MA1200 – Calc & Basic Linear Algebra I
Mid-term Test for CA1, CB1, CC1 and CD1 Solution

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1. *Solution.*

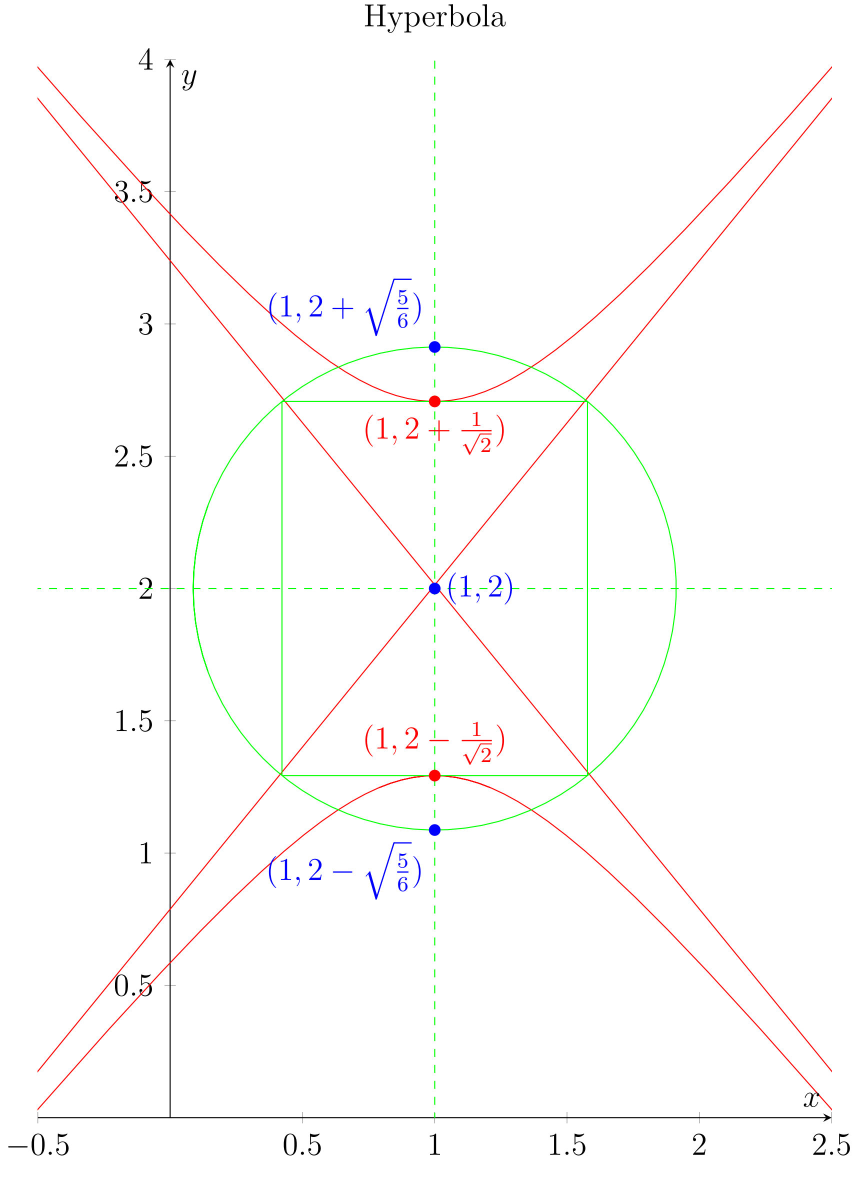
   (a) $3x^2 - 2y^2 - 6x + 8y - 4 = 0 \implies -\frac{(x-1)^2}{\frac{1}{3}} + \frac{(y-2)^2}{\frac{1}{2}} = 1.$

   (b) Center:$(1, 2)$.

   (c) Vertices:$(1, 2 + \frac{1}{\sqrt{2}}), (1, 2 - \frac{1}{\sqrt{2}})$.

   (d) $c^2 = 1/3 + 1/2 \implies c = \sqrt{\frac{5}{6}}$, foci:$(1, 2 + \sqrt{\frac{5}{6}}), (1, 2 - \sqrt{\frac{5}{6}})$

   (e)



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2. Express the following rational functions in partial fraction.

