MA1200 Calculus and Basic Linear Algebra I

Lecture Note 1

Coordinate Geometry and Conic Sections

Topic Covered

- Two representations of coordinate systems:

  Cartesian coordinates [(x, y)-coordinates] and

  Polar coordinates [(r, θ)-coordinates].

- Conic Sections: Circle, Ellipse, Parabola and Hyperbola.

- Classify the conic section in 2-D plane.
  - ✓ General equation of conic section
  - ✓ Identify the conic section in 2-D plane

    -Useful technique: Rotation of Axes

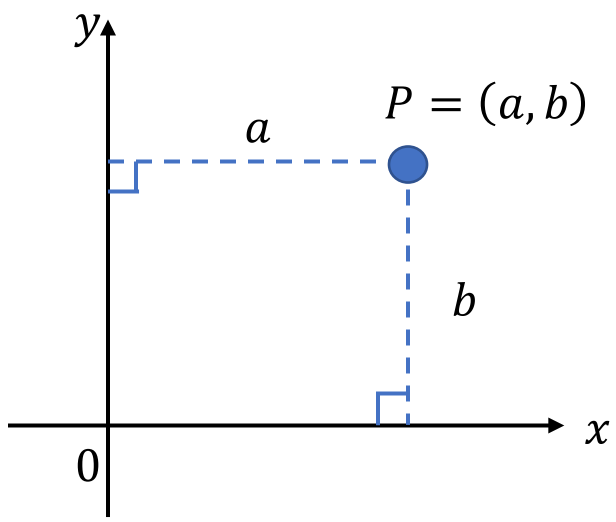
    -General results

Representations of coordinate systems in 2-D

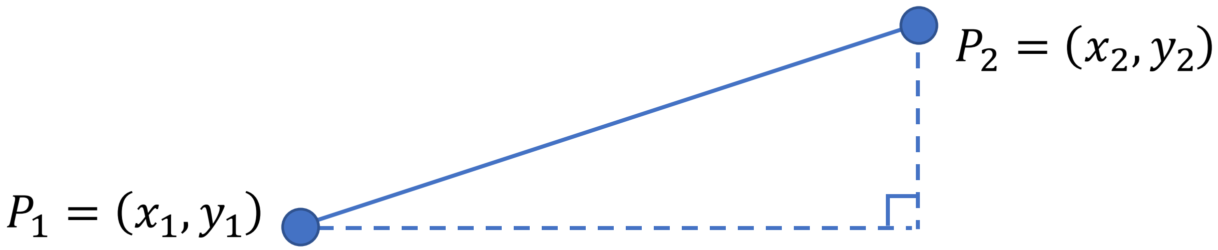
There are two different types of coordinate systems used in locating the position of a point in 2-D.

First representation: Cartesian coordinates

We describe the position of a given point by considering the (directed) distance between the point and x-axis and the distance between the point and y-axis.



Here, a is called "x-coordinate" of P and b is called "y-coordinate" of P.



Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we learned that

- the distance between P_1 and P_2: $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
- the slope of line segment joining P_1 and P_2:

$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1},$$

- the equation of the straight line P_1P_2:

$$\frac{y - y_1}{x - x_1} = m = \frac{y_2 - y_1}{x_2 - x_1} \quad (Point - slope\ form),$$

- The equation can be expressed in the following form:

$$y = mx + c \quad (slope - intercept\ form)$$

where m and c are the slope and y-intercept respectively.

Example 1

Find the equation of straight line L which satisfies each of the following conditions:

(a) Passing through $P = (3, 4)$ and $Q = (1, -1)$.

(b) Perpendicular to the line $L_1: 3x - 2y + 5 = 0$ and cuts the x-axis at $(5, 0)$.

☺ Solution:

(a) The slope of $PQ = \frac{-1-4}{1-3} = \frac{5}{2}$. The equation of PQ is then given by

$$\frac{y-4}{x-3} = \frac{5}{2} \Rightarrow 2(y-4) = 5(x-3) \Rightarrow 5x - 2y - 7 = 0.$$

(b) IDEA: In order to find the equation of L, we have to find the slope of L. Since L and L_1 are perpendicular, then we can obtain this by first finding the slope of L_1.

To obtain the slope of L_1, one has to express the equation of L_1 in the slope-intercept form:

$$3x - 2y + 5 = 0 \Rightarrow 2y = 3x + 5 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}.$$

We get the slope of L_1 is $m_1 = \frac{3}{2}$.

Since the line L and L_1 are perpendicular, we have

$$m \times m_1 = -1 \Rightarrow \frac{3}{2}m = -1 \Rightarrow m = -\frac{2}{3}.$$

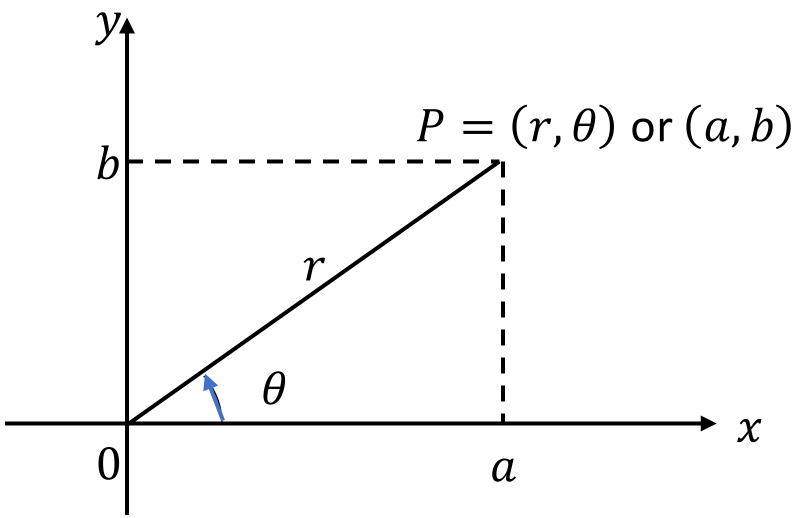
(Here, m is the slope of L)

Then the equation of L is found to be

$$\frac{y - 0}{x - 5} = -\frac{2}{3} \Rightarrow 3y = -2(x - 5) \Rightarrow 2x + 3y - 10 = 0.$$

Second coordinates: Polar coordinates

Alternatively, one can describe the position of a point by considering the distance between the point P and the origin (r) and the (directed) angle between OP and the positive x-axis (θ).



Note:

$\theta > 0$: anti-clockwise direction

$\theta < 0$: clockwise direction

The relations between Polar (r, θ) and Cartesian (x, y) Coordinates are given by

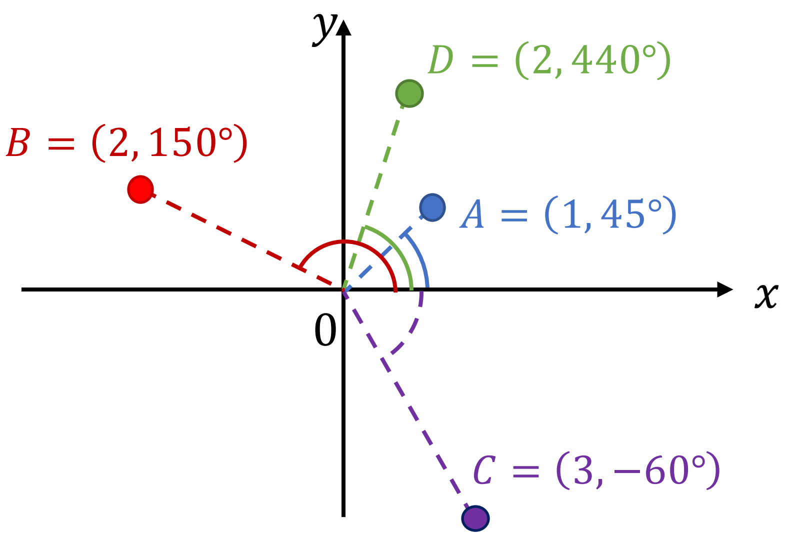
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}.$$

Example 2

Given the points $A = (1, 45°)$, $B = (2, 150°)$, $C = (3, -60°)$ and $D = (2, 440°)$.

Locate the points in the xy-plane. Find the corresponding Cartesian coordinates of these points.

☺ Solution:



The corresponding Cartesian coordinates of these points are given by

$$A = (1\cos 45°, 1\sin 45°) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad B = (2\cos 150°, 2\sin 150°) = \left(-\sqrt{3}, 1\right),$$

$$C = (3\cos -60°, 3\sin -60°) = \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right), \qquad D = (2\cos 440°, 2\sin 440°) \approx (0.35, 1.97).$$

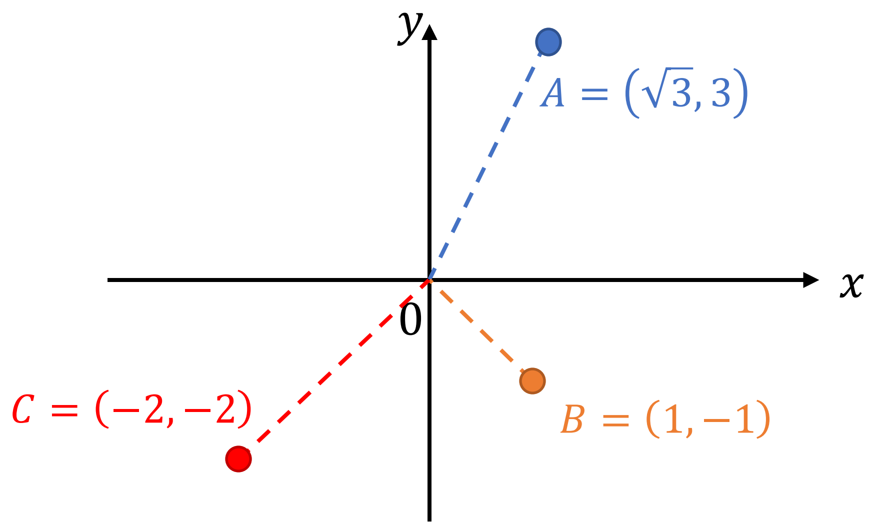
Example 3

Let $A = (\sqrt{3}, 3)$, $B = (1, -1)$, $C = (-2, -2)$. Find the corresponding polar coordinates of A, B, C.

(Here, choose $-180° < \theta \leq 180°$)

☺ Solution:

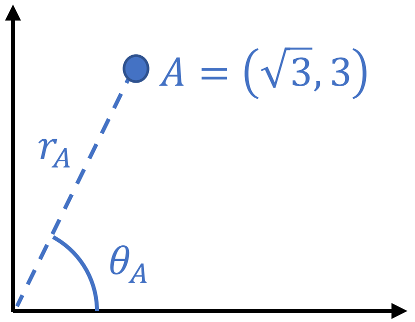
Tips: It is always a good idea to locate the points on x, y so that you can obtain the values of r and θ correctly.



