MA1200 Hand-in Assignment #3 due at 3:00PM on November 27, 2024

Instructions to students:

1. *Please submit it via Canvas in a PDF file (you can handwrite the answers and take photos by your phone, then make it into a PDF file, see, for example, https://www.wikihow.com/Convert-JPG-to-PDF for how to combine JPG files to a PDF; you can also do it by note-taking apps on an iPad or a Surface)*

2. *The assignment is due on* **3:00PM of November 27, 2024**. *Your score of this assignment is only based on what appears on Canvas as a successful submission. Any unsuccessful submissions will* ***NOT*** *be marked, which results in your getting zero point.*

3. *Please write down your name and student ID.*

10 points for every question below. There are totally ten questions.

Questions:

1. Compute the following limits:

(a) $\lim\limits_{x\to -1} \dfrac{x^4+2x+1}{x^3+5x^2+7x+3}$.

Solution: We first have

$$\begin{aligned}
&\lim_{x\to -1} \frac{x^4+2x+1}{x^3+5x^2+7x+3} = \lim_{x\to -1} \frac{x(x^3+1)+(x+1)}{(x^3+1)+5(x^2+x)+2(x+1)} \\
&= \lim_{x\to -1} \frac{x(x+1)(x^2-x+1)+(x+1)}{(x+1)(x^2-x+1)+5x(x+1)+2(x+1)} \\
&= \lim_{x\to -1} \frac{x(x^2-x+1)+1}{x^2-x+1+5x+2} \\
&= \lim_{x\to -1} \frac{x^3-x^2+x+1}{(x+1)(x+3)}.
\end{aligned}$$

Then, we get

$$\lim_{x\to -1^+} \frac{x^3-x^2+x+1}{(x+1)(x+3)} = -\infty, \quad \lim_{x\to -1^-} \frac{x^3-x^2+x+1}{(x+1)(x+3)} = +\infty,$$

which implies that the limit does not exist.

(b) $\lim\limits_{x\to +\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x-\sqrt{x+\sqrt{x}}}\right)$.

Solution:

$$\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x-\sqrt{x+\sqrt{x}}}\right)=\lim_{x\to+\infty}\frac{2\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x-\sqrt{x+\sqrt{x}}}}$$

$$=\lim_{x\to+\infty}\frac{2\sqrt{1+\sqrt{\frac{1}{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}+\sqrt{1-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}=1.$$

(c) $\lim_{x\to 0}\frac{\cos(3x)-\cos(2x)}{\sin(3x)-\sin(2x)}$.

Solution: By the L' Hospital's rule, we have

$$\lim_{x\to 0}\frac{\cos(3x)-\cos(2x)}{\sin(3x)-\sin(2x)}=\lim_{x\to 0}\frac{-3\sin(3x)+2\sin(2x)}{3\cos(3x)-2\cos(2x)}=0$$

(d) $\lim_{x\to 0}\frac{\tan(x)}{2\sin(x+\frac{\pi}{6})-1}$

Solution: By the L' Hospital's rule, we have

$$\lim_{x\to 0}\frac{\tan(x)}{2\sin(x+\frac{\pi}{6})-1}=\lim_{x\to 0}\frac{1}{2\cos(x+\frac{\pi}{6})\cos^2 x}=\frac{1}{\sqrt{3}}$$

2. Compute the following limits:

(a) $\lim_{x\to+\infty}\frac{1+a^x+(2a)^x}{1-a^x-(3a)^x}$ where (i) $a=0.4$ (ii) $a=0.8$, (b) $\lim_{x\to+\infty}\left(1-\frac{1}{2x}\right)^{3x}$.

Solution: (a) For $a\neq 0$, we divide the numerator and denominator by $(3a)^x$ and get

$$\lim_{x\to+\infty}\frac{1+a^x+(2a)^x}{1-a^x-(3a)^x}=\lim_{x\to+\infty}\frac{(\frac{1}{3a})^x+(\frac{1}{3})^x+(\frac{2}{3})^x}{(\frac{1}{3a})^x-(\frac{1}{3})^x-1}$$

For both $a=0.4$ and $a=0.8$, we have

$$\lim_{x\to+\infty}(\frac{1}{3a})^x=0,\quad \lim_{x\to+\infty}(\frac{1}{3})^x=0,\quad \lim_{x\to+\infty}(\frac{2}{3})^x=0.$$

Hence, the limit is 0 when $a=0.4$ and $a=0.8$.

