MATH3204 MIDSEM PROOFS & QUESTIONS

1. Prove an intersection of subspaces is always a subspace.
2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of a vector space $\mathcal{V}$ over a field $\mathbb{F}$. Prove the sum of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the smallest subspace containing $\mathcal{S}_1$ and $\mathcal{S}_2$.
3. Prove any $\mathbf{w} \in \mathcal{S}_1 \oplus \mathcal{S}_2$ can be uniquely represented as $\mathbf{w} = \mathbf{u} + \mathbf{v},\ \mathbf{u} \in \mathcal{S}_1,\ \mathbf{v} \in \mathcal{S}_2$.
4. Prove an invertible map has a unique inverse.
5. (ex.) Show that a Hermitian matrix has real main diagonal entries.
6. (ex.) Show that a skew-Hermitian matrix has pure imaginary main diagonal entries.
7. (ex.) What are the main diagonal entries of a real skew-symmetric matrix?
8. For any unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$, show that $|\det(\mathbf{U})| = 1$.
9. (n.p.) Let $\mathbf{f} : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Then $\exists$ a unique matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x},\ \forall \mathbf{x} \in \mathbb{F}^n$. Conversely, if $\mathbf{A} \in \mathbb{F}^{m\times n}$ then the function defined above is a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$.
10. For any $\mathbf{A} \in \mathbb{F}^{m\times n}$, we have $\dim(\mathrm{Range}(\mathbf{A})) + \dim(\mathrm{Null}(\mathbf{A})) = n$.
11. If $\mathbf{A} \in \mathbb{C}^{m\times n}$, then $\mathrm{Null}(\mathbf{A}) = \mathrm{Range}(\mathbf{A}^*)^\perp$ and $\mathrm{Null}(\mathbf{A}^*) = \mathrm{Range}(\mathbf{A})^\perp$.
12. Any norm satisfies $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$.
13. (exmp) Let $\mathbf{W}$ be a diagonal matrix with positive diagonal elements. Show that $\|\mathbf{x}\|_\mathbf{W} \triangleq \sqrt{\langle \mathbf{x}, \mathbf{W}\mathbf{x}\rangle}$ defines a norm, known as the weighted Euclidean norm.
14. Given any matrix $\mathbf{U} \in \mathbb{C}^{m\times d}$ with $m \ge d$ and orthonormal columns, $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$.
15. Given any matrix $\mathbf{U} \in \mathbb{C}^{p\times m}$ with $p \ge m$ and orthonormal columns, we have $\|\mathbf{U}\mathbf{A}\|_\mathbf{F} = \|\mathbf{A}\|_\mathbf{F}$
16. (ex.) Prove the Frobenius and entry-wise ℓ_1 norm are sub-multiplicative, but the entry-wise max-norm is not sub-multiplicative.
17. Given any matrix $\mathbf{U} \in \mathbb{F}^{p\times m}$ orthonormal columns, we have $\|\mathbf{U}\mathbf{A}\|_2 = \|\mathbf{A}\|_2$ where the norm is the induced 2-norm.
18. Let $\|\cdot\|_p, \|\cdot\|_q, \|\cdot\|_r$ be vector norms on, respectively, domain of $\mathbf{B}$, range of $\mathbf{B}$ and range of $\mathbf{A}$. We have $\|\mathbf{A}\mathbf{B}\|_{p,r} \le \|\mathbf{A}\|_{q,r}\|\mathbf{B}\|_{p,q}$
19. $\kappa(\mathbf{A}) \ge 1$.
20. What is the condition number of a unitary matrix?
21. (fact) Prove that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
22. (fact) Prove that $\mathrm{Trace}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$.
23. (fact) Prove that $\rho(\mathbf{A}) = \rho(\mathbf{A}^*)$.
24. If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A} \in \mathbb{C}^{n\times n}$, then $(p(\lambda), \mathbf{v})$ is an eigenpair of $p(\mathbf{A})$. Conversely, if $k \ge 1$ and μ is an eigenvalue of $p(\mathbf{A})$, then there is some eigenvalue λ of $\mathbf{A}$ such that $\mu = p(\lambda)$.
