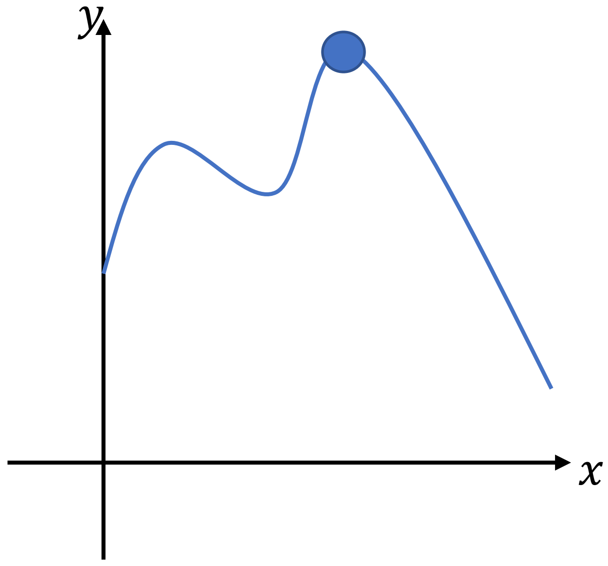
MA1200 Calculus and Basic Linear Algebra I

Lecture Note 7

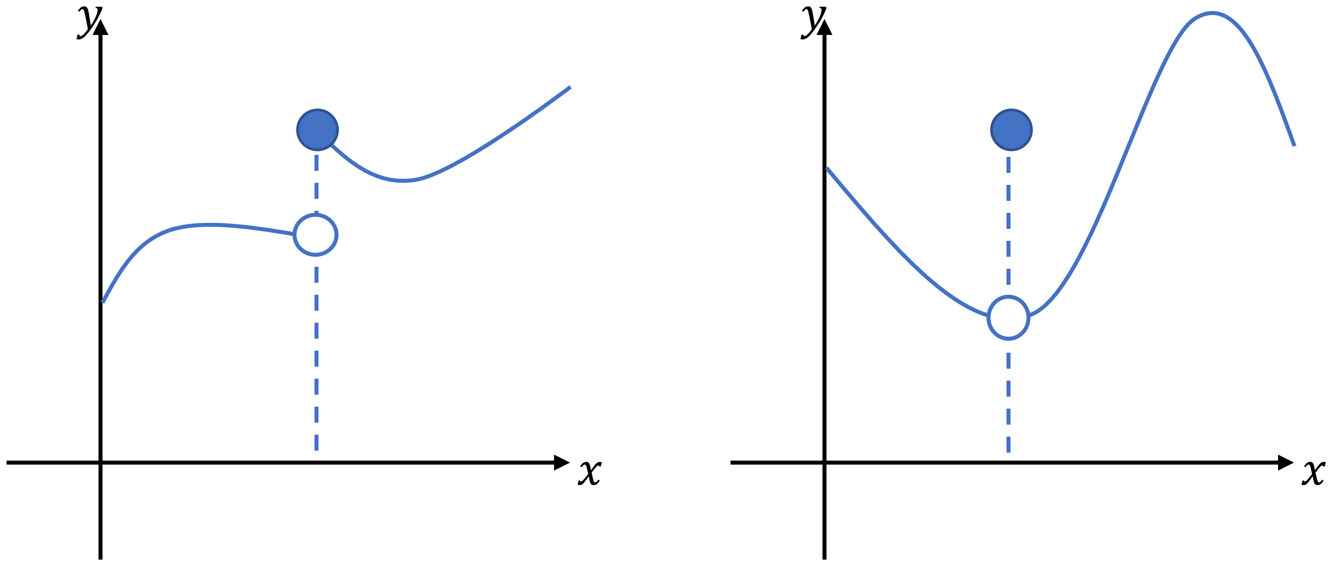
Continuity of functions

What is continuous function?

Roughly speaking, a continuous function $f(x)$ is a function which the graph $y = f(x)$ is continuous (no jumps, no breaks).