Solution.

$$\frac{3x^4+5x^2+2x-10}{6x^2+x-2} = \frac{x^2}{2} - \frac{x}{12} + \frac{73}{72} + \frac{\frac{59}{72}x - \frac{287}{36}}{(3x+2)(2x-1)}$$

$$\begin{aligned}
\text{Let } \frac{\frac{59}{72}x - \frac{287}{36}}{(3x+2)(2x-1)} &= \frac{A}{(3x+2)} + \frac{B}{(2x-1)} \\
\frac{59}{72}x - \frac{287}{36} &= A(2x-1) + B(3x+2) \qquad (*) \\
(*) \text{ with } x = -\frac{2}{3}: \qquad -\frac{230}{27} &= A\left(-\frac{7}{3}\right) \implies A = \frac{230}{63} \\
(*) \text{ with } x = \frac{1}{2}: \qquad -\frac{121}{16} &= B\left(\frac{7}{2}\right) \implies B = -\frac{121}{56} \\
\text{Hence, } \frac{3x^4+5x^2+2x-10}{6x^2+x-2} &= \frac{x^2}{2} - \frac{x}{12} + \frac{73}{72} + \frac{230}{63(3x+2)} - \frac{121}{56(2x-1)}
\end{aligned}$$

Or

$$\frac{3x^4+5x^2+2x-10}{6x^2+x-2} = \frac{1}{72}\left(36x^2 - 6x + 73 + \frac{59x-574}{(3x+2)(2x-1)}\right)$$

$$\begin{aligned}
\text{Let } \frac{59x-574}{(3x+2)(2x-1)} &= \frac{A}{(3x+2)} + \frac{B}{(2x-1)} \\
59x - 574 &= A(2x-1) + B(3x+2) \qquad (*) \\
(*) \text{ with } x = -\frac{2}{3}: \qquad -\frac{1840}{3} &= A\left(-\frac{7}{3}\right) \implies A = \frac{1840}{7} \\
(*) \text{ with } x = \frac{1}{2}: \qquad -\frac{1089}{2} &= B\left(\frac{7}{2}\right) \implies B = -\frac{1089}{7} \\
\text{Hence, } \frac{3x^4+5x^2+2x-10}{6x^2+x-2} &= \frac{1}{72}\left(36x^2 - 6x + 73 + \frac{1840}{7(3x+2)} - \frac{1089}{7(2x-1)}\right)
\end{aligned}$$

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3. Solve the equation:

Solution.

$$\begin{aligned}
2\tan x + \sec x &= 2 \\
2\sin x + 1 &= 2\cos x \\
\sin x - \cos x &= -\frac{1}{2}
\end{aligned}$$

Let $\sin x - \cos x = A\cos(x-\alpha) = A\cos x\cos\alpha + A\sin x\sin\alpha \implies$

$$\begin{aligned} A\sin\alpha &= 1 && \text{(i)}\\ A\cos\alpha &= -1 && \text{(ii)}\\ \frac{\text{(i)}}{\text{(ii)}}: \qquad \tan\alpha &= -1\\ \text{take} \qquad \alpha &= -\frac{\pi}{4}\\ \text{hence,} \qquad A &= -\sqrt{2}\\ \text{hence, } -\sqrt{2}\cos(x+\pi/4) &= -\frac{1}{2}\\ \cos(x+\pi/4) &= \frac{1}{2\sqrt{2}}\\ (x+\pi/4) &= \pm\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right) + 2n\pi \qquad n\in\mathbf{Z}\\ x &= -\pi/4 \pm \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right) + 2n\pi \qquad n\in\mathbf{Z}\\ x &\approx \ldots, -5.859, -1.9948, 0.424, 4.288, \ldots \end{aligned}$$

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4. Solve the equation: $2^{x+1} - 2^{-x} = 1$.

Solution.

$$2^{x+1} - 2^{-x} = 1 \Rightarrow 2^{2x+1} - 1 = 2^x \Rightarrow 2\times(2^x)^2 - 2^x - 1 = 0$$

By letting $y = 2^x$, we have

$$2y^2 - y - 1 = 0 \Rightarrow (2y+1)(y-1) = 0$$

$$\Rightarrow y = 1 \text{ or } y = -\frac{1}{2} \text{ (rejected).}$$

As a result, $2^x = y = 1 \Rightarrow x = 0$. ■

5. Find the following limits

Solution.

