Advanced Functions Exam - Part A

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Definitions:
V.A: Vertical Asymptotes
H.A: Horizontal Asymptote
O.A: Oblique Asymptote

Part A: Short Answer

1. List the asymptotes of the following:

a) $y = \frac{2+3x}{6-x}$

V.A: $x - 6 = 0, x = 6$ (Denominator equals zero)

H.A: $y = \dfrac{2}{-6} = -3$ (Degree of numerator equals degree of denominator, ratio of leading coefficients)

b) $y = \frac{2+3x}{x^2+5x-14}$

$$y = \frac{2+3x}{(x+7)(x-2)} \text{ (Factor out the denominator)}$$

V.A: $x = x + 7 = 0, x = -7, x - 2 = 0, x = 2$ (Denominator equals zero)

H.A: $y = 0$ (Degree of denominator is greater than numerator)

c) $y = \frac{x^2-5x}{x-1}$

$$y = \frac{x(x-5)}{x-1} \text{ (Factor out numerator)}$$

V.A: $x - 1 = 0, x = 1$ (Denominator equals zero)

$$\begin{array}{r|l} & x - 4 \\ \hline x-1 & x^2 - 5x \\ & -x^2 + x \\ \hline & -4x \\ & 4x - 4 \\ \hline & -4 \end{array}$$

O.A: $y = x - 4$ (Long division since numerator degree is higher)

2. Solve: $5 = \frac{2+3x}{6-x}$

$$\begin{aligned} 5(6-x) &= 2+3x && \text{Expand using distributive property} \\ 30-5x &= 2+3x && \text{Perform multiplication} \\ 30-2 &= 5x+3x && \text{Combine like terms} \\ 28 &= 5x+3x && \text{Subtract 2 from both sides} \\ 28 &= 8x && \text{Combine like terms} \\ x &= \frac{28}{8} && \text{Divide both sides by 8} \\ x &= \frac{7}{2} && \text{Simplify the fraction} \end{aligned}$$

3. Describe the function $y = \frac{x^3-5x}{6x^7-4x^3}$ as even, odd or neither.

$$\begin{aligned} y(-x) &= \frac{(-x)^3-5(-x)}{6(-x)^7-4(-x)^3} \\ &= \frac{-x^3+5x}{-6x^7+4x^3} \\ &= -\frac{x^3-5x}{6x^7-4x^3} \\ &= -y(x) \end{aligned}$$

Therefore, the function is odd.

4. An odd function has 3 vertical asymptotes, one is $x = 3$. What are the other two?

For an odd function, the asymptotes should be symmetric about the origin. Thus, if $x = 3$ is a vertical asymptote, then the other two are $x = -3$ and $x = 0$.

5. True or False? An even non-constant function that is continuous at $x = 0$ has a local max or min there.

True. By symmetry, an even function continuous at $x = 0$ has a local extremum at $x = 0$.

6. True or False? A reciprocal function has no roots.

True. A reciprocal function of the form $y = \dfrac{1}{f(x)}$ is undefined where $f(x) = 0$.

7. Convert to radians

a) $225°$ exact

$$\begin{aligned} 225° &= 225 \times \frac{\pi}{180} \\ &= \frac{225\pi}{180} \\ &= \frac{5\pi}{4} \end{aligned}$$

b) $164°$ to 3 decimal places

$$\begin{aligned} 164° &= 164 \times \frac{\pi}{180} \\ &= \frac{164\pi}{180} \\ &= \frac{41\pi}{45} \approx 2.862 \end{aligned}$$

8. Convert to degrees

a) $\frac{7\pi}{12}$

$$\begin{aligned} \frac{7\pi}{12} &= \frac{7\pi}{12} \times \frac{180}{\pi} \\ &= \frac{7 \times 180}{12} \\ &= 105° \end{aligned}$$

b) 2.34 (to 1 decimal place)

$$\begin{aligned} 2.34 &= 2.34 \times \frac{180}{\pi} \\ &= \frac{2.34 \times 180}{\pi} \\ &\approx 134.1° \end{aligned}$$

9. Determine the angle $x \in [0, 360°]$ and $\theta \in [0, 2\pi]$

a) $\sin x = -0.5$

$$x = 210°, 330°$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

b) $\cot x = -\sqrt{3}$

$$\cot x = \frac{\cos x}{\sin x} = -\sqrt{3}$$
$$x = 120°, 300°$$
$$\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$$

c) $\sec x = 2.5$ (to 1 decimal place)

$$\sec x = \frac{1}{\cos x} = 2.5 \implies \cos x = 0.4$$
$$x \approx 66.4°, 293.6°$$
$$\theta \approx 1.16, 5.12$$

d) $\cos\theta = \frac{1}{\sqrt{2}}$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4}$$

e) $\csc\theta = 2$

$$\csc\theta = \frac{1}{\sin\theta} \implies \sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

f) $\cot\theta = 2.5$ (to 1 decimal place)

$$\cot\theta = \frac{1}{\tan\theta} \implies \tan\theta = \frac{1}{2.5} = 0.4$$
$$\theta \approx 0.38, 3.52$$

10. Solve for the angle $x \in [0, 360°]$ and $\theta \in [0, 2\pi]$

a) $\csc^2 x = 2$

$$\csc^2 x = 2 \implies \sin^2 x = \frac{1}{2} \implies \sin x = \pm\frac{1}{\sqrt{2}}$$
$$x = 45°, 135°, 225°, 315°$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

b) $3\sec^2\theta - 4 = 0$

$$3\sec^2\theta = 4 \implies \sec^2\theta = \frac{4}{3} \implies \cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

c) $\sin 2x = 0.5$

$$2x = 30^\circ, 150^\circ, 390^\circ, 510^\circ$$

$$x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

11. Express as a simple trig function of the angle x.

a) $\sin\left(\frac{\pi}{2} + x\right)$

$$= \cos x \text{ (Using co-function identity)}$$

b) $\sec(-x)$

$$= \sec x \text{ (Even function property of secant)}$$

c) $\tan\left(x - \frac{3\pi}{2}\right)$

$$= \cot x \text{ (Using period property of tangent)}$$

12. Give the period, amplitude, phase shift and axis of $y = 5\sin(3x - \pi) - 7$

Period: $\frac{2\pi}{3}$ (Coefficient of x)

Amplitude: 5 (Coefficient of sine)

Phase shift: $\frac{\pi}{3}$ (Solving $3x - \pi = 0$)

Axis: $y = -7$ (Vertical shift down 7 units)

13. Simplify

a) $\sin 13 \cos 25 + \sin 25 \cos 13$

$$= \sin(13 + 25) \text{ (Using sum-to-product identities)}$$
$$= \sin 38$$

b) $2 \sin 50 \cos 50$

$$= \sin 100 \text{ (Using double angle identity for sine)}$$

c) $\cos^2 x - 1$

$$= -\sin^2 x \text{ (Using Pythagorean identity)}$$

d) $\cos(\alpha + 2b)\cos(\alpha + b)\sin(\alpha + b)$

$$= \frac{1}{2}[\cos((\alpha + 2b) - (\alpha + b)) + \cos((\alpha + 2b) + (\alpha + b))]\sin(\alpha + b) \qquad \text{(Product-to-sum identity)}$$
$$= \frac{1}{2}[\cos(b) + \cos(2\alpha + 3b)]\sin(\alpha + b) \qquad \text{(Simplify the cosine arguments)}$$
$$= \frac{1}{2}\cos(b)\sin(\alpha + b) + \frac{1}{2}\cos(2\alpha + 3b)\sin(\alpha + b) \qquad \text{(Distribute the sine term)}$$

14. Give an exact value for $\csc 15°$

$$\begin{aligned}
\csc 15° &= \frac{1}{\sin 15°} && \text{Definition of cosecant.} \\
&= \frac{1}{\sin(45° - 30°)} && \text{Expressing } 15° \text{ as } 45° - 30°. \\
&= \frac{1}{\sin 45° \cos 30° - \cos 45° \sin 30°} && \text{Applying the subtraction formula for sine.} \\
&= \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} && \text{Substituting values for sine and cosine.} \\
&= \frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}} && \text{Simplifying the expression in the denominator.} \\
&= \frac{4}{\sqrt{6} - \sqrt{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}} && \text{Rationalizing the denominator.} \\
&= \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} && \text{Simplifying after multiplication.} \\
&= \sqrt{6} + \sqrt{2} && \text{Final simplification.}
\end{aligned}$$

15. The population of trout in a river is given by $N(t) = 1000 + \frac{1000t^2}{t^2+100}$, $t \geq 0$.

a) What size will the trout population be after a long time?

