HW 1 Reference solution

Q1. Rewrite L_1 into point-slope form: $2x+3y=1+y$

$$2y=-2x+1$$
$$y=-x+\frac{1}{2}$$

$\therefore$ The slope of L_1 is $m_1=-1$

Since L is perpendicular to L_1, so the slope of L: $m\cdot m_1=-1$

$\therefore m=1$

Also, L intersects L_2 at $(w,3)$. L_2 is $6y-5x+2=0$

$(w,3)$ is on $6y-5x+2=0$ $\Rightarrow$ $6\times3-5w+2=0$

$\Rightarrow$ $w=4$

Use point-slope form, we have $y-3=1\cdot(x-4)$

$\Rightarrow$ $y=x-1$

$\therefore$ L is $y=x-1$ or $x-y-1=0$

Q2 We know line L passes through $(2t,t)$ and the slope is $-\frac{4}{3}$

$$y-t=-\frac{4}{3}(x-2t) \Rightarrow y=-\frac{4}{3}x+\frac{8}{3}t+t$$
$$\Rightarrow y=-\frac{4}{3}x+\frac{11}{3}t \qquad (t<0)$$

The interception point on x-axis: $0=-\frac{4}{3}x+\frac{11}{3}t$

$\Rightarrow$ $x=\frac{11}{4}t$

$\Rightarrow$ $(\frac{11}{4}t,0)$

The interception point on y-axis: $y=-\frac{4}{3}\cdot0+\frac{11}{3}t$

$\Rightarrow$ $y=\frac{11}{3}t$

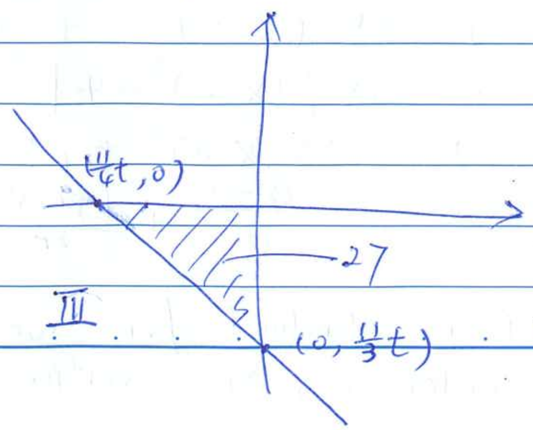
$\Rightarrow$ $(0,\frac{11}{3}t)$

The area of the triangle is 27

so $27=\frac{1}{2}\cdot\left|\frac{11}{4}t\right|\cdot\left|\frac{11}{3}t\right|$

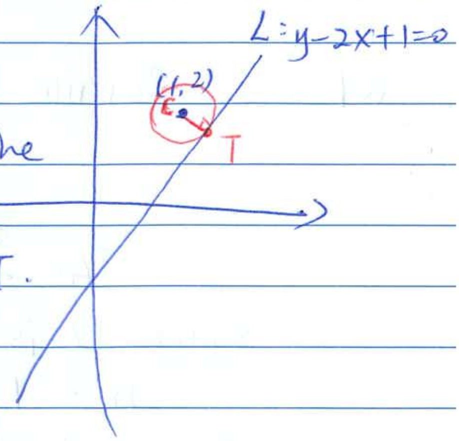
$\Rightarrow$ $54=\frac{11^2}{4\times3}t^2$

$\Rightarrow$ $t^2=\frac{648}{121}$ $\Rightarrow$ $t=-\frac{18}{11}\sqrt{2}<0$



Q3.

Since the Line L intersects with the circle only at one point, the Line is a tangent line. Let's call the centre point $(1,2)$ point C. the toughing point (intersection point point T.



Line CT is perpenticular to Line L.

As the slope of L is 2 as $y=2x-1$

so the slope of CT is $-\frac{1}{2}$

we can find $y-2 = -\frac{1}{2}(x-1)$

$\Rightarrow \quad y = -\frac{1}{2}x+\frac{1}{2}+2$

$\Rightarrow \quad y = -\frac{1}{2}x+\frac{5}{2}$

$\begin{cases} y=2x-1 \\ y=-\frac{1}{2}x+\frac{5}{2} \end{cases}$ $\underset{\text{intersection of 2 perpenticular lines}}{\overset{\text{find point } T}{\Rightarrow}}$ point T is $(\frac{7}{5}, \frac{9}{5})$

The radius of circle is the distance between C and T.

