will learn how to make use of derivative to obtain the maximum/ minimum value of $f(x)$ in a computational way.

How to find the maximum and minimum of a function? An intuitive approach.

Suppose you are given the graph of the $y = f(x)$ on the interval $[a, b]$ and we would like the maximum value and/or $f(x)$ on $[a, b]$



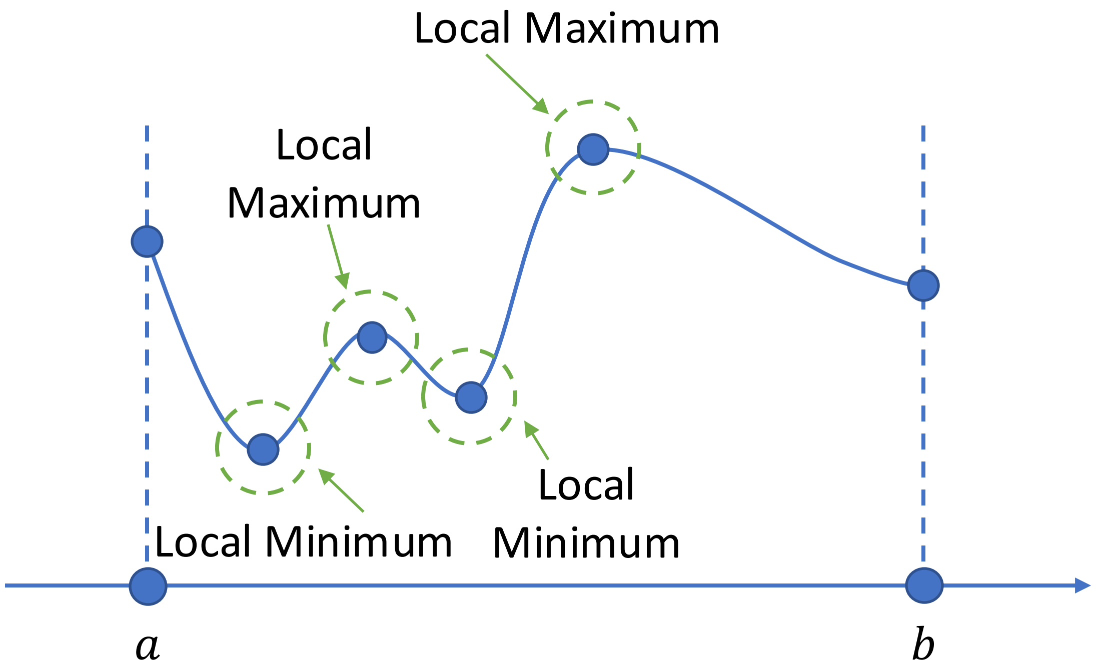
We first narrow the range by finding all potential candidate of maximum point and minimum point (circled points) and compare the value one by one.

Local maximum and Local Minimum (Local extrema)

We consider the function $f(x)$ defined on $[a, b]$. We say

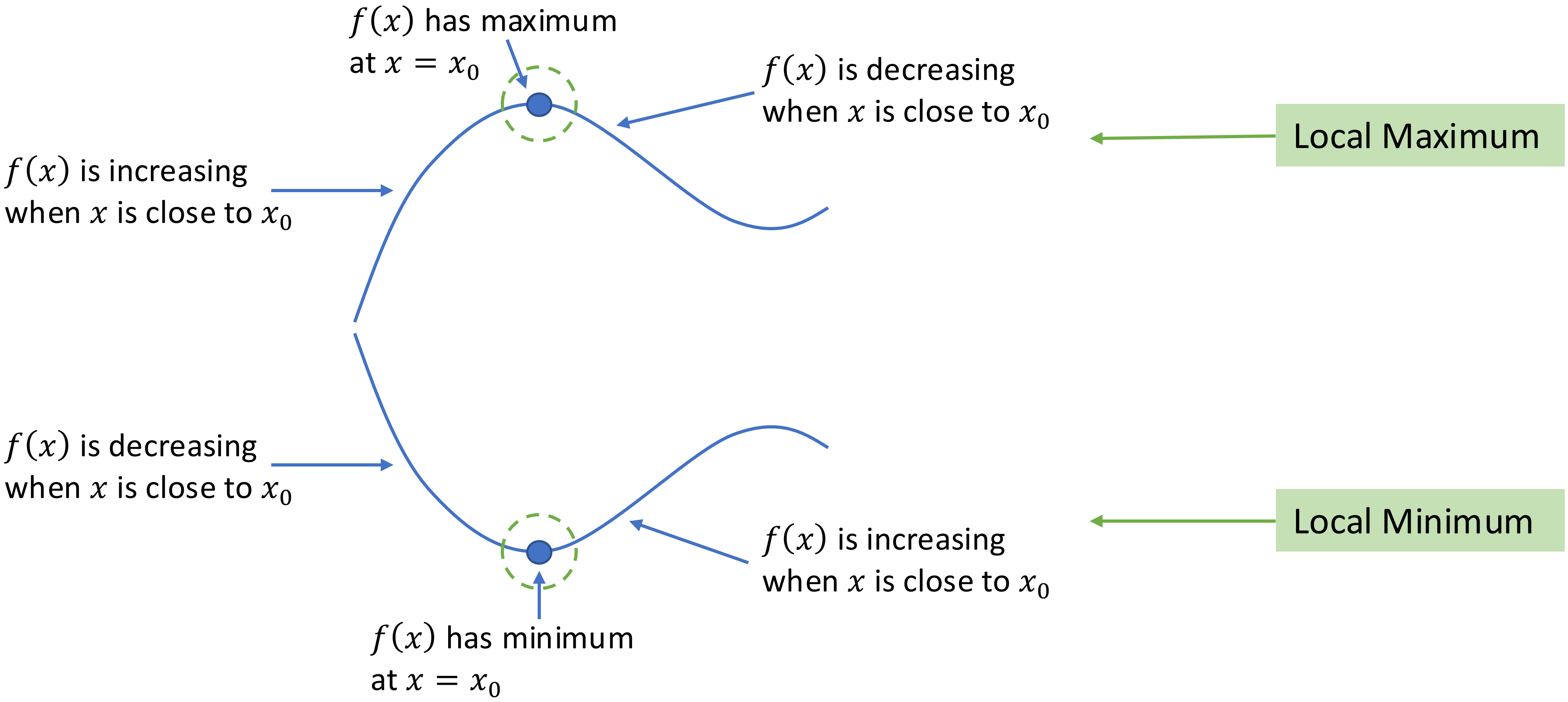
- $f(x)$ has local maximum (resp. minimum) at $x = x^*$ if $f(x^*) \geq f(x)$ (resp. $f(x^*) \leq f(x)$) for all x near $x = x^*$.

