An approach to the theoretical background of statistical energy analysis applied to structural vibration

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Rayleigh's classical approach to the study of vibration of systems having a finite number of degrees of freedom is applied to the problem of coupling of subsystems in a complicated structure, in order to probe the regions of applicability of the approach to vibration analysis usually known as statistical energy analysis (SEA). The classical method has advantages of simplicity and rigor over previous approaches to the background of SEA in certain cases, and provides extensions and simplifications in several areas of the theory. It also suggests modifications to SEA modeling strategy depending on the type of coupling involved, even when that coupling is weak, so that earlier analyses might be thought to apply.

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INTRODUCTION

A great number of papers have been written with the aim, at least in part, of investigating the scope of validity of the approach to vibration problems which is commonly called statistical energy analysis (SEA). Before adding to this literature, some apology and explanation is perhaps called for. First we must recall the basic issue. We consider a complicated structure which for some reason we wish to regard as made up of a number of simpler substructures, coupled together in some way. The problem of vibration analysis for such a structure is in principle amenable to exact calculation, but in practice exact studies of realistic structures are only possible for the lowest few vibration modes. If we are to make useful progress, we must therefore take the formal exact expressions for the vibrational response of the structure to specified driving forces, and approximate these in some way to produce useful information from reasonably simple calculations.

The SEA approach to the problem of vibration analysis for such structures involves adopting one particular point of view, and thus making approximations of a certain general type. It is supposed that energy in the form of vibration in the structure behaves in the same way as energy in the form of heat: it diffuses from one substructure to another at a rate proportional to the difference in "temperature" of the substructures, and it is dissipated internally in each substructure at a rate proportional to the temperature of that substructure. The SEA approach, when applied appropriately, gives a very simple means of predicting the distribution of mean-square vibration amplitudes among the substructures in response to a known pattern of external driving: once we have measured or calculated the various constants of proportionality, we have only to solve a few simple simultaneous equations representing the power balance for the various subsystems.

A major theoretical problem of SEA is thus to investigate the approximations implicit in these two fundamental assumptions, and the extent to which they are justified in various practical situations. The assumption that the rate of internal dissipation of energy is proportional to the kinetic energy of the substructure is a familiar idealization; its scope and the method of extending this scope are discussed extensively in Lord Rayleigh's famous book *The Theory of Sound*[1] (Chaps. 4 and 5). In Rayleigh's language, it amounts to assuming that the dissipation function is simultaneously diagonalizable with the kinetic and potential energy functions. Rayleigh shows that this assumption does not lead to serious problems provided the damping is small, as it usually is in practical problems. We shall be content to make this first assumption: our main object is to study in some detail the second assumption.

Previous studies of the second SEA assumption, that the rate of power flow between substructures is proportional to the difference in their energies, have used model systems ranging from the very special to the fairly general. These studies have generally shown that in the limit of weak coupling the proportionality relation is always a good approximation. As the coupling becomes stronger the relation holds true exactly in certain special circumstances, and more or less approximately in others. The fact that such similar behavior has been found in a variety of models suggests that there might be some simple and general results underlying it. However, the normal method of treatment in previous studies does not seem to the present author to be especially appropriate to a search for any such simple and general results, whereas the method outlined here, based again on Chaps. 4 and 5 of Rayleigh's *Theory of Sound* seems more promising for many applications. Some rather general statements about the scope of the proportionality result emerge quite easily from this approach. This simplicity increases insight into the meaning of the approximations made. It seems likely that the method might lend itself to finding more such results: this account is only a preliminary description covering the simplest application of the method. In particular, in this initial exposition we deal only with coupling between subsystems which are structures: coupling between a structure and its surrounding fluid requires a somewhat different treatment, for reasons which will be touched on in Sec. V.

Familiarity with the conventional treatment of the basis of the SEA assumptions is not likely to be of great help in understanding the treatment given here. (This is not to decry the conventional treatment, but simply

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means that the account given here is reasonably self-contained.) Some discussion of how the present approach compares with the conventional treatment is given at various points, but usually after an exposition of the proposed method. Had Rayleigh considered our problem in his book, he would only have needed perhaps a few pages to present this entire discussion. However, to give an account in isolation we must first be reminded of some aspects of his general formulation. The forbearance of the reader is asked if he is already familiar with this formulation.

This report is not concerned in the main with the statistical aspects of SEA. The aim is instead to clarify the circumstances under which the two underlying assumptions described above are true. There are two aspects to this study. One is to show how the present formulation can yield the well-known exact results of Lyon and Maidanik[2] and Lyon and Scharton,[3] as well as an interesting extension to them. It is, of course, of some interest before resorting to approximate treatments to chart the scope of usefulness of exact studies. The other aspect of this study, more significant in practice, is to seek wider conditions under which the assumptions are approximately satisfied. It turns out that such conditions are quite commonly achieved, but that the usual SEA method of proceeding from the basic assumptions to obtain useful statistical predictions needs to be modified under some circumstances, which are elucidated.

I. FORMULATION

The method we use is a simple application of the general theory of vibration of linear viscoelastic bodies. It will be recalled that the definition[4] of such a material is that it is one whose behavior can be modeled by a microstructure consisting entirely of linear elements—springs, masses, and dashpots. The theory treats systems having a finite number of degrees of freedom via matrix methods, rather than using continuous normal mode functions as the basis of discussion. It is frequently forgotten that this is the formulation used to demonstrate many general results now taken for granted, such as stationarity of the Rayleigh quotient and the various reciprocity theorems.

The assumption of a finite number of degrees of freedom does not restrict the usefulness of the results. The limit in which such a discrete system tends to a continuous system is generally very well-behaved in situations of interest. In any case, a real body has in fact only a finite number of atoms, so that in the last analysis a treatment assuming a finite number of degrees of freedom is perhaps more appropriate than one based on the continuum hypothesis!

