MA1200 Calculus and Basic Linear Algebra

Lecture Note 6

Limits

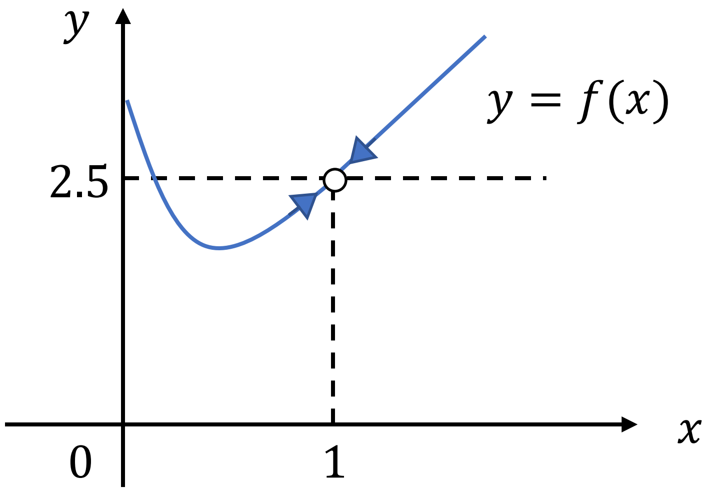
Motivation of Limit

We consider the following function

$$f(x) = \frac{x^3 + 2x - 3}{x^2 - 1}.$$

- The function $f(x)$ is not defined at $x = 1$ since the denominator equals to 0.
- Although we have no idea about the value of $f(1)$, we still wish investigate the value of $f(x)$ when x is closed to 1.

$\boldsymbol{x}$	$f(x)$	$\boldsymbol{x}$	$f(x)$
$\mathbf{0.99}$	2.493578	$\mathbf{1.01}$	2.502537
$\mathbf{0.999}$	2.49975	$\mathbf{1.001}$	2.50025
$\mathbf{0.9999}$	2.499975	$\mathbf{1.0001}$	2.500025
$\mathbf{0.99999}$	2.499998	$\mathbf{1.00001}$	2.500003



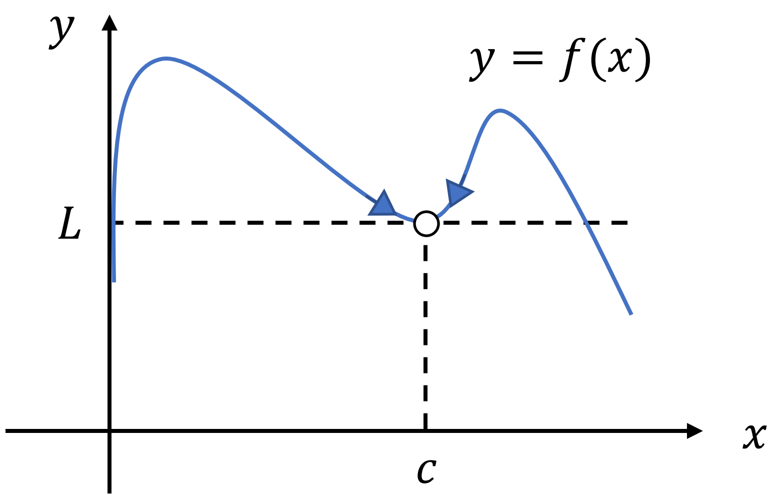
- We observe from the above table that $f(x)$ is closed to 2.5 when x is closed to 1. This leads to the concept called “limits”.

Definition of Limit

Intuitive definition of limit (informal one)

We say $f(x)$ has the limit L when x tends to c if the value of $f(x)$ gets close to L (as close as we like) if x is sufficiently close to c. Mathematically, we write

$$\lim_{x \to c} f(x) = L.$$

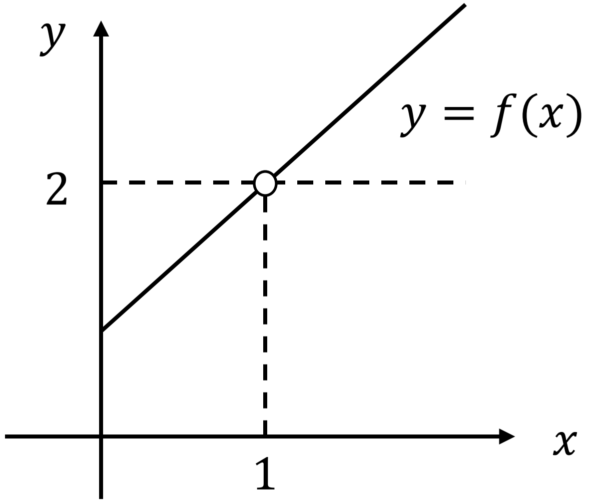


Example 1

Let $f(x) = \frac{x^2-1}{x-1}$, find the value of $\lim_{x\to 1} f(x)$ using the graph of $f(x)$.

☺ Solution:

The following figure shows the graph of $y = f(x)$. (Remark: $f(1)$ is NOT defined!)



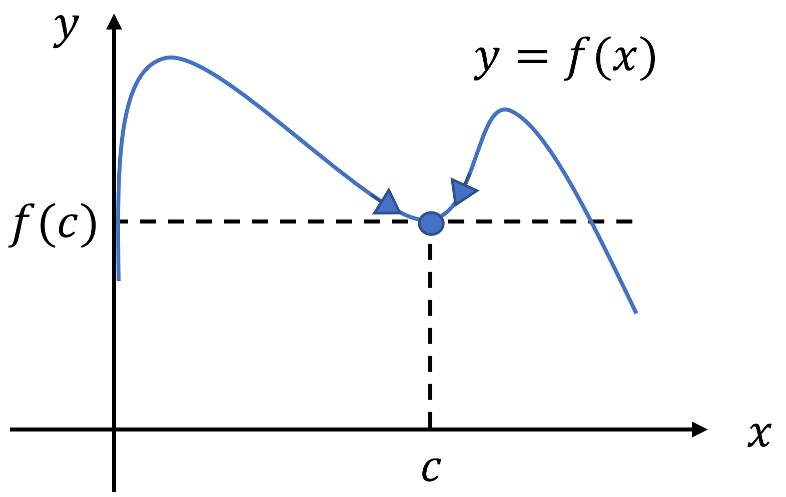
x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
1	???
1.0001	2.0001
1.001	2.001
1.01	2.01

From the graph and the table, we conjecture that $\lim_{x\to 1} f(x) = 2$.

Limit of continuous function (Useful Fact)

If a given function $f(x)$ is continuous[1] (no "break" and no "jump"), then

$$\lim_{x \to c} f(x) = f(c).$$



Example of continuous function:

$$y = c, \qquad y = x^n, \qquad y = e^x, \qquad y = \cos x\,, \qquad y = \sqrt{x}.$$

[1] We will give a precise definition of "continuous function" in later Chapter.

Example 2

Find the value of

$$\lim_{x\to 4}(5x-3)\,, \qquad \lim_{x\to 2}\cos(x^2)\,.$$

☺ Solution:

One can plot the graphs and observe that both functions are continuous.

So using the above fact, we get

$$\lim_{x\to 4}(5x-3) = 5(4)-3 = 17,$$

$$\lim_{x\to 2}\cos(x^2) = \cos(2^2) = \cos 4 \approx -0.6536.$$

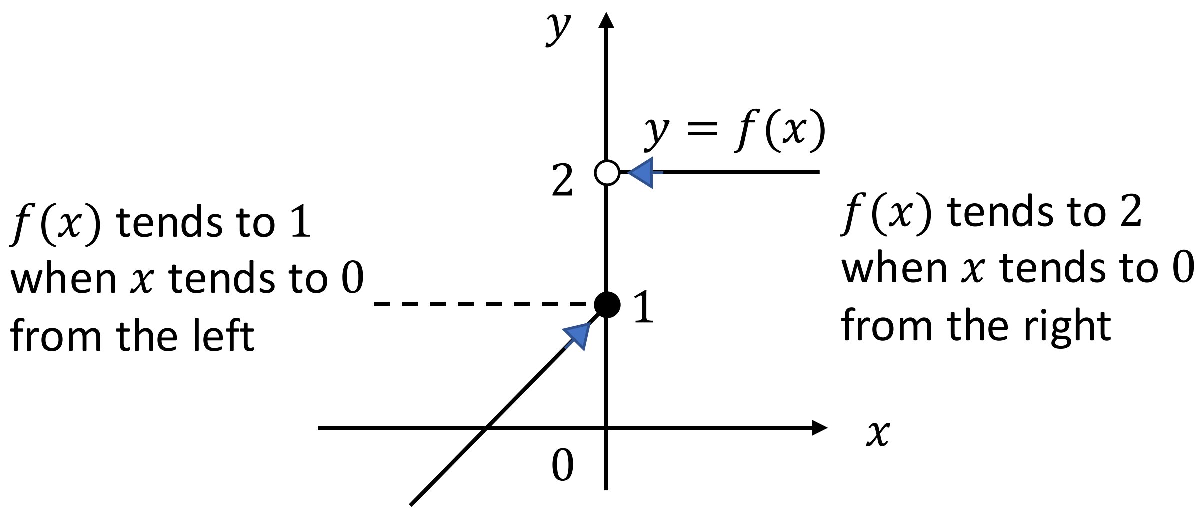
Example 3 (The limit may not exist)

We define the function $f(x)$ as

$$f(x) = \begin{cases} x+1 & if\ x \le 0 \\ 2 & if\ x > 0 \end{cases}.$$

What is $\lim_{x \to 0} f(x)$?