*Note: Local Maximum (minimum) $\neq$ True Maximum (minimum)



Mathematical Properties of local maximum and local minimum

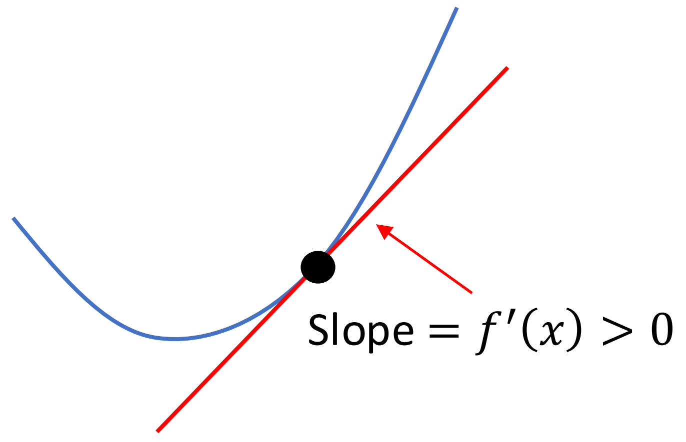
Next, we should investigate the properties of local maximum and local minimum (behavior of $f(x)$) in order to develop a suitable mathematical tool to find the maximum and minimum value of the function.



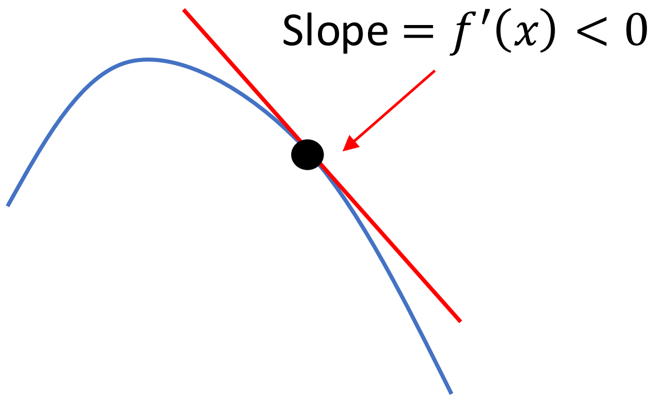
How do we describe the monotonicity (increasing or decreasing) using mathematics?

In fact, the first derivative $f'(x) = \frac{df}{dx}$ can capture the monotonicity of the function. It is because $f'(x)$ represents the slope of the tangent to the graph of $y = f(x)$ and the slope of tangent can indicate the "trend" of the function.

- If $f'(x) > 0$ over (a, b), then we see $f(x)$ is increasing over (a, b)
- If $f'(x) < 0$ over (a, b), then we see $f(x)$ is decreasing over (a, b).



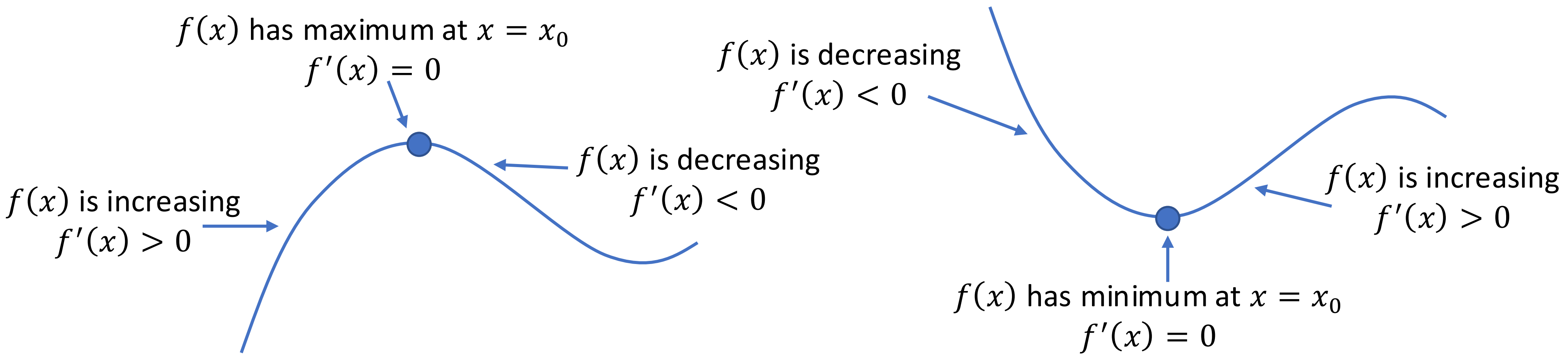
Increasing



Decreasing

Combining the ideas in P.32 and P.33, we come up the following tool in finding the local maxima and local minima.

It is called first-derivative test.



First Derivative Test

Let $f(x)$ be a differentiable function, then we say

- f has local maximum at $x = x_0$ if $f'(x_0) = 0$, $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$.
- f has local minimum at $x = x_0$ if $f'(x_0) = 0$, $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$.

Example 8

Find the local maxima and minima of the function $f(x) = e^{12x-x^3}$.

☺ Solution:

Since both local maxima and local minimum satisfies $f'(x) = 0$, one can find all local maxima and minima by solving

$$f'(x) = 0 \Rightarrow (12 - 3x^2)e^{12x-x^3} = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow (2 - x)(2 + x) = 0$$

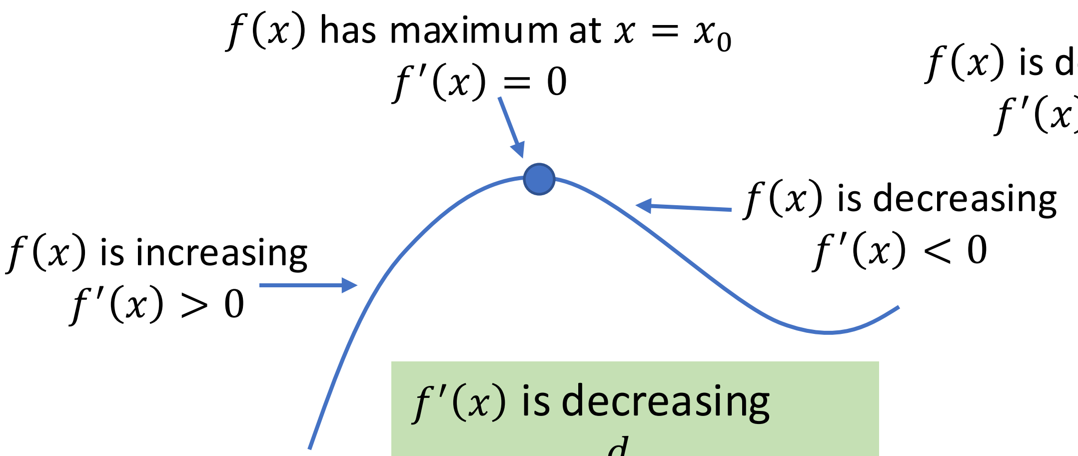
$$\Rightarrow x = 2 \ \ or \ \ x = -2.$$

Next, to identity whether these turning points are maxima or minima, we consider

	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	< 0	0	> 0	0	< 0
Graph	decreasing	Local min.	increasing	Local max.	decreasing

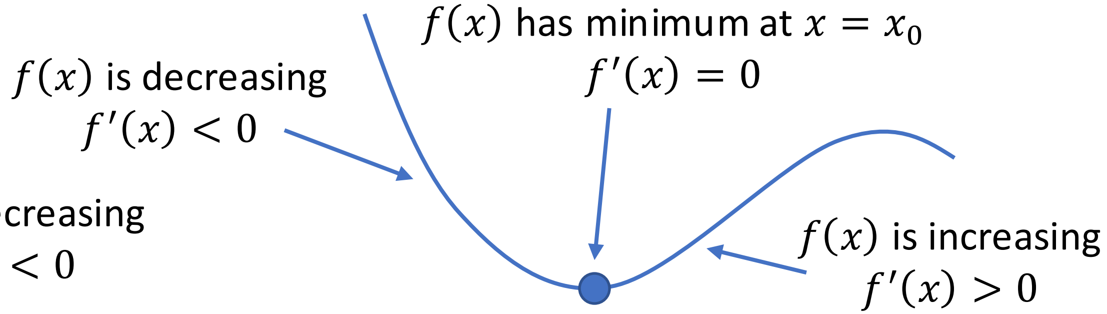
Hence, $f(x)$ has local minimum at $x = -2$ and has local maximum at $x = 2$.