(b)

$$\lim_{x\to+\infty}\left(1-\frac{1}{2x}\right)^{3x}=\lim_{x\to+\infty}e^{\ln\left(1-\frac{1}{2x}\right)^{3x}}=\lim_{x\to+\infty}e^{3x\ln\left(1-\frac{1}{2x}\right)}=e^{\lim_{x\to+\infty}3x\ln\left(1-\frac{1}{2x}\right)}.$$

By L'Hospital's rule, we get

$$\lim_{x\to+\infty} 3x\ln\left(1-\frac{1}{2x}\right) = \lim_{x\to+\infty}\frac{\ln(1-\frac{1}{2x})}{\frac{1}{3x}} = \lim_{x\to+\infty}\frac{\frac{2x}{2x-1}\frac{1}{2x^2}}{-\frac{1}{3x^2}} = -\frac{3}{2}.$$

Hence, we have

$$\lim_{x\to+\infty}\left(1-\frac{1}{2x}\right)^{3x} = e^{-\frac{3}{2}}$$

3. Compute the following limits:

(a) $\displaystyle\lim_{n\to+\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+1}+\frac{3}{n^2+1}+\cdots+\frac{n}{n^2+1}\right)$,

Solution: We first note that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Then, we have

$$\lim_{n\to+\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+1}+\frac{3}{n^2+1}+\cdots+\frac{n}{n^2+1}\right) = \lim_{n\to+\infty}\frac{n(n+1)}{2(n^2+1)} = \frac{1}{2}.$$

(b) $\displaystyle\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$.

Solution: Note that for any $1\le k\le n$ we have

$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{\sqrt{n^2+1}}.$$

We thus get

$$\frac{n}{\sqrt{n^2+n}} \le \left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right) \le \frac{n}{\sqrt{n^2+1}}$$

It is easy to obtain that

$$\lim_{n\to+\infty}\frac{n}{\sqrt{n^2+n}} = 1,\ \lim_{n\to+\infty}\frac{n}{\sqrt{n^2+1}} = 1.$$

By the Sandwich Theorem, we have

$$\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right) = 1.$$

4. Let

$$f(x) = \begin{cases} x^2+1, & \text{if } x<0, \\ c(e^{2x})+d, & \text{if } 0 \le x \le 1, \\ (x+7)^{1/3}, & \text{if } x>1. \end{cases}$$

Determine the values of c and d, such that $f(x)$ is continuous everywhere.

Solution: Since $f(x)$ is continuous when $x<0$, $0<x<1$, and $x>1$, it is sufficient to ensure that $f(x)$ is continuous at $x=0$ and $x=1$. To this end, we let

$$\lim_{x\to 0} f(x) = f(0) = c+d, \text{ and } \lim_{x\to 1} f(x) = f(1) = ce^2+d.$$

For the above limits, we have

$$\begin{cases} \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} c(e^{2x})+d = c+d \\ \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^2+1 = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x+7)^{1/3} = 2 \\ \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} c(e^{2x})+d = ce^2+d \end{cases}$$

We thus take

$$\begin{cases} c+d=1 \\ ce^2+d=2. \end{cases}$$

Solving the above equation we obtain

$$c = \frac{1}{e^2-1}, d = \frac{e^2-2}{e^2-1}.$$

5. Which of the following functions are differentiable at $x=0$?

$$f(x) = |x|\sin(x), \quad g(x) = \ln(x^2), \quad h(x) = x+|x|, \quad j(x) = \begin{cases} x & \text{if } x<0, \\ \ln(1+x+3x^2) & \text{if } x \ge 0. \end{cases}$$

Solution:

(a) $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|\sin h}{h} = 0$. By the first principle, $f(x)$ is differentiable at $x=0$.

(b) The domain for $g(x) = \ln x^2$ is $\{x \in \mathbb{R} \mid x \neq 0\}$. Hence, $g(x)$ has no definition at $x=0$ and is not differentiable.

(c) We first have $\lim_{h\to 0} \frac{h(0+h)-h(0)}{h} = \lim_{h\to 0} \frac{h+|h|}{h}$. Then, we get $\lim_{h\to 0^+} \frac{h+|h|}{h} = 2$ and $\lim_{h\to 0^-} \frac{h+|h|}{h} = 0$. Hence, $\lim_{h\to 0} \frac{h(0+h)-h(0)}{h}$ does NOT exist and, by the first principle, the function $h(x)$ is NOT differentiable at $x=0$.