25. (exmp) Suppose that $\mathbf{A} \in \mathbb{C}^{n\times n}$. If $\mathrm{spec}(\mathbf{A}) = \{-1, 1\}$, what is $\mathrm{spec}(\mathbf{A}^2)$?
26. $\mathbf{A}$ is singular if and only if $0 \in \mathrm{spec}(\mathbf{A})$.
27. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\lambda, \mu \in \mathbb{C}$. Then $\lambda \in \mathrm{spec}(\mathbf{A}) \iff \lambda + \mu \in \mathrm{spec}(\mathbf{A} + \mu\mathbf{I})$.
28. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian, then all its eigenvalues are real.
29. (ex.) Show that skew-Hermitian matrices have pure imaginary eigenvalues.
30. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian, then eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.
31. (exmp) Compute the eigenpair of $\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$
32. Let $m(\lambda)$ be the algebraic multiplicity of λ. Then $1 \le \dim(\mathcal{E}_\lambda(\mathbf{A})) \le m(\lambda)$.
33. If $\mathbf{A}$ and $\mathbf{B}$ are similar, then they have the same characteristic polynomial.
34. $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ if and only if $(\lambda, \mathbf{S}^{-1}\mathbf{v})$ is an eigenpair for $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.
35. (n.p.) The matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is diagonalisable if and only if it has n linearly independent eigenvectors. In other words, $\mathbf{A}$ is diagonalisable if and only if it is not defective.
36. Find all possible Jordan canonical forms for a 3×3 matrix whose eigenvalues are $\{-2, 3, 3\}$.
37. Prove a matrix is unitarily diagonalisable if and only if it is normal.
38. (fact) Prove a normal matrix is Hermitian if and only if all its eigenvalues are real.
39. (fact) The singular values of A are uniquely determined by the eigenvalues of $\mathbf{A}^*\mathbf{A}$ or equivalently by the eigenvalues of $\mathbf{A}\mathbf{A}^*$.
40. For any $\mathbf{A} \in \mathbb{C}^{m\times n}$, we have $\sigma_i = \sqrt{\lambda_i(\mathbf{A}^*\mathbf{A})} = \sqrt{\lambda_i(\mathbf{A}\mathbf{A}^*)},\ i = 1, \ldots, \mathrm{Rank}(\mathbf{A})$.
41. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a normal matrix whose (not necessarily distinct eigenvalues) are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that the singular values of $\mathbf{A}$ are $|\lambda_1|, \ldots, |\lambda_n|$.
42. Prove (in the context of singular value decomposition) that $\mathcal{E} = \mathbf{U}\mathcal{E}_0$ where $\mathcal{E}_0 = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{y_i^2}{\sigma_i^2} = 1 \right\}$
43. Prove that if $\mathbf{P}$ is a projection, then $\mathbf{P}\mathbf{v} = \mathbf{v} \iff \mathbf{v} \in \mathrm{Range}(\mathbf{P})$
44. Prove that if $\mathbf{P}$ is a projection, then so is $\mathbf{I} - \mathbf{P}$.
45. Prove that if $\mathbf{P}$ is a projection, then $\mathrm{Range}(\mathbf{I} - \mathbf{P}) = \mathrm{Null}(\mathbf{P})$.
46. Prove that if $\mathbf{P}$ is a projection, then $\mathrm{Range}(\mathbf{P}) \cap \mathrm{Range}(\mathbf{I} - \mathbf{P}) = \{\mathbf{0}\}$.
47. (ex) Prove that for orthogonal projections, we have the Pythagorean theorem $\|\mathbf{v}\|^2 = \|\mathbf{P}\mathbf{v}\|^2 + \|(\mathbf{I} - \mathbf{P})\mathbf{v}\|^2$.