Sometimes, one may prefer to use second derivative test to test whether the turning point is local maxima and local minima



$f'(x)$ is decreasing

$$f''(x_0) = \frac{d}{dx} f'(x_0) < 0$$



$f'(x)$ is increasing

$$f''(x_0) = \frac{d}{dx} f'(x_0) > 0$$

Second Derivative Test

Let $f(x)$ be a (at least two-times) differentiable function, then we say

- f has local maximum at $x = x_0$ if $f'(x_0) = 0$, $f''(x_0) < 0$.
- f has local minimum at $x = x_0$ if $f'(x_0) = 0$, $f''(x_0) > 0$.

Remark

In some cases which $f''(x_0) = 0$. The second derivative test fails to determine whether a given local extrema is local minimum or local maximum. In this case, one has to use the first-derivative test.

Example 9

One can consider the Example 11 again. We know that $x = -2$ and $x = 2$ are the turning point of the function. Then we consider

$$f''(x) = \frac{d^2}{dx^2} e^{12x-x^3} = \cdots = (9x^4 - 72x^2 - 6x + 144)e^{12x-x^3}.$$

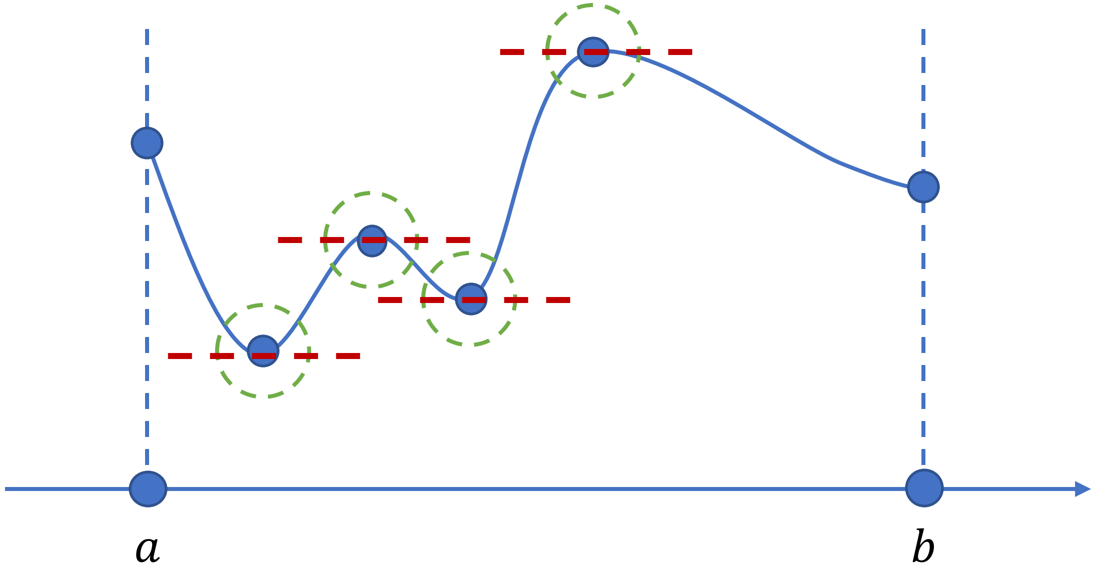
Note that

- $f''(2) = -12e^{16} < 0$, so $x = 2$ is the local maxima.
- $f''(-2) = 12e^{-16} > 0$, so $x = -2$ is the local minima.

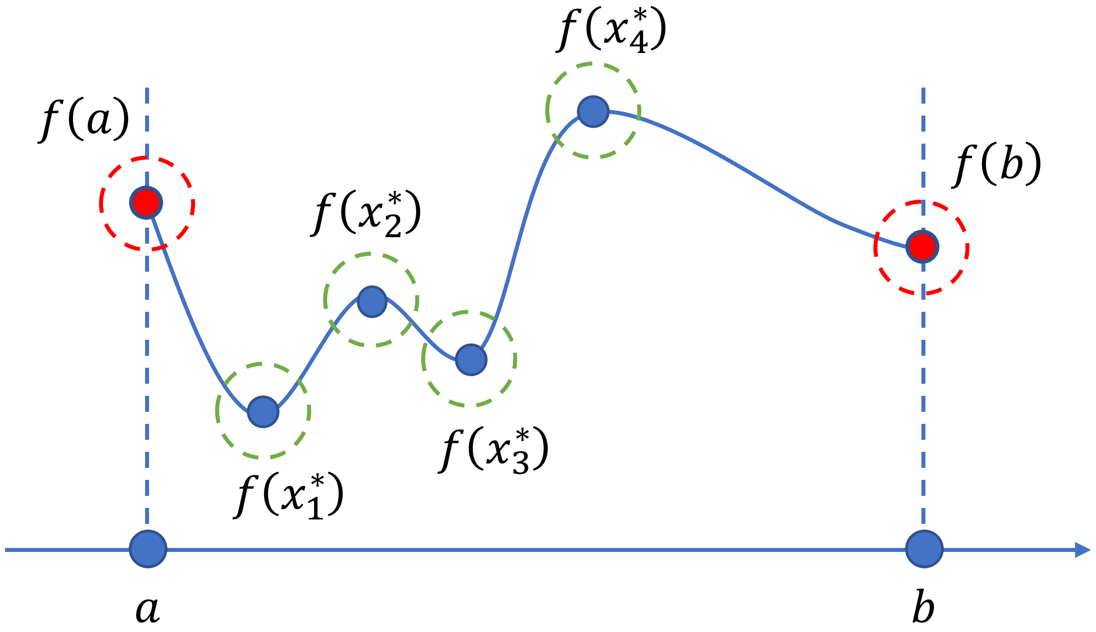
A general procedure of finding the maximum and minimum value of a function

One can follow the steps below to obtain the maximum and minimum value of $f(x)$ over $[a, b]$.

1. Find all possible local extrema. This can be done by solving $f'(x) = 0$.
2. For each turning points x^* obtained in Step 1, determine whether it is local maximum or local minimum by using either first derivative test or second derivative test. Find the value of $f(x^*)$ also.



