Graph Theory and Algorithms

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Question 1:

Prove that every graph has an even number of vertices of odd degree

Solution:

By handshaking lemma:

$\implies \sum\limits_{v \in V} deg(v) = 2|E_G|$

Let V_{odd} denoted as the set of vertices with odd degrees and V_{even} denoted as the set of vertices with even degrees.

$\implies \sum\limits_{v \in V_{odd}} deg(v) + \sum\limits_{v' \in V_{even}} deg(v') = 2|E_G|$

Since, $\sum\limits_{v' \in V_{even}} deg(v')$ is even by the fact that V_{even} is a collection of vertices with even degrees. Thus, $deg(v')$ is even and the summation is even.

Therefore, $\sum\limits_{v \in V_{odd}} deg(v)$ has to be even to satisfies the handshaking lemma.

Let $V_{odd} = \{v_1, v_2, ..., v_n\}$ for some n $\in \mathbb{N}$

$\sum\limits_{v \in V_{odd}} deg(v)$ is even

$\implies$ $|V_{odd}|$ is even because if not, then $\sum\limits_{v \in (V_{odd} - \{v_n\})} deg(v)$ is even. (Trivial to show because sum of two odd number is even, split the set $V_{odd} - \{v_n\}$ into pairs, then the sum of the set is the sum of k even number for some k $\in \mathbb{N}$, which is even)

$\implies \sum\limits_{v \in (V_{odd} - \{v_n\})} deg(v) + deg(v_n)$ is odd. Hence, contradicts with the condition that $\sum\limits_{v \in V_{odd}} deg(v)$ is even

$\implies$ $|V_{odd}|$ is even. ■

Question 2:

Let $n \in \mathbb{N}$. How many edges does K_n have? Use the Handshaking Lemma to justify your answer.

Solution:

By definition, K_n is a simple complete graph on n vertices.

$\implies$ $\forall$ (u,v) where u, v $\in V_{K_n}$, $\exists$ e $\in E_{K_n}$ that connects u,v.

$\implies \forall v \in V_{K_n}$, $deg(v) = n - 1$

By handshaking lemma,

$\implies \sum\limits_{v \in V_G} deg(v) = 2|E_G|$

$\implies \sum\limits_{v \in V_{K_n}} deg(v) = \sum\limits_{i=1}^{n} (n-1)$ since, there are n vertices

$\implies \sum\limits_{i=1}^{n} (n-1) = n(n-1) = 2 \times |E_{K_n}|$

$\implies |E_{K_n}| = \frac{n(n-1)}{2}$ ■

Question 3:

Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$

Solution:

Adjacency matrix for $K_{m,n}$

	m	n
m	0	1
n	1	0

By the property of bipartite graph, all the elements in the set m is not connected and all the elements in the set n is not connected. Thus, 0 for m,m and n,n, and 1 for m,n and n,m.

Question 4:

Let G and G' be graphs. Prove that $G \cong G'$ iff $\bar{G} \cong \bar{G}'$

Solution:

Let G = (V_G, E_G, f) and G' = $(V_{G'}, E_{G'}, f)$

($\implies$)

$G \cong G' \implies \exists f_V{:}V_G \to V_{G'}$ and $f_E{:}E_G \to E_{G'}$

Also, f must preserve the adjacency relation

Let u,v $\in V_{\bar{G}}$ and suppose $\{u, v\} \in E_{\bar{G}}$

$\iff \{u, v\} \notin E_G$ (definition of complement graph)

$\iff f(u)f(v) \notin E_{G'}$(definition of isomorphism b/c f is bijective)

$\iff f(u)f(v) \in E_{\bar{G}'}$

$\implies \bar{G} \cong \bar{G}'$

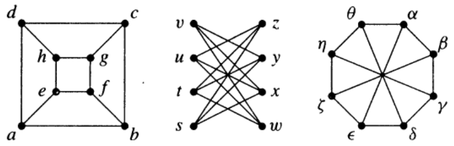
Therefore, if $G \cong G'$, then $\bar{G} \cong \bar{G}'$

($\impliedby$)

It's trivial to show this is true. By letting G = $\bar{G}$ and G' = $\bar{G}'$. The above proof follows. ■

Question 5:

Determine which pairs of graphs below are isomorphic:



Solution:

The third graph has 5-cycle where the first and second don't have it. Thus, the third graph is not isomorphic to the first two graph in the diagram.