[a]

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\times\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)}$$
$$= \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

[b] We need to consider this question by evaluating left hand side limit and right hand side limite separately:

As $-1 \leq \sin\frac{1}{x} \leq 1$, and $\sin 5x$ is positive when $x \to 0^+$, we have $-(\sin 5x) \leq (\sin 5x)\cdot(\sin\frac{1}{x}) \leq (\sin 5x)$. Moreover, we can show that

$$\lim_{x\to 0^+}(\sin 5x) = 0 \text{ and } \lim_{x\to 0^+} -(\sin 5x) = 0.$$

By sandwich theorem, we can prove that $\lim_{x\to 0^+}(\sin 5x)\cdot(\sin\frac{1}{x}) = 0$.

Similarly, as $-1 \leq \sin\frac{1}{x} \leq 1$, and $\sin 5x$ is negative when $x \to 0^-$, we have $(\sin 5x) \leq (\sin 5x)\cdot(\sin\frac{1}{x}) \leq -(\sin 5x)$. Moreover, we can show that

$$\lim_{x\to 0^-}(\sin 5x) = 0 \text{ and } \lim_{x\to 0^-} -(\sin 5x) = 0.$$

By sandwich theorem, we can prove that $\lim_{x\to 0^-}(\sin 5x)\cdot(\sin\frac{1}{x}) = 0$.

Therefore,

$$\lim_{x\to 0^+}(\sin 5x)\cdot(\sin\frac{1}{x}) = 0 = \lim_{x\to 0^-}(\sin 5x)\cdot(\sin\frac{1}{x}) \Rightarrow \lim_{x\to 0}(\sin 5x)\cdot(\sin\frac{1}{x}) = 0.$$

[c]

$$\lim_{x\to+\infty}\frac{x^2+2x+\sin 2x}{2x^2-3\cos x-2} = \lim_{x\to+\infty}\frac{1+\frac{2}{x}+\frac{\sin 2x}{x^2}}{2-\frac{3\cos x}{x^2}-\frac{2}{x^2}}.$$

As $-1 \leq \sin 2x \leq 1$ and $-1 \leq \cos x \leq 1$, we can easily show that $-\frac{1}{x^2} \leq \frac{\sin 2x}{x^2} \leq \frac{1}{x^2}$ and $-\frac{1}{x^2} \leq \frac{\cos x}{x^2} \leq \frac{1}{x^2}$. By sandwich theorem that $\lim_{x\to+\infty} -\frac{1}{x^2} = \lim_{x\to+\infty}\frac{1}{x^2} = 0$, we can show $\frac{\sin 2x}{x^2} = 0$ and $\frac{\cos x}{x^2} = 0$. Combining the fact that $\lim_{x\to+\infty}\frac{2}{x} = 0$ and $\lim_{x\to+\infty}\frac{2}{x^2} = 0$, we have

$$\lim_{x\to+\infty}\frac{x^2+2x+\sin 2x}{2x^2-3\cos x-2} = \lim_{x\to+\infty}\frac{1+\frac{2}{x}+\frac{\sin 2x}{x^2}}{2-\frac{3\cos x}{x^2}-\frac{2}{x^2}} = \frac{1+0+0}{2-0-0} = \frac{1}{2}.$$

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6. Let $f(x) = \begin{cases} \frac{\sin(3(x-2))\sec(2(x-2))}{x-2} & \text{if } x < 2; \\ 7-x^2 & \text{if } x \geq 2. \end{cases}$ Determine whether the function is continuous at $x = 2$. Justify your answer.

Solution. Firstly, $f(2) = 7 - 2^2 = 3$. Then,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 7 - x^2 = 7 - 2^2 = 3.$$

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}\frac{\sin(3(x-2))\sec(2(x-2))}{x-2} \\ &= \lim_{x\to 2^-} 3\frac{\sin(3(x-2))}{3(x-2)}\cdot\frac{1}{\cos(2(x-2))} = 3\times 1\times 1 = 3.\end{aligned}$$

Therefore, $f(2) = \lim_{x\to 2} f(x) = 3$ and it is continuous at $x = 2$. ■

— *End* —