☺ Solution:

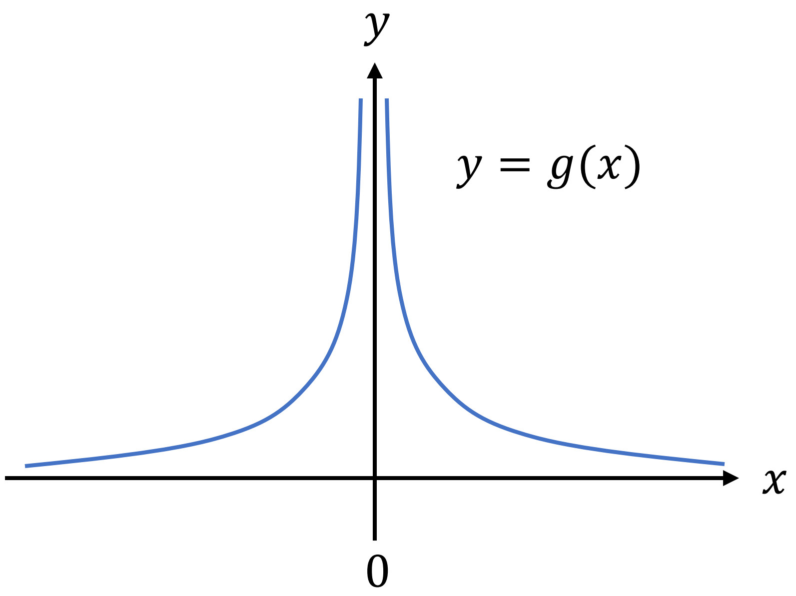


Hence, $\lim_{x \to 0} f(x)$ does not exist (since L cannot be found).

Example 4 (The limit is not real number)

We let $g(x) = \frac{1}{x^2}$ for $x \neq 0$, what is $\lim_{x \to 0} g(x)$?

☺ Solution:

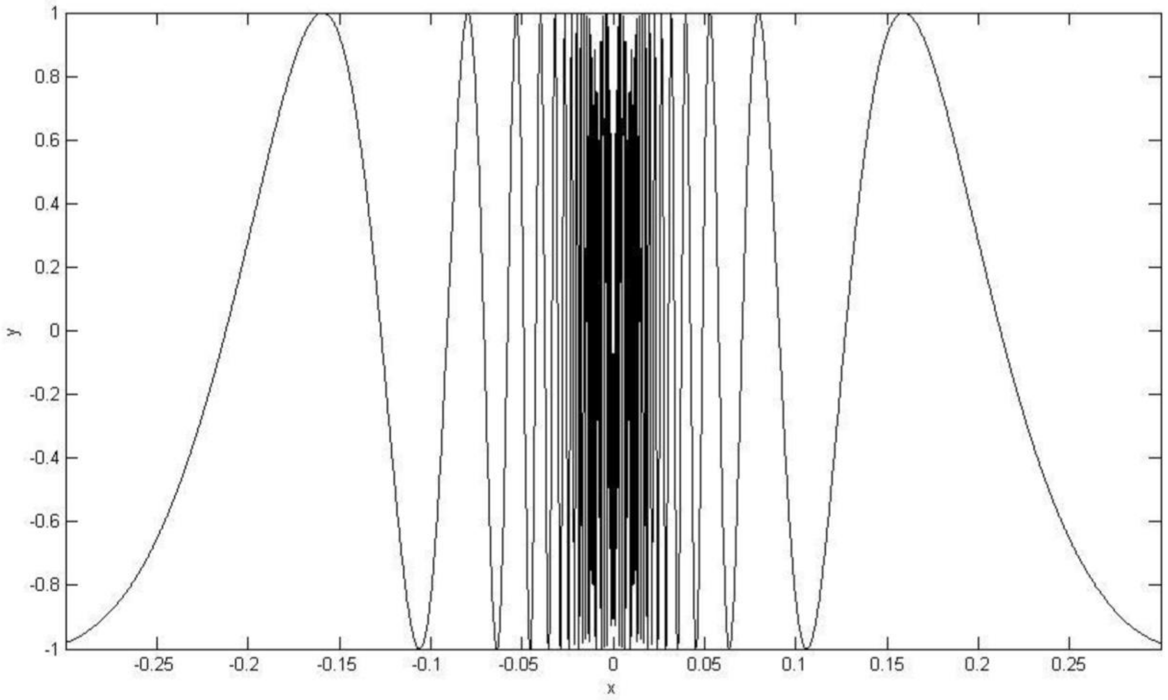


One can see that $g(x)$ tends to very large values when $x \to 0$. So $\lim_{x \to 0} g(x)$ cannot be real number and therefore does not exist (We may write $\lim_{x \to 0} g(x) = \infty$).

Example 5

Does the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ exist?

☺ Solution:



Since $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 when $x \to 0$, thus $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Some insights about limits

- The computation of $\lim_{x \to c} f(x)$ does not require the value of $f(c)$ (It just needs the value of $f(x)$ near x).

  Therefore, $\lim_{x \to c} f(x)$ may exist even though $f(c)$ is not defined (see Example 1).

- If the given function is continuous, one can obtain the limit $\lim_{x \to c} f(x)$ by substituting $x = c$ into $f(x)$ directly.

- The limit of a function may not exist (as a real number) at some points.

  Some possible reasons are

  i. $f(x)$ is not “continuous” at that point (see Example 3),

  ii. $f(x)$ tends to infinity when x tends to a particular point (see Example 4),

  iii. $f(x)$ continues to oscillate when x tends to this point (see Example 5).

So far, we compute the limits or check the existence of limits by drawing graphs or by computing the values of $f(x)$ near $x = c$ and estimating the limit by investigating the trend.

Two Questions

- How to check the existence of limit mathematically (instead of drawing graph)?
  - ➢ One-side limit: Left-hand limit and Right-hand limit.

- How to compute the limits for more complicated functions?
  - ➢ Properties of Limits.
  - ➢ A few algebraic tricks (when "Properties of Limits" cannot be applied directly).

Existence of limits

According to the discussions in Example 1 – Example 4, we have the following observations.

- The limits $\lim_{x \to c} f(x) = L$ exists (as a number) if $f(x)$ tends to L when x tends to c either from the left or from the right. (See Example 1 and 2).
- The limits $\lim_{x \to c} f(x)$ does not exist if either
  - ✓ $f(x)$ tends to different limits when x tends to c from the left and from the right (Example 3) or
  - ✓ $f(x)$ tends to very large number when x tends to c from the left and from the right (Example 4) or
  - ✓ $f(x)$ does not tend to any number when x tends to c either from the left or from the right.

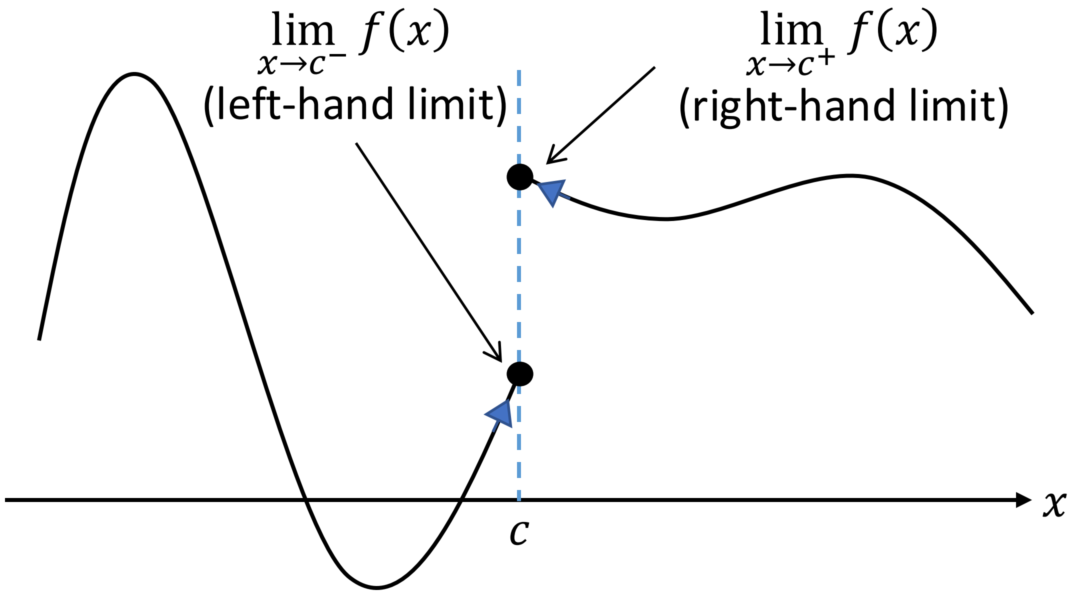
Therefore, the existence of limit can be examined by investigating the limits of $f(x)$ from one direction only (left or right).