3. Calculate the value of $f(a), f(b)$ (value at boundary). (*Note: If $b = \infty$, then we compute $\lim_{x \to \infty} f(x)$ instead of $f(b)$. If $a = -\infty$, then we compute $\lim_{x \to -\infty} f(x)$ instead of $f(a)$.)



4. Compare the value of $f(a), f(b)$ and $f(x_i^*)s$ to obtain the max. and min..

Example 10

Find the maximum and minimum value of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 7$$

over the interval $[-3, 3]$.

☺ Solution:

Step 1: Finding all turning points

We need to solve $f'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2 \;\; or \;\; x = -1.$

Step 2: Determine whether the turning points are local maxima or local minima

Method 1: First Derivative Test

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	> 0	0	< 0	0	> 0
Graph	increasing	Local max.	decreasing	Local min.	increasing

Method 2: Second derivative test

Note that $f''(x) = 12x - 6$.

$f''(-1) = -18 < 0 \Rightarrow x = -1$ is local maximum.

$f''(2) = 12(2) - 6 = 18 > 0 \Rightarrow x = 2$ is local minimum.

By some calculation, one can see that $f(-1) = 14$ and $f(2) = -13$.

Step 3: Compute the value of $f(3)$ and $f(-3)$ (value at boundary)

One can see that $f(3) = -2$ and $f(-3) = -38$.

Step 4: Determine the maximum and minimum value

By comparing the values of $f(-1)$, $f(2)$, $f(3)$, $f(-3)$, we conclude that

- The maximum value of $f(x)$ is 14 at $x = -1$.
- The minimum value of $f(x) = -38$ at $x = -3$.

Example 11

Find the maximum and minimum value of $f(x) = \frac{(x+1)^2}{4(x-2)}$ for $3 \leq x \leq 8$.

Step 1: Find all local extrema. We need to solve $f'(x) = 0$

$$\Rightarrow f'(x) = \frac{1}{4}\left[\frac{(x-2)\frac{d}{dx}(x+1)^2 - (x+1)^2\frac{d}{dx}(x-2)}{(x-2)^2}\right] = 0$$

$$\Rightarrow f'(x) = \frac{1}{4}\left[\frac{(x-2)[2(x+1)] - (x+1)^2}{(x-2)^2}\right] = 0$$

$$\Rightarrow f'(x) = \frac{1}{4}\left[\frac{\overbrace{\overbrace{(x+1)(x-5)}^{=(x+1)(x-5)}}^{=(x+1)[2(x-2)-(x+1)]}}{(x-2)^2}\right] = 0$$

$\Rightarrow x = -1$ (rejected) or $x = 5$ (for $x \neq 2$).

(*Here, we reject $x = -1$ since we are considering $f(x)$ within the interval $[3, 8]$ only!)

Step 2: Determine whether the turning points is local maximum or local minimum

Method 1: First Derivative Test

Recall that $f'(x) = \frac{1}{4}\frac{(x+1)(x+5)}{(x-2)^2}$

	$3 \leq x < 5$	$x = 5$	$5 < x \leq 8$
$f'(x)$	< 0	0	> 0
Graph	decreasing	local min.	increasing

Method 2: Second derivative test

Note that $f''(x) = \frac{9}{2(x-2)^3}$

$f''(5) = \frac{1}{6} > 0 \Rightarrow f(x)$ has local maximum at $x = 5$.

By some calculation, one can see that $f(5) = 3$.

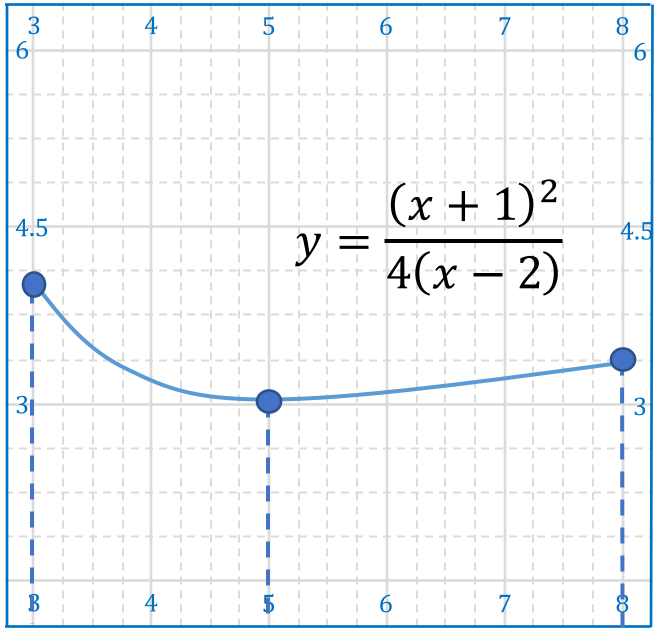
Step 3: Compute the value of $f(3)$ and $f(8)$ (values at boundary)

One can see that $f(3) = 4$ and $f(8) = \frac{27}{8} \approx 3.375$.

Step 4: Determine the maximum and minimum value

By comparing the values of $f(3)$, $f(5)$, $f(8)$, we conclude that

- The maximum value of $f(x)$ is 4 at $x = 3$.
- The minimum value of $f(x)$ is 3 at $x = 5$.



Example 13

Find the maximum value and minimum value of $f(x) = (x-1)x^{\frac{2}{3}}$ for $-1 \leq x \leq 1$.

☺ Solution:

Step 1: Find all local extrema. We need to solve $f'(x) = 0$

$$\Rightarrow f'(x) = x^{\frac{2}{3}} + (x-1)\left(\frac{2}{3}x^{-\frac{1}{3}}\right) = 0, \qquad x \neq 0$$

$$\Rightarrow f'(x) = \frac{x + \frac{2}{3}(x-1)}{x^{\frac{1}{3}}} = 0, \qquad x \neq 0$$

$$\Rightarrow f'(x) = \frac{\frac{5}{3}x - \frac{2}{3}}{x^{\frac{1}{3}}} = 0, \qquad x \neq 0$$

$$\Rightarrow x = \frac{2}{5}.$$

Step 2: Determine whether the turning points is local maximum or local minimum

Here, we use first derivative test. One has to be careful that <u>the first derivative does not exist at $x = 0$</u>.

