MA1200 Calculus and Basic Linear Algebra I

Lecture Note 8

Differentiation

Definition (Differentiability of function)

We say a function $f(x)$ is differentiable at $x = c$ if the limits

$$\lim_{h\to 0} \frac{\overbrace{f(c+h)-f(c)}^{change\ of\ f(x)\ from\ c\ to\ c+h}}{\underbrace{h}_{=(c+h)-c}}$$

exists as a real number.

- If $f(x)$ is differentiable $x = c$, then the limits $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h}$ is called the *derivative of* $f(x)$ and usually denoted by $\frac{df}{dx}(c)$ or $f'(c)$., i.e.

$$\frac{df}{dx}|_{x=c} = f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}.$$

- The most direct way to find the derivative is to evaluate the limits $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h}$. This method is called *differentiation from the first principle*.

Differentiability and derivative of some elementary functions

Example 1 (Constant function)

Show that the constant function $f(x) = c$ is always differentiable and find the derivative $\frac{df}{dx}$ (or $f'(x)$).

☺ Solution:

Using the first principle, we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Therefore, $f(x) = c$ is differentiable and

$$\frac{df}{dx} = f'(x) = 0.$$

Example 2 (Polynomial)

Show that the function $f(x) = 2x^3$ is differentiable and find the derivative $f'(x)$.

☺ Solution:

Using the first principle again, we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^3-2x^3}{h} = 2\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= 2\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} = 2\lim_{h\to 0}(3x^2+3xh+h^2) = 2(3x^2+3x(0)+0^2)$$

$$= 6x^2.$$

Hence $f(x) = 2x^3$ is differentiable and

$$\frac{df}{dx} = f'(x) = 6x^2.$$

Remark

In general, one can use the similar method to show that for any positive integer n, the function $f(x) = x^n$ is differentiable at any x and

$$\frac{df}{dx} = f'(x) = nx^{n-1}.$$

☺Proof :

Using the first principle and Binomial theorem (see Chapter 3), we have

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + C_1^n x^{n-1}h + C_2^n x^{n-2}h^2 + \cdots + C_n^n h^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{C_1^n x^{n-1}h + C_2^n x^{n-2}h^2 + \cdots + C_n^n h^n}{h}$$

$$= \lim_{h\to 0} \left(C_1^n x^{n-1} + \underbrace{C_2^n x^{n-2}h + \cdots + C_n^n h^{n-1}}_{\to 0 \ as \ h\to 0} \right) = C_1^n x^{n-1} = nx^{n-1}.$$

Therefore $f(x) = x^n$ is differentiable at any x and $\frac{df}{dx} = nx^{n-1}$.

Example 3 (Trigonometric Functions)

Show that the function $f(x) = \sin x$ is differentiable at any x and find $\frac{df}{dx}$.

☺ Solution:

Using the first principle again, we have

$$\lim_{h\to 0}\left[\frac{\sin(x+h)-\sin x}{h}\right] \overset{\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{=} \lim_{h\to 0}\frac{2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2}}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} = \lim_{h\to 0}\underbrace{\cos\left(x+\frac{h}{2}\right)}_{\to \cos x \text{ as } h\to 0}\underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\to 1 \text{ as } h\to 0} = \cos x\,.$$

Hence $f(x) = \sin x$ is differentiable and $\frac{df}{dx} = \cos x$.

(The derivation of $\cos x$ is similar and left as exercise)

Example 4

Show that $f(x) = \tan x$ is also differentiable and find $\frac{df}{dx}$.

☺ Solution:

Using the first principle again, we have

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \overset{\tan\theta=\frac{\sin\theta}{\cos\theta}}{\cong} \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0}\frac{\overbrace{\sin(x+h)\cos x-\cos(x+h)\sin x}^{\sin A\cos B-\cos A\sin B}}{h\cos(x+h)\cos x} = \lim_{h\to 0}\frac{\overbrace{\sin[(x+h)-x]}^{\substack{\sin(A-B)\\ =\sin A\cos B-\cos A\sin B}}}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\underbrace{\left(\frac{\sin h}{h}\right)}_{\to 1}\left(\frac{1}{\underbrace{\cos(x+h)}_{\to\cos x}\cos x}\right) = \frac{1}{\cos^2 x} = \sec^2 x\,.$$

So $f(x) = \tan x$ is differentiable and $\frac{df}{dx} = \sec^2 x$.

Example 5 (Exponential Function)

We define a number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (i.e. $e = 2.71828\ldots$) and consider the exponential function $f(x) = e^x$.

Show that the $f(x) = e^x$ is differentiable and find $\frac{d}{dx}e^x$.

