

CONFINEMENT OF VIBRATION BY ONE-DIMENSIONAL DISORDER, I: THEORY OF ENSEMBLE AVERAGING

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The use of ensemble averaging procedures in studying the dynamics of structurally irregular mechanical systems is discussed. It is found that taking a linear average over the ensemble does not always yield the behaviour of a “typical” member of that ensemble. Under certain circumstances there are occasional members with properties so exceptional as to weight the average appreciably, and the linear ensemble average can then be seriously misleading. The effect is illustrated by the response and transmission properties of two simple mechanical models, a system of coupled oscillators with varying natural frequencies, and a vibrating string or bending beam with irregularly spaced mass and spring constraints. The distinction between the ensemble average and the typical value is important in understanding the phenomenon of Anderson localization described previously [1–3] by using these two models. This paper provides some analytic results, while an accompanying paper [4] provides complementary results from a numerical study of the coupled-oscillator model.

1. INTRODUCTION

When investigating the dynamics of complex mechanical structures it is sometimes convenient to consider an ensemble of systems of the same type and to try to calculate the average over the ensemble of the properties desired. For example, in Statistical Energy Analysis [3, 5] it is common to assume that the energy flow between subsystems does not depend crucially on the details of the modal distributions in those subsystems, provided that the modes are sufficiently dense in frequency. One then takes the ensemble average of the coupling loss factor over all modal distributions to obtain an estimate of the energy flow in terms of the average modal densities of the subsystems [3, 5].

This sort of approach is also often used in discussing the effect of random inhomogeneities or extended structural irregularity on the propagation of waves. To take a specific example, suppose one is concerned with the transmission of waves on a vibrating string or bending beam across a number of constraints whose impedances or spacing varies along the string or beam in some kind of irregular fashion. Suppose further that one is not interested in the detailed dependence of the transmission on the particular constraint configuration or set of impedances. One can then try to predict the ensemble average of the transmitted intensity over all possible constraint configurations or sets of impedance values consistent with a particular type and degree of disorder.

Of course, for practical applications one is really interested in estimating the behaviour of *typical* individual members of the ensemble. If one wishes to interpret the ensemble average as this typical behaviour, one has to assume that any members of the ensemble

† This paper describes work on which Dr Hodges was engaged up to the time of his tragic death in 1986.

with exceptional properties are sufficiently rare that they do not appreciably weight the average. The problem of transmission across irregularly spaced constraints on a string or beam can produce situations where this assumption breaks down quite dramatically. For example, in reference [1], section 3, we considered the case of N identical constraints with fluctuations in the spacing between adjacent constraints which are much larger than the wavelength on the string or beam (the case of "random phase factors"). The analysis there showed that typical values of the transmitted intensity are exponentially small, of order $|t|^{2N}$, where $|t|^2$ is the transmitted intensity across an isolated constraint on an infinite string or beam. This remains true whatever the boundary conditions at the far end of the string or beam, and is a manifestation of the phenomenon of Anderson localization [1, 6]. However, it was pointed out [7] that the ensemble average of the expression leading to this result gives unity when conservative boundary conditions are imposed at the far end of the string or beam and there is no dissipation. Localization somehow disappears altogether in the ensemble average!

So in-grained is the ensemble averaging philosophy into the way of thinking amongst theorists of random media that the result for the average could be (and was) taken as casting doubt on the existence of Anderson localization. But Anderson has stressed right from the outset [6], and again more recently [8], that localization is a typical property. For this problem one cannot be concerned with ensemble averages alone; one must also consider probability distributions and typical values. Evidently he has had at the back of his mind the possibility of the sort of situation exemplified by our non-dissipative vibrating string or beam, where the ensemble average for the transmission differs radically from the typical value. It was, incidentally, perhaps not sufficiently emphasized in references [1] and [2] that the analysis there concerns typical individual disordered configurations of the constraints.

The distinction we are concerned with here is between the mean and the "mode" of a probability distribution. If the distribution is sharply peaked but has a long tail, as shown schematically in Figure 1, it is possible for the mean (i.e., the ensemble average) to differ significantly from the mode (i.e., the peak of the distribution). Moreover, at the same time it is quite possible that the probability in the tail is small so that almost all members of the ensemble lie in the vicinity of the mode, which therefore gives a well-defined sense to the phrase "typical value".

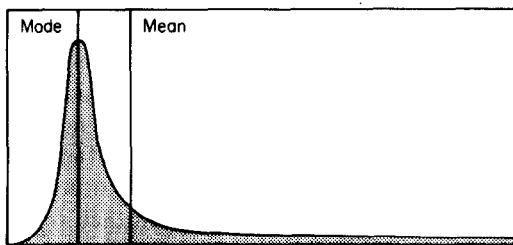


Figure 1. The difference between the mean (the centre of mass) and the typical value (the mode or peak) of a probability distribution with a long tail.

It may be important from a practical point of view to understand the reason for any difference between typical values and the ensemble average. For example, in the case mentioned above there must be some anomalous configurations with such exceptionally

large transmission values that they significantly weight the ensemble average. Clearly, in any practical application of Anderson localization to noise control one would want to avoid any such anomalous members of the ensemble when choosing a specific realization of randomness to build into a system.

In this paper and in a companion one [4] we investigate in more detail the relation between the ensemble average and the typical value, and we seek to identify those anomalous configurations which account for any difference between these quantities. By “ensemble average” one means the arithmetic mean or linear average over the configurations in the ensemble. We shall contrast this with the geometric mean or logarithmic average over the ensemble. This latter averaging procedure is less sensitive to anomalous contributions, and turns out to give a measure of the typical value in the ensemble, as we shall illustrate.

