Graph Theory and Algorithms

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Note: All the reference theorems are at the end of the assignment

Question 1:

Let v be a cut-vertex of a graph G. Prove that $\bar{G} - v$ is connected

Solution: Let $V_1, V_2, ..., V_n \in V_{G-v}$ where each V_i is the set of vertices of each component in G-v. v is a cut-vertex in G, removing v will form at least two connected components. [Definition of cut-vertex]
$\implies$ n $\geq i \geq$ 2 for some i and n $\in \mathbb{N}$
Note: the vertex sets $V_1, V_2, ..., V_n \in V_{G-v}$ are in $V_{\bar{G}-v}$ [Definition of complement graph]
$\implies$ $\bar{G} - v$ is a n-partite graph with partite sets $V_1, V_2, ..., V_n$
$\implies$ $\bar{G} - v$ is connected [Property of multipartite graph]■

Question 2:

Prove that a graph with only even degree vertices has no cut-edge.

Solution:
Suppose a graph G is a graph with only even degree and there exist a cut-edge between vertices i,j in G.
$\implies$ deleting the cut-edge will give us two induced sub-graphs namely, G' and G"
$\implies$ $\sum\limits_{v \in V_{G'}-\{i\}} deg(v) + deg(i) = 2|E_{G'}|$ and $\sum\limits_{v \in V_{G''}-\{j\}} deg(v) + deg(j) = 2|E_{G''}|$
Since the degree of each vertices is even except deg(i) and deg(j)(one edge connecting them is removed)
$\implies$ handshaking lemma doesn't hold because number of edges has to be in $\mathbb{N}$ and the quotient of odd number divide by two $\notin \mathbb{N}$
$\implies$ contradiction. ■

Question 3:

A connected graph G has a Eulerian trail if and only if the number of vertices in G whose degree is odd is zero or two.

Solution:
($\implies$)
Case 1: The Eulerian trail in the connected graph is a circuit
$\implies$ G is a Eulerian graph
$\implies$ All the vertices have even degree [Reference theorem 1]

$\implies$ the number of vertices in G whose degree is odd is 0

Case 2: The Eulerian trail in the connected graph isn't a circuit

$\implies$ The Eulerian trail is not a closed walk

Let s,e be represents the starting vertex and ending vertex respectively of the Eulerian trail

Let $u \in V_G - \{s, e\}$

For any u, there should be an edge going in and an edge going out of each vertices along the walk based on the definition of the walk and the fact there exist an Eulerian trail in G.

$\implies$ the degree for any u is divisible by 2.

Since, s is not the ending vertex, therefore, there exist one edge exiting s and that edge can't pair with any edge entering s along the walk. The same argument applies to e. Since e is not the starting vertex, therefore, there exist one edge entering e and that edge can't pair with any edge exiting edge.

$\implies$ The graph G has two odd degree vertices

From case 1 and case 2, we can conclude if a connected graph G has a Eulerian trail, then the number of vertices in G whose degree is odd is zero or two.

($\impliedby$)

Case 1: The number of vertices whose degree is odd is zero

$\implies$ G is an Eulerian graph [Reference theorem 1]

$\implies$ there exist an Eulerian circuit [Definition of Eulerian graph]

$\implies$ there exist an Eulerian trail [Definition of Eulerian circuit]

Case 2: The number of vertices whose degree is odd is two, namely, u,v.

$\implies$ either u,v are adjacent or not

Case 2.1: The two vertices u,v are adjacent

If u,v is adjacent, then removing one edge that connects them.

$\implies$ every vertices in G has even degree.

$\implies$ G is an Eulerian graph [Reference theorem 1]

$\implies$ there exists an Eulerian circuit in G without the edge uv. [Definition of Eulerian graph]

$\implies$ if we start at u and connect u with v by the edge uv that was remove previously, then we will end up with an Eulerian trail. This can be seen in reverse order. By definition of Eulerian circuit, if the circuit starts at v, it will end at v. The Eulerian trail can be formed by adding an extra edge uv after an Eulerian circuit starting at v. So, the Eulerian trail starts at u, then connects to a Eulerian circuit which will end end at v.