The first and second graph are isomorphic because they are bipartite graphs. Consider the following mapping:

$a \to v$ $\quad$ $b \to z$

$c \to u$ $\quad$ $d \to y$

$e \to x$ $\quad$ $f \to t$

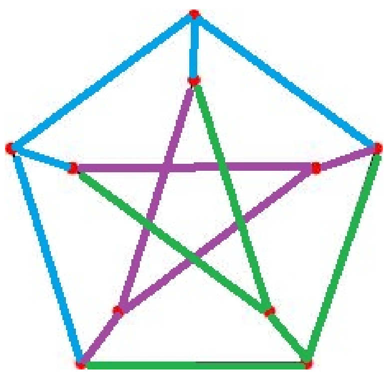
$g \to w$ $\quad$ $h \to s$

Observe that the degree of each vertices in the graphs is the same and the number of element in each vertices set for the first two graph is equal and the adjacent relationship is preserved because of the property of bipartite graph. Thus, we can conclude the pair is isomorphic.

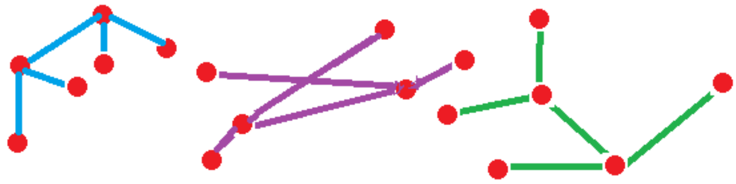
Question 6:

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Make sure to fully justify your solution.

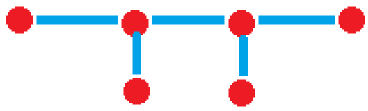
Solution:



Petersen Graph



Observe the three highlighted sub-graphs can all transforms into the sub-graph in the form below. Thus, it's trivial to show they are pairwise isomorphic.(b/c they are the same graph)



Question 7:

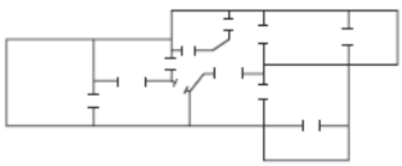
Let m,n $\in \mathbb{N}$. Prove that $K_{m,n}$ decomposes into two isomorphic sub-graphs iff m and n are not both odd.

Solution:
($\Longrightarrow$) If $K_{m,n}$ decomposes into two isomorphic sub-graphs, then m and n are not both odd.
The number of edges in $K_{m,n} = m * n$
Since $K_{m,n}$ can be decompose to two isomorphic sub-graphs, therefore $m * n$ are even. (because by the bijection property of isomorphism, the number of edges of the sub-graphs has to be equal)
$\Longrightarrow$ m,n are not both odd.

($\Longleftarrow$) If m and n are not both odd, then $K_{m,n}$ decomposes into two isomorphic sub-graphs.
If m,n are not both odd, then m * n are even.
$\Longrightarrow$ the number of edges in $K_{m,n}$ is even.
WLOG, assume m is even.
$\Longrightarrow$ $K_{m,n}$ can be decompose to two isomorphic sub-graphs. i.e $K_{m,n}$ can be decompose into $K_{m/2,n}$ graphs.(trivial to show) ■

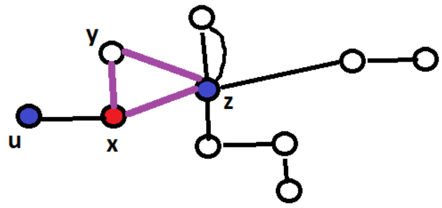
Question 8:

Oh no! My dog Firefly has got into my coloured paints. His favourite colours are blue and red. Firefly wants to paint each room in our apartment (including the hallway) either blue or red such that every time he waddles from one room to an adjacent room, the colour changes. Is this possible? Justify



Solution:

Below is the apartment floor plan in a graph where each vertex is a room and each hall way connecting the room are edges.