We consider two different models in this investigation. In section 2 we discuss a finite chain of coupled oscillators of the type investigated previously [1]. The chain is driven with frequency ω at an end site and the response is determined at sites further down the chain. In section 3 we consider a different model, in which identical constraints are spaced irregularly on a vibrating string or bending beam. The dynamics of this model are closely related to those of the coupled oscillator model, particularly when the constraints on the string or beam are strong in an appropriate sense. There are a number of cases to consider, depending on the boundary conditions applied to the ends of the string or beam. We shall be concerned with the case where the boundaries are conservative, and the case where they are radiative, so that no reflection occurs.

These two models typify the two general approaches to vibration problems, via normal modes or via travelling waves. The coupled oscillator model illustrates the modal approach, as used for example by Lyon in discussing statistical energy analysis [5]. The problem of constraints spaced along a string or beam is the simplest relevant problem formulated in terms of travelling waves. This idealized model has been discussed by us previously [2], and a closely related model in which the irregularity along the string or beam is continuous rather than discrete has been discussed in the present context in considerable detail by Scott [9]. These two approaches suggest slightly different quantities as being of prime interest in the averaging process. The modal approach tends to focus attention on the response of the whole system to driving at a single point—for example, on the transfer admittance function. From the wave approach, on the other hand, it is more natural to consider the transmission of an incident wave through a length of the system. We will consider both quantities, response and transmission. For the oscillator model, we define the transmission as the ratio of response at the remote site to response at the driven site. For the problem of the string or beam, we note in passing that if there is no reflection behind the driving point, the transmission is the same as the response to driving.

In the companion paper [4], the model of coupled oscillators to be discussed in section 2 is investigated numerically. Random numbers were used to give different sets of oscillator frequencies corresponding to different physical realizations of the disordered chain. Various quantities of interest have been calculated numerically for each realization, and the arithmetic and geometric means of each of these quantities have been determined over a large number of realizations. The results complement those obtained here by analytic means: analytic results can reveal more of the parameter dependence of the phenomena discussed, but not all cases of interest are amenable to analysis. The numerical results give a more systematic coverage of the range of cases. Some readers may perhaps prefer to break off at this point and read the companion paper first, to be convinced of the main points of qualitative behaviour before returning to the detailed analysis contained in the remainder of this paper.

2. THE CHAIN OF COUPLED OSCILLATORS

In this section we study a one-dimensional chain of coupled oscillators of the sort described elsewhere [1-3]. We assume constant coupling V between nearest-neighbour sites on the chain. The equation of motion for the k th oscillator in the chain in response to driving at frequency ω elsewhere on the chain can therefore be written as

$$V(x_{k+1} + x_{k-1}) = (\omega_k^2 - \omega^2)x_k, \quad (1)$$

where x_k is the response amplitude of the oscillator and ω_k its natural frequency. We neglect for the moment any effects of dissipation. We suppose that the system is driven with a force $F e^{i\omega t}$ at site zero, where equation (1) becomes

$$Vx_1 = (\omega_0^2 - \omega^2)x_0 - F. \quad (2)$$

We shall consider an ensemble of chains of $N + 1$ sites defined by the following statistical properties, which are most conveniently expressed in terms of the distributions of the squared-frequency parameters $\lambda_k = \omega_k^2$. We assume that, in the ensemble, each λ_k is distributed independently about a common mean $\bar{\lambda}$ with a common distribution $f(\lambda - \bar{\lambda})$. The distribution f is characterized by a half-width W which is a measure of the degree of disorder. It is conventional [6] to take f constant over an interval of width $2W$ so that each λ_k lies with uniform probability within the interval

$$\bar{\lambda} - W < \lambda_k < \bar{\lambda} + W. \quad (3)$$

If we specify a set of λ_k ($k = 0, \dots, N$) consistent with the bounds in equation (3), we define what may be termed a physical realization of the chain.

It is clear that most members of the ensemble of realizations will have λ_k varying randomly with k down the chain. We may say that a set $\{\lambda_k\}$ is a "typical" one if the discrete distribution of the $N + 1$ values of λ_k in some sense approximates the smooth *a priori* distribution f . This notion becomes clearer as the number N becomes larger; it has obvious connections with the concept of ergodicity used in statistical mechanics. A set which is atypical would occur if for example all λ_k were equal. This would represent a periodic or ordered chain, and it is clear that such ordered chains must occur, albeit infrequently, in the ensemble defined above. Such atypical sets $\{\lambda_k\}$ may well produce some anomalous weighting of the ensemble average. However, we must not assume that any such anomalous weighting necessarily comes from atypical sets of λ_k . Even typical sets of λ_k (in the sense just defined) may sometimes represent chains with unusual properties: for example a chain whose λ 's were symmetrically placed about the centre could still be "typical" in our present sense.

The effect of disorder on the dynamics of a chain depends only on the ratio of disorder width W to coupling V , as may be seen by writing equation (1) in the dimensionless form

$$(x_{k+1} + x_{k-1}) = \alpha_k x_k \quad (4)$$

where $\alpha_k = (\lambda_k - \omega^2)/V$. From the inequalities (3), α_k is distributed uniformly between $\bar{\alpha} \pm W/V$, where $\bar{\alpha} = (\bar{\lambda} - \omega^2)/V$.