☺ Solution:

Note that

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} \ldots\ldots (*)$$

To compute $\lim_{h\to 0}\frac{e^h-1}{h}$, we recall from Example 29 of Lecture Note 6 (Limits) that (replacing x by h)

$$e^h = \lim_{n\to\infty}\left(1+\frac{h}{n}\right)^n.$$

Using the binomial theorem to expand $\left(1+\frac{h}{n}\right)^n$, we have

$$\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{h\to 0}\frac{\lim\limits_{n\to\infty}\left(1+\frac{h}{n}\right)^n-1}{h}=\lim_{h\to 0}\lim_{n\to\infty}\frac{\left(1+\frac{h}{n}\right)^n-1}{h}$$

$$=\lim_{h\to 0}\lim_{n\to\infty}\frac{\left(1+C_1^n\frac{h}{n}+C_2^n\left(\frac{h}{n}\right)^2+\cdots+C_n^n\left(\frac{h}{n}\right)^n\right)-1}{h}$$

$$=\lim_{h\to 0}\lim_{n\to\infty}\frac{\left[C_1^n\frac{h}{n}+C_2^n\left(\frac{h}{n}\right)^2+\cdots+C_n^n\left(\frac{h}{n}\right)^n\right]}{h}$$

$$=\lim_{h\to 0}\lim_{n\to\infty}\left[n\frac{1}{n}+\left(\frac{n(n-1)}{2}\right)\frac{h}{n^2}+\left(\frac{n(n-1)(n-2)}{3!}\right)\frac{h^2}{n^3}\ldots+\frac{h^{n-1}}{n^n}\right]$$

$$= \lim_{h\to 0}\lim_{n\to\infty}\left[1+\underbrace{\frac{1}{2}(1)\left(1-\frac{1}{n}\right)h+\frac{1}{3!}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)h^2+\cdots+\frac{h^{n-1}}{n^n}}_{\to 0\ when\ n\to\infty, h\to 0}\right]$$

$= 1.$

Therefore from the result of $(*)$, we have

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x(1) = e^x.$$

So $f(x) = e^x$ and $\frac{df}{dx} = e^x$.

☹ Caution:

If $f(x) = a^x$ and $a \neq e$, then $\frac{df}{dx} = \frac{d}{dx}a^x \neq a^x$!!!!! In fact, we will show that $\frac{d}{dx}a^x = a^x \ln a$ for any real number a.

Summary of Derivatives of Elementary Functions

As shown in the previous example, most of the elementary functions (say polynomial, exponential functions, trigonometric functions) are differentiable in its domain. The following table summarizes the derivatives of these functions:

$y = f(x)$	The derivative $f'(x)$
$y = c \quad (c\ is\ constant)$	$\frac{dy}{dx} = 0$
$y = x^a \quad (a\ is\ real)$	$\frac{dy}{dx} = ax^{a-1}$
$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x \quad (Careful!)$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x = \frac{1}{\cos^2 x}$
$y = e^x$	$\frac{dy}{dx} = e^x$

Example 6 (Absolute value function)

We consider the function $f(x) = |x| \left(= \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases} \right)$.

(a) Determine if $f(x)$ is differentiable at $x = 2$ and find $f'(2)$.

(b) Determine if $f(x)$ is differentiable at $x = 0$.

☺ Solution:

(a) Using the first principle, we get

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{|2+h| - |2|}{h} = \lim_{h \to 0} \frac{\overbrace{2+h}^{\substack{When\ h \to 0 \\ 2+h\ is \\ positive}} - 2}{h}$$

$$= \lim_{h \to 0} 1 = 1.$$

So $f(x)$ is differentiable at $x = 2$ and $f'(2) = 1$.

(b) Note that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}.$$

To compute the limits, we need to consider the left hand limit and right hand limit

$$\lim_{h\to 0^+}\frac{|h|}{h}\overset{\substack{h>0\\|h|=h}}{\widehat{=}}\lim_{h\to 0^+}\frac{h}{h}=1,\qquad \lim_{h\to 0^-}\frac{|h|}{h}\overset{\substack{h<0\\|h|=-h}}{\widehat{=}}\lim_{h\to 0^+}\frac{-h}{h}=-1.$$

Since $\lim_{h\to 0^+}\frac{|h|}{h}\neq\lim_{h\to 0^-}\frac{|h|}{h}$, so the limits $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence $f(x)$ is <u>not</u> differentiable at $x=0$.

Example 7

We let

$$f(x) = \begin{cases} x & if\ x \geq 1 \\ x^2 & if\ x < 1 \end{cases}.$$

Determine if the function is differentiable at $x = 1$. How about the case for $x = 0$?

☺ Solution:

At $x = 1$

To check the differentiability, we consider the limits

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{f(1+h) - 1}{h}.$$

Since $1 + h$ can be greater or less that 1 and $(1 + h)$ takes different form for each case, we need to consider Left-hand limits and Right-hand limits.

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{\overbrace{1+h}^{\substack{1+h>1\\ \Rightarrow f(x)=x}}-1}{h}=\lim_{h\to 0^+}\frac{h}{h}=1,$$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{\overbrace{(1+h)^2}^{\substack{1+h<1\\ \Rightarrow f(x)=x^2}}-1}{h}=\lim_{h\to 0^-}\frac{2h+h^2}{h}=\lim_{h\to 0^-}(2+h)=2.$$

Since $\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}\neq\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$, so the limits $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ does not exist and $f(x)$ is not differentiable at $x=1$.