Initial Expression

$$\lim_{t\to\infty} N(t) = 1000 + \frac{1000}{1 + \lim_{t\to\infty}\left[\frac{100}{t^2}\right]} \quad \text{(Evaluate the Inner Limit)}$$

As t approaches infinity, the term $\dfrac{100}{t^2}$ approaches 0.

$$\lim_{t\to\infty}\left[\frac{100}{t^2}\right] = 0$$

Substitute the Inner Limit Result

Substitute 0 for $\displaystyle\lim_{t\to\infty}\left[\frac{100}{t^2}\right]$ in the initial expression.

$$\lim_{t\to\infty} N(t) = 1000 + \frac{1000}{1+0} = 1000 + 1000$$

Final Result:

The limit of $N(t)$ as t approaches infinity is 2000 Trout.

$$\lim_{t\to\infty} N(t) = 2000 \text{ Trout}$$

b) How many trout were in the river to begin with?

$$\begin{aligned} N(0) &= 1000 + \frac{1000\cdot 0^2}{0^2+100} \\ &= 1000 + 0 \\ &= 1000 \end{aligned}$$

Therefore, the trout were in the river to begin with 1000

c) How fast is the trout population growing at three years?

$$N(3) = 1000 + \frac{1000\cdot 3}{3^2+100} = 1000 + \frac{9000}{109} \approx 1082.569$$

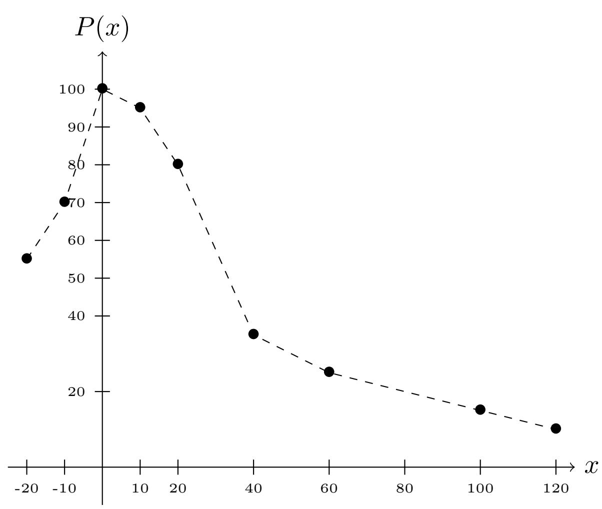
d) What is the average population growth for the first three years?

$$\begin{aligned}
\text{Average growth rate} &= \frac{N(3) - N(0)}{3 - 0} \\
N(3) &= 1000 + \frac{1000 \cdot 3^2}{3^2 + 100} \\
&= 1000 + \frac{9000}{109} \\
&= 1000 + 82.57 \\
&= 1082.57 \\
\text{Average growth rate} &= \frac{1082.57 - 1000}{3} \\
&= \frac{82.57}{3} \\
&\approx 27.52 \text{ trout per year}
\end{aligned}$$

16. The probability, P of hitting a target x feet away is graphed below.

a) What is the rate of change of P as player moves 10 ft to 90 ft away?

b) How fast is the probability changing when the participant is 50 ft away?



a) The average rate of change of P from 10 ft to 90 ft is calculated as follows:

$$\text{Average rate of change} = \frac{P(90) - P(10)}{90 - 10}$$

After plugging in the values from the graph, we get:

$$\text{Average rate of change} = \frac{80 - 30}{90 - 10} = \frac{50}{80} = 0.625$$

So, the average rate of change is 0.625 approximately.

b) The instantaneous rate of change of P at 50 ft is the derivative of P at that point:

$$P'(50) = \lim_{h \to 0} \frac{P(50 + h) - P(50)}{h}$$

$$\begin{aligned}
P'(50) &= \lim_{h \to 0} \frac{P(50 + h) - P(50)}{h} \\
&= \lim_{h \to 0} \frac{P(50) + P'(50)h - P(50)}{h} \quad \text{(Using the definition of the derivative)} \\
&= \lim_{h \to 0} \frac{P'(50)h}{h} \quad \text{(Canceling out } P(50)\text{)} \\
&= \lim_{h \to 0} P'(50) \\
&= P'(50)
\end{aligned}$$

Or

$$P'(50) = \frac{d}{dx} P(x) \bigg|_{x=50}$$

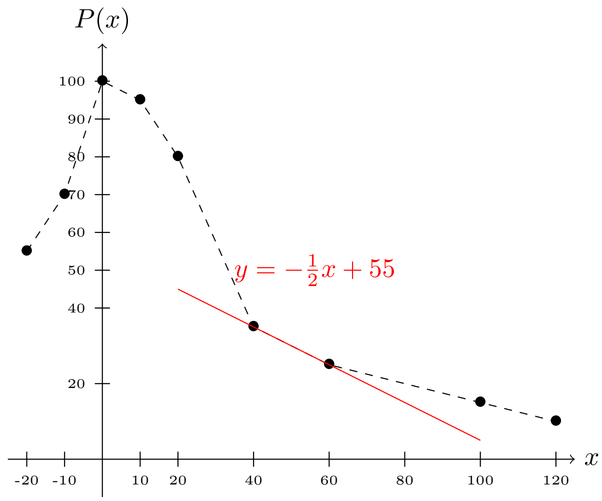
In simpler terms, we're figuring out how fast the probability changes at 50 ft by looking at the slope of the curve at that point. We do this by drawing a line that just touches the curve at 50 ft and finding its steepness. The result is $P'(50)$, which tells us the rate of change at that exact spot.

$$\begin{aligned}
\text{Slope} &= \frac{\text{Change in } y}{\text{Change in } x} \\
&= \frac{-10}{20} \\
&= -\frac{1}{2}
\end{aligned}$$

Using the point-slope form of a line:

$$\begin{aligned}
y - 35 &= -\frac{1}{2}(x - 40) \\
y - 35 &= -\frac{1}{2}x + 20 \\
y &= -\frac{1}{2}x + 55
\end{aligned}$$

So, the equation of the tangent line passing through the points $(40, 35)$ and $(60, 25)$ is $y = -\frac{1}{2}x + 55$.



Advanced Functions Exam - Part B

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May 23rd, 2024

Part B: Long Answer

1. Solve ($\theta \in [0, 2\pi]$)

   a) $2x^4 + 4x + 4 = x^3 + 9x^2$

$$\begin{aligned} 2x^4 + 4x + 4 = x^3 + 9x^2 \quad &\text{(Rearrange all terms to one side)} \\ 2x^4 - x^3 + 9x^2 + 4x + 4 = 0 \quad &\text{(Form polynomial equation)} \end{aligned}$$

   b) $x^4 + x^3 + 3x + 7 > 5x^2 + 7$

$$\begin{aligned} x^4 + x^3 + 3x + 7 > 5x^2 + 7 \quad &\text{(Rearrange all terms to one side)} \\ x^4 + x^3 - 5x^2 + 3x > 0 & \\ x(x^3 + x^2 - 5x + 3) = 0 \quad &\text{(Factor out } x\text{)} \\ x = 0 \quad &\text{(One solution)} \end{aligned}$$

