MA1200 Calculus and Basic Linear Algebra

Final Review

Chapter 1: Conic Section

Things you need to know:

1. Four conic sections (Circle, Ellipse, Hyperbola and Parabola) and their equations.
    - See P.13, 22, 26, 27, 29 and P.34 of Note 1

2. Identify the conic section (including the details of conic section)
    - Using completing square techniques

      (for equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$).
    - Using rotation of axes techniques

      (for equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$).

Remember (Rotation of axes formula)

$$x = x' \cos\theta - y' \sin\theta$$

$$y = y' \cos\theta + x' \sin\theta$$

Example 1

It is given that the equation of conic section is $3x^2 - y^2 + 4y - 7 = 0$. Identify this conic section and sketch its graph.

☺ Solution:

Since there is no xy-term in the equation, one can use complete square technique to identify the conic section.

$3x^2 - y^2 + 4y - 7 = 0 \Rightarrow 3x^2 - (y^2 - 4y) = 7$

$\Rightarrow 3x^2 - (y-2)^2 = 3$

$$\Rightarrow \frac{x^2}{1^2} - \frac{(y-2)^2}{(\sqrt{3})^2} = 1.$$

The conic section is a hyperbola with centre $(0, 2)$.

Chapter 2: Functions

Things you need to know:

- Basic Concept of functions: Domain and Range.

  - Identify the domain and range of some simple functions, e.g. $\ln(1+x)$, $\sqrt{1-x^2}$ etc.

- Some special functions: Increasing function, Decreasing function, periodic function.

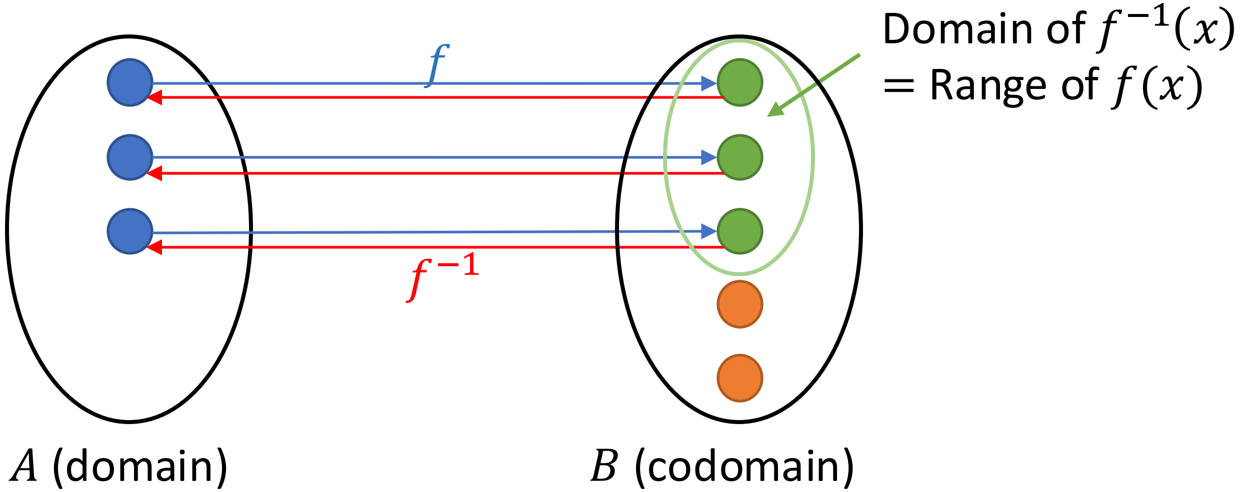
- Inverse function $f^{-1}(x)$

  - Existence of f^{-1}: One-to-one

    For and $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

  - Domain of $f^{-1}(x) =$ Range of $f(x)$

  - Range of $f^{-1}(x) =$ Domain of $f(x)$



Example 2

We consider the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=x+\frac{1}{x}$.

(a) Show that the value of $f(x)$ cannot be any number between 0 and 2 (excluding 2).

(b) Compute $\lim\limits_{x\to\infty} f(x)$ and $f(1)$.

(c) Using (a) and (b), find the range of $f(x)$.

(d) Show that the inverse of $f(x)$ does not exist by showing the function is not one-to-one.

☺ Solution:

(a) It suffices to show the equation $f(x)=x+\frac{1}{x}=a$ has no solution for any $a\in(0,2)$. Note that

$$x+\frac{1}{x}=a\Rightarrow x^2+1=ax\Rightarrow x^2-ax+1=0.$$

Since the discriminant $\Delta=(-a)^2-4(1)(1)=a^2-4<0$, so the equation has no real solution.

(b) It is easy to see that $f(1) = 1 + \frac{1}{1} = 2$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x + \frac{1}{x} = \infty + 0 = \infty.$$

(c) Note that $x + \frac{1}{x}$ is positive for any $x \in (0, \infty)$ and $x + \frac{1}{x}$ is continuous, then we can conclude from its graph that

$$\text{Range of } f(x) = [2, \infty).$$

(d) To show $f(x)$ is not one-to-one, we pick $x = \frac{1}{3}$ and $x = 3$, then we see

$$f\left(\frac{1}{3}\right) = \frac{1}{3} + 3 \;\; and \;\; f(3) = 3 + \frac{1}{3}.$$

Since $f\left(\frac{1}{3}\right) = f(3)$, thus the function is not one-to-one and the inverse does not exist.

Example 3

We consider the function $f:(0,\infty)\to\mathbb{R}$ defined by

$$f(x)=e^{1-\frac{1}{x}}.$$

(a) Calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} e^{1-\frac{1}{x}}$.

(b) Show that $f(x)$ is strictly increasing.

(c) Hence, show that the inverse of $f(x)$ exists and find $f^{-1}(x)$.

(d) Find the range and domain of $f^{-1}(x)$. (Hint: Consider the graph of $y=f(x)$)

☺ Solution:

(a) Note that $\frac{1}{x}\to\infty$ as $x\to 0^+$ and $\frac{1}{x}\to 0$ as $x\to\infty$, we have

$$\lim_{x\to 0^+} f(x)=e^{1-\infty}=e^{-\infty}=0 \quad and \quad \lim_{x\to\infty} e^{1-\frac{1}{x}}=e^1=e.$$

(b) One can show this by considering its derivative

$$f'(x) = \frac{d\left(e^{1-\frac{1}{x}}\right)}{d\left(1-\frac{1}{x}\right)} \frac{d\left(1-\frac{1}{x}\right)}{dx} \overset{\frac{1}{x}=x^{-1}}{\cong} \frac{1}{x^2} e^{1-\frac{1}{x}}.$$

Since $e^{1-\frac{1}{x}} > 0$ and $\frac{1}{x^2} > 0$, thus $f'(x) > 0$ and $f(x)$ is strictly increasing.