Case 2.2: The two vertices u,v are not adjacent

By adding an edge connecting u,v, to the graph G and formed G'

$\implies$ all the vertices in the new graph G' will have even degree

$\implies$ G' is an Eulerian graph [Reference theorem 1]

$\implies$ there exists an Eulerian circuit in G' [Definition of Eulerian graph]

WLOG, let u be the starting vertex and v be the second last vertex in the Eulerian circuit in G'.

$\implies$ removing the edge uv from the Eulerian circuit defined above will result an Eulerian trail.

$\implies$ there exist an Eulerian trail in the graph G.

$\implies$ A connected graph G has a Eulerian trail if and only if the number of vertices in G whose degree is odd is zero or two. ■

Question 4:

(a) If G is a graph all of whose vertices have even degree, then provide a complete proof that G decomposes into cycles.

(b) What would be the digraph analogue to the statement in 4(a) above?

Solution:
(a) Assuming cycles refers to at least one cycle, if not this is not a true statement. i.e a rectangle with four edges and four vertices can only be decompose into one cycle.
G is a graph where all of whose vertices have even degree
$\implies$ G is an Eulerian graph [Reference theorem 1]
Prove by contradiction:
Suppose G doesn't decompose into at least one cycle
$\implies$ $\forall e \in E_G$, e is a cut edge [Reference theorem 2]
$\implies$ contradiction [Q2]
$\implies$ If G is a graph all of whose vertices have even degree, then provide G decomposes into cycles. ■

(b) If G is a digraph all of whose vertices' indegree and outdegree is equivalent, then G can be decompose into cycles.

Question 5:

(a) For what values of n is K_n Eulerian? Justify

(b) For what values of n is K_n Hamilton? Justify

Solution:
(a) Assumption: empty graph is excluded (vacuously true if included).
Case 1: n is odd
Every vertices in K_n has degree n-1. [Definition of complete graph]
$\implies$ the degree for every vertices in K_n is even. [n is odd]
$\implies$ K_n is Eulerian when n is odd [Reference theorem 1]
Case 2: n is even
Every vertices in K_n has degree n-1. [Definition of complete graph]
$\implies$ the degree for every vertices in $K - n$ is odd. [n is even]
$\implies$ K_n is not Eulerian when n is even [Negate both side of reference theorem 1]

$\implies$ K_n is Eulerian when n is odd ■

(b) For all n $\in \mathbb{N}$, K_n is Hamilton

Base case:

For n = 1, vacuously true.

For n = 2, let s be any one of the two vertices. $K_2 - s$ has only one component.

$\implies$ K_2 has a Hamiltonian cycle [Reference theorem 3]

$\implies$ K_2 is a Hamilton graph [Definition of Hamilton graph]

IH: Suppose for some n, K_n is Hamilton

WTS K_{n+1}:

The graph K_{n+1} is equivalent to an vertex u with n edges connect to every vertices in K_n by the definition of complete graph.

K_n is Hamilton [IH]

$\implies$ there exists an Hamilton cycle [Definition of Hamilton graph]

For any $S \subseteq V_{K_{n+1}}$,

Case 1: $u \in S$

Let $S' = S - u, S' \in V_{K_n}$

$\implies$ $K_{n+1} - S = K_n + u - S' - u = K_n - S'$

$\implies$ By IH, the Hamiltonian cycle's condition is satisfied

Case 2: $u \notin S$

u is connected to all of the vertices in K_n. [Definition of complete graph]

$\implies$ the component of $(K_n + u) - S = K_n - S$ when $u \notin S$

$\implies$ K_{n+1} has at most |S| component by IH, since $S \in V_{K_n}$

$\implies$ There exist an Hamilton cycle in K_{n+1}

$\implies$ K_{n+1} is a Hamilton graph [Definition of Hamilton graph]

By induction principle, for all n $\in \mathbb{N}$, K_n is Hamilton. ■

Question 6:

(a) Prove that if G is a bipartite graph with an odd number of vertices, then G is not Hamiltonian.

(b) On a chessboard, a knight always moves two squares in a horizontal or vertical direction and one square in a perpendicular direction. Show that a knight cannot visit all the squares of a 7 × 7 or 5 × 5 chessboard exactly once by knight's moves and return to its starting point. Can a knight visit all the squares of a 8 × 8 chessboard?

