MHF4U

Advanced Functions
Grade 12 University

Unit 6 Logs & Exponentials

8 Video Lessons

Allow **no more** than **15** class days for this unit!
This includes time for review and to write the test.

Lesson #	Title	Practice Questions	Date Completed
1	Logarithms	Page 328 #1-4, 6-8, 10, 11, 13, 14, 15	
2	Power Law of Logarithms	Page 347 #1-10, 12, 13, 15, 17, 20	
3	Equivalent Exponential Expressions	Page 368 #(1-6)eop 7, 10, 11, 13, 14	
4	Techniques for Solving Exponential Equations	Page 375 #2-4, 7, 8, 11, 15, 16	
5	Log Laws	Page 384 #(1-10)eop, 17, 18, 20	
6	Techniques for Solving Log and Exponential Equations More examples	***NOTE - Leave more time for these lessons - the questions are harder!*** Page 391 #1-3 (pick 'n choose), 5-8, 10	
7	The Logarithmic Scale in the Sciences	Page 353 #1ac, 2ac, 3, 4, 7-10, 12-15	
8	Modeling with Exponentials	Page 405 #7, 8, 10 Page 408 #8, 16, 17	

Test Written on : ____________________________________

Topic : Logarithms

Goal : I know what the definition of a logarithm is, and how to evaluate them.

Logarithms

As we saw last class, the logarithm function is the inverse of an exponential. But exactly does this mean?

$\log_2 16$

- is read "log base 2 of 16"
- asks the question "If I have a base of 2, what exponent will give me 16?"

$2^x = 16$

In function terms...

$f(x) = 2^x$ — returns the value of a power when given an exponent.

$f(x) = \log_2 x$ — returns the exponent when given the value of a power

Example 1. Going between log and exponential form. Write the equivalent log or exponential equation.

a) $\log_2 16 = 4$ ⬌ $2^4 = 16$

b) $7^2 = 49$ ⬌ $\log_7 49 = 2$

c) $5^x = 11$ ⬌ $\log_5 11 = x$

d) $\log_x 42 = 7$ ⬌ $x^7 = 42$

$(x^7)^{1/7} = (42)^{1/7}$

$x \cong 1.7$

Example 2. Evaluate the following logs...

a) $\log_{10}100 = 2$ $10^x = 100$

b) $\log_2 64 = 6$ $2^x = 64$

c) $\log_2(1/2) = -1$ $2^x = 1/2$

d) $\log_3(1/27) = -3$ $3^x = \frac{1}{27}$

Example 3. Estimate $\log_3 10$

$3^x = 10$

$3^2 = 9$ $3^3 = 27$ $2 < x < 3$

$x \cong 2.1$

The calculator has two log buttons

log and ln

2.7183

Example 4. Use a calculator to evaluate, then write and equivalent exponential equation.

a) log 52 $= 1.716$ $10^{1.716} = 52$

b) log 24 $= 1.3979$ $10^{1.3979} = 24$

c) ln 12 $= 2.4849$ $10^{2.4849} = 12$

Practice Questions - Page 328 #1-4, 6-8, 10, 11, 13, 14, 15

Topic : Power Law of Logarithms

Goal : I know the power law of logarithms and how I can use it to evaluate logarithmic expressions.

Power Law of Logarithms

Evaluate each pair of logarithms on your calculator - What do you notice?

$2^4 = 16$

a) 4 log2 $= 1.2$ log16 $= 1.2$

$3^2 = 9$

b) 2 log3 $= 0.95$ log 9 $= 0.95$

$5^2 = 25$

c) 2 log 5 $= 1.40$ log 25 $= 1.40$

$\log 3^5 = 5 \log 3$

What do you notice? Can you come up with a relationship?