(d) First, we have

$$\lim_{h\to0^+}\frac{j(0+h)-j(0)}{h}=\lim_{h\to0^+}\frac{\ln(1+h+3h^2)}{h}\overset{L'Hospital's\ rule}{=}\lim_{h\to0^+}\frac{1+6h}{1+h+3h^2}=1.$$
$$\lim_{h\to0^-}\frac{j(0+h)-j(0)}{h}=\lim_{h\to0^-}\frac{h}{h}=1.$$

Hence, we conclude that $\lim_{h\to0}\frac{j(0+h)-j(0)}{h}$ exist and the function $j(x)$ is differentiable at $x=0$.

6. Find derivatives of the following functions $y=f(x)$:

(a) $f(x)=x[\sin(\ln(2x))-\cos(\ln(3x))]$, (b) $f(x)=\dfrac{x}{\sqrt{1+x^2}}$, (c) $f(x)=\tan^{-1}(x+\sqrt{1+x^2})$,

(d) $f(x)=(\sin x)^{\tan x}$, (e) $x^{1/3}+y^{1/3}=a^{1/3}$ $(a\neq0)$, (f) $\begin{cases}x=a\cos^3 t\\ y=a\tan^3 t\end{cases}$.

Solution:

(a) $f'(x)=\sin(\ln 2x)+\sin(\ln 3x)+\cos(\ln 2x)-\cos(\ln 3x)$

(b) $f'(x)=\frac{1}{(1+x^2)\sqrt{1+x^2}}$

(c) $f'(x)=\frac{1}{(\sqrt{1+x^2}+x)^2+1}(\frac{x}{\sqrt{1+x^2}}+1)$

(d) $f'(x)=(\frac{\ln\sin x}{\cos^2 x}+1)(\sin x)^{\tan x}$

(e) $f'(x)=-(a^{\frac{1}{3}}-x^{\frac{1}{3}})^2x^{-\frac{2}{3}}$

(f) $f'(x)=-\frac{\sin t}{\cos^6 t}$

7. Find the tangent line of the curve $y^2-3x^2+6x+2y=0$ at the point $(2,0)$.

Solution: First, by the implicit differentiation, we have

$$2y\frac{dy}{dx}-6x+6+2\frac{dy}{dx}=0,$$

which implies that

$$\frac{dy}{dx}=\frac{3x-3}{y+1}$$

Then we find the slope of the tangent line at $(2,0)$, note that

$$\text{Slope of tangent line}=\frac{dy}{dx}\mid_{x=2,y=0}=3.$$

Since the tangent line passes through the point $(2,0)$, so the equation of tangent line is given by

$$\frac{y-0}{x-2}=3,\ \text{i.e., } y=3x-6.$$

8. Find the tangent line of the curve $\begin{cases} x = 2t - \sqrt{3t}, \\ y = 3t^2 - t^3, \end{cases}$ at the point when $t = 1$.

   Solution: We first have

$$\frac{dx}{dt} = 2 - \frac{3}{2\sqrt{3t}}, \quad \frac{dy}{dt} = 6t - 3t^2,$$

   which implies that

$$\frac{dy}{dx} = \frac{6t - 3t^2}{2 - \frac{3}{2\sqrt{3t}}}.$$

   The slope of the tangent line is $\frac{dy}{dx} \mid_{t=1} = \frac{6}{4-\sqrt{3}}$. Moreover, when $t = 1$, we have $x = 2 - \sqrt{3}$ and $y = 2$. Hence, the equation of the tangent line is

$$\frac{y - 2}{x - (2 - \sqrt{3})} = \frac{6}{4 - \sqrt{3}}$$

9. Find two nonnegative numbers whose sum is 10 and so that the product of one number and the square of the other number is a minimum.