c) $\frac{x^2}{2x^2-5x+3}+\frac{2x}{4x^2-9}=\frac{5}{4x^3-4x^2-9x+9}$

$$\frac{x^2}{2x^2-5x+3}+\frac{2x}{4x^2-9}=\frac{5}{4x^3-4x^2-9x+9}$$
$$2x^2-5x+3=(2x-3)(x-1)$$
$$4x^2-9=(2x-3)(2x+3)$$
$$\frac{x^2}{(2x-3)(x-1)}+\frac{2x}{(2x-3)(2x+3)}=\frac{5}{4x^3-4x^2-9x+9}$$
$$\frac{x^2(2x+3)}{(2x-3)(x-1)(2x+3)}+\frac{2x(x-1)}{(2x-3)(x-1)(2x+3)}=\frac{5}{4x^3-4x^2-9x+9}$$
$$\frac{2x^3+3x^2+2x^2-2x}{(2x-3)(x-1)(2x+3)}=\frac{5}{4x^3-4x^2-9x+9}$$
$$\frac{2x^3+5x^2-2x}{(2x-3)(x-1)(2x+3)}=\frac{5}{4x^3-4x^2-9x+9}$$
$$\frac{2x^3+5x^2-2x}{(2x-3)(x-1)(2x+3)}=\frac{5}{(2x-1)(x-3)}$$

d) $\frac{-4x+1}{x^2-4}<\frac{2x+1}{x+2}$

$$\frac{-4x+1}{(x-2)(x+2)}<\frac{2x+1}{x+2}$$
$$\frac{-4x+1-(2x+1)(x-2)}{(x-2)(x+2)}<0$$
$$\frac{-4x+1-2x^2+4x-2x+2}{(x-2)(x+2)}<0$$
$$\frac{-2x^2+3}{(x-2)(x+2)}<0$$
$$x=\pm 2,\sqrt{\frac{3}{2}}$$
$$\text{Test intervals:} x\in(-\infty,-2)\cup(2,\sqrt{\frac{3}{2}})$$

e) $8.72(0.93)^{x+3} + 17 = 22$

$$\begin{aligned}
8.72(0.93)^{x+3} &= 5 \quad \text{(Isolate the exponential term)} \\
(0.93)^{x+3} &= \frac{5}{8.72} \\
\ln((0.93)^{x+3}) &= \ln\left(\frac{5}{8.72}\right) \\
(x+3)\ln(0.93) &= \ln\left(\frac{5}{8.72}\right) \\
x+3 &= \frac{\ln\left(\frac{5}{8.72}\right)}{\ln(0.93)} \\
x &= \frac{\ln(\frac{5}{8.72})}{\ln(0.93)} - 3 \\
x &\approx 2.10
\end{aligned}$$

f) $\log_{12}(x-3) + \log_{12}(x+1) = 1$

$$\begin{aligned}
\log_{12}((x-3)(x+1)) &= 1 \\
(x-3)(x+1) &= 12 \\
x^2 - 2x - 3 &= 12 \\
x^2 - 2x - 15 &= 0 \\
(x-5)(x+3) &= 0 \\
x &= 5, -3 \\
\text{Check solutions:} x &= 5 \quad \text{(valid)}, \quad x = -3 \quad \text{(invalid)}
\end{aligned}$$

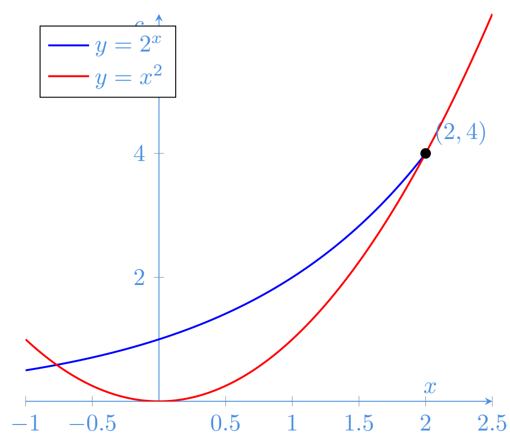
g) $\cos(2\theta) + 5 = 4\sin^2(\theta) + \cos(\theta) + 2$

$$\begin{aligned}
\cos(2\theta) + 5 &= 4\sin^2(\theta) + \cos(\theta) + 2 \\
\cos(2\theta) &= 2\cos^2(\theta) - 1 \quad \text{(Double Angle)} \\
2\cos^2(\theta) - 1 + 5 &= 4(1 - \cos^2(\theta)) + \cos(\theta) + 2 \\
2\cos^2(\theta) + 4 &= 4 - 4\cos^2(\theta) + \cos(\theta) + 2 \\
6\cos^2(\theta) - \cos(\theta) &= 2 \\
6\cos^2(\theta) - \cos(\theta) - 2 &= 0 \quad \text{(Let u } = \cos\theta) \\
6u^2 - u - 2 &= 0
\end{aligned}$$

Solve quadratic in $\cos(\theta)$

$$\begin{aligned}
\cos(\theta) &= \frac{1 \pm \sqrt{1+48}}{12} \\
\cos(\theta) &= \frac{1 \pm 7}{12} \\
\cos(\theta) &= \frac{8}{12}, \frac{-6}{12} \\
\cos(\theta) &= \frac{2}{3}, -\frac{1}{2} \\
\theta = \cos^{-1}\left(\frac{2}{3}\right), \cos^{-1}\left(-\frac{1}{2}\right) &\implies \theta = 53.5^\circ, 133.3^\circ
\end{aligned}$$

h) $2^x - x^2 = 0$



Logarithmic form:

$$x \log_2(2) = 2\log_2(x)$$

Integers Solutions:

$$x = 2, x = 4$$

More possible solutions can be found here. Lambert W function

2. Show that the line $y = -10x + 20$ is tangent to the curve $y = x^4 - 4x^3 - 5x^2 + 26x - 16$

Given the functions $f(x) = x^4 - 4x^3 - 5x^2 + 26x - 16$ and $g(x) = -10x + 20$, we find:

$$f(x) - g(x) = x^4 - 4x^3 - 5x^2 + 36x - 36$$

We factor $f(x) - g(x)$ using synthetic division. Possible factors are $\pm 1, 2, 3, 6, 12, 18, 36$. First, we try $x = 2$:

$$\begin{array}{c|ccccc} & 1 & -4 & -5 & 36 & -36 \\ 2 & & 2 & -4 & -18 & 36 \\ \hline & 1 & -2 & -9 & 18 & 0 \end{array}$$

This gives us $(x-2)(x^3 - 2x^2 - 9x + 18)$. We use synthetic division again on $x^3 - 2x^2 - 9x + 18$:

$$\begin{array}{c|cccc} & 1 & -2 & -9 & 18 \\ 2 & & 2 & 0 & -18 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

This results in:

$$(x-2)^2(x^2 - 9) = (x-2)^2(x+3)(x-3)$$

Finally, we find the tangent line $y = -10x + 20$ at $x = 2$:

$$y = -10(2) + 20 = 0$$

$\therefore$ The tangent points are $(2, 0)$.

Additional way using Calculus

This question seems to be a calculus question too, so I decided to start with the curve:

First, find the first derivative of the curve:

$y' = 4x^3 - 12x^2 - 10x + 26$

Then, set the derivative equal to the slope of the line:

$-10 = 4x^3 - 12x^2 - 10x + 26$

$4x^3 - 12x^2 - 10x + 36 = 0$

Greatest Common Factor is 2:

$2(2x^3 - 6x^2 - 5x + 18) = 0$

I found possible numbers for this function is $(x-2)=0$, which is 2 because I plugged and I got 0. Now, the next step comes the synthetic division,

$$\begin{array}{c|cccc} & 2 & -6 & -5 & 18 \\ 2 & & 4 & -4 & -18 \\ \hline & 2 & -2 & -9 & 0 \end{array}$$

$\therefore 2(x-2)(2x^2-2x-9)=0$

Verify the point using the original function:

$y = x^4 - 4x^3 - 5x^2 + 26x - 16 \quad \text{at } x = 2$

$y = (2)^4 - 4(2)^3 - 5(2)^2 + 26(2) - 16$

$y = 16 - 32 - 20 + 52 - 16 = 0$

Point of tangency: $(2, 0)$

3. Assume the cosine addition formula has been proven graphically.

a) Prove the sine addition formula.

$$\begin{aligned}
\cos(a) &= \cos(a+b-b) \\
&= \cos(a+b)\cos(b) + \sin(a+b)\sin(b) \quad \text{(Subtracting } b \text{ from both sides)} \\
&= (\cos(a)\cos(b) - \sin(a)\sin(b))\cos(b) + \sin(a+b)\sin(b) \quad \text{(Cosine addition formula)} \\
&= \sin(a+b)\sin(b) \quad \text{(Expanding)}
\end{aligned}$$

Trig identity: $\cos(a)\cos(b) - \sin(a)\sin(b) = \sin(a+b)\sin(b)$.

$$\begin{aligned}
&= \cos(a) - \cos(a)\cos^2(b) + \sin(a)\sin(b)\cos(b) \\
&= (1-\cos^2(b))\cos(a) + \sin(a)\sin(b)\cos(b) \quad \text{(Using identities)} \\
&= \sin(b)\cdot(\cos(a)\sin(b) + \sin(a)\cos(b)) \quad \text{(Substituting } \sin^2(b) = 1-\cos^2(b)\text{)}
\end{aligned}$$

Hence,

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

Note: If $\sin(b) = 0$, then $b = \pi \cdot n$, where n is an integer. This implies $\cos(b) = (-1)^n$, as cosine alternates between -1 and 1 at integer multiples of π, effectively alternating between ends of the circle.