They are equivalent expression

$\log_b x^n = n \log_b x,\ b>0,\ b \neq 1,\ x>0,\ n \in R$

Proof : Let w = $\log_b x$

$$\log_a(x^n) = n\log_a(x)$$

Proof : Let $\lambda = \log_a(x)$, thus $a^\lambda = x$

$$\log_a(x^n) = k$$

Hence we have $a^k = x^n$

Using $a^\lambda = x$, we have $a^k = a^{\lambda n}$

$$a^k = a^{\lambda n} \implies k = \lambda n \implies k = n\log_a(x)$$

$$\therefore k = n\log_a(x) \implies \log_a(x^n) = n\log_a(x)$$

What use is the power law?

1. It can help to simplify an expression involving logs

2. It gives us a way to get a variable out of the exponent and turns it into a factor.

Example 1. Evaluate the following logarithms

a) $\log_2 8^{10}$

Method 1: change 8 to 2^3

$\log_2 (2^3)^{10}$

$= \log_2 2^{30}$

$= 30$

Method 2: power rule

$\log_2 8^{10}$

$10 \log_2 8$

$= 10(3)$

$= 30$

a) $\log_3 \sqrt{27}$

$= \log_3 27^{1/2}$

$= \frac{1}{2} \log_3 27$

$= \frac{1}{2}(3) \Rightarrow \frac{3}{2}$

Example 2. Solve by first taking the log of both sides and using the power law of logarithms

a) $5^t = 15625$

Taking $\log_{10}$ of both sides

$\log 5^t = \log 15625$

$\frac{t \log 5}{\log 5} = \frac{\log 15625}{\log 5}$

$t = 6$

b) $1000 = 2000(1+0.2)^n$

$\frac{1000}{2000} = \frac{2000(1+0.2)^n}{200}$

$0.5 = (1.2)^n$

$\log 0.5 = \log (1.2)^n$

$\frac{\log 0.5}{\log(1.2)} = \frac{n \log(1.2)}{\log(1.2)}$

$n = -3.8$

Example 3. Evaluate $\log_3 54$

First let the expression equal a variable and turn it into the equivalent exponential expression.

for $\log_3 54 = x$

$3^x = 54$

$\log 3^x = \log 54$

$x \log 3 = \log 54$

$x = \frac{\log 54}{\log 3} \Rightarrow \log_3 54 = \frac{\log 54}{\log 3}$

$x = 3.6$

This actually gives us the change of base formula...

$$\log_b m = \frac{\log m}{\log b}$$

Example 4. Use the change of base formula to evaluate $\log_8 254$

$= \frac{\log 254}{\log 8}$

$= 2.7$

Example 5. An investment of 2000 earns 2% interest, compounded yearly. A formula to represent this situation is

principal years

$$A = 2000\,(1.02)^n$$

Rate

where **A** is the amount of the investment and **n** is the number of years of the investment. How long before the investment doubles?

$$\frac{4000}{2000} = \frac{\cancel{2000}(1.02)^n}{\cancel{2000}}$$

$$2 = (1.02)^n$$

$$\log 2 = \log (1.02)^n$$

$$\log 2 = n \log (1.02)$$

$$n = \frac{\log 2}{\log(1.02)}$$

$$n \approx 35$$

Practice Questions - Page 347 #1-10, 12, 13, 15, 17, 20

Topic : Equivalent Forms of Exponential Equations

Goal : I can rewrite powers with the same base in order to simplify or solve for exponents.

Equivalent Forms of Exponential Equations

Example 1. Rewrite the following using a base of 3.

$$81$$

$$= 9(9)$$

$$= 3^2(3^2)$$

$$= 3^4$$

$$\sqrt[4]{27}^{\,3}$$

$$= (3^3)^{\frac{3}{4}}$$

$$= 3^{\frac{9}{4}}$$

$$\sqrt{81} \times \sqrt[3]{27}^{\,2}$$

$$= (3^{\cancel{4}})^{\frac{1}{\cancel{2}}} \times (3^{\cancel{3}})^{\frac{2}{\cancel{3}}}$$

$$= 3^2 \times 3^2$$

$$= 3^4$$

$$36$$

$$3^x = 36$$

$$\log 3^x = \log 36$$

$$x \log 3 = \log 36$$

$$x = \frac{\log 36}{\log 3}$$

$$x = \frac{\log 6^2}{\log 3}$$

$$x = \frac{2 \log 6}{\log 3}$$

$$36 = 3^{\frac{2\log 6}{\log 3}}$$

Example 2. Solve for x by first writing as powers with the same base.