For Point A

$$r_A = \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = \sqrt{12}.$$

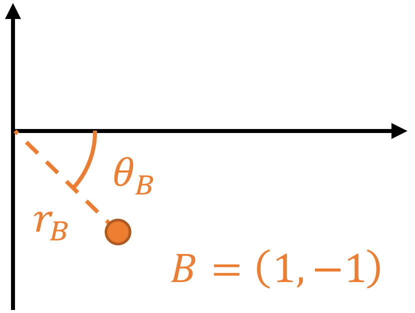
$$\tan\theta_A = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \theta_A = 60°.$$



For Point B

$$r_B = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

$$\tan\theta_B = \frac{-1}{1} = -1 \Rightarrow \theta_B = -45°.$$

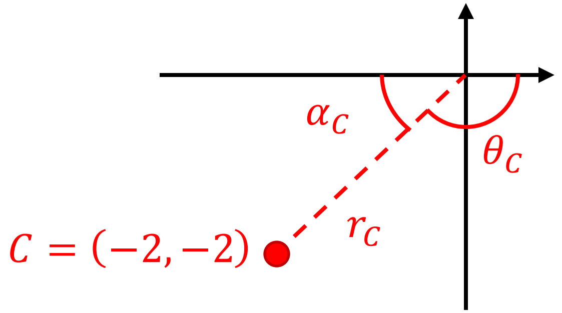


For Point C

$$r_C = \sqrt{2^2 + (-2)^2} = \sqrt{8}.$$

$$\tan\alpha_C = \frac{2}{2} = 1 \Rightarrow \alpha_C = 45°.$$

So $\theta_C = -(180° - 45°) = -135°.$



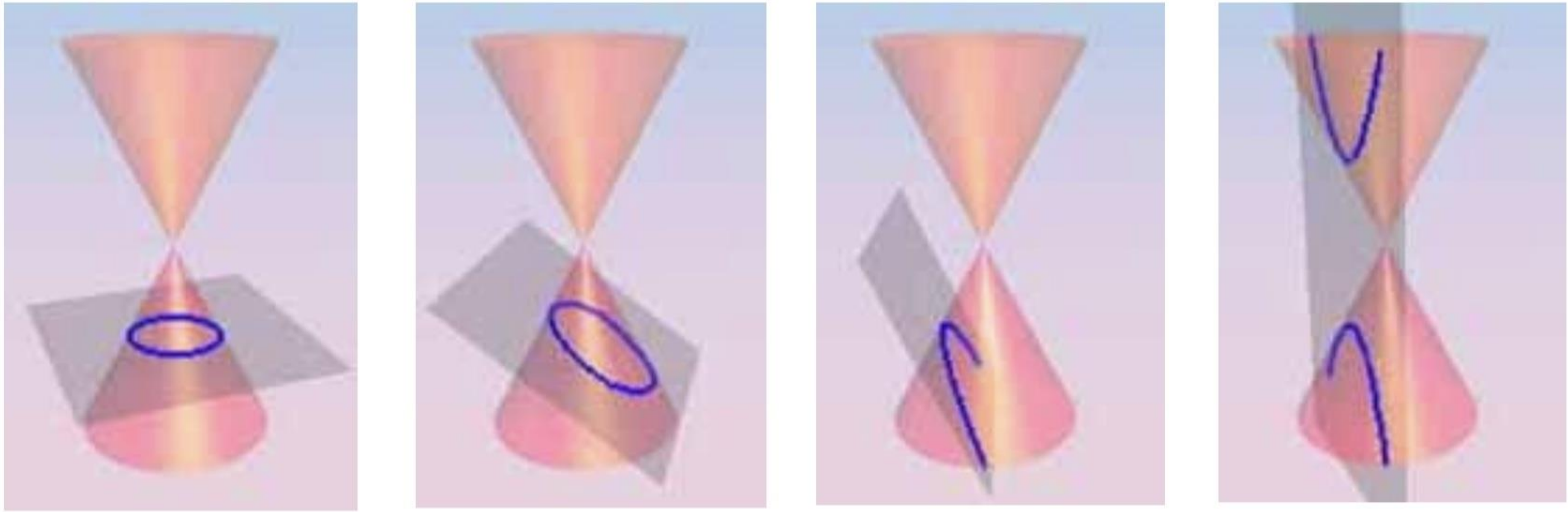
$$\therefore A = \left(\sqrt{12}, 60°\right), \qquad B = \left(\sqrt{2}, -45°\right), \qquad C = \left(\sqrt{8}, -135°\right).$$

☺ Note: In the remaining section, we will adopt the Cartesian coordinates system unless otherwise specified.

Conic Sections

Conic sections, by definition, are curves that result from intersecting a right circular cone with a plane.

The following figure shows the four conic sections:

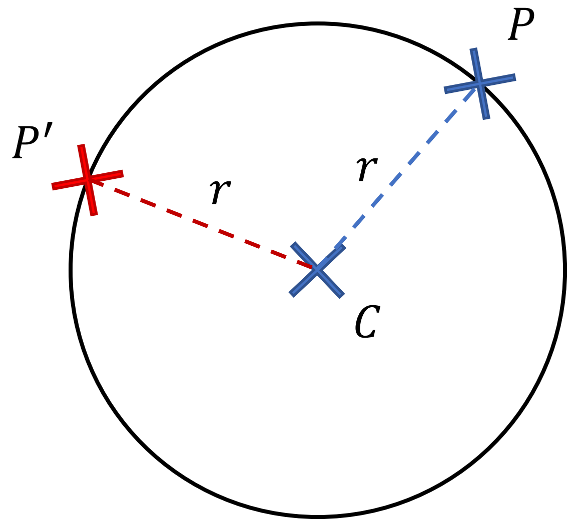


Circle Ellipse Parabola Hyperbola

Conic Section #1: Circle

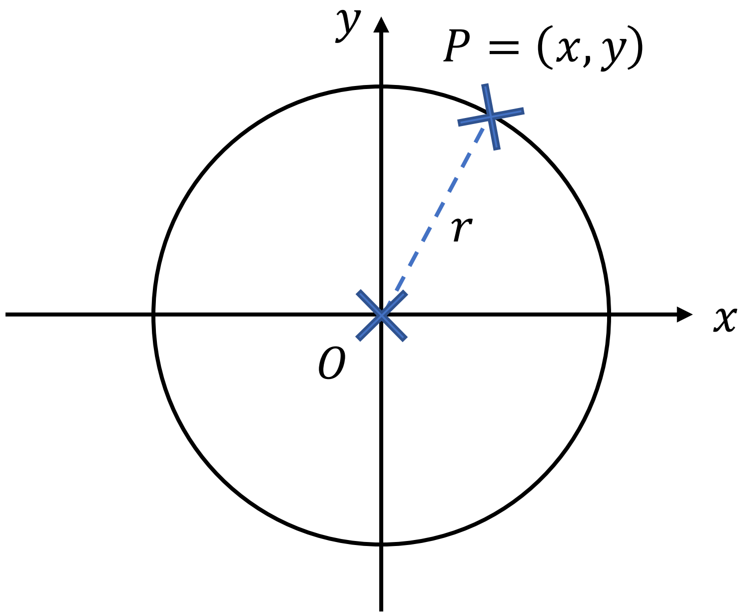
Given a fixed point C, a circle is a set of all points such that the distance between the point and C is always fixed.

($*$ C is called centre of the circle, the fixed distance is called radius of the circle and is denoted by r.)



The equation of circle

<u>The circle with centre at the origin $O = (0,0)$</u>



For any point P lying on the circle, we have

$$OP = r \Rightarrow \sqrt{(x-0)^2 + (y-0)^2} = r$$

$$\Rightarrow x^2 + y^2 = r^2$$

which is the equation of circle (with centre origin).

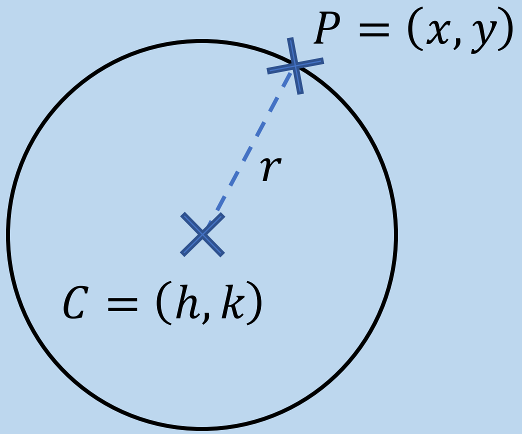
Using similar method, we can derive the following general case:

Equation of Circle (General Case)

The equation of circle with centre $C = (h, k)$

and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2.$$



Example 4

(a) Find the equation of the circle with centre $(1,-1)$ and radius $\sqrt{2}$.

(b) The equation of a circle is given by $(x+3)^2+(y-1)^2=7$. Find the centre and the radius of this circle.

☺ Solution:

(a) The equation of the circle is

$$(x-1)^2+\left(y-(-1)\right)^2=\left(\sqrt{2}\right)^2 \Rightarrow (x-1)^2+(y+1)^2=2$$

$$\Rightarrow x^2+y^2-2x+2y=0.$$

(b) In order to find the centre and radius, one has to express the equation into the form

$$(x-h)^2+(y-k)^2=r^2.$$

Note that

$$(x+3)^2+(y-1)^2=7 \Rightarrow \left(x-(-3)\right)^2+(y-1)^2=\left(\sqrt{7}\right)^2.$$

So, the circle has centre $C=(-3,1)$ and radius $\sqrt{7}$.

Example 5 (Useful technique: Completing Square)

The equation of a circle is given by $x^2+y^2+8x-10y-8=0$. Find the centre and radius of this circle.

☺ Solution:

Note that

$x^2+y^2+8x-10y-8=0 \Rightarrow (x^2+8x)+(y^2-10y)-8=0$

$\Rightarrow (x^2+2(4)x)+(y^2-2(5)y)-8=0$

$\Rightarrow \underbrace{(x^2+2(4)x+4^2)}_{a^2+2ab+b^2=(a+b)^2}-4^2+\underbrace{(y^2-2(5)y+5^2)}_{a^2-2ab+b^2=(a-b)^2}-5^2-8=0$

$\Rightarrow (x+4)^2+(y-5)^2=8+5^2+4^2=49$

$\Rightarrow (x-(-4))^2+(y-5)^2=7^2.$

So, the centre is $C=(-4,5)$ and the radius is 7.

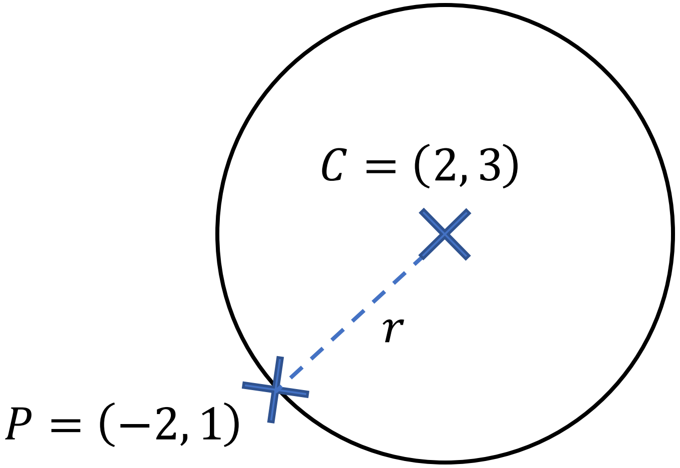
Example 6

Find the equation of the circle centered at $C = (2, 3)$ and passing through the point $P = (-2,1)$.

☺ Solution:

We need to find the radius of the circle r.

From the figure on the right, we have



$$r = CP = \sqrt{\left(2-(-2)\right)^2 + (3-1)^2} = \sqrt{20}.$$

Then the equation of circle is then given by

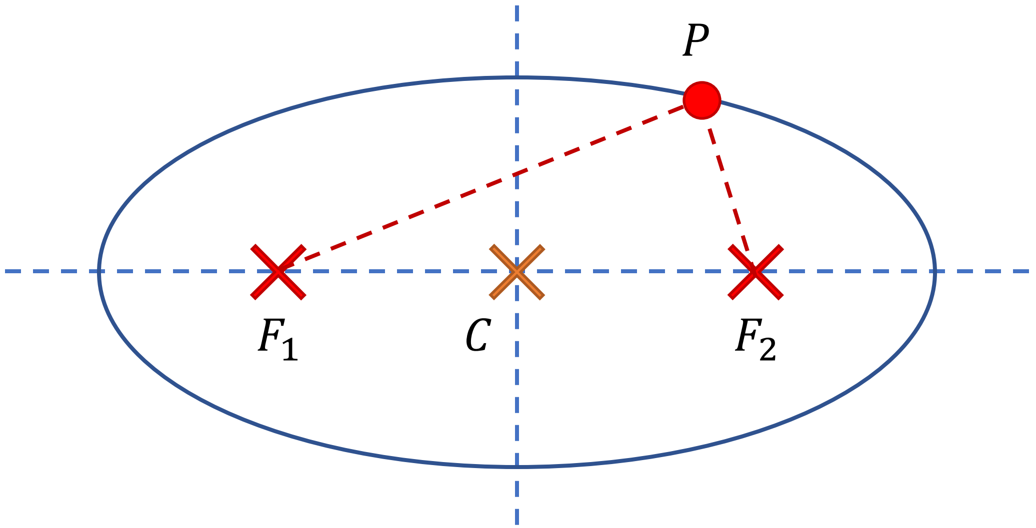
$$(x-2)^2 + (y-3)^2 = \left(\sqrt{20}\right)^2 \Rightarrow x^2 + y^2 - 4x - 6y - 7 = 0.$$

Conic Section #2: Ellipse

Given two fixed points F_1 and F_2 (called foci) on the plane, the ellipse is the set of all points P such that

$$PF_1 + PF_2 = constant = 2a.$$

(Note: The purpose of choosing the constant to be $2a$ (instead of a) is to make the equation of ellipse looks "nicer").