$r = \sqrt{(\frac{7}{5}-1)^2+(\frac{9}{5}-2)^2} = \sqrt{\frac{4+1}{25}} = \frac{\sqrt{5}}{5}$

$\therefore$ The equation of circle is $(x-1)^2+(y-2)^2 = \frac{1}{5}$

Q4

(a) $25x^2+y^2-100x+4y+74=0$

$\Rightarrow \quad 25x^2-100x \quad +y^2+4y \quad +74=0$

$\Rightarrow \quad 25(x^2-4x) + y^2+4y+74=0$

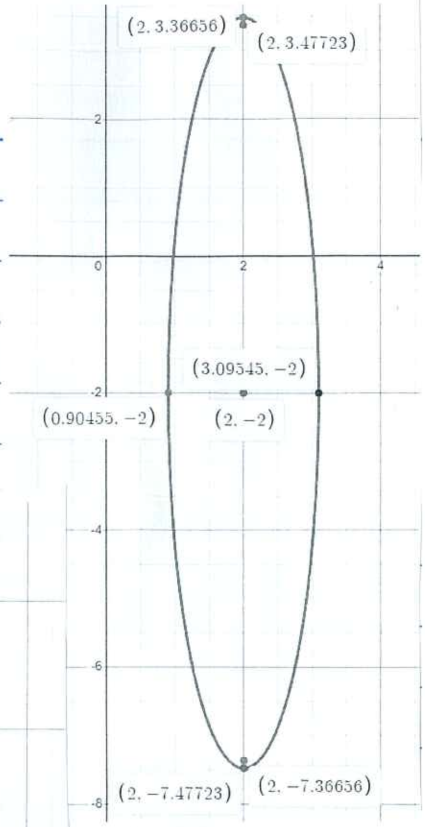
$\Rightarrow \quad 25[(x-2)^2-4] + (y+2)^2-4+74=0$

$\Rightarrow \quad 25(x-2)^2+(y+2)^2 = 30$

$\Rightarrow \quad \frac{(x-2)^2}{\frac{30}{25}} + \frac{(y+2)^2}{30} = 1 \quad \Rightarrow \quad \frac{(x-2)^2}{\left(\frac{\sqrt{30}}{5}\right)^2} + \frac{(y+2)^2}{(\sqrt{30})^2} = 1$

From standard form we know $b < a$

center: $(2,-2)$, vertices: $(2,-2+\sqrt{30})$, $(2,-2-\sqrt{30})$, $(2-\frac{\sqrt{30}}{5},-2)$ $(2+\frac{\sqrt{30}}{5},-2)$



$c=\sqrt{a^2-b^2} \quad =\sqrt{30-\frac{30}{25}} \quad = \sqrt{\frac{720}{25}} \quad = \frac{6}{5}\sqrt{20}$

foci : $\left(2, -2+\frac{6\sqrt{20}}{5}\right)$, $\left(2, -2-\frac{6\sqrt{20}}{5}\right)$