48. (ex) Given any matrix $\mathbf{Q} \in \mathbb{C}^{m\times n}$ with orthonormal columns, prove that $\mathbf{P} = \mathbf{Q}\mathbf{Q}^*$ is an orthogonal projection onto $\mathrm{Range}(\mathbf{Q})$.
49. (ex) Given any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{P} = \mathbf{A}\mathbf{A}^\dagger$ is an orthogonal projection onto $\mathrm{Range}(\mathbf{A})$.
50. (ex) Prove $\mathbf{A} \succ \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle > 0, \quad \forall \mathbf{x} \ne \mathbf{0} \in \mathbb{R}^n$.
51. (ex) Prove $\mathbf{A} \succeq \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \ge 0, \quad \forall \mathbf{x} \in \mathbb{R}^n$.
52. (ex) Prove $\mathbf{A} \succ \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle > 0, \quad \forall \mathbf{x} \ne \mathbf{0} \in \mathbb{C}^n$.
53. (ex) Prove $\mathbf{A} \succeq \mathbf{0} \iff \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \ge 0, \quad \forall \mathbf{x} \in \mathbb{C}^n$.
54. (ex) Prove that $\mathbf{A} \in \mathbb{C}^{n\times n}$ being either PD or PSD implies Hermitian.
55. (ex) Prove that for any matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, we have $\mathbf{A}^*\mathbf{A} \succeq \mathbf{0}$ and $\mathbf{A}\mathbf{A}^* \succeq \mathbf{0}$.
56. (ex) Prove that $\mathrm{Rank}(\mathbf{A}) = m \Longrightarrow \mathbf{A}\mathbf{A}^* \succ \mathbf{0}$.
57. (ex) Prove that $\mathrm{Rank}(\mathbf{A}) = n \Rightarrow \mathbf{A}^*\mathbf{A} \succ \mathbf{0}$
58. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succ \mathbf{0} \iff \lambda_i(\mathbf{A}) > 0, i = 1, \ldots n$.
59. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succeq \mathbf{0} \iff \lambda_i(\mathbf{A}) \ge 0, i = 1, \ldots n$.
60. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \prec \mathbf{0} \iff \lambda_i(\mathbf{A}) < 0, i = 1, \ldots n$.
61. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \preceq \mathbf{0} \iff \lambda_i(\mathbf{A}) \le 0, i = 1, \ldots n$.
62. (ex) Prove for Hermitian $\mathbf{A}$ that every PD matrix is invertible and its inverse is also PD.
63. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A}, \mathbf{B} \succ \mathbf{0}$ and $\alpha > 0$ implies $\alpha\mathbf{A} \succ \mathbf{0}$ and $\mathbf{A} + \mathbf{B} \succ \mathbf{0}$.
64. (ex) Prove for Hermitian $\mathbf{A}$ that $\mathbf{A} \succeq \mathbf{0} \iff \exists! \mathbf{B} \succeq \mathbf{0}$ s.t. $\mathbf{B}^2 = \mathbf{A}$ (square root, optional)
65. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succ \mathbf{0}$, the Schur, spectral and SV decompositions all coincide.
66. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succeq \mathbf{0}$, then $\mathbf{B}^*\mathbf{A}\mathbf{B} \succeq \mathbf{0}, \forall \mathbf{B} \in \mathbb{C}^{n\times n}$.
67. (ex) Prove for Hermitian $\mathbf{A}$ that if $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{B}$ has full column rank, then $\mathbf{B}^*\mathbf{A}\mathbf{B} \succ \mathbf{0}$.
68. (fact) Prove that a strictly diagonally dominant matrix is non-singular (Levy-Desplanques theorem).
69. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be full column rank. Prove $\kappa(\mathbf{A}^\top\mathbf{A}) = \kappa^2(\mathbf{A})$.
70. Prove that $\|\mathbf{T}\| < 1 \implies \lim_{k\to\infty} \mathbf{e}_k = 0$, where $\mathbf{T} \triangleq \mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ is the iteration matrix and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^\star$ is the error vector.