$$25^{x-1} = 5^{3x}$$

$$= (5^2)^{x-1} = 5^{3x}$$

$$5^{2x-2} = 5^{3x}$$

Equating the exponents..

$$\cancel{2x} - 1 = 3x \quad (-2x)$$

$$-1 = x$$

Example 3. Solve for x by first writing as powers with the same base.

$$27^x = 9^{2x-3}$$

$$(3^3)^x = (3^2)^{x-3}$$

$$3^{3x} = 3^{4x-6}$$

Equating exponent...

$$3x = 4x - 6$$

$$-x = -6$$

$$x = 6$$

Since the base are the same, the exponet must be equal

Practice Questions - Page 368 #(1-6)eop 7, 10, 11, 13, 14

Topic : Techniques for Solving Exponential Equations

Goal : I know various strategies to help me solve exponential equations.

Techniques for Solving Exponential Equations

Last lesson we solved exponential equations by forcing them to have an equal base and equating their exponents. But what about a situation like this...

Problem :

Example 1.

$$5^{2x+4} = 3^{x-7}$$

Solution :

$$(2x+4)\log 5 = (x-7)\log 3$$

$$2x\log 5 + 4\log 5 = x\log 3 - 7\log 3$$

$$2x\log 5 - x\log 3 = -7\log 3 + 4\log 5$$

$$\frac{x\cancel{(2\log 5 - \log 3)}}{\cancel{2\log 5 - \log 3}} = \frac{-7\log 3 + 4\log 5}{-2\log 5 - \log 3}$$

$$x = \frac{-7\log 3 - 4\log 5}{2\log 5 - \log 3}$$

$$x \approx -6.7$$

Example 2. Solve for x.

$$2^x - 2^{-x} = 4$$

There's no immediate method to solve this, we have to make a couple of adjustments in order to solve. Notice it's somewhat similar to a QUADRATIC in that the variables are in decreasing order. We are going to multiply through by 2^x and this will become more apparent.

$$2^2$$

$$2^x(2^x) - 2^x(2^{-x}) = 2^x(4)$$

$$2^{2x} - 2^0 = 4(2^x)$$

$$(2^x)^2 - 4(2^x) - 1 = 0$$

let $u = 2^x$

$$n^2 - 4n - 1 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{4 \pm \sqrt{16 - 4(1)(-1)}}{2}$$

$$= \frac{4 \pm \sqrt{20}}{2}$$

$$= \frac{4 \pm \cancel{2}\sqrt{5}}{\cancel{2}}$$

$$= 2 \pm \sqrt{5}$$

$\therefore 2^x = 2 + \sqrt{5}$

or $2^x = 2 - \sqrt{5}$

$$\log 2^x = \log 2 + \sqrt{5}$$

$$x \log 2 = \log 2 + \sqrt{5}$$

$$x = \frac{\log 2 + \sqrt{5}}{\log 2}$$

$$x \approx 2.08$$

Example 3. Using a Half - Life

An archeological discovery of an unknown plant fossil, has 1/8 the amount of radioactive carbon as plants have today. If the half-life of the carbon is 5730 years, how old is the fossil?

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Equating the exponent

$$3 = \frac{t}{5730}$$

$t = 17190$ years

Practice Questions - Page 375 #2-4, 7, 8, 11, 15, 16

Topic : Product and Quotient Laws of Logarithms

Goal : I know how to use the product and quotient laws of logarithms to simplify logarithmic expressions.