(c) Note that the function is strictly increasing, thus for any $x_1 \neq x_2$, we must have $f(x_1) \neq f(x_2)$.

So $f(x)$ is one-to-one and its inverse exists.

To find the inverse, we set

$$y = e^{1-\frac{1}{x}} \Rightarrow \ln y = 1 - \frac{1}{x} \Rightarrow x = \frac{1}{1-\ln y} \Rightarrow f^{-1}(x) = \frac{1}{1-\ln x}.$$

(d) Using the graph of $y = f(x)$ (you can sketch it using (a) and (b))

Domain of $f^{-1}(x)$ = Range of $f(x) = [0, e)$

Range of $f^{-1}(x)$ = Domain of $f(x) = (0, \infty)$.

Chapter 3: Polynomial and Rational Function

Things you need to know:

1. Factorize the polynomial using factor theorem and cross multiplication (for quadratic function).

2. How to decompose a rational function using method of partial fraction
   - Step 1: Factorize the denominator
   - Step 2: Choose the decomposition based on the factorization of denominator
     - ✓ Distinct linear factor
     - ✓ Repeated factor
     - ✓ Quadratic factor
     - ✓ Improper rational function
   - Step 3: Find the values of the unknown in your decomposition (using method of substitution or comparing coefficients)

Example 4

(a) Factorize the polynomial $P(x) = 2x^4 - 6x^3 + 5x^2 - 3x + 2$ into the product of two linear factor and one quadratic factor.

(b) Decompose the rational function

$$f(x) = \frac{5x^3 - 10x^2 + 7x - 5}{2x^4 - 6x^3 + 5x^2 - 3x + 2}.$$

using method of partial fraction.

☺ Solution:

(a) One should identify some linear factors by trial and error. Find x_0 such that $P(x_0) = 0$. To do this, one can first consider the integer values:

	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$P(x)$	108	18	2	0	0

Immediately, we see $(x - 1)$ and $(x - 2)$ are (linear) factor of $P(x)$.

By long division, we get

$$2x^4 - 6x^3 + 5x^2 - 3x + 2 = (x - 1)(x - 2)(2x^2 + 1).$$

(b) Note that there is a quadratic factor and other factors are linear and distinct, we should try the following decomposition:

$$\frac{5x^3-10x^2+7x-5}{(x-1)(x-2)(2x^2+1)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{Cx+D}{2x^2+1}.$$

Taking common factor, we get

$5x^3-10x^2+7x-5=A(x-2)(2x^2+1)+B(x-1)(2x^2+1)+(Cx+D)\,(x-1)(x-2).$

To find the constants, we use method of substitution:

- Put $x=1$, we get $-3=A(-1)(3)\Rightarrow A=1.$
- Put $x=2$, we get $9=B(1)(9)\Rightarrow B=1.$
- Put $x=0$, we get $-5=-2A-B+2D\Rightarrow D=-1.$
- Put $x=-1$, we get $-27=-9A-6B+(-C+D)(6)\Rightarrow C=1.$

$$\frac{5x^3-10x^2+7x-5}{(x-1)(x-2)(2x^2+1)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{x-1}{2x^2+1}.$$

Example 5

Decompose the function

$$f(x) = \frac{x+2}{(x+1)^2(x-2)^2}$$

using method of partial fraction.

☺ Solution:

Since there exists repeated factor, we shall consider the following decomposition:

$$\frac{x+2}{(x+1)^2(x-2)^2} = \underbrace{\frac{A}{x+1} + \frac{B}{(x+1)^2}}_{\substack{repeated\, factors \\ (x+1)^2}} + \underbrace{\frac{C}{x-2} + \frac{D}{(x-2)^2}}_{\substack{repeated\, factors \\ (x-2)^2}}$$

To determine the unknown, we observe that

$$x + 2 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2$$

- Put $x = -1$, we get $1 = 9B \Rightarrow B = \frac{1}{9}$.
- Put $x = 2$, we get $4 = D(9) \Rightarrow D = \frac{4}{9}$.
- Put $x = 0$, we get $2 = 4A + 4B - 2C + D \Rightarrow 2A - C = \frac{5}{9}$.
- Put $x = 1$, we get $3 = 2A + B - 4C + 4D \Rightarrow A - 2C = \frac{5}{9}$.

Solving the last two equations, we get

$$A = \frac{5}{27} \quad and \quad C = -\frac{5}{27}.$$

Therefore, we obtain

$$\frac{x+2}{(x+1)^2(x-2)^2} = \frac{5}{27(x+1)} + \frac{1}{9(x+1)^2} - \frac{5}{27(x-2)} + \frac{4}{9(x-2)^2}.$$

Chapter 4: Trigonometric Function

Things you need to know:

- Six trigonometric functions and some simple identities and properties:

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta\,.$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\,, \qquad \cos(\pi - \theta) = -\cos\theta \qquad etc.$$

- Compound angle formulae, Sum-to-product formula and product-to-sum formula.

  (See P.18, P.26 and P.44 of Lecture Note 4)

- Inverse functions $\sin^{-1}x\,, \cos^{-1}x\;\; and\;\; \tan^{-1}x$. What are the domain and range of these functions?

- (Important) General Solution of trigonometric equations
  - $\cos\theta = k \Rightarrow \theta = 360°n \pm \cos^{-1} k$.
  - $\sin\theta = k \Rightarrow \theta = 180°n + (-1)^n \sin^{-1} k$.
  - $\tan\theta = k \Rightarrow \theta = 180°n + \tan^{-1} k$.

  where n is positive integer.