Definition (Left-hand limit and Right-hand limit)

We say $f(x)$ has the left (right) hand limit L when x tends to c if the value of $f(x)$ gets arbitrarily close to L if x tends to c from the left (right).

Mathematically, we write

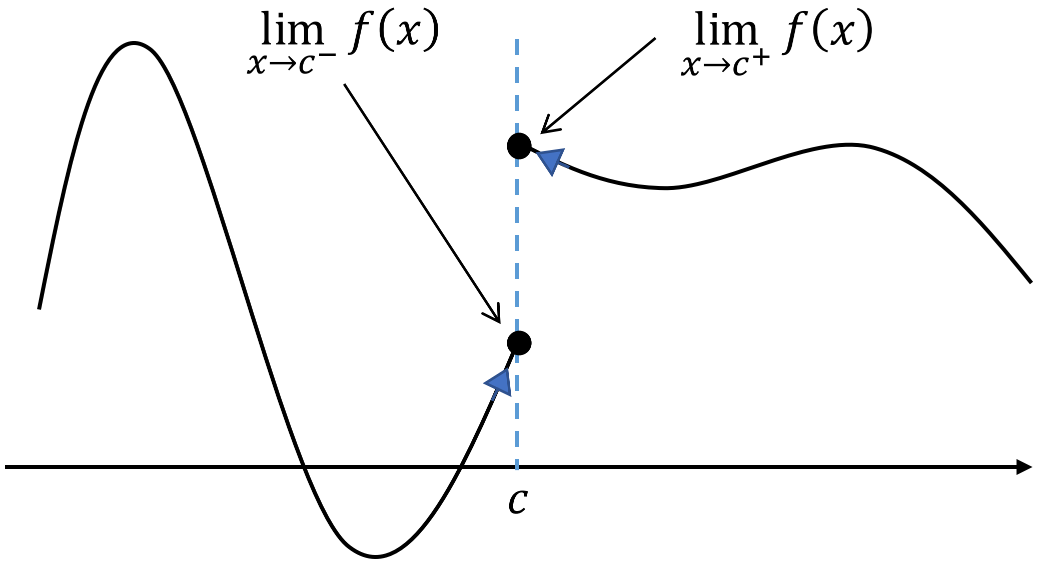
$$\lim_{x \to c^-} f(x) = L \quad \left(\lim_{x \to c^+} f(x) = L \right).$$



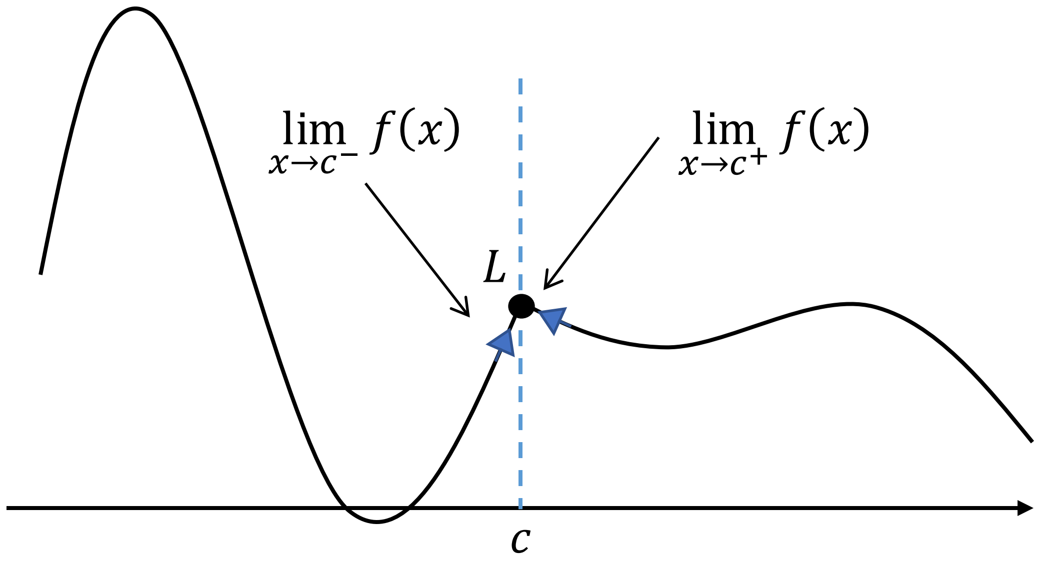
Theorem (Existence of limits)

The limits $\lim_{x \to c} f(x)$ exists if and only if ($\Leftrightarrow$) both left-hand limits and right-hand limits exist and

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) .$$



$\lim_{x \to c} f(x)$ does not exist.



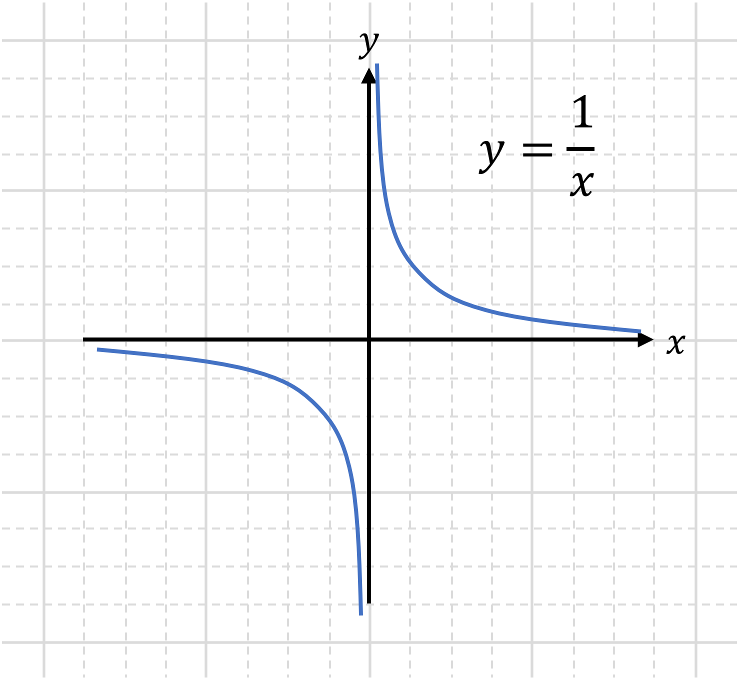
$\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

Example 6

Given a function $f(x) = \frac{1}{x}$, determine whether the limit $\lim_{x \to 0} \frac{1}{x}$ exists.

☺ Solution:

We consider the left-hand limit and right-hand limit. Note that when $x \to 0$, the denominator becomes very small and the fraction $\frac{1}{x}$ becomes very large. So we expect that



$$\lim_{x \to 0^+} \frac{1}{x} = \underbrace{+\infty}_{\because x > 0}$$

$$\lim_{x \to 0^-} \frac{1}{x} = \underbrace{-\infty}_{\because x < 0}$$

Since $\lim_{x \to 0^+} \frac{1}{x} \neq \lim_{x \to 0^-} \frac{1}{x}$, hence the limit $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Example 7

Consider the function $f(x) = \begin{cases} x^3 & if\ x \neq 1 \\ 0 & if\ x = 1 \end{cases}$, check whether the limits $\lim_{x \to 1} f(x)$ exist.

☺ Solution:

One can check the existence by computing $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$:

$$\lim_{x \to 1^+} f(x) = \underbrace{\lim_{x \to 1^+} x^3}_{x \neq 1 \Rightarrow f(x) = x^3} = 1^3 = 1.$$

Similarly, we have $\lim_{x \to 1^-} f(x) = 1$.