Recall that

$$f'(x) = \frac{\frac{5}{3}x - \frac{2}{3}}{x^{\frac{1}{3}}}, \qquad x \neq 0$$

	$-1 \leq x < 0$	$x = 0$	$0 < x < \frac{2}{5}$	$x = \frac{2}{5}$	$\frac{2}{5} < x \leq 1$
$f'(x)$	> 0	X	< 0	0	> 0
Graph	increasing	local max.	decreasing	local min.	increasing

By some calculation, we get

$$f(0) = 0, \qquad f\left(\frac{2}{5}\right) = -\frac{3}{5}\left(\frac{2}{5}\right)^{\frac{2}{3}} = -0.326.$$

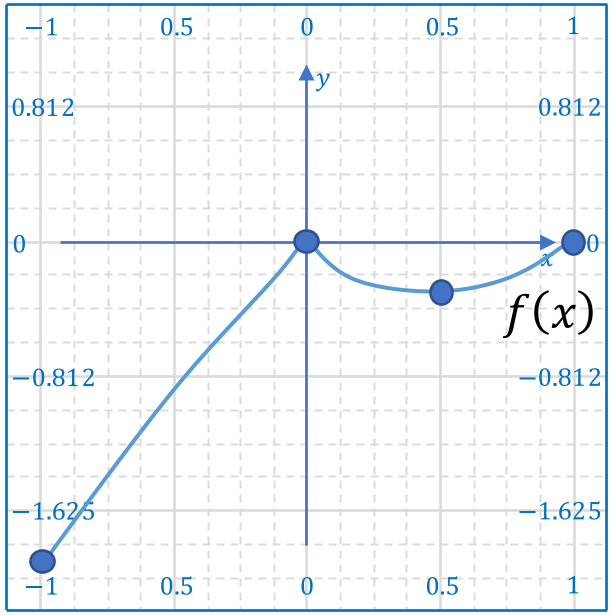
Step 3: Compute the value of $f(-1)$ and $f(1)$ (values at boundary)

One can see that $f(-1) = -2$ and $f(1) = 0$.

Step 4: Determine the maximum and minimum value

By comparing the values of $f(-1), f(0), f\left(\frac{2}{5}\right), f(1)$, we conclude that

- The maximum value of $f(x)$ is 0 at $x = 0$ or $x = 1$.
- The minimum value of $f(x)$ is -2 at $x = -1$.



$f(x) = (x-1)x^{\frac{2}{3}}$

Example 14

Find the maximum and minimum values of the function $f(x) = \sin^3 x + \cos^3 x$ for $0 \leq x \leq \pi$.

Step 1: Find all local extrema.

We need to solve $f'(x) = 0$

$$\Rightarrow f'(x) = 3\sin^2 x \cos x + 3\cos^2 x\,(-\sin x) = 0,$$

$$\Rightarrow f'(x) = 3\sin x \cos x\,(\sin x - \cos x) = 0,$$

$$\Rightarrow \sin x = 0 \;\; or \;\; \cos x = 0 \;\; or \;\; \sin x - \cos x = 0$$

$$\Rightarrow \sin x = 0 \;\; or \;\; \cos x = 0 \;\; or \;\; \tan x = 1.$$

$$\Rightarrow x = \underbrace{0, \pi}_{\sin x = 0}, \underbrace{\frac{\pi}{2}}_{\cos x = 0}, \underbrace{\frac{\pi}{4}}_{\tan x = 1}.$$

Step 2: Determine whether the turning point is local maximum or local minimum

Here, we use first derivative test.

Recall that

$$f'(x) = 3\sin x\cos x\,(\sin x - \cos x),$$

	$x = 0$	$0 < x < \frac{\pi}{4}$	$x = \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{\pi}{2}$	$x = \frac{\pi}{2}$	$\frac{\pi}{2} < x < \pi$	$x = \pi$
$f'(x)$	0	< 0	0	> 0	0	< 0	0
Graph	Local Max	decreasing	Local min	increasing	Local max	decreasing	Local Min

By some calculation, we get

$$f(0) = 1, \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \qquad f\left(\frac{\pi}{2}\right) = 1, \qquad f(\pi) = -1.$$

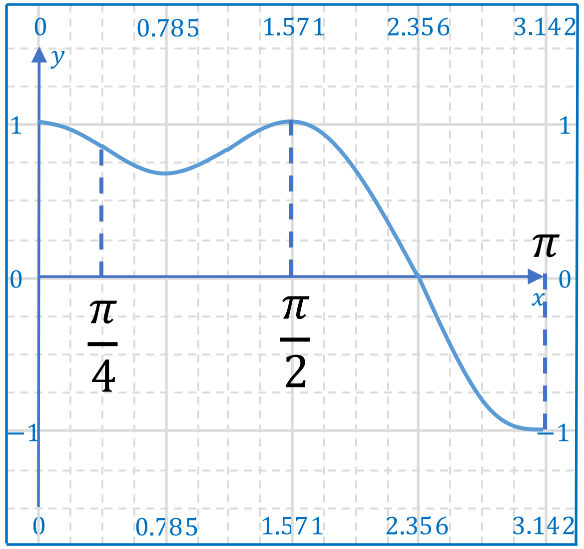
Step 3: Compute the value of $f(0)$ and $f(\pi)$ (values at boundary)

One can see that $f(0) = 1$ and $f(\pi) = -1$.
(*This step may be skipped since the values are computed in Step 2)

Step 4: Determine the maximum and minimum value

By comparing the values of $f(0)$, $f\left(\frac{\pi}{4}\right)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$, we conclude that

- The maximum value of $f(x)$ is 1 at $x = \frac{\pi}{2}$ or $x = 0$.
- The minimum value of $f(x)$ is -1 at $x = \pi$.



A power formula for computing limits: L' Hopital's Rule

In Chapter 6 (Limits), we learned various techniques to compute the limits.

$$\lim_{x\to 0}\frac{x^2-3x}{x^2-5x}=\lim_{x\to 0}\frac{x-3}{x-5}=\frac{3}{5},\qquad \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right)=\cdots=2.$$

$$\lim_{x\to 0}\frac{\sin 2x\cos x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}\cos x=2.$$

$$\lim_{x\to\infty}\frac{x^2-3x}{3x^2-2}=\lim_{x\to\infty}\frac{1-\frac{3}{x}}{3-\frac{2}{x^2}}=\frac{1}{3},\qquad \lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{2x}=\lim_{x\to\infty}\left[\left(1-\frac{3}{x}\right)^{x}\right]^2=(e^{-3})^2=e^{-6}.$$

Each of these techniques is useful in some special types of limits only.

One cannot compute the following limits using simple algebraic tricks:

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x},\qquad \lim_{x\to 0}\frac{e^{2x^2}-1}{x^2}.$$

- Observe that these two limits assumes the form $0/0$ (second limits) or ∞/∞ (first limits).
- The following theorem, called L' Hopital Rule, provides an efficient method to compute these kinds of limits.

Theorem 1A (L' Hopital's Rule, $\left(\frac{\mathbf{0}}{\mathbf{0}}\right)$ type)

Let $f(x)$ and $g(x)$ be two differentiable functions. Suppose that

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
2. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ $exists$ $\left(i.e. \lim_{x\to a} \frac{f'(x)}{g'(x)} = L \;\; or \;\; \infty \;\; or \;\; -\infty\right)$

Then, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Theorem 1B (L' Hopital's Rule, $\left(\frac{\infty}{\infty}\right)$ type)

Let $f(x)$ and $g(x)$ be two differentiable functions. Suppose that

1. $\lim_{x \to a} f(x) = \infty \;\; or \;\; -\infty \;\; and \;\; \lim_{x \to a} g(x) = \infty \;\; or \;\; -\infty$

2. $\lim_{x \to a} \frac{f'(x)}{g'(x)} \; exists \; \left(i.e. \lim_{x \to a} \frac{f'(x)}{g'(x)} = L \;\; or \;\; \infty \;\; or \;\; -\infty\right)$

Then, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example 15

Compute the limits

$$(a) \;\; \lim_{x \to \infty} \frac{\ln(1+x)}{x}, \qquad (b) \;\; \lim_{x \to 0} \frac{e^{2x^2} - 1}{x^2}.$$

☺ Solution:

(a) Note that $f(x) = \ln(1+x) \to \infty$ and $g(x) = x \to \infty$ as $x \to \infty$, then the limits is of the type $\frac{\infty}{\infty}$.