Example 6

Solve the equation

$$\tan x + 1 = \sec^2 x.$$

☺ Solution:

$\tan x + 1 = \sec^2 x \Rightarrow \tan x + 1 = 1 + \tan^2 x \Rightarrow \tan^2 x = \tan x$

$\Rightarrow \tan x\,(\tan x - 1) = 0 \Rightarrow \tan x = 0 \;\; or \;\; \tan x = 1.$

$\Rightarrow x = n\pi + \tan^{-1} 0 = n\pi \;\; or \;\; x = n\pi + \tan^{-1} 1 = n\pi + \frac{\pi}{4}, \qquad n \in \mathbb{Z}.$

Example 7

Solve the equation

$$\frac{1+\sin x}{\cos x}+\frac{\cos x}{1+\sin x}=4.$$

☺ Solution:

By taking the common factor, we get

$$\frac{(1+\sin x)^2+\cos^2 x}{\cos x\,(1+\sin x)}=4 \Rightarrow \frac{1+2\sin x+\overbrace{\sin^2 x+\cos^2 x}^{=1}}{\cos x\,(1+\sin x)}=4$$

$$\Rightarrow \overbrace{2+2\sin x}^{2(1+\sin x)}=4\cos x\,(1+\sin x) \Rightarrow (1+\sin x)(2-4\cos x)=0$$

$$\Rightarrow (1+\sin x)=0 \;\; or \;\; (2-4\cos x)=0$$

$$\Rightarrow \cos x=\frac{1}{2} \Rightarrow x=360°n \pm \cos^{-1}\frac{1}{2} \Rightarrow 360°n \pm 60°.$$

$$\sin x=-1 \Rightarrow x=180°n+(-1)^n\sin^{-1}(-1) \Rightarrow 180°n+(-1)^n(-90°).$$

Example 8

Find the largest possible domain of the function

$$f(\theta) = \frac{1}{\sin^3\theta + 3\cos^2\theta - 1}.$$

☺ Solution:

The function is not defined when

$\sin^3\theta + 3\cos^2\theta - 1 = 0 \Rightarrow \sin^3\theta + 3(1 - \sin^2\theta) - 1 = 0$

$\Rightarrow \sin^3\theta - 3\sin^2\theta + 2 = 0$

$\Rightarrow (\sin\theta - 1)(\sin^2\theta - 2\sin\theta - 2) = 0$

$\Rightarrow \sin\theta = 1 \;\; or \;\; \sin\theta = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2} = 1 + \sqrt{3}\,(rej.)\; or\; 1 - \sqrt{3}.$

$\Rightarrow \theta = n\pi + (-1)^n \dfrac{\pi}{2} \;\; or \;\; \theta = n\pi + (-1)^n \sin^{-1}\left(1 - \sqrt{3}\right).$

Hence, the domain is $\mathbb{R} \backslash \left\{n\pi + (-1)^n \frac{\pi}{2},\; n \in \mathbb{Z}\right\} \cup \left\{n\pi + (-1)^n \sin^{-1}\left(1 - \sqrt{3}\right),\; n \in \mathbb{Z}\right\}$.

Chapter 6, 7 and 8: Limits, Continuity and Differentiability

Things you need to know:

1. How to check whether a given limits exists $\lim_{x\to a} f(x)$?

   - Compute $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$

     If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, then the limits exists.

     If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then the limits does not exist.

2. Computation of limits via various techniques

   - Algebraic Trick (See Lecture Note 6) or
   - L' Hopital Rule (See Lecture Note 10)

     (Unless otherwise specified, you may use either one of these methods in computing limits)

3. How to check the continuity and differentiability of a function at $x = c$.

- Continuity

  We need to check whether $\lim_{x \to c} f(x) = f(c)$.

- Differentiability

  We need to use method of first principle and compute

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (REMEMBER\ THIS!!!)$$

(Note: As mentioned in lecture, the use of L'Hopital rule may lead to some logic problems. If possible, avoid to use this unless you have no other choices)

Example 9

Compute the limits

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-4x}.$$

☺ Solution:

Method 1: Use classical method

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-4x}=\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-4x}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)=\lim_{x\to 4}\frac{x-4}{x(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{x(\sqrt{x}+2)}=\frac{1}{4\left(\sqrt{4}+2\right)}=\frac{1}{16}.$$

Method 2: Use L' Hopital Rule (Since the limits is $\frac{0}{0}$ type)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-4x}\overset{\sqrt{x}=x^{\frac{1}{2}}}{\underset{}{\stackrel{\text{LH}}{=}}}\lim_{x\to 4}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{2x-4}=\frac{\frac{1}{2}(4)^{-\frac{1}{2}}}{2(4)-4}=\frac{1}{16}.$$

Example 10

Compute the limits

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1}.$$

☺ Solution:

Method 1: Use classical method

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \frac{1}{\sqrt{2}}}{\sin x - \frac{1}{\sqrt{2}}} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \cos\frac{\pi}{4}}{\sin x - \sin\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{-2\sin\frac{x+\frac{\pi}{4}}{2}\sin\frac{x-\frac{\pi}{4}}{2}}{2\sin\frac{x-\frac{\pi}{4}}{2}\cos\frac{x+\frac{\pi}{4}}{2}} = -\lim_{x\to\frac{\pi}{4}} \frac{\sin\frac{x+\frac{\pi}{4}}{2}}{\cos\frac{x+\frac{\pi}{4}}{2}}$$

$$= -\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = -1.$$

Method 2: Use L' Hopital Rule

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1} = \lim_{x\to\frac{\pi}{4}} \frac{-\sqrt{2}\sin x}{\sqrt{2}\cos x} = -\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = -1.$$

Example 11

Compute the limits

$$\lim_{x\to 0}(\tan x)\left(\sin\frac{1}{x^2}\right).$$

☺ Solution:

Note that as $x \to 0$, the limits $\lim_{x\to 0}\sin\frac{1}{x^2} = \sin\infty$ does not exist. Thus one has to *use sandwich theorem* to eliminate this term $\sin\frac{1}{x^2}$.

(☺ Tips: This techniques can be applied when you see the terms such as $\sin\left(\frac{1}{x}\right)$, $\cos\left(\frac{1}{x}\right)$ ... etc. which their limits do not exist as $x \to 0$.)

Using the fact that $-1 \le \sin\theta \le 1$, we get

$$-\tan x \le (\tan x)\left(\sin\frac{1}{x^2}\right) \le \tan x$$

$$\Rightarrow -0 = -\lim_{x\to 0}\tan x \le \lim_{x\to 0}(\tan x)\left(\sin\frac{1}{x^2}\right) \le \lim_{x\to 0}\tan x = 0 \Rightarrow \lim_{x\to 0}(\tan x)\left(\sin\frac{1}{x^2}\right) = 0.$$

Example 12

Compute the limits

$$\lim_{x\to\infty} \frac{x^3 - x^2 \sin x}{4x^3 + 1}.$$

☺ Solution:

$$\lim_{x\to\infty} \frac{x^3 - x^2 \sin x}{4x^3 + 1} = \lim_{x\to\infty} \frac{\frac{x^3 - x^2 \sin x}{x^3}}{\frac{4x^3 + 1}{x^3}} = \lim_{x\to\infty} \frac{1 - \frac{\sin x}{x}}{4 + \frac{1}{x^3}} = \frac{1-0}{4+0} = \frac{1}{4}.$$

☺ Remark:

Here, you may not able to use L'Hopital Rule since the limits $\lim_{x\to\infty} \sin x = \sin \infty$ does not exist.