The first step in the formulation is to choose a set of generalized coordinates which are sufficient to describe a given configuration of the system. For purposes of visualization, it is convenient to use the ordinary position coordinates of the elemental masses of which our finite-degree-of-freedom model is composed. Call these coordinates $q_1, q_2, \ldots, q_N$, or in vectorial form $\mathbf{q}$. Now we know from Lagrange's method that the dynamical behavior of the system is completely specified by three functions: the potential energy, the kinetic energy, and the dissipation function or rate of loss of energy. Within linear theory, each of these functions can be written as a symmetric, positive-definite quadratic form: define

$$\text{potential energy} = \mathbf{q}' V \mathbf{q}; \tag{1}$$

$$\text{kinetic energy} = \dot{\mathbf{q}}' T \dot{\mathbf{q}}; \tag{2}$$

and

$$\text{dissipation function} = \dot{\mathbf{q}}' F \dot{\mathbf{q}}, \tag{3}$$

where $\dot{\mathbf{q}}$ is the vector of time derivatives of the coordinates, prime ($'$) denotes transpose, and V, T, and F are all $N \times N$ real, symmetric matrices.

To treat vibration problems, we usually want to transform to what Rayleigh called normal coordinates, now often called normal modes. These are a new set of coordinates made up of linear combinations of the original ones, chosen to make the matrices T and V both diagonal, and thus to make the kinetic and potential energies into sums of squares. It is a standard result of linear algebra that we can always choose coordinates in this way to simultaneously diagonalize two real, symmetric matrices. However, it cannot, in general, be done for three matrices simultaneously. This makes a thorough treatment of the dissipation function rather messy, and we will be content to make sweeping assumptions about the form of the dissipation function to explore the issues in question.

Of course, the normal modes have a simple physical interpretation: they are the spatial configurations w_r, $r = 1, \cdots, N$ in which the system, in the absence of

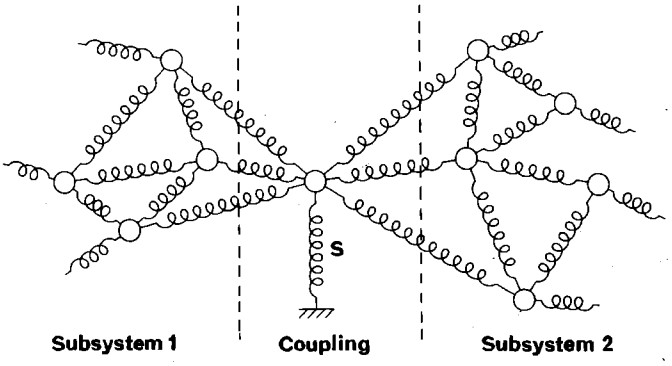


FIG. 1. Two finite-degree-of-freedom mechanical systems are coupled together via a single extra degree of freedom, corresponding to movement in one direction only of the mass m which is restrained by the spring S. For simplicity only the elastic structures are shown: dashpots are omitted. The coupling can be made weak in a number of ways, the one we discuss first being to make the spring S very strong compared with the other springs in the system. The systems can be completely decoupled by letting S become infinitely stiff. As an example of realistic coupled subsystems which can be represented in discrete form like this, consider two sections of cylinder separated by a rib; if in the frequency range of interest the ring of rib had just one resonance, its motion could be adequately described by a single degree of freedom representing the corresponding mode shape. We would certainly expect that such a resonance in the coupling would have a significant effect on power flow between the subsystems.

damping, can vibrate regularly with $q_r = w_r e^{i\omega t}$ for all r. There is a characteristic frequency ω associated with each mode, the set of these frequencies being given by the roots of the equation

$$\det(V - \omega^2 T) = 0. \tag{4}$$

Having found these frequencies, the mode corresponding to a given frequency ω is given by solving the set of simultaneous equations

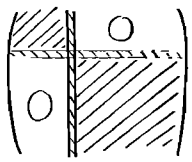
$$V\mathbf{w} - \omega^2 T\mathbf{w} = 0, \tag{5}$$

for the vector w whose components are w_r. When damping is present, the system still has a set of characteristic frequencies, now complex, given by the roots of

$$\det(V + i\omega F - \omega^2 T) = 0. \tag{6}$$

However, in this case the characteristic frequencies are not naturally associated with a mutually orthogonal set of configurations like the normal coordinates of the no-damping case.

We now examine the form of these matrices for the simplest example of coupling: consider the situation illustrated in Fig. 1, in which two subsystems are coupled together via a single degree of freedom. In this case, the potential energy matrix V takes the form shown schematically as

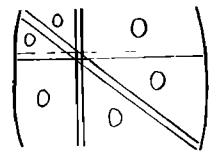


(Diag. I)

where we have ordered the coordinates q_r such that $r = 1, \ldots, n-1$ correspond to the degrees of freedom of subsystem 1, q_n corresponds to the coupling degree of freedom, and $r = n+1, \ldots, N$ correspond to the degrees of freedom of subsystem 2. The kinetic energy matrix T will obviously be diagonal if we have described the three degrees of freedom of each component mass by Cartesian coordinates (or indeed any other orthogonal coordinates). The dissipation matrix we are going to treat in a rather cavalier manner in any case, so we will not follow its fate in detail at this stage.

Our aim is to discuss power flow between the two subsystems through the coupling. Before we can do this, we need a clear definition of what we mean by the separate subsystems. There are various possibilities, and we choose one which is physically appealing and which is a variant of one often used in SEA[5]: we define our separate subsystems to be those which result from decoupling by immobilizing or "blocking" the coupling point, in this case by making the spring S infinitely stiff. In Sec. V we discuss briefly the implications of this type of blocking, and how it compares with the more usual SEA definition.

The natural way to describe these subsystems is then to transform to the normal coordinates of the blocked subsystems, essentially as done by Lyon[5] and others. This results in the two square submatrices of V being diagonalized, leaving the form



(Diag. II)

The matrix T is still diagonal, of course, and it is convenient for notational simplicity if at this stage we scale each of our individual coordinates to absorb the square root of the corresponding diagonal element of T so that T reduces to the identity matrix. The diagonal elements of each submatrix of V are then just the squared-frequency eigenvalues of that subsystem, that is the roots for ω^2 of Eq. (5) for that subsystem.