Using the L' Hopital Rule, we get

$$\overbrace{\lim_{x\to\infty} \frac{\ln(1+x)}{x}}^{\lim_{x\to\infty}\frac{f(x)}{g(x)}} \overset{\frac{\infty}{\infty}\,type}{\cong} \overbrace{\lim_{x\to\infty} \frac{\frac{1}{1+x}}{1}}^{\lim_{x\to\infty}\frac{f'(x)}{g'(x)}} = \lim_{x\to\infty} \frac{1}{1+x} \overset{\to\frac{1}{\infty}}{\cong} 0.$$

(b) On the other hand, note that $f(x) = e^{2x^2} - 1 \to 0$ and $g(x) = x^2 \to 0$ as $x \to 0$ and the limits is of the type $\frac{0}{0}$.

Using L' Hopital Rule, we get

$$\overbrace{\lim_{x\to 0} \frac{e^{2x^2}-1}{x^2}}^{\lim_{x\to 0}\frac{f(x)}{g(x)}} \overset{\frac{0}{0}\,type}{\cong} \overbrace{\lim_{x\to 0} \frac{4xe^{2x^2}}{2x}}^{\lim_{x\to 0}\frac{f'(x)}{g'(x)}} = \lim_{x\to 0} 2e^{2x^2} = 2e^{2(0)^2} = 2.$$

Example 16A

Compute the limits

$$\lim_{x\to 0}\frac{1-\cos 3x}{x^2}.$$

☺ Solution:

Since $f(x) = 1 - \cos 3x \to 0$ and $g(x) = x^2 \to 0$ as $x \to 0$. Using L' Hopital Rule, we get

$$\lim_{x\to 0}\frac{1-\cos 3x}{x^2} \overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong} \lim_{x\to 0}\frac{\overbrace{3\sin 3x}^{\to\frac{0}{0}}}{2x} \overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong} \lim_{x\to 0}\frac{9\cos 3x}{2} = \frac{9\cos 3(0)}{2} = \frac{9}{2}.$$

In fact, L' Hopital provides an "extremely" effective way for us to compute the limits.

Consider the Example 16 again.

Before Use

$$\lim_{x\to 0}\frac{1-\cos 3x}{x^2}$$

$$=\lim_{x\to 0}\frac{\cos 0-\cos 3x}{x^2}$$

$$=\lim_{x\to 0}\frac{-2\sin\frac{0+3x}{2}\sin\frac{0-3x}{2}}{x^2}$$

$$=\lim_{x\to 0}2\frac{\sin^2\left(\frac{3x}{2}\right)}{x^2}$$

$$=\lim_{x\to 0}2\left(\frac{9}{4}\right)\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2=\frac{9}{2}.$$

After Use

$$\lim_{x\to 0}\frac{1-\cos 3x}{x^2}$$

$$=\lim_{x\to 0}\frac{3\sin 3x}{2x}$$

$$=\lim_{x\to 0}\frac{9\cos 3x}{2}$$

$$=\frac{9\cos 3(0)}{2}=\frac{9}{2}.$$

Example 16B

Compute the following limits

$$\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}.$$

☺ Solution:

Note that both $f(x) = \tan x - x \to 0$ and $g(x) = x - \sin x \to 0$ when $x \to 0$.

Thus, one can use the L' Hopital Rule and get:

$$\lim_{x\to 0}\frac{\tan x - x}{x - \sin x} \overset{\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}}{\triangleq} \lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x} \quad \left(\frac{0}{0} - type\right)$$

$$\overset{\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}}{\triangleq} \lim_{x\to 0}\frac{2\sec x\,(\sec x \tan x)}{\sin x} = \lim_{x\to 0}\frac{2}{\cos^3 x} = \frac{2}{\cos^3 0} = 2.$$

Remarks about L' Hopital Rule

1. One can use L' Hopital Rule only when the limits is of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$. This rule cannot be applied when the limits is not of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example:

$$\lim_{x\to 0}\frac{\overbrace{2x+1}^{\to\frac{1}{1}=1}}{x+1} \neq \lim_{x\to 0}\frac{2}{1} = 2 \qquad \left(True\ limits = \frac{2(0)+1}{0+1} = 1\right).$$

2. L' Hopital Rule can be applied <u>ONLY WHEN</u> the limits $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists (either a real number or infinity). For example: It is INCORRECT to say that

$$\lim_{x\to\infty}\frac{\overbrace{x+\sin x}^{\to\frac{\infty}{\infty}}}{x-3} = \lim_{x\to\infty}\frac{1+\cos x}{1} = \lim_{x\to\infty}(1+\overbrace{\cos x}^{\to ??}) = does\ not\ exist.$$

In fact, this limits exists and $= 1$.

3. One has to be careful that the use of L' Hopital Rule in problem related to differentiability.

Example:

Suppose we would like to check the differentiability of $f(x) = \sin 2x$ at $x = \pi$ using first principle, it is INCORRECT to say that

$$\lim_{h \to 0} \frac{\sin(2(\pi + h)) - \sin 2\pi}{h} = \lim_{h \to 0} \frac{\sin 2h}{h} = \lim_{h \to 0} \frac{2 \cos 2h}{1} = 2 \cos 0 = 2$$

since the above argument assumes that $\sin x$ is differentiable at $x = 2\pi$ which is something that you need to show. To obtain the correct solution, one has to use sum-to-product formula and the fact that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Indeterminate forms

In this sub-section, we will discuss the computation of the following limits:

$$(1)\ \lim_{x\to 0} x^x, \qquad (\to 0^0\ as\ x\to 0)$$

$$(2)\ \lim_{x\to\infty}\left(\sqrt{x^2+2x+2}-\sqrt{x^2+1}\right), \qquad (\to \infty-\infty\ as\ x\to\infty)$$

$$(3)\ \lim_{x\to 0}\frac{1}{x}\tan\frac{x}{2}, \qquad (\to 0\ \times\ \infty\ as\ x\to 0)$$

$$(4)\ \lim_{x\to 0}(1+x)^{\frac{1}{x}}, \qquad (\to 1^\infty \neq 1\ as\ x\to 0)$$

$$(5)\ \lim_{x\to 0}\left(\frac{1}{x^2}\right)^x, \qquad (\to \infty^0\ as\ x\to 0)$$

Such kinds of limits are said to be of *indeterminate forms*. Those limits can be computed using L' Hopital Rule. In order to apply this rule, one has to rewrite the above limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that L' Hopital can be applied.

Type 1: $0 \times \infty$ type

Example 17

Compute the limits

$$\lim_{x\to 0^+} x\ln x\,.$$

☺ Note:

$x\ln x \to 0 \times (-\infty)$ as $x \to 0$.

