(fact) means something on the slides that looks provable, (exmp) means an example, (ex.) means an exercise and definitely do it, (n.p.) means a theorem that wasn't proven, (long) means the proof is longer than one page

1. Prove an intersection of subspaces is always a subspace.
2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of a vector space $\mathcal{V}$ over a field $\mathbb{F}$. Prove the sum of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the smallest subspace containing $\mathcal{S}_1$ and $\mathcal{S}_2$.
3. Prove any $\mathbf{w} \in \mathcal{S}_1 \oplus \mathcal{S}_2$ can be uniquely represented as $\mathbf{w} = \mathbf{u} + \mathbf{v},\ \mathbf{u} \in \mathcal{S}_1,\ \mathbf{v} \in \mathcal{S}_2$.
4. Prove an invertible map has a unique inverse.
5. (ex.) Show that a Hermitian matrix has real main diagonal entries.
6. (ex.) Show that a skew-Hermitian matrix has pure imaginary main diagonal entries.
7. (ex.) What are the main diagonal entries of a real skew-symmetric matrix?
8. For any unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$, show that $|\det(\mathbf{U})| = 1$.
9. (n.p.) Let $\mathbf{f} : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Then $\exists$ a unique matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\forall \mathbf{x} \in \mathbb{F}^n$. Conversely, if $\mathbf{A} \in \mathbb{F}^{m\times n}$ then the function defined above is a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$.
10. For any $\mathbf{A} \in \mathbb{F}^{m\times n}$, we have $\dim(\text{Range}(\mathbf{A})) + \dim(\text{Null}(\mathbf{A})) = n$.
11. If $\mathbf{A} \in \mathbb{C}^{m\times n}$, then $\text{Null}(\mathbf{A}) = \text{Range}(\mathbf{A}^*)^\perp$ and $\text{Null}(\mathbf{A}^*) = \text{Range}(\mathbf{A})^\perp$.
12. Any norm satisfies $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$.
13. (exmp) Let $\mathbf{W}$ be a diagonal matrix with positive diagonal elements. Show that $\|\mathbf{x}\|_{\mathbf{W}} \triangleq \sqrt{\langle \mathbf{x}, \mathbf{W}\mathbf{x}\rangle}$ defines a norm, known as the weighted Euclidean norm.
14. Given any matrix $\mathbf{U} \in \mathbb{C}^{m\times d}$ with $m \ge d$ and orthonormal columns, $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$.
15. Given any matrix $\mathbf{U} \in \mathbb{C}^{p\times m}$ with $p \ge m$ and orthonormal columns, we have $\|\mathbf{U}\mathbf{A}\|_{\mathbf{F}} = \|\mathbf{A}\|_{\mathbf{F}}$
16. (ex.) Prove the Frobenius and entry-wise ℓ_1 norm are sub-multiplicative, but the entry-wise max-norm is not sub-multiplicative.
17. Given any matrix $\mathbf{U} \in \mathbb{F}^{p\times m}$ orthonormal columns, we have $\|\mathbf{U}\mathbf{A}\|_2 = \|\mathbf{A}\|_2$ where the norm is the induced 2-norm.
18. Let $\|\cdot\|_p, \|\cdot\|_q, \|\cdot\|_r$ be vector norms on, respectively, domain of $\mathbf{B}$, range of $\mathbf{B}$ and range of $\mathbf{A}$. We have $\|\mathbf{A}\mathbf{B}\|_{p,r} \le \|\mathbf{A}\|_{q,r}\|\mathbf{B}\|_{p,q}$
19. $\kappa(\mathbf{A}) \ge 1$.
20. What is the condition number of a unitary matrix?
21. (fact) Prove that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
22. (fact) Prove that $\text{Trace}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$.
23. (fact) Prove that $\rho(\mathbf{A}) = \rho(\mathbf{A}^*)$.
24. If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A} \in \mathbb{C}^{n\times n}$, then $(p(\lambda), \mathbf{v})$ is an eigenpair of $p(\mathbf{A})$. Conversely, if $k \ge 1$ and μ is an eigenvalue of $p(\mathbf{A})$, then there is some eigenvalue λ of $\mathbf{A}$ such that $\mu = p(\lambda)$.
25. (exmp) Suppose that $\mathbf{A} \in \mathbb{C}^{n\times n}$. If $\text{spec}(\mathbf{A}) = \{-1, 1\}$, what is $\text{spec}(\mathbf{A}^2)$?
26. $\mathbf{A}$ is singular if and only if $0 \in \text{spec}(\mathbf{A})$.
27. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\lambda, \mu \in \mathbb{C}$. Then $\lambda \in \text{spec}(\mathbf{A}) \iff \lambda + \mu \in \text{spec}(\mathbf{A} + \mu\mathbf{I})$.
28. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian, then all its eigenvalues are real.
29. (ex.) Show that skew-Hermitian matrices have pure imaginary eigenvalues.
30. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian, then eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.
31. (exmp) Compute the eigenpair of $\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$
32. Let $m(\lambda)$ be the algebraic multiplicity of λ. Then $1 \le \dim(\mathcal{E}_\lambda(\mathbf{A})) \le m(\lambda)$.
33. If $\mathbf{A}$ and $\mathbf{B}$ are similar, then they have the same characteristic polynomial.
34. $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ if and only if $(\lambda, \mathbf{S}^{-1}\mathbf{v})$ is an eigenpair for $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.
35. (n.p.) The matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is diagonalisable if and only if it has n linearly independent eigenvectors. In other words, $\mathbf{A}$ is diagonalisable if and only if it is not defective.
36. (exmp) Show that $\text{Trace}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$ by Schur Triangularisation Theorem.
37. Prove that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
38. Find all possible Jordan canonical forms for a 3×3 matrix whose eigenvalues are $\{-2, 3, 3\}$.
39. Prove a matrix is unitarily diagonalisable if and only if it is normal.
40. (fact) Prove a normal matrix is Hermitian if and only if all its eigenvalues are real.
41. (fact) The singular values of A are uniquely determined by the eigenvalues of $\mathbf{A}^*\mathbf{A}$ or equivalently by the eigenvalues of $\mathbf{A}\mathbf{A}^*$.
42. For any $\mathbf{A} \in \mathbb{C}^{m\times n}$, we have $\sigma_i = \sqrt{\lambda_i(\mathbf{A}^*\mathbf{A})} = \sqrt{\lambda_i(\mathbf{A}\mathbf{A}^*)}$, $i = 1, \ldots, \text{Rank}(\mathbf{A})$.
43. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a normal matrix whose (not necessarily distinct eigenvalues) are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that the singular values of $\mathbf{A}$ are $|\lambda_1|, \ldots, |\lambda_n|$.
44. Prove (in the context of singular value decomposition) that $\mathcal{E} = \mathbf{U}\mathcal{E}_0$ where $\mathcal{E}_0 = \left\{\mathbf{y} \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{y_i^2}{\sigma_i^2} = 1\right\}$
45. Prove that if $\mathbf{P}$ is a projection, then $\mathbf{P}\mathbf{v} = \mathbf{v} \iff \mathbf{v} \in \text{Range}(\mathbf{P})$
46. Prove that if $\mathbf{P}$ is a projection, then so is $\mathbf{I} - \mathbf{P}$.
47. Prove that if $\mathbf{P}$ is a projection, then $\text{Range}(\mathbf{I} - \mathbf{P}) = \text{Null}(\mathbf{P})$.
48. Prove that if $\mathbf{P}$ is a projection, then $\text{Range}(\mathbf{P}) \cap \text{Range}(\mathbf{I} - \mathbf{P}) = \{\mathbf{0}\}$.
49. (ex) Prove that for orthogonal projections, we have the Pythagorean theorem $\|\mathbf{v}\|^2 = \|\mathbf{P}\mathbf{v}\|^2 + \|(\mathbf{I} - \mathbf{P})\mathbf{v}\|^2$.
50. (ex) Given any matrix $\mathbf{Q} \in \mathbb{C}^{m\times n}$ with orthonormal columns, prove that $\mathbf{P} = \mathbf{Q}\mathbf{Q}^*$ is an orthogonal projection onto $\text{Range}(\mathbf{Q})$.
51. (ex) Given any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{P} = \mathbf{A}\mathbf{A}^\dagger$ is an orthogonal projection onto $\text{Range}(\mathbf{A})$.
52. (ex) Prove $\mathbf{A} \succ \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle > 0, \quad \forall \mathbf{x} \ne \mathbf{0} \in \mathbb{R}^n$.
53. (ex) Prove $\mathbf{A} \succeq \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \ge 0, \quad \forall \mathbf{x} \in \mathbb{R}^n$.
54. (ex) Prove $\mathbf{A} \succ \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle > 0, \quad \forall \mathbf{x} \ne \mathbf{0} \in \mathbb{C}^n$.
55. (ex) Prove $\mathbf{A} \succeq \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \ge 0, \quad \forall \mathbf{x} \in \mathbb{C}^n$.
56. (ex) Prove that $\mathbf{A} \in \mathbb{C}^{n\times n}$ being either PD or PSD implies Hermitian.
57. (ex) Prove that for any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, we have $\mathbf{A}^*\mathbf{A} \succeq \mathbf{0}$ and $\mathbf{A}\mathbf{A}^* \succeq \mathbf{0}$.
58. (ex) Prove that $\text{Rank}(\mathbf{A}) = m \Longrightarrow \mathbf{A}\mathbf{A}^* \succ \mathbf{0}$.