Example 13

Compute the limits

$$\lim_{x\to\infty}\left(1-\frac{3}{x^3}\right)^{2x^3-3x}.$$

☺ Solution:

Method 1: Classical method

$$\lim_{x\to\infty}\left(1-\frac{3}{x^3}\right)^{2x^3-3x} = \lim_{x\to\infty}\left[\left(1+\frac{-3}{x^3}\right)^{x^3}\right]^{\frac{2x^3-3x}{x^3}} \overset{\substack{\lim\limits_{y\to\infty}\left(1+\frac{a}{y}\right)^y=e^a \\ \lim\limits_{x\to\infty}\frac{2x^3-3x}{x^3}=\lim\limits_{x\to\infty}\frac{2-\frac{3}{x^2}}{1}=2}}{=} (e^{-3})^2 = e^{-6}.$$

Method 2: L' Hopital Rule

Note that the limits is of the type 1^∞.

Let $y = \left(1-\frac{3}{x^3}\right)^{2x^3-3x} \Rightarrow \ln y = \underbrace{(2x^3-3x)}_{\to\infty}\underbrace{\ln\left(1-\frac{3}{x^3}\right)}_{\to\ln 1=0}$

Then

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}(2x^3-3x)\ln\left(1-\frac{3}{x^3}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x^3}\right)}{\frac{1}{2x^3-3x}} \quad \left(\frac{0}{0}\ type\right)$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1-\frac{3}{x^3}}\left(\frac{9}{x^4}\right)}{-\frac{6x^2-3}{(2x^3-3x)^2}} = \lim_{x\to\infty}\frac{-9(2x^3-3x)^2}{(x^4-3x)(6x^2-3)} = -9\lim_{x\to\infty}\frac{\frac{(2x^3-3x)^2}{x^6}}{\frac{(x^4-3x)(6x^2-3)}{x^6}}$$

$$= -9\lim_{x\to\infty}\frac{\left(2-\frac{3}{x^2}\right)^2}{\left(1-\frac{3}{x^3}\right)\left(6-\frac{3}{x^2}\right)} = -9\frac{4}{1\times 6} = -6.$$

Thus $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{-6}$.

Example 14 (Continuity Problem)

We consider the following functions

$$f(x)=\begin{cases}\dfrac{\ln x}{x^2-1} & if\ x>1\\ x+1 & if\ x\le 1\end{cases}.$$

Determine whether the function $f(x)$ is continuous at $x=1$.

☺Solution:

We need to consider whether $\lim_{x\to 1} f(x)=f(1)=1+1=2$.

It remains to compute $\lim_{x\to 1} f(x)$. Note that

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{\ln x}{x^2-1}\overset{\to\frac{\ln 1}{0}=\frac{0}{0}}{\overset{\text{L}}{=}}\lim_{x\to 1^+}\frac{\frac{1}{x}}{2x}=\lim_{x\to 1^+}\frac{1}{2x^2}=\frac{1}{2(1)^2}=\frac{1}{2}.$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+1)=2.$$

Since $\lim_{x\to 1^+} f(x)\neq\lim_{x\to 1^-} f(x)$, thus the limits $\lim_{x\to 1^+} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x=1$.

Example 15 (Continuity Problem 2)

Let $f(x)$ be a differentiable function with $f(0) = 0$ and $f'(0) = 2$. We consider another function defined by

$$F(x) = \begin{cases} \dfrac{f(x) - x}{f(2x)} & if\ x \neq 0 \\ \dfrac{1}{4} & if\ x = 0 \end{cases}.$$

Determine whether the function $F(x)$ is continuous at $x = 0$.

☺ Solution:

Again, we need to check whether $\lim_{x\to 0} F(x) = F(0) = \frac{1}{4}$. Note that

$$\lim_{x\to 0} F(x) \overset{x\neq 0}{\cong} \lim_{x\to 0} \frac{f(x) - x}{f(2x)} \overset{\to \frac{f(0)-0}{f(0)} = \frac{0}{0}}{\cong} \lim_{x\to 0} \frac{f'(x) - 1}{2f'(2x)} = \frac{f'(0) - 1}{2f'(0)} = \frac{2 - 1}{2(2)} = \frac{1}{4}.$$

Since $\lim_{x\to 0} F(x) = F(0) = \frac{1}{4}$, thus $F(x)$ is continuous at $x = 0$.

Example 16 (Differentiability Problem)

Using the method of the first principle, determine whether the function $f(x) = \sqrt{2x+1}$ is differentiable. What is the corresponding derivative?

☺ Solution:

We consider the limits

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h}\left(\frac{\sqrt{2x+2h+1}+\sqrt{2x+1}}{\sqrt{2x+2h+1}+\sqrt{2x+1}}\right)$$

$$= \lim_{h\to 0}\frac{\overbrace{2x+2h+1-(2x+1)}^{2h}}{h\left(\sqrt{2x+2h+1}+\sqrt{2x+1}\right)} = \lim_{h\to 0}\frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

Thus $f(x)$ is differentiable and $f'(x) = \frac{1}{\sqrt{2x+1}}$.

Example 17 (Differentiability Problem 2)

Using the method of first principle, find the derivative of $f(x) = \ln x$.

(Hint: You can consider the following series for $\ln(1+y)$):

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \cdots.$$

☺ Solution:

Using the method of first principle, we consider

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \cdots}{h} = \lim_{h\to 0} \left(\frac{1}{x} \overbrace{- \frac{h}{2x^2} + \frac{h^2}{3x^3} - \cdots}^{=0}\right) = \frac{1}{x}.$$

☺ Exercise: You can try this for $f(x) = e^x$. (Hint: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$).

Example 18 (Differentiability Problem 3)

Let $f(x)$ be a (twice) differentiable function, we define the following new function:

$$F(x) = \begin{cases} \dfrac{xf(x)}{\sin x} & if\ x \neq 0 \\ f(0) & if\ x = 0 \end{cases}.$$

Determine whether $F(x)$ is differentiable at $x = 0$.

☺ Solution: Again, we use method of first principle and consider

$$\lim_{h\to 0} \frac{F(0+h) - F(0)}{h} \overset{h\neq 0}{\widehat{=}} \lim_{h\to 0} \frac{\frac{hf(h)}{\sin h} - f(0)}{h} = \lim_{h\to 0} \frac{hf(h) - f(0)\sin h}{h\sin h} \quad \left(\frac{0}{0}\ type\right)$$

$$= \lim_{h\to 0} \frac{f(h) + hf'(h) - f(0)\cos h}{\sin h + h\cos h} \quad \left(\frac{0}{0}\ type\right)$$

$$= \lim_{h\to 0} \frac{f'(h) + f'(h) + hf''(h) + f(0)\sin h}{\cos h + \cos h - h\sin h} = \frac{2f'(0)}{2} = f'(0).$$

Hence, $F(x)$ is differentiable at $x = 0$ and $F'(0) = f'(0)$.