An interesting fact is now apparent: with these forms of the matrices T and V, we can no longer tell that we started with two subsystems coupled together. All we have is the general form for a system to which one extra degree of freedom has been added. This means that our ordering of the coordinates is no longer useful, since the current order reflects the two-subsystem structure. A more natural way to order the coordinates is to put the new degree of freedom first, and sort the others so that the remaining diagonal elements are in ascending order. V then takes the form

$$\begin{bmatrix} A & a_1 & a_2 & \cdots & a_{N-1} \\ a_1 & \lambda_1 & & & \\ a_2 & & \lambda_2 & & \\ \vdots & & & \ddots & \\ a_{N-1} & & & & \lambda_{N-1} \end{bmatrix}, \tag{7}$$

where A and a_r $(r = 1, \ldots, N-1)$ describe the spring strengths of the coupling elements, and λ_r $(r = 1, \ldots, N-1)$ are the squares of the natural frequencies of the blocked system in ascending order. T is still the identity matrix.

Before proceeding to analyze the behavior arising from this form of the potential energy matrix, we should note that generalization of the formulation to cope with the case of coupling of any number of subsystems through any number of degrees of freedom is very straightforward. All that happens is that we introduce a coordinate to describe each degree of freedom involved in the coupling mechanism, and perform the same sequence of manipulations of the matrices as above. The result is that after once again ordering the coordinates to put the eigenvalues in ascending order, our matrix V is of exactly the form (7), but with the single row and column of coupling elements replaced by *bands* of coupling elements.

Now we should note that *any* structure–structure coupling can be modeled in this way in terms of a finite number of degrees of freedom, as emphasized by Rayleigh. The implication is that when formulated in this way, the general problem of coupling together any num-

ber of structural subsystems by any number and type of structural coupling elements is the same, regardless of whether we are talking of two subsystems coupled in a complicated way or many subsystems coupled in simpler ways. The information about the number of subsystems is hidden in the details of the bands of coupling elements in the potential energy matrix. We thus do not necessarily have to treat separately, as has been done in the past, the problems of power flow between two arbitrarily coupled systems and of power sharing among many systems—the two problems are formally identical. Also, once we have studied our simplest example of coupling through one degree of freedom in sufficient detail, the process of adapting the analysis to more general cases should be reasonably straightforward.

Before we apply this formulation based on the classical work of Rayleigh, we note for comparison the conventional SEA formulation of the problem. There are many references for this approach, perhaps the most authoritative being Lyon's recent textbook.[5] We take the formulation of Lyon's Sec. 3.2 "Energy exchange in multidegree-of-freedom systems," but we adapt his notation for consistency.

We imagine two continuous systems coupled together in some way. We treat these initially in much the same way as above, that is to say we describe them in terms of normal modes of the blocked subsystems (but notice that we are using a slightly different definition of blocking than Lyon, as will be discussed later). The first difference is that these normal modes are now continuous rather than discrete. If we call them $v_n(\mathbf{x})$ and $w_m(\mathbf{x})$ for the two subsystems, respectively, then the differential equations governing the behavior in time of these modal displacements in the blocked subsystems are

$$\frac{\partial^2 v_n}{\partial t^2} + \delta_n \frac{\partial v_n}{\partial t} + \Gamma_n v_n = F_n \tag{8}$$

and a similar equation for the modes w_m, where F_n is the external driving force applied to the nth mode, and δ_n and Γ_n describe the internal damping and squared frequency of that mode. We now "unblock" the systems, and we suppose that the nth mode of one subsystem "feels" the modes of the other subsystem according to an assumed form

$$\frac{\partial^2 v_n}{\partial t^2} + \delta_n \frac{\partial v_n}{\partial t} + \Gamma_n v_n = F_n + \sum_m M_{nm} \frac{\partial^2 w_m}{\partial t^2} + \sum_m G_{nm} \frac{\partial w_m}{\partial t} + \sum_m S_{nm} w_m \tag{9}$$

and a similar set of equations for the modes w_m, where three matrices of coupling terms are introduced and named as follows: the term in M_{nm} is called "mass coupling," the term in G_{nm} is called "gyroscopic coupling," and the term in S_{nm} is called "spring coupling." (However, we may note that the classical term for this form of coupling is "gyrostatic,"[1,6] surely a better term since "gyroscopic" has other connotations.) It is not clear to the present author how one can claim to allow "mass coupling" and "spring coupling" simultaneously without allowing additional degrees of freedom in the coupling structure, which the conventional method does not do, in contrast to the present approach. Gyrostatic coupling seems to have no relevance to structure–structure coupling.

We can already see disadvantages to this approach. It is more complicated and yet less physically graspable and in some ways less general than ours. Whereas we can describe completely all physically realizable structural coupling in terms of elements of just one matrix (the potential energy), Eq. (9) introduces three matrices. For any given application of our approach, for example, to a plate and rib structure, we can readily imagine a finite-degree-of-freedom approximation to the structure to which our model would correspond exactly. (Not that one would construct such a model explicitly, of course, in applying the SEA methods which rest on the considerations given here: the issue is rather to know in advance whether a particular application of SEA *does* come within the area covered by the theoretical discussion.) However, given the same physical system, it is hard to see how one could, even in principle, identify all the terms of the three matrices of Eq. (9) to relate the model to general calculations. The result is that in our approach any approximations we may make will at least be explicit, and therefore analyzable, whereas in the conventional SEA treatment one starts with an equation which is itself approximate in a rather ill-defined way.

II. A NECESSARY CONDITION FOR THE POWER-FLOW PROPORTIONALITY RELATIONSHIP

We now apply our formulation to the power-flow problem. We are seeking to relate power flow between the two subsystems to the difference of their energies. It is clear from the formulation that any such result cannot refer to the subsystems as such, since the form of V shown in Diag. (II) conceals the fact that we started with two subsystems: it shows only one system to which one extra degree of freedom has been added. Notice also that although the matrix V contains coupling terms between the coordinates, T is diagonal so that it is meaningful to talk about "the" kinetic energy associated with a particular coordinate. The total kinetic energy is simply the sum of squares of velocity responses of the separate coordinates. For brevity we may refer to "the energy of the coordinate." The only possible result which we can now seek relates to power flow between pairs of our coordinates describing the blocked modes, whether we choose two modes from the same subsystem or one from each subsystem. In other words, coupling mixes energy among the modes of either individual subsystem just as much as it mixes it between subsystems. We therefore seek conditions under which power flow in the coupled system between two of our coordinates is proportional to the difference of energy between those two coordinates. Such results relating to power flow between blocked modal coordinates are of course just what have been previously discussed; for example, Refs. 2, 3, and 5. Having established conditions under which this proportionality is exactly or approximately true, we then return in Sec. V to a

discussion of the implications for actual power flow between the two subsystems, involving many modes.