Since $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$, we conclude that $\lim_{x \to 1} f(x)$ exists and

$$\lim_{x \to 1} f(x) = 1.$$

Example 8

Consider the function $f(x) = \begin{cases} x^3 & if\ x < 1 \\ \frac{1}{x^3} & if\ x > 1 \end{cases}$, check whether the limits $\lim_{x \to 1} f(x)$ exist.

☺ Solution:

One can check the existence by computing $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$:

$$\lim_{x \to 1^-} f(x) = \underbrace{\lim_{x \to 1^-} x^3}_{x<1 \Rightarrow f(x)=x^3} = 1^3 = 1.$$

$$\lim_{x \to 1^+} f(x) = \underbrace{\lim_{x \to 1^+} \frac{1}{x^3}}_{x>1 \Rightarrow f(x)=\frac{1}{x^3}} = 1.$$

Since $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$, we conclude that $\lim_{x \to 1} f(x)$ exists and $\lim_{x \to 1} f(x) = 1$.

Example 9

We let $f(x) = \frac{|x|}{x}$. Does the limit $\lim_{x\to 0} f(x)$ exist?

☺ Solution:

Recall that $|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$. Again, we consider the left hand limit and right hand limit:

$$\lim_{x\to 0^+} \frac{|x|}{x} = \underbrace{\lim_{x\to 0^+} \frac{x}{x}}_{x>0\Rightarrow|x|=x} \underbrace{=}_{x\neq 0} \lim_{x\to 0^+} 1 = 1,$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \underbrace{\lim_{x\to 0^-} \frac{-x}{x}}_{x<0\Rightarrow|x|=-x} = \lim_{x\to 0^-} (-1) = -1.$$

Since $\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$, so $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 10

Let $[x]$ be the greatest integer less or equal to x (e.g. $[7.2] = 7,\ [7.9] = 7,\ [7] = 7$). Determine whether the limits $\lim\limits_{x \to 3} [x]$ exist.

☺ Solution:

We consider the left-hand limits and the right-hand limits again.

When x approaches to 3 from the left, $x \to 2.\mathrm{xx}$ ... so that $[x] = 2$. Then

$$\lim_{x \to 3^-} [x] = \lim_{x \to 3^-} 2 = 2.$$

When x approaches to 3 from the right, $x \to 3.\mathrm{xx}$ ... so that $[x] = 3$. Then

$$\lim_{x \to 3^+} [x] = \lim_{x \to 3^+} 3 = 3.$$

Since $\lim\limits_{x \to 3^-} [x] \neq \lim\limits_{x \to 3^+} [x]$, we conclude that $\lim\limits_{x \to 3} [x]$ does not exist.

Note: The values of $f(x)$ near $x = 3$ is enough to compute the limits.

Example 11

We consider the function

$$f(x) = \begin{cases} 2\cos x & if\ x \geq 0 \\ -\cos x & if\ x < 0 \end{cases}.$$

Determine whether the limits $\lim_{x\to 0} f(x)$ exist. How about the limits $\lim_{x\to\frac{\pi}{4}} f(x)$?

☺ Solution:

For $\lim_{x\to 0} f(x)$, we consider the left-hand limits and right-hand limits:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\cos x = 2\cos(0) = 2,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -\cos x = -\cos 0 = -1.$$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, so the limits $\lim_{x\to 0} f(x)$ does not exist.

For $\lim_{x\to\frac{\pi}{4}} f(x)$, we can again consider the left-hand limits and right-hand limits:

$$\lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{x\to\frac{\pi}{4}^+} 2\cos x = 2\cos\frac{\pi}{4} = \sqrt{2},$$
$$\lim_{x\to\frac{\pi}{4}^-} f(x) = \lim_{x\to\frac{\pi}{4}^-} 2\cos x = 2\cos\frac{\pi}{4} = \sqrt{2}.$$

Since $\lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{x\to\frac{\pi}{4}^-} f(x)$, hence the limits $\lim_{x\to\frac{\pi}{4}} f(x)$ exists.

Remark:

Since $f(x) = 2\cos x$ when x is closed to $\frac{\pi}{4}$ and $2\cos x$ is a continuous function, hence it is always true that

$$\lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{x\to\frac{\pi}{4}^-} f(x) = \lim_{x\to\frac{\pi}{4}} f(x).$$

Actually, we do not need to consider the right-hand limit and left-hand limit when computing the limits $\lim_{x\to\frac{\pi}{4}} f(x)$.

Computation of Limits

Theorem (Properties of Limits)

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, where L and M are two real numbers. Then

1. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$.
2. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm M$.
3. $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = LM$.
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$, provided that $\lim_{x\to c} g(x) = M \neq 0$.
5. $\lim_{x\to c}[f(x)]^{\alpha} = \left[\lim_{x\to c} f(x)\right]^{\alpha} = L^{\alpha}$, α is real.

   (Note: Property (5) holds only when L^{α} is real)

Example 12

Find the limits $\lim_{x\to 2} x^{\frac{2}{3}} \sin x$ and $\lim_{x\to 0} \frac{x^4+\cos x}{1+x^2}$.

☺ Solution:

For the first limits, using the properties of limits, we have

$$\lim_{x\to 2} x^{\frac{2}{3}} \sin x = \left(\lim_{x\to 2} x^{\frac{2}{3}}\right)\left(\lim_{x\to 2} \sin x\right) = \left(\lim_{x\to 2} x\right)^{\frac{2}{3}}\left(\lim_{x\to 2} \sin x\right) = 2^{\frac{2}{3}} \sin 2 \approx 1.4434.$$

For the second limits, note that $\lim_{x\to 0}(1+x^2) = 1 \neq 0$. Using the properties of limits again, we get

$$\lim_{x\to 0} \frac{x^4+\cos x}{1+x^2} = \frac{\lim_{x\to 0}(x^4+\cos x)}{\lim_{x\to 0}(1+x^2)} = \frac{\lim_{x\to 0} x^4 + \lim_{x\to 0} \cos x}{\lim_{x\to 0} 1 + \lim_{x\to 0} x^2} = \frac{0+1}{1+0} = 1.$$

Some algebraic tricks in computing limits

Suppose we would like to compute the following limits,

$$\lim_{x \to 1} \frac{x^2 + x + 2}{x^2 - x}, \qquad \lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}, \qquad \lim_{\theta \to 0} \frac{\sin \theta}{\theta}.$$

One can see that the properties of limits cannot be applied since the limit of denominators ($x^2 - x$, x^2, θ) tends to 0 and the properties of limits require the limit of denominator to be nonzero.

In order to compute the limit, one has to "transform" the limits into another form so that the above problem does not happen.

There are two major methods:

- Cancelling/ Creating a common factor
- Using inequalities method (sandwich theorem)

Method 1: Cancelling/ Creating a common factor

- The method tries to cancel all factors with zero limits in the denominator. One way is to factorize both numerator and denominator of the expression and cancel some common factors.

Example 13 (Direct Factorization)

Compute

$$\lim_{x\to 0}\frac{x^2+3x}{x}, \qquad \lim_{x\to 2}\frac{x^2+x-6}{x^2-4}.$$

☺ Solution

<u>1st limit</u>

(Step 1: Factorize numerator and denominator and make cancelation)

For $x \neq 0$, we have $\frac{x^2+3x}{x} = \frac{x(x+3)}{x} = x+3.$

(Step 2: Compute limits)

$$\lim_{x\to 0}\frac{x^2+3x}{x}=\lim_{x\to 0}(x+3)=\lim_{x\to 0}x+\lim_{x\to 0}3=0+3=3.$$

Note: Recall that the computation of limits only require the value of $f(x)$ near $x=0$ but not $f(0)$. So we can cancel the factor x in the calculation.

<u>2nd limit</u>

(Step 1: Factorize numerator and denominator and make cancelation)

For $x\neq 2$, we have $\frac{x^2+x-6}{x^2-4}=\frac{(x+3)(x-2)}{(x+2)(x-2)}=\frac{x+3}{x+2}$.

(Step 2: Compute limits)

Since $\lim_{x\to 2}(x+2)=4\neq 0$, then

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}=\lim_{x\to 2}\frac{x+3}{x+2}=\frac{\lim_{x\to 2}(x+3)}{\lim_{x\to 2}(x+2)}=\frac{5}{4}.$$

Example 14

Compute the limits

$$\lim_{x \to -1} \frac{x^3+1}{x^2-x-2}.$$

☺ Solution

Since $\lim_{x \to -1}(x^2-x-2)=0$, so we cannot use properties of limits directly.

