Question 1.

a). $9x^2 + 4y^2 - 18x + 8y - 23 = 0.$

$\Rightarrow 9(x^2 - 2x + 1) - 9 + 4(y^2 + 2y + 1) - 4 - 23 = 0$

$\Rightarrow 9(x-1)^2 + 4(y+1)^2 = 36$

$\Rightarrow \frac{1}{4}(x-1)^2 + \frac{1}{9}(y+1)^2 = 1$

$\Rightarrow \frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1$: Elliptic.

(b). let $X = x-1$. $Y = y+1$. $a=3$. $b=2$.

then $\frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1$ becomes

$\frac{Y^2}{a^2} + \frac{X^2}{b^2} = 1$. Then $c^2 = a^2 - b^2 = 9 - 4 = 5 \Rightarrow c = \pm\sqrt{5}$.

foci : $X=0$. $Y = \pm\sqrt{5} \Rightarrow x = 1$, $y = \pm\sqrt{5} - 1$.

$\therefore (1, \sqrt{5}-1)$, $(1, -\sqrt{5}-1)$.

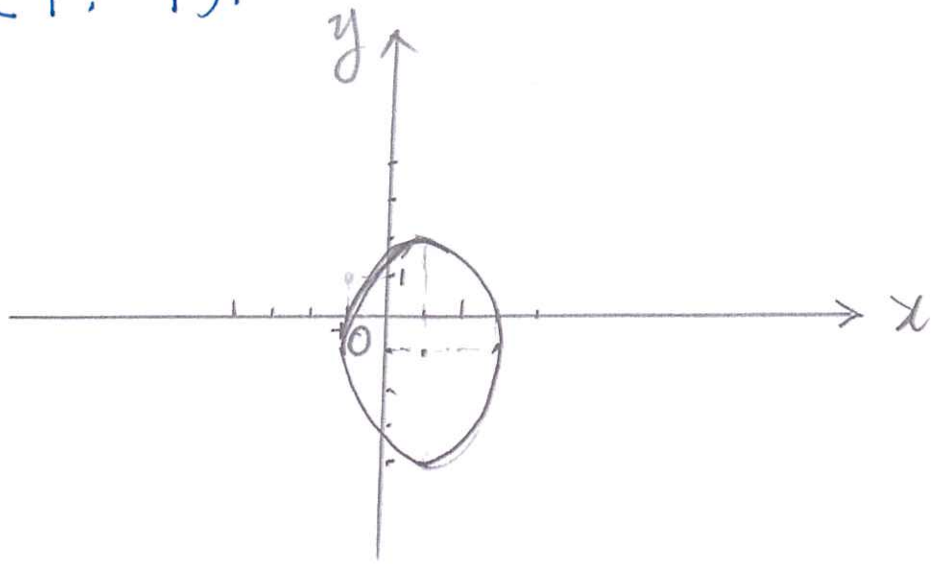
vertices: $X = \pm b = \pm 2$. $Y = 0 \Rightarrow x = \pm 2 + 1$, $y = -1$

$X = 0$, $Y = \pm a = \pm 3 \Rightarrow x = 1$. $y = \pm 3 - 1$

$\therefore (3, -1)$, $(-1, -1)$, $(1, 2)$, $(1, -4)$.

Center: $(1, -1)$.

(c).



Question 2.

let $2+x-x^2=0. \Rightarrow x=2, \quad x=-1$

let $2+x-x^2 \geq 0 \Rightarrow x \in [-1,2]$

$\therefore$ (a). $x \in [-1, 2]$

(b). $f'(x) = \dfrac{1-2x}{2\sqrt{2+x-x^2}}$

when $x \in [-1, \frac{1}{2}]$. $f'(x) \geq 0 \Rightarrow f(x)$ increasing.