$\therefore$ $\sin(a+b)$ simplifies to $(-1)^n \sin(a)$, which equals $\cos(b)\sin(a)$. Hence, the formula $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$ still holds true.

b) Derive the tangent addition formula.

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} \quad \text{(Definition of tangent)} \\
&= \frac{\sin a\cos b + \cos a\sin b}{\cos a\cos b - \sin a\sin b} \quad \text{(Expansion of sine and cosine)} \\
&= \frac{\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b}}{\frac{\cos a\cos b-\sin a\sin b}{\cos a\cos b}} \quad \text{(Divide numerator and denominator by } \cos a\cos b\text{)} \\
&= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a}{\cos a}\frac{\sin b}{\cos b}} \quad \text{(Rewrite in terms of tangent)} \\
&= \frac{\tan a + \tan b}{1 - \tan a\tan b} \quad \therefore \text{(Definition of tangent)}
\end{aligned}$$

4. Stan invests \$1000 at 3% compounded semi-annually and \$1500 at 1.8% compounded annually. When will the two investments be equal?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

For \$1000 compounded semi-annually:

$$\begin{aligned}
A_1 &= 1000\left(1+\frac{0.03}{2}\right)^{2t} \\
&= 1000\,(1+0.015)^{2t}
\end{aligned}$$

For \$1500 compounded annually:

$$\begin{aligned}
A_2 &= 1500\,(1+0.018)^{t} \\
&= 1500(1.018)^{t}
\end{aligned}$$

$$\begin{aligned}
\text{Set } A_1 &= A_2 \\
1000(1.015)^{2t} &= 1500(1.018)^{t} \\
\left(\frac{1.015^{2t}}{1.018^{t}}\right) &= \frac{1500}{1000} \\
\left(\frac{(1.015)^2}{1.018}\right)^{t} &= 1.5 \\
\left(\frac{1.030225}{1.018}\right)^{t} &= 1.5 \\
(1.012085)^{t} &= 1.5 \\
t &= \frac{\ln(1.5)}{\ln(1.012085)} \\
\therefore t &\approx 34.56 \text{ years}
\end{aligned}$$

5. Determine whether the following are equations or identities. If they are equations, solve them; otherwise, prove the identity. $x \in [0, 360°]$

a) $4\cos^2 x = 3 - 2\sin^2 x$

$$\begin{aligned}
4\cos^2 x &= 3 - 2\sin^2 x \\
4\cos^2 x + 2\sin^2 x &= 3 \\
4\cos^2 x + 2(1-\cos^2 x) &= 3 \\
4\cos^2 x + 2 - 2\cos^2 x &= 3 \\
2\cos^2 x &= 1 \\
\cos^2 x &= \frac{1}{2} \\
\cos x &= \pm\frac{1}{\sqrt{2}} \\
\therefore x &= 45^\circ, 135^\circ, 225^\circ, 315^\circ
\end{aligned}$$

b) $\sin^4 x + \cos^4 x = \sin^2 x(\csc^2 x - 2\cos^2 x)$

$$\begin{aligned}
LHS \quad & RHS \\
\sin^4 x + \cos^4 x &= \sin^2 x(\csc^2 x - 2\cos^2 x) \\
&= \sin^2 x\left(\frac{1}{\sin^2 x} - 2\cos^2 x\right) \\
&= \sin^2 x\left(\frac{1 - 2\cos^2 x\sin^2 x}{\sin^2 x}\right) \\
&= 1 - 2\cos^2 x\sin^2 x \\
&= \sin^4 x + \cos^4 x \quad \text{(Identity)} \\
\therefore RHS = LHS
\end{aligned}$$

c) $\frac{1-\sin 2x}{\cos 2x} = \frac{\cos 2x}{1+\sin 2x}$

$$\begin{aligned}
\frac{1-\sin 2x}{\cos 2x} &= \frac{\cos 2x}{1+\sin 2x} \\
(1-\sin 2x)(1+\sin 2x) &= (\cos 2x)^2 \quad \text{(Cross-multiply)} \\
1 - \sin^2 2x &= \cos^2 2x \\
\cos^2 2x &= \cos^2 2x \quad \text{(Identity)}
\end{aligned}$$

d) $(\cot x)(\csc x)(\tan x)(\cos x) = \cos 2x + 2\sin^2 x$

$$\begin{aligned}
(\cot x)(\csc x)(\tan x)(\cos x) &= \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\cdot\frac{\sin x}{\cos x}\cdot\cos x \\
&= \frac{\cos x \cdot 1 \cdot \sin x \cdot \cos x}{\sin x \cdot \sin x \cdot \cos x} \\
&= \cos 2x + 2\sin^2 x \quad \text{(Prove identity)}
\end{aligned}$$

6. If $\sin x = \frac{12}{13}$, $\quad 0 < x < \frac{\pi}{2}$ and $\cos y = \frac{4}{5}$, $\quad -\frac{\pi}{2} < y < 0$, determine an exact value for $\sin[2(x-y)]$.

$$\begin{aligned}
\sin x &= \frac{12}{13}, \quad \cos x = \frac{5}{13} \\
\cos y &= \frac{4}{5}, \quad \sin y = -\frac{3}{5} \\
\sin[2(x-y)] &= \sin 2x \cos 2y - \cos 2x \sin 2y \\
&= 2 \sin x \cos x \cdot 2\cos^2 y - 2\cos^2 x \cdot 2 \sin^2 y \\
&= 2\left(\frac{12}{13} \cdot \frac{5}{13}\right) \cdot 2\left(\frac{4}{5}\right)^2 - 2\left(\frac{5}{13}\right)^2 \cdot 2\left(\frac{3}{5}\right)^2 \\
&= \frac{120}{169} \cdot \frac{32}{25} - \frac{50}{169} \cdot \frac{18}{25} \\
&= \frac{3840}{4225} - \frac{900}{4225} \\
&= \frac{2940}{4225}
\end{aligned}$$

7. A mass on the end of a spring is pulled so that its distance from the rest position is initially 3cm. After being released, the mass oscillates while the spring contracts and expands. The motion of the mass is sinusoidal with a period of 3 seconds.

a) Give the theoretical equation for the mass if it is distance from rest in terms of the time t, in seconds, assuming no energy is lost with each cycle.

$$y(t) = 3\cos\left(\frac{2\pi}{3}t\right)$$

b) If the spring loses 5% of its energy with each cycle, give the equation for this motion.

$$y(t) = 3\,(0.95)^{\frac{t}{3}} \cos\left(\frac{2\pi}{3}t\right)$$

8. The frets on a guitar are placed so that they make the correct vibrating string length for the note of music. We are interested in how the vibrating string length changes for each fret position. Below is the length from the bridge to each fret position.

Fret Number	0	1	2	3	4	5	6	7	8	9	10	11	12
Length (mm)	660	623	588	555	524	494	467	440	416	392	370	350	330

Using one of the methods from class determine the relationship between the fret number and the length

The formula to calculate the distance from the bridge to the nth fret (D_n) is given by:

$$D_n = D_0 \times \left(\frac{1}{2}\right)^{\frac{n}{12}}$$

Where:

- D_0 is the scale length (the distance from the bridge to the nut, which is the 0th fret).
- n is the fret number.