<u>At $x=0$</u>, we consider

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\overbrace{h^2}^{\substack{0+h<1\\ \Rightarrow f(x)=x^2}}-0}{h}=\lim_{h\to 0^+}h=0.$$

Hence, $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

Example 8

Consider the function $f(x) = \sqrt[3]{x}$, determine whether the function is differentiable at $x = c$ $(c \neq 0)$ and $x = 0$ respectively.

☺ Solution:

<u>Differentiability at $x = c$ $(c \neq 0)$</u>

Using first principle, we get

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{c+h}-\sqrt[3]{c}}{h}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt[3]{c+h}-\sqrt[3]{c}\right)\left[\left(\sqrt[3]{c+h}\right)^2+\left(\sqrt[3]{c+h}\right)\left(\sqrt[3]{c}\right)+\left(\sqrt[3]{c}\right)^2\right]}{h\left[\left(\sqrt[3]{c+h}\right)^2+\left(\sqrt[3]{c+h}\right)\left(\sqrt[3]{c}\right)+\left(\sqrt[3]{c}\right)^2\right]}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt[3]{c+h}\right)^3-\left(\sqrt[3]{c}\right)^3}{h\left[\left(\sqrt[3]{c+h}\right)^2+\left(\sqrt[3]{c+h}\right)\left(\sqrt[3]{c}\right)+\left(\sqrt[3]{c}\right)^2\right]}$$

$$= \lim_{h \to 0} \frac{c + h - c}{h\left[\left(\sqrt[3]{c+h}\right)^2 + \left(\sqrt[3]{c+h}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2\right]}$$

$$= \lim_{h \to 0} \frac{1}{\underbrace{\left(\sqrt[3]{c+h}\right)^2 + \left(\sqrt[3]{c+h}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2}_{denominator \to \left(\sqrt[3]{c}\right)^2 + \left(\sqrt[3]{c}\right)^2 + \left(\sqrt[3]{c}\right)^2 = 3\left(\sqrt[3]{c}\right)^2}}$$

It is clear that the limits exists only when $c \neq 0$.

The limits tends to $+\infty$ when $c = 0$ (since the denominator $\left(\sqrt[3]{h}\right)^2$ is positive when $c = 0$).

Hence, the function is differentiable at $x = c \neq 0$ and is not differentiable at $x = 0$.

☺ Note

Recall that the function is differentiable when the limits $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ is <u>a real number</u>.

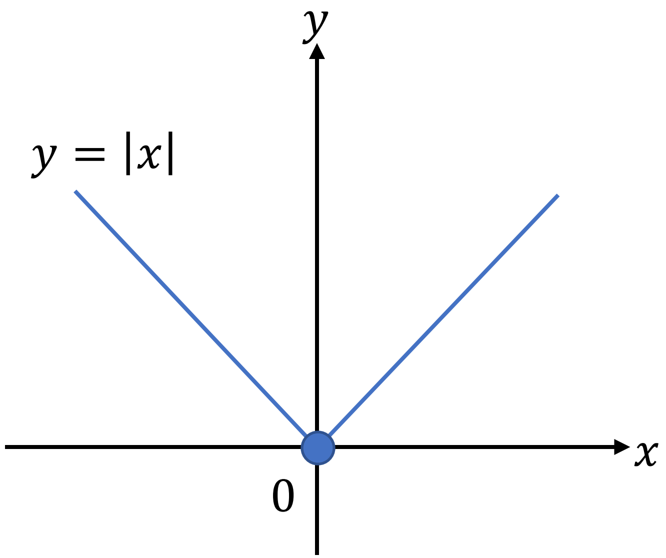
Insights about differentiability

1. *Relationship between continuity and differentiability*

   If the function $f(x)$ is differentiable at $x = c$, then $f(x)$ is also continuous at $x = c$.

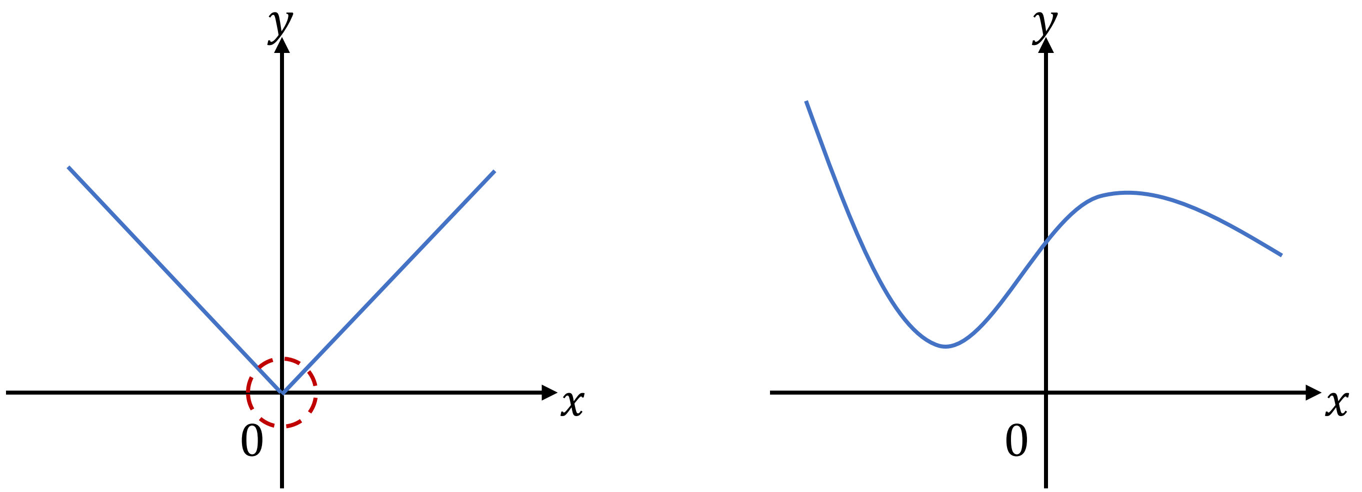
   However, the converse is not true in general.