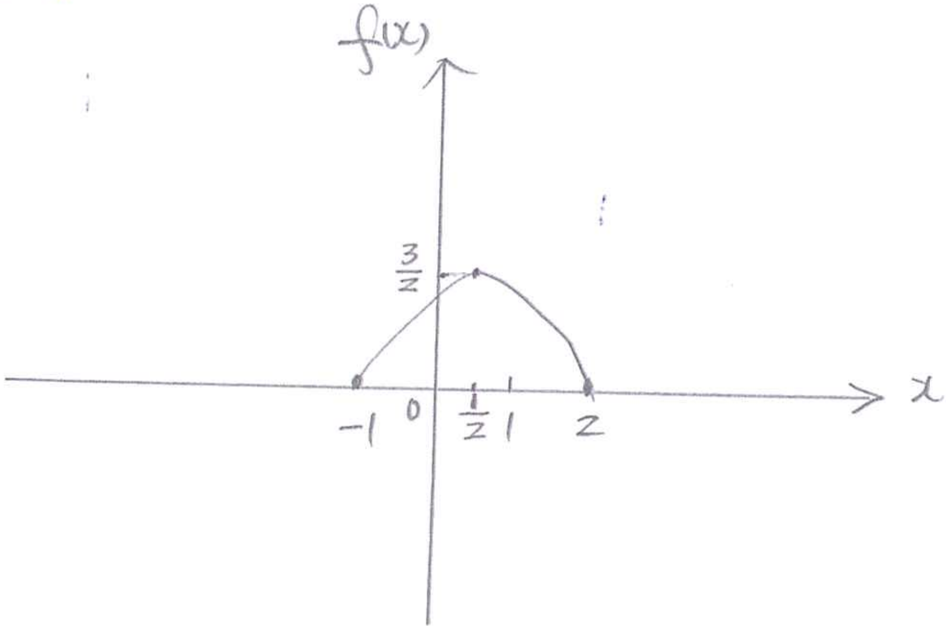
when $x \in [\frac{1}{2}, 2]$. $f'(x) \leq 0 \Rightarrow f(x)$ decreasing.

(c). $\max\limits_{x\in[-1,2]} f(x) = f(\frac{1}{2}) = \sqrt{2+\frac{1}{2}-\frac{1}{4}} = \frac{3}{2}$.

$\min\limits_{x\in[-1,2]} f(x) = 0$

$\therefore f(x) \in [0, \frac{3}{2}]$

(d).



Question 5.

(a). Let $A := \tan^{-1}(\tan 2)$

$\Rightarrow \tan A = \tan 2$, By $A \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\therefore A = 2-\pi$.

(b) Let $A = \sin^{-1}\frac{1}{3}$, $B = \sin^{-1}\frac{2}{3} \Rightarrow \sin A = \frac{1}{3}$, $\sin B = \frac{2}{3}$.

$\sin(\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3}) = \sin(A+B) = \sin A\cos B + \cos A \sin B$

$= \frac{1}{3}\cdot\frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\cdot\frac{2}{3} = \frac{\sqrt{5}+4\sqrt{2}}{9}$.

(c). $3-4\cos^2\theta = 0 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

If $\cos\theta = \frac{\sqrt{3}}{2}$, then $\begin{cases} \theta = \frac{\pi}{6} + 2k\pi, & k \in \mathbb{Z} \\ \theta = -\frac{\pi}{6} + 2k\pi, & k \in \mathbb{Z} \end{cases}$

If $\cos\theta = -\frac{\sqrt{3}}{2}$, then $\begin{cases} \theta = \frac{7\pi}{6} + 2k\pi. & k \in \mathbb{Z} \\ \theta = -\frac{7\pi}{6} + 2k\pi. & k \in \mathbb{Z} \end{cases}$

Question 4.

(a). $f(1)=1-5=-4<0,\quad f(2)=8-5=3>0$

Thus, $\exists\ f(x_0)=0,\quad x_0 \in (1,2)$.