To determine the response we must apply a boundary condition at the far end of the chain. For simplicity, we shall use the condition $x_{N+1} = 0$, which gives $x_{N-1}/x_N = \alpha_N$. We do not expect essentially different behaviour from any other conservative boundary condition. Given this boundary condition, we can solve equation (4) by iterating it backwards, determining x_{k-1} in terms of x_k and x_{k+1} for each value of k in turn starting from $k = N - 1$, for the moment keeping x_N as an undetermined parameter. The value of this parameter is finally fixed by normalizing the solution to satisfy equation (2). It is

most convenient for this process to write equation (4) in the form of a simple recursion relation

$$x_{k-1}/x_k = \alpha_k - (x_k/x_{k+1})^{-1}. \quad (5)$$

The transmission from site zero to some site n is then $|T_{0n}|^2$, where

$$T_{0n} = x_n/x_0 = \prod_{k=1}^n (x_k/x_{k-1}) = \prod_{k=1}^n [\alpha_k - (x_{k+1}/x_k)]^{-1}. \quad (6)$$

The response at site n is $|Y_{0n}|^2$, where

$$Y_{0n} = T_{0n}x_0 = \frac{F}{V} \prod_{k=0}^n [\alpha_k - (x_{k+1}/x_k)]^{-1} \quad (7)$$

from equation (2). Note that the product in equation (7) is the same as the transmission (equation (6)) across a chain with one extra site added. Thus for the model discussed in this section, transmission and response have the same behaviour.

We first analyze the case of large disorder, $W/V \gg 1$. This treatment extends the analysis given in section II of reference [2]. We restrict ourselves for simplicity to the midband frequency $\omega^2 = \bar{\lambda}$. Thus $\bar{\alpha}_k = 0$, and for a typical set (see above) α_k is more or less uniformly distributed between $\pm(W/V)$ down the chain; most of the α_k are larger than unity in absolute value. This suggests neglecting the term (x_k/x_{k+1}) in equation (5) and approximating equation (6) as

$$T_{0n} \simeq \prod_{k=1}^n (\alpha_k)^{-1} \quad (8)$$

for large disorder. We discuss the validity of this approximation in more detail below.

Equation (8) may be expressed in terms of the geometric mean of the factors $\alpha_1, \alpha_2, \dots, \alpha_n$. Defining

$$\hat{\alpha}^n = \left| \prod_{k=1}^n \alpha_k \right| \quad (9)$$

we have

$$|T_{0n}| = (\hat{\alpha})^{-n}. \quad (10)$$

For typical sets $\{\alpha_k\}$, in the sense defined above, this amounts to determining the geometric mean of numbers uniformly distributed over the interval $0 < \alpha_k < W/V$. The result, most readily obtained by averaging $\ln \alpha$ over this interval, is

$$\hat{\alpha} = W/(eV) \quad (11)$$

and hence

$$|T_{0n}|^2 \sim (eV/W)^{2n}. \quad (12)$$

This gives the typical decay rate of transmission (or response) for strong disorder.

It is interesting to note that this typical value for the transmission may also be obtained by taking the logarithmic average (or geometric mean) of equation (8) over the ensemble. Indeed, mathematically, this amounts to repeating the calculation just carried out. Thus

$$\langle \ln |T_{0n}| \rangle = - \sum_{k=1}^n \langle \ln |x_k| \rangle = -n \ln \hat{\alpha}, \quad (13)$$

with \hat{a} given by equation (11). Evidently, taking the logarithm much reduces the sensitivity of the average to anomalous contributions. This point has been discussed recently in the solid state physics literature [8]. One can make an interesting link at this point with the earlier discussion of "typical" behaviour in terms of the mode of the probability distribution. By the central limit theorem, the sum in equation (13) will tend to be normally distributed whatever the detailed distribution of the individual terms. For a normal distribution, of course, the mode coincides with the mean.

Equation (8) is also of some relevance to ensemble averaging. We cannot use it to average $|T_{0n}|^2$ since this quantity is heavily weighted by small values of α_k where the approximation leading to equation (8) breaks down. We shall return to this problem at the end of the section. But we can use equation (8) to average the *inverse* transmission, a quantity which is also of some interest (in the solid state problem of electron conduction, this relates to the resistance of the sample):

$$|T_{0n}|^{-2} \simeq \prod_{k=1}^n (\alpha_k^2). \quad (14)$$

Since the factors α_k^2 in expression (14) are distributed independently we obtain

$$\langle |T_{0n}|^{-2} \rangle \simeq \langle \alpha_k^2 \rangle^n = (W^2/3V^2)^n, \quad (15)$$

where $\langle \dots \rangle$ denotes the ensemble average. On the other hand, typical values of $|T_{0n}|^{-2}$ are clearly the inverse of those of $|T_{0n}|^2$, i.e.

$$|T_{0n}|^{-2} \simeq [(1/e^2)(W^2/V^2)]^n \quad (16)$$

from equation (12). This is qualitatively similar behaviour to that of the ensemble average, equation (15), but the growth rate is significantly less rapid.

We see here our first example of a case where the ensemble average is different from the typical value. Evidently, on averaging equation (14) we include atypical sets of α_k whose product is much larger than the typical product. The values of α_k in these atypical sets crowd towards the extremal values $\pm W/V$ instead of being uniformly distributed over the interval.

We now return to discuss the validity of equation (8). We note first of all a kind of paradox. We can rewrite equation (5) in a form suitable for iterating in the opposite direction, towards increasing k instead of towards decreasing k :

$$x_{k+1}/x_k = \alpha_k - (x_k/x_{k-1})^{-1}. \quad (17)$$

Were we to apply the same reasoning as before it would now lead us to deduce

$$x_{k+1}/x_k \simeq \alpha_k \quad \text{and} \quad T_{0n} = x_n/x_0 = \prod_{k=1}^n (x_k/x_{k-1}) = \prod_{k=1}^n \alpha_k, \quad (18, 19)$$

a very different result from equation (8). The solution seems to grow in whichever direction we choose to iterate!

