MA1200 Calculus and Basic Linear Algebra I

Lecture Note 5

Exponential Function and Logarithmic Function

Exponential Function

An exponential function $f(x)$ is a function of the following form:

$$f(x) = a^x$$

where a is a real constant which $a > 0$ and $a \neq 1$.

Remark:

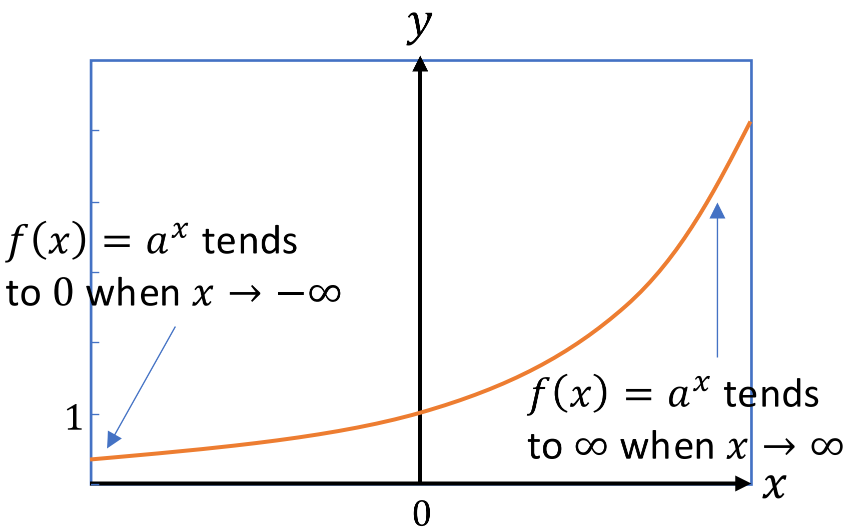
- Here, the domain of $f(x)$ is $\mathbb{R}$. (i.e. you can input any real number x into the function $f(x) = a^x$.)
- Here, we do not allow $a \leq 0$ since the function is then undefined for some values of x. For example, one cannot define the values of $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ and $(-2)^{\frac{1}{2}} = \sqrt{-2}$.
- When $a = 1$, the function simply becomes $f(x) = 1^x = 1$ which is a constant function.

Graph of Exponential Function

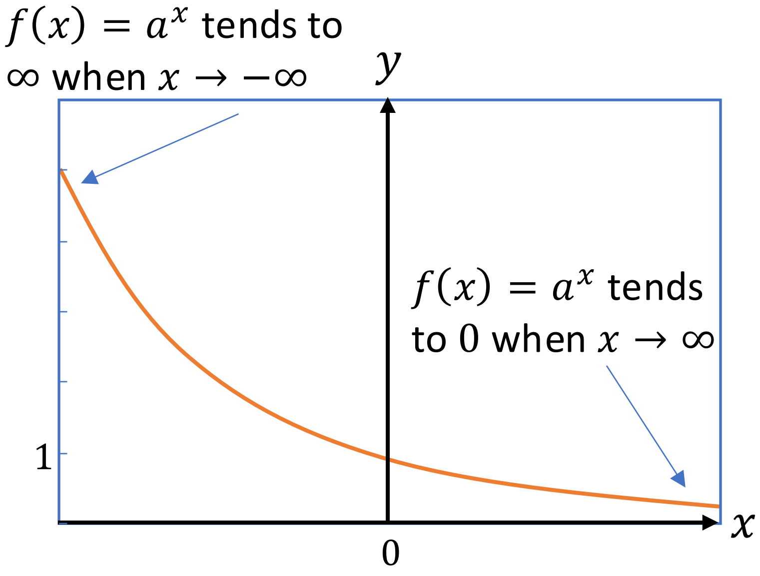
From the graphs, we observe that

- The range of the function $f(x)$ is always $(0, \infty)$ (i.e. $0 < f(x) < \infty$).
- For $0 < a < 1$, the function $f(x) = a^x$ is *strictly decreasing*.

  For $a > 1$, the function $f(x) = a^x$ is *strictly increasing*.



$f(x) = a^x$
$a > 1$



$f(x) = a^x$
$0 < a < 1$

Basic Operation of Exponential Functions

(1) $a^x \cdot a^y = a^{x+y}$, (2) $\dfrac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}$.

(3) $(a^x)^y = a^{xy}$

(4) $a^x \cdot b^x = (ab)^x$

Example 1

Compute $\frac{3^{2x+1}5^{2x-1}}{15^{2x}}$ and $\frac{100^{2x+1}}{25^{2x}}$.