Using the given scale length of 660 mm, the lengths for the first few frets are calculated as follows:

$$D_2 = 660 \times \left(\frac{1}{2}\right)^{\frac{2}{12}} \approx 588.00 \text{ mm}$$

$$D_5 = 660 \times \left(\frac{1}{2}\right)^{\frac{5}{12}} \approx 494.00 \text{ mm}$$

$$D_9 = 660 \times \left(\frac{1}{2}\right)^{\frac{9}{12}} \approx 416.00 \text{ mm}$$

$$D_{12} = 660 \times \left(\frac{1}{2}\right)^{\frac{12}{12}} = 330.00 \text{ mm}$$

In other words, the formula to calculate the distance from the bridge to the nth fret (D_n) is also can be log, which is:

$$D_n = D_0 \times 2^{\frac{-n}{12}}$$

Where:

- D_0 is the scale length (the distance from the bridge to the nut, which is the 0th fret).
- n is the fret number.

This can also be expressed using logarithms as follows:

$$D_n = D_0 \times 2^{\log_2\left(\frac{1}{2}\right)^{\frac{n}{12}}}$$

Same thing, using the given scale length of 660 mm, the lengths for the first few frets are calculated as follows:

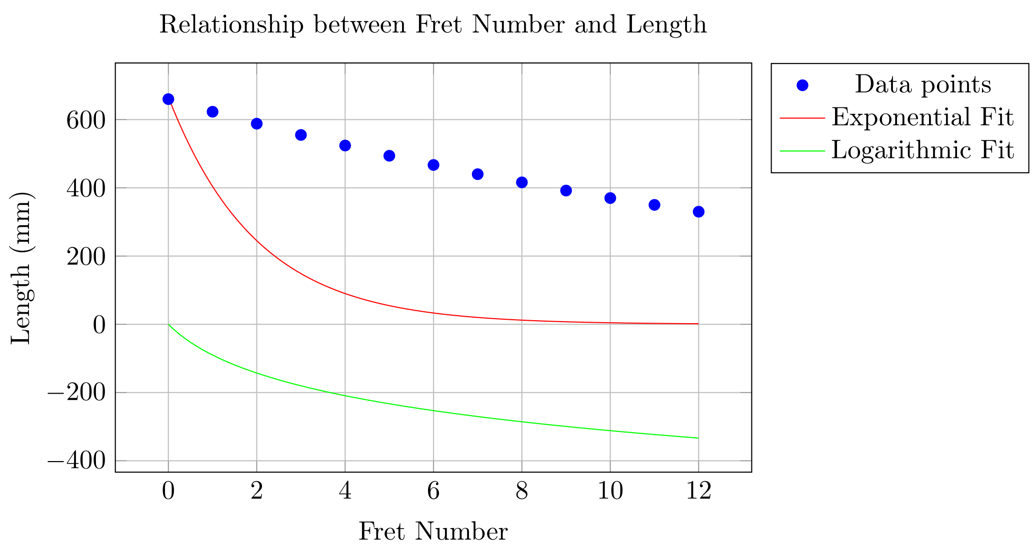
$$D_2 = 660 \times 2^{\log_2\left(\frac{1}{2}\right)^{\frac{2}{12}}} \approx 588.00 \text{ mm}$$

$$D_5 = 660 \times 2^{\log_2\left(\frac{1}{2}\right)^{\frac{5}{12}}} \approx 494.00 \text{ mm}$$

$$D_9 = 660 \times 2^{\log_2\left(\frac{1}{2}\right)^{\frac{9}{12}}} \approx 416.00 \text{ mm}$$

$$D_{12} = 660 \times 2^{\log_2\left(\frac{1}{2}\right)^{\frac{12}{12}}} = 330.00 \text{ mm}$$

Graph - Relationship between Fret Number and Length



9. I have \$100 dollars to invest, and I want to know how to allocate it between two possible entrepreneurs, Alpha and Beta, to maximize my total annual return.

Alpha: If I give x dollars to Alpha, my annual return is $A(x) = \frac{x(200-x)}{1000}$.

Beta: If I give the remaining $(100 - x)$ dollars to Beta, my return is $B(x) = r(100 - x)$ dollars per year. That is, Beta simply pays me annual interest at rate r.

a) Take $r = 12\%$ (that is, $r = 0.12$). Using the given formula for $A(x)$, find the allocation that maximizes the sum of my returns through Alpha and Beta. Illustrate your solution on a graph.

$$
\begin{aligned}
\text{Total return: } R(x) &= A(x) + B(x) \\
&= \frac{x(200-x)}{1000} + 0.12(100-x) \\
&= \frac{200x - x^2}{1000} + 12 - 0.12x \\
&= \frac{200x - x^2 + 12000 - 120x}{1000} \\
&= \frac{-x^2 + 80x + 12000}{1000} \\
&= -\frac{x^2}{1000} + \frac{80x}{1000} + 12 \\
&= -\frac{x^2}{1000} + 0.08x + 12 \\
\text{My next idea comes take derivative:} & \\
R'(x) &= -\frac{2x}{1000} + 0.08 \\
&= -0.002x + 0.08 \\
\text{Set derivative to zero to find critical points:} & \\
0 &= -0.002x + 0.08 \\
0.002x &= 0.08 \\
x &= \frac{0.08}{0.002} \\
x &= 40 \\
\text{Check second derivative:} & \\
R''(x) &= -0.002 \\
&< 0 \quad \text{(Maximum)}
\end{aligned}
$$

$\therefore$ The optimal allocation is $x = 40$ dollars to Alpha and 60 dollars to Beta.

b) In case the optimal allocation is split, find a formula for the optimal allocation x in terms of the interest rate r. What interest rates r would compel me to give everything to Beta?

$$
\begin{aligned}
\text{Total return: } R(x) &= A(x) + B(x) \\
&= \frac{x(200-x)}{1000} + r(100-x) \\
&= \frac{200x - x^2}{1000} + 100r - rx \\
&= \frac{-x^2 + (200 - 1000r)x + 100000r}{1000}
\end{aligned}
$$

My next idea comes take derivative for this too:

$$
\begin{aligned}
R'(x) &= -\frac{2x}{1000} + \frac{200 - 1000r}{1000} \quad \text{(Divide)} \\
&= -0.002x + 0.2 - r
\end{aligned}
$$

Set derivative to zero to find critical points:

$$
\begin{aligned}
0 &= -0.002x + 0.2 - r \\
0.002x &= 0.2 - r \\
x &= \frac{0.2 - r}{0.002} \\
x &= \frac{200(0.2 - r)}{0.002} \\
\text{For } x \leq 0 &: \\
0.2 - r &\leq 0 \\
r &\geq 0.2 \\
\text{For } x \geq 100 &: \\
0.2 - r &\geq 0.002 \cdot 100 \\
r &\leq 0.2
\end{aligned}
$$

$\therefore$ If $r \geq 0.2$, I should invest everything in Beta.

10. After you eat something that contains sugar, the pH or acid level in your mouth changes. This can be modeled by the function

$$
L(m) = \frac{-20.4m}{m^2 + 36} + 6.5,
$$

where **L** is the **pH** level and **m** is the number of minutes that have elapsed since eating. Find the average rate of change from 1.5 minutes to 3 minutes, and find the instantaneous rate of change at 3 minutes.