In fact, for $x \neq -1$, we have

$$\frac{x^3+1}{x^2-x-2} = \frac{(x+1)(x^2-x+1)}{(x-2)(x+1)} \overset{\substack{x\neq -1 \\ \Rightarrow x+1\neq 0}}{=} \frac{x^2-x+1}{x-2}.$$

Now $\lim_{x \to -1}(x-2) = -3 \neq 0$, so by properties of limits, we get

$$\lim_{x \to -1} \frac{x^3+1}{x^2-x-2} = \lim_{x \to -1} \frac{x^2-x+1}{x-2} = \frac{\lim_{x \to -1} x^2-x+1}{\lim_{x \to -1}(x-2)} = \frac{(-1)^2-(-1)+1}{(-1)-2} = -\frac{1}{3}.$$

Example 15

Compute

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}.$$

☺ Solution:

Since $\sin^3 x \to 0$ as $x \to 0$, so the property of limits cannot be applied. Note that for $x \neq 0$, we have

$$\frac{\tan x - \sin x}{\sin^3 x} = \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \frac{\frac{1}{\cos x} - 1}{\sin^2 x} = \frac{1 - \cos x}{\sin^2 x \cos x} = \frac{1 - \cos x}{(1 - \cos^2 x) \cos x}$$

$$= \frac{1 - \cos x}{(1 - \cos x)(1 + \cos x) \cos x} = \frac{1}{(1 + \cos x) \cos x}.$$

Since $(1 + \cos x) \cos x \to 2 \neq 0$ as $x \to 0$, by the property of limits, we have

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{1}{(1 + \cos x) \cos x} = \frac{1}{(1 + \cos 0) \cos 0} = \frac{1}{2}.$$

Example 16 (Rationalization)

Compute

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

☺ Solution:

It is infeasible for us to factorize the numerator and make cancellation (the existence of square root). Instead, we transform the expression by "creating" a new factor

Note that for $x \neq 0$,

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\left(\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}\right)$$

$$\overset{\substack{(a-b)(a+b)\\=a^2-b^2}}{=} \frac{x^2+100-10^2}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{1}{\sqrt{x^2+100}+10}.$$

Since $\lim_{x\to 0}\sqrt{x^2+100}+10=20\neq 0$, using properties of limits, we get

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{20}.$$

Example 17

Compute the limit

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}.$$

☺ Solution

We multiply both numerator and denominator by a common factor:

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}=\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)=\lim_{x\to 1}\frac{(x^2-1)(\sqrt{x+3}+2)}{x+3-2^2}$$

$$=\lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{x+3}+2)}{x-1}=\lim_{x\to 1}(x+1)\left(\sqrt{x+3}+2\right)=(1+1)\left(\sqrt{1+3}+2\right)=8.$$

Example 18 (Application of Sum-to-Product Formula)

Compute the limit

$$\lim_{x\to 0}\frac{\sin 5x-\sin x}{\cos(2x+1)-\cos(2x-1)},\qquad \lim_{x\to 0}\frac{\sin 2x}{\sin 7x-\sin 3x}.$$

☺ Solution:

IDEA:

Note that the denominator of both expressions when $x\to 0$, the property of limits can be applied. In order to do some algebra, one has to transform the expression into the product of the functions using product-to-sum formula.

For the first limits, we have

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 7x-\sin 3x}=\lim_{x\to 0}\frac{\sin 2x}{\underbrace{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}_{\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{2\cos 5x \sin 2x} = \lim_{x\to 0} \frac{1}{2\cos 5x} = \frac{1}{2\cos 5(0)} = \frac{1}{2}.$$

Consider the second limit, we have

$$\lim_{x\to 0} \frac{\sin 5x - \sin x}{\cos(2x+1) - \cos(2x-1)}$$

$$= \lim_{x\to 0} \frac{\overbrace{2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)}^{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}}{\underbrace{-2\sin\left[\frac{(2x+1)+(2x-1)}{2}\right]\sin\left[\frac{(2x+1)-(2x-1)}{2}\right]}_{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}}$$

$$= \lim_{x\to 0} -\frac{\cos(3x)\sin(2x)}{\sin(2x)\sin 1} = -\lim_{x\to 0} \frac{\cos 3x}{\sin 1} = -\frac{1}{\sin 1}.$$

Method 2: Using inequality method (sandwich theorem)

This method consists of two steps:

We wish to compute $\lim_{x \to c} f(x)$,

Step 1: Find two functions $g(x)$ and $h(x)$ such that

$$g(x) \leq f(x) \leq h(x).$$

Step 2: Suppose $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then from the above inequality, we have

$$L = \lim_{x \to c} g(x) \leq \lim_{x \to c} f(x) \leq \lim_{x \to c} h(x) = L.$$

Using this inequality, one can conjecture that

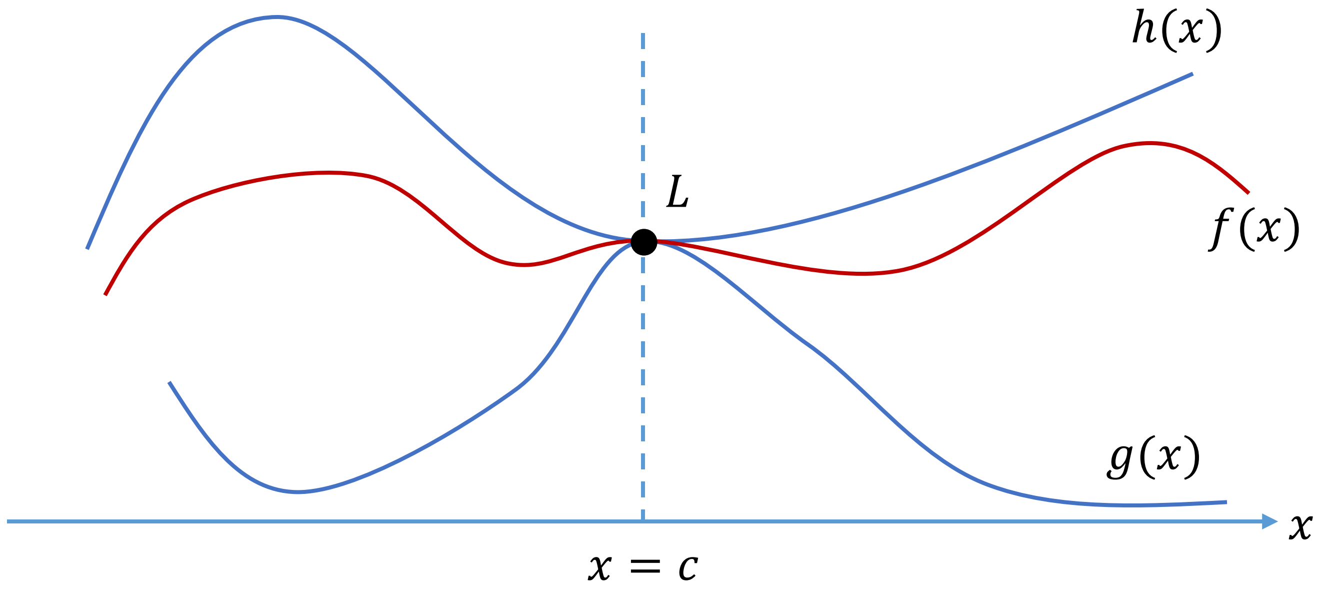
$$\lim_{x \to c} f(x) = L.$$

Formally, this result is known as sandwich theorem (or squeeze theorem).

Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then the limits $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L.$$



Example 19

Compute the limit

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right).$$

Note: Since the limits $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist, the property of limits cannot be applied directly.

☺ Solution:

Note that $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$, then we have

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Since $\lim_{x \to 0}(-x) = \lim_{x \to 0} x = 0$, then the sandwich theorem suggests that

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

Example 20 (Useful Fact)

Show that

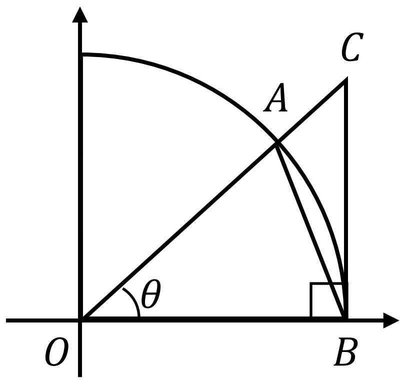
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

☺ Solution:

(Step 1: Find $\boldsymbol{g(\theta)}$ and $\boldsymbol{h(\theta)}$ such that $\boldsymbol{g(\theta) \leq \frac{\sin\theta}{\theta} \leq h(\theta)}$)