Study Tips:

- Try to practice how to evaluate the limits using either algebraic trick or L'Hopital Rule. We will provide you some instructions if we ask you to compute the limits in certain way.

- ALWAYS WRITE DOWN THE DEFINITION when doing continuity or differentiability problem. Try to memorize them CAREFULLY. You may write down the formal definition in the question paper after the start of exam (before you attempt the problem).

- You may need to memorize some identities (say trigonometric identities, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, $\lim_{y\to\infty}\left(1+\frac{a}{y}\right)^y = e^a$. etc.) If you need them.

  (P.S. Perhaps, most of you may prefer to memorize the L'Hopital Rule which is "INEVITABLE").

Chapter 8, 9: Differentiation

Things you need to know:

- How to compute the derivative of a given function using various rules
  - Product Rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
  - Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$.
  - Chain Rule: $\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx}$

  (Remark: The derivatives of some functions are provided in final)

- How to compute the derivative (or higher-order derivative) using various tricks
  - Implicit Differentiation: Find $\frac{dy}{dx}$ for $x^2 - y^3 + 2xy = 0$
  - Inverse Differentiation: $\frac{d}{dx} f^{-1}(x)$
  - Logarithmic Differentiation: $\frac{d}{dx} x^x$
  - Differentiation of parametric equations: $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ for $x = t^2$ and $y = \cos t$.
  - Leibniz's Rule: $\frac{d^n}{dx^n} f(x)g(x) = \sum_{r=0}^{n} C_r^n f^{(r)}(x) g^{(n-r)}(x)$.

Example 19

Compute the following derivatives

$$(a)\ \frac{d}{dx}\cos 2x \ln(x^2+1)$$

$$(b)\ \frac{d}{dx}\tan\left(e^{1+\frac{1}{x}}\right)$$

$$(c)\ \frac{d}{dx}\cos^{-1}\left(\frac{1}{(2x+3)^2}\right) \qquad (WARNING\colon \cos^{-1} y \neq (\cos y)^{-1}!!!)$$

$$(d)\ \frac{d}{dx}x^{\cosh(x^2)}.$$

☺ Solution:

$$(a)\ \frac{d}{dx}\cos 2x \ln(x^2+1) = \ln(x^2+1)\frac{d}{dx}\cos 2x + \cos 2x\frac{d}{dx}\ln(x^2+1)$$

$$= \ln(x^2+1)\frac{d(\cos 2x)}{d(2x)}\frac{d(2x)}{dx} + \cos 2x\frac{d(\ln(x^2+1))}{d(x^2+1)}\frac{d(x^2+1)}{dx}$$

$$= \ln(x^2+1)\,(-\sin 2x)(2) + \cos 2x \,\frac{1}{x^2+1}\,(2x)$$

$$= -2\sin 2x \ln(x^2+1) + \frac{2x}{x^2+1}\cos 2x\,.$$

$$(b)\quad \frac{d}{dx}\tan\left(e^{1+\frac{1}{x}}\right) = \frac{d\tan\left(e^{1+\frac{1}{x}}\right)}{d\left(e^{1+\frac{1}{x}}\right)}\frac{de^{1+\frac{1}{x}}}{d\left(1+\frac{1}{x}\right)}\frac{d\left(1+\frac{1}{x}\right)}{dx}$$

$$\overset{\frac{1}{x}=x^{-1}}{\triangleq} \quad \sec^2\left(e^{1+\frac{1}{x}}\right) e^{1+\frac{1}{x}}(-x^{-2}) = -x^{-2}e^{1+\frac{1}{x}}\sec^2\left(e^{1+\frac{1}{x}}\right).$$

$$(c)\quad \frac{d}{dx}\cos^{-1}\left(\frac{1}{(2x+3)^2}\right) = \frac{d(\cos^{-1}[(2x+3)^{-2}])}{d\left((2x+3)^{-2}\right)}\frac{d[(2x+3)^{-2}]}{d(2x+3)}\frac{d(2x+3)}{dx}$$

$$= \frac{1}{\sqrt{1-[(2x+3)^{-2}]^2}}[-2(2x+3)^{-3}](2) = \frac{-4(2x+3)^{-3}}{\sqrt{1-(2x+3)^{-4}}}.$$

(d) Let $y = x^{\cosh(x^2)}$, then $\ln y = \cosh(x^2) \ln x$.

Differentiating both sides of the equation with respect to x, we have

$$\frac{d}{dx} \ln y = \frac{d}{dx} \cosh(x^2) \ln x$$

$$\Rightarrow \frac{d \ln y}{dy} \frac{dy}{dx} = \ln x \frac{d}{dx} \cosh(x^2) + \cosh(x^2) \frac{d}{dx} \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x \left[\frac{d \cosh(x^2)}{d(x^2)} \frac{d(x^2)}{dx} \right] + \cosh(x^2) \left(\frac{1}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \overbrace{x^{\cosh(x^2)}}^{y} \left[\ln x \,(\sinh x^2)(2x) + \frac{\cosh(x^2)}{x} \right].$$

Note:

$$\frac{d}{dy} \ln y = \frac{1}{y} \quad and \quad \frac{d}{dy} \cosh y = \sinh y$$

Example 20

The equation of a graph is $x^2 + 2x\cos y + 3y = 3$.

(a) Find the equation of tangent to the graph at $(x, y) = (1, 0)$.

(b) Find the equation of normal to the graph at $(x, y) = (1, 0)$.

☺ Solution:

We differentiate the equation with respect to x and obtain

$$2x + 2\cos y - 2x\sin y\frac{dy}{dx} + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x - 2\cos y}{-2x\sin y + 3}$$

Then the slope of tangent is given by $\frac{dy}{dx}|_{(x,y)=(1,0)} = \frac{-4}{3}$ and the equation of tangent is given by

$$\frac{y-0}{x-1} = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}x + \frac{4}{3}.$$

On the other hand, (slope of tangent)×(slope of normal) $= -1 \Rightarrow$ slope of normal $= \frac{3}{4}$.