(b) $-y^2-8x+2y+9=0$

$\Rightarrow \quad -(y-2y) \quad -8x+9=0$

$\Rightarrow \quad -[(y-1)^2-1] -8x+9=0$

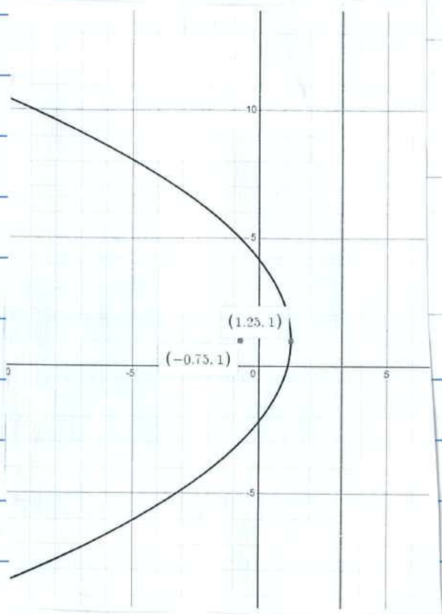
$\Rightarrow \quad -(y-1)^2 = 8x-10$

$\Rightarrow \quad (y-1)^2 = -8\left(x - \frac{5}{4}\right)$

$\Rightarrow \quad a=2$

~~cente~~ vertex : $\left(\frac{5}{4}, 1\right)$

Directrix : $x=\frac{13}{4}$



(c) $16x^2-9y^2-32x+36y-151=0$

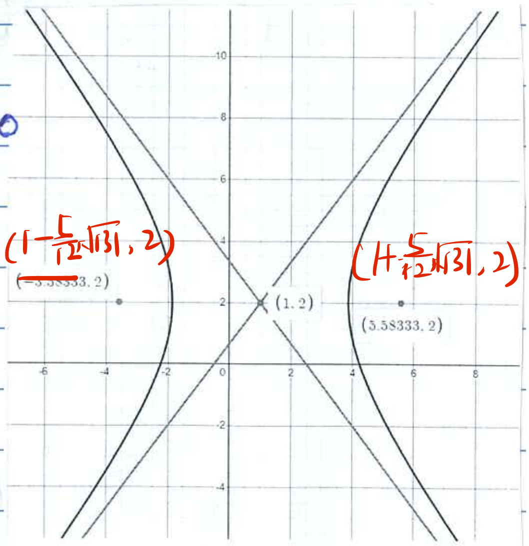
$\Rightarrow \quad 16(x^2-2x) + (-9)(y^2-4y)-151=0$

$\Rightarrow \quad 16[(x-1)^2-1] - 9[(y-2)^2-4] - 151=0$

$\Rightarrow \quad 16(x-1)^2 - 9(y-2)^2 = \cancel{121}\ 131$

$\Rightarrow \quad \dfrac{(x-1)^2}{\frac{\cancel{121}\,131}{16}} - \dfrac{(y-2)^2}{\frac{\cancel{121}\,131}{9}} = 1$

$\Rightarrow \quad \dfrac{(x-1)^2}{\underset{\uparrow\, a}{\cancel{\left(\frac{11}{4}\right)^2}\left(\frac{\sqrt{131}}{4}\right)^2}} - \dfrac{(y-2)^2}{\underset{\uparrow\, b}{\cancel{\left(\frac{11}{3}\right)^2}\left(\frac{\sqrt{131}}{3}\right)^2}} = 1$

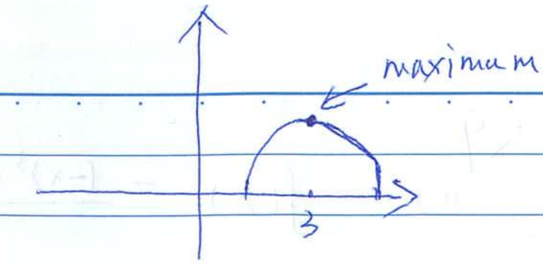


$c=\sqrt{a^2+b^2} \quad = \sqrt{\frac{131}{16}+\frac{131}{9}} \quad = \sqrt{\frac{3275}{144}} \quad = \cancel{\frac{55}{12}} \quad \frac{5}{12}\sqrt{131}$

Foci : $\left(1-\frac{55}{12}\sqrt{131}, 2\right)$, $\left(1+\frac{55}{12}\sqrt{131}, 2\right)$

center : $(1, 2)$

Asymptotes : $y-2=-\frac{4}{3}(x-1)$ and $y-2=\frac{4}{3}(x-1)$



Q5

(a) $F(x) = \sqrt{-x^2+6x-5}$

$-x^2+6x-5 \geq 0 \Rightarrow -(x^2-6x+5) \geq 0 \Rightarrow (x-5)(x-1) \leq 0$
$\Rightarrow 1 \leq x \leq 5$

Domain: $[1, 5]$

Since $-x^2+6x-5$ is a parabola (downward), the maximum value is achieved at vertex. $(3, 4)$. and the minimum value is 0. Square root function is an increasing function. As a result, the range of $F(x)$ is $[0, \sqrt{4}] = [0, 2]$.