71. Prove that $\rho(\mathbf{T}) < 1 \iff \lim_{k\to\infty} \mathbf{e}_k = 0$.
72. (long, for understanding) There is a positive integer $t \triangleq t(\mathbf{v}, \mathbf{A})$ called the grade of $\mathbf{v}$ with respect to $\mathbf{A}$ such that $\dim(\mathcal{K}_k(\mathbf{A}, \mathbf{v})) = \begin{cases} k & k \le t \\ t & k \ge t \end{cases}$.
73. Prove $t = \min\{k \mid \mathbf{A}^{-1}\mathbf{v} \in \mathcal{K}_k(\mathbf{A}, \mathbf{v})\}$.
74. Assume Arnoldi process does not terminate before k steps. Then the vectors $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$ form an orthonormal basis for $\mathcal{K}_k(\mathbf{A}, \mathbf{r}_0)$.
75. Arnoldi process breaks down at step j ($h_{j+1,j} = 0$) if and only if the grade of $\mathbf{r}_0$ with respect to $\mathbf{A}$ is j, i.e. $t(\mathbf{r}_0, \mathbf{A}) = j$.
76. The matrix $\mathbf{L}^\top\mathbf{A}\mathbf{K}$ is non-singular if either $\mathbf{A} \succ \mathbf{0}$ and $\mathcal{L} = \mathcal{K}$ or $\det(\mathbf{A}) \ne 0$ and $\mathcal{L} = \mathbf{A}\mathcal{K}$.
77. The case where $\mathbf{A} \succ \mathbf{0}$ and $\mathcal{L}_k = \mathcal{K}_k$ is equivalent to $\mathbf{x}_k = \mathrm{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{x} - \mathbf{x}^\star\|_\mathbf{A} = \mathrm{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \frac{1}{2}\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$.
78. (sort-of ass) Prove $\mathrm{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{x} - \mathbf{x}^\star\|_\mathbf{A} = \mathrm{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \frac{1}{2}\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$.
79. (slightly long) The case where $\det(\mathbf{A}) \ne 0$ and $\mathcal{L}_k = \mathbf{A}\mathcal{K}_k$ is equivalent to $\mathbf{x}_k = \mathrm{argmin}_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$.
80. (long) If Arnoldi (or Lanczos) process breaks down at step $t = t(\mathbf{A}, \mathbf{r}_0)$, then $\mathbf{x}_t$ from any projection method onto $\mathcal{K}_t(\mathbf{A}, \mathbf{r}_0)$ or $\mathbf{A} \cdot \mathcal{K}_t(\mathbf{A}, \mathbf{r}_0)$ would be exact.
81. (*Assignment questions from now onwards*) Suppose $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{W}$ are all subspaces of $\mathcal{V}$ such that $\mathcal{U}_1 + \mathcal{W} = \mathcal{U}_2 + \mathcal{W}$. Can we conclude that $\mathcal{U}_1 = \mathcal{U}_2$?
82. Suppose $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{W}$ are all subspaces of $\mathcal{V}$ such that $\mathcal{U}_1 \oplus \mathcal{W} = \mathcal{U}_2 \oplus \mathcal{W}$. Can we conclude that $\mathcal{U}_1 = \mathcal{U}_2$?
83. Let $\mathbf{f} : \mathcal{V} \to \mathcal{W}$ be a linear map. Prove that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.
84. $\left(\mathbf{A} + \mathbf{B}\mathbf{C}^\top\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{I} + \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}^\top\mathbf{A}^{-1}$.
85. (long) Prove the equivalence of norms in $\mathbb{C}^d$: $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{d}\|\mathbf{x}\|_2 \le d\|\mathbf{x}\|_\infty$
86. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n, \mathbf{u} \ne \mathbf{0}, \mathbf{v} \ne \mathbf{0}, \mathbf{A} = \mathbf{u}\mathbf{v}^*$. Show that $\|\mathbf{A}\|_2 = \|\mathbf{u}\|_2\|\mathbf{v}\|_2$;
87. Show that $\|\mathbf{A}\|_\mathbf{F} = \|\mathbf{u}\|_2\|\mathbf{v}\|_2$;
88. Let $\lambda \in \mathrm{spec}(\mathbf{A})$ where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{A}^k = \mathbf{0}$ for some positive integer k. Show that $\lambda = 0$.