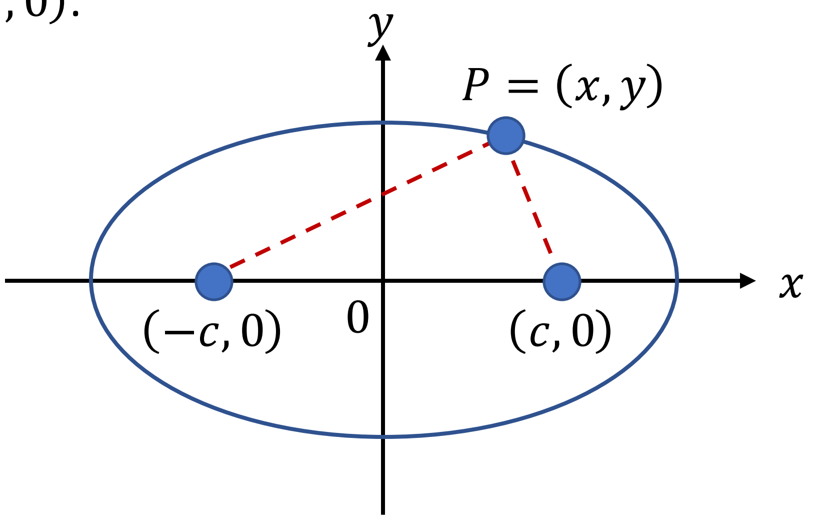


Here, C is the "centre" of ellipse and is also the "mid-point" between two foci F_1, F_2.

Equation of Ellipse

For simplicity, we consider the ellipse with foci $F_1 = (-c, 0)$ and $F_2 = (c, 0)$.

To derive the equation of this ellipses, note that



$$PF_1 + PF_2 = 2a$$

$$\Rightarrow \underbrace{\sqrt{\left(x-(-c)\right)^2 + (y-0)^2}}_{PF_1} + \underbrace{\sqrt{(x-c)^2 + (y-0)^2}}_{PF_2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}.$$

Taking square on both sides and expand the term, we get

$$x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + \underbrace{x^2 - 2xc + c^2 + y^2}_{(x-c)^2 + y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4xc$$

$$\Rightarrow a\sqrt{(x-c)^2 + y^2} = a^2 - xc.$$

Taking square on both sides again, we get

$$a^2[(x-c)^2 + y^2] = a^4 - 2a^2xc + x^2c^2$$

Expanding the terms and rearranging them, we finally get

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

We define $b^2 = a^2 - c^2 > 0$, then

This is OK since

$$PF_1 + PF_2 > F_1F_2 \Rightarrow 2a > 2c$$

$$\Rightarrow a > c.$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

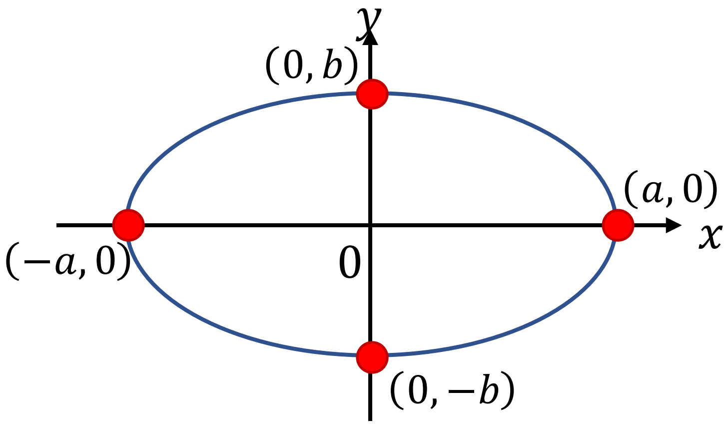
Equation of Ellipse (Standard Case)

For $a, c > 0$, the equation of ellipse with foci $(c, 0)$ and $(-c, 0)$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad where\ b^2 = a^2 - c^2, a > b > 0.$$

Here, $2a$ is the sum of distance from any point on ellipse to the two foci.

Some remarks on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



- The ellipse passes through the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$

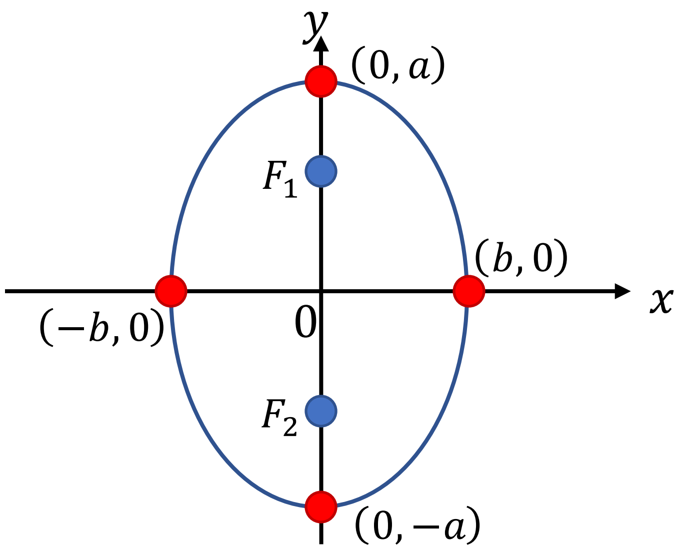
  (they are called vertices).

- The ellipse is centered at the origin $(0, 0)$.

- If the foci $F_1 = (0, c)$ and $F_2 = (0, -c)$ lies on y-axis instead of x-axis, the corresponding equation of the ellipse formed is seen to be

  $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ \textit{where}\ b^2 = a^2 - c^2.$$

  (The role of a and b are reversed)



- When $a = b$, then the equation of ellipse becomes

  $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Rightarrow x^2 + y^2 = a^2$ which is the equation of circle.

Example 7

The equation of an ellipse is $4x^2 + 9y^2 = 36$. Sketch the graph. Find the coordinates of vertices and the foci of this ellipse.

☺ Solution:

$$4x^2 + 9y^2 = 36 \Rightarrow \frac{4x^2 + 9y^2}{36} = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

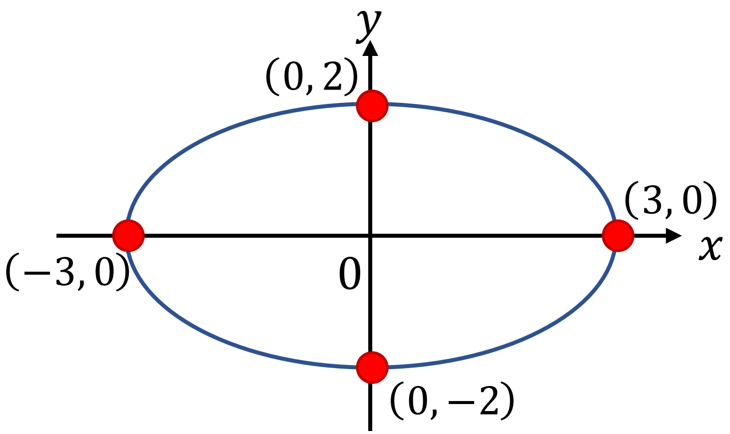
The ellipse passes through the vertices $(3,0)$, $(-3,0)$, $(0,2)$, $(0,-2)$.

From the graph of ellipse, we see the foci lies on x-axis and

$c^2 = 3^2 - 2^2 \Rightarrow c = \sqrt{5}.$

Hence, the foci of this ellipse is

$F_1 = \left(-\sqrt{5}, 0\right)$ and $F_2 = \left(\sqrt{5}, 0\right)$.



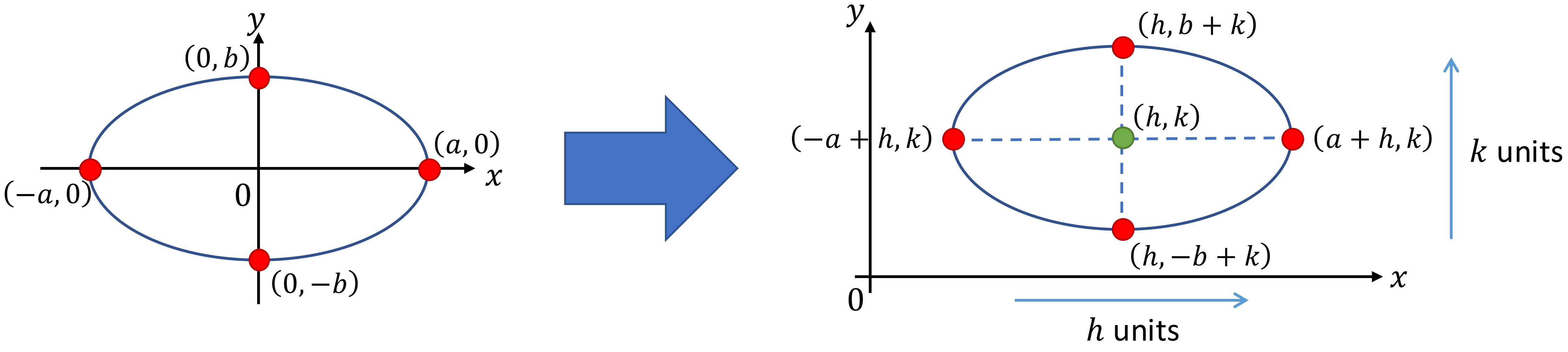
General equation of ellipse

In general, if the centre of the ellipse is $C = (h, k)$ instead of origin $(0, 0)$, then the equation of ellipse is given by

Equation of Ellipse (General Case)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where $C = (h, k)$ is the centre of the ellipse.



Example 8

The equation of the ellipse is $4x^2 + 9y^2 - 8x + 36y - 9 = 0$. Sketch the graph of this ellipse and indicate the coordinates of the vertexes.

☺ Solution:

Similar to Example 5, we rewrite the equation into the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ using completing square technique.

$$4x^2 + 9y^2 - 8x + 36y - 9 = 0$$

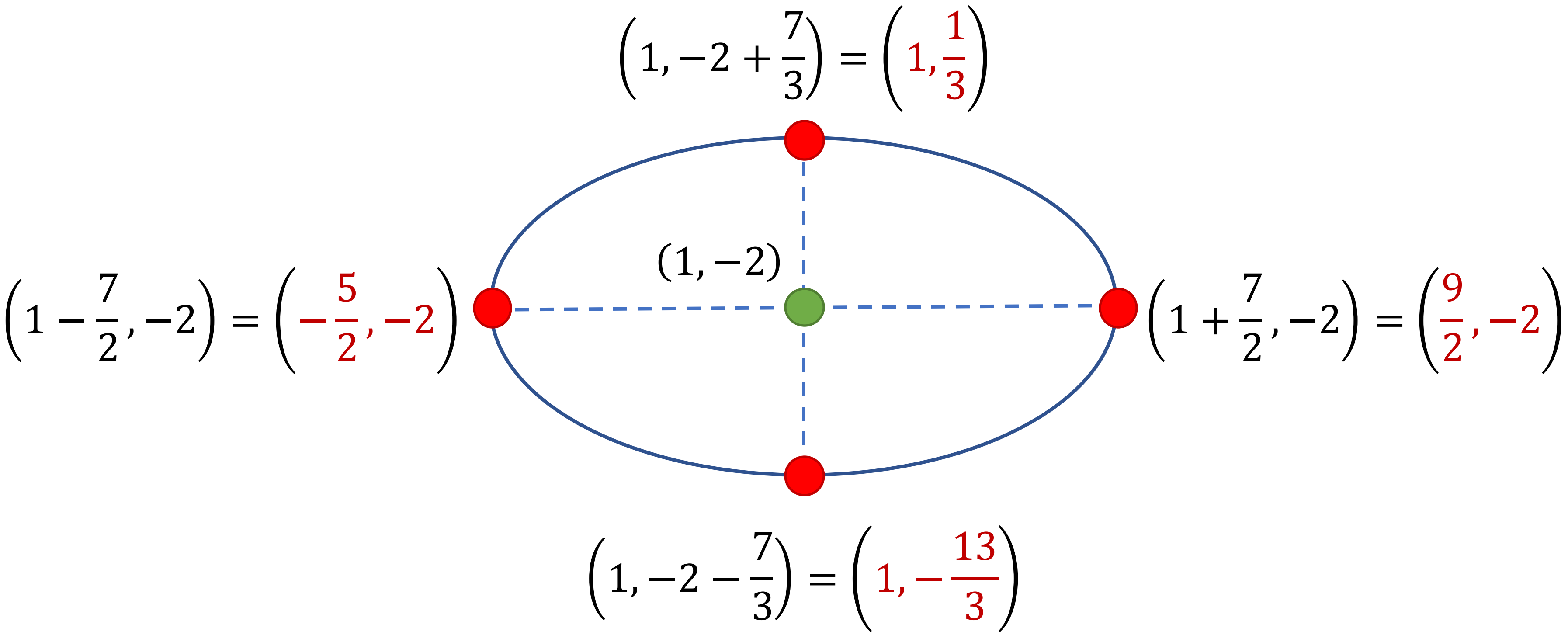
$$\Rightarrow 4(x^2 - 2x) + 9(y^2 + 4y) - 9 = 0$$

$$\Rightarrow 4(x^2 - 2(1)x + 1^2 - 1^2) + 9(y^2 + 2(2)y + 2^2 - 2^2) - 9 = 0$$

$$\Rightarrow 4(x-1)^2 + 9(y+2)^2 = 49$$

$$\Rightarrow \frac{(x-1)^2}{\left(\frac{7}{2}\right)^2} + \frac{\big(y-(-2)\big)^2}{\left(\frac{7}{3}\right)^2} = 1.$$