(continuous function)



(not continuous function)

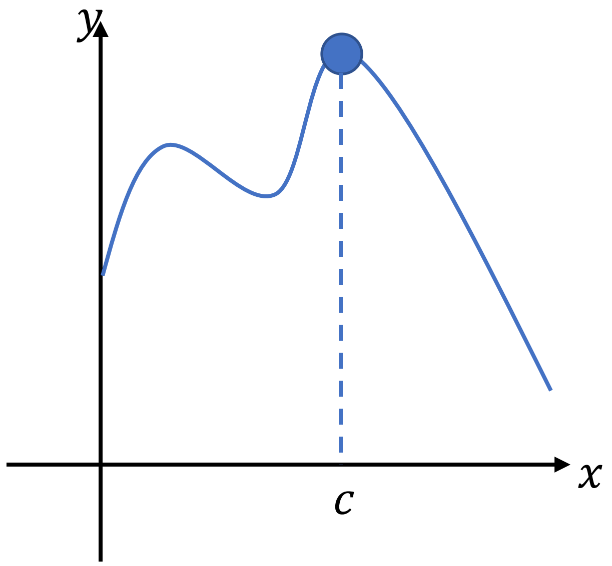
Mathematical definition of continuity

Definition (Continuity of function $f(x)$)

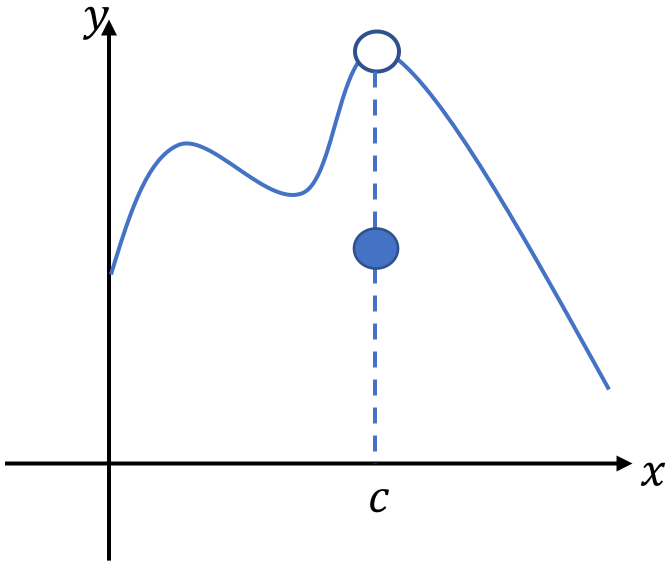
We say a function is <u>continuous at $x = c$</u> if both $\lim_{x\to c} f(x)$ and $f(c)$ exist and

$$\lim_{x\to c} f(x) = f(c).$$

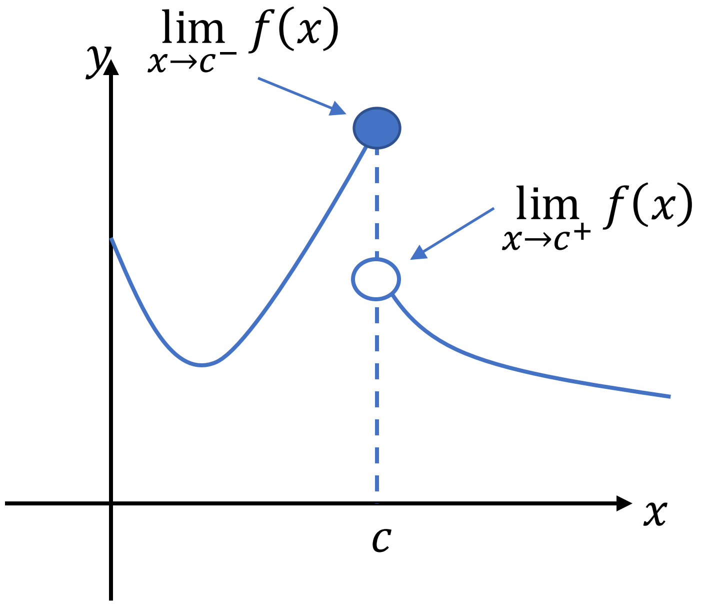
Furthermore, we say a function is <u>continuous</u> on its domain if it is continuous at every point of its domain.



If the condition "$\lim_{x \to c} f(x) = f(c)$" does not satisfy, we say the function is not continuous at $x = c$. For example



$$\lim_{x \to c} f(x) \neq f(c)$$



$$\lim_{x \to c} f(x) \ \textit{does not exist}$$
$$\left(\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)\right)$$

Notes on continuity

- Most of the elementary functions such as $y = x^3$, $y = e^x$, $y = \cos x$, $y = \sqrt{x}$, $y = |x|$ are all continuous on its domain.
- To check the continuity of a function at $x = c$, we may follow the following procedure:

  Step 1: Compute $f(c)$

  Step 2: Compute $\lim_{x \to c} f(x)$

  (Note: If necessary, one needs to consider the left-hand limit and right-hand limit when computing $\lim_{x \to c} f(x)$)

  Step 3: Compare the limits with $f(c)$.