89. Let $\lambda \in \mathrm{spec}(\mathbf{A})$ where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{A}$ is unitary. Show that $|\lambda| = 1$.
90. Find the eigenvalues of $\mathbf{A} = \mathbf{u}\mathbf{v}^*, \mathbf{u}, \mathbf{v} \in \mathbb{C}^n, n \ge 2, \mathbf{u} \ne \mathbf{0}, \mathbf{v} \ne \mathbf{0}$.
91. Find conditions on $\mathbf{u}, \mathbf{v}$ such that $\mathbf{A}$ is not defective.
92. Find conditions on $\mathbf{u}, \mathbf{v}$ such that $\mathbf{A}$ is normal.
93. Find the singular values of $\mathbf{A}$.
94. What is $\mathrm{Rank}(\mathbf{A})$?
95. Find the left and right singular vectors corresponding to the largest singular value of $\mathbf{A}$.
96. Is a skew Hermitian matrix normal?
97. Is a skew Hermitian matrix diagonalisable?
98. Prove that the eigenvalues of a skew-Hermitian matrix are all pure imaginary.
99. Show that $\mathbf{I} - \mathbf{A}$ is nonsingular.
100. Use SVD to prove that $\mathrm{Rank}(\mathbf{A}) = \mathrm{Rank}(\mathbf{A}^*)$.
101. Use SVD to prove that $\mathrm{Rank}(\mathbf{A}) + \dim(\mathrm{Null}(\mathbf{A})) = n$
102. Use SVD to prove that $\mathrm{Rank}(\mathbf{A}) + \dim\left(\mathrm{Null}\left(\mathbf{A}^*\right)\right) = m$
103. Use SVD to prove that $\mathrm{Rank}(\mathbf{A}) = \mathrm{Rank}\left(\mathbf{A}^*\mathbf{A}\right) = \mathrm{Rank}\left(\mathbf{A}\mathbf{A}^*\right)$
104. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be skew-symmetric, and denote its singular values by $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n \ge 0$. Show that if n is even, then $\sigma_{2k} = \sigma_{2k-1}, k = 1, 2, \ldots, n/2$. If n is odd, then the same holds up to $k = (n-1)/2$ and also $\sigma_n = 0$.
105. Show that eigenvalues of $\mathbf{A}$ can be written as $\lambda_j = (-1)^j i\sigma_j, j = 1, \ldots, n$ where i is the imaginary unit.
106. Show that if $A \in \mathbb{C}^{n\times n}$ and λ is an eigenvalue of A, then $\overline{\lambda}$ is an eigenvalue of A^*.
107. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. For an arbitrary matrix induced norm show that $\rho(\mathbf{A}) \le \left\|\mathbf{A}^k\right\|^{1/k}, \quad k = 1, 2, \ldots$ where $\rho(\mathbf{A})$ is the spectral radius of $\mathbf{A}$.
108. Consider $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$. Prove that if one of $\mathbf{A}$ or $\mathbf{B}$ is non-singular, then $\mathbf{A}\mathbf{B}$ and $\mathbf{B}\mathbf{A}$ are similar.
109. Let $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathrm{Rank}(\mathbf{A}) = r$ and the non-zero singular values of $\mathbf{A}$ be $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ Show that the Hermitian matrix $\mathbf{B} = \begin{bmatrix} \mathbf{0}_{m\times m} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0}_{n\times n} \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$ has non-zero eigenvalues $\{\pm\sigma_1, \pm\sigma_2, \ldots, \pm\sigma_r\}$.