The essential qualitative feature explaining this behaviour may be seen in the very simple case $\alpha_k = \text{constant} = \alpha \gg 1$. (This corresponds to a periodic system whose pass band has been shifted by a large frequency margin, so that our frequency now lies in a stop band.) The general solution for the response then takes the approximate form

$$x_k \simeq A\alpha^k + B\alpha^{-k}, \quad (20)$$

where A and B are constants: it is the sum of a growing and a decaying component. Applying the boundary conditions $x_{N+1} = 0$ and equation (2) yields

$$x_k \simeq (F/\alpha V)[(1/\alpha^k) - (\alpha^k/\alpha^{2N+2})], \quad (21)$$

so that while there is indeed a growing component, it has extremely small amplitude. On the other hand, if we were simply to specify x_1/x_0 arbitrarily, we would expect growing and decaying components of comparable magnitude so that the growing solution would soon dominate as we moved along the chain.

The resolution of the paradox for the case of disorder has been discussed by Borland [10]. We still expect to find one solution growing, on average, in either direction. In terms of our coupled oscillator model, Borland's argument is as follows. If we impose boundary conditions at the left-hand end of the chain by specifying x_1/x_0 and then iterate equation (17), the solution grows towards the right-hand end. At the same time this equation shows that successive values of x_{k+1}/x_k become more and more precisely defined; i.e., insensitive to the initial value x_1/x_0 . This means that if, instead, we impose boundary conditions at the right-hand end by specifying x_{N+1}/x_N we are very unlikely to choose a value which matches the solution obtained by iterating from the left-hand end. The approximation equation (8) is therefore to be interpreted in a probabilistic sense. It is almost always good when iterating towards the driving point, i.e., in the sense indicated by equation (5), but occasionally breaks down very badly. If the reader is not satisfied by this heuristic argument when the original paradox was of a mathematical nature, note that a formal proof of the result will be given shortly which does not rely on the strong disorder assumption.

The preceding discussion is of some relevance to understanding how localized normal modes arise from solving the equations of motion [10]. These modes occur at very special frequencies where the two solutions obtained by iterating in from the left-hand and right-hand boundaries happen to match at some interior site, say, j . It is plausible that this is most likely to happen when the driving frequency is approximately equal to the natural frequency of one oscillator, say the j th. Now, for the two solutions to match at that point we must have

$$\alpha_j = (x_{j-1}/x_j) + (x_{j+1}/x_j). \quad (22)$$

The first term on the right is determined by iterating equation (17) from $j=1$ and has a value of approximately α_{j-1}^{-1} . The second term is determined by equation (5) and is approximately α_{j+1}^{-1} . Equation (22) may therefore be written as $\alpha_j \approx \alpha_{j-1}^{-1} + \alpha_{j+1}^{-1}$, and from equation (4) the condition for matching at site j becomes

$$\omega^2 \approx \lambda_j + [V^2/(\omega^2 - \lambda_{j-1})] + [V^2/(\omega^2 - \lambda_{j+1})]. \quad (23)$$

This gives precisely the frequency of the normal mode localized at site j , when calculated allowing for nearest-neighbour interactions only. This may be verified by putting the trial vector $\{0, \dots, 0, \delta, 1, \epsilon, 0, \dots, 0\}$ (with the unity entry in the j th position) in the Rayleigh quotient for this problem, and minimizing over δ and ϵ .

So far the discussion has been restricted to the case of strong disorder $W/V \gg 1$. In the remainder of this section we derive an exact result for the ensemble-averaged inverse transmission as in equation (15), which extends to the case of weak disorder. It is quite easy to perform the average exactly [11] without making the strong disorder approximation, equation (8). We give a simple derivation due to Dr F. Wray [12]. We write equation (4) in the form suitable for iterating towards $k=0$: $x_{k-1} = \alpha_k x_k - x_{k+1}$. Now square this and take the ensemble average. The amplitude x_k depends only on $\alpha_{k+1}, \alpha_{k+2}, \dots$, etc., and is independent of α_k . We have $\langle \alpha_k \rangle = 0$ since we are at midband, and so

$$\langle x_{k-1}^2 \rangle = \langle \alpha_k^2 \rangle \langle x_k^2 \rangle + \langle x_{k+1}^2 \rangle = (W^2/3V^2) \langle x_k^2 \rangle + \langle x_{k+1}^2 \rangle.$$

This is a simple recurrence relation, the general solution of which is

$$\langle x_k^2 \rangle = A \alpha_+^{N-k} + B \alpha_-^{N-k} \quad (24)$$

with

$$\alpha_{\pm} = -b/2 \pm [(b^2/4) + 1]^{1/2} \quad \text{and} \quad b = \langle \alpha_k^2 \rangle = W^2/3V^2. \quad (25, 26)$$

Note that $\alpha_+ \alpha_- = -1$. The coefficients A and B are determined by specifying boundary values x_N^2, x_{N+1}^2 . Since α_- is negative A can never be zero. Whatever the boundary conditions, the average is eventually dominated by α_+^{N-k} as k decreases. We therefore have

$$\langle |T_{0N}|^{-2} \rangle = x_N^{-2} \langle x_0^2 \rangle \sim \alpha_+^N. \quad (27)$$

For strong disorder $W/V \gg 1$ this result is identical to equation (15). For weak disorder $W/V < 1$ it may be written as

$$\langle |T_{0N}|^{-2} \rangle \sim (1 + b/2)^N \simeq \exp(NW^2/6V^2). \quad (28)$$

Space does not permit the derivation here of typical transmission values for the case of weak disorder. This is mathematically more subtle and has been discussed elsewhere [2]. One finds that, as for strong disorder, typical values are less extreme than the ensemble averages. In equation (28) one would have to divide the argument of the exponential by 2 to obtain the typical inverse transmitted intensity. The same behaviour will be revealed by the model to be described in the next section when the constraints are weakly reflecting, so that the disorder is weak. This may be seen by comparing equations (31) and (39) when the reflection coefficient $|r|$ is small, and noting that $|t|^2 = 1 - |r|^2$.