The equation of the normal is given by

$$\frac{y-0}{x-1} = \frac{3}{4} \Rightarrow y = \frac{3}{4}x - \frac{3}{4}$$

Example 21

Compute the following derivatives

$$\frac{d^3}{dx^3}\frac{1}{(2x-3)^3}.$$

☺ Solution:

We write $\frac{1}{(2x-3)^3} = (2x-3)^{-3}$. Then we have

$$\frac{d}{dx}(2x-3)^{-3} = \frac{d(2x-3)^{-3}}{d(2x-3)}\frac{d(2x-3)}{dx} = -3(2x-3)^{-4}(2) = -6(2x-3)^{-4}.$$

$$\frac{d^2}{dx^2}(2x-3)^{-3} = -6\frac{d}{dx}(2x-3)^{-4} = -6\frac{d(2x-3)^{-4}}{d(2x-3)}\frac{d(2x-3)}{dx} = 48(2x-3)^{-5}.$$

$$\frac{d^3}{dx^3}(2x-3)^{-3} = 48\frac{d}{dx}(2x-3)^{-5} = 48\frac{d(2x-3)^{-5}}{d(2x-3)}\frac{d(2x-3)}{dx} = -480(2x-3)^{-6}.$$

Example 22

Let $x(t) = 3t^2 + \frac{1}{t}$ and $y(t) = t - \frac{1}{t^2}$, compute the derivative

$$\frac{dy}{dx} \quad and \quad \frac{d^2y}{dx^2}.$$

☺ Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(t - \frac{1}{t^2}\right)}{\frac{d}{dt}\left(3t^2 + \frac{1}{t}\right)} = \frac{1 + 2t^{-3}}{6t - t^{-2}} = \frac{1 + \frac{2}{t^3}}{6t - \frac{1}{t^2}} = \frac{t^3 + 2}{6t^4 - t}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\overbrace{\frac{t^3 + 2}{6t^4 - t}}^{z(t)} = \frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{t^3 + 2}{6t^4 - t}\right)}{\frac{d}{dt}\left(3t^2 + \frac{1}{t}\right)}$$

$$= \frac{\frac{(6t^4 - t)\frac{d}{dt}(t^3 + 2) - (t^3 + 2)\frac{d}{dt}(6t^4 - t)}{(6t^4 - t)^2}}{6t - \frac{1}{t^2}} = \frac{(6t^4 - t)(3t^2) - (t^3 + 2)(24t^3 - 1)}{\left(6t - \frac{1}{t^2}\right)(6t^4 - t)^2}.$$

Example 23

Using Leibniz's Rule to compute

$$\frac{d^4}{dx^4}\sqrt{x}\cos(1+4x)\,.$$

☺ Solution:

Using the Leibniz's Rule with $f(x)=\sqrt{x}$ and $g(x)=\cos(1+4x)$, we have

$$\frac{d^4}{dx^4}\sqrt{x}\cos(1+4x)=\frac{d^4}{dx^4}f(x)g(x)=\sum_{r=0}^{4}C_r^4 f^{(r)}(x)g^{(4-r)}(x)$$

$$=\overbrace{C_0^4}^{1}\overbrace{f^{(0)}(x)}^{=x^{\frac{1}{2}}}\overbrace{g^{(4)}(x)}^{256\cos(1+4x)}+\overbrace{C_1^4}^{4}\overbrace{f^{(1)}(x)}^{=\frac{1}{2}x^{-\frac{1}{2}}}\overbrace{g^{(3)}(x)}^{64\sin(1+4x)}+\overbrace{C_2^4}^{6}\overbrace{f^{(2)}(x)}^{-\frac{1}{4}x^{-\frac{3}{2}}}\overbrace{g^{(2)}(x)}^{-16\cos(1+4x)}$$

$$+\overbrace{C_3^4}^{4}\overbrace{f^{(3)}(x)}^{\frac{3}{8}x^{-\frac{5}{2}}}\overbrace{g^{(1)}(x)}^{-4\sin(1+4x)}+\overbrace{C_4^4}^{1}\overbrace{f^{(4)}(x)}^{-\frac{15}{16}x^{-\frac{7}{2}}}\overbrace{g^{(0)}(x)}^{\cos(1+4x)}$$

$$=256x^{\frac{1}{2}}\cos(1+4x)+128x^{-\frac{1}{2}}\sin(1+4x)+24x^{-\frac{3}{2}}\cos(1+4x)-6x^{-\frac{5}{2}}\sin(1+4x)-\frac{15}{16}x^{-\frac{7}{2}}\cos(1+4x)\,.$$

Example 24

For any positive integer n, compute the derivative

$$\frac{d^n}{dx^n}(x^3 - 2x + 2)e^{4x}.$$

☺ Solution:

Using the Leibniz's Rule with $f(x) = x^3 - 2x + 2$ and $g(x) = e^{4x}$, we have

$$\frac{d^n}{dx^n}(x^3 - 2x + 2)e^{4x} = \frac{d^n}{dx^n} f(x)g(x) = \sum_{r=0}^{n} C_r^n f^{(r)}(x) g^{(n-r)}(x)$$

$$= C_0^n \overbrace{f^{(0)}(x)}^{x^3-2x+2} \overbrace{g^{(n)}(x)}^{4^n e^{4x}} + C_1^n \overbrace{f^{(1)}(x)}^{3x^2-2} \overbrace{g^{(n-1)}(x)}^{4^{n-1}e^{4x}} + C_2^n \overbrace{f^{(2)}(x)}^{6x} \overbrace{g^{(n-2)}(x)}^{4^{n-2}e^{4x}}$$

$$+C_3^n \overbrace{f^{(3)}(x)}^{6} \overbrace{g^{(n-3)}(x)}^{4^{n-3}e^{4x}} + \cancel{C_4^n \overbrace{f^{(4)}(x)}^{0} g^{(n-4)}(x) + \cdots}$$

$$\cancel{+C_n^n \overbrace{f^{(n)}(x)}^{0} g^{(0)}(x)}$$

The binomial coefficient can be computed as (Recall that $C_r^n = \frac{n!}{r!(n-r)!}$)

$$C_0^n = 1, \qquad C_1^n = n, \qquad C_2^n = \frac{n(n-1)}{2}, \qquad C_3^n = \frac{n(n-1)(n-2)}{6}.$$

Combining the result, we finally get

$$\frac{d^n}{dx^n}(x^3 - 2x + 2)e^{4x}$$

$$= 4^n(x^3 - 2x + 2)e^{4x} + n4^{n-1}(3x^2 - 2)e^{4x} + 3n(n-1)4^{n-2}xe^{4x} + n(n-1)(n-2)4^{n-3}e^{4x}.$$

Example 25

Let a be a real number and define $y = \left(\frac{x+1}{x-1}\right)^r$ for $x > 1$.

(a) Show that $(x^2 - 1)\frac{dy}{dx} + 2ry = 0$.