Product and Quotient law of Logarithms

Remember that there were laws for multiplying and dividing powers with the same base-- we need to adapt these for use with logarithms.

Let's let $x = \log_b m$ and $y = \log_b n$

By definition that means

$b^x = m$ and $b^y = n$

So what happens when we multiply them?

$mn = b^x b^y$

$mn = b^{x+y}$ (exp rules)

$\log_b mn = \log_b b^{x+y}$ (taking the log both sides)

$\log_b mn = (x+y)\log_b b$ (power rule)

$\log_b mn = x+y$ (def$\underline{n}$ of logarithm)

$\log_b mn = \log_b m + \log_b n$

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

MHF4U U6L5 Log Laws

Example 1. Simplify using the laws of logarithms

a) log 5 + log 10 $= \log(5 \times 10)$

$= \log 50$

b) log 12 - log 2 $= \log\left(\frac{12}{2}\right)$

$= \log 6$

Example 2. Simplify each expression.

a) log (5a) + log 10 - log(2b) $= \log(50a) - \log(2b)$

$= \log\left(\frac{50a}{2b}\right)$

$= \log\left(\frac{25a}{b}\right)$

b) log x + 6log y + 3log y $= \log x + \log y^6 + \log y^3$

$= \log(x y^6 y^3)$

$= \log(x y^9)$

Example 3. Evaluate

a) log 50 + log 10 - log 5 $= \log 500 - \log 5$

$= \log 100$

$= 2$

$\log\left(\frac{50(10)}{5}\right)$

b) $4\log_{12} 2 + 2\log_{12} 3$ $= \log_{12} 2^4(3^2)$

$= \log_{12} 16(9)$

$= \log_{12} 144$

$= 2$

Example 4. Simplify and state any restrictions

a) log $(2x^2 + 9x - 5)$ - log$(x + 5)$

$= \log\left(\frac{2x^2+9x-5}{x+5}\right)$

$= \log \frac{(2x-1)\cancel{(x+5)}}{\cancel{x+5}}$, $x \neq -5$, $x > -5$

$= \log 2x-1$

b) log (x+3) + log (2x-5)

$= \log\left[(x+3)(2x-5)\right]$

$= \log(2x^2-9x-15)$

$x+3>0$
$x>-3$

$2x-5>0$
$2x>5$
$x>\frac{5}{2}$

Practice Questions - Page 384 #(1-10)eop, 17, 18, 20

Topic : Techniques for Solving Logarithmic Equations

Goal : I can apply various methods of solving logarithmic equations using the laws of logarithms.

Techniques for Solving Logarithmic Equations

There are three main techniques involved in solving logarithmic equations.

1. Use the definition of a logarithm to rewrite the equation as an exponential. Then solve using the techniques for exponential equations.

2. First simplify using the laws of logarithms, and then rewrite as an exponential to solve.

3. First simplify using the laws of logarithms and then equate the arguments of the logs on both sides of the equal sign.

Example 1. Use the definition to change into an exponential.

Solve for n : $\log_3 (n^2 - 3n + 5) = 2$

$3^2 = n^2 - 3n + 5$ (by defn)

$9 = n^2 - 3n + 5$

$0 = n^2 - 3n - 4$ (rearranging)

$0 = (n+1)(n-4)$ (factoring)

$\therefore n = -1, n = 4$

Example 2. Simplify and then apply the definition.

Solve for p : $\log(p + 5) - \log(p + 1) = 3$

$\log\left(\frac{p+5}{p+1}\right) = 3$ (by Quot. law)

$10^3 = \frac{p+5}{p+1}$ (by defn)

$1000 = \frac{p+5}{p+1}$

$100p + 1000 = p + 5$

$999p = -995$

$p = -\frac{995}{999}$

$= -0.996$

Example 3. Equating the arguments.