So far we have avoided discussing the linear ensemble average of transmitted intensity $|T_{0n}|^2$. This quantity is more difficult to determine, but some facts about it may be deduced quite easily. First, we must introduce some damping. We define modal damping factors Δ as usual, supposed the same for all modes. If Δ is allowed to tend towards zero, the ensemble average response $\langle |Y_{0n}|^2 \rangle$ must diverge like Δ^{-1} . This is because as the parameters α_k vary through the ensemble, normal mode frequencies will sweep through the driving frequency. We therefore obtain the result of integrating the intensity through resonances of width Δ , which varies as Δ^{-1} . An explicit expression for the ensemble average in terms of normal modes is given by equation (44) in the next section. In view of the relation between equations (6) and (7), the same divergence must occur in the ensemble-averaged transmission.

What is not certain is the behaviour of the coefficient of Δ^{-1} , i.e., the ensemble average of the product of squared mode amplitudes in equation (44). There seems to be no simple exact expression for this quantity. Nevertheless, one can obtain some idea of its behaviour from the following argument. First, note that just as the ensemble average $\langle |T_{0n}|^{-2} \rangle$ was weighted by sets $\{\alpha_k\}$ with large disorder in the sense defined above, so the average $\langle |T_{0n}|^2 \rangle$ is dominated by sets with small disorder. Let

$$\frac{w^2}{3} = \frac{1}{n} \sum_{k=1}^n \alpha_k^2 \quad (29)$$

define the degree of disorder w of a set $\{\alpha_k\}$. Let also $eV \ll w < W$. There is then a fraction $(w/W)^n$ of the ensemble whose members have typical transmission values at least as large as $(eV/w)^{2n}$ (from equation (12)). The increase in transmission more than compensates for the small proportion of members with reduced disorder. However, these expressions should not be used incautiously as a basis for estimating the ensemble average, since when $w \rightarrow eV$ the strong disorder result equation (12) breaks down. The transmission does not become greater than unity for $w < eV$, as suggested by the strong disorder result, but rather takes the form f^n with f of order unity. We conclude that the ensemble average is dominated by a fraction $(V/W)^n$ of the ensemble with transmission coefficients of order unity, so that we might expect

$$\langle |T_{0n}|^2 \rangle \sim \Delta^{-1} (V/W)^n. \quad (30)$$

While we should not attribute too much significance to this estimate, it is reassuring to note that numerical averaging lends some qualitative support to equation (30): see Appendix B of the accompanying paper [4].

3. IRREGULARLY SPACED CONSTRAINTS ON A STRING OR BEAM

In this section we discuss the model the behaviour of which first drew our attention to the possibility that the ensemble average and the typical value of transmission might be different. This is the case of identical constraints spaced irregularly on a vibrating string or beam (see Figure 2). We analyze the total transmission in terms of reflection and transmission of propagating waves at the constraints (see Figure 3). Each constraint is defined to have a reflection coefficient r and a transmission coefficient t as in Figure 3(a). Conservation of flux requires $|r|^2 + |t|^2 = 1$. For the bending beam we assume that the spacing between constraints is large enough for the near-field components not to influence the answer. We also assume that the variation in constraint spacing is much larger than the wavelength λ on the string or beam, so that the phase factors $\exp(ikX_n)$ are essentially random and uncorrelated. (X_n is the spacing between adjacent constraints and $k = 2\pi/\lambda$.)

Under these conditions it may be shown that the typical value of the total transmission across N constraints is of order

$$T_N \sim t^N \quad (31)$$

and the corresponding value of the transmitted intensity of order

$$I_N = |T_N|^2 \sim |t|^{2N}. \quad (32)$$

The arguments have been given in detail elsewhere [1] and we only summarize the essential points here. We associate a net reflection coefficient R_n with the totality of constraints to the right of x_n . We then define a renormalized transmission coefficient t_n at x_n according to Figure 3(b). This is given by

$$t_n = t(1 - rR_n)^{-1}, \quad (33)$$

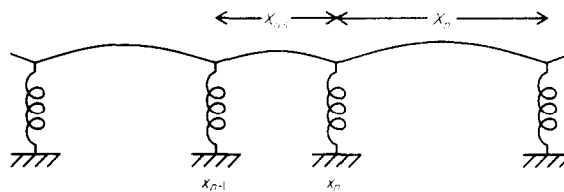


Figure 2. A vibrating string constrained in its motion by attached masses and springs.

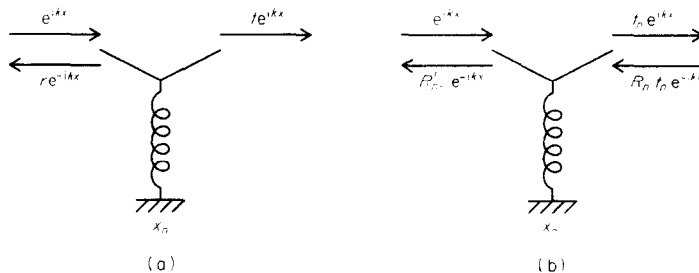


Figure 3. (a) Reflected and transmitted waves at an isolated constraint n on an infinite string; (b) modification of these reflection and transmission processes arising from waves reflected from constraints to the right of n .

