MA1200 – Calc & Basic Linear Algebra I

Hand-in Assignment 2 Solution

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1. Factorize

   (a) $x^4 - 1$

   Solution.
   $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$ ■

   (b) $x^5 - 1$

   Solution.
   $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ ■

2. Express the following rational functions in partial fraction.

   (a) $\dfrac{x^3 + x^2 + 6x - 1}{x^3 - 1}$

   Solution.

$$
\begin{aligned}
\frac{x^3 + x^2 + 6x - 1}{x^3 - 1} &= 1 + \frac{x^2 + 6x}{x^3 - 1} \\
&= 1 + \frac{x^2 + 6x}{(x-1)(x^2 + x + 1)} \\
\text{Let } \frac{x^2 + 6x}{(x-1)(x^2 + x + 1)} &= \frac{A}{(x-1)} + \frac{Bx + C}{(x^2 + x + 1)} \\
x^2 + 6x &= A(x^2 + x + 1) + (Bx + C)(x - 1) \qquad (*) \\
(*) \text{ with } x = 1: \qquad 7 &= 3A \implies A = \frac{7}{3} \\
(*) \text{ with } x = 0: \qquad 0 &= A - C \implies C = \frac{7}{3} \\
(*) \text{ with } x = -1: \qquad -5 &= A + (-B + C)(-2) \implies B = -\frac{4}{3} \\
\text{Hence, } \frac{x^3 + x^2 + 6x - 1}{x^3 - 1} &= 1 + \frac{7}{3(x-1)} + \frac{-4x + 7}{3(x^2 + x + 1)}
\end{aligned}
$$

■

   (b) $\dfrac{11x + 10}{(x+1)^2 x^2}$

Solution.

$$
\begin{aligned}
\text{Let } \frac{11x+10}{(x+1)^2x^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2} \\
11x+10 &= Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2 && (**) \\
(**) \text{ with } x=0: \qquad 10 &= B \\
(**) \text{ with } x=-1: \qquad -1 &= D \\
(**) \text{ with } x=1: \qquad 21 &= 4A+4B+2C+D \implies -9 = 2A+C && \text{(i)} \\
(**) \text{ with } x=2: \qquad 32 &= 18A+9B+12C+4D \implies -54 = 18A+12C && \text{(ii)} \\
\text{(i)} * 9 - \text{(i)}: \qquad -27 &= -3C \implies C = 9 \\
\text{(i) with } C=9: \qquad -9 &= 2A+9 \implies A=-9 \\
\text{Hence, } \frac{11x+10}{(x+1)^2x^2} &= -\frac{9}{x} + \frac{10}{x^2} + \frac{9}{(x+1)} - \frac{1}{(x+1)^2}
\end{aligned}
$$

■

3. Calculate

   (a) $C_3^9 C_9^{12}$

   Solution.
   $C_3^9 C_9^{12} = \dfrac{9!}{3! * 6!} \cdot \dfrac{12!}{9! * 3!} = \dfrac{9*8*7}{3*2*1} \cdot \dfrac{12*11*10}{3*2*1} = 18480$ ■

   (b) C_{n-3}^m (($n \geq 3$ is an integer)

   Solution.
   $C_{n-3}^m = \dfrac{m!}{(n-3)! * (m-(n-3))!} = C_{n-3}^m = \dfrac{m*(m-1)*(m-2)*\cdots*(m-(n-3)+1)}{(n-3)!}$

   ■

4. Simplify

   (a) $(\tan\alpha + \sec\alpha)(\tan\alpha - \sec\alpha)$

   Solution.
   $(\tan\alpha + \sec\alpha)(\tan\alpha - \sec\alpha) = \tan^2\alpha - \sec^2\alpha = -1$ ■

   (b) $1 - \dfrac{\cos^2\alpha}{1+\sin\alpha}$

   Solution.
   $1 - \dfrac{\cos^2\alpha}{1+\sin\alpha} = \dfrac{(1+\sin\alpha) - (1-\sin^2\alpha)}{1+\sin\alpha} = \dfrac{\sin\alpha + \sin^2\alpha}{1+\sin\alpha} = \sin\alpha$ ■

5. It is given that $\sin A = -\frac{1}{3}$ with $-\pi < A < -\frac{\pi}{2}$, and that $\cos B = \frac{2}{5}$ with $-\frac{\pi}{2} < B < 0$. Calculate the exact values of