   In Example 6, we see that $|x|$ is continuous at $x = 0$ and it is not differentiable at $x = 0$.



In other words, the “differentiable” is stronger than “continuous”.

- Continuity simply requires that the function has no break and no jump.
- Differentiability requires that the function is continuous. In addition, it also requires the graph of the function has no “sharp corner”.

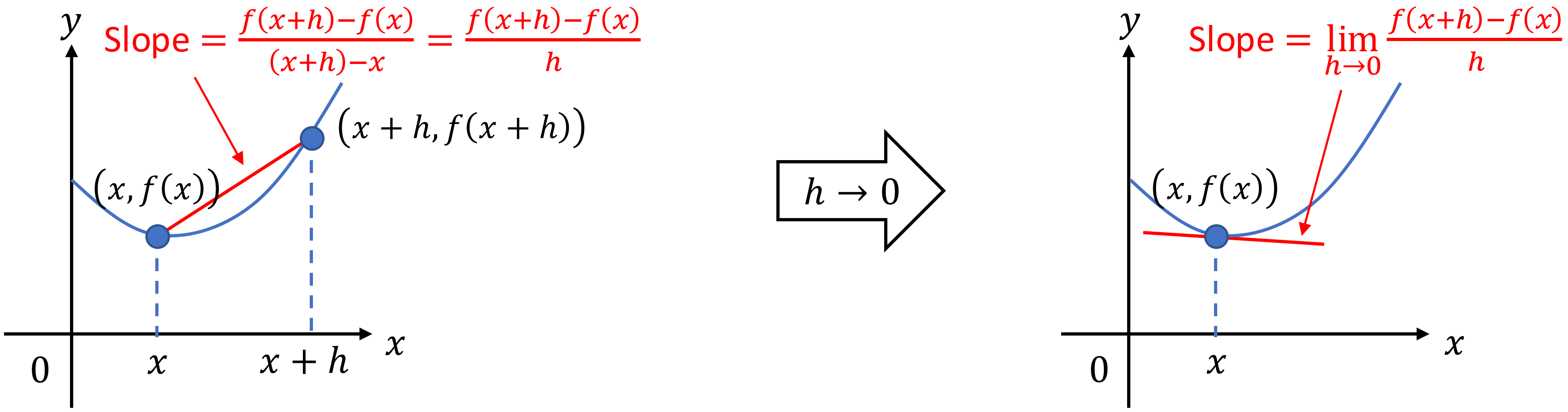


Not differentiable Differentiable

- Therefore, when the function can be differentiated as many as we like (will be discussed in details later), the function is said to be *smooth*.

2. *Geometric interpretation of derivative*

The quantity $\frac{f(x+h)-f(x)}{h}\left(=\frac{f(x+h)-f(x)}{(x+h)-x}\right)$ can be treated as the slope of the line segment joining $\big(x, f(x)\big)$ and $\big(x+h, f(x+h)\big)$.



When $h \to 0$, the line segment becomes the *tangent* at x (i.e. a line segment intersecting the graph of $f(x)$ at one point $\big(x, f(x)\big)$ only). *The limits* $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ *can then represent the slope of tangent at* x.

Example 9

Given the graph of $y = f(x) = x^5$, what is the equation of the tangent line at $x = 1$.

☺ Solution:

We first find the slope of the tangent line at $x = 1$, note that

$$Slope\ of\ tangent = \frac{dy}{dx}|_{x=1} = \frac{d}{dx}x^5|_{x=1} \overset{\frac{d}{dx}x^a = ax^{a-1}}{\cong} 5x^4|_{x=1} = 5.$$