and in terms of t_n the total transmission is

$$T_n = \prod_n^N t_n = t^N \prod_n^N \{(1 - rR_n)^{-1}\}. \quad (34)$$

The variation of the reflection coefficients R_n is deduced from the recursion relations

$$R_{n-1} = \exp(2ikX_{n-1})R'_{n-1} \quad \text{and} \quad R'_{n-1} = r + t^2 R_n (1 - rR_n)^{-1}. \quad (35, 36)$$

Boundary conditions are applied at the far end of the string by specifying R_N . When equations (35) and (36) are iterated towards $n=0$ the phase of R_n varies randomly with n for a typical solution because of the random phase factor in equation (35). On the other hand, equation (36) does not map the phase of R_n uniformly randomly onto that of R'_{n-1} . The phase of R'_{n-1} , the reflection coefficient immediately to the left of x_n , does not vary randomly for iteration in this sense. This solution is therefore a very atypical one for iteration in the opposite sense, towards $n=N$. The non-uniform mapping in equation (36) means that the phase of R_n becomes ever more precisely defined (i.e., insensitive to the starting value) as n gets smaller. When iterating in the opposite sense, we would therefore be unlikely to choose a starting value which would enable us to "retrace our steps" [10]. The situation bears a strong similarity to that in statistical mechanics, where microscopic reversibility of the equations of motion in time is consistent with macroscopic irreversibility. This argument is the "travelling wave" analogue of the discussion in section 2 of essentially the same phenomenon in the coupled-oscillator model.

Typical values of T_N may be deduced by taking the logarithm of equation (34) and expanding:

$$\ln T_N = N \ln t - \sum_n^N \ln(1 - rR_n) = N \ln t - \sum_{p=1}^{\infty} \frac{r^p}{p} \sum_n^N (R_n)^p. \quad (37)$$

(Since $|r| < 1$ the expansion is absolutely convergent so that the orders of summation may be interchanged in this way.) For typical solutions R_n has random phase and so the sum over n above is of order \sqrt{N} . We therefore have

$$\ln T_N = N \ln t \pm O(\sqrt{N}) \quad (38)$$

which gives our basic results in equations (31) and (32) to within a fluctuation factor $\exp(\pm\sqrt{N})$. If we ensemble average equation (38) the fluctuation term would disappear. Thus the logarithmic average of T_N gives the typical decay rate, as was found to be the case in the discussion of the coupled oscillator model in the previous section.

We now discuss the question of linear ensemble averaging the transmitted intensity $|T_N|^2$ or its inverse $|T_N|^{-2}$. We wish to ensemble average an expression of the form

$$|T_N|^2 = \prod_n^N \left\{ \frac{|t|^2}{1 + |r|^2 |R_n|^2 - 2|r| |R_n| \cos \phi_n} \right\}, \quad (39)$$

or its inverse, where ϕ_n is the phase of rR_n . These expressions cannot be averaged without knowing the distribution of $|R_n|$, which is in general difficult to determine. However, one particular case is easy. If we take a string or beam with zero dissipation and conservative boundary conditions, all magnitudes $|R_n|$ are unity by energy flux conservation. The phase factors ϕ_n in equation (39) are random and uncorrelated in the ensemble. We therefore have

$$\langle |T_N|^2 \rangle = \prod_n^N \left\langle \left\{ \frac{|t|^2}{1 + |r|^2 - 2|r| \cos \phi_n} \right\} \right\rangle_{\phi_n} = 1 \quad (40)$$

and

$$\langle |T_N|^{-2} \rangle = \prod_n \left\langle \left\{ \frac{1 + |r|^2 - 2|r| \cos \phi_n}{|t|^2} \right\} \right\rangle_{\phi_n} = \left(\frac{1 + |r|^2}{|t|^2} \right)^N = \left(\frac{1 + |r|^2}{1 - |r|^2} \right)^N. \quad (41)$$

Equation (41) is the equivalent of the results given by equations (25) to (27) in the previous section. It represents a special case of a result derived by Landauer some time ago [13]. As in the previous section we find that the ensemble average of $|T_N|^{-2}$ grows more rapidly with N than does the typical value, which is the inverse of equation (32). Evidently, in the ensemble there are values of $|T_N|^2$ much smaller as well as much greater than the typical value. The discrepancy between equations (32) and (41) is due to the factor $1 + |r|^2 = 2 - |t|^2$. When the constraints are strong $|t|$ is small and the ensemble and typical growth factors differ by a factor of 2. This should be compared to the discrepancy between equations (15) and (16) for strong disorder which is a factor of $e^2/3 = 2.46$ per site. The difference between the ensemble average and typical value must be due to special constraint configurations associated with sets of ϕ_n the values of which are weighted towards $\cos \phi_n = -1$. We give a physical description of these special configurations below.

The result given by equation (40) was first pointed out to the authors by M. S. Howe [7]. Here the behaviour of the ensemble average is qualitatively as well as quantitatively different from the typical value; localization does not manifest itself in this average. Since the average is unity and the typical value is exponentially small it is clear that the anomalous weightings come from a small proportion of the ensemble with very large transmission factors. Note that our conservative boundary conditions at the far end allow values of $|T_N|^2 > 1$, but for radiative boundary conditions $|T_N|^2$ must be less than unity. The result is therefore very sensitive to the boundary conditions.

Because the anomalous weighting in equation (40) comes from very large transmission values it does not seem to be especially associated with quasi-periodic configurations of small disorder. These will raise the transmission towards unity but not, in general, much more. The origin of the effect must be sought elsewhere, and in fact arises from the reversibility of the equations of motion when there is no dissipation present. Thus typical transmission values of the order $|t|^{2N}$ imply the existence of other atypical values $|t|^{-2N}$ in the ensemble, as has been discussed in section 2 and references [1] and [10]. These atypical values are very rare but they are so large as to give the dominant contribution to equation (40).

