ASSIGNMENT 2: ANSWERS

1. Answers

1: Factorize $4x^3 - 20x^2 - 24x$.

$$4x^3 - 20x^2 - 24x = 4x(x^2 - 5x - 6) = 4x(x-6)(x+1). \tag{1.1}$$

2. Express the following rational functions in partial fraction
(a).

$$\begin{aligned}\frac{x^3 - x^2 + 9x - 1}{x^4 - 1} &= \frac{x^3 - x^2 + 9x - 1}{(x-1)(x+1)(x^2+1)} := \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \\ &= \frac{A(x^3 + x^2 + x + 1) + B(x^3 - x^2 + x - 1) + Cx^3 + Dx^2 - x + D}{(x-1)(x+1)(x^2+1)}.\end{aligned} \tag{1.2}$$

Then

$$A + B + C = 1, \tag{1.3}$$
$$A - B + D = -1, \tag{1.4}$$
$$A + B - C = 9, \tag{1.5}$$
$$A - B - D = -1. \tag{1.6}$$

Then

$$A = 2, \quad B = 3, \quad C = -4, \quad D = 0. \tag{1.7}$$

Thus,

$$\frac{x^3 - x^2 + 9x - 1}{x^4 - 1} = \frac{2}{x-1} + \frac{3}{x+1} - \frac{4x}{x^2+1}. \tag{1.8}$$

(b).

$$\frac{11x - 10}{x^2 - 2x} = \frac{11x - 10}{x(x-2)} := \frac{A}{x} + \frac{B}{x-2} = \frac{(A+B)x - 2A}{x(x-2)}. \tag{1.9}$$

Then

$$A + B = 11, \quad -2A = -10, \tag{1.10}$$

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which yields that

$$A = 5, \quad B = 6. \tag{1.11}$$

Thus,

$$\frac{11x - 10}{x^2 - 2x} = \frac{5}{x} + \frac{6}{x-2}. \tag{1.12}$$

(c).

$$\begin{aligned}
&\frac{-x^5 - x^4 + 3x^3 + 5x^2 + 6x + 6}{x^4 + x^3} \\
=& -\frac{x^5 + x^4}{x^3(x+1)} + \frac{3x^3}{x^3(x+1)} + \frac{5x^2}{x^3(x+1)} + \frac{6}{x^3} \\
=& -x + \frac{3}{x+1} + \frac{5}{x(x+1)} + \frac{6}{x^3} \\
=& -x + \frac{3}{x+1} + 5\Big(\frac{1}{x} - \frac{1}{x+1}\Big) + \frac{6}{x^3} \\
=& -x + \frac{5}{x} - \frac{2}{x+1} + \frac{6}{x^3}.
\end{aligned} \tag{1.13}$$

3. Simplify

(a).

$$\begin{aligned}
&(\cot\theta + \csc\theta)(\cot\theta - \csc\theta) \\
=&\cot^2\theta - \csc^2\theta = \frac{\cos^2\theta}{\sin^2\theta} - \frac{1}{\sin^2\theta} = \frac{\cos^2\theta - 1}{\sin^2\theta} = \frac{-\sin^2\theta}{\sin^2\theta} = -1.
\end{aligned} \tag{1.14}$$

(b).

$$1 - \frac{\sin^2 x}{1+\cos x} = \frac{1 + \cos x - \sin^2 x}{1 + \cos x} = \frac{\cos^2 x + \cos x}{1 + \cos x} = \cos x. \tag{1.15}$$

4. It is given that $\sin A = -\frac{1}{2}$ with $-\frac{\pi}{2} < A < 0$, and that $\cos B = \frac{3}{5}$ with $0 < B < \frac{\pi}{2}$. Calculate the exact values of $\sin(A+B)$ and $\cos(A-B)$

Note

$$\sin(A+B) = \sin A\cos B + \cos A \sin B. \tag{1.16}$$

$$\cos(A-B) = \cos A\cos B + \sin A \sin B. \tag{1.17}$$

By $\sin A = -\frac{1}{2}$ with $-\frac{\pi}{2} < A < 0$, one has $\cos A > 0$ and

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}. \tag{1.18}$$

By $\cos B = \frac{3}{5}$ with $0 < B < \frac{\pi}{2}$, one has $\sin B > 0$ and

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}. \tag{1.19}$$

Thus,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B = -\frac{1}{2}\frac{3}{5} + \frac{\sqrt{3}}{2}\frac{4}{5} = \frac{4\sqrt{3}-3}{10}. \tag{1.20}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B = \frac{\sqrt{3}}{2}\frac{3}{5} - \frac{1}{2}\frac{4}{5} = \frac{3\sqrt{3}-4}{10}. \tag{1.21}$$

5. Calculate

(a). Let $t = \sin^{-1}(\sin(-\frac{5\pi}{4}))$, then $\sin t = -\sin\frac{5\pi}{4} = \frac{\sqrt{2}}{2}$. By $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, one has $t = \frac{\pi}{4}$. That is, $\sin^{-1}(\sin(-\frac{5\pi}{4})) = \frac{\pi}{4}$.