In order to tackle the power-flow problem we first derive a simple *necessary* condition for the proportionality result to be true, leaving the question of sufficiency for the moment. Consider the two situations illustrated in Figs. 2(a) and (b), in which one coordinate only is externally driven and we examine the resulting energy balance between that coordinate and another one when a steady state has been reached. In case (a) the power-balance equation for the coordinate q which is not being externally driven is simply

$$T_{pq} - \Delta_q E_q = 0, \tag{10}$$

in the notation defined in the figure and caption (where we have assumed the dissipation matrix to be diagonal, as will be explained in more detail in Sec. III), so that if

$$T_{pq} = \alpha_{pq}(E_p - E_q), \tag{11}$$

for some constant α_{pq}, then

$$\alpha_{pq} = \Delta_q E_q/(E_p - E_q). \tag{12}$$

Similarly for case (b), we require

$$\alpha_{qp} = \Delta_p E_p'/(E_q' - E_p'). \tag{13}$$

Thus, for these two situations to yield the same proportionality constant (i.e., $\alpha_{pq} = \alpha_{qp}$), we require

$$\Delta_q E_q/(E_p - E_q) = \Delta_p E_p'/(E_q' - E_p'), \tag{14}$$

which is the necessary condition for the proportionality result between a pair of our coordinates.

What is needed to turn this necessary condition into a sufficient condition? This is a complicated question in general, and for this preliminary account of the general method we will be content to note only the simplest answer, as usually done in the past.[2,3,5] If the driving forces, and thus the responses, in the two cases of Fig. 2 (a) and (b) are statistically independent, then we may superimpose the two situations and drive both coordinates at once, and the contributions to power and power flow will simply add (i.e., no cross terms are generated), and the necessary condition becomes sufficient. This remains true if we now drive all the coordinates of the system: if all these driving forces are statistically independent of each other, then power flow between any pair of coordinates will be proportional to their energy difference if condition (14) is satisfied. The extent to which such an assumption of independent driving forces is satisfied in practice will vary from problem to problem, but we do not pursue the question here.

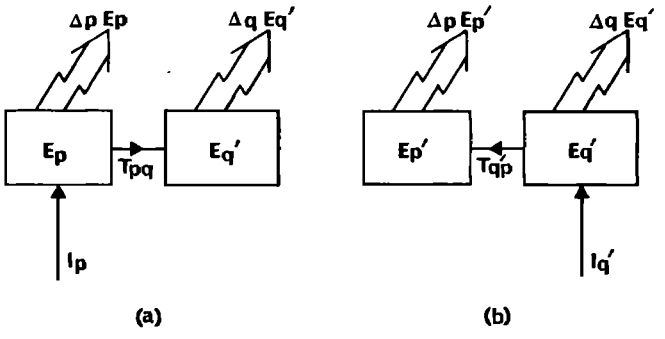


FIG. 2. Power flows and energies associated with two coordinates of our model. In case (a) coordinate p is driven, power being injected at a rate I_p. The resulting energies of the two coordinates are E_p and E_q, and the internal losses associated with the coordinates are proportional to these energies with proportionality constants Δ_p and Δ_q. The power flow between the coordinates is T_{pq}. In case (b) the situation is reversed, and all the variables acquire primes to denote this.

We can conveniently express the condition (14) in terms of the *admittance matrix* of the system. The admittance matrix $Y_{pq}(\omega)$ is the velocity response of coordinate q to a force of the form $e^{i\omega t}$ applied to coordinate p. For simplicity we consider the case in which each external driving force is random, with a spectrum which is flat in the region of interest—this is the normal assumption in SEA investigations. For case (a), we imagine applying such a random force with a spectrum of unit magnitude: the total energy E_p is then[5] just

$$E_p = \int |Y_{pp}(\omega)|^2 \, d\omega, \tag{15}$$

and the energy of the other coordinate is

$$E_q = \int |Y_{pq}(\omega)|^2 \, d\omega. \tag{16}$$

All integrals run from zero to infinity. For case (b) we apply a similar random force to coordinate q, obtaining

$$E_q' = \int |Y_{qq}(\omega)|^2 \, d\omega \tag{17}$$

and

$$E_p' = \int |Y_{qp}(\omega)|^2 \, d\omega. \tag{18}$$

But transfer admittances have the well-known reciprocal property

$$Y_{pq}(\omega) = Y_{qp}(\omega), \tag{19}$$

so that

$$E_p' = E_q. \tag{20}$$

Thus our condition (14) now reads

$$\Delta_p \int |Y_{pp}(\omega)|^2 \, d\omega - \Delta_q \int |Y_{qq}(\omega)|^2 \, d\omega = (\Delta_p - \Delta_q) \int |Y_{pq}(\omega)|^2 \, d\omega. \tag{21}$$

(It is not claimed, of course, that the application of reciprocity to SEA results is new: see, for example, Ref. 5. However, the precise use made of it here and in the next section does not seem to have appeared in print before.)

III. THE SCOPE OF EXACT RESULTS ON POWER FLOW

We now consider under what circumstances the condition (21) can be satisfied. In this section we probe the rather restricted conditions under which it is *exactly*

satisfied, and then in Sec. IV we make use of these exact results to provide useful approximations in the more general case. We first give a simple and general argument which is sufficient to show that (21) is satisfied under all conditions for which existing exact results apply, as well as for a significantly more general class of problems. Only then do we need to evaluate the admittance matrix explicitly for our model, to calculate the proportionality constant α_{pq}. We shall thus provide demonstrations of the well-known results of Lyon and Maidanik[2] relating to power flow between two oscillators, and of Lyon and Scharton[3] relating to many identical oscillators, which are greatly simpler than the original calculations.