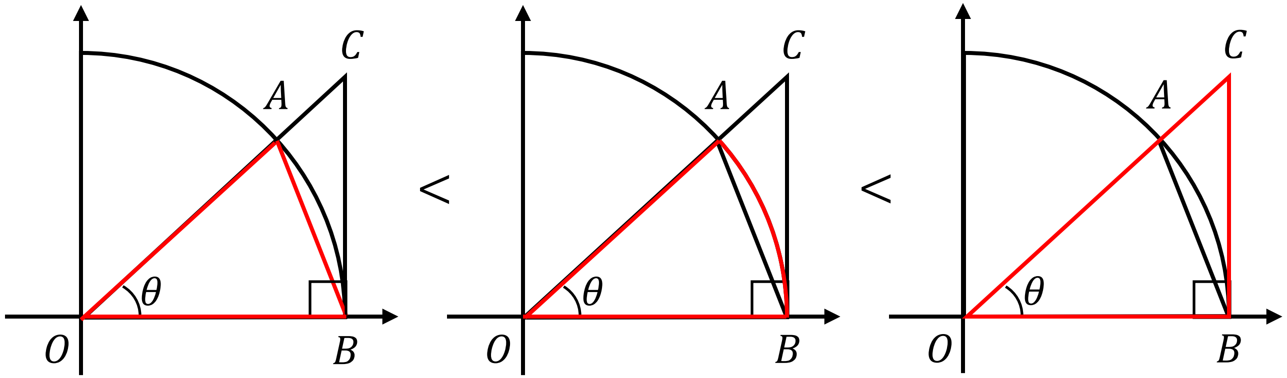
To do this, we first consider the case when $\theta \geq 0$. We consider the following diagram:

(The circle has radius (OA, OB) 1).



It is easy to see that

$$\underbrace{Area\ of\ \Delta OAB}_{\substack{=\frac{1}{2}OA\times OB\times\sin\theta \\ =\frac{1}{2}\sin\theta}} < \underbrace{Area\ of\ sector\ OAB}_{\substack{=\frac{1}{2}(1^2)\theta \\ =\frac{1}{2}\theta}} < \underbrace{Area\ of\ \Delta OCB}_{\substack{=\frac{1}{2}CB\times OB \\ =\frac{1}{2}\tan\theta}}$$



This implies

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta \Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta \quad for\ \theta \geq 0.$$

For the case when $\theta < 0$, we write $\theta = -\varphi$ where $\varphi > 0$. Then replacing θ by φ in the above inequalities, we have

$$1 > \frac{\sin\varphi}{\varphi} > \cos\varphi\,.$$

Substitute $\varphi = -\theta$ and note that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, we have

$$1 > \frac{\sin(-\theta)}{(-\theta)} > \cos(-\theta) \Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta \quad for\ \theta < 0.$$

Combining the two cases, we deduce the following inequalities

$$\boxed{\underbrace{1}_{h(\theta)} > \frac{\sin\theta}{\theta} > \underbrace{\cos\theta}_{g(\theta)}\,.}$$

(Step 2: Use Sandwich theorem)

Since $\lim\limits_{\theta\to 0} 1 = 1$ and $\lim\limits_{\theta\to 0}\cos\theta = \cos 0 = 1$, then by sandwich theorem, we get

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Example 21

Using the fact that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, compute

$$\lim_{x \to 0} \frac{\sin 2x}{3x} \quad \textit{and} \quad \lim_{x \to 1} \frac{\sin(x-1)}{x^2 - 4x + 3}.$$

☺ Solution:

$$\lim_{x \to 0} \frac{\sin 2x}{3x} = \lim_{x \to 0} \frac{2x}{3x} \left(\frac{\sin 2x}{2x} \right) = \lim_{x \to 0} \frac{2}{3} \frac{\sin 2x}{2x} \overset{take\ \theta = 2x}{\cong} \frac{2}{3}.$$

$$\lim_{x \to 1} \frac{\sin(x-1)}{x^2 - 4x + 3} = \lim_{x \to 1} \frac{\sin(x-1)}{(x-1)(x-3)} = \lim_{x \to 1} \left[\frac{\sin(x-1)}{x-1} \right] \left(\frac{1}{x-3} \right) \overset{take\ \theta = x-1}{\cong} 1 \times \frac{1}{1-3} = -\frac{1}{2}.$$

Example 22

Compute the limits

$$\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x}.$$

☺ Solution:

Note that $\cos x$ tends to 1 as $x \to 0$ and it does not give us any trouble in computing the limits. So we just need to concentrate on $(\sin 3x)^2$ and x^2.

Using the fact that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$,

$$\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x} = \lim_{x\to 0}\left[\left(\frac{\sin 3x}{x}\right)\left(\frac{\sin 3x}{x}\right)\left(\frac{1}{\cos x}\right)\right]$$

$$= \lim_{x\to 0} 9\left[\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{\cos x}\right)\right] \overset{take\ \theta=3x}{\cong} 9 \times 1 \times 1 \times 1 = 9.$$

Example 23

Compute the limit

$$\lim_{h\to 0}\frac{\sin(\frac{\pi}{4}+h)-\sin\frac{\pi}{4}}{h}.$$

☺ Solution:

We try to combine the two sine functions in the numerator using sum-to-product formula:

$$\boxed{\sin A-\sin B=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.}$$

$$\lim_{h\to 0}\frac{\sin(\frac{\pi}{4}+h)-\sin\frac{\pi}{4}}{h}=\lim_{h\to 0}\frac{2\cos\frac{\frac{\pi}{4}+h+\frac{\pi}{4}}{2}\sin\frac{\frac{\pi}{4}+h-\frac{\pi}{4}}{2}}{h}=\lim_{h\to 0}\frac{2\cos\left(\frac{\pi}{4}+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$=\lim_{h\to 0}\underbrace{\cos\left(\frac{\pi}{4}+\frac{h}{2}\right)}_{\to\cos\frac{\pi}{4}}\underbrace{\frac{\sin(h/2)}{h/2}}_{\to 1}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}.$$

Example 24

Compute the limits

$$\lim_{x\to 0}\frac{\cos x - 1}{x}.$$

☺ Solution

In order to apply the given result, we first express the numerator in terms of $\sin x$. There are two different ways to do this:

1st method (taught in old AL syllabus)

Using compound angle formula ($\cos(A+B) = \cos A\cos B - \sin A\sin B$), we have

$$\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \left(1-\sin^2\frac{x}{2}\right) - \sin^2\frac{x}{2} = 1 - 2\sin^2\frac{x}{2}.$$

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{1 - 2\sin^2\frac{x}{2} - 1}{x} = \lim_{x\to 0} -2\frac{\sin^2\frac{x}{2}}{x} = \cdots = \lim_{x\to 0} -\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right]\sin\left(\frac{x}{2}\right) \overset{take\ \theta=\frac{x}{2}}{\triangleq} -1 \times 0 = 0.$$

2nd method: Using sum-to-product formula

Note that $1 = \cos 0$, then we have

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{\cos x - \cos 0}{x} \overset{\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}{\triangleq} \lim_{x\to 0}\frac{-2\sin\frac{x+0}{2}\sin\frac{x-0}{2}}{x} = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{x}$$

$$= \lim_{x\to 0} -\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right]\sin\left(\frac{x}{2}\right) \overset{take\ \theta=\frac{x}{2}}{\triangleq} -1 \times 0 = 0$$

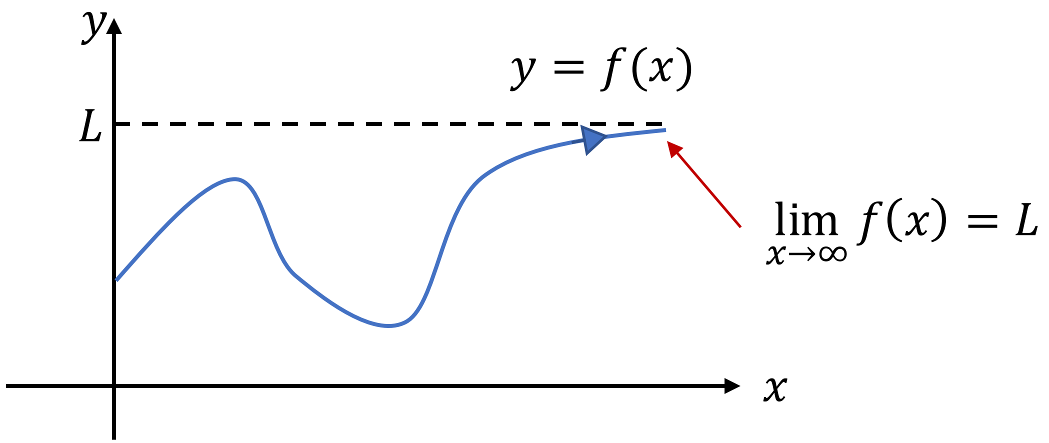
Limit at infinity

Previously, we look at the limits of a function when x tends to some real number c. In many applications, one may be interested in investigating the limits of a function when x tends to infinity (∞ or $-\infty$).