Example 1

Consider the function

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x} & if\ x \neq 0 \\ 2 & if\ x = 0 \end{cases}.$$

Is $f(x)$ continuous at $x = 0$?

☺ Solution:

Step 1: First, note that $f(0) = 2$ by definition.

Step 2: $\lim_{x\to 0} f(x) \overset{x\neq 0}{\widehat{=}} \lim_{x\to 0} \dfrac{\sin 3x}{x} = \lim_{x\to 0} 3\left(\dfrac{\sin 3x}{3x}\right) \overset{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}{\widehat{=}} 3 \times 1 = 3.$

Step 3: $\lim_{x\to 0} f(x) = 3 \neq 2 = f(0).$

Therefore, we conclude that $f(x)$ is not continuous at $x = 0$.

Example 2

Consider the function

$$f(x) = \begin{cases} 2x+1 & if\ x<1 \\ 3x^2 & if\ x \geq 1 \end{cases}.$$

Determine whether the function is continuous at $x = 1$.

☺ Solution:

Step 1: First, note that $f(1) = 3(1)^2 = 3$.

Step 2: Note that

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2x+1) = 3, \qquad \lim_{x\to1^+} f(x) = \lim_{x\to1^+} 3x^2 = 3.$$

So the limits $\lim_{x\to1} f(x)$ exists and $\lim_{x\to1} f(x) = 3$.

Step 3: $\lim_{x\to1} f(x) = 3 = f(1)$.

Therefore, we conclude that $f(x)$ is continuous at $x = 1$.

Example 3

Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 - 3x - 10}{x - 5} & if\ x \neq 5 \\ a & if\ x = 5 \end{cases}$$

where a is real number.

(a) If $a = 4$, is $f(x)$ continuous at $x = 5$?

(b) What is the value of a so that $f(x)$ is continuous at $x = 5$?

☺ Solution:

(a) Step 1: By definition, we get $f(5) = a = 4$.

Step 2: For $x \neq 5$, we have $f(x) = \frac{x^2-3x-10}{x-5}$.

$$\lim_{x\to5} f(x) = \lim_{x\to5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x\to5} \frac{(x-5)(x+2)}{x-5} = \lim_{x\to5}(x+2) = 7.$$

Step 3: $\lim\limits_{x\to5} f(x) = 7 \neq 4 = f(5)$.

Hence, the function is not continuous at $x = 5$ in this case.

(b) If $f(x)$ is continuous at $x = 5$, then we must have

$$\lim_{x\to5} f(x) = f(5).$$

Using the result in (a), we obtain

$$\underbrace{a}_{f(5)} = \underbrace{7}_{\lim\limits_{x\to5} f(x)}.$$

Some properties of continuous functions

Theorem 1 (Basic algebraic operation of continuous functions)

If $f(x)$ and $g(x)$ be continuous at $x = c$, then the function

$$kf(x), \qquad f(x) + g(x), \qquad f(x) - g(x), \qquad f(x)g(x), \qquad \frac{f(x)}{g(x)}(if\ g(c) \neq 0), \qquad |f(x)|$$

are all continuous at $x = c$.

Theorem 2 (Composition of continuous functions)

If $f(x)$ is continuous at c and $g(x)$ is continuous at $f(c)$, then the composition $(g \circ f)(x)$ is also continuous at $x = c$

$$\lim_{x \to c} (g \circ f)(x) = \lim_{x \to c} g\big(f(x)\big) = g\left(\lim_{x \to c} f(x)\right) = g\big(f(c)\big).$$

(*Note: Theorem 2 is quite useful in computing limits)

Example 4

Let $f(x) = \cos x$ and $g(x) = e^x$ are continuous function over the real number. Using Theorem 1 and 2, we can conclude that the following functions

$$kf(x) = k\cos x\,, \qquad f(x) \pm g(x) = \cos x \pm e^x$$

$$f(x)g(x) = e^x \cos x\,, \qquad \frac{f(x)}{g(x)} = \frac{\cos x}{e^x}\,(e^x \neq 0)$$

$$|f(x)| = |\cos x|, \qquad (g \circ f)(x) = g\big(f(x)\big) = g(\cos x) = e^{\cos x}$$

are all continuous over real number.