The graph of this ellipse is sketched below:

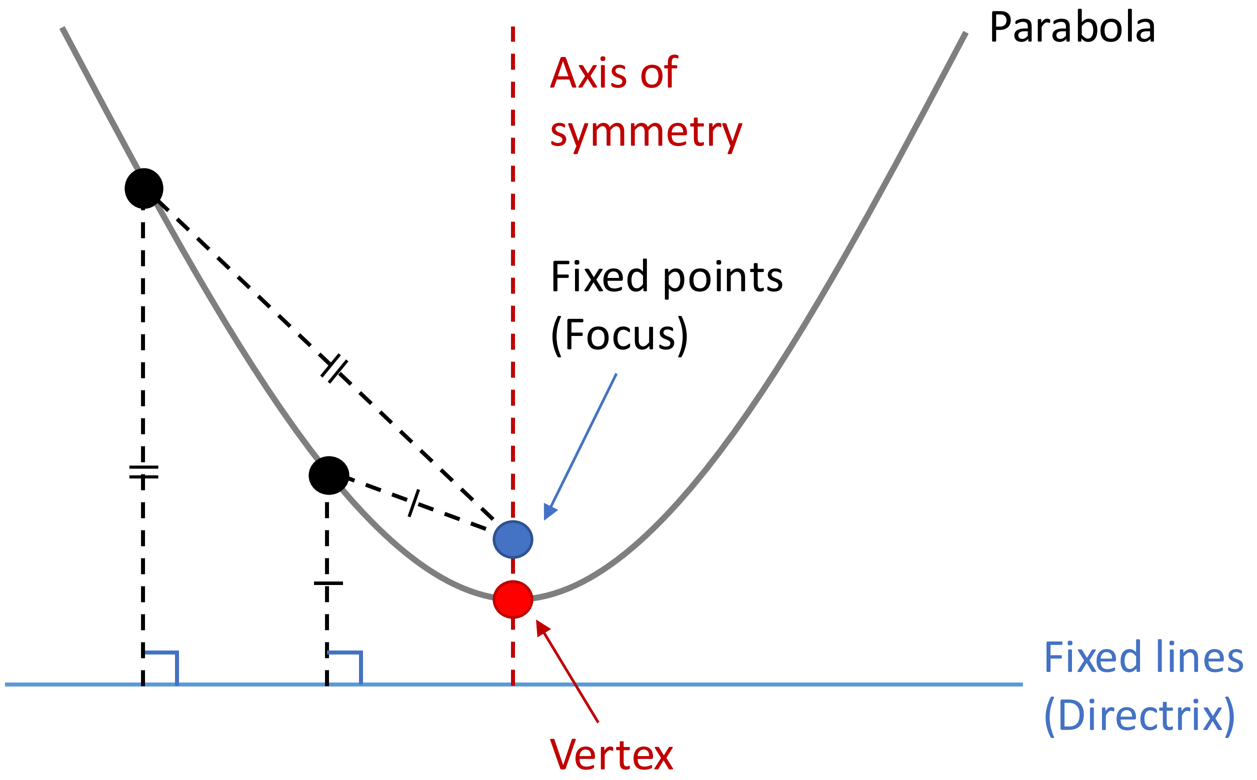


The coordinates of the vertexes are given by

$$\left(1, \frac{1}{3}\right), \left(1, -\frac{13}{3}\right), \left(-\frac{5}{2}, -2\right) \; and \; \left(\frac{9}{2}, -2\right).$$

Conic Section #3: Parabola

A parabola is the set of all points in a plane that are equidistant from a fixed line (called directrix) and a fixed point (called focus).



Equation of Parabola (Standard Case)

We consider the parabola with focus at $F = (0, a)$ and the directrix $y = -a$, where $a > 0$.

Let $P = (x, y)$ be any point on the parabola.

According to the definition, we must have

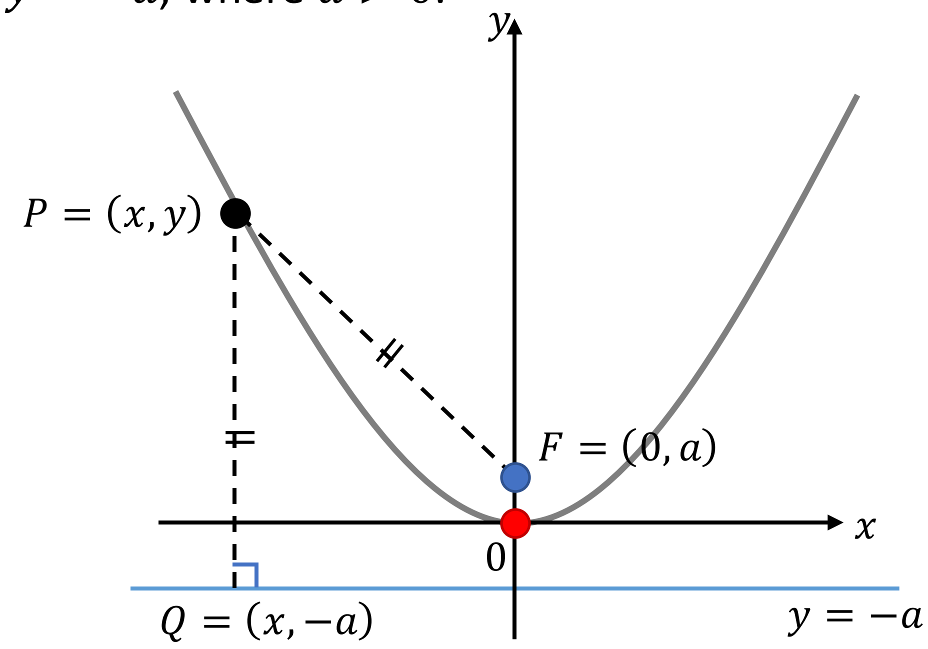
$PQ = PF$

$\Rightarrow y - (-a) = \sqrt{(x-0)^2 + (y-a)^2}$

$\Rightarrow (y+a)^2 = x^2 + (y-a)^2$

$\Rightarrow y^2 + 2ay + a^2 = x^2 + y^2 - 2ay + a^2$

$\Rightarrow x^2 = 4ay.$



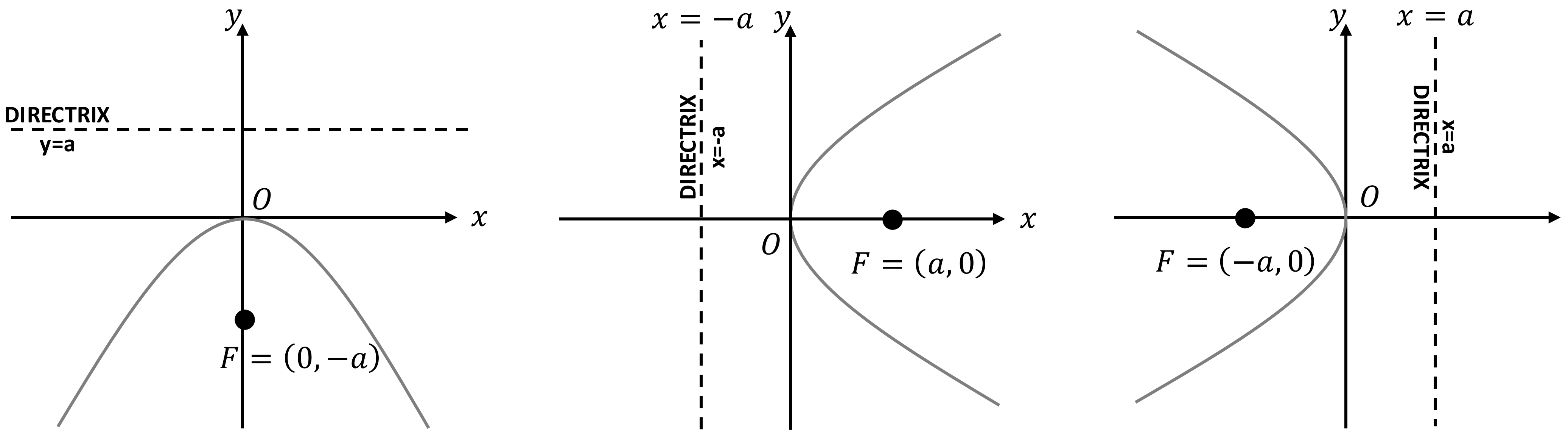
Equation of Parabola (Standard Case)

For $a > 0$, the equation of parabola with focus $(0, a)$ and directrix $y = -a$ is given by

$$x^2 = 4ay.$$

Remark on standard equation of parabola

- The vertex of the parabola is simply the origin $O = (0,0)$ and the y-axis is axis of symmetry.
- One can obtain different parabola with different focus and the directrix, some common types of parabola are shown below:



$x^2 = -4ay$
(a becomes $-a$)

$y^2 = 4ax$
(Interchange x and y)

$y^2 = -4ax$

Example 9

The vertex and the axis of symmetry of parabola are the origin and the x-axis respectively. If the parabola passes through the point $(6, 3)$. Find the equation of this parabola.

☺ Solution:

Note that the parabola is symmetry about x-axis and has vertex at the origin, the equation of parabola is of the form $y^2 = 4ax$.

Since the parabola passes through the point $(6, 3)$, we substitute $(x, y) = (6, 3)$ into the equation and get

$$3^2 = 4a(6) \Rightarrow a = \frac{3}{8}.$$

Then the equation of parabola is given by

$$y^2 = 4\left(\frac{3}{8}\right)x \Rightarrow y^2 = \frac{3}{2}x.$$

More general equation of parabola

In general, the graph of a parabola can have the vertex (h,k) (instead of origin $(0,0)$). Depending on the axis of symmetry, one can derive the following general equation of parabola for this case:

Equation of Parabola (More General Case)

Take $a \neq 0$, the equation of parabola with vertex (h,k) is given by

(i) If the axis of symmetry is $x = h$,

$$(x-h)^2 = 4a(y-k).$$

(ii) If the axis of symmetry is $y = k$,

$$(y-k)^2 = 4a(x-h).$$

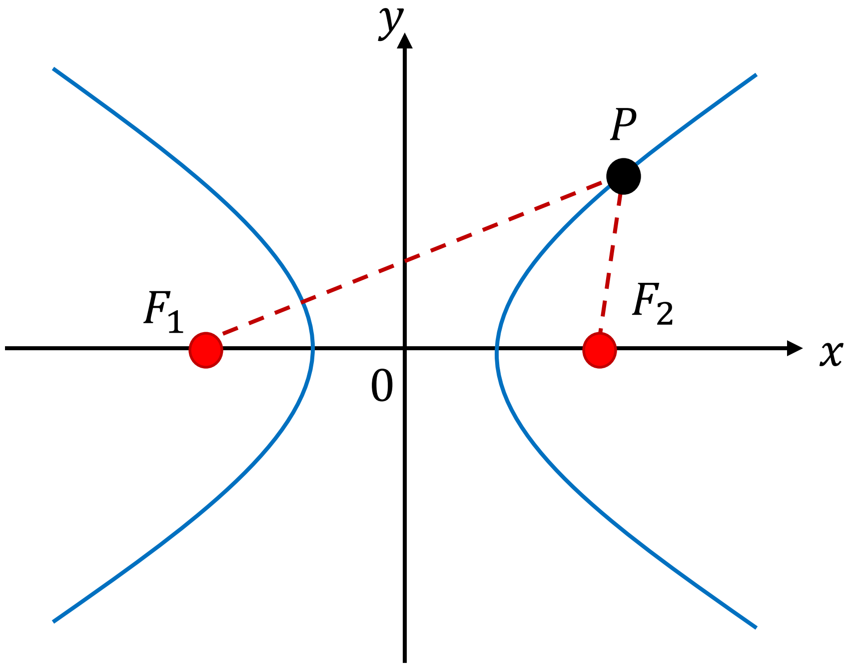
Roughly speaking, the equation in general case can be obtained by replacing x with $x - h$ and y with $y - k$ in the standard form of the equation.