Average rate of change:

$$\begin{aligned}
L(3) &= \frac{-20.4 \cdot 3}{3^2 + 36} + 6.5 \\
&= \frac{-61.2}{45} + 6.5 \\
&= -1.36 + 6.5 \\
&= 5.14 \\
L(1.5) &= \frac{-20.4 \cdot 1.5}{1.5^2 + 36} + 6.5 \\
&= \frac{-30.6}{38.25} + 6.5 \\
&= -0.8 + 6.5 \\
&= 5.7
\end{aligned}$$

$\therefore$ Average rate of change:

$$\begin{aligned}
&= \frac{L(3) - L(1.5)}{3 - 1.5} \\
&= \frac{5.14 - 5.7}{1.5} \\
&= \frac{-0.56}{1.5} \\
&= -0.373
\end{aligned}$$

Instantaneous rate of change(Literally calculus):

$$\begin{aligned}
L'(m) &= \frac{d}{dm}\left(\frac{-20.4m}{m^2 + 36} + 6.5\right) && \text{(Find derivative)} \\
&= \frac{-20.4(m^2 + 36) - (-20.4m) \cdot 2m}{(m^2 + 36)^2} && \text{(Quotient rule)} \\
&= \frac{-20.4m^2 - 734.4 + 40.8m^2}{(m^2 + 36)^2} && \text{(Simplify)} \\
&= \frac{20.4m^2 - 734.4}{(m^2 + 36)^2} \\
L'(3) &= \frac{20.4(3)^2 - 734.4}{(3^2 + 36)^2} && \text{(Plug 3)} \\
&= \frac{183.6 - 734.4}{45^2} \\
&= \frac{-550.8}{2025} \\
&= -0.272
\end{aligned}$$

$\therefore$ Instantaneous rate of change is (3, -0.272).

Advanced Functions Exam - Part C & D

Kensukeken

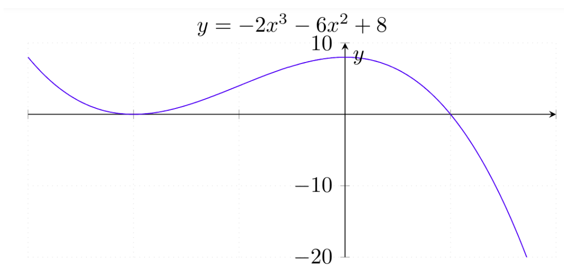
May 23rd, 2024

Part C: Graphing

1. List the key properties of each graph. Use these to create a sketch. Indicate at least two key points on each curve.

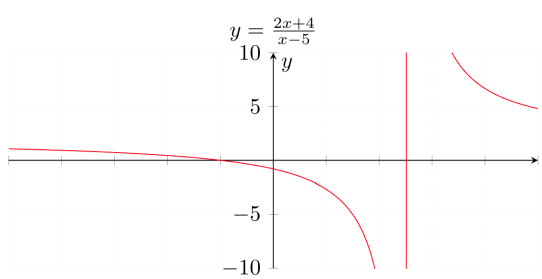
a) $y = -2x^3 - 6x^2 + 8$

- Cubic function with leading coefficient -2, facing downwards.
- Key points: $(0, 8)$ (y-intercept), $(-1, 0)$ (crosses x-axis).



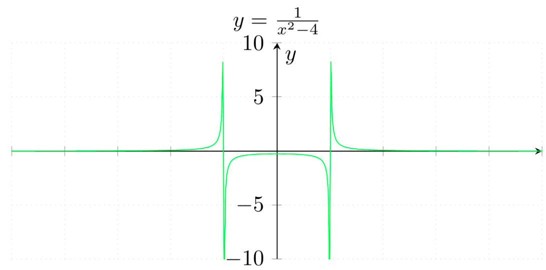
b) $y = \frac{2x+4}{x-5}$

- Rational function with vertical asymptote at $x = 5$.
- Key points: $(0, -0.8)$ (y-intercept), $(5, \text{undefined})$ (vertical asymptote).



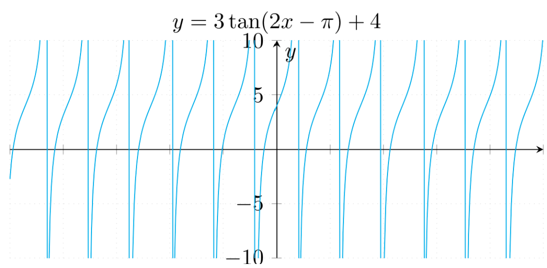
c) $y = \frac{1}{x^2-4}$

- Rational function with vertical asymptotes at $x = \pm 2$.
- Key points: $(0, 0.25)$ (y-intercept), $(2, \text{undefined})$ (vertical asymptote).



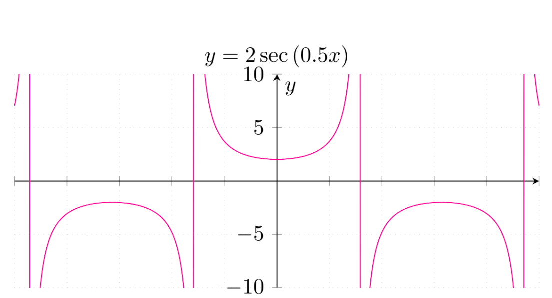
e) $y = 3\tan(2x - \pi) + 4$

- Tangent function with vertical asymptotes.
- Key points: $\left(\frac{\pi}{4}, 4\right)$, $\left(\frac{3\pi}{4}, 4\right)$.



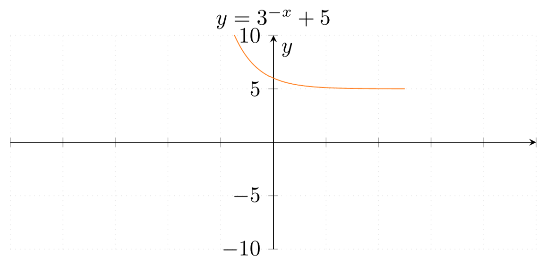
f) $y = 2\sec(0.5x)$

- Secant function with vertical asymptotes.
- Key points: $(0, 2)$ (y-intercept), $(4\pi, 2)$.



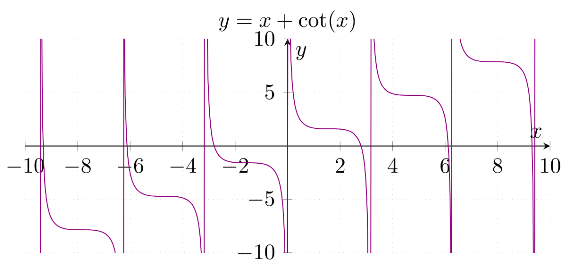
g) $y = 3^{-x} + 5$

- Exponential function approaching $y = 5$ as $x \to \infty$ and $y = 6$ as $x \to -\infty$.
- Key points: $(0, 6)$ (y-intercept), $(1, \frac{8}{3})$.

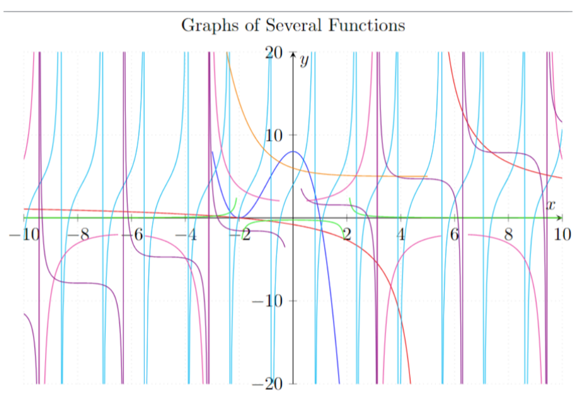


h) $y = x + \cot(x)$

- Sum of linear and cotangent functions.
- Key points: $(0, 0)$ (origin), (π, π).



Mixed Graphs (All Together)



Part D: Reverse Graphing

1. For each sketch below;
   a) List the key properties of the graph.
   b) What type of relationship is being shown?
   c) Use a guess and check method to determine the equation of the curve.

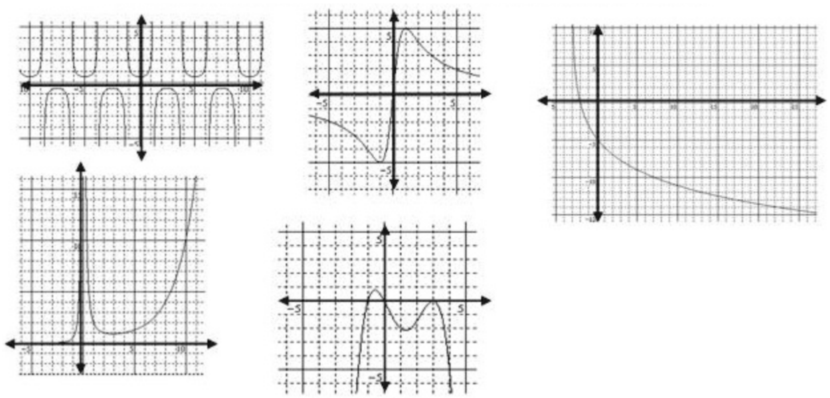


Figure 1: Part D: Revere Graphing

Sources

Dear Ms. G. Jaksic,

I hope this letter finds you well. I wanted to provide some additional resources regarding the graphs we've been discussing in class. Below, you'll find helpful links that I utilized to create graphs using the LaTeX typesetting language.