Solution:

a) G is a bipartite graph

$\implies$ G has vertex sets L,R where L and R are the partite sets.

Since, the number of vertices in G is odd. WLOG, assume $|L| < |R|$

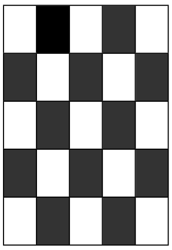
$\implies$ G-L will have |R| component which is greater than |L|

$\implies$ G doesn't have a Hamiltonian cycle [Reference theorem 3]

$\implies$ G is not a Hamilton graph ■

b) Formulate a graph G, where each square on the chessboard are vertices in G and there exists an edge between two vertices iff it's a valid move for the knight on the chess board.

For 5x5 chess board



Let G be the graph formulate for the chess board as described above.

Let S be vertex set represents with shaded squares in the 5x5 chess board above.

$\implies$ G-S will result 13 components in 5x5 chess board because there doesn't exist a valid move for the knight between any vertices in G-S. So, each vertices in G-S is disconnected, hence represents one component.

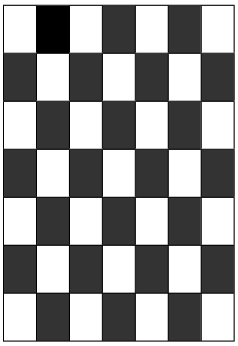
$\implies$ G-S has more than |S| $(13 > 12)$

$\implies$ There doesn't exist a Hamiltonian cycle in G formulated for 5x5chessboard. [Reference theorem 3]

$\implies$ G are not Hamilton [Definition of Hamilton Graph]

$\implies$ It's impossible for a knight to visit all square in 5x5

For 7x7 chess board:



Let G be the graph formulate for the chess board as described above.

Let S be vertex set represents shaded squares in the 7x7 chess board above.

$\implies$ G-S will result 25 components in 7x7 chess board because there doesn't exist a valid move for the knight between any vertices in G-S. So, each vertices in G-S is disconnected, hence represents one component.

$\implies$ G-S has more than |S| $(25 > 24)$

$\implies$ There doesn't exist a Hamiltonian cycle in G formulated for 7x7 chessboard. [Reference theorem 3]

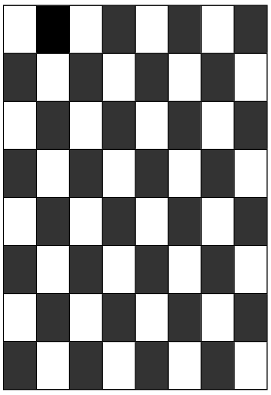
$\implies$ G are not Hamilton [Definition of Hamilton Graph]

$\implies$ It's impossible for a knight to visit all square in 7x7

For 8x8 chess board:

Formulate the graph G in the same way above.

Let S be any vertex set in G



The vertex set above(shaded squares) will result the most component in G-S because if not it implies at least one of the shaded square should not be shaded. It's trivial to see, if any of the shaded squares is unshaded, then the number of components in G-S will decrease at least by one while |S| decrease by one i.e the shaded square in the corner.

$\implies$ $\exists$ a Hamiltonian cycle in the graph G, since the condition is satisfied [Reference theorem 3]

$\implies$ G is a Hamilton graph [Definition of Hamilton Graph]

$\implies$ It's possible for a knight to visit all square in a 8x8 chessboard.

Question 7:

Describe a relation in the real world whose digraph model has no cycles.

Solution:

Family trees, the directed graph model can only goes downward because you can't give birth to your parent. Assuming religion belief isn't included. In other word, reincarnation isn't in the consideration.

Question 8:

Prove that every u,v walk in a digraph contains a u, v path.

Solution:

By definition, a digraph is a path if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering.

Therefore, in any walk W, let the first two vertices in the walk be u,v. Then, the u, v walk contained in W is a u, v path by the definition.

References:

Theorem proved in class or from the textbook

1. A connected graph G is Eulerian iff its vertices must all have even degree.
2. An edge is a cut-edge(in G) iff it does not belong to any cycle (in G)
3. If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph G - S has at most $|S|$ components. [Proposition 7.2.3 textbook]