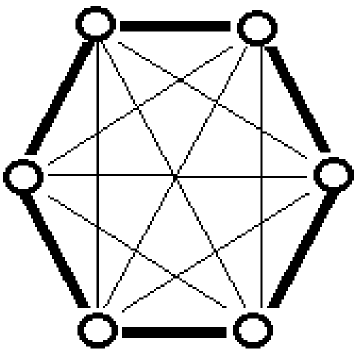


It's not possible. WLOG, assume the dog, firefly, starts from the vertex u noted in the graph with the color blue. As the dogs travels to the cycle colored in purple and follows the rule that the color in the room has to be different to the adjacent room. We can observe that in the cycle colored in purple, the vertex y can't be red or blue because if it's red it will contradict the rule with vertex x and the same argument applies if the vertex y is blue with the vertex z.

Question 9:

Prove that, in any gathering of six people, there are either three people who all know each other, or three people who are all strangers to each other.

Solution:



Let the graph above denoted as G and represents the relationship between six people where each vertices represent one person in the gathering and the edge represents the relationship between two people(endpoints).

Let K for Kathleen be an arbitrary person in the gathering.

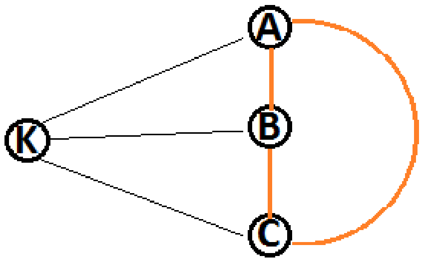
By the construction of the graph, the degree of K is 5.

$\implies$ There will be two conditions, Kathleen knows at least three people or there are at least three people Kathleen doesn't know.

WLOG: Assume Kathleen knows at least three people denoted them as A,B,C

A should not know B and B shouldn't know C or else we are done.

If A knows C, then there are three people(A,C,K) knows each other. If A doesn't know C, then there are three people(A,B,C) that doesn't know each other. As indicated as the image below.



By pigeonhole principle, we conclude if Kathleen knows at least three people there are either three people who all know each other, or three people who are all strangers to each other.
The same argument can apply to the second condition where there are at least three people Kathleen doesn't know.
$\Longrightarrow$ In any gathering of six people, there are either three people who all know each other, or three people who are all strangers to each other. ■

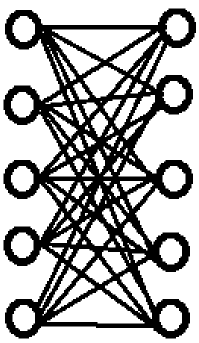
Question 10:

A graph G is called regular if $deg_G(v_i) = deg_G(v_j) \forall$ i,j i.e each vertex has the same number of neighbours. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k.

(a) Give an example of a bipartite graph that is 5-regular. Provide brief justification that your example satisfies the desired properties specified in the question.

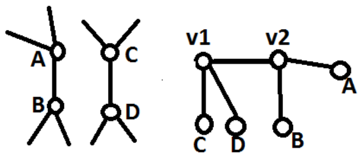
(b) Let n $\in \mathbb{N}$. For what n values does there exist a simple, 3-regular graph with exactly n vertices. Prove your answer.

Solution:
a)



This is a $K_{5,5}$ bipartite graph. It's a 5-regular graph because the degree of all the vertices of the right and left hand side in the bipartite graph is 5 by the construction of the graph.
b) Claim: The number of vertices for a 3-regular graph is 2(k+2), for some k $\in \mathbb{N}$
Base Case: k = 0

Consider a tetrahedron. (3-regular graph with 4 vertices)
Suppose the claim holds for k. [IH]
Consider k + 1



Suppose A,B,C,D are vertices in the graph and such that there exist an edge connecting A,B and C,D. For the case, k+1, the number of vertices is 2((k+1)+2) = 2k + 4 + 2. By induction hypothesis, we can assume there exist a 3-regular graph with 2k+4 vertices. Consider the diagram above, we can remove the edges between A,B and C,D, and connect A,B,C,D to the two new vertices v1,v2 respectively and connect the two new vertices together. We can now form a new 3-regular graph because the two new vertices have degree 3 and A,B,C,D still maintain degree 3 and the rest of the graph is degree 3 by IH. In addition, we can verifies its true, by verifying the fact we proved in Q1. Every graph has even number of vertices with odd degree. It's easy to observe that 2k+4+2 is even.
By induction principle, the claim is true.
Therefore, the number of vertices for a 3-regular graph is 2(k+2), for some $k \in \mathbb{N}$ ■