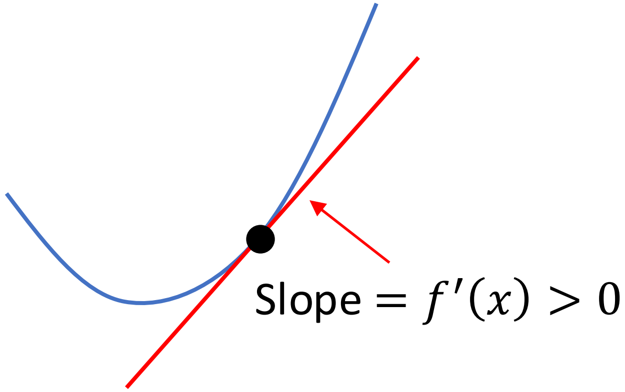
Since the tangent passes through the point $(1, f(1)) = (1,1)$, so the equation of tangent line at $x = 1$ is given by

$$\frac{y-1}{x-1} = 5 \Rightarrow y = 1 + 5(x-1) \Rightarrow y = 5x - 4.$$

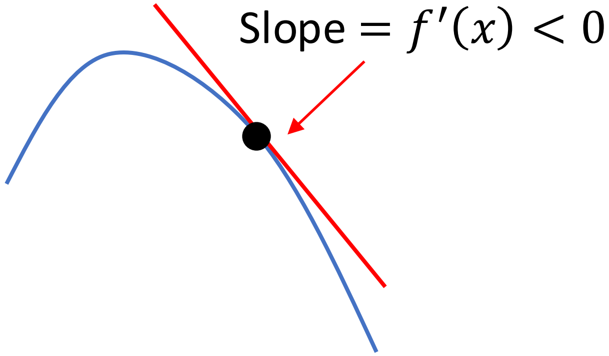
Importance of tangent line

In fact, the slope of tangent line (or more precisely, the first derivative $f'(x) = \frac{dy}{dx}$) can reflect the trend of the graph of function.

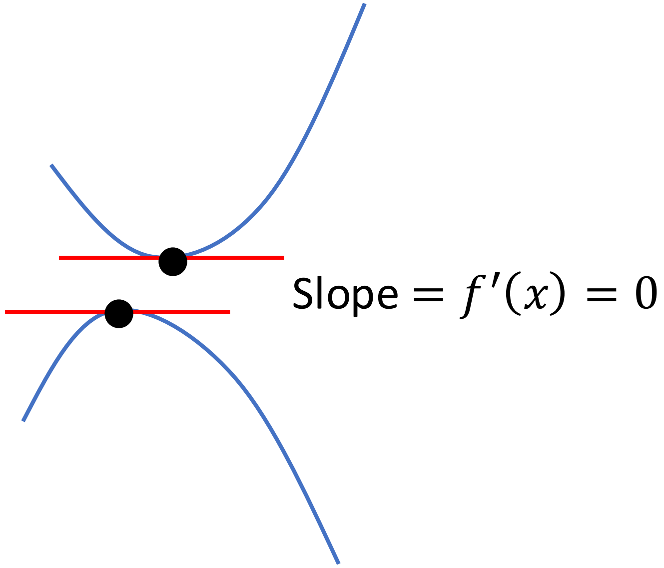
- If $f'(x) > 0$ over (a, b), then we see $f(x)$ is increasing over (a, b)
- If $f'(x) < 0$ over (a, b), then we see $f(x)$ is decreasing over (a, b).



Increasing



Decreasing



Reach Max./Min.