Conic Section #4: Hyperbola

Let F_1 and F_2 be two fixed points (foci) in the plane, a hyperbola is the set of all points P in a plane such that

$$|PF_1 - PF_2| = constant = 2a$$

i.e. the difference between PF_1 and PF_2 is always fixed.



Equation of Hyperbola (Standard Form)

For simplicity, we consider the case when the foci are $F_1 = (-c, 0)$ and $F_2 = (c, 0)$.

According to the definition of hyperbola:

$|PF_1 - PF_2| = 2a$

$\Rightarrow (PF_1 - PF_2)^2 = (2a)^2 = 4a^2$

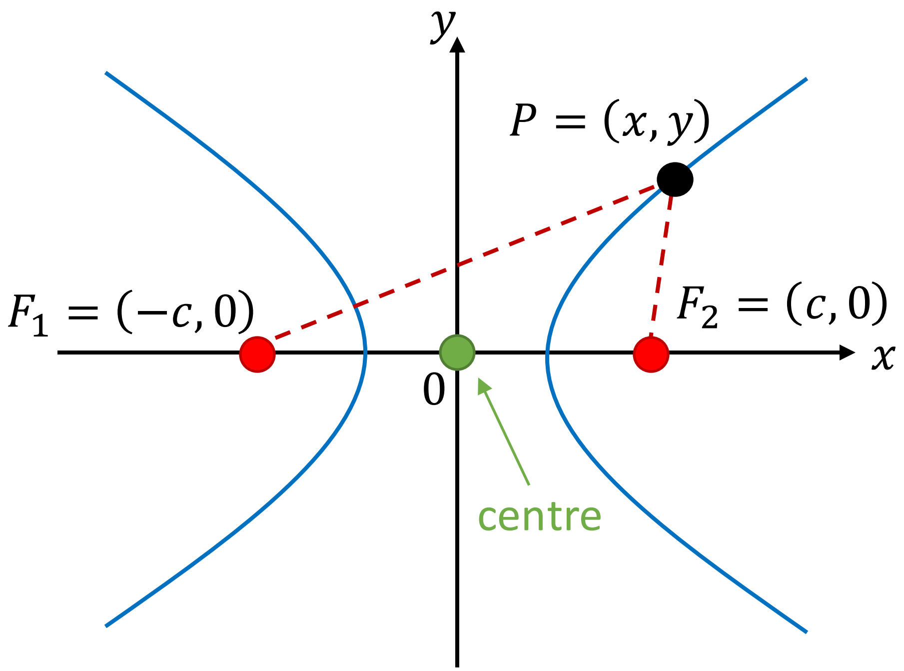
$$\Rightarrow \left[\sqrt{\left(x - (-c)\right)^2 + y^2} - \sqrt{(x - c)^2 + y^2}\right]^2 = 4a^2.$$

After a tedious calculation, one can obtain

$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$

We let $b^2 = c^2 - a^2 > 0$ ($c > a$, why??), then

$$b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



Equation of Hyperbola (Standard Case)

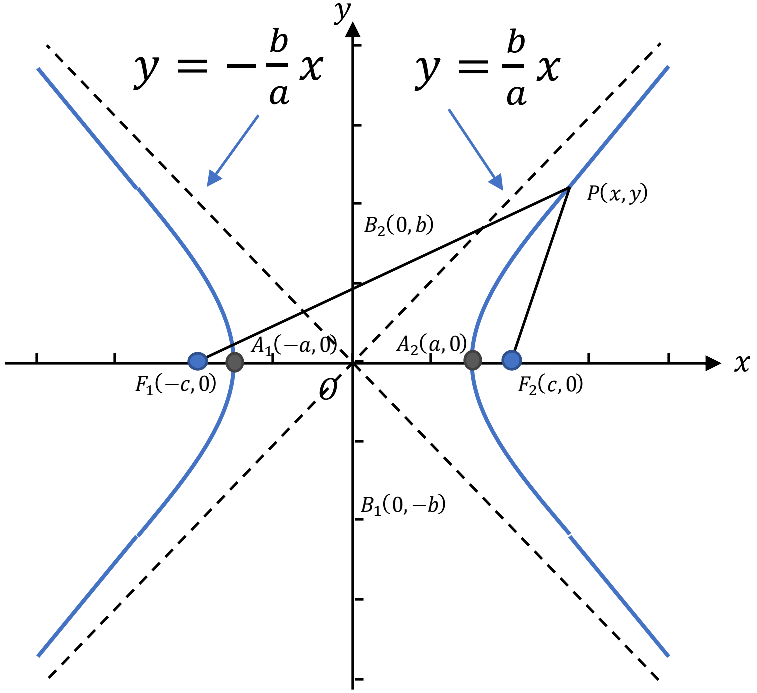
For $c > 0$, the equation of hyperbola with foci $(c, 0)$ and $(-c, 0)$ is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad where\ b^2 = c^2 - a^2, \qquad a, b > 0.$$

Here, $2a$ is the difference between distance from any point on hyperbola to the two foci.

Remark on standard equation of parabola

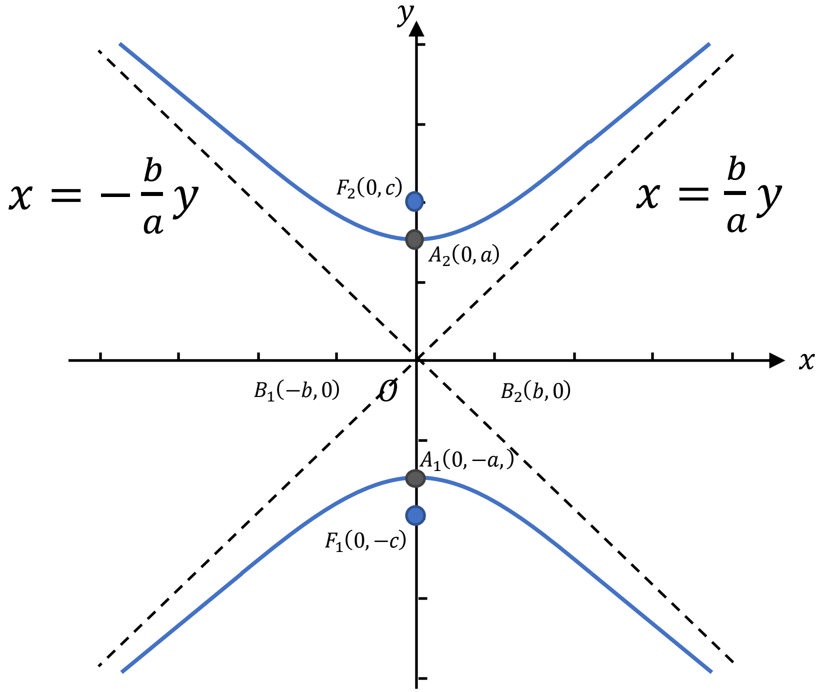
- The origin (mid-point of foci F_1 and F_2) is the *centre* of the hyperbola.
- $A_1 = (-a, 0)$, $A_2 = (a, 0)$ are called *vertices* of the hyperbola.
- When x, y get larger, the graph gets close to the pair of straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ which are called asymptotes.



- If the foci are $F_1 = (0, c)$ and $F_2 = (0, -c)$, then the corresponding equation of the hyperbola is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad b^2 = c^2 - a^2.$$

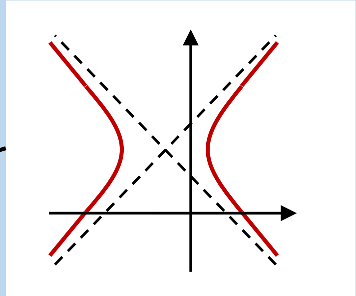
The centre remains to be the origin and the vertices are $A_1 = (0, -a)$, $A_2 = (0, a)$.



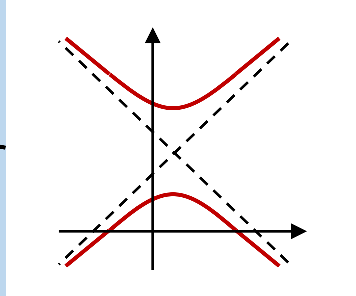
- In general, the equation of hyperbola is of the following general form.

Equation of Hyperbola (General Form)

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$



$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



where the centre is $C = (h, k)$.

Example 10

It is given that the equation of a conic section is $-x^2 + 4y^2 + 4x - 24y + 28 = 0$. Show that the conic section is hyperbola. Find the coordinates of the centre and the vertices of the hyperbola.

☺ Solution:

$$-x^2 + 4y^2 + 4x - 24y + 28 = 0$$

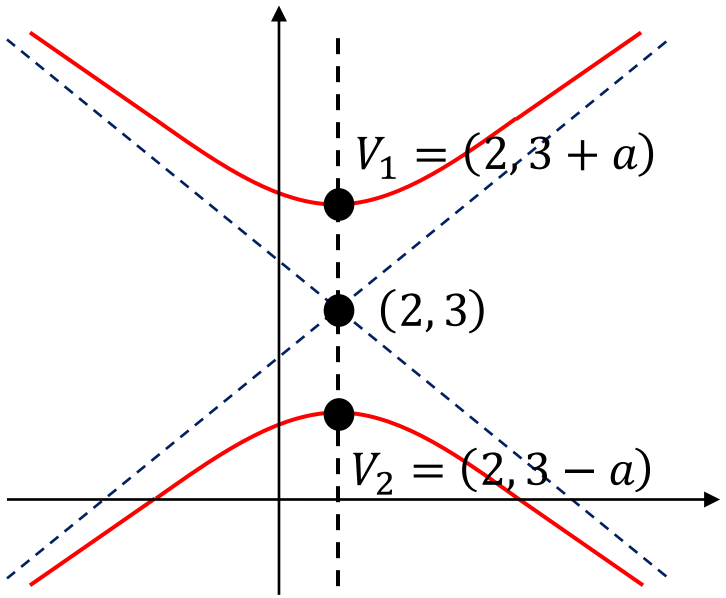
$$\Rightarrow -(x^2 - 4x) + 4(y^2 - 6y) + 28 = 0$$

$$\Rightarrow -(x^2 - 2(2)x + 2^2 - 2^2) + 4(y^2 - 2(3)y + 3^2 - 3^2) + 28 = 0$$

$$\Rightarrow -(x-2)^2 + 4(y-3)^2 - 4 = 0$$

$$\Rightarrow (y-3)^2 - \frac{(x-2)^2}{4} = 1$$

$$\Rightarrow \frac{(y-3)^2}{1^2} - \frac{(x-2)^2}{2^2} = 1.$$



So the conic section is a hyperbola.

The centre of hyperbola is $(2, 3)$.

With $a = 1$, the coordinates of the vertices are $\big(2, (3+1)\big) = (2, 4)$ and $\big(2, (3-1)\big) = (2, 2)$.