Theorem 3

If $f(x)$ is a continuous function and $g(x)$ is a function (may not be continuous), then we have

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$$

provided that the limits $\lim_{x\to c} g(x)$ exists.

Example 5

Compute $\lim_{x\to\pi} \sin(x+\cos x)$ and $\lim_{x\to 1} \cos\frac{\sqrt{x+3}-2}{x-1}$.

☺ Solution:

<u>1st limit</u>

Note that $f(x) = \sin x$ is continuous, then

$$\lim_{x\to\pi} \sin(x+\cos x) = \sin\left(\lim_{x\to\pi}(x+\cos x)\right) = \sin(\pi-1) \approx 0.8415.$$

2nd limit

Note that $g(x) = \cos x$ is continuous, then

$$\lim_{x\to 1} \cos \frac{\sqrt{x+3}-2}{x-1} = \cos\left(\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}\right) \dots\dots (*)$$

Note that

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1} \frac{\overbrace{x+3-2^2}^{=x-1}}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2} = \frac{1}{4}.$$

From $(*)$, we conclude that

$$\lim_{x\to 1} \cos \frac{\sqrt{x+3}-2}{x-1} = \cos\left(\frac{1}{4}\right) = 0.9689.$$

Example 6

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function (may or may not continuous) such that $\lim_{x\to 0} \frac{f(x)}{x^3} = \frac{\pi}{2}$. Compute the limits

$(a) \lim_{x\to 0} f(x) \quad and \quad (b) \lim_{x\to 0} e^{\cos\left(\frac{f(x)}{x^2}\right)}$.

☺ Solution:

(a) Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x^3} x^3 = \frac{\pi}{2} \times 0 = 0.$

(b) Note that the function $e^{\cos x}$ is continuous (see Example 4) and

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{f(x)}{x^3} x = \frac{\pi}{2} \times 0 = 0.$$

So using Theorem 3, we have

$$\lim_{x\to 0} e^{\cos\left(\frac{f(x)}{x^2}\right)} = e^{\cos\left(\lim_{x\to 0} \frac{f(x)}{x^2}\right)} = e^{\cos 0} = e^1 = e.$$

Example 7

(a) Find the limits $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$.

(b) Hence, determine if the limits $\lim_{x\to 0} \frac{1+2^{1/x}}{3+2^{1/x}}$ exists.

☺ Solution:

(a) Using the graph of $y = \frac{1}{x}$, one can see that $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

(b) We consider the left-hand limits and right-hand limits.

- <u>Left-hand limits</u>

  When $x \to 0^-$, then $\frac{1}{x} \to -\infty$. So we have

$$\lim_{x\to 0^-} \frac{1+2^{1/x}}{3+2^{1/x}} = \frac{1+2^{\lim_{x\to 0^-}\frac{1}{x}}}{3+2^{\lim_{x\to 0^-}\frac{1}{x}}} = \frac{1+2^{-\infty}}{3+2^{-\infty}} \overset{2^{-\infty}=\frac{1}{2^{\infty}}\to 0}{\cong} \frac{1}{3}.$$

- <u>Right-hand limits</u>

  When $x \to 0^+$, then $\frac{1}{x} \to +\infty$, $2^{\frac{1}{x}} \to 2^{+\infty} = +\infty$ and hence $\frac{1}{2^{\frac{1}{x}}} = 0$.

  So we have

$$\lim_{x\to 0^+} \frac{1 + \overbrace{2^{1/x}}^{\to\infty}}{3 + \underbrace{2^{1/x}}_{\to\infty}} = \lim_{x\to 0^+} \frac{\frac{1}{2^{1/x}} + 1}{\frac{3}{2^{1/x}} + 1} = \frac{0+1}{0+1} = 1.$$