☺Solution:

$$\frac{3^{2x+1}5^{2x-1}}{15^{2x}} = \frac{3^{2x+1} \cdot 5^{2x-1}}{3^{2x} \cdot 5^{2x}} = 3^{2x+1-2x} \cdot 5^{2x-1-2x} = 3 \cdot 5^{-1} = \frac{3}{5},$$

$$\frac{100^{2x+1}}{25^{2x}} = \frac{100 \cdot 100^{2x}}{25^{2x}} = \frac{100 \cdot (25^{2x} \cdot 4^{2x})}{25^{2x}} = 100 \cdot 4^{2x}.$$

Special exponential function e^x

The number e

Consider the expression $\left(1+\frac{1}{n}\right)^n$ where n is real number, this expression has a special property that $\left(1+\frac{1}{n}\right)^n$ tends to a finite number (≈ 2.71828) where n gets larger ($n \to \infty$):

n	100	1,000	10,000	100,000	1,000,000	10,000,000
$\left(1+\frac{1}{n}\right)^n$	2.70481	2.71692	2.71815	2.71827	2.71828	2.71828

This finite number is denoted by e $\overbrace{\left(=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)}^{value\ of\ \left(1+\frac{1}{n}\right)^n\ where\ n=\infty}$, is called *natural base*. The corresponding exponential function $f(x)=e^x$ is called *natural exponential function* (or simply exponential function).

Importance of exponential function e^x

- Practically, exponential function in many modeling such as population growth in biology and continuously compound interest in finance.
- In calculus, the exponential function e^x has a special property that

$$\frac{d}{dx}e^x = e^x, \qquad \int e^x dx = e^x + C.$$

  This property makes the exponential function useful in theoretical aspect (say in the study of differential equation).
- In algebra, the exponential function e^x is an important element in the Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

  where $i = \sqrt{-1}$. (You will learn more in MA1201)

Logarithmic Functions

Given a real number a (with $a > 0$ and $a \neq 1$), the *logarithmic function with base* a,

denoted by $y = f(x) = \log_a x$, is the number such that

$$a^y = x.$$

From the definition, one can see that the function $\log_a x$ is the inverse function of a^x. So we expect that

$$\log_a(a^x) = x \ \ and \ \ a^{\log_a x} = x.$$

Two commonly used logarithmic functions

- When $a = 10$, the function $\log_{10} x$ (or simply $\log x$) is called *common logarithm* which is commonly used in actual computation (say solving equation involving exponential function),
- When $a = e$, the function $\log_e x$ is called *natural logarithm* which is commonly used in calculus (e.g. differentiation and integration). We usually denote this function by $\ln x$.

Example 2

Compute $\log_5 25$, $\log_3 \sqrt[3]{9}$, $\log_2 0.5$ and $\ln 1$ using definition.

☺ Solution:

$y = \log_5 25 \Rightarrow 5^y = 25 \Rightarrow 5^y = 5^2 \Rightarrow y = \log_5 25 = 2.$

$y = \log_3 \sqrt[3]{9} \Rightarrow 3^y = \sqrt[3]{9} \Rightarrow 3^y = 9^{\frac{1}{3}} \Rightarrow 3^y = 3^{\frac{2}{3}} \Rightarrow y = \log_3 \sqrt[3]{9} = \frac{2}{3}.$

$y = \log_2 0.5 \Rightarrow 2^y = 0.5 \Rightarrow 2^y = \frac{1}{2} \Rightarrow 2^y = 2^{-1} \Rightarrow y = \log_2 0.5 = -1.$

