MA1200.

1. (a) $\lim\limits_{x\to 3} \dfrac{x^2-5x+6}{x^2-8x+15} = \lim\limits_{x\to 3} \dfrac{(x-2)(x-3)}{(x-5)(x-3)} = \lim\limits_{x\to 3} \dfrac{x-2}{x-5} = -\dfrac{1}{2}$

(b). Let $t = \frac{1}{x}$.

$$\lim_{t\to\infty} \frac{2\sqrt{t+\sqrt{t}}}{\sqrt{t+\sqrt{t+\sqrt{t}}}+\sqrt{t-\sqrt{t+\sqrt{t}}}} = \lim_{t\to\infty} \frac{2\sqrt{1+\frac{1}{\sqrt{t}}}}{\sqrt{1+\frac{\sqrt{t+\sqrt{t}}}{t}}+\sqrt{1-\frac{\sqrt{t+\sqrt{t}}}{t}}} = 1.$$

(c). $\lim\limits_{x\to\infty} \left(\dfrac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}} = \lim\limits_{x\to\infty} \left(\dfrac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\right)^{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}} = 1.$

(d). $\lim\limits_{x\to 0} \dfrac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$= \lim\limits_{x\to 0} \dfrac{2\cos(a+2x)-2\cos(a+x)}{2x} = \lim\limits_{x\to 0} \dfrac{-2\sin(a+2x)+\sin(a+x)}{1}$

$= -\sin a.$

2. Let $(cx+d)\big|_{x=0} = (3x^2+1)\big|_{x=0} = 1$

$\therefore d = 1.$

Let $(cx+1)\big|_{x=1} = \sqrt{x+3}\,\big|_{x=1} = \sqrt{4} = 2.$

$\therefore c = 1.$

3. $f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0. \end{cases}$

Since $(2x)\big|_{x=0} = -(2x)\big|_{x=0} = 0.$

$\therefore f(x)$ is differentiable at $x=0$.

$$g(x) = \ln|x| = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0. \end{cases}$$

$g'(x) = \frac{1}{x}$ Not differentiable at $x=0$.

$$h(x) = x-|x| = \begin{cases} 0, & x \geq 0 \\ 2x, & x \leq 0 \end{cases}$$

$0 \neq 2$. Not differentiable

$$j(x) = \begin{cases} x, & \text{if } x<0 \\ \ln(1+x), & \text{if } x \geq 0 \end{cases}$$

Note $(\ln(1+x))'\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1$

$\therefore j(x)$ is differentiable at $x=0$.

4.
(a). $f(x) = x[\sin(\ln x) - \cos(\ln x)]$

$$\begin{aligned}\therefore f'(x) &= \sin(\ln x) - \cos(\ln x) \\ &\quad + x\left[\cos(\ln x)\cdot\frac{1}{x} + \sin(\ln x)\cdot\frac{1}{x}\right] \\ &= \sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 2\sin(\ln x).\end{aligned}$$

(b). $f(x) = \dfrac{x}{\sqrt{a^2-x^2}}$

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{a^2-x^2}} - \frac{x\cdot(-2x)}{2(a^2-x^2)^{\frac{3}{2}}} \\ &= \frac{1}{\sqrt{a^2-x^2}} + \frac{x^2}{(a^2-x^2)^{\frac{3}{2}}}.\end{aligned}$$

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(c). $f(x) = \tan^{-1}(x + \sqrt{1+x^2})$

$\therefore f'(x) = \frac{1}{1+(x+\sqrt{1+x^2})^2} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$

(d). $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

$:= f_1(x) + f_2(x).$

$f_1(x) = (\sin x)^{\cos x}$

$\therefore \ln f_1(x) = \cos x \cdot \ln \sin x$

$\therefore \frac{f_1'(x)}{f_1(x)} = -\sin x \cdot \ln \sin x + \cos x \cdot \frac{\cos x}{\sin x}$

$\therefore f_1'(x) = (\sin x)^{\cos x} \left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln \sin x\right).$

Similarly. $f_2'(x) = (\cos x)^{\sin x} \cdot \left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right).$

(e). $\sqrt{x} + \sqrt{y} = \sqrt{a}.$

$\therefore \frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2}\frac{1}{\sqrt{y}} \cdot y_x' = 0 \Rightarrow y_x' = -\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}} = 1 - \frac{\sqrt{a}}{\sqrt{x}}.$

(f). $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t. \end{cases}$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3a\sin^2 t\cos t}{-3a\sin t\cos^2 t} = \frac{-\sin t}{\cos t} = -\tan t.$

5. $\vec{v} = \left(\frac{y}{32}, -\frac{x}{50}\right).$

At $(6, \frac{32}{5})$, $\vec{v} = \left(\frac{1}{5}, -\frac{3}{25}\right).$

let $y = -\frac{3}{5}x + b. \Rightarrow \frac{32}{5} = -\frac{3}{5} \times 6 + b \Rightarrow b = 10.$

$\therefore y = -\frac{3}{5}x + 10.$

6. $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3-3t^2}{2-2t} = \frac{3(1-t^2)}{2(1-t)} = \frac{3(1+t)}{2}$

$\therefore \left.\frac{dy}{dx}\right|_{t=1} = 3.$ when $t=1$, $x=1$, $y=2.$

$\therefore y = 3x + b \Rightarrow b = -1 \Rightarrow y = 3x - 1.$

7. $\begin{cases} a+b=9. \quad a\geq 0. \quad b\geq 0. \\ F(a):=ab^2=a(9-a)^2, \quad a\in[0,9]. \end{cases}$

$F'(a)=(9-a)(9-3a). \Rightarrow F'(a)=0 \Rightarrow a=3. \; a=9.$

$F(3)=3\cdot(9-3)^2=3\times 36=108.$

8. (a). By $f(x)=(1+x^2)^{-1}$, $\Rightarrow f'(x)=-(1+x^2)^{-2}\cdot 2x.$

$\therefore (1+x^2)f'(x)+2xf(x)=-2x(1+x^2)^{-1}+2x(1+x^2)^{-1}=0.$

(b). By (a): $(1+x^2)f'(x)+2xf(x)=0$

$\Rightarrow (1+x^2)f''(x)+2xf'(x)+2xf'(x)+2f(x)=0$

$\Rightarrow (1+x^2)f''(x)+4xf'(x)+2f(x)=0.$

$\cdots \Rightarrow (1+x^2)f^{(n+1)}(x)+(2n+2)xf^{(n)}(x)+n(n+1)f^{(n-1)}(x)=$

(c). $f(0)=1. \quad f'(0)=0$

By (b):

$f''(0)=-2f(0)=-2$

$f'''(0)=-2(2+1)f'(0)=0$

$f''''(0)=-3(3+1)f''(0)=24.$

$\therefore (1+x^2)^{-1}=1+\frac{-2}{2!}x^2+\frac{24}{4!}x^4+\cdots$

$=1-x^2+x^4+\cdots$

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