   Solution: We denote such two numbers as x and y. Then, we have $x \geq 0$, $y \geq 0$, and $x + y = 10$. To determine x and y, we solve the minimization problem

$$\min xy^2, \text{ subject to } x \geq 0, y \geq 0, x + y = 10.$$

   Since $y = 10 - x$ and $y \geq 0$, we have $10 \geq x \geq 0$ the above minimization problem can be equivalently written as

$$\min f(x) := x(10 - x)^2, \text{ subject to } 10 \geq x \geq 0$$

   To find the minimum of $x(10 - x)^2$ over $[0, 10]$, we first find all the turning points in $[0, 10]$. For this purpose, we solve

$$f'(x) = (10 - x)(10 - 3x) = 0,$$

   and get $x = 10$ or $x = \frac{10}{3}$. Then it is easy to check that $f(\frac{10}{3})$ has local maximum, and it is sufficient to check the function values at $x = 0$ and $x = 10$. We have that

$$f(0) = f(10) = 0.$$

   Hence, over $[0, 10]$, the function $f(x)$ has minimum at $x = 0$ or $x = 10$. In both cases, the desired two numbers are 0 and 10.

10. Let $f(x) = \dfrac{e^{-2x}}{(1 - x)^2}$.

(a) Show that
$$(1-x)f'(x) - 2xf(x) = 0.$$

Solution: One can compute that
$$f'(x) = \frac{2xe^{-2x}}{(1-x)^3},$$
and hence $(1-x)f'(x) = 2xf(x)$.

(b) Let n be a positive integer, show that
$$(1-x)f^{(n+1)}(x) - (n+2x)f^{(n)}(x) - 2nf^{(n-1)}(x) = 0.$$

Solution: To derive the desired result, we differentiate the equation obtained in (a) with respect to x for n times and have
$$\frac{d^n}{dx^n}[(1-x)f'(x)] - \frac{d^n}{dx^n}[2xf(x)] = 0. \tag{1}$$
To compute the derivatives, we apply the Leibnitz's rule and have
$$\frac{d^n}{dx^n}[(1-x)f'(x)] = \sum_{r=0}^{n} C_r^n \frac{d^r}{dx^r}(1-x)\frac{d^{n-r}}{dx^{n-r}}f'(x)$$
Since $\frac{d^r}{dx^r}(1-x) = 0$ when $r \geq 2$, we get
$$\begin{aligned} \frac{d^n}{dx^n}[(1-x)f'(x)] &= C_0^n(1-x)\frac{d^n}{dx^n}f'(x) + C_1^n\frac{d}{dx}(1-x)\frac{d^{n-1}}{dx^{n-1}}f'(x) \\ &= (1-x)f^{(n+1)}(x) - nf^{(n)}(x). \end{aligned} \tag{2}$$
In a similar way, we have
$$\frac{d^n}{dx^n}[2xf(x)] = 2xf^{(n)}(x) + 2nf^{(n-1)}(x). \tag{3}$$
Substitute the results (2) and (3) into equation (1), we get the desired result.

(c) Hence, or otherwise, find the Taylor series of $f(x)$ at $a = 3$ up to the term $(x-3)^3$.

Solution: The Taylor series of $f(x)$ at $a = 3$ up to the term $(x-3)^3$ is given by
$$f(x) = f(3) + f'(3)(x-3) + \frac{f^{(2)}(3)}{2!}(x-3)^2 + \frac{f^{(3)}(3)}{3!}(x-3)^3 + \dots$$
First, it is easy to get
$$f(3) = \frac{1}{4}e^{-6}, \quad f'(3) = -\frac{3}{4}e^{-6}.$$

To compute $f^{(2)}(3)$ and $f^{(3)}(3)$, we use the equation obtained in (b). To do so, we first take $x = 3$ and have

$$-2f^{(n+1)}(3) - (n+6)f^{(n)}(3) - 2nf^{(n-1)}(3) = 0.$$

Let $n = 1$, we have

$$-2f^{(2)}(3) - 7f'(3) - 2f(3) = 0,$$

which implies that

$$f^{(2)}(3) = \frac{19}{8}e^{-6}.$$

Let $n = 2$, we have

$$-2f^{(3)}(3) - 8f^{(2)}(3) - 4f'(3) = 0,$$

and thus

$$f^{(3)}(3) = -8e^{-6}.$$

Hence, the Taylor series of $f(x)$ at $a = 3$ up to the term $(x-3)^3$ is specified as

$$f(x) = \frac{1}{4}e^{-6} - \frac{3}{4}e^{-6}(x-3) + \frac{\frac{19}{8}e^{-6}}{2!}(x-3)^2 - \frac{8e^{-6}}{3!}(x-3)^3 + \ldots$$

End