Definition (Limit to infinity)

We say $f(x)$ has the limit L when x tends to ∞ ($-\infty$) if the value of $f(x)$ gets close to L when x is sufficiently large (sufficiently small). Mathematically, we write

$$\lim_{x\to\infty} f(x) = L \ \left(\lim_{x\to-\infty} f(x) = L\right).$$



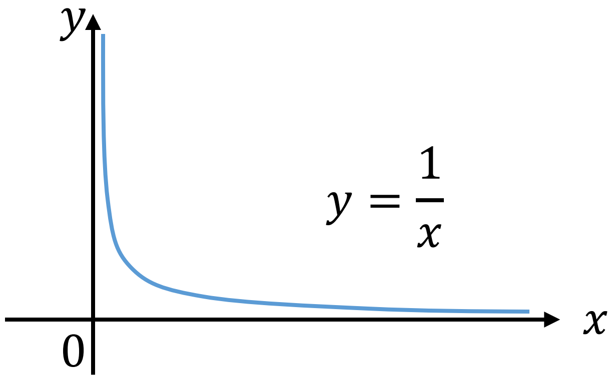
Example 25

Compute

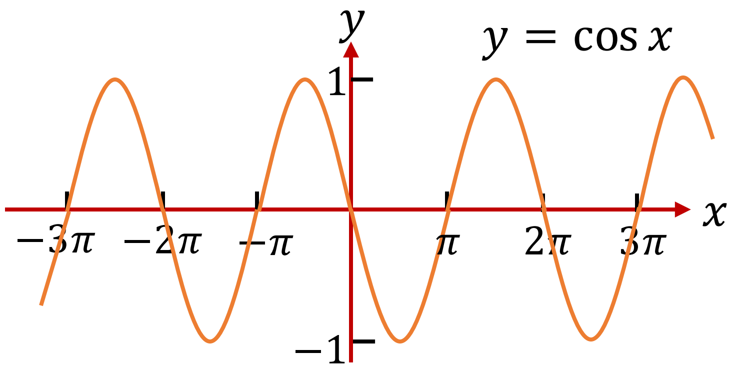
$$\lim_{x\to\infty}\frac{1}{x},\qquad \lim_{x\to\infty}\cos x\,.$$

☺ Solution:

(a) The denominator becomes very large when $x\to\infty$ so that $\frac{1}{x}$ becomes small (close to 0). Hence $\lim_{x\to\infty}\frac{1}{x}=0$.



(b) One can observe from the figure that the function still oscillates when x is large. So $\lim_{x\to\infty}\cos x$ does not exist.



How to compute limits at infinity?

Even $f(x)$ is nice (continuous), one cannot compute the limits $\lim_{x\to\infty} f(x) = L$ by direct substitution since we cannot perform any algebraic operation on the $\infty, -\infty$. For example

$$\lim_{x\to\infty}\frac{x^2-x-1}{2x^2+x+3} = \underbrace{\frac{\infty^2-\infty-1}{2\infty^2+\infty+3}}_{cannot\ calculate\ anymore} = ???.$$

The following facts are useful in computing the limit to infinity.

Some useful facts for limit to infinity

(1) $\lim_{x\to\infty}\frac{1}{x} = 0, \quad \lim_{x\to\infty}\frac{1}{x} = 0, \quad \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

(2) $For\ a > 1, \quad \lim_{x\to\infty} a^x = \infty \ \ and \ \ \lim_{x\to-\infty} a^x = 0.$

$For\ 0 < a < 1, \quad \lim_{x\to\infty} a^x = 0 \ \ and \ \ \lim_{x\to-\infty} a^x = \infty.$

Example 26 (Using properties of limits)

Compute

$$\lim_{x\to\infty}\frac{3x^6+x^4-3x^2+6}{2x^6-2x-1}.$$

☺ Solution:

$$\lim_{x\to\infty}\frac{3x^6+x^4-3x^2+6}{2x^6-2x-1}=\lim_{x\to\infty}\frac{\frac{3x^6+x^4-3x^2+6}{x^6}}{\frac{2x^6-2x-1}{x^6}}=\lim_{x\to\infty}\frac{3+\overbrace{\frac{1}{x^2}}^{\to 0}-\overbrace{\frac{3}{x^4}}^{\to 0}+\overbrace{\frac{6}{x^6}}^{\to 0}}{2-\underbrace{\frac{2}{x^5}}_{\to 0}-\underbrace{\frac{1}{x^6}}_{\to 0}}=\frac{3}{2}.$$

Note: One cannot calculate the limits by using properties of limits directly.

$$\lim_{x\to\infty}\frac{3x^6+x^4-3x^2+6}{2x^6-2x-1}=\frac{\lim\limits_{x\to\infty}3x^6+\lim\limits_{x\to\infty}x^4-\lim\limits_{x\to\infty}3x^2+\lim\limits_{x\to\infty}6}{\lim\limits_{x\to\infty}2x^6-\lim\limits_{x\to\infty}2x-\lim\limits_{x\to\infty}1}.$$

It is because the limits of each term does not exist as a number.

Example 27

Compute the limits

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right), \qquad \lim_{x\to\infty}\sqrt{x+2\sqrt{x}}-\sqrt{x}.$$

☺ Solution:

$$\lim_{x\to\infty}\overbrace{\left(\sqrt{x+1}-\sqrt{x}\right)}^{\infty-\infty}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\underbrace{\sqrt{x+1}+\sqrt{x}}_{denominator\to\infty}}=0.$$

$$\lim_{x\to\infty}\sqrt{x+2\sqrt{x}}-\sqrt{x}=\lim_{x\to\infty}\frac{\left(\sqrt{x+2\sqrt{x}}-\sqrt{x}\right)\left(\sqrt{x+2\sqrt{x}}+\sqrt{x}\right)}{\sqrt{x+2\sqrt{x}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{(x+2\sqrt{x})-x}{\sqrt{x+2\sqrt{x}}+\sqrt{x}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{\sqrt{x+2\sqrt{x}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{2\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x+2\sqrt{x}}+\sqrt{x}}{\sqrt{x}}}$$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{\sqrt{x}}}+1}=\frac{2}{\sqrt{1+0}+1}=1.$$

Example 28 (Using Sandwich Theorem)

Compute

$$\lim_{x\to\infty} \frac{\cos x}{x^2}.$$

Solution:

Note: Although $\lim\limits_{x\to\infty} \cos x$ does not exist, note that the numerator is bounded (varies from -1 to 1) and the denominator tends to very large number (as $x \to \infty$), hence, we expect that $\lim\limits_{x\to\infty} \frac{\cos x}{x^2} = 0$.

Note that $-1 \le \cos x \le 1$, so we have $-\frac{1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$.

Since $\lim\limits_{x\to\infty} -\frac{1}{x^2} = \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$, we use Sandwich theorem and conclude that

$$\lim_{x\to\infty} \frac{\cos x}{x^2} = 0.$$

Example 29

Show that for any real number $x \neq 0$,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}.$$

☺ Solution:

Using the fact that $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y} = e$, we have

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}(x)} = \left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^{x}$$

$$\overset{\substack{take\ y=\frac{n}{x} \\ n\to\infty\Rightarrow y=\frac{n}{x}\to 0}}{\equiv} e^{x}.$$

Extra Example 1 (Another application of sandwich theorem)

Let $f(x)$ be a function such that the value of $f(x)$ is always a single-digit positive prime number for every $x \in \mathbb{R}$.

Compute the limits

$$\lim_{x\to 0} \frac{x^6 + \sin^2 x}{f(x)}.$$

☺ Solution:

One cannot compute the limits by considering

$$\lim_{x\to 0} \frac{x^6 + \sin^2 x}{f(x)} = \frac{\lim_{x\to 0} x^6 + \lim_{x\to 0} \sin^2 x}{\lim_{x\to 0} f(x)}.$$

Since the limits $\lim_{x\to 0} f(x)$ may not exist and we do not know the value of $\lim_{x\to 0} f(x)$ even it exists.

Because of this, one has to "eliminate" $f(x)$ by using some inequality technique, so we will compute the limits using sandwich theorem.

Since $f(x)$ is a single-digit positive prime number (i.e. $f(x) = 2, 3, 5, 7$), so we must have $2 \leq f(x) \leq 10$. Then we have (note that $x^6 + \sin^2 x \geq 0$)

$$\frac{x^6 + \sin^2 x}{10} \leq \frac{x^6 + \sin^2 x}{f(x)} \leq \frac{x^6 + \sin^2 x}{2}.$$

Taking limits on both sides, we have

$$0 = \lim_{x \to 0} \frac{x^6 + \sin^2 x}{10} \leq \lim_{x \to 0} \frac{x^6 + \sin^2 x}{f(x)} \leq \lim_{x \to 0} \frac{x^6 + \sin^2 x}{2} = 0.$$

Therefore, by sandwich theorem, we conclude that

$$\lim_{x \to 0} \frac{x^6 + \sin^2 x}{f(x)} = 0.$$

Extra Example 2

Compute the limits

$$\lim_{x\to\pi} \sin 2x \cos\left[\frac{2x}{\pi}\right]$$

where $[y]$ denote the greatest integer less than or equal to y.