Further light is shed on equation (40) by thinking of the problem as that of a finite system subject to a driving force, the incident wave, and to the radiation damping due to the reflected wave. Then the transmission is proportional to the response which may be analyzed in terms of the normal modes ψ_α suitably normalized. It is convenient to use the discrete model of section 2 and write

$$T_{0n} \propto Y_{0n} = \sum_\alpha \frac{\psi_\alpha(0)\psi_\alpha(n)}{\omega_\alpha^2 - \omega^2 + 2i\omega\Delta_\alpha}, \quad (42)$$

where n labels the bays of our constrained string or beam. When there is no dissipation the modal damping Δ_α is due entirely to radiation damping, and in our case varies among modes proportional to the squared mode amplitude on the radiating site 0:

$$\Delta_\alpha \propto [\psi_\alpha(0)]^2. \quad (43)$$

The anomalous average arises because modes ψ_α localized towards the far end of the string (i.e., with small $\psi_\alpha(0)$) are weakly driven but also weakly damped. They may therefore be strongly excited at the resonant frequencies $\omega = \omega_\alpha$. The anomalous contributions come from these special, very precisely defined frequencies rather than periodic configurations.

The net effect of these contributions is to make the average response constant along the string. To show this we take the squared modulus of equation (42) and average over the narrow resonances by varying ω_α^2 and using an ergodicity assumption. We disregard cross terms: i.e., in SEA jargon we neglect modal overlap [5]. The result is

$$\langle |T_{0n}|^2 \rangle \propto \left\langle \sum_\alpha \frac{\psi_\alpha(0)^2 \psi_\alpha(n)^2}{\Delta_\alpha} \delta(\omega^2 - \omega_\alpha^2) \right\rangle \quad (44)$$

which becomes, on using equation (43),

$$\langle |T_{0n}|^2 \rangle \propto \left\langle \sum_\alpha \psi_\alpha(n)^2 \delta(\omega^2 - \omega_\alpha^2) \right\rangle. \quad (45)$$

The quantity to be averaged on the right side of equation (45) is called the local spectral density of modes of site n . It is known that this quantity depends only on the properties of the chain in the region of site n and so, far enough along the chain, it becomes insensitive to the boundary conditions at site zero. On taking the ensemble average the transmitted intensity then becomes independent of n because of the statistical homogeneity of the chain.

This confirms our previous conclusion that Anderson localization does not manifest itself in the ensemble average under the conditions assumed here. The argument above makes it quite clear that this conclusion is valid only in the absence of dissipation or of radiation damping at the far end of the string or beam, when the terms $\psi_\alpha(0)^2$ and Δ_α in equation (44) cancel. When, on the other hand, the damping is dissipation-dominated, Δ_α is mode-independent and no such cancellation takes place. Anderson localization will then certainly affect the remaining product of the mode intensities at sites 0 and n even under ensemble averaging. The modes which give the anomalous contributions to equation (40) are localized towards the far end of the string or beam and have exponentially small radiation damping due to the reflected wave as in equation (43). Only exponentially small values of the dissipation are needed to dominate this radiation damping and consequently to wipe out the effects of anomalous contributions from narrow, radiation damped resonances. Once this threshold has been passed we return to the situation described at the end of section 2 where anomalous weighting of the ensemble-averaged transmission comes from configurations with small disorder. Thus for most values of the dissipation constant we expect the ensemble average to show localization, but with a slower decay than the typical value as in equation (29).

In this section we have so far considered only the transmission of a wave incident on the left-hand constraint. We should in conclusion discuss the response of the system to a driving force. We now suppose that the wave incident on the left-hand constraint is due to a force F exciting the string as illustrated in Figure 4. If the boundary conditions to the left of the driving point are radiative, then the response of the string is just the transmission times the admittance Y_0 of the unconstrained string. If, on the other hand,

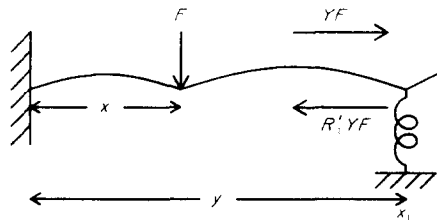


Figure 4. Response to driving near the end of a finite string, showing the net waves at the leftmost constraint.

the left-hand boundary has a non-zero reflection coefficient r_b , the net wave incident on the left-hand constraint is

$$YF = \left\{ \frac{\exp(-ikx) + r_b \exp(ikx)}{\exp(-iky) - r_b R'_1 \exp(iky)} \right\} Y_0 F, \quad (46)$$

where x and y are the distances and R'_1 is the net reflection coefficient shown in Figure 4. The response further down the string is the transmission times the incident wave YF . The quantity Y given by equation (46) is essentially the driving point response of the system as modified by the constraints.

We may now consider the average response over constraint configurations. This includes varying the constraint positions relative to the boundaries. We first consider the partial ensemble average over the distance y in Figure 4, keeping x and all constraint spacings constant. From equation (46) we may obtain

$$\langle |Y|^2 \rangle_y = \frac{|Y_0|^2 |1 + r_b \exp(2ikx)|^2}{1 - |r_b R'_1|^2}. \quad (47)$$

Suppose that the left-hand boundary is conservative, i.e. $|r_b| = 1$, and average over x . Equation (47) then becomes

$$\langle |Y|^2 \rangle_{y,x} = 2|Y_0|^2 / (1 - |R'_1|^2). \quad (48)$$

Suppose further that the string is non-dissipative and the right-hand boundary conditions are radiative. The partial average above now becomes

$$\langle |Y|^2 \rangle_{y,x} = 2|Y_0|^2 / |T_N|^2, \quad (49)$$

by flux conservation. Small values of the transmission $|T_N|$ have the effect of increasing the driving point response very substantially. Moreover, we see that the response at the far end of the string, and thus the intensity radiated from the right-hand constraint, is $2|Y_0|^2$, of order unity.