You can access the links here:

- Desmos.com..
- My work on Desmons.
- MHF4U - Advanced Functions Class Notes
- Learn Tikzpicture here.

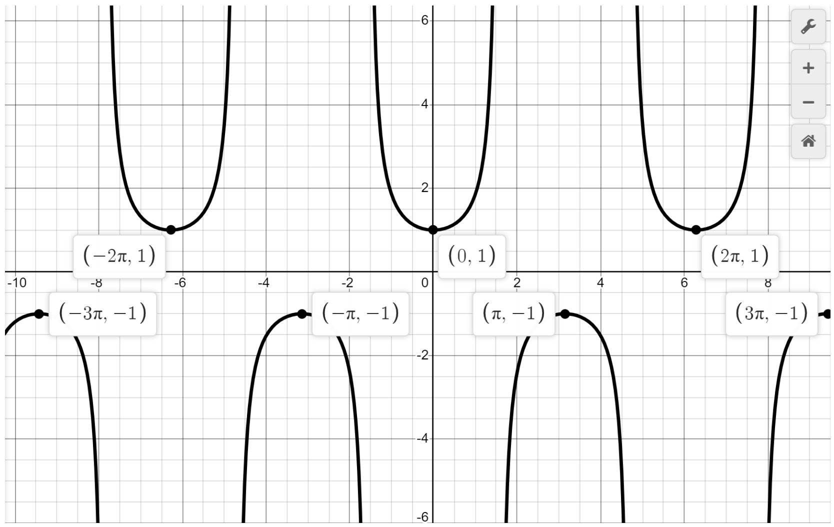
Also, I made a graph in Desmos to make sure it's correct. I've attached the pictures for you to see as well.

Thank you for your guidance and support throughout the course.

Best regards,

Hia Al Saleh

0.1 Graph 1

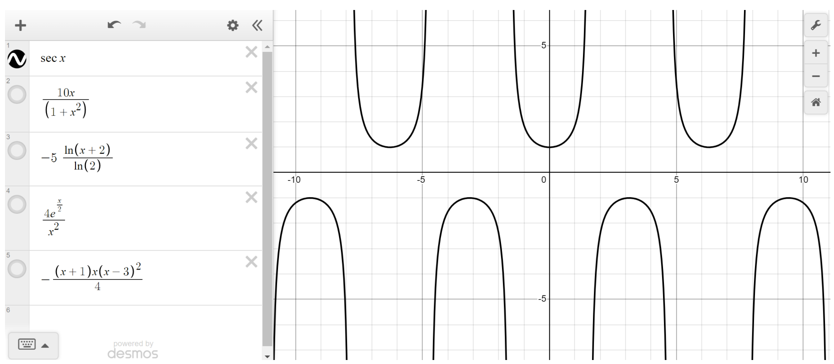


a) **Key properties:** Periodic function with asymptotes where $\cos x = 0$.

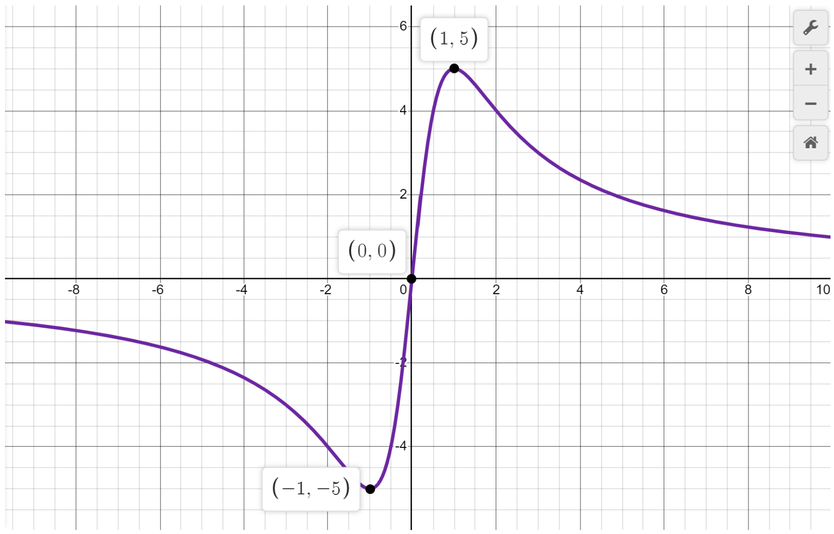
b) **Relationship:** Reciprocal of cosine function.

c) **Guess The Equation:** I found my equation for this graph is

$$y = \sec x$$



0.2 Graph 2

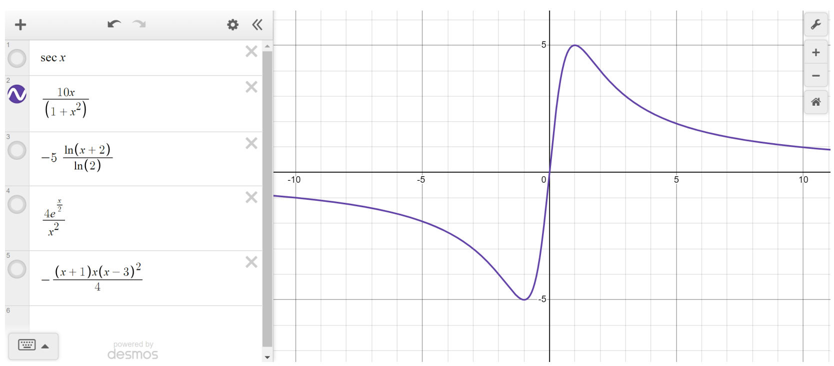


a) **Key properties:** Rational function with vertical asymptotes at $x = \pm\sqrt{1}$.

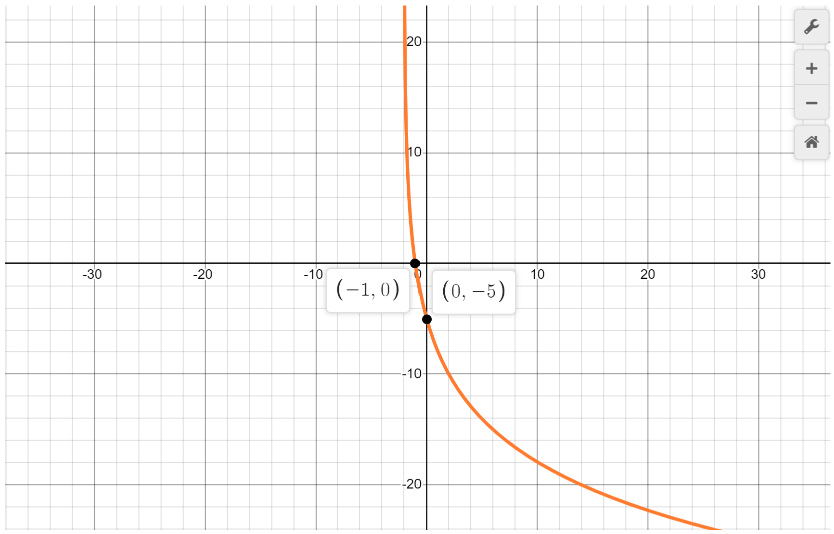
b) **Relationship:** Rational function.

c) **Guess The Equation:** I found my equation for this graph is

$$y = \frac{10x}{(1 + x^2)}$$



0.3 Graph 3

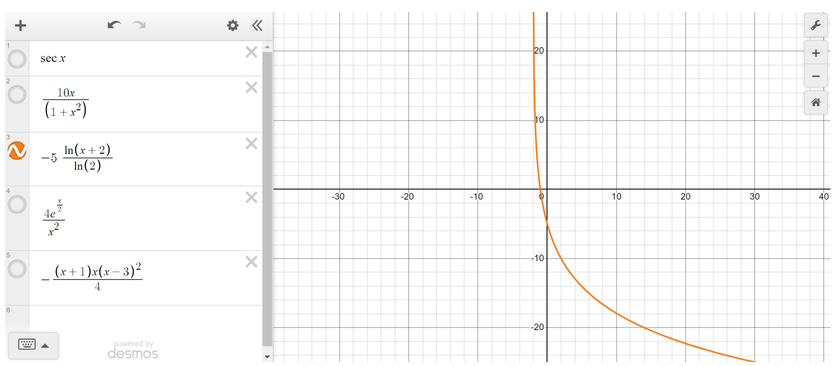


a) **Key properties:** Logarithmic function with domain $x > -2$.