(☺Note: Here we consider the limits when $x \to 0^+$ since $\ln x$ is defined for $x > 0$ only)

☺ Solution:

By rewriting the limits and using L'Hopital Rule, we have

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\overbrace{\ln x}^{\frac{-\infty}{\infty}}}{\frac{1}{x}} \overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0.$$

Type 2: $\infty - \infty$ type

Example 18

Compute the limits

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right).$$

☺ Note:

Limits $\to \infty - \infty$ as $x \to 1^+$.

☺ Solution:

One can take the common factor and get

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{x\ln x-(x-1)}{(x-1)\ln x}$$

$$x\ln x-(x-1)\to 0$$

$$(x-1)\ln x \to 0$$

$$\overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong}\ \lim_{x\to 1^+}\frac{\ln x+x\left(\frac{1}{x}\right)-1}{\ln x+(x-1)\left(\frac{1}{x}\right)}=\lim_{x\to 1^+}\frac{\ln x}{\ln x+\frac{x-1}{x}}$$

$$\lim_{x\to 1^+}\frac{x\ln x}{x\ln x + x - 1} \qquad \left(\frac{0}{0}\ type\right)$$

$$\overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong} \lim_{x\to 1^+}\frac{\ln x + x\left(\frac{1}{x}\right)}{\ln x + x\left(\frac{1}{x}\right) + 1}$$

$$= \lim_{x\to 1^+}\frac{\ln x + 1}{\ln x + 2} = \frac{\ln 1 + 1}{\ln 1 + 2} = \frac{1}{2}.$$

☺ Note:

$$\frac{x}{\sin^3 x} = \overbrace{\left(\frac{x}{\sin x}\right)}^{\to 1}\overbrace{\left(\frac{1}{\sin^2 x}\right)}^{\to\infty} \to \infty$$

Example 19

Compute the limits

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{x}{\sin^3 x}\right).$$

(☺ Note: Again, this limits is of $\infty - \infty$ type.)

☺ Solution:

Note that

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{x}{\sin^3 x}\right)=\lim_{x\to 0}\frac{\sin x - x}{\sin^3 x}\quad \left(\frac{0}{0}\ type\right)$$

$$\overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong}\lim_{x\to 0}\frac{\cos x - 1}{3\sin^2 x\cos x}\quad \left(\frac{0}{0}\ type\right)$$

$$\overset{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}}{\cong}\lim_{x\to 0}\frac{-\sin x}{6\sin x\cos^2 x - 3\sin^3 x}$$

$$=\lim_{x\to 0}\frac{-1}{6\cos^2 x - 3\sin^2 x}$$

$$=\frac{-1}{6\cos^2 0 - 3\sin^2 0}=-\frac{1}{6}.$$

Type 3: $0^0, \infty^0$ and 1^∞ type

One can compute these limits (via L'Hopital Rule) by first transforming the limits into the form of $0 \times \infty$ so that one can compute the limits using the method described in Type 1. This can be done by taking natural logarithm, i.e.

$$y = 0^0 \Rightarrow \ln y = \underbrace{0}_{0}\underbrace{\ln 0}_{-\infty}$$

$$y = \infty^0 \Rightarrow \ln y = \underbrace{0}_{0}\underbrace{\ln \infty}_{\infty}$$

$$y = 1^\infty \Rightarrow \ln y = \underbrace{\infty}_{\infty}\underbrace{\ln 1}_{0}$$

To compute the limits $\lim_{x\to a} y$, we first compute $\lim_{x\to a} \ln y$ and get $\lim_{x\to a} \ln y = L$. Then the limits can be obtained by

$$\lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^{\lim_{x\to a} \ln y} = e^L.$$

Example 20

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{2x}}.$$

$(1+x)^{\frac{1}{2x}} \to 1^{\infty}$

$(1^{\infty} - type)$

☺ Solution:

Let $y=(1+x)^{\frac{1}{2x}} \Rightarrow \ln y = \ln(1+x)^{\frac{1}{2x}} = \underbrace{\frac{1}{2x}}_{\to\infty}\underbrace{\ln(1+x)}_{\to 0}.$

Taking limits on both sides, we have

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{1}{2x}\ln(1+x) = \lim_{x\to 0^+}\frac{\ln(1+x)}{2x} \quad \left(\frac{0}{0}\ type\right)$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{2} = \lim_{x\to 0^+}\frac{1}{2(1+x)} = \frac{1}{2(1+0)} = \frac{1}{2}.$$

Thus we get

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{2x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+}\ln y} = e^{\frac{1}{2}}.$$

Example 21

Evaluate the limits

$$\lim_{x\to 0^+} x^{\sin x}.$$

$$x^{\sin x} \to 0^0$$

$$(0^0 - type)$$

☺ Solution:

Let $y = x^{\sin x}$, then $\ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x$.

Taking limits on both sides, we have

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \quad \left(\frac{\infty}{\infty}\ type\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{(\sin x)^2}\cos x} = -\lim_{x\to 0^+} \frac{(\sin x)^2}{x\cos x} = -\lim_{x\to 0^+} \frac{(\sin x)^2}{x\cos x}$$

$$= -\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right) \sin x \cos x = -1 \times 0 \times 1 = 0$$

$$\Rightarrow \lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1.$$

Example 22

$$(\cot x)^x \to \infty^0$$

$$(\infty^0 - type)$$

Compute the limits

$$\lim_{x\to 0^+} (\cot x)^x .$$

☺ Solution:

Let $y = (\cot x)^x$, then $\ln y = x \ln(\cot x)$.

Taking limits on both sides, we get

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln(\cot x) = \lim_{x\to 0^+} \frac{\ln(\cot x)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\ type\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{-\csc^2 x}{\cot x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-\frac{\sin x}{\cos x \sin^2 x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{\cos x \sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} \left(\frac{x}{\sin x}\right) x$$

$$= 1 \times 1 \times 0 = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\cot x)^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1.$$

Some Harder Examples

Example 23

Let $f(x)$ be a differentiable function with $f(0) = 0 \; and \; f'(0) = 1$, we consider another function

$$g(x) = \begin{cases} \dfrac{f(x)}{2\sin x} & if \; x \neq 0 \\ a & if \; x = 0 \end{cases}.$$

Find the possible a such that the function $g(x)$ is continuous at $x = 0$.

☺ Solution:

$g(x)$ is continuous at $x = 0$

$\Leftrightarrow \lim_{x \to 0} g(x) = g(0)$.

Using L'Hopital Rule, we get

$$\lim_{x\to 0} g(x) \overset{x \neq 0}{\widehat{=}} \lim_{x \to 0} \frac{\overbrace{f(x)}^{\to \frac{f(0)}{2\sin 0} = \frac{0}{0}}}{2\sin x} = \lim_{x\to 0} \frac{f'(x)}{2\cos x} = \frac{f'(0)}{2\cos 0} = \frac{f'(0)}{2} = \frac{1}{2}.$$

Since $g(x)$ is continuous at $x = 0$, then we have

$$\lim_{x\to 0} g(x) = g(0) = a \Rightarrow a = \frac{1}{2}.$$

Example 24

Let $f(x)$ and $g(x)$ be two differentiable functions over $\mathbb{R}$. Suppose that

- $f(0) = 0,\ f'(0) = 1$
- $g(0) = 0,\ g'(0) = 0\ \ and\ \ g''(0) = 1.$

We consider the following the function:

$$P(x) = \begin{cases} f(2x) & if\ x \geq 0 \\ \dfrac{g(x)}{x} & if\ x < 0 \end{cases}.$$

(a) Determine whether the function $P(x)$ is continuous at $x = 0$.