It is possible to perform the remaining average of equation (49) over all constraint configurations with radiative boundary conditions at the far end [13]. One obtains

$$\langle |T_N|^{-2} \rangle = 1 + [(1 + |r|^2)/|t|^2]^N. \quad (50)$$

When N is large this result is essentially the same as equation (41), which holds for conservative boundary conditions at the far end. At intermediate positions along the string the ensemble average response is

$$\langle |Y_{on}|^2 \rangle = \langle |Y|^2 |T_{on}|^2 \rangle = 2|Y_0|^2 \{1 + [(1 + |r|^2)/|t|^2]^{N-n}\}. \quad (51)$$

It is of order unity at the far end and grows exponentially towards the driving point.

This example illustrates the build-up of the response towards the driving point which occurs with localization in the absence of dissipation. This feature, which has been stressed previously [1, 3], would indicate that irregularity has an adverse effect in controlling noise levels. However, exponentially small values of dissipation counter the build-up. Once the internal dissipation is larger than the energy flux radiated, the overall response will begin to drop dramatically.

Let us summarize the results obtained in this section. Firstly, we have derived equations (31) and (32) for the typical transmission along the string or beam. The results correspond to equation (12) for the oscillator model, which is, however, valid only in the strong disorder limit. The random phase factor model studied in this section has enabled us to derive results for the typical transmission which extend to the regime where the constraints scatter weakly, which is in some sense a case of weak disorder. However, we should point

out that the oscillator model with weak disorder is most closely equivalent to a string or beam with strong constraints which are almost regularly spaced, the theory of which is given elsewhere [2].

Secondly, we have derived a general expression for the ensemble-averaged inverse transmission, equation (41). This result is the equivalent of equations (25) to (27) for the oscillator model. As for the oscillator model the ensemble average is larger than the typical value. It was pointed out above that the anomalous weighting comes from constraint configurations where reflection coefficient phases ϕ_n are not random but are weighted towards odd multiples of π . For strong constraints this corresponds to a configuration where the driving frequency passes close to an anomalously large number of anti-resonance frequencies of the string or beam segments when isolated from their neighbours by clamping.

Thirdly, we have derived the ensemble average transmission (equation (40)) when the far end boundary conditions are conservative and there is no dissipation. This average does not show localization. The anomalous weighting is not specially associated with quasi-periodic constraint configurations. It comes instead from the resonant response, limited only by radiation damping, when a normal mode localized at the far end of the string lies precisely at the exciting frequency. Exponentially small values of the internal dissipation constant are sufficient to counter this effect and the ensemble average will then return to localized form.

Finally, we have considered the response of the system to a driving force when there are conservative boundary conditions at the left-hand end and radiative ones at the right-hand end. The ensemble-averaged response now shows localization, but in the form of an exponential energy build-up due to reflection by the structural irregularity. This is shown by equation (51) which holds for a non-dissipative string or beam. Once again, exponentially small values of the internal dissipation are sufficient to counter the build-up and reduce the overall response.

4. CONCLUSIONS

We have discussed the use of ensemble averaging in the study of the dynamics of structurally irregular mechanical systems. We have found repeatedly that the linear average over an ensemble does not automatically give the property of a typical disordered member of this ensemble. We have illustrated this in terms of the models previously used [1-3] to demonstrate the phenomenon of Anderson localization [6]. The point is one which is not always appreciated, even by those solid state theorists who have made extensive investigations of Anderson localization. In other fields workers concerned with the theory of wave propagation in random media seem hardly aware of it at all.

We have given a definition of the typical value as the mode of the statistical distribution over the ensemble when this distribution is sharply peaked. For the ensembles and transmission and response properties considered here we have found that the logarithmic ensemble average determines the typical value.

We have calculated typical values (or logarithmic averages) for the transmission or response along a chain of coupled oscillators with disordered frequencies and across irregularly spaced constraints on a string or beam. We have also determined the linear ensemble average of these quantities where this was possible analytically; numerical calculations of these averages are described elsewhere [4]. Where the linear and logarithmic ensemble averages differ we have endeavoured to determine the nature of those ensemble members which anomalously weight the former. In practical applications to noise control of localization effects, one would wish to avoid any anomalously high values

of the transmission. It would therefore be important to be able to identify those configurations of the system responsible for such anomalous values if any occur.

We have found that anomalous weighting of the linear ensemble average can occur for two different reasons. First, in the type of ensemble usually considered in the theory of disordered systems one automatically includes some members which are obviously atypical, for example a small number of regular or periodic configurations. There will always be members which are less disordered (in the sense of equation (31)) and some which are more disordered than the degree of disorder characteristic of the ensemble as a whole. We have found that the former can weight the average transmitted intensity whereas the latter can weight the average inverse intensity. However, these members are readily identified and should not pose problems from a practical point of view.

The second reason is more subtle. Anomalous high transmission can occur in systems whose sets of oscillator frequencies or constraint spacings are not obviously atypical, provided the internal dissipation rate is small enough. In fact, such anomalous transmission values occur in most members of the ensemble at special frequencies associated with normal modes localized away from the driving point. They are very sensitive to the internal dissipation and the boundary conditions. Exponentially small internal damping factors are sufficient to render their contribution unimportant. We therefore expect that they could pose practical problems only in rather special circumstances.

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