As an example, we consider the following quadratic function

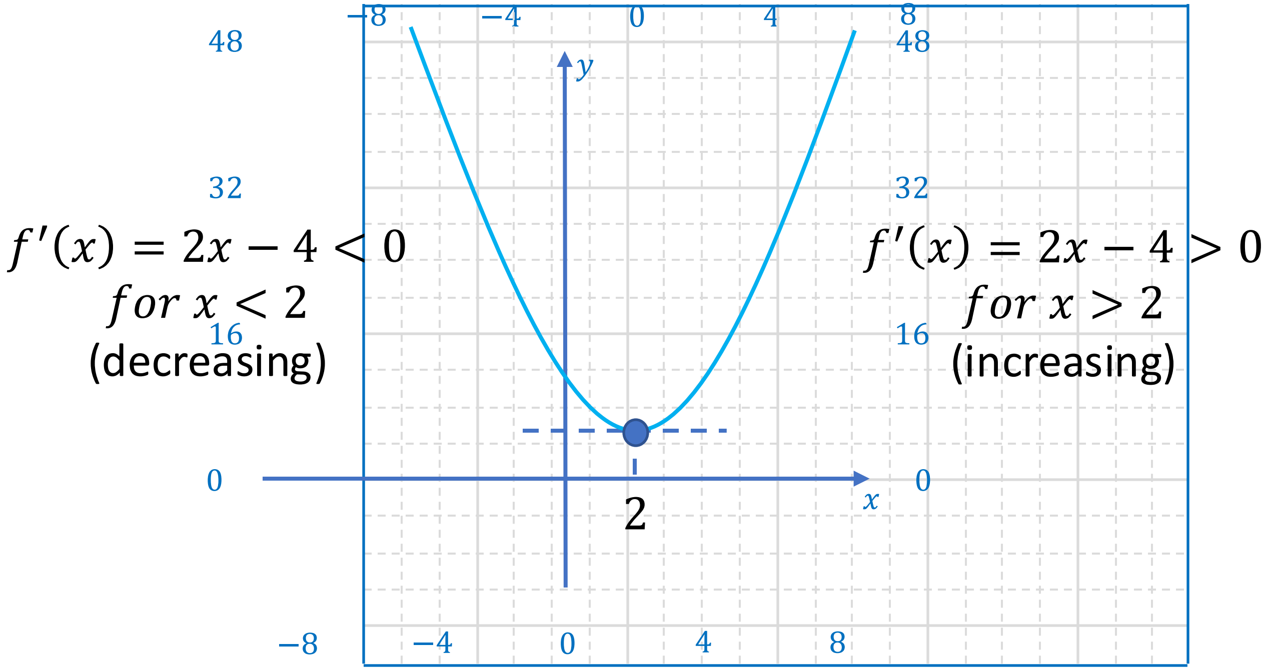
$$f(x) = x^2 - 4x + 9$$

- Using completing square, we can rewrite $f(x)$ as

$$f(x) = x^2 - 4x + 9 = (x-2)^2 + 5.$$

- Using the method of first principle, one can find its derivative as

$$f'(x) = 2x - 4.$$

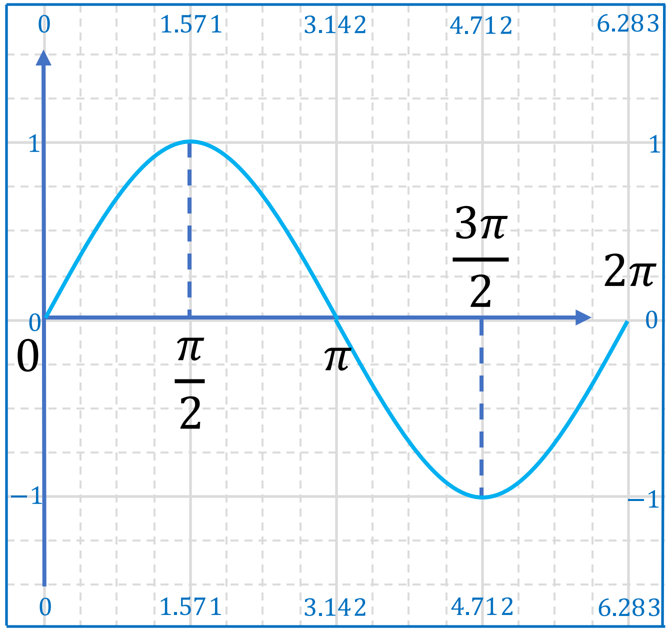


Consider another function

$$g(x) = \sin x \quad on\ [0, 2\pi]$$

In Example 3, we have shown that $g'(x) = \cos x$. We observe that

- $g'(x) = \cos x > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$ or $x \in \left(\frac{3\pi}{2}, 2\pi\right)$. So $g(x)$ is increasing in these two intervals
- $g'(x) = \cos x < 0$ for $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. So $g(x)$ is decreasing in this interval



Simple differentiation rule

Theorem (Properties of derivative)

Let $f(x)$ and $g(x)$ be two differentiable functions, then we have

1. $\dfrac{d}{dx}kf(x) = k\dfrac{d}{dx}f(x)$.
2. $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$.
3. $\dfrac{d}{dx}[f(x) - g(x)] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$.
4. $\dfrac{d}{dx}[f(x)g(x)] = g(x)\dfrac{d}{dx}f(x) + f(x)\dfrac{d}{dx}g(x)$.
5. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$.

These properties can be obtained using the first principle.

Example 10

Compute the derivatives

$$\frac{d}{dx}\left(x^5 - 3x^2 + \frac{1}{x}\right) \ \ and \ \ \frac{d}{dx}(\sin x - \sqrt[3]{x}).$$

☺ Solution:

Using the properties of derivatives, we get

$$\frac{d}{dx}\left(x^5 - 3x^2 + \frac{1}{x}\right) = \frac{d}{dx}x^5 - 3\frac{d}{dx}x^2 + \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^5 - 3\frac{d}{dx}x^2 + \frac{d}{dx}x^{-1}$$

$$\overset{\frac{d}{dx}x^a = ax^{a-1}}{\triangleq} \quad 5x^{5-1} - 3(2x^{2-1}) + (-1x^{-1-1}) = 5x^4 - 6x - x^{-2}.$$

$$\frac{d}{dx}(\sin x - \sqrt[3]{x}) = \frac{d}{dx}\sin x - \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}\sin x - \frac{d}{dx}x^{\frac{1}{3}} = \cos x - \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= \cos x - \frac{1}{3}x^{-\frac{2}{3}}.$$

Example 11

Compute the following derivatives

$$\frac{d}{dx}e^x\cos x\,,\qquad \frac{d}{dx}\left(\frac{x+\sin x}{x^2}\right).$$

☺Solution:

$$\frac{d}{dx}e^x\cos x = e^x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}e^x = e^x(-\sin x) + \cos x\,(e^x)$$

$$= e^x\cos x - e^x\sin x\,.$$

$$\frac{d}{dx}\left(\frac{\overbrace{x+\sin x}^{f(x)}}{\underbrace{x^2}_{g(x)}}\right) = \frac{x^2\frac{d}{dx}(x+\sin x) - (x+\sin x)\frac{d}{dx}x^2}{(x^2)^2}$$

$$= \frac{x^2(1+\cos x) - (x+\sin x)(2x)}{x^4} = \frac{x^2\cos x - 2x\sin x - x^2}{x^4}.$$