$G(x) = \frac{3}{x^3-3x^2+3x-1} = \frac{3}{(x-1)^3}$

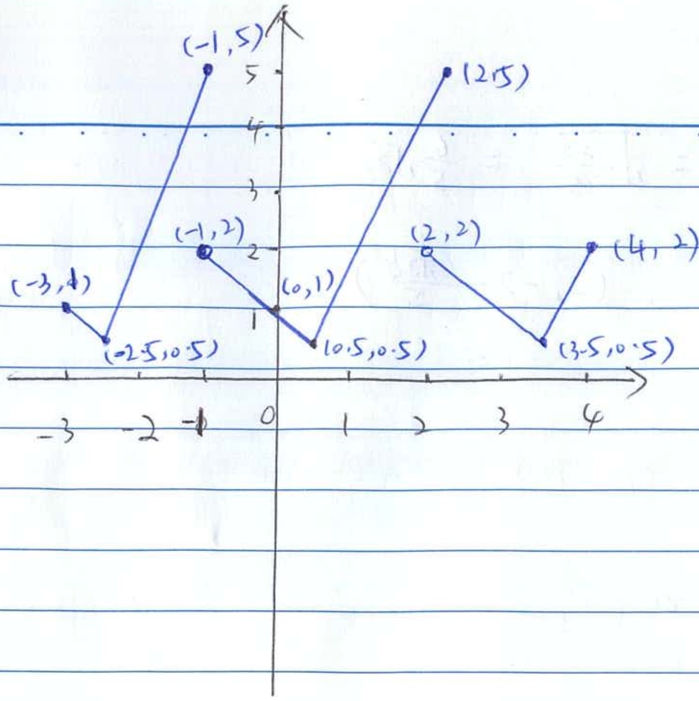
$(x-1)^3$ is at the denominator and it cannot be zero.
$(x-1)^3 \neq 0 \Rightarrow x \neq 1$
Domain: $(-\infty, 1) \cup (1, \infty)$

We know $(x-1)^3$ is an increasing function. $\frac{3}{(x-1)^3}$ will be very big if $(x-1)^3 \to 0^+$, and it will go to $-\infty$ if $(x-1)^3 \to 0^-$
And $\frac{3}{(x-1)^3} \neq 0$ as $3 \neq 0$

So the range of $G(x)$ is $(-\infty, 0) \cup (0, \infty)$

(b) $(F \circ G)(x) = F(G(x)) = \sqrt{-\left(\frac{3}{(x-1)^3}\right)^2 + 6\left(\frac{3}{(x-1)^3}\right) - 5}$
$G(x)$ should satisfy the restriction of $F(x)$ on domain i.e. $\frac{3}{(x-1)^3} \in [1, 5]$. As $\frac{3}{(x-1)^3}$ is decreasing for $x > 1$. so we can find that when $x = \sqrt[3]{\frac{3}{5}} + 1$, $\frac{3}{(x-1)^3} = 5$; when $x = \sqrt[3]{3} + 1$, $\frac{3}{(x-1)^3} = 1$.
So the largest possible domain is $\left[\sqrt[3]{\frac{3}{5}} + 1, \sqrt[3]{3} + 1\right]$.