Solve for x : $\log(2x^2 - 7x - 4) = \log(2x + 16)$

Equating the arguments . . .

$$2x^2 - 7x - 4 = 2x + 16$$

$$2x^2 - 9x - 20 = 0$$

using Quadratic formula . .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{9 \pm \sqrt{81 - 4(2)(-20)}}{4}$$

$$= \frac{9 \pm \sqrt{241}}{4}$$

$$\approx 6.1, -1.6$$

Practice Questions - Page 391 #1-3 (pick 'n choose), 5-8, 10

Topic : Solving More Log and Exponential Equations

Goal : to explore some more complicated examples of log and exponential functions

More Log and Exponential Equations

Solve the following equations...

Example 1. An exponential with a product in it.

$$7(1.06^x) = 5.2$$

Taking the log of both sides

$$\log[7(1.06^x)] = \log 5.2$$

$$\log 7 + x\log 1.06 = \log 5.2$$

$$\frac{x\cancel{\log 1.06}}{\cancel{\log 1.06}} = \frac{\log 5.2 - \log 7}{\log 1.06}$$

$$x = \frac{\log 5.2 - \log 7}{\log 1.06}$$

$$x \approx -5.1$$

Example 2. Logarithms where an answer doesn't work out.

$$\log_6 x + \log_6(x+1) = 1$$

$$\log_6[x(x+1)] = 1$$

$$6^1 = x(x+1)$$

$$6 = x^2 + x$$

$$0 = x^2 + x - 6$$

$$0 = (x+3)(x-2)$$

$$x = -3, \quad x = 2$$

inadmissible

Practice Questions - Handout Page

MHF 4U More Practice with Log & Exponential Equations

1. **Solve the following:**
   - a. $\log_2 x = 2\log_2 4$ b. $\log_3 x = 4\log_3 3$
   - c. $2\log_5 x = \log_5 36$ d. $2\log x = 4\log 7$

2. **Solve the following. Give the answer correct to two decimal places.**
   - a. $3^x = 5$ b. $5^x = 6$ c. $2^x - 1 = 4$ d. $7 = 12 - 4^x$

3. **Solve for x.**
   - a. $\log x = 2\log 3 + 3\log 2$ b. $\log x + \log 3 = \log 1 + \log 4$
   - c. $\log x^2 = 3\log 4 - 2\log 2$ d. $\log\sqrt{x} = \log 1 - 2\log 3$
   - e. $\log x^{\frac{1}{2}} - \log x^{\frac{1}{3}} = \log 2$ f. $\log_4(x+2) + \log_4(x-3) = \log_4 9$

4. **Solve the following:**
   - a. $\log_6(x+1) + \log_6(x+2) = 1$
   - b. $\log_7(x+2) + \log_7(x-4) = 1$
   - c. $\log_2(x+2) = 3 - \log_2 x$
   - d. $\log_4 x + \log_4(x+6) = 2$
   - e. $\log_5(2x+2) - \log_5(x-1) = \log_5(x+1)$

1. a. 16 **b.** 81 **c.** 6 **d.** 49
2. a. 1.46 **b.** 1.11 **c.** 2.32 **d.** 1.16
3. a. 72 **b.** $\frac{4}{3}$ **c.** ± 4 **d.** $\frac{1}{81}$ **e.** 64 **f.** $\frac{1+\sqrt{61}}{2}$
4. a. 1 **b.** 5 **c.** 2 **d.** 2 **e.** 3

12. Solve each of the following:
    - a. $12^{2x-3} = 144$ b. $7^{x+9} = 56$ c. $5^{3x+4} = 25$
    - d. $10^{2x+1} = 95$ e. $6^{x+5} = 71.4$ f. $3^{5-2x} = 875$

13. Solve each of the following. Write your answers correct to two decimal places.
    - a. $2 \times 3^x = 7 \times 5^x$ b. $12^x = 4 \times 8^{2x}$
    - c. $4.6 \times 1.06^{2x+3} = 5 \times 3^x$ d. $2.67 \times 7.38^x = 9.36^{5x-2}$
    - e. $12 \times 6^{2x-1} = 11^{x+3}$ f. $7 \times 0.43^{2x} = 9 \times 6^{-x}$
    - g. $5^x + 3^{2x} = 92$ h. $4 \times 5^x - 3(0.4)^{2x} = 11$