59. (ex) Prove that $\text{Rank}(\mathbf{A}) = n \Rightarrow \mathbf{A}^*\mathbf{A} \succ \mathbf{0}$
60. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succ \mathbf{0} \iff \lambda_i(\mathbf{A}) > 0, i = 1, \ldots n$.
61. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succeq \mathbf{0} \iff \lambda_i(\mathbf{A}) \ge 0, i = 1, \ldots n$.
62. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \prec \mathbf{0} \iff \lambda_i(\mathbf{A}) < 0, i = 1, \ldots n$.
63. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \preceq \mathbf{0} \iff \lambda_i(\mathbf{A}) \le 0, i = 1, \ldots n$.
64. (ex) Prove for Hermitian $\mathbf{A}$ that every PD matrix is invertible and its inverse is also PD.
65. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A}, \mathbf{B} \succ \mathbf{0}$ and $\alpha > 0$ implies $\alpha\mathbf{A} \succ \mathbf{0}$ and $\mathbf{A} + \mathbf{B} \succ \mathbf{0}$.
66. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succeq \mathbf{0} \iff \exists! \mathbf{B} \succeq \mathbf{0}$ s.t. $\mathbf{B}^2 = \mathbf{A}$ (square root, optional)
67. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succ \mathbf{0}$, the Schur, spectral and SV decompositions all coincide.
68. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succeq \mathbf{0}$, then $\mathbf{B}^*\mathbf{A}\mathbf{B} \succeq \mathbf{0}, \forall \mathbf{B} \in \mathbb{C}^{n\times n}$.
69. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{B}$ has full column rank, then $\mathbf{B}^*\mathbf{A}\mathbf{B} \succ \mathbf{0}$.
70. (fact) Prove that a strictly diagonally dominant matrix is non-singular (Levy-Desplanques theorem).
71. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be full column rank. Prove $\kappa(\mathbf{A}^\top\mathbf{A}) = \kappa^2(\mathbf{A})$.
72. Prove that $\|\mathbf{T}\| < 1 \implies \lim_{k\to\infty} \mathbf{e}_k = 0$, where $\mathbf{T} \triangleq \mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ is the iteration matrix and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^\star$ is the error vector.
73. Prove that $\rho(\mathbf{T}) < 1 \iff \lim_{k\to\infty} \mathbf{e}_k = 0$.
74. (long, for understanding) There is a positive integer $t \triangleq t(\mathbf{v}, \mathbf{A})$ called the grade of $\mathbf{v}$ with respect to $\mathbf{A}$ such that $\dim(\mathcal{K}_k(\mathbf{A}, \mathbf{v})) = \begin{cases} k & k \le t \\ t & k \ge t \end{cases}$.
75. Prove $t = \min\{k \mid \mathbf{A}^{-1}\mathbf{v} \in \mathcal{K}_k(\mathbf{A}, \mathbf{v})\}$.
76. Assume Arnoldi process does not terminate before k steps. Then the vectors $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$ form an orthonormal basis for $\mathcal{K}_k(\mathbf{A}, \mathbf{r}_0)$.
77. Arnoldi process breaks down at step j ($h_{j+1,j} = 0$) if and only if the grade of $\mathbf{r}_0$ with respect to $\mathbf{A}$ is j, i.e. $t(\mathbf{r}_0, \mathbf{A}) = j$.
78. The matrix $\mathbf{L}^\top\mathbf{A}\mathbf{K}$ is non-singular if either $\mathbf{A} \succ \mathbf{0}$ and $\mathcal{L} = \mathcal{K}$ or $\det(\mathbf{A}) \ne 0$ and $\mathcal{L} = \mathbf{A}\mathcal{K}$.
79. The case where $\mathbf{A} \succ \mathbf{0}$ and $\mathcal{L}_k = \mathcal{K}_k$ is equivalent to $\mathbf{x}_k = \text{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{x} - \mathbf{x}^\star\|_{\mathbf{A}} = \text{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \frac{1}{2}\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$.
80. (sort-of ass) Prove $\text{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{x} - \mathbf{x}^\star\|_{\mathbf{A}} = \text{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \frac{1}{2}\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$.
81. (slightly long) The case where $\det(\mathbf{A}) \ne 0$ and $\mathcal{L}_k = \mathbf{A}\mathcal{K}_k$ is equivalent to $\mathbf{x}_k = \text{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$.
82. (long) If Arnoldi (or Lanczos) process breaks down at step $t = t(\mathbf{A}, \mathbf{r}_0)$, then $\mathbf{x}_t$ from any projection method onto $\mathcal{K}_t(\mathbf{A}, \mathbf{r}_0)$ or $\mathbf{A} \cdot \mathcal{K}_t(\mathbf{A}, \mathbf{r}_0)$ would be exact.