We first note a result, discussed by Lyon in a different context (Ref. 5, p. 32) and pointed out to the author by Dr. C. Hodges,[7] relating to the total rate of power injection Π into the complete system when driven on one coordinate only. This is simply the mean of the product of driving force $f(t)$ there and the velocity response $v(t)$ of that coordinate:

$$\Pi = \frac{1}{T}\int_0^T f(t)v(t)\,dt\,. \tag{22}$$

But we can express $v(t)$ as the convolution of $f(t)$ with the Green's function (impulse response) $g(t)$ of the system at that coordinate, so that

$$\Pi = \frac{1}{T}\int_0^T \int_0^T f(t)f(\tau)g(t-\tau)\,dt\,d\tau\,, \tag{23}$$

i.e.,

$$\Pi = \int_0^T g(t)A(t)\,dt\,, \tag{24}$$

where

$$A(t) = \frac{1}{T}\int_0^T f(\tau)f(\tau - t)\,d\tau \tag{25}$$

is the autocorrelation of the force $f(t)$.

But if $f(t)$ is a random force, $A(t)$ will be a delta function $\delta(t)$, or in practice a narrow spike approximating a delta function. Thus the only aspect of the system behavior which influences power injection is the behavior of the Green's function $g(t)$ at $t = 0+$, which is well known to be a jump whose magnitude is the inverse of the mass at the driving point. Since all our coordinate masses are scaled to unity, this says that the rate of power injection Π when driving any one coordinate is independent of which coordinate is driven, and also independent of damping constants.

Thus we deduce that the total power *dissipated* by the whole system when one coordinate is driven is independent of coordinate and damping factors, so that in the notation of the previous section

$$\Delta_p \int |Y_{pp}(\omega)|^2\,d\omega + \sum_{r\neq p}\Delta_r \int |Y_{pr}(\omega)|^2\,d\omega = \Delta_q \int |Y_{qq}(\omega)|^2\,d\omega + \sum_{r\neq q}\Delta_r \int |Y_{qr}(\omega)|^2\,d\omega\,, \tag{26}$$

for any pair of coordinates p and q. Combining this with the condition (21), we deduce that our condition for the power flow proportionality relation may be written

$$\sum_{r\neq p,q}\Delta_r \int |Y_{pr}(\omega)|^2\,d\omega = \sum_{r\neq p,q}\Delta_r \int |Y_{qr}(\omega)|^2\,d\omega\,. \tag{27}$$

This condition is trivially satisfied under certain circumstances. If there are only two coordinates, then the sums in (27) become null. Thus we deduce that the power flow proportionality relation is satisfied exactly for power flow between two oscillators connected by *any* conservative coupling. This includes the case investigated by Lyon and Maidanik,[2] but also goes significantly beyond it. The other existing exact result is that due to Lyon and Scharton,[3] for a number of identical oscillators, identically coupled. Since for two such oscillators p and q

$$Y_{pr}(\omega) = Y_{qr}(\omega) \tag{28}$$

is clearly true for any other oscillator r, we see that the condition (27) is again trivially satisfied. Thus we have proved with minimal algebra all previous exact results on the proportionality relation. It is of course no criticism of the earlier results that they were derived more long-windedly: this made them no less true. However, it is surely instructive to see these results in the new light of this simple derivation.

While the bald statement that the two-oscillator result remains true for the more general case may not seem very impressive, this does represent a significant advance on the earlier result. It is possible[8] that one reason for poor results from SEA predictions in some situations is that if the coupling mechanism has degrees of freedom, but not enough to treat as another statistical group, this might impair the validity of the fundamental SEA assumption about power flow. [This point has been touched on by Smith,[8] although apparently that author did not intend the implication drawn here.[9]] However, we see that this is not so: for example, the result for flow between two oscillators coupled through a third remains true, even in the case of the "triple coincidence" when all three oscillators when uncoupled would have the same frequencies. SEA predictions may well be less accurate in such cases, but the reason is presumably to be sought in the statistical properties of the ensemble average which must be taken to produce the SEA model of a multimodal system. We discuss in Sec. V how conventional SEA modeling can be modified to deal with such coupling resonances.

Before considering whether there are any more exact results, we calculate the proportionality constant α_{pq} for the two-oscillator problem for which we have just proved the proportionality result. To do this, we must evaluate the admittance matrix Y_{pq} for our model. First, we must add to our existing forms for the kinetic and potential energy matrices a specification of the dissipation matrix F. Since our condition (21) already contains tacit assumptions that this dissipation matrix is diagonal and that there is no dissipation associated with the coupling degree of freedom, the most general relevant form is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2\Delta_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2\Delta_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 2\Delta_{N-1} \end{bmatrix}, \tag{29}$$

where Δ_j $(j=1,\ldots,N-1)$ are the same as those used above in Fig. 2 and ensuing calculations.

Now, from Lagrange's equations for the system,[1] it is not hard to see that the admittance matrix is given by

$$Y_{pq}=i\omega(V_{pq}+i\omega F_{pq}-\omega^2 T_{pq})^{-1}=i\omega\begin{bmatrix} A-m\omega^2 & a_1 & a_2 & \cdots & a_{N-1} \\ a_1 & \lambda_1+2i\omega\Delta_1-\omega^2 & & & \\ a_2 & & \lambda_2+2i\omega\Delta_2-\omega^2 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \\ a_{N-1} & & & & \lambda_{N-1}+2i\omega\Delta_{N-1}-\omega^2 \end{bmatrix}^{-1}, \tag{30}$$

where we have introduced the factor m for the mass of the coupling degree of freedom: this is a temporary departure from our previous notation where all masses were normalized out, but we shall soon want to consider the case $m=0$ to check our method against that of Lyon and Maidanik.

This matrix is sufficiently simple that its inverse may be written down by the method of cofactors. We consider only the case $N=3$; as we shall see later, we are not interested in the case of general N here. Thus consider

$$Y_{pq}(\omega)=i\omega\begin{bmatrix} A-m\omega^2 & a_1 & a_2 \\ a_1 & \lambda_1+2i\omega\Delta_1-\omega^2 & \\ a_2 & & \lambda_2+2i\omega\Delta_2-\omega^2 \end{bmatrix}^{-1}. \tag{31}$$

Hence

$$Y_{11}(\omega)=[i\omega(A-m\omega^2)(\lambda_2+2i\omega\Delta_2-\omega^2)-a_2^2]/D, \tag{32}$$

and

$$Y_{12}(\omega)=i\omega a_1 a_2/D, \tag{33}$$

where the determinant

$$D=(A-m\omega^2)(\lambda_2+2i\omega\Delta_2-\omega^2)(\lambda_1+2i\omega\Delta_1-\omega^2) - a_1^2(\lambda_2+2i\omega\Delta_2-\omega^2)-a_2^2(\lambda_1+2i\omega\Delta_1-\omega^2). \tag{34}$$

Y_{22} is of course obtained from Y_{11} by interchanging 1's and 2's. Y_{12} is symmetric in 1's and 2's as expected.