14. Write each of the following as a single logarithm.
    - a. $\frac{1}{3}\log_a x + \frac{1}{4}\log_a y - \frac{2}{5}\log_a w$ b. $(4\log_5 x - 2\log_5 y) \div 3\log_5 w$

12. a. 2.5 **b.** −6.93 **c.** $-\frac{2}{3}$ **d.** 0.4889 **e.** −2.6178 **f.** −0.5831
13. a. −2.45 **b.** −0.83 **c.** 0.09 **d.** 0.59 **e.** 5.5 **f.** 2.4 **g.** 1.93
h. 0.64
14. a. $\log_a\left[\frac{\sqrt[3]{x}\sqrt[4]{y}}{\sqrt[5]{w^2}}\right]$ **b.** $\log_5\left(\frac{x^4}{y^2}\right) \div \log_5 w^3$

MHF4U U6L7 The Logarithmic Scale in Science

Topic : Applications of Logarithms

Goal : I can extend my knowledge of logarithms to real life applications in the various sciences.

The Logarithm Scale in the Physical Sciences

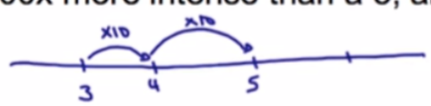
A. EarthQuakes

The Richter Scale defines magnitude of an earthquake as

$$M = \log\left(\frac{I}{I_0}\right)$$

Where I is the earthquake measured and I_0 is the intensity of a reference quake.

Put simply, each step on the scale represents a 10x intensity change. An earthquake that was a 7 on the scale is 10x more intense than one that was a 6, 100x more intense than a 5, and 1000x more intense than a 4.



Example 1. The California earthquake that interrupted the world series in 1989 measured 6.9 on the Richter scale. The quake that caused the 2004 tsunami in Indonesia measured 9.2. How much more powerful was that Indonesian quake?

Intuitively : $9.2 - 6.9 = 2.3$

$10^{2.3} = 200X$ more intense

Using the Definition :

$$M_2 - M_1 = \log\left(\frac{I_2}{I_1}\right)$$

$$9.2 - 6.9 = \log\left(\frac{I_2 \rightarrow 2004}{I_1 \rightarrow 1989}\right)$$

$$2.3 = \log\left(\frac{I_2}{I_1}\right)$$

$$10^{2.3} = \frac{I_2}{I_1} \Rightarrow 200 = \frac{I_2}{I_1}$$

$$200 I_1 = I_2$$

B. Sound

The decibel scale compares sound

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I is the intensity of the sound being measured and I_0 is the threshold of human hearing (the quietest sound we can hear)

Example 2. A sound is 5000 times more intense than one that is just audible. How many decibels is the sound?

$\underline{I = 5000 I_0}$ *

$$L = 10\log\left(\frac{I}{I_0}\right)$$

$$= 10\log\left(\frac{5000\cancel{I_0}}{\cancel{I_0}}\right) \text{ from } *$$

$$= 10\log 5000$$

$$\approx 37 \text{ dB}$$

Example 3. A jet engine emits a 160 dB sound, while Niagara Falls is 90 dB. How many times louder than Niagara falls is a jet engine?