$y = \ln 1 \Rightarrow e^y = 1 \Rightarrow e^y = e^0 \Rightarrow y = \ln 1 = 0.$

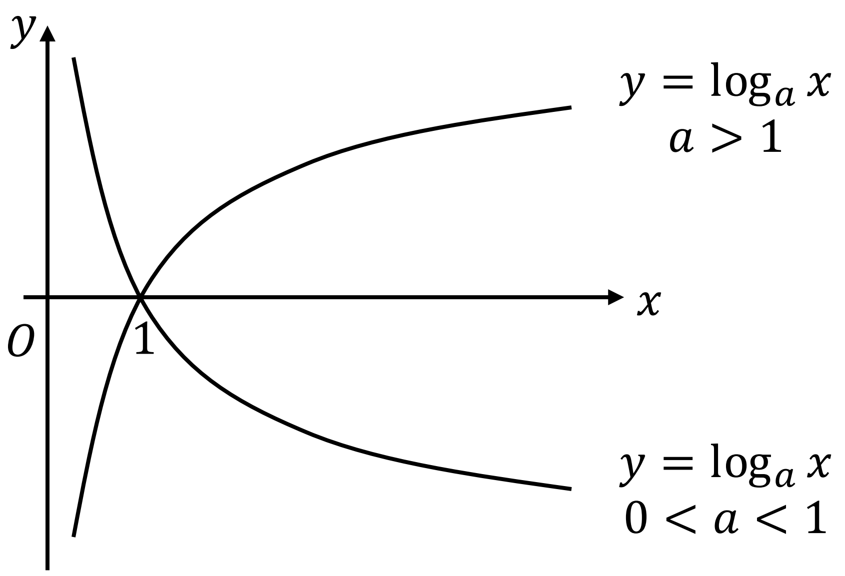
Example 3

Can we find the value of $\log(-2)$?

☺ Solution:

$y = \log(-2) \Rightarrow \underbrace{10^y}_{positive} = \underbrace{-2}_{negative}$. So $y = \log(-2)$ does not exist.

Graph of logarithmic functions and its properties



- The domain of $f(x) = \log_a x$ is $(0, \infty)$ (i.e. $0 < x < \infty$).
- The range of $f(x) = \log_a x$ is $\mathbb{R}$.
- For $0 < a < 1$, the function $f(x) = \log_a x$ is decreasing.

  For $a > 1$, the function $f(x) = \log_a x$ is increasing.

Operation of Logarithmic Functions

(The proof can be found in Appendix A)

1. $\log_a a^x = x \ \ and \ \ a^{\log_a x} = x$ (Properties of inverse functions)
2. $\log_a M^n = n \log_a M$
3. $\log_a MN = \log_a M + \log_a N$
4. $\log_a \frac{M}{N} = \log_a M - \log_a N$
5. $\log_a M = \frac{\log M}{\log a}$.

Example 4

$$2\log 8 + 3\log 5 - 3\log 2 = \log 8^2 + \log 5^3 - \log 2^3$$

$$= \log \frac{8^2 \times 5^3}{2^3} \overset{8=2^3}{=} \log \frac{2^6 \times 5^3}{2^3} = \log 10^3 = 3\log 10 \overset{\log 10=1}{=} 3.$$

Example 5

(a) Solve the equation $3 \cdot 5^{2x-1} + 2 = 23$.

(b) Solve the equation $2^{2x+2} - 17 \cdot 2^{x} + 4 = 0$.
(Hint: Note that $2^{2x+2} = 2^2 \cdot 2^{2x} = 4 \cdot 2^{2x}$)

☺ Solution:

(a) $3 \cdot 5^{2x-1} + 2 = 23 \Rightarrow 5^{2x-1} = 7 \Rightarrow (2x-1)\log 5 = \log 7$

$\Rightarrow x = \frac{1}{2}\left(\frac{\log 7}{\log 5} + 1\right) \approx 1.1045.$

(b) Let $y = 2^x$, then

$2^{2x+2} - 17 \cdot 2^x + 4 = 0 \Rightarrow 4 \cdot (2^x)^2 - 17 \cdot 2^x + 4 = 0$

$\Rightarrow 4y^2 - 17y + 4 = 0 \Rightarrow (4y-1)(y-4) = 0$

$\Rightarrow y = \frac{1}{4} \;\; or \;\; y = 4$

$\Rightarrow 2^x = \frac{1}{4} = 2^{-2} \;\; or \;\; 2^x = 4 = 2^2$

$\Rightarrow x = -2 \;\; or \;\; x = 2.$

Example 6

(a) Solve $9^{x+1} = 12^{x-1}$.

(b) $\log(x+1) + \log(2x-1) = \log 14$.

☺ Solution:

(a) By taking logarithm on both sides, we have

$$9^{x+1} = 12^{x-1} \Rightarrow \log 9^{x+1} = \log 12^{x-1}$$

$$\Rightarrow (x+1)\log 9 = (x-1)\log 12 \Rightarrow (\log 12 - \log 9)x = \log 9 + \log 12$$