Now in order to calculate

$$\int |Y_{pq}(\omega)|^2 d\omega$$

for the various values of p and q we can make use, as does Lyon, of the remarkable tabulated values of integrals of this form due to G. R. Maclane.[10] We first do this for the case $m=0$ since it is much simpler than the case of general m and since it reduces our model to a case of that considered by Lyon and Maidanik (who treated blocking in a different way and did not allow degrees of freedom in the coupling, as we shall discuss in Sec. V). This yields

$$\frac{d}{\pi}\int |Y_{11}|^2 d\omega=\Delta_1[a_1^2a_2^2+4A\Delta_2(\Delta_1+\Delta_2)(A\lambda_2-a_2^2)] + \Delta_2[a_1^4+a_2^4-a_1^2a_2^2+A^2(\lambda_1-\lambda_2)^2 + 4A\Delta_2(\Delta_1+\Delta_2)(A\lambda_1-a_1^2) + 2A(\lambda_1-\lambda_2)(a_1^2-a_2^2)] \tag{35}$$

and

$$\frac{d}{\pi}\int |Y_{12}|^2 d\omega=a_1^2a_2^2A(\Delta_1+\Delta_2), \tag{36}$$

where

$$d=4(\Delta_1+\Delta_2)^2a_1^2a_2^2+4\Delta_1\Delta_2\{A^2(\lambda_1-\lambda_2)^2+(a_1^2-a_2^2)^2 + 2A(\lambda_1-\lambda_2)(a_2^2-a_1^2)+4A(\Delta_1+\Delta_2)[(A\lambda_1-a_1^2)\Delta_2 + (A\lambda_2-a_2^2)\Delta_1]\}. \tag{37}$$

From these results it is apparent that the necessary condition (21) is satisfied exactly by this model, confirming our general result in this case. We can then calculate the value for the constant of proportionality in the power-flow relation to obtain

$$\alpha_{12}=\frac{a_1^2a_2^2(\Delta_1+\Delta_2)}{[(a_1^2-a_2^2)+A(\lambda_2-\lambda_1)]^2+4A(\Delta_1+\Delta_2)[\Delta_1(A\lambda_2-a_2^2)+\Delta_2(A\lambda_1-a_1^2)]}. \tag{38}$$

We can now restore m in the expressions for $Y_{pq}(\omega)$, and again use the tabulated values for the integrals for the more general case in which the coupling degree of freedom is fully accounted for. As the expressions for the integrals for this case (in which the determinant D is 6th order in ω) are rather messy, the calculations are hard to do by hand. However, the problem lends itself to computer symbolic manipulation. Thus the algebra language CAMAL on the Cambridge University IBM 370/165 was used, and the result confirmed that the condition (21) is still exactly satisfied in this more general case. The individual expressions analogous to Eqs. (35)–(38) are rather too long to be worth reproducing here, but we use part of the computed results in the next two sections, when we extract weak-coupling approximations from them.

Having seen that the power flow condition (27) is satisfied exactly for certain cases, it is natural to ask whether this exact result extends to more cases: we show by a simple argument that this is not so, at least for the most natural extension within our formulation. Suppose we add one more row and column to the matrix of Eq. (36), introducing λ_3, Δ_3, and a_3 in the obvious way. Now if the condition (27) were satisfied exactly for power flow between any two of the three coordinates of this system, it is easy to see that the three quantities

$$\int |Y_{pq}(\omega)|^2 d\omega ,$$

with $p \neq q$ would all have to be equal. But this condition cannot be satisfied identically: from the expression (30) for Y_{pq} we obtain

$$\int |Y_{pq}(\omega)|^2 d\omega = a_p^2 a_q^2 \int \frac{\omega^2 d\omega}{|B(\omega)|^2} , \tag{39}$$

with

$$B(\omega) = (\lambda_p + 2i\omega\Delta_p - \omega^2)(\lambda_q + 2i\omega\Delta_q - \omega^2)\left(A - m\omega^2 + \frac{a_1^2}{(\lambda_1 + 2i\omega\Delta_1 - \omega^2)} + \frac{a_2^2}{(\lambda_2 + 2i\omega\Delta_2 - \omega^2)} + \frac{a_3^2}{(\lambda_3 + 2i\omega\Delta_3 - \omega^2)}\right).$$

Thus if we consider the case in which a_1, say, tends to zero, two of the three quantities tend to zero without changing the third, so that they clearly cannot all three remain equal. Thus exact results analogous to that of Lyon and Maidanik do not extend beyond the case of two coupled oscillators (since the argument given above is not particularly special to the case of three oscillators, but can be readily extended to larger numbers). To learn about the validity of SEA estimates in realistic structures we must go on to discuss approximate results.

IV. APPROXIMATE TREATMENT IN TERMS OF THE MODES OF THE COUPLED SYSTEM

The method we have used in Sec. III to test the necessary condition (21) treated dissipation exactly within a particular model, in which the dissipation matrix was assumed diagonal. The method we now develop treats the problem approximately, ignoring dissipation entirely at first, then allowing for it in a rather *ad hoc* way later. This approach represents a valid approximation under most circumstances when dissipation is small, as it usually is in practice. The method involves calculating the normal modes of the *coupled* system. This turns out to be very easy. Since T is just the identity matrix, we see from Eq. (5) that we have merely to calculate the eigenvalues and eigenvectors of V in the form shown in Eq. (7). If we call the eigenvalues μ_r, $r = 1, \ldots, N$ then the rth eigenvector has components

$$x_j^r = -(x_0^r a_j/\lambda_j - \mu_r), \quad j = 1, \ldots, N, \tag{40}$$

in terms of the component x_0^r, representing the amount of movement in the rth mode of the coupling point. The magnitude of this component is fixed by normalizing the eigenvector in the usual way so that the sum of squares of its components is unity: this yields

$$x_0^r = \left(1 + \sum_j \frac{a_j^2}{(\lambda_j - \mu_r)^2}\right)^{-1/2} . \tag{41}$$