Example 12

Compute the derivative

$$\frac{d}{dx}\frac{\sin x}{\sqrt[4]{x}}.$$

☺ Solution:

Method 1 (Using quotient rule):

$$\frac{d}{dx}\frac{\sin x}{\sqrt[4]{x}} = \frac{d}{dx}\frac{\sin x}{x^{\frac{1}{4}}} = \frac{x^{\frac{1}{4}}\frac{d}{dx}\sin x - \sin x\frac{d}{dx}x^{\frac{1}{4}}}{\left(x^{\frac{1}{4}}\right)^2} = \frac{x^{\frac{1}{4}}\cos x - \sin x\left(\frac{1}{4}x^{\frac{1}{4}-1}\right)}{\left(x^{\frac{1}{4}}\right)^2}$$

$$= \frac{x^{\frac{1}{4}}\cos x - \frac{1}{4}x^{-\frac{3}{4}}\sin x}{x^{\frac{2}{4}}} = x^{-\frac{1}{4}}\cos x - \frac{1}{4}x^{-\frac{5}{4}}\sin x.$$

Mehtod 2 (Using Product Rule):

$$\frac{d}{dx}\frac{\sin x}{\sqrt[4]{x}} = \frac{d}{dx}x^{-\frac{1}{4}}\sin x = \sin x\frac{d}{dx}x^{-\frac{1}{4}} + x^{-\frac{1}{4}}\frac{d}{dx}\sin x$$

$$= \sin x\left(-\frac{1}{4}x^{-\frac{1}{4}-1}\right) + x^{-\frac{1}{4}}\cos x = -\frac{1}{4}x^{-\frac{5}{4}}\sin x + x^{-\frac{1}{4}}\cos x.$$

Differentiation of composite function -- Chain Rule

Suppose we would like to compute the derivative: $\frac{d}{dx}\sin(x^2+x+1)$, one cannot apply the result $\frac{d}{dy}\sin y=\cos y$ and conclude that $\frac{d}{dx}\sin(x^2+x+1)=\cos(x^2+x+1)$ because the derivative is NOT of the form $\frac{d}{dy}\sin y$.

Question: How do we compute $\frac{d}{dx}\sin(x^2+x+1)$?

Theorem (Chain Rule)

If $f(x)$ and $u(x)$ be two differentiable functions, then

$$\frac{d}{dx}f\big(u(x)\big)=\frac{df\big(u(x)\big)}{du(x)}\frac{du(x)}{dx}\quad\left(or\,\frac{df}{du}\frac{du}{dx}\right).$$

Going back to the problem $\frac{d}{dx}\sin(x^2+x+1)$, the function $\sin(x^2+x+1)$ can be expressed as the composition of two functions $f\big(u(x)\big)$ where $f(x)=\sin x$ and $u(x)=x^2+x+1$.

$$\boxed{x} \xrightarrow{u(x)=x^2+x+1} \boxed{u(x)=x^2+x+1} \xrightarrow{f(x)=\sin x} \boxed{f\big(u(x)\big)=\sin(x^2+x+1)}$$

Using the Chain Rule, we then have

$$\frac{d}{dx}\sin(x^2+x+1)=\frac{d}{dx}f\big(u(x)\big)=\frac{df\big(u(x)\big)}{du(x)}\frac{du(x)}{dx}$$

$$=\frac{d\sin(x^2+x+1)}{d(x^2+x+1)}\frac{d(x^2+x+1)}{dx}$$

$$\overset{\frac{d}{dy}\sin y=\cos y}{\underset{}{\overset{\mathit{take}\ y=x^2+x+1}{\triangleq}}}\cos(x^2+x+1)\,[2x+1]=(2x+1)\cos(x^2+x+1)\,.$$

Example 13

Compute the derivative

$$\frac{d}{dx}e^{2x+1}, \qquad \frac{d}{dx}\cos(3e^x).$$

☺ Solution:

Using Chain Rule, we get

$$\frac{d}{dx}e^{\overbrace{2x+1}^{u(x)}} = \frac{de^{2x+1}}{d(2x+1)}\frac{d(2x+1)}{dx} \overset{\frac{de^y}{dy}=e^y}{\underset{}{\overset{Take\ y=2x+1}{\widehat{=}}}} \frac{de^y}{dy}\frac{d(2x+1)}{dx} = e^y(2)$$

$$= 2e^{2x+1}.$$

$$\frac{d}{dx}\cos\left(\overbrace{3e^x}^{u(x)}\right) = \frac{d\cos(3e^x)}{d(3e^x)}\frac{d(3e^x)}{dx} \overset{\frac{d}{dy}\cos y=-\sin y}{\underset{}{\overset{Take\ y=3e^x}{\widehat{=}}}} = -3e^x\sin(3e^x).$$

Example 14 (A bit harder example)