Classification of conic section

Recall that the equations of the above four conic sections are given by

- Circle -- $(x-h)^2+(y-k)^2=r^2$,
- Ellipse -- $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$,
- Parabola -- $(x-h)^2=4a(y-k)$ or $(y-k)^2=4a(x-h)$,
- Hyperbola -- $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.

By expanding the terms, one can see that each of the equations can be expressed in the following form:

$$Ax^2+Cy^2+Dx+Ey+F=0 \ldots\ldots(*)$$

where A, C, D, E and F are some numbers.

In many cases, one may be only given the equation of the conic section of the form $(*)$. In order to classify it (circle, ellipse, parabola or hyperbola), one can adopt "completing square technique" [see Example 5, 8, 10] to rewrite the equation into the "standard form".

Example 11

Classify the type of conic section described by each of the following equations.

(a) $4x^2 - 16x + 25y^2 - 84 = 0.$

(b) $4x^2 + 4y^2 + 8x - 24y + 15 = 0.$

☺ Solution:

(a) Note that

$$4x^2 - 16x + 25y^2 - 84 = 0$$

$$\Rightarrow 4(x^2 - 4x) + 25y^2 - 84 = 0$$

$$\Rightarrow 4(x^2 - 2(2)x + 2^2 - 2^2) + 25y^2 - 84 = 0$$

$$\Rightarrow 4(x-2)^2 + 25y^2 = 100$$

$$\Rightarrow \frac{(x-2)^2}{25} + \frac{y^2}{4} = 1 \Rightarrow \frac{(x-2)^2}{5^2} + \frac{y^2}{2^2} = 1.$$

So this equation represents an ellipse.

(b) Note that

$$4x^2 + 4y^2 + 8x - 24y + 15 = 0$$

$$\Rightarrow 4(x^2 + 2x) + 4(y^2 - 6y) + 15 = 0$$

$$\Rightarrow 4(x^2 + 2(1)x + 1^2 - 1^2) + 4(y^2 - 2(3)y + 3^2 - 3^2) + 15 = 0$$

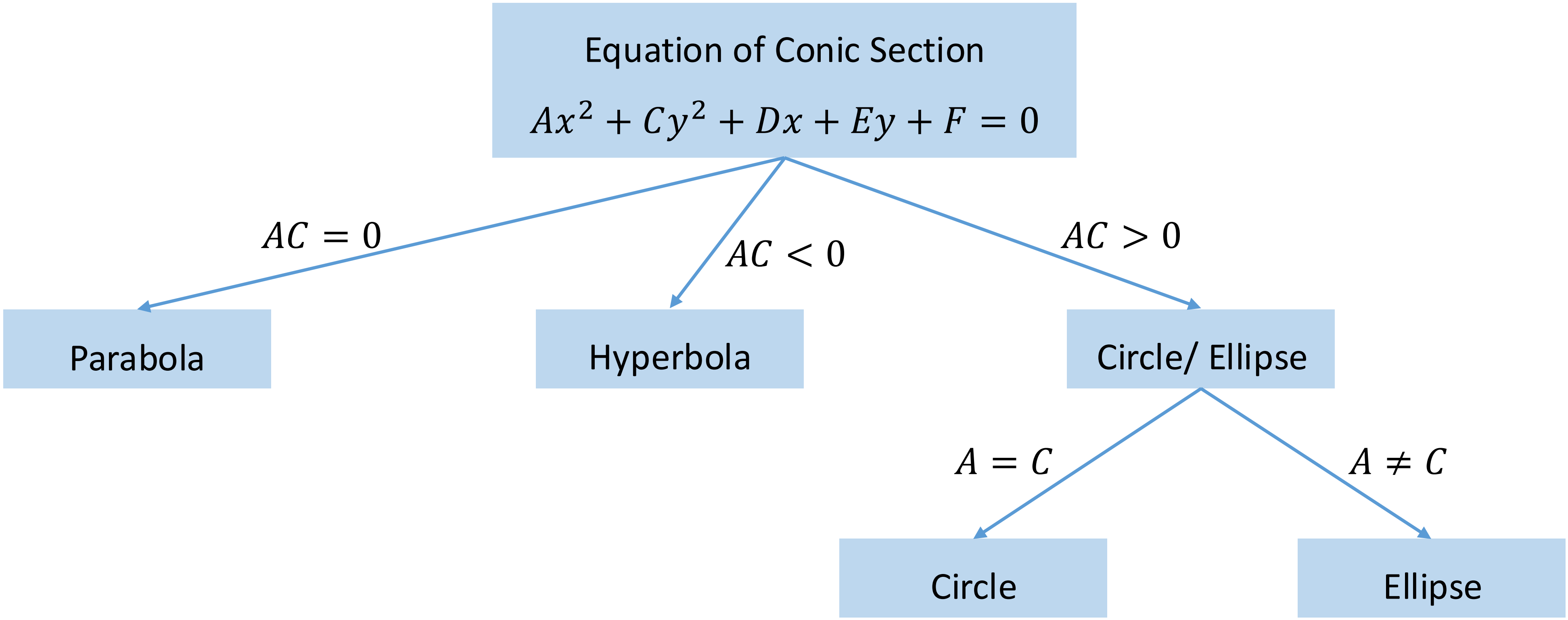
$$\Rightarrow 4(x+1)^2 + 4(y-3)^2 = 25$$

$$\Rightarrow (x+1)^2 + (y-3)^2 = \frac{25}{4}$$

$$\Rightarrow \left(x-(-1)\right)^2 + (y-3)^2 = \left(\frac{5}{2}\right)^2.$$

So the equation represents a circle.

In general, given the equation of conic section $Ax^2 + Cy^2 + Dx + Ey + F = 0$, one can adopt the similar technique to classify the conic section. The flow chart below summarizes the result: (assuming at least one of A, C is non-zero).



Rough reason of the above result:

- If $AC = 0$, then either $A = 0$ or $C = 0$.

  So the equation must be of the forms

  $Cy^2 + Dx + Ey + F = 0 \ \ (D \neq 0) \ \ or \ \ Ax^2 + Dx + Ey + F = 0 \ \ (E \neq 0)$

  Using completing square, one can rewrite the equations as

$$(y-?)^2 = 4a(x-?) \ \ or \ \ (x-?)^2 = 4a(y-?)$$

  which are equation of parabola.

Remark:

In the unlikely case which $D = 0$ (for 1st equation) or $E = 0$ (for two equations). Then the 1st equation become

$$Cy^2 + Ey + F = 0 \ldots \ldots (**)$$

Depending on the number of roots of equation $(**)$, the conic section will be

$$\begin{cases} y = y_1, y = y_2 & if\ (**)\ has\ two\ solutions \\ y = y_0 & if\ (**)\ has\ one\ solution \\ no\ line & if\ (**)\ has\ no\ solution \end{cases}.$$

- If $AC < 0$, then either $\underline{A > 0, C < 0}$ or $\underline{A < 0, C > 0}$.

  Then after completing square, the equation can be rewritten as

$$A(x-?)^2 + C(y-?)^2 = F' \Rightarrow \begin{cases} \dfrac{(x-?)^2}{?^2} - \dfrac{(y-?)^2}{?^2} = 1 \\ \dfrac{(y-?)^2}{?^2} - \dfrac{(x-?)^2}{?^2} = 1 \end{cases} \quad where\ (F' \neq 0)$$

  which is equation of hyperbola.

Remark: (Degenerate Case)

In the unlikely case when $F' = 0$, the equation becomes

$$A(x-?)^2 + C(y-?)^2 = 0 \Rightarrow y-? = \pm\sqrt{\underbrace{-\frac{A}{C}}_{>0}}(x-?) \Rightarrow \begin{cases} y = \sqrt{\dfrac{A}{C}}x+? \\ y = -\sqrt{\dfrac{A}{C}}x+? \end{cases}$$

which represents a pair of (unparallel) straight lines.

- If $AC > 0$, then either $\underline{A, C \text{ have the same sign}}$ (both positive or both negative).

  After completing square, the equation can be rewritten as

  $$A(x-?)^2 + C(y-?)^2 = F' \Rightarrow \frac{(x-?)^2}{\left(\sqrt{\frac{F'}{A}}\right)^2} - \frac{(y-?)^2}{\left(\sqrt{\frac{F'}{C}}\right)^2} = 1, \qquad where\ (F' \neq 0)$$

  which is the equation of circle or ellipse.

  - ❖ If $A = C$, then $\frac{F'}{A} = \frac{F'}{C} = r^2$. Hence the equation can be expressed as

    $$(x-?)^2 + (y-?)^2 = r^2$$

    which is the equation of circle.

  - ❖ If $A \neq C$, then $\frac{F'}{A} \neq \frac{F'}{C}$. Then the equation is simply the equation of ellipse ($\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$).

> Remark: (Degenerate Case)
>
> In the unlikely case when $F' = 0$, then the conic section will become a single point.

Example 12

Identify the graph of the following functions

(a) $3x^2 - 2y^2 + 5x - y - 5 = 0$

(b) $2x^2 + 2y^2 - x + y - 7 = 0$

(c) $y^2 - 4x + 2y - 1 = 0$

☺ Solution:

(a) Note that $AC = 3 \times (-2) = -6 < 0$, the graph is a hyperbola.

(b) Note that $AC = 2 \times 2 = 4 > 0$, the graph is either circle or ellipse. Since $A = C = 2$, we conclude that the graph is a circle.

(c) Note that $AC = 0 \times 1 = 0$ and coefficient of x $(= -4)$ is non-zero, the graph is a parabola.

General Conic Section

In general, a conic section in 2-D can be expressed in the following form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E and F are constants.

- It is important for us to rewrite the equation into the form

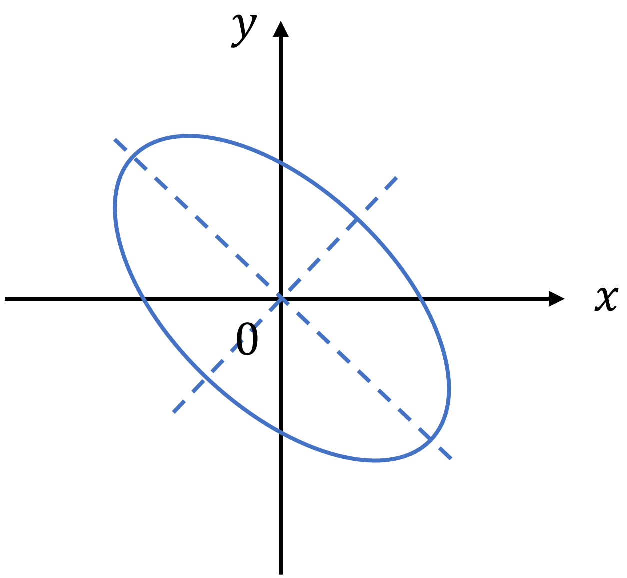
$$\frac{(x-h)^2}{p} - \frac{(y-k)^2}{q} = 1$$

  because of the existence of the term Bxy.

- The term Bxy exists because the conic section (circle, ellipse, parabola or hyperbola) is being rotated from its "standard position".

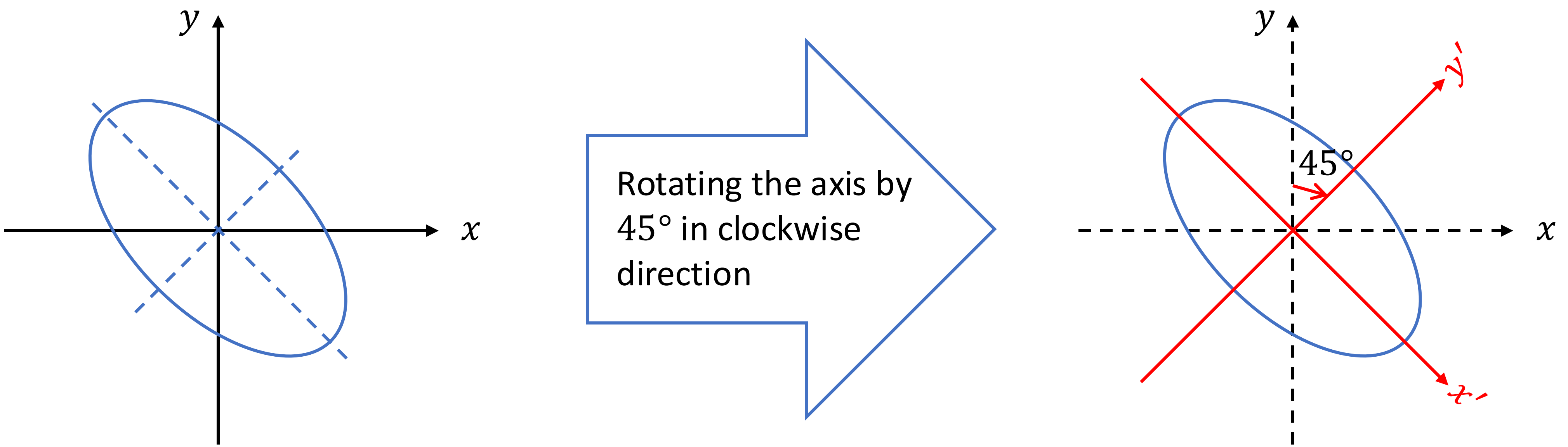
As an example, we suppose that the equation of a conic section is given by $13x^2 + 10xy + 13y^2 = 72$. We may try to identify the conic section by first sketching the graph of this conic section.