Since $\lim_{x\to 0^-} \frac{1+2^{1/x}}{3+2^{1/x}} \neq \lim_{x\to 0^+} \frac{1+2^{1/x}}{3+2^{1/x}}$, so we conclude that the limits

$$\lim_{x\to 0} \frac{1 + 2^{1/x}}{3 + 2^{1/x}}$$

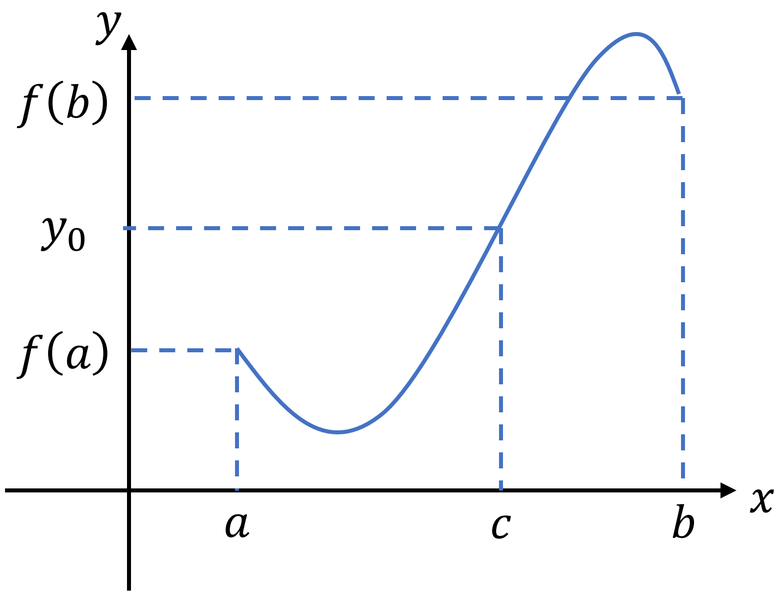
does not exist.

Important property of continuous function

Theorem 4 (Intermediate Value Theorem)

If $f(x)$ is a continuous function on an interval $[a, b]$ and y_0 is a real number between $f(a)$ and $f(b)$, then there is a number c $(a \leq c \leq b)$ such that

$$f(c) = y_0$$



The intermediate value theorem is useful in checking whether a given equation has solution. It provides a way to find the solution of the equation.

Example 8

Consider the equation $x^5 + 2x - 1 = 0$, show that there is a solution between 0 and 1.

☺ Solution:

One may rephrase the statement as

"There is a number $0 \leq z \leq 1$ such that $z^5 + 2z - 1 = 0$."

We let $f(x) = x^5 + 2x - 1$ and $f(x)$ is continuous. By simple calculation, we get $f(0) = -1 < 0$ and $f(1) = 2 > 0$.

By intermediate value theorem, there is z $(0 \leq z \leq 1)$ such that

$$f(z) = z^5 + 2z - 1 = 0.$$

Application of intermediate value theorem: Method of Bisection

- It is a root-finding technique by using intermediate value theorem repeatedly.
- In Example 8, we have shown that the solution lies between 0 and 1. The bisection method aims to obtain the solution by narrowing this range.

Step 1:

We pick the mid-point between 0 and 1. That is, $x = 0.5$. We compute the value of $f(0.5)$.

Since $f(0.5) = 0.03125 > 0$, then the solution lies between 0 and 0.5.

Step 2:

We pick the mid-point between 0 and 0.5. That is, $x = 0.25$. We compute the value of $f(0.25)$.

Since $f(0.25) = -0.4990 < 0$, then the solution lies between 0.25 and 0.5.

One can repeat this process and obtain the approximated solution of the equation:

Midpoint x	$f(x)$	Updated range of z
0.5	0.03125	$0 \leq z \leq 0.5$
0.25	-0.4990	$0.25 \leq z \leq 0.5$
0.375	-0.24258	$0.375 \leq z \leq 0.5$
0.4375	-0.10897	$0.4375 \leq z \leq 0.5$
0.46875	-0.03987	$0.46875 \leq z \leq 0.5$
0.484375	-0.00459	$0.484375 \leq z \leq 0.5$
0.492188	0.01326	$0.484375 \leq z \leq 0.492188$
0.488282	0.004319	$0.484375 \leq z \leq 0.488282$
0.486329	-0.00014	$0.486329 \leq z \leq 0.488282$
0.487306	0.00209	$0.486329 \leq z \leq 0.487306$
0.486818	0.000977	$0.486329 \leq z \leq 0.486818$
0.486574	0.000421	$0.486329 \leq z \leq 0.486574$
0.486452	0.000142	$0.486329 \leq z \leq 0.486452$

The approximated solution is $x \approx 0.486$.