Q6



Q7 $|4-3x| = 2|2x+1|$

$\Rightarrow (4-3x)^2 = [2(2x+1)]^2$

$\Rightarrow 16+9x^2-24x = 4(4x^2+1+4x)$

$\Rightarrow 7x^2+16x+24x+4-16=0$

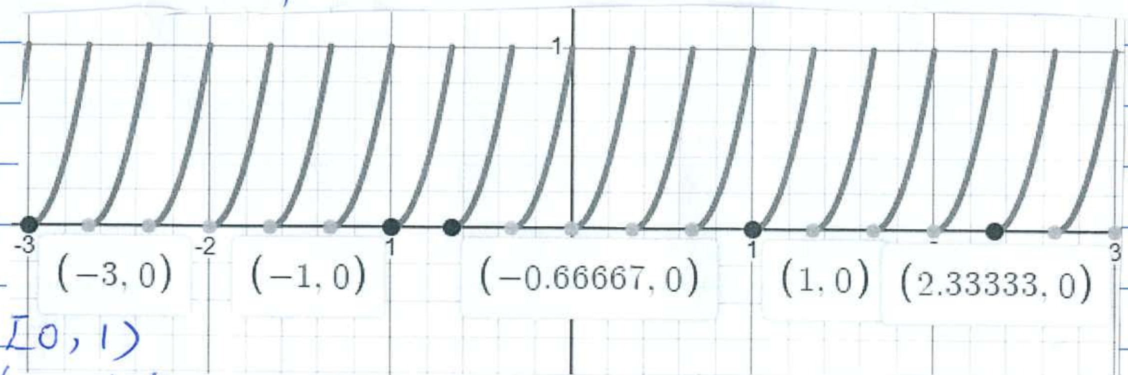
$\Rightarrow 7x^2+40x-12=0$

$\Rightarrow x = \frac{-40 \pm \sqrt{40^2-4\times7\times(-12)}}{14} = -\frac{40}{14} \pm \frac{44}{14} = -6$ or $\frac{4}{14}$

$\therefore x=-6$ or $x=\frac{4}{14}$

Q8

(a)



(b) range : $[0, 1)$

(c) Yes, it's periodic.

$[3x+2]$ is a ~~per~~ ~~step wise~~ piecewise constant function

The difference between $3x+2$ and $[3x+2]$ cannot be equal or bigger than 1. As a resule, the period should be $1/3$ as the graph.

Q9.

(a) $f(-x) = \dfrac{(-x)^3 + 6(-x)^2 + 12(-x) + 8}{(-x)^4 + 1}$

$= \dfrac{-x^3 + 6x^2 - 12x + 8}{x^4 + 1} \neq f(x)$

or $\neq -f(x)$

so it's neither even or odd.

(b) $f(-x) = \dfrac{(-x)^4}{(-x)^3 + 3} = \dfrac{x^4}{-x^3 + 3} \neq \dfrac{x^4}{x^3 + 3} = f(x)$

or $\neq \dfrac{-x^4}{x^3 + 3} = -f(x)$

so it's neither even or odd.

Q10 $F(x) = x^3 + 3x^2 + 3x + 1 = (x+1)^3$ Domain : $[-1, \infty)$

For any

$-1 \le x_1 < x_2$

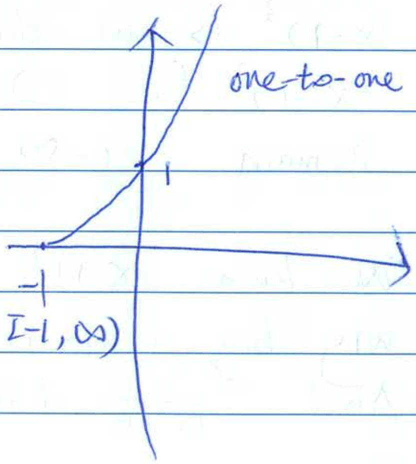
$\Rightarrow 0 \le (x_1 + 1) < (x_2 + 1)$

$\Rightarrow (x_1 + 1)^3 < (x_2 + 1)^3$

$\Rightarrow F(x_1) < F(x_2)$

$\therefore$ $F(x)$ is strictly increasing function in $[-1, \infty)$

$\therefore$ $F(x)$ is one-to-one



Or For any $-1 \le x_1 \neq x_2$

$\Rightarrow 0 \le x_1 + 1 \neq x_2 + 1$

$\Rightarrow (x_1 + 1)^3 \neq (x_2 + 1)^3$

$\Rightarrow F(x_1) \neq F(x_2)$

$\Rightarrow$ one-to-one.

Let $y = (x+1)^3 \Rightarrow \sqrt[3]{y} = x + 1 \Rightarrow \sqrt[3]{y} - 1 = x \Rightarrow F^{-1}(x) = \sqrt[3]{x} - 1$

As $F(x)$ is an increasing function, it's minimum is at $x = -1$, $F(-1) = 0$

The range of $F(x)$ is the domain of $F^{-1}(x)$, which is $[0, \infty)$

The domain of $F(x)$ is the range of $F^{-1}(x)$, which is $[-1, \infty)$.