$$\underset{\text{Jet}}{L_2} - \underset{\text{N.F}}{L_1} = 10\log\left(\frac{I_2 \ (\text{Jet})}{I_1 \ (\text{N.F})}\right)$$

$$160 - 90 = 10\log\left(\frac{I_2}{I_1}\right)$$

$$70 = 10\log\left(\frac{I_2}{I_1}\right)$$

$$7 = \log\left(\frac{I_2}{I_1}\right)$$

$$10^7 = \frac{I_2}{I_1}$$

$$10^2 I_{NF} = I_{jet}$$

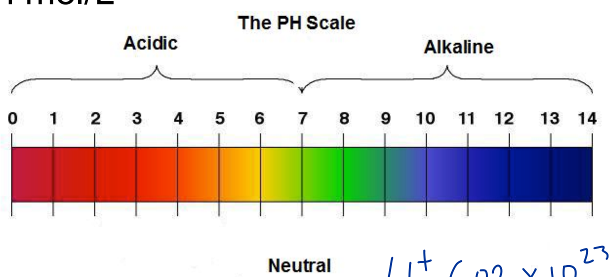
$$10\,000\,000\, I_{NF} = I_{jet}$$

C. The pH Scale

The acidity or alkalinity of a solution is given by the pH scale.

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of hydronium ions in the solutions in mol/L



H^+ 6.02×10^{23}

Example 4. The hydronium ions in blood measure in at a concentration of 4×10^{-7} mol/L. What is the pH of blood?

0.000 0004 mol/L

$$pH = -\log[H^+]$$
$$= -\log[4 \times 10^{-7}]$$
$$= 6.4$$

Example 5. What is the concentration of hydronium ions in a pool if the pH is 8.2?

$$8.2 = -\log[H^+]$$
$$-8.2 = \log_{10}[H^+]$$
$$10^{-8.2} = [H^+]$$
$$10^{-8.2} \text{ mol/L} = [H^+]$$

Practice Questions - Page 353 #1ac, 2ac, 3, 4, 7-10, 12-15

Topic : Modelling with Exponentials

Goal : I can solve problems related to exponential growth or decay

Modelling with Exponential Functions

There are many things that follow an exponential growth or decay model, among them

* value of a vehicle
* population
* investments with compound interest
* radioactivity (i.e. half-life)

The basic growth or decay formula is

$$A = A_0(1 \pm r)^n$$

Where A is the amount of whatever is growing or decaying, A_0 is the amount that was there at the start, r is the rate of growth or decay and n is the number of growth or decay periods that have passed. If something is growing we use the plus sign, if it's decaying we use the minus.

Example 1. Blue jeans fade when washed due to the loss of blue dye from the fabric. If each washing removes about 2.2% of the original dye from the fabric, how many washings are required to give a pair of jeans a well-worn look (30% dye left)

$D = D_0(1 - r)^n$

$D = D_0(1 - 0.022)^n$

$\frac{D}{D_0} = (0.978)^n$

$0.3 = (0.978)^n$

$\log 3 = n \log(0.978)$

$n = \frac{\log 0.3}{\log 0.978}$

$54 = n$

∴ It will take 54 washer

Example 2. How long does it take for $2500 to grow to $4000 if it is invested at an interest rate of 6.5% per year, and compounded semi-annually (twice a year)?

$r = 0.065/2$

$$A = A_0(1+r)^n$$
$$4000 = 2500\left(1 + \frac{0.065}{2}\right)^n$$
$$4000 = 2500(1.0325)^n$$
$$1.6 = 1.0325^n$$

Taking the log of both sides...

$\log 1.6 = n \log 1.0325$ (power law)

$\dfrac{\log 1.6}{\log 1.0325} = n$

$15 \doteq n$

$15/2 = 7.5$ years.

Example 3. Assume that the increasing inflation rate stays constant at 3.1% per year. If you pay $4.99 for a bag of milk right now, what would you theoretically pay in 5 years? How long would it take for milk to cost $10/bag?

$P = P_0(1+0.031)^n$

$= 4.99(1.031)^5$

$= \$5.81$

$P = 4.99(1.031)^n$

$10 = 4.99(1.031)^n$

$2 = 1.031^n$

Taking the log of both sides...

$\log 2 = n \log 1.031$ (power law)

$\dfrac{\log 2}{\log 1.031} = n$

$23 = n$

$\therefore$ In 23 years we may be paying $10 for a bag of milk!

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