The figure below shows the graph of the conic section



One can guess that the conic section is an ellipse since one can see the "standard ellipse" by rotating our "view point" by certain degree.

Mathematically, rotating our “view point” is equivalent to rotating the xy-axis (change to $x'y'$-axis).



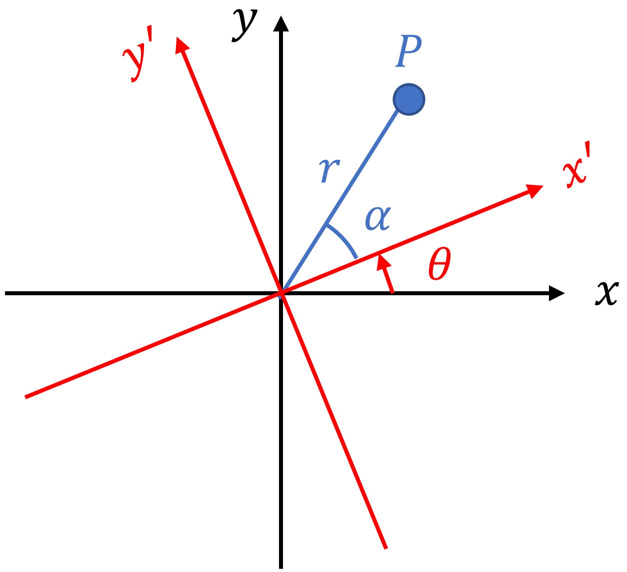
$$13x^2 + 10xy + 13y^2 = 72$$
(xy-coordinate)

$$8(x')^2 + 8(y')^2 = 72$$
$$\Rightarrow \frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$
($x'y'$-coordinate)

This technique is called “Rotation of Axes”.

Rotation of Axes

Suppose the x-axis and y-axis are rotated through a "positive" angle (anti-clockwise direction), we obtain a new x'-axis and y'-axis and this forms a $x'y'$-coordinate system:



In (x, y)-coordinate, the coordinate of P is $x = r\cos(\theta + \alpha)$, $y = r\sin(\theta + \alpha)$.

In (x', y')-coordinate, the coordinate of P is $x' = r\cos\alpha$, $y' = r\sin\alpha$.

Using the compound angle formula (details will be discussed in later Chapter),

$$\cos(\alpha+\theta)=\cos\alpha\cos\theta-\sin\alpha\sin\theta\,,$$

$$\sin(\alpha+\theta)=\sin\alpha\cos\theta+\cos\alpha\sin\theta\,,$$

we have

$$x=r\cos(\alpha+\theta)=r\cos\alpha\cos\theta-r\sin\alpha\sin\theta=x'\cos\theta-y'\sin\theta\,,$$

$$y=r\sin(\alpha+\theta)=r\sin\alpha\cos\theta+r\cos\alpha\sin\theta=y'\cos\theta+x'\sin\theta\,.$$

Summing up, we have the following relation between xy-coordinate and $x'y'$-coordinate:

$$x=x'\cos\theta-y'\sin\theta\,,$$

$$y=y'\cos\theta+x'\sin\theta\,.$$

Example 13

It is given that the equation of a conic section is $xy = 1$. Suppose the new $x'y'$-system is formed by rotating x-axis and y-axis by $45°$ in anti-clockwise direction.

(a) Find the equation of this conic section in $x'y'$-system.

(b) Hence, identify the surface and sketch the graph.

☺ Solution:

(a) Using the transformation formulae:

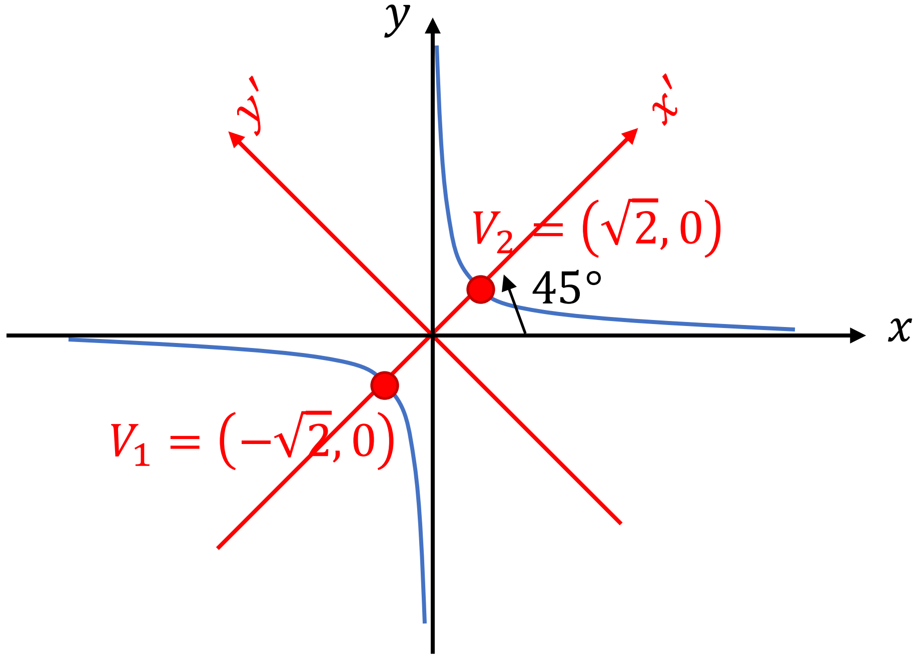
$$x = x' \cos 45° - y' \sin 45° = \frac{1}{\sqrt{2}}(x' - y'),$$
$$y = y' \cos 45° + x' \sin 45° = \frac{1}{\sqrt{2}}(x' + y').$$

Substitute these into $xy = 1$, we get

$$xy = 1 \Rightarrow \frac{1}{\sqrt{2}}(x' - y')\frac{1}{\sqrt{2}}(x' + y') = 1 \Rightarrow \frac{1}{2}\left(x'^2 - y'^2\right) = 1$$

$$\Rightarrow \frac{x'^2}{2} - \frac{y'^2}{2} = 1 \quad \left(or \ \frac{x'^2}{(\sqrt{2})^2} - \frac{y'^2}{(\sqrt{2})^2} = 1 \right).$$

(b) From (a), we see that the conic section is a hyperbola (see P.34). The graph (blue line) of this conic section is shown below:



Example 14

It is given that the equation of a conic section is

$$5x^2 + 6xy + 5y^2 - 18\sqrt{2}x - 14\sqrt{2}y + 26 = 0.$$

Suppose that the new $x'y'$-system is formed by rotating x-axis and y-axis by $45°$ in clockwise direction, find

(a) The equation of the conic section in $x'y'$-coordinates.

(b) Hence, identify the surface and sketch the graph.

☺ Solution:

(a) Using the transformation formula:

$$x = x'\cos(-45°) - y'\sin(-45°) = \frac{1}{\sqrt{2}}(x' + y'),$$
$$y = x'\sin(-45°) + y'\cos(-45°) = \frac{1}{\sqrt{2}}(-x' + y').$$

Substitute the formulae into the equation, we have

$$\frac{5}{2}(x'+y')^2+\frac{6}{2}(x'+y')(-x'+y')+\frac{5}{2}(-x'+y')^2-\frac{18\sqrt{2}}{\sqrt{2}}(x'+y')-\frac{14\sqrt{2}}{\sqrt{2}}(-x'+y')+26=0$$

$$\Rightarrow \frac{5}{2}\left(x'^2+2x'y'+y'^2\right)+3\left(-x'^2+y'^2\right)+\frac{5}{2}\left(x'^2-2x'y'+y'^2\right)-18(x'+y')-14(-x'+y')+26=0$$

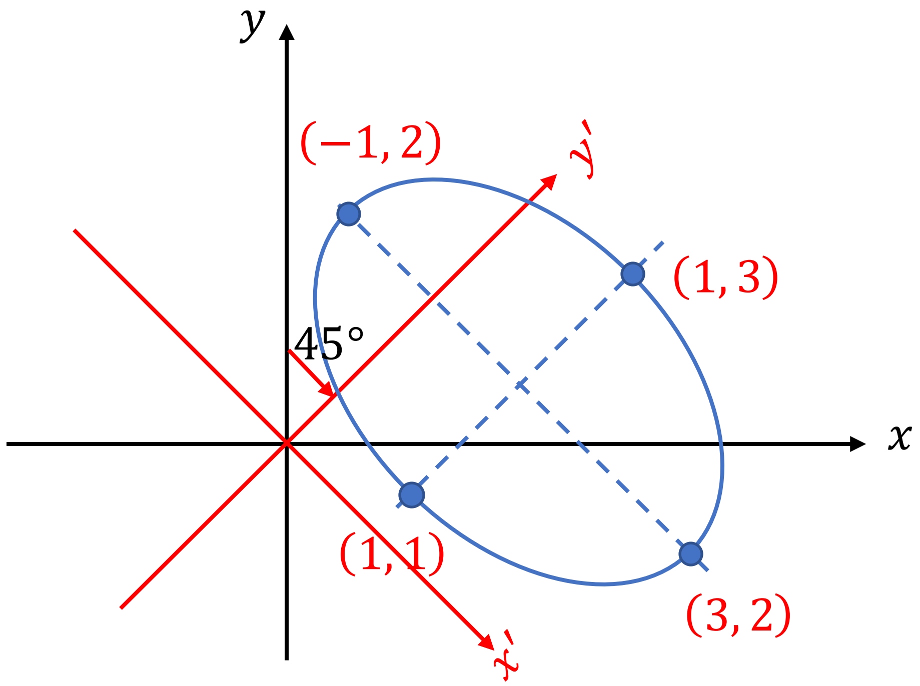
$$\Rightarrow 2x'^2+8y'^2-4x'-32y'+26=0$$

$$\Rightarrow x'^2+4y'^2-2x'-16y'+13=0$$

$$\Rightarrow (x'-1)^2+4(y'-2)^2=4$$

$$\Rightarrow \frac{(x'-1)^2}{2^2}-\frac{(y'-2)^2}{1^2}=1.$$

(b) From the result of (a), we observe that the graph is a standard ellipse (with centre $(1, 2)$ in $x'y'$-plane). The graph of the conic section is presented below:



(Here, coordinates highlighted in red are the coordinates in $x'y'$-plane!!)