Compute the derivative

$$\frac{d}{dx}\sin\left(e^{x^3-3x+1}\right)$$

☺ Solution:

$$\frac{d}{dx}\sin\left(\overbrace{e^{x^3-3x+1}}^{u(x)}\right)=\frac{d\sin\left(e^{x^3-3x+1}\right)}{d\left(e^{x^3-3x+1}\right)}\frac{de^{\overbrace{x^3-3x+1}^{v(x)}}}{dx}$$

$$=\frac{d\sin\left(e^{x^3-3x+1}\right)}{d\left(e^{x^3-3x+1}\right)}\frac{de^{x^3-3x+1}}{d(x^3-3x+1)}\frac{d(x^3-3x+1)}{dx}$$

$$\overset{\substack{y=e^{x^3-3x+1}\\ z=x^3-3x+1}}{\cong}\frac{d\sin y}{dy}\frac{de^z}{dz}\frac{d(x^3-3x+1)}{dx}=(\cos y)(e^z)\overbrace{(3x^{3-1}-3(1)x^{1-1})}^{\frac{d}{dx}x^a=ax^{a-1}}$$

$$=\cos\left(e^{x^3-3x+1}\right)e^{x^3-3x+1}(3x^2-3)$$

$$=(3x^2-3)e^{x^3-3x+1}\cos\left(e^{x^3-3x+1}\right).$$

Example 15

Compute the derivative $\frac{d}{dx}\cos^4(\sin x)$.

☺Solution:

$$\frac{d}{dx}\cos^4(\sin x)=\frac{d}{dx}\Big[\overbrace{\cos(\sin x)}^{u(x)}\Big]^4$$

$$=\frac{d[\cos(\sin x)]^4}{d[\cos(\sin x)]}\frac{d\left[\cos\left(\overbrace{\sin x}^{v(x)}\right)\right]}{dx}=\frac{d[\cos(\sin x)]^4}{d[\cos(\sin x)]}\frac{d[\cos(\sin x)]}{d(\sin x)}\frac{d(\sin x)}{dx}$$

$$\overset{\substack{y=\cos(\sin x)\\ z=\sin x}}{\underset{\smile}{=}}\frac{dy^4}{dy}\frac{d\cos z}{dz}\frac{d\sin x}{dx}=4y^3(-\sin z)\cos x$$

$$=-4[\cos(\sin x)]^3[\sin(\sin x)]\cos x\,.$$

Example 16

Compute the derivative $\frac{d}{dx}\sqrt{x\cos(x^2)}$.

☺ Solution:

$$\frac{d}{dx}\sqrt{x\cos(x^2)} = \frac{d}{dx}\Bigg[\overbrace{x\cos(x^2)}^{u(x)}\Bigg]^{\frac{1}{2}} = \frac{d[x\cos(x^2)]^{\frac{1}{2}}}{d[x\cos(x^2)]}\frac{d[x\cos(x^2)]}{dx}$$

$$= \frac{d[x\cos(x^2)]^{\frac{1}{2}}}{d[x\cos(x^2)]}\left[\cos(x^2)\frac{dx}{dx} + x\frac{d\cos\overbrace{(x^2)}^{v(x)}}{dx}\right]$$

$$= \frac{d[x\cos(x^2)]^{\frac{1}{2}}}{d[x\cos(x^2)]}\left[\cos(x^2)\frac{dx}{dx} + x\frac{d\cos(x^2)}{d(x^2)}\frac{d(x^2)}{dx}\right]$$

$$= \frac{1}{2}[x\cos(x^2)]^{\frac{1}{2}-1}[\cos(x^2) + x(-\sin(x^2))(2x)]$$

$$= \frac{1}{2}[x\cos(x^2)]^{-\frac{1}{2}}[\cos(x^2) - 2x^2\sin(x^2)].$$

Derivative of $f(x) = \sec x$, $g(x) = \csc x$ and $h(x) = \cot x$

The functions $\sec x$, $\csc x$ and $\cot x$ are defined as

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}, \qquad \cot x = \frac{1}{\tan x}.$$

Using chain rule and the properties of derivatives, one can see that

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\mathbf{sec}\,\boldsymbol{x} = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d(\cos x)^{-1}}{d(\cos x)}\frac{d(\cos x)}{dx} = -\frac{1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \mathbf{tan}\,\boldsymbol{x}\,\mathbf{sec}\,\boldsymbol{x}.$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\mathbf{csc}\,\boldsymbol{x} = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d(\sin x)^{-1}}{d(\sin x)}\frac{d(\sin x)}{dx} = -\frac{1}{\sin^2 x}\cos x = -\frac{1}{\frac{\sin x}{\cos x}}\frac{1}{\sin x} = -\frac{1}{\tan x}\csc x = -\,\mathbf{cot}\,\boldsymbol{x}\,\mathbf{csc}\,\boldsymbol{x}.$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\mathbf{cot}\,\boldsymbol{x} = \frac{d}{dx}\frac{1}{\tan x} = \frac{d(\tan x)^{-1}}{d(\tan x)}\frac{d(\tan x)}{dx} = \cdots = -\,\mathbf{csc}^{\mathbf{2}}\,\boldsymbol{x}.$$