(b) Using Leibniz's Rule and the result of (a), show that

$$(x^2 - 1)y^{(n+1)} + 2(nx + r)y^{(n)} + (n^2 - n)y^{(n-1)} = 0,$$

where $y^{(k)} = \frac{d^k y}{dx^k}$.

Chapter 10: Application of Derivatives

Things you need to know:

- Taylor series and Maclaurin Series of a function

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

(Maclaurin Series)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(Taylor Series)

- Application of Taylor series in computing limits (though we will provide you hints if we ask you to use this approach).

- Finding the maximum value and minimum value of a function
  - ✓ Find all local extrema: Local maximum and Local Minimum

    (First derivative test or second derivative test)
  - ✓ Compute the value of $f(x)$ at the boundary points

- Find the tangent and normal to the graph at certain point $X = (a, b)$
  - ✓ Tangent: Straight line which touches the graph at the point X.

$$Slope\ of\ tangent = \left.\frac{dy}{dx}\right|_{(x,y)=(a,b)}$$

  - ✓ Normal: Straight line which passes through X and perpendicular to the tangent.

- L'Hopital Rule and its application.

  (*Note: You have seen some related examples on the "limits" sections).

Example 26

Find the Maclaurin series of the function $f(x) = \ln(1+e^x)$ up to the terms x^3.

☺ Solution:

Note that the Maclaurin series of $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \ldots (*)$$

Since $f(0) = \ln(1+e^0) = \ln 2$,

$$f'(0) = \left.\frac{e^x}{1+e^x}\right|_{x=0} = \frac{1}{2}, \qquad f''(0) = \left.\frac{e^x}{1+e^x} - \frac{e^{2x}}{(1+e^x)^2}\right|_{x=0} = \frac{1}{4}.$$

$$f^{(3)}(0) = \left.\frac{e^x}{1+e^x} - \frac{e^{2x}}{(1+e^x)^2} - \frac{2e^{2x}}{(1+e^x)^2} + \frac{2e^{3x}}{(1+e^x)^3}\right|_{x=0} = 0.$$

Hence, the Maclaurin series is then given by

$$f(x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \cdots$$

Example 27

Find the Taylor expansion of the function $g(x) = \cos\sqrt{x}$ at $x = 1$ up to $(x-1)^3$.

☺ Solution:

The Taylor Series of $g(x)$ at $x = 1$ is given by

$$g(x) = g(1) + g'(1)(x-1) + \frac{g^{(2)}(1)}{2!}(x-1)^2 + \frac{g^{(3)}(1)}{3!}(x-1)^3 + \cdots$$

Since $g(1) = \cos\sqrt{1} = \cos 1$ and

$$g'(1) = -\frac{1}{2}x^{-\frac{1}{2}}\sin\sqrt{x}\Big|_{x=1} = -\frac{1}{2}\sin 1\,,$$

$$g^{(2)}(1) = \frac{1}{4}x^{-\frac{3}{2}}\sin\sqrt{x} - \frac{1}{4}x^{-1}\cos\sqrt{x}\Big|_{x=1} = \frac{1}{4}\sin 1 - \frac{1}{4}\cos 1\,,$$

$$g^{(3)}(1) = -\frac{3}{8}x^{-\frac{5}{2}}\sin\sqrt{x} + \frac{1}{8}x^{-2}\cos\sqrt{x} + \frac{1}{4}x^{-2}\cos\sqrt{x} + \frac{1}{8}x^{-\frac{3}{2}}\sin\sqrt{x}\Big|_{x=1} = -\frac{1}{4}\sin 1 + \frac{3}{8}\cos 1\,.$$

Hence, the Taylor series is then given by

$$g(x) = \cos 1 - \frac{1}{2}(\sin 1)x + \left(\frac{1}{8}\sin 1 - \frac{1}{8}\cos 1\right)x^2 + \left(-\frac{1}{24}\sin 1 + \frac{1}{16}\cos 1\right)x^3.$$

Example 28

(a) Find the Maclaurin's expansion of the function $f(x) = \ln(1 + \sin x)$ up to the term in x^4.

(b) Hence, compute the limits

$$\lim_{x\to 0} \frac{\ln(1 + \sin(2x^2))}{x^2}$$

Example 29

Find the maximum and minimum of the function $f(x) = e^{2x-x^2}$ over the interval $[-2, 2]$.

☺ Solution:

Step 1: Find all local extrema (or turning point)

We need to solve $f'(x) = 0$

$$\Rightarrow (2-2x)e^{2x-x^2} = 0 \quad \overset{e^{2x-x^2}>0}{\Rightarrow} \quad 2 - 2x = 0 \Rightarrow x = 1$$

Step 2: Check whether the given local extrema is local maxima or local minima

Note that $f(x) = (2-2x)e^{2x-x^2}$

	$-2 \le x < 1$	$x = 1$	$1 < x \le 2$
$f'(x)$	> 0	0	< 0
$Graph$	$increasing$	$local\ max.$	$decreasing$

By direct substitution, we get $f(1) = e^{2-1} = e$.

Step 3: Compute the function values at the boundary points

By direct substitution again, we get

$$f(2) = e^{2(2)-2^2} = e^0 = 1, \qquad f(-2) = e^{2(-2)-(-2)^2} = e^{-8}.$$

Step 4: Find the maximum value and minimum value of the function

By comparing the $f(2)$, $f(1)$ and $f(-2)$, we get

- The maximum value of $f(x)$ is e (at $x = 1$).
- The minimum value of $f(x)$ is e^{-8} (at $x = -2$).

Example 30

Find the maximum and minimum value of the function

$$f(x) = x^{\frac{1}{x}}$$

over the interval $[1, \infty)$.

☺ Solution:

Step 1: Find all local extrema (or turning point)

We need to solve $f'(x) = 0$

$$\Rightarrow x^{\frac{1}{x}}\left(\frac{1 - \ln x}{x^2}\right) = 0 \overset{\substack{x^{\frac{1}{x}}>0 \\ x^2>0}}{\Rightarrow} 1 - \ln x = 0 \Rightarrow x = e.$$

Step 2: Check whether the given local extrema is local maxima or local minima

Note that $f(x) = x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right)$,

	$1 \leq x < e$	$x = e$	$e < x < \infty$
$f'(x)$	> 0	0	< 0
$Graph$	$increasing$	$local\ max.$	$decreasing$

By direct substitution, we get $f(e) = e^{\frac{1}{e}}$.

Step 3: Compute the function values at the boundary points

By direct substitution, we get $f(1) = 1^{\frac{1}{1}} = 1$. On the other hand, we need to consider $\lim_{x\to\infty} x^{\frac{1}{x}}$ (since we need to compute the value at $x = \infty$).