$$\Rightarrow x = \frac{\log 9 + \log 12}{\log 12 - \log 9}.$$

(b) Note that

$$\log(x+1) + \log(2x-1) = \log 14 \Rightarrow \log[(x+1)(2x-1)] = \log 14$$

$$\Rightarrow (x+1)(2x-1) = 14 \Rightarrow 2x^2 + x - 15 = 0$$

$$\Rightarrow (x+3)(2x-5) = 0$$

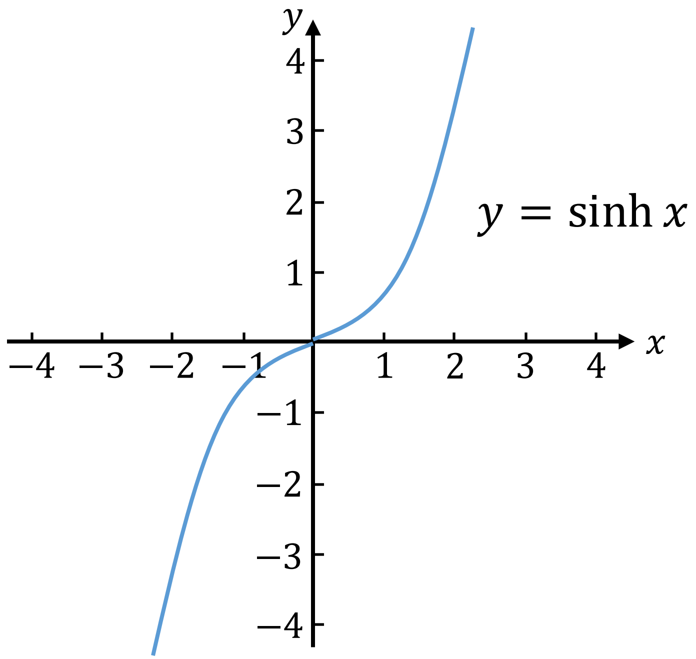
$$\Rightarrow x = -3\ (rejected)\ \ or\ \ x = \frac{5}{2}.$$

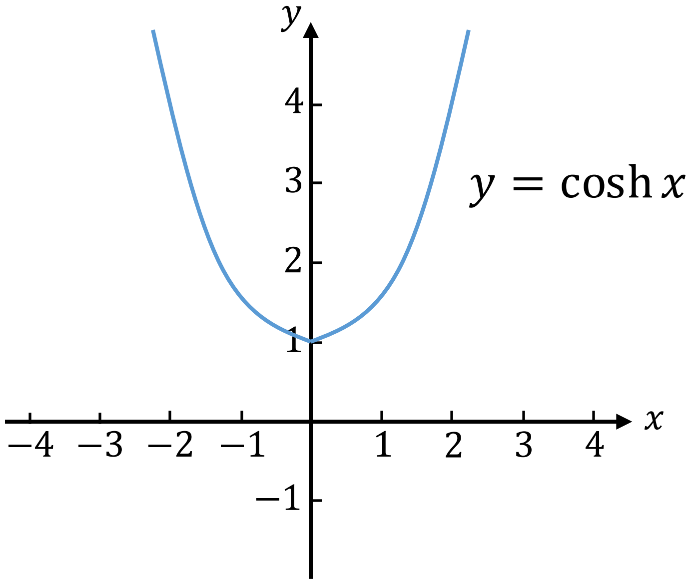
Hyperbolic Sine and Hyperbolic Cosine Functions

The hyperbolic sine and hyperbolic cosine function are defined as the algebraic combination of some exponential functions:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

The graphs of these functions are shown below:





Example 7

Solve the equation

$$\sinh x = 2.$$

☺ Solution:

Using the definition of $\sinh x$, we have

$$\sinh x = 2 \Rightarrow \frac{1}{2}(e^x - e^{-x}) = 2 \Rightarrow e^x - e^{-x} = 4$$

$$\Rightarrow e^{2x} - 1 = 4e^x \Rightarrow e^{2x} - 4e^x - 1 = 0.$$

We let $y = e^x$, then the equation becomes

$$y^2 - 4y - 1 = 0 \Rightarrow y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

$$\Rightarrow e^x = 2 + \sqrt{5} \ \ or \ \ 2 - \sqrt{5}\ (rej.)$$

$$\Rightarrow x = \ln\left(2 + \sqrt{5}\right).$$

Example 8

(a) Using the definition of $\sinh x$ and $\cosh x$, show that

(i) $\cosh^2 x - \sinh^2 x = 1.$

(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x.$

(b) Hence, or otherwise, solve the equation $3\sinh x + \cosh 2x = 0.$

☺ Solution of (a)

$$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}\left[e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x} - \left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right)\right] = \frac{1}{4}(4) = 1.$$

$$\cosh^2 x + \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 + \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}\left[e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x} + \left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right)\right] = \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x\,.$$

☺ Solution of (b)