(b). $x^4-x^3-3x^2+17x-30 := f(x)$

Then $f(2)=0$. So $(x-2)$ is a factor of $f(x)=x^4-x^3-3x^2+17x-$

(c). (i) $$f(x)=\frac{x^4+3x^2-x-8}{x^3+4x}=\frac{x^2(x^2+4)-x^2-x-8}{x(x^2+4)}$$

$$=x-\frac{x^2+x+8}{x(x^2+4)}.$$

Note: $$\frac{x^2+x+8}{x(x^2+4)} := \frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$=\frac{ax^2+4a+bx^2+cx}{x(x^2+4)}$$

$\therefore\ a+b=1,\quad c=1,\quad a=2 \Rightarrow a=2,\ b=-1,\ c=1$

$$\therefore f(x)=x-\frac{2}{x}-\frac{1-x}{x^2+4}.$$

(ii) $$f'(x)=1+\frac{2}{x^2}+\frac{1}{x^2+4}+\frac{2x(1-x)}{(x^2+4)^2}.$$

Question 5.

(a). $f(x) = \dfrac{x^2-1}{2x^2-x-1} = \dfrac{(x-1)(x+1)}{(x-1)(2x+1)} = \dfrac{x+1}{2x+1}$

(i) $\lim\limits_{x\to 0} f(x) = \dfrac{1}{1} = 1$

(ii) $\lim\limits_{x\to 1} f(x) = \dfrac{1+1}{2+1} = \dfrac{2}{3}$.

(iii). $\lim\limits_{x\to \infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{2+\frac{1}{x}} = \dfrac{1}{2}$.

(b). $\lim\limits_{x\to 4} \dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{\frac{1}{2}(1+2x)^{-\frac{1}{2}}\cdot 2}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim\limits_{x\to 4} \dfrac{2\sqrt{x}}{\sqrt{1+2x}} =$

(c). (i). $\lim\limits_{x\to 0} \dfrac{x(e^x+1)-2(e^x-1)}{x^3} = \lim\limits_{x\to 0} \dfrac{e^x+1+xe^x-2e^x}{3x^2}$

$= \lim\limits_{x\to 0} \dfrac{e^x+xe^x-e^x}{6x} = \lim\limits_{x\to 0} \dfrac{e^x}{6} = \dfrac{1}{6}$.

(ii) let $F(x) = x^{\frac{k}{1+\ln x}} \Rightarrow \ln F(x) = k\dfrac{\ln x}{1+\ln x}$

$\therefore \lim\limits_{x\to 0^+} \ln F(x) = k \Rightarrow \lim\limits_{x\to 0^+} F(x) = e^k$.

Question 6.

(a). $(\cos 2x - 2\sin x)' = -2\sin 2x - 2\cos x.$

(b). Let $F(x) = x^{\frac{1}{x}} \Rightarrow \ln F(x) = \frac{1}{x}\ln x$

$$\Rightarrow \frac{F'(x)}{F(x)} = -\frac{1}{x^2}\ln x + \frac{1}{x^2}$$

$$\therefore F'(x) = \left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right)\cdot F(x) = \left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right)\cdot x^{\frac{1}{x}}.$$

(c) $$\left(\ln\sqrt{\frac{1-\sin x}{1+\sin x}}\right)' = \sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\frac{-\cos x(1+\sin x)-(1-\sin x)\cos x}{(1+\sin x)^2}$$

$$= \frac{1}{2}\,\frac{1+\sin x}{1-\sin x}\cdot\frac{-2\cos x}{(1+\sin x)^2}$$

$$= \frac{1}{2}\,\frac{-2\cos x}{\cos^2 x} = -\frac{1}{\cos x} = -\sec x.$$

(d). $$\left(x\sqrt{\frac{1-x}{1+x}}\right)' = \sqrt{\frac{1-x}{1+x}} + \frac{1}{2}x\sqrt{\frac{1+x}{1-x}}\cdot\frac{-(1+x)-(1-x)}{(1+x)^2}$$

$$= \sqrt{\frac{1-x}{1+x}} - x\sqrt{\frac{1+x}{1-x}}\cdot\frac{1}{(1+x)^2}$$

$$= \sqrt{\frac{1-x}{1+x}}\left(1 - x\,\frac{1+x}{1-x}\,\frac{1}{(1+x)^2}\right)$$

$$= \sqrt{\frac{1-x}{1+x}}\left(1 - x\,\frac{1}{1-x^2}\right)$$

$$= \sqrt{\frac{1-x}{1+x}}\cdot\frac{1-x^2-x}{1-x^2}.$$

Question 7.