Identifying the conic section

Given the equation of conic section $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, one can identify the conic section using the following procedure:

1. Use the transformation formula

$$x = x' \cos\theta - y' \sin\theta$$

$$y = x' \sin\theta + y' \cos\theta$$

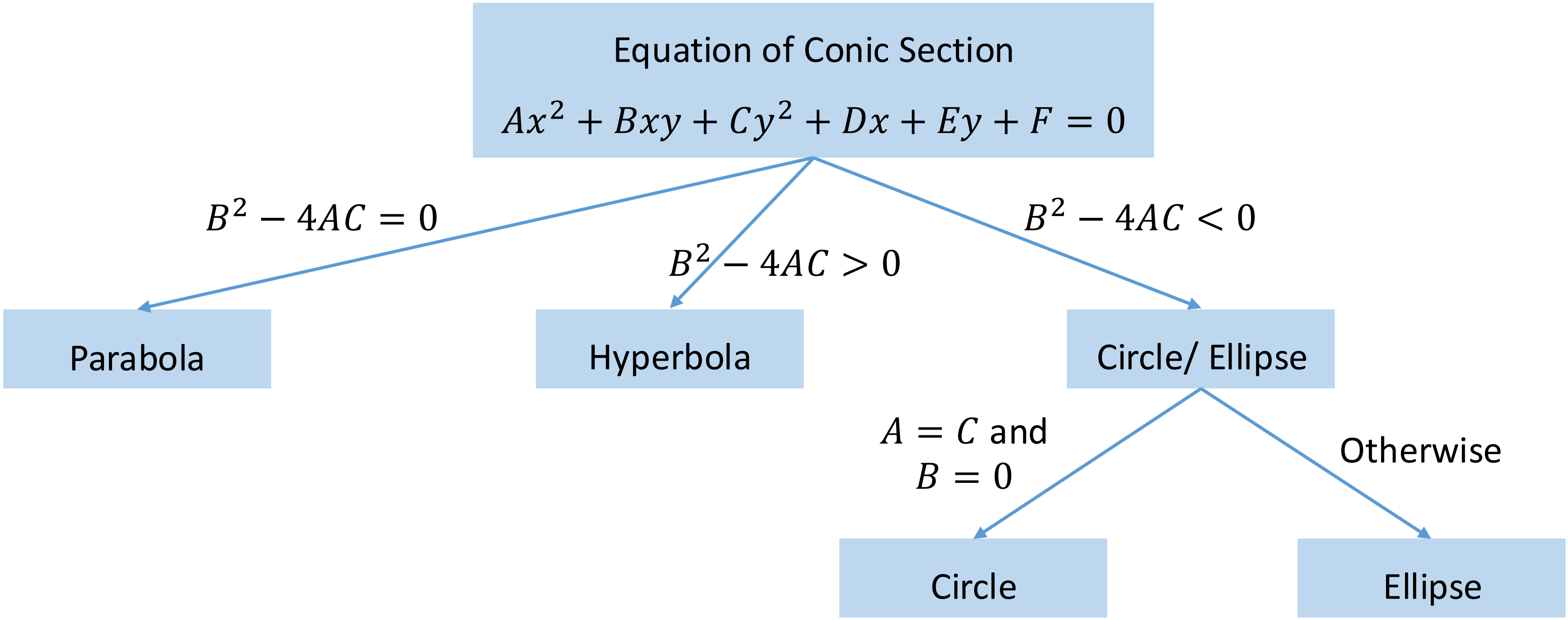
   and rewrite the equation into the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

   where A', B', C', D', E', F' are some numbers in terms of θ.

2. Choose θ such that $B'x'y' = 0$ or $B' = 0$.
3. The equation becomes $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$. One can identify the conic section by using the result in P.39.

Using this procedure and the fact that $\overbrace{B'^2}^{=0} - 4A'C' = B^2 - 4AC$, one can establish the following characterization of conic section:



Example 15

Identify the graph of each of the following equations:

(a) $x^2 - 2xy - 3y^2 - 3x + 6y - 5 = 0.$

(b) $4x^2 - 4xy + y + 2 = 0.$

(c) $3x^2 - xy + 12y^2 - 5x - 7y + 12 = 0.$

☺ Solution:

(a) Since $B^2 - 4AC = (-2)^2 - 4(1)(-3) = 16 > 0$, so the graph is hyperbola.

(b) Since $B^2 - 4AC = (-4)^2 - 4(4)(1) = 0$, so the graph is parabola.

(c) Since $B^2 - 4AC = (-1)^2 - 4(3)(12) = -143 < 0$, so the graph can be either ellipse or circle.

Furthermore, since $B = -1 \neq 0$, we conclude the graph is ellipse.

Parametric Equations

In Physics, we usually describe the motion of a moving particle by the following pair of equations:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in [a, b]$$

where t is a parameter (which may be interpreted as time) and $f(t)$, $g(t)$ are some functions of t. This pair of equation is called parametric equations.

Summary – Things you need to know

- Two basic coordinate systems – Cartesian coordinated (xy-coordinate) and Polar coordinate ($r\theta$-coordinate)
  - ✓ Interchanging between Cartesian coordinate and Polar coordinate.
- Four conic sections – Circle, Ellipse, Parabola, Hyperbola
  - ✓ Know the standard equations of four conic sections.
  - ✓ Know how to identify the conic section using completing square technique.
- Classification of general conic sections
  - ✓ Classification of $Ax^2 + Cy^2 + Dx + Ey + F = 0$
  - ✓ Classification of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
    - Using rotation of axes.
    - Using $B^2 - 4AC$.

Appendix – Detailed description of the classification procedure of

$\boldsymbol{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$. (For $\boldsymbol{B \neq 0}$)

This appendix provides a detailed explanation of the classification procedure described in P.54.

Step 1: Transform the equation

Apply the transformation formula

$$x = x' \cos\theta - y' \sin\theta \quad and \quad y = x' \sin\theta + y' \cos\theta\,,$$

one can transform the equation into the alternative form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

where

$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$,

$B' = -2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta$,

$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta,$$

$$D' = D\cos\theta + E\sin\theta\,, \qquad E' = -D\sin\theta + E\cos\theta\,, \qquad F' = F.$$

Step 2:

Choose θ such that $B' = 0$ so that the equation becomes $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ and the result in P.39 can be applied.

We need to solve

$$B' = -2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta = 0.$$

Using the fact that

$$2\sin\theta\cos\theta = \sin 2\theta \quad and \quad \cos^2\theta - \sin^2\theta = \cos 2\theta\,,$$

we get

$$-A\sin 2\theta + B\cos 2\theta + C\sin 2\theta = 0$$
$$\Rightarrow (A - C)\sin 2\theta = B\cos 2\theta$$

$$\Rightarrow \begin{cases} \tan 2\theta = \dfrac{B}{A - C} & if\ A \neq C \\ \cos 2\theta = 0 & if\ A = C \end{cases}$$

With this θ, the equation becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

The result in P.39 suggests that

$A'C' = 0 \Rightarrow$ The conic section is parabola

$A'C' > 0 \Rightarrow$ The conic section is ellipse or circle

$A'C' < 0 \Rightarrow$ The conic section is hyperbola

(* Here, we do not consider those "degenerate" cases)

Step 4: Establish the relationship between A, B, C and A', B', C'.

One can show, by direct (but tedious) calculation, that

$$B'^2 - 4A'C' = B^2 - 4AC \quad for\ any\ \theta.$$

Since the θ is chosen such that $B' = 0$, so we have

$$B^2 - 4AC = -4A'C'.$$

Therefore, we can conclude that

$B^2 - 4AC = 0 \Rightarrow -4A'C' = 0 \Rightarrow A'C' = 0 \Rightarrow$ parabola,

$B^2 - 4AC > 0 \Rightarrow -4A'C' > 0 \Rightarrow A'C' < 0 \Rightarrow$ hyperbola,

$B^2 - 4AC < 0 \Rightarrow -4A'C' < 0 \Rightarrow A'C' > 0 \Rightarrow$ circle or ellipse.

(* In the last case, one can check that the conic is circle only when $B = 0$ and $A = C$)

One more examples on hyperbola and parabola

Extra Example 1 (Parabola: Degenerate Case)

It is given that the equation of a parabola is $y^2 + ax - 4y + 3 = 0$.

(a) If $a = 1$, sketch the graph and locate the vertex of parabola.

(b) What will happen when $a = 0$?

☺ Solution:

(a) If $a = 1$, the equation becomes

$$y^2 + x - 4y + 3 = 0$$

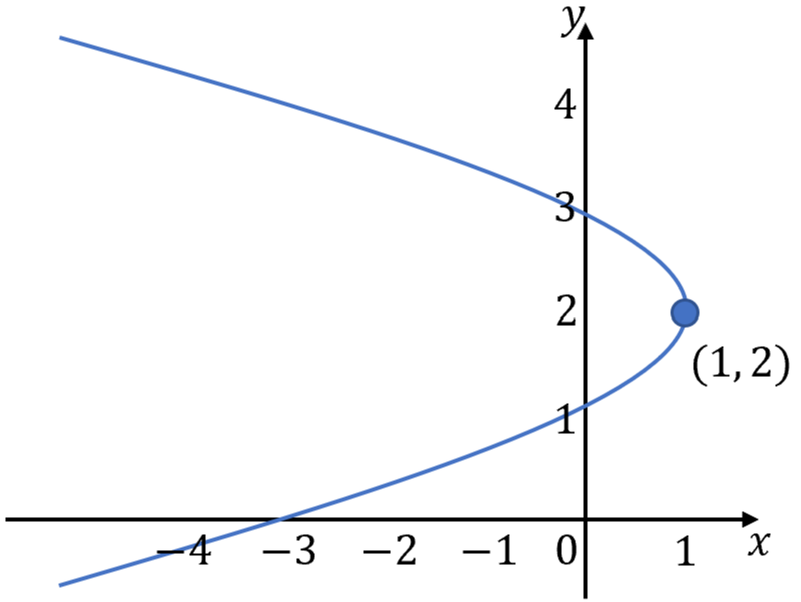
$$\Rightarrow (y^2 - 4y) + x + 3 = 0$$

$$\Rightarrow (y^2 - 2(2)y + 2^2 - 2^2) + x + 3 = 0$$

$$\Rightarrow (y-2)^2 + x - 1 = 0 \Rightarrow (y-2)^2 = -(x-1)$$

$$\Rightarrow \underbrace{(y-2)^2}_{(y-k)^2} = \underbrace{4\left(-\frac{1}{4}\right)}_{a}\underbrace{(x-1)}_{(x-h)}.$$

This parabola has vertex $(h, k) = (1, 2)$.



(b) If $a = 0$, the equation becomes

$$y^2 - 4y + 3 = 0$$

One can solve this equation and get

$$y^2 - 4y + 3 = 0 \Rightarrow (y-1)(y-3) = 0 \Rightarrow y = 1 \; or \; y = 3.$$

Then the conic section becomes two separated horizontal lines on 2-D plane.

☺Note:

The above case is called “degenerate case”.

Extra Example 2

It is given that the equation of a conic section is $3x^2 + 2\sqrt{3}xy + y^2 - x + \sqrt{3}y + 4 = 0$.

(a) Identify the conic section (Circle/ Ellipse/ Hyperbola or Parabola).

(b) Sketch the graph and indicate the necessary details.

☺ Solution:

(a) Note that

$$B^2 - 4AC = \left(2\sqrt{3}\right)^2 - 4(3)(1) = 12 - 12 = 0.$$

So the conic section is parabola.

(b) We need to transform the equation into the form: $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ (using rotation of axes).

Since $\tan 2\theta = \frac{B}{A-C} \Rightarrow \tan 2\theta = \frac{2\sqrt{3}}{3-1} = \sqrt{3} \Rightarrow 2\theta = 60° \Rightarrow \theta = 30°$.

Therefore one can transform the equation into the desired form by rotating xy-axes by $30°$ in <u>anti-clockwise direction</u>.

Using the transformation formula:

$$x = x' \cos(30°) - y' \sin(30°) = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$$
$$y = x' \sin(30°) + y' \cos(30°) = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y',$$

and substitute into the original equation, we get

$$3\left(\tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y'\right)^2 + 2\sqrt{3}\left(\tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y'\right)\left(\tfrac{1}{2}x' + \tfrac{\sqrt{3}}{2}y'\right) + \left(\tfrac{1}{2}x' + \tfrac{\sqrt{3}}{2}y'\right)^2 - \left(\tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y'\right) + \sqrt{3}\left(\tfrac{1}{2}x' + \tfrac{\sqrt{3}}{2}y'\right) + 4 = 0$$

$\Rightarrow \cdots$

$\Rightarrow 4x'^2 + 2y' + 4 = 0$

$$\Rightarrow x'^2 = -\frac{1}{4}(2y' + 4)$$

$$\Rightarrow x'^2 = -\frac{1}{2}(y' + 2)$$

$$\Rightarrow \underbrace{x'^2}_{(x'-h)^2} = -4\underbrace{\left(\frac{1}{8}\right)}_{a}\underbrace{\left(y' - (-2)\right)}_{(y'-k)}.$$

So the vertex of the parabola is $(h, k) = (0, -2)$

and it is the inverted U-shape curve in $x'y'$-plane.