(b). Let $t = \cos^{-1}(\cos(-\frac{5\pi}{4}))$, then $\cos t = \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$. By $t \in (0, \pi)$, one has $t = \frac{3\pi}{4}$. That is, $\cos^{-1}(\cos(-\frac{5\pi}{4})) = \frac{3\pi}{4}$.

(c). Let $t = \tan^{-1}(\tan(-\frac{5\pi}{4}))$, then $\tan t = \tan\frac{5\pi}{4} = -1$. By $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, one has $t = -\frac{\pi}{4}$. That is, $\tan^{-1}(\tan(-\frac{5\pi}{4})) = -\frac{\pi}{4}$.

(d)

$$\sin^{-1}(\sin(2)) = \sin^{-1}\left(\sin(\pi - (\pi - 2))\right) = \sin^{-1}(\sin(\pi - 2)) = \pi - 2. \tag{1.22}$$

(e)

$$\cos^{-1}(\cos(2)) = 2. \tag{1.23}$$

(f) Let $t = \tan^{-1}(\tan(2)) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\tan t = \tan 2$, then $t = 2 + k\pi$, by $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $t = 2 - \pi$, thus

$$\tan^{-1}(\tan(2)) = 2 - \pi. \tag{1.24}$$

6. Does $\cos(\cos^{-1}(3))$ exist? If yes, find its value. If no, give your reason. How about $\tan(\tan^{-1}(3))$?

$\cos(\cos^{-1}(3))$ not exist, since $\cos\theta \in [-1, 1]$. Moreover, $\tan(\tan^{-1}(3)) = 3$.

7. Given $\sec t = -\frac{13}{5}$ and t in the second quadrant, find the other five trigonometric functions of t.

Since $\sec t = -\frac{13}{5}$ with $t \in (\frac{\pi}{2}, \pi)$, then one has

$$\cos t = \frac{1}{\sec t} = -\frac{5}{13}, \tag{1.25}$$

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - (\frac{5}{13})^2} = \frac{12}{13}, \tag{1.26}$$

$$\tan t = \frac{\sin t}{\cos t} = -\frac{12}{13}\frac{13}{5} = -\frac{12}{5}, \tag{1.27}$$

$$\cot t = = \frac{1}{\tan t} = -\frac{5}{12}, \tag{1.28}$$

$$\csc t = \frac{1}{\sin t} = \frac{13}{12}. \tag{1.29}$$

8. Express $\cos x + \sin x$ in the form of $A\cos(x - \alpha)$ with $0 < \alpha < \frac{\pi}{2}$.

$$\begin{aligned}\cos x+\sin x =&\sqrt{2}\Big(\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x\Big)=\sqrt{2}\Big(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\Big)\\ =&\sqrt{2}\cos(x-\frac{\pi}{4}).\end{aligned}\tag{1.30}$$

9. Find the general solution of $2\cos^2 4\theta = 1$.

$2\cos^2 4\theta = 1$ yields that $\cos^2 4\theta = \frac{1}{2}$, then $\cos 4\theta = \pm\frac{\sqrt{2}}{2}$. If $\cos 4\theta = \frac{\sqrt{2}}{2}$, then

$$4\theta = \frac{\pi}{4} + 2k\pi, \quad 4\theta = -\frac{\pi}{4} + 2k\pi. \tag{1.31}$$

If $\cos 4\theta = -\frac{\sqrt{2}}{2}$, then

$$4\theta = \frac{3\pi}{4} + 2k\pi, \quad 4\theta = -\frac{3\pi}{4} + 2k\pi. \tag{1.32}$$

Thus

$$\theta = \frac{\pi}{16} + \frac{k\pi}{2}, \quad \theta = -\frac{\pi}{16} + \frac{k\pi}{2}, \tag{1.33}$$

$$\theta = \frac{3\pi}{16} + \frac{k\pi}{2}, \quad \theta = -\frac{3\pi}{16} + \frac{k\pi}{2}, \quad k \in \mathbb{R} \tag{1.34}$$