(a). (i). $1+3\cdot 1=4$.

(ii). $\vec{\tau}=(3y^2+3,\ 1)$

At. (4,1). $\vec{\tau}=(3+3,1)=(6,1)$.

let $y=\frac{1}{6}x+b \Rightarrow 1=\frac{4}{6}+b \Rightarrow b=1-\frac{2}{3}=\frac{1}{3}$

$\therefore\ y=\frac{1}{6}x+\frac{1}{3}$

b). $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{3-3t^2}{2-2t}=\frac{3(1-t^2)}{2(1-t)}=\frac{3(1+t)}{2}$

$\lim_{t\to 0}\frac{3(1+t)}{2}=\frac{3}{2}$.

At $t=0$, $x=0$. $y=0$

$\therefore\ y=\frac{3}{2}x$.

Question 8.

let $ab = S$.

let $F(a) = a+b = a + \frac{S}{a}$

$\therefore F'(a) = 1 - \frac{S}{a^2}$. Let $F'(a) = 0 \Rightarrow a = \sqrt{S}$.

Thus. $a = \sqrt{S}$. $b = \frac{S}{\sqrt{S}} = \sqrt{S}$.

$\therefore 2(a+b) = 4\sqrt{S}$.

Question 4.

(a). let $F(x) = \sin^{-1} x$ $\therefore$ $\sin F(x) = x \Rightarrow \cos F(x) \cdot F'(x) = 1$

$$\therefore F'(x) = \frac{1}{\cos F(x)} = \frac{1}{\cos(\sin^{-1} x)}$$

$$f'(x) = \left(\cos(m \sin^{-1} x)\right)' = -\sin(m \sin^{-1} x) \cdot \frac{m}{\cos(\sin^{-1} x)}.$$

$$f''(x) = -\cos(m \sin^{-1} x)\left(\frac{m}{\cos(\sin^{-1} x)}\right)^2 - \sin(m \sin^{-1} x) \cdot \frac{m \sin(\sin^{-1} x)}{\cos^3(\sin^{-1} x)}$$

$$= -\frac{1}{\cos^2(\sin^{-1} x)}\left(m^2 \cos(m \sin^{-1} x) + \sin(m \sin^{-1} x) \frac{m x}{\cos(\sin^{-1} x)}\right)$$

By $\cos^2(\sin^{-1} x) = 1 - x^2$, one has (a).

(b). By (a) $\Rightarrow$ $((1-x^2) f''(x))^{(n)} - (x f'(x))^{(n)} + m^2 f^{(n)}(x) = 0$. By the Leibniz rule,

$$(1-x^2) f^{(n+2)}(x) + n(-2x) f^{(n+1)}(x) + \frac{n(n-1)}{2}(-2) f^{(n)}(x) - x f^{(n+1)}(x) - n f^{(n)}(x) + m^2 f^{(n)}(x) = 0$$

$$\Rightarrow (1-x^2) f^{(n+2)}(x) - (2n+1) x f^{(n+1)}(x) + (m^2 - n^2) f^{(n)}(x) = 0.$$

(c). $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

Note $f(0) = \cos 0 = 1$.

$f'(0) = 0$

$f''(0) = -m^2$

By (b). $f^{(3)}(0) = 0$

$f^{(4)}(0) = -(m^2 - 4) f''(0) = m^2(m^2 - 4)$

$$\therefore f(x) = 1 - \frac{m^2}{2} x^2 + \frac{m^2(m^2-4)}{24} x^4 + \cdots$$