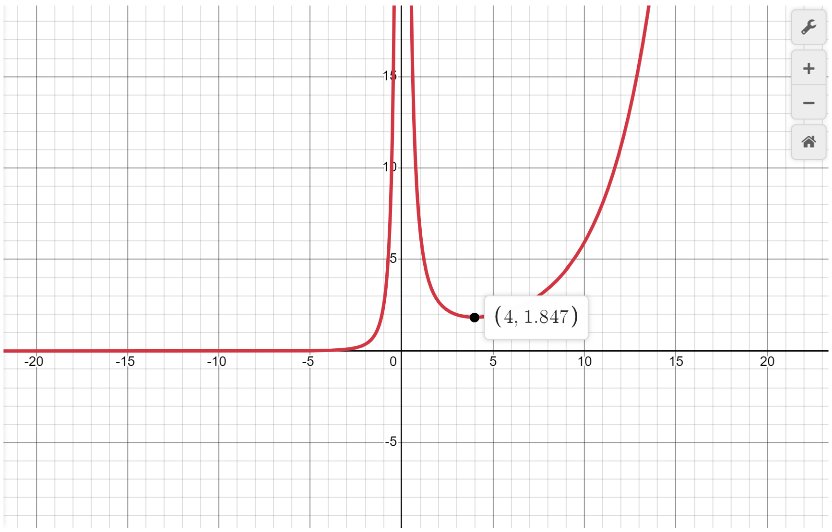
b) **Relationship:** Logarithmic function.

c) **Guess The Equation:** I found my equation for this graph is

$$y = -5\,\frac{\ln(x+2)}{\ln(2)}$$



0.4 Graph 4

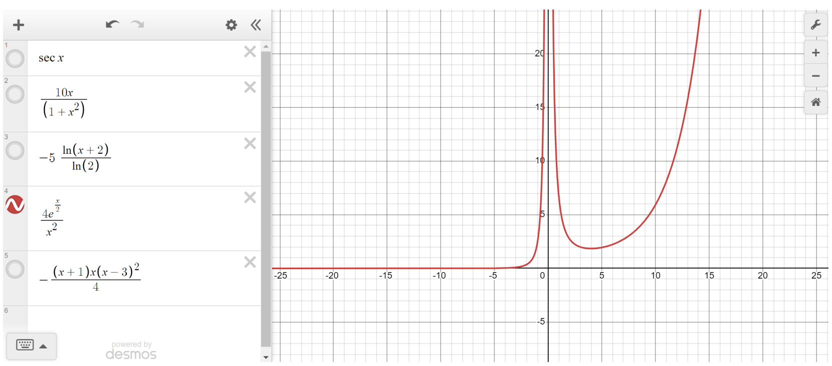


a) **Key properties:** Exponential function with vertical asymptote at $x = 0$.

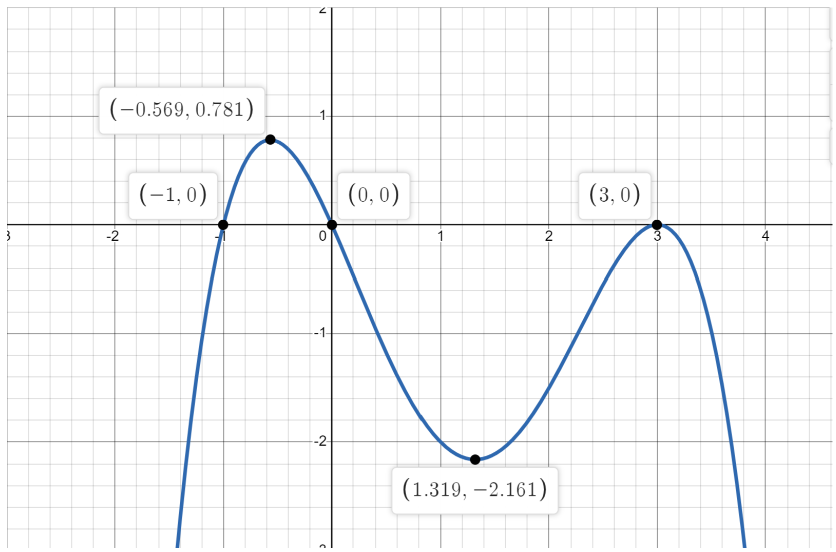
b) **Relationship:** Natural exponential function.

c) **Guess The Equation:** I found my equation for this graph using e ☺

$$y = \frac{4e^{\frac{x}{2}}}{x^2}$$



0.5 Graph 5

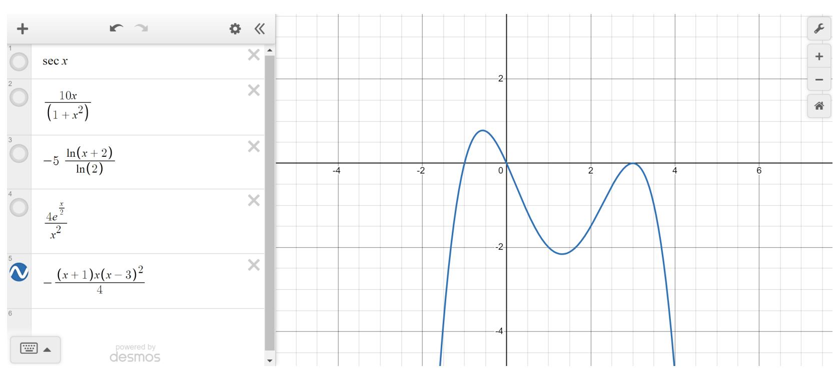


a) **Key properties:** Polynomial function with roots at $x = -1$ and $x = 3$.

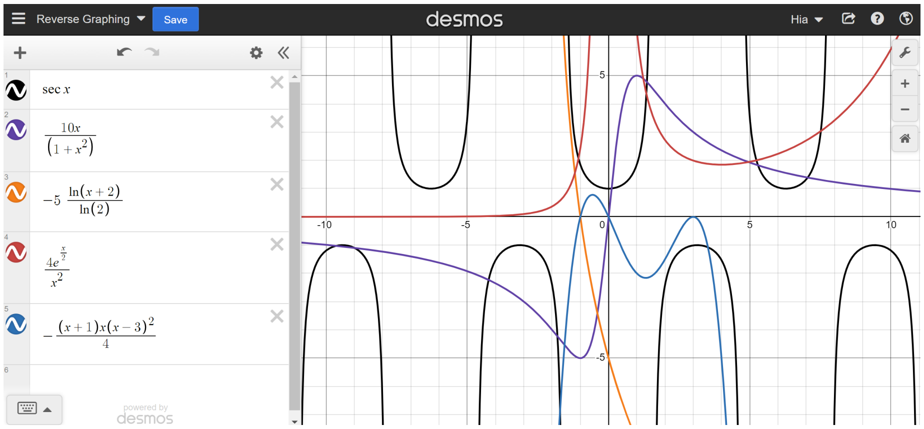
b) **Relationship:** Polynomial function.

c) **Guess The Equation:** I found my equation for this graph using e :)

$$y = -\frac{(x+1)\,x\,(x-3)^2}{4}$$



All equations guessed from graphs are added together at the end.



FSE - Part A: 10 pages
FSE - Part B: 14 pages
FSE - Part C & D: 10 pages
In total 34 pages :)

Get full course notes access here at
https://github.com/Kensukeken/MHF4U-Advanced-Functions-Notes