   (a) $\tan(A + B)$

   Solution.
   $\sin A = -\dfrac{1}{3}$ with $-\pi < A < -\frac{\pi}{2} \implies A = -\pi - \sin^{-1}\left(-\frac{1}{3}\right)$ and $\cos A = -\frac{2\sqrt{2}}{3}$
   $\cos B = \dfrac{2}{5}$ with $-\frac{\pi}{2} < B < 0 \implies B = -\cos^{-1}\left(\frac{2}{5}\right)$ and $\sin B = -\frac{\sqrt{21}}{5}$
   $$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{(-\frac{1}{3})(\frac{2}{5}) + (-\frac{2\sqrt{2}}{3})(-\frac{\sqrt{21}}{5})}{(-\frac{2\sqrt{2}}{3})(\frac{2}{5}) - (-\frac{1}{3})(-\frac{\sqrt{21}}{5})} = \frac{-2 + 2\sqrt{42}}{-4\sqrt{2} - \sqrt{21}}$$
   or $\dfrac{2}{11}(25\sqrt{2} - 9\sqrt{21})$, ($\approx -1.0705$) ■

   (b) $\cot(A - B)$

   Solution.
   $$\cot(A - B) = \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{-4\sqrt{2} + \sqrt{21}}{-2 - 2\sqrt{42}}$$
   or $\dfrac{\sqrt{21} - 4\sqrt{2}}{-2 - 2\sqrt{42}}$, ($\approx -0.0718$) ■

6. Calculate the values of the following formulas. Caution! You need to present your result in the principal ranges of appropriate inverse of trigonometric functions.

   Solution.

   (a) $\cos^{-1}\left(\cos\left(-\dfrac{6\pi}{5}\right)\right) = \cos^{-1}\left(\cos\left(2\pi - \dfrac{6\pi}{5}\right)\right) = \dfrac{4\pi}{5}$

   (b) $\cos^{-1}\left(\sin\left(-\dfrac{2\pi}{3}\right)\right) = \cos^{-1}\left(\sin\left(-\pi + \dfrac{\pi}{3}\right)\right) = \dfrac{5\pi}{6}$

   (c) $\sin^{-1}\left(\cos\left(\dfrac{5\pi}{6}\right)\right) = \sin^{-1}\left(\cos\left(\pi - \dfrac{\pi}{6}\right)\right) = -\dfrac{\pi}{2} + \dfrac{\pi}{6}$

   (d) $\tan^{-1}(\tan(3)) = 3 - \pi$

   ■

7. Does $\cos\left(\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)\right)$ exists? If yes, find its value. If no, give your reason. How about $\cot\left(\tan^{-1}(0)\right)$?

   Solution.
   $\cos\left(\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)\right) = \cos(\pi/3) = 1/2$, $\cot\left(\tan^{-1}(0)\right) = \cot(0)$ is undefined, since the domain of cot does not contain 0. ■

8. Given $\csc\alpha = \frac{13}{5}$ and α in the second quadrant, find the other five trigonometric functions of α.

Solution.
$\sin\alpha = \frac{5}{13} \implies \alpha = \pi - \beta$ where $\beta = \sin^{-1}\frac{5}{13}$ is in the first quadrant.

(a) $\cos\alpha = -\cos\beta = -\dfrac{12}{13}$

(b) $\tan\alpha = -\tan\beta = -\dfrac{5}{12}$

(c) $\sec\alpha = -\dfrac{13}{12}$

(d) $\cot\alpha = -\dfrac{12}{5}$

■

9. Express $\cos x + \sqrt{3}\sin x$ in the form of $A\cos(x-\alpha)$ with α in the third quadrant and $A < 0$.

Solution.
$A\cos(x-\alpha) = A\cos x\cos\alpha + A\sin x\sin\alpha \implies$

$$A\sin\alpha = \sqrt{3} \qquad \text{(i)}$$
$$A\cos\alpha = 1 \qquad \text{(ii)}$$
$$\frac{\text{(i)}}{\text{(ii)}}: \qquad \tan\alpha = \sqrt{3}$$
$$\alpha = \frac{\pi}{3} + \pi \qquad \text{third quadrant}$$
$$\text{hence,} \qquad A = -2$$

■

10. Find the general solution of $3(\tan(3\theta))^2 = 1$. The unknown is θ.

$$\tan(3\theta) = \pm 1/\sqrt{3}$$
$$3\theta = \pm\arctan\left(1/\sqrt{3}\right) + n\pi \qquad n \in \mathbf{Z}$$
$$\theta = \pm\frac{\pi}{18} + \frac{n\pi}{3} \qquad n \in \mathbf{Z}$$
$$\theta = \pm\frac{\pi}{18}, \pm\frac{5\pi}{18}, \pm\frac{7\pi}{18}, \pm\frac{11\pi}{18}, \pm\frac{13\pi}{18}, \ldots$$

— *End* —