The eigenvalue equation for the values μ_r is

$$A - \mu_r = \sum_j \frac{a_j^2}{\lambda_j - \mu_r} . \tag{42}$$

The physical interpretation of these eigenvectors and eigenvalues is straightforward. The new eigenvalues μ_r are known to interleave one for one the old eigenvalues λ_r, according to a general theorem given by Rayleigh.[1] (This behavior is also called the "Sturm sequence property.") In the case of weak coupling, we shall see shortly that there is just one μ_r very close to each λ_r, as again we would expect since weak coupling should not move the mode frequencies much. Now each eigenvector is dominated by the contribution from the coordinate corresponding to the value λ_r which is closest to μ_r, and there are decreasing contributions to the eigenvector from coordinates corresponding to frequencies further and further from μ_r. This is the source of the power flow between coordinates which we are seeking to study: when we excite one coordinate with some external driving force, the modes of the coupled system which are actually excited each involve a certain amount of motion of all the other coordinates.

The expression for the admittance matrix in terms of these eigenvectors is well known: it is most familiar in the statement appropriate to continuous systems with normal modes $w_r(\mathbf{x})$ and corresponding squared-frequency eigenvalues Λ_r, where $\mathbf{x}$ refers to position in the system. The result is then

$$Y(\mathbf{x}, \xi; \omega) = \sum_r \frac{i\omega w_r(\mathbf{x}) w_r(\xi)}{\Lambda_r - \omega^2} , \tag{43}$$

for the velocity response at the point ξ to the application of a force $e^{i\omega t}$ at point $\mathbf{x}$. We can directly translate this expression into the terms of our discrete system, and the fact that our coordinates are blocked normal modes rather than position coordinates does not affect its validity. Thus

$$Y_{pq}(\omega) = \sum_r \frac{i\omega x_0^{r2} a_p a_q}{(\lambda_p - \mu_r)(\lambda_q - \mu_r)(\mu_r - \omega^2)} . \tag{44}$$

Since our system is linear, we can superpose contributions to velocity of coordinate q arising from driving all the coordinates p in specified ways, and thus we have now a formal statement of the complete solution to the problem of the response of a coupled *elastic* system to known external driving (we have not yet taken any account of dissipation).

Equation (44) is an approximate version of Eq. (30), and we should note in what sense it represents an improvement over Eq. (30). The essential thing is that when we evaluate the inverse matrix of (30), we obtain algebraic expressions like Eqs. (32) and (33) which have the determinant (34) in the denominator. As the number of degrees of freedom is increased, such expressions rapidly become too large for analytic treatment. Equation (44) gives a version of these expressions expanded in partial fraction form, taking advantage of the fact that without dissipation the coefficients of that partial fraction expansion are simply given in terms of the coupled modes, which in turn are easy to write down. It is much easier to use this partial fraction series than it is to approximate the exact expressions when we want to study particular cases.

As the first and most important case, we consider the simplified forms of Eqs. (41), (42), and (44) which arise from an assumption of weak coupling. There are various ways in which weak coupling can be modeled in our system, but we shall initially discuss only the one which arises when the constant A in matrix (7) is much larger than all the other terms of that matrix. This corresponds physically to the spring S in Fig. 1 being much stiffer than other springs in the system. (We return in the next section to other ways in which coupling can be made weak.) In that case, the eigenvalue Eq. (42) reduces approximately to

$$A = a_r^2/(\lambda_r - \mu_r), \tag{45}$$

so that, as stated above, the frequencies are perturbed very little by weakly coupling the systems together. Now, from Eq. (41), we have

$$x_0^r \simeq (\lambda_r - \mu_r)/a_r \simeq a_r/A. \tag{46}$$

Finally we can write down weak-coupling approximations to the admittance from Eq. (44). We need two different versions of this: the driving-point admittance (but note that, although this is the conventional phrase, it is a somewhat peculiar use of the word "point" with our coordinates) $Y_{pp}(\omega)$ and the transfer admittance $Y_{pq}(\omega)$ $(p \neq q)$. These are

$$Y_{pp}(\omega) = i\omega/(\mu_p - \omega^2) + O(A^{-2}) \tag{47}$$

and

$$Y_{pq}(\omega) = i\omega a_p a_q/A\{[(\lambda_q - \mu_p)(\mu_p - \omega^2)]^{-1} + [(\lambda_p - \mu_q)(\mu_q - \omega^2)]^{-1}\} + O(A^{-2}), \tag{48}$$

where we have approximated the series (44) by its dominant terms, $r = p$ for Eq. (47) and $r = p$, $r = q$ for Eq. (48) on the assumption that the shift in each eigenvalue due to coupling $\lambda_p - \mu_p$ is much smaller than the eigenvalue spacing $\lambda_p - \lambda_q$.

We now perceive another disadvantage of the conventional SEA approach compared with the present approach: since the conventional approach is formulated entirely in terms of the eigenfunctions of the blocked subsystems, it can never tell us anything about how much the coupling points themselves move. We on the other hand explicitly recognize that the act of blocking reduces the number of degrees of freedom of the system, and we assign new coordinates to these degrees of freedom when we unblock the constraint. Now, when applying the theory we obviously need some way of determining in practice the proportionality constant α_{pq} of Eq. (11), or more properly the averaged value of this appropriate to a statistical discussion of power flow when many modes are involved. Various methods for doing this have been discussed,[11] but no one appears to have discussed the most obvious of all because of the shortcoming in the theoretical background associated with not explicitly allowing for coupling degrees of freedom. We would surely expect the power flow through the coupling elements to be related to the movement of those elements, since it is this movement which is zero for the blocked subsystems, and is small for weak coupling. In our formulation, we see quite explicitly that the strength of coupling, in the weak-coupling approximation, is measured by the movement of the coupling point $[x_r^0$ in Eq. (46)]. In the same equation (46) we see the basis of another method of measuring coupling strength, by eigenvalue shift produced by the coupling: this is related to the measure introduced by Newland[12] and discussed further by subsequent workers.[11] We return later to consider the usefulness of this approach to estimating the proportionality constant in practice.