☺ Solution:

Again, the limits $\lim_{x\to\pi} \cos\left[\frac{2x}{\pi}\right]$ does not exist (since $\lim_{x\to\pi^-} \cos\left[\frac{2x}{\pi}\right] = \cos 1$ and $\lim_{x\to\pi^+} \cos\left[\frac{2x}{\pi}\right] = \cos 2$).

Using the fact that $-1 \le \cos y \le 1$, we have $-\sin 2x \le \sin 2x \cos\left[\frac{2x}{\pi}\right] \le \sin 2x$

$$\Rightarrow 0 = -\sin 2\pi = \lim_{x\to\pi}(-\sin 2x) \le \lim_{x\to\pi} \sin 2x \cos\left[\frac{2x}{\pi}\right] = \lim_{x\to\pi} \sin 2x = \sin 2\pi = 0.$$

By sandwich theorem, we conclude that

$$\lim_{x\to\pi} \sin 2x \cos\left[\frac{2x}{\pi}\right] = 0.$$

Final Remark about the use of sandwich theorem

- The main purpose of sandwich theorem is to "eliminate" some terms which the limits does not exist. One may consider sandwich theorem when he encounters the following limits:

$$\lim_{x\to 0} x \underbrace{\cos\left(\frac{1}{x}\right)}_{\substack{\textit{limit not}\\ \textit{exist}}}, \qquad \lim_{x\to\pi} \sin 2x \underbrace{\cos\left[\frac{2x}{\pi}\right]}_{\substack{\textit{limit not}\\ \textit{exist}}}, \qquad \lim_{x\to 0} x \underbrace{f(x)}_{\substack{\textit{No idea}\\ \textit{about } f(x)}}$$

- However, one should NOT use sandwich theorem and should stick to the properties of limits if the limits of every terms exist, say

$$\lim_{x\to\pi} \underbrace{x}_{\to\pi} \underbrace{\sin x}_{\to 0}, \qquad \lim_{x\to 2\pi} \underbrace{e^x}_{\to e^{2\pi}} \underbrace{\cos[x]}_{\to 1}.$$

- Also, the use of sandwich theorem requires some inequality techniques. One should not rely too much on this theorem if he/she is not skillful in inequalities.

More Examples on Classical Limits

Extra Example 3

Compute the limits

$$\lim_{x\to 0}\frac{\sin 3x \sin 5x}{\sin 2x \sin 4x}.$$

☺ Solution:

Note that

$$\lim_{x\to 0}\frac{\sin 3x \sin 5x}{\sin 2x \sin 4x}=\lim_{x\to 0}\frac{\left(\frac{\sin 3x}{x}\right)\left(\frac{\sin 5x}{x}\right)}{\left(\frac{\sin 2x}{x}\right)\left(\frac{\sin 4x}{x}\right)}=\lim_{x\to 0}\frac{3\times 5}{2\times 4}\frac{\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 5x}{5x}\right)}{\left(\frac{\sin 2x}{2x}\right)\left(\frac{\sin 4x}{4x}\right)}\underbrace{=}_{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}\frac{15}{8}.$$

Extra Example 4

(a) Show that

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

(b) Hence, compute the limits

$$\lim_{x \to 0} \frac{\sin(2 \tan 3x)}{x}.$$

☺ Solution:

(a) Using the identity $\tan x = \frac{\sin x}{\cos x}$, we get

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right) = 1 \times \left(\frac{1}{\cos 0}\right) = 1.$$

(b) Note that $2\tan 3x \to 2\tan 0 = 0$ when $x \to 0$, thus

$$\lim_{x\to 0}\frac{\sin(2\tan 3x)}{x} = \lim_{x\to 0}\frac{\sin(2\tan 3x)}{2\tan 3x}\left(\frac{2\tan 3x}{x}\right) = \lim_{x\to 0}\underbrace{\frac{\sin(2\tan 3x)}{2\tan 3x}}_{\to 1\ as\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}(6)\underbrace{\left(\frac{\tan 3x}{3x}\right)}_{\to 1\ as\ \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1} = 1\times 6\times 1 = 6.$$

Extra Example 5

Compute the limit

$$\lim_{x\to a}\frac{\tan x - \tan a}{x-a}$$

where a is real number such that $\cos a \neq 0$ (so that $\tan a$ is defined).

☺ Solution:

Note that

$$\lim_{x\to a}\frac{\tan x-\tan a}{x-a}=\lim_{x\to a}\frac{\frac{\sin x}{\cos x}-\frac{\sin a}{\cos a}}{x-a}=\lim_{x\to a}\frac{\sin x\cos a-\sin a\cos x}{(x-a)\cos x\cos a}$$

By compound angle formula

$$\sin(A-B)=\sin A\cos B-\sin B\cos A\,.$$

$$=\lim_{x\to a}\frac{\sin(x-a)}{(x-a)\cos x\cos a}=\frac{1}{\cos a}\lim_{x\to a}\underbrace{\left(\frac{\sin(x-a)}{x-a}\right)}_{\substack{\to 1\ as\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\\ and\ x-a\to 0}}\left(\frac{1}{\cos x}\right)=\frac{1}{\cos a}\times 1\times\frac{1}{\cos a}=\frac{1}{\cos^2 a}(or=\sec^2 a).$$

More Examples on Limits to infinity

Extra Example 6

Compute the limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n.$$

☺ Solution:

The expression looks like the form $\left(1+\frac{1}{??}\right)^{??}$ and one may consider to compute the limits using $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

So firstly, we should rewrite the expression a bit.

Note that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{n+1}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{n+1}}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{n+1}}\right)^{\frac{n^2}{n+1}\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n^2}{n+1}}\right)^{\frac{n^2}{n+1}}\right]^{\frac{n+1}{n}}$$

As $\lim_{n\to\infty}\frac{n^2}{n+1} = \lim_{n\to\infty}\frac{\overbrace{n}^{\to\infty}}{\underbrace{1+\frac{1}{n}}_{\to 1}} = \infty$ and $\lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$, using the result $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (with $x = \frac{n^2}{n+1}$), we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\underbrace{\left[\left(1+\frac{1}{\frac{n^2}{n+1}}\right)^{\frac{n^2}{n+1}}\right]}_{\to e}^{\overbrace{\frac{n+1}{n}}^{\to 1}} = e^1 = e.$$

Extra Example 7

(a) Compute the limits

$$\lim_{x\to\infty}\frac{\cos x}{x^3} \quad \textit{and} \quad \lim_{x\to\infty}\frac{\sin x}{x^3}.$$

(b) Hence, compute the limits

$$\lim_{x\to\infty}\frac{x^3+2\cos x}{2x^3+x-\sin x}.$$

☺ Solution

Note that both limits $\lim\limits_{x\to\infty}\cos x$ and $\lim\limits_{x\to\infty}\sin x$ do not exist, one should "eliminate" $\cos x$ (in the first limit) and $\sin x$ (in the second limit using sandwich theorem).

(a) Note that $-1 \le \cos x \le 1$, so that

$$-\frac{1}{x^3} \le \frac{\cos x}{x^3} \le \frac{1}{x^3}. \quad (Since\ x \to \infty, so\ x > 0)$$

Taking limits on both sides, we have

$$\underbrace{\lim_{x\to\infty} -\frac{1}{x^3}}_{=0} \leq \lim_{x\to\infty} \frac{\cos x}{x^3} \leq \underbrace{\lim_{x\to\infty} \frac{1}{x^3}}_{=0}.$$

So $\lim_{x\to\infty} \frac{\cos x}{x^3} = 0$ by sandwich theorem. Similarly, one can find that $\lim_{x\to\infty} \frac{\sin x}{x^3} = 0$.

(b) By dividing both numerator and denominator by x^3, we get

$$\lim_{x\to\infty} \frac{x^3 + 2\cos x}{2x^3 + x - \sin x} = \lim_{x\to\infty} \frac{1 + 2\overbrace{\left(\frac{\cos x}{x^3}\right)}^{\to 0}}{2 + \underbrace{\frac{1}{x^2}}_{\to 0} - \underbrace{\frac{\sin x}{x^3}}_{\to 0}} = \frac{1+0}{2+0-0} = \frac{1}{2}.$$