Let $y = x^{\frac{1}{x}}$ (∞^0-type), then $\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$ and

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty} - type\right) = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = 1.$$

Step 4: Find the maximum value and minimum value of the function

By comparing the $f(1)$, $f(e)$ and $f(\infty) = \lim_{x\to\infty} f(x)$, we get

- The maximum value of $f(x)$ is $e^{\frac{1}{e}}$ (at $x = e$).
- The minimum value of $f(x)$ is 1 (at $x = 1, \infty$).

Example 31

The graph of the curve $y = f(x) = \frac{x^2+ax+b}{(x+1)^2}$ has a turning point at $Q = \left(-3, \frac{3}{4}\right)$.

Find the values of a and b. Show that Q is the local minimum point of the curve.

☺ Solution:

Note that $Q = (x, y) = \left(-3, \frac{3}{4}\right)$ is the turning point, it must be that

$$f'(-3) = 0 \Rightarrow \frac{2x + a}{(x+1)^2} - \frac{2(x^2 + ax + b)}{(x+1)^3}\bigg|_{x=-3} = 0 \Rightarrow 2a - b = 3.$$

On the other hand, the graph of $y = f(x)$ passes through the point Q, we have

$$f(-3) = \frac{3}{4} \Rightarrow \frac{9 - 3a + b}{4} = \frac{3}{4} \Rightarrow 3a - b = 6.$$

Solve these two equations, we get $a = 3$ and $b = 3$.

To verify Q is the local minimum, one can consider the second derivative test, note that

$$f'(x) = \frac{2x + 3}{(x+1)^2} - \frac{2(x^2 + 3x + 3)}{(x+1)^3} = \frac{-x - 3}{(x+1)^3} = -(x+3)(x+1)^{-3}.$$

$$\Rightarrow f''(x) = -(x+1)^{-3} + 3(x+3)(x+1)^{-4}.$$

Since $f''(-3) = \frac{1}{8} + 3(0)\left(\frac{1}{16}\right) = \frac{1}{8} > 0$, Q is local minimum by second derivative test.

Harder Example

It is given that $x = \tan t$ and $y = (\sec t + \tan t)^k$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and k is positive integer. In this problem, we would like to compute the derivative

$$U_n = \frac{d^n y}{dx^n}, \qquad n \text{ is positive integer}$$

(a) Find the equation of tangent to the graph at $t = 0$ (i.e. $(x, y) = (0, 1)$)

(b) Show that

$$(1 + x^2)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - k^2 y = 0.$$

(c) Using Leibniz's Rule, show that

$$(1 + x^2)y^{(n+2)} + (2n + 1)xy^{(n+1)} + (n^2 - k^2)y^{(n)} = 0$$

where $y^{(n)} = \frac{d^n y}{dx^n}$.

(d) Hence, conclude that

$$(1 + x^2)\frac{d^2 U_n}{dx^2} + (2n + 1)x\frac{dU_n}{dx} + (n^2 - k^2)U_n = 0.$$

☺ Solution:

(a) Note that the slope of tangent is given by

$$\left.\frac{dy}{dx}\right|_{t=0}=\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0}=\left.\frac{k(\sec t+\tan t)^{k-1}[\sec t\tan t+\sec^2 t]}{\sec^2 t}\right|_{t=0}$$

$$=\left.\frac{k(\sec t+\tan t)^{k}}{\sec t}\right|_{t=0}=k.$$

Hence the equation of tangent is given by

$$\frac{y-1}{x-0}=k\Rightarrow y=kx+1$$

(b) Note that $\frac{dy}{dx}=\frac{k(\sec t+\tan t)^k}{\sec t}$, it remains to compute $\frac{d^2y}{dx^2}$. Note that

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d\left(\frac{k(\sec t+\tan t)^k}{\sec t}\right)}{dx}=\frac{\frac{d}{dt}\left(\frac{k(\sec t+\tan t)^k}{\sec t}\right)}{\frac{dx}{dt}}=\cdots$$

$$=\frac{k^2\sec t\,(\sec t+\tan t)^k-k\tan t\,(\sec t+\tan t)^k}{\sec^3 t}$$

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-k^2y$$

$$=\overbrace{(1+\tan^2 t)}^{=\sec^2 t}\left(\frac{k^2\sec t\,(\sec t+\tan t)^k-k\tan t\,(\sec t+\tan t)^k}{\sec^3 t}\right)$$

$$+\tan t\left(\frac{k(\sec t+\tan t)^k}{\sec t}\right)-k^2(\sec t+\tan t)^k=0$$

(c) We differentiate the equation in (b) with respect to x for n-times, we have

$$\frac{d^n}{dx^n}\left[(1+x^2)y^{(2)}\right]+\frac{d^n}{dx^n}xy^{(1)}-k^2\overbrace{\frac{d^n}{dx^n}y}^{=y^{(n)}}=0\ldots(*)$$

Using Leibniz's Rule, we have

$$\frac{d^n}{dx^n}\left[\overbrace{(1+x^2)}^{f(x)}\overbrace{y^{(2)}}^{g(x)}\right]=\sum_{r=0}^{n}C_r^n f^{(r)}(x)g^{(n-r)}(x)$$

$$= \overbrace{C_0^n}^{1} \overbrace{f^{(0)}}^{(1+x^2)} \overbrace{g^{(n)}}^{y^{(n+2)}} + \overbrace{C_1^n}^{n} \overbrace{f^{(1)}}^{2x} \overbrace{g^{(n-1)}}^{y^{(n+1)}} + \overbrace{C_2^n}^{\frac{n(n-1)}{2}} \overbrace{f^{(2)}}^{2} \overbrace{g^{(n-2)}}^{y^{(n)}} + \cancel{C_3^n \overbrace{f^{(3)}}^{=0} g^{(n-3)} + \cdots + C_n^n \overbrace{f^{(n)}}^{=0} g^{(0)}}$$

$$= (1+x^2)y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)}.$$

Similarly, we get

$$\frac{d^n}{dx^n} xy^{(1)} = xy^{(n+1)} + ny^{(n)}.$$

Substitute the result in $(*)$, we get

$$(1+x^2)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2 - k^2)y^{(n)} = 0.$$

(d) Note that $\frac{d^2U_n}{dx^2} = y^{(n+2)}$ and $\frac{dU}{dx} = y^{(n+1)}$, substitute them into (c), we get

$$(1+x^2)\frac{d^2U_n}{dx^2} + (2n+1)x\frac{dU_n}{dx} + (n^2 - k^2)U_n = 0.$$

Hence, one can obtain U_n by solving this differential equation.