(b) Determine whether the function $P(x)$ is differentiable at $x = 0$.

☺ Solution:

(a) We need to check whether $\lim_{x\to 0} P(x) = P(0) = f(0) = 0$. To compute the limits on the left-hand sides, we need to consider left-hand limits and right-hand limits:

$$\lim_{x\to 0^+} P(x) = \lim_{x\to 0^+} f(2x) = f\big(2(0)\big) = 0$$

$$\lim_{x\to 0^-} P(x) = \lim_{x\to 0^-} \frac{g(x)}{x} \overset{\frac{g(x)}{x}\to\frac{g(0)}{0}=\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to 0^-} \frac{g'(x)}{1} = g'(0) = 0.$$

Thus $\lim_{x\to 0} P(x) = 0$. Since $\lim_{x\to 0} P(x) = 0 = P(0)$, so $P(x)$ is continuous at $x = 0$.

(b) Using first principle, we consider (put $x = 0$)

$$\lim_{h\to 0} \frac{P(0+h) - P(0)}{h} = \lim_{h\to 0} \frac{P(h) - 0}{h} = \lim_{h\to 0} \frac{P(h)}{h}.$$

Again, we need to consider the left-hand limits and right-hand limits:

$$\lim_{h\to 0^+} \frac{P(h)}{h} = \lim_{h\to 0^+} \frac{f(2h)}{h} \overset{\frac{f(2h)}{h}\to\frac{f(0)}{0}=\frac{0}{0}}{\overset{L}{=}} \lim_{h\to 0^+} \frac{\overbrace{2f'(2h)}^{\frac{d}{dh}f(2h)=\frac{df(2h)}{d(2h)}\frac{d(2h)}{dh}}}{1} = 2f'(0) = 2(1) = 2.$$

$$\lim_{h\to0^-}\frac{P(h)}{h}=\lim_{h\to0^-}\frac{\frac{g(h)}{h}}{h}=\lim_{h\to0^-}\frac{g(h)}{h^2}\overset{\frac{g(h)}{h^2}\to\frac{g(0)}{0^2}=\frac{0}{0}}{\overset{L'H}{=}}\lim_{h\to0^-}\frac{g'(h)}{2h}$$

$$\overset{\frac{g'(h)}{2h}\to\frac{g'(0)}{2(0)}=\frac{0}{0}}{\overset{L'H}{=}}\lim_{h\to0^-}\frac{g''(h)}{2}$$

$$=\frac{g''(0)}{2}=\frac{1}{2}.$$

Since the left-hand limits does not equal to right-hand limits and the limits $\lim_{h\to0}\frac{P(0+h)-P(0)}{h}=\lim_{h\to0}\frac{P(h)}{h}$ does not exist, thus the function $P(x)$ is not differentiable at $x=0$.

Remark:

It is OK to apply the L'Hopital Rule to check the differentiability since it just requires the differentiability of $f(x)$ and $g(x)$ which are given in the problem.

Extra Example 1

Compute the limits

$$\lim_{x\to\infty}\frac{x^2+1}{x} \quad and \quad \lim_{x\to 0}\frac{\tan^{-1}(3x^2)}{x^3+x^2}$$

(Note: Here, $\tan^{-1} y$ takes the value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

☺ Solution:

The first limits is of the type $\frac{x^2+1}{x} \to \frac{\infty}{\infty}$ (as $x \to \infty$). Using L'Hopital Rule, we get

$$\lim_{x\to\infty}\frac{x^2+1}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(x^2+1)}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{2x}{1} = \lim_{x\to\infty} 2x = \infty.$$

On the other hand, the second limits is of the type $\frac{\tan^{-1}(3x^2)}{x^3+x^2} \to \frac{\tan^{-1} 0}{0} = \frac{0}{0}$. By L'Hopital Rule again, we have

$$\lim_{x\to 0} \frac{\tan^{-1}(3x^2)}{x^3+x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\tan^{-1}(3x^2)}{\frac{d}{dx}x^3+x^2}$$

$$\frac{d}{dx}\tan^{-1}(3x^2) = \frac{d\tan^{-1}(3x^2)}{d(3x^2)}\frac{d(3x^2)}{dx}$$

$$\lim_{x\to 0} \frac{\frac{1}{1+(3x^2)^2}(6x)}{3x^2+2x} = \lim_{x\to 0} \frac{6x}{x(3x+2)(1+(3x^2)^2)}$$

$$= \lim_{x\to 0} \frac{6}{(3x+2)(1+(3x^2)^2)}$$

$$= \frac{6}{2(1)} = 3.$$

Extra Example 2

Compute the limits

$$\lim_{x\to\infty} x^2e^{-3x}.$$

☺ Solution:

Note that the limits is of the type $x^2e^{-3x} \to \infty^2 e^{-\infty} = \infty \times 0$, one has to transform the limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in order to apply L'Hopital Rule.

$$\lim_{x\to\infty} x^2e^{-3x} = \lim_{x\to\infty} \frac{x^2}{e^{3x}} \quad \left(\frac{\infty}{\infty}\ type\right)$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{3x}} = \lim_{x\to\infty} \frac{2x}{3e^{3x}} \quad \left(\frac{\infty}{\infty}\ type\right)$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}2x}{\frac{d}{dx}3e^{3x}} = \lim_{x\to\infty} \frac{2}{9e^{3x}} = \frac{2}{9 \times \infty} = 0.$$

Extra Example 3

Compute the limits

$$\lim_{x\to 0^+}\left(\frac{1}{\tan x}-\frac{1}{x}\right).$$

Since $\left(\frac{1}{\tan x}-\frac{1}{x}\right)\to\frac{1}{\tan 0}-\frac{1}{0}=\infty-\infty$ when $x\to 0^+$, we need to transform the limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in order to apply L'Hopital Rule. Note that

$$\lim_{x\to 0^+}\left(\frac{1}{\tan x}-\frac{1}{x}\right)=\lim_{x\to 0^+}\frac{x-\tan x}{x\tan x}\quad\left(\frac{0}{0}-type\right)$$

$$=\lim_{x\to 0^+}\frac{\frac{d}{dx}(x-\tan x)}{\frac{d}{dx}x\tan x}=\lim_{x\to 0^+}\frac{1-\sec^2 x}{\tan x+x\sec^2 x}\quad\left(\frac{0}{0}-type\right)$$

$$=\lim_{x\to 0^+}\frac{-2\sec x\,(\sec x\tan x)}{\sec^2 x+\sec^2 x+2x\sec x\,(\sec x\tan x)}=\frac{0}{1+1+0}=0.$$