Note that

$3\sinh x + \cosh 2x = 0 \Rightarrow 3\sinh x + (\cosh^2 x + \sinh^2 x) = 0$

$\Rightarrow 3\sinh x + (1 + \sinh^2 x) + \sinh^2 x = 0$

$\Rightarrow 2\sinh^2 x + 3\sinh x + 1 = 0.$

Let $y = \sinh x$, then the equation becomes

$$2y^2 + 3y + 1 = 0 \Rightarrow (2y+1)(y+1) = 0 \Rightarrow y = -\frac{1}{2} \ \ or\ y = -1.$$

$$\Rightarrow \sinh x = -\frac{1}{2} \ \ or\ \ \sinh x = -1.$$

$$\Rightarrow \frac{1}{2}(e^x - e^{-x}) = -\frac{1}{2} \ \ and\ \ \frac{1}{2}(e^x - e^{-x}) = -1.$$

Using similar techniques as in Example 7, we get

$$x = \ln\frac{-1+\sqrt{5}}{2} \ \ or\ \ x = \ln\left(-1+\sqrt{2}\right).$$

Summary of Chapter 3, 4 and 5

In these 3 chapters, we learn a lot about the properties of some elementary functions such as polynomials, rational functions, trigonometric functions, exponential and logarithmic functions. These properties will be useful when we study the calculus.

Things you need to know

Chapter 3 – Polynomial and rational functions

- Factorize the polynomial using remainder theorem and factor theorem.
- Method of partial functions – "Decompose" a complicated rational functions into a sum of simpler fractions:

$$\frac{5x+3}{x^3-2x^2-3x} = -\frac{1}{x} - \frac{1}{2(x+1)} + \frac{3}{2(x-3)}.$$

Chapter 4 – Trigonometric Functions

- Properties of trigonometric functions
- Compound angle formula $\cos(A+B)$, $\sin(A+B)$ and $\tan(A+B)$
- Sum-to-product formula (useful in computing limits) and Product-to-sum (useful in differentiation/ integration)

$$\underbrace{\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta}}_{\substack{difficult\ to \\ calculate}} \overset{\substack{sum-to-product \\ formula}}{\cong} -\underbrace{\frac{\sin 4\theta \cos\theta}{\sin 4\theta \sin\theta}}_{\substack{easy\ to \\ calculate}}$$

$$\underbrace{\sin^2\theta\cos^3\theta}_{difficult\ for\ calculus} \overset{\substack{product-to-sum \\ formula}}{\cong} \underbrace{\frac{1}{8}\cos\theta - \frac{1}{16}\cos 5\theta - \frac{1}{16}\cos 3\theta}_{easy\ for\ calculus}$$

Chapter 5 – Exponential and Logarithmic Function

- Basic properties of exponential and logarithmic function
- The natural exponential function $f(x) = e^x$ and natural logarithmic function $g(x) = \ln x$.

Appendix A – Proof of the "Operation of Logarithmic Function"

Property 1

It follows from the fact that $\log_a x$ is the inverse function of a^x.

Property 2

We let $A = \log_a M \Leftrightarrow M = a^A$.

Let $y = \log_a M^n$, then by the definition of logarithmic function, we have

$a^y = M^n \Rightarrow a^y = a^{An} \Rightarrow y = nA \Rightarrow \log_a M^n = n \log_a M.$

Property 3 & 4

We let $A = \log_a M \Leftrightarrow M = a^A$ and $B = \log_a N \Leftrightarrow N = a^B$.

Let $y = \log_a MN$, then by the definition of logarithmic function, we have

$a^y = MN \Rightarrow a^y = a^A a^B \Rightarrow a^y = a^{A+B} \Rightarrow y = A + B \Rightarrow \log_a MN = \log_a M + \log_a N.$

Let $z = \log_a \frac{M}{N}$, then we have

$$a^z = \frac{M}{N} \Rightarrow a^z = \frac{a^A}{a^B} \Rightarrow a^z = a^{A-B} \Rightarrow z = A - B$$

$$\Rightarrow \log_a \frac{M}{N} = \log_a M - \log_a N\,.$$

<u>Property 5</u>

We let $A = \log M$ and $B = \log a$ and we let $y = \log_a M$, then we have

$$a^y = M \Rightarrow a^y = 10^A \Rightarrow \log a^y = \log 10^A$$

$$\overset{\substack{From\\ Property\,2}}{\Rightarrow} \quad y \log a = A \Rightarrow y = \frac{A}{\log a} \Rightarrow \log_a M = \frac{A}{B}.$$