110. Show that the Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is positive semi-definite.
111. Find an eigenvector corresponding to the smallest eigenvalue of $\mathbf{L}$.
112. Show that if $\lambda_{min} < 0 < \lambda_{max}$ then Richardson iteration will always be divergent for some initial iterate. (First compute the iteration matrix)
113. Compute the spectral radius of the iteration matrix in terms of $\alpha, \lambda_{min}, \lambda_{max}$.
114. Find $\alpha_{min} < \alpha_{max}$ such that the method converges for any $\alpha_{min} < \alpha < \alpha_{max}$.
115. Compute α_{opt}.
116. Obtain the optimal convergence rate if $\mathbf{A} \succ \mathbf{0}$ in terms of the matrix condition number.
117. Consider the least squares problem $\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is rank deficient. Show that among all possible solutions to this problem, $\mathbf{A}^\dagger\mathbf{b}$ has the smallest Euclidean norm.
118. We know $\mathbf{A}\mathbf{B}$ and $\mathbf{B}\mathbf{A}$ are not in general similar. Suppose $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{n\times m}, m \le n$ Show the n eigenvalues of $\mathbf{B}\mathbf{A}$ are the m eigenvalues of $\mathbf{A}\mathbf{B}$ together with $n - m$ zeros.
119. **Using question 118**, if $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, show that $\det\left(\mathbf{A} + \mathbf{x}\mathbf{y}^\top\right) = \det(\mathbf{A})\left(1 + \mathbf{y}^\top\mathbf{A}^{-1}\mathbf{x}\right)$.
120. Prove the six properties of the Krylov subspace.
121. Consider the generalized minimum residual (GMRES) method. Let $\mathbf{H}_{k+1,k} = \mathbf{U}_{k+1,k}\mathbf{R}_k$ be the reduced QR factorization of $\mathbf{H}_{k+1,k}$. Show that the least-squares sub-problems of GMRES $\min_\mathbf{y} \left\|\mathbf{H}_{k+1,k}\mathbf{y} - \right\| \mathbf{r}_0 \|\mathbf{e}_1\|$ is solved using $\mathbf{y} = \mathbf{R}_k^{-1}\mathbf{U}_{k+1,k}^\top \|\mathbf{r}_0\|\mathbf{e}_1$ where $\mathbf{r}_0$ is the initial residual $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$ and $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]^\top$.
122. Show that the residual at the k^{th} iteration can be computed as $\|\mathbf{b} - \mathbf{A}\mathbf{x}_k\| = \|\mathbf{r}_0\|\sqrt{1 - \left\|\mathbf{U}_{k+1,k}^\top\mathbf{e}_1\right\|^2}$
123. Consider the subproblems of conjugate gradient i.e. $\min_{\mathbf{x}\in\mathbf{x}_0+\mathcal{K}_k(\mathbf{A},\mathbf{r}_0)} \left\|\mathbf{x} - \mathbf{x}^\star\right\|_\mathbf{A}$ where $\|\mathbf{v}\|_\mathbf{A} \triangleq \sqrt{\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle}$ and $\mathbf{A}$ is PD. SHow that the solution to this sub problem is the minimizer of $\min_{\mathbf{x}\in\mathbf{x}0+\mathcal{K}_k(\mathbf{A},\mathbf{r}_0)} \frac{1}{2}\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$.
124. Consider the derivation of CG motivated by the Cholesky factorization of the tridiagonal matrix $\mathbf{T}_k = \mathbf{L}_k\mathbf{D}_k\mathbf{L}_k^\top$ Recall that we defined $\tilde{\mathbf{P}}_k \triangleq \mathbf{Q}_k\mathbf{L}_k^{-\top}$, where $\mathbf{Q}_k$ is obtained as part of the Lancsoz process such that $\mathbf{Q}_k^\top\mathbf{A}\mathbf{Q}_k = \mathbf{T}_k$. Show that $\tilde{\mathbf{P}}_k^\top\mathbf{A}\tilde{\mathbf{P}}_k = \mathbf{D}_k$