We now test our necessary condition with these admittance functions calculated for the weak-coupling case. From Eq. (46) we evaluate the integrals of the squared driving point admittances. In doing this and similar integrals we must take account of the internal damping of the modes, which up to now we have ignored. An approximate treatment of damping is simply to allow the eigenvalues μ_r to be complex, with an imaginary part which must be small for the approximation to be valid. Thus write

$$\mu_r = (\omega_r + i\Delta_r)^2, \tag{49}$$

where Δ_r is in fact the same quantity as appears in Fig. 2 and the ensuing discussion. It is then a simple exercise to evaluate the integral to obtain

$$\int |Y_{pp}(\omega)|^2 d\omega = \frac{\pi}{4\Delta_p} + O(A^{-2}). \tag{50}$$

Now we need only note that

$$\int |Y_{pq}(\omega)|^2 d\omega = O(A^{-2}), \tag{51}$$

for the result (21) to be seen to be satisfied to dominant order in weak coupling. The proportionality constant α_{pq}, from Eqs. (12), (15), and (16) is then

$$\alpha_{pq} \simeq \frac{4\Delta_p\Delta_q}{\pi} \int |Y_{pq}(\omega)|^2 d\omega. \tag{52}$$

It is clear from Eq. (52) that our expression for α_{pq} is indeed symmetric in p and q, which is the essence of the necessary condition (14).

Having seen that to demonstrate the proportionality result for this particular form of weak coupling requires only very simple calculation, it is not surprising that we can generalize this result quite easily to the case where the coupling has many degrees of freedom. We have noted above, at the end of Sec. I, how the formulation of the problem is modified to take account of many degrees of freedom: the single row and column of coupling terms become bands of terms. But we also noted that one can no longer distinguish directly between two subsystems coupled in a complicated way, and many subsystems coupled in simpler ways. However, under weak-coupling approximations the results (50) and (51) are still appropriate if we take A to represent a typical diagonal term in the potential energy matrix, provided p and q refer to two coordinates which are directly coupled together. (If we have a *chain* of subsystems as shown in Fig. 3, we naturally have to do a little more calculation to discuss power flow between coordinates in nonadjacent systems.) Thus the proportionality result holds true in the limit of weak coupling, whatever the detailed nature of that coupling, between pairs of normal coordinates of blocked subsystems which are directly coupled together, when the driving forces are statistically independent.

It is perhaps now natural to ask whether the condition (14) or (21) can be satisfied when the coupling is not weak, other than in the two-oscillator case already considered. The first observation to make is that the weak-coupling approximation given above is quite robust, in the sense that we have approximated the admittance series by their largest terms, and these remain the largest terms for arbitrarily strong coupling—the eigenvalue-interlacing theorem mentioned earlier guarantees this for the single-degree-of-freedom coupling case. Thus we would not perhaps expect the result to go disastrously wrong when the coupling ceases to be extremely small.

It is in any case questionable whether one is really interested in SEA-type methods when coupling is very strong. The SEA equations say that distribution of mean-square vibration amplitude in a structure is governed by the balance between internal energy loss in the substructures and energy flow between substructures. Now, if flow through the couplings is much stronger than internal loss, then the result is simply approximate equipartition of energy among modes of the substructures, and this will happen with or without a strict proportionality relation between power flow and energy differences. Thus strong coupling is only interesting if internal loss is also high, and then other approximations we have made will break down, especially our assumed diagonal form of the dissipation function.

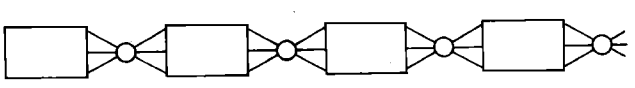


FIG. 3. Schematic representation of a chain of finite-degree-of-freedom subsystems coupled together through finite-degree-of-freedom coupling networks. This is a generalization of the case illustrated in Fig. 1.

V. IMPLICATIONS OF VARIOUS MODELS OF WEAK COUPLING

In the previous section we found that for the model of weak coupling represented by Fig. 1 with the spring S very stiff, a simple argument demonstrates the approximate validity of the power-flow proportionality relation. This calculation serves as an example of a more general phenomenon: the approximation used can be adapted to a number of other models of weak coupling. This approximation amounts simply to noting that when coupling is weak, the power flow between two coordinates is dominated by the direct pathway between those coordinates, so that we can approximate the many-degree-of-freedom problem by a two-degree-of-freedom problem for the purpose of calculating power flow. Since we have already shown that the proportionality relation is exactly true for the two-oscillator problem, it is not surprising that we were able to show readily that the relation is approximately true for this weakly coupled case.

Now the basic meaning of "weak" coupling of any kind is surely that the rate of power flow from a directly to an indirectly excited coordinate is small compared with the rate of power input to that coordinate. Thus it will always be the case, for any type of weak coupling, that a power-flow pathway from one coordinate to another via one or more intermediary coordinates will carry less power than the direct pathway, and hence the reduction to a two-oscillator problem will be a universally valid approximation for any weak coupling. The approximate correctness of the power-flow proportionality result follows automatically. Thus we can obtain leading-order approximations to the power-flow proportionality constant in various models of weak coupling, from the results of the computer algebra used above for the problem of two oscillators coupled via a third.

Before doing this, we note a further important point about the range of applicability of results obtained from the general two-oscillator calculation. If in a real problem our coupling has insufficient degrees of freedom to be treated statistically as a subsystem in its own right, then we are surely justified in allowing for only one degree of freedom in this coupling when investigating power flow: the probability of more than one coupling degree of freedom being simultaneously important (when there are not many of these degrees of freedom in total) will be small, and the approximation made in neglecting these additional degrees of freedom will be unimportant in comparison with other approximations we need to make in deriving a statistical model. Thus we can regard values of the proportionality constant derived from the two-oscillator problem as being valid approximations for *any* type of weak coupling permitted within our general model. (We discuss later in this section the major type of weak coupling which is *not* contained in this model.)

Thus we calculate from the computed expressions for integrated admittances, leading-order approximations to the proportionality constant in various cases of weak coupling. First let us be reminded of the notation in the two-oscillator