

CITY UNIVERSITY OF HONG KONG

Department of Mathematics

Course Code & Title : MA1200 Calculus and Basic Linear Algebra I
Session : Semester A, 2023-2024
Time Allowed : Three Hours

This paper has **FIVE** pages. (including this cover page)

A brief table of useful formulas is attached on page 5.

Instructions to candidates:

1. Answer **all** questions.
 2. Start each main question on a new page.
 3. Show all step.
-

*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable portable battery operated calculator.

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

Question 1

[11 marks]

Consider the conic section described by the equation $9x^2 + 4y^2 - 18x + 8y - 23 = 0$.

- (a) Classify its type. (2 marks)
- (b) Find its center, vertices, and foci. (6 marks)
- (c) Sketch its graph. (3 marks)

Question 2

[11 marks]

Let $f(x) = \sqrt{2 + x - x^2}$.

- (a) Find the largest possible domain of $f(x)$. (2 marks)
- (b) Study the increasing or decreasing property of $f(x)$ in the largest possible domain. (4 marks)
- (c) Find the range of $y = f(x)$. (3 marks)
- (d) Sketch the graph of the curve $y = f(x)$. (2 marks)

Question 3

[11 marks]

- (a) Find the exact value of $\tan^{-1}(\tan 2)$. (3 marks)
- (b) Find the exact value of $\sin\left(\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3}\right)$. (3 marks)
- (c) Find, in radians, the general solutions of the equation (5 marks)

$$3 - 4\cos^2\theta = 0.$$

Question 4

[11 marks]

- (a) Show that $f(x) = x^3 - 5$ has a zero between 1 and 2. (2 marks)
- (b) Show that $x - 2$ is a factor of $x^4 - x^3 - 3x^2 + 17x - 30$. (2marks)
- (c) Let $f(x) = \frac{x^4 + 3x^2 - x - 8}{x^3 + 4x}$.
 - (i) Express $f(x)$ in partial fractions. (5 marks)
 - (ii) Find $f'(x)$. (2 marks)

Question 5

[11 marks]

- (a) Let $f(x) = \frac{x^2 - 1}{2x^2 - x - 1}$. Find the limits
(i) $\lim_{x \rightarrow 0} f(x)$; (ii) $\lim_{x \rightarrow 1} f(x)$; (iii) $\lim_{x \rightarrow \infty} f(x)$. (3 marks)
- (b) Find the limit $\lim_{x \rightarrow 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$. (2 marks)
- (c) Find the limits
(i) $\lim_{x \rightarrow 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$; (ii) $\lim_{x \rightarrow 0+} x^{\frac{k}{1+\ln x}}$. (6 marks)

Question 6

[11 marks]

Differentiate the following functions about the variable x :

- (a) $\cos 2x - 2 \sin x$; (2 marks)
- (b) $x^{\frac{1}{x}}$, where $x > 0$; (3 marks)
- (c) $\ln \sqrt{\frac{1 - \sin x}{1 + \sin x}}$; (3 marks)
- (d) $x \sqrt{\frac{1 - x}{1 + x}}$. (3 marks)

Question 7

[11 marks]

- (a) Let C be a curve given by the equation $y^3 + 3y = x$.
(i) Verify that $(4, 1)$ is a point on the curve C ; (2 marks)
(ii) Find the tangent line of the curve C at the point $(4, 1)$. (5 marks)
- (b) Let C be a curve given by $\begin{cases} x = 2t - t^2, \\ y = 3t - t^3. \end{cases}$

Find the tangent line of the curve C at $t = 0$. (4 marks)

Question 8

[11 marks]

Find the smallest perimeter of rectangulars, whose areas are S .

Question 9

[12 marks]

Let $f(x) = \cos(m \sin^{-1} x)$.

- (a) Show that (4 marks)

$$(1 - x^2)f''(x) - xf'(x) + m^2f(x) = 0.$$

- (b) Let n be a positive integer, show that (5 marks)

$$(1 - x^2)f^{(n+2)}(x) - (2n + 1)xf^{(n+1)}(x) + (m^2 - n^2)f^{(n)}(x) = 0.$$

- (c) Hence, or otherwise, find the Maclaurin series of $\cos(m \sin^{-1} x)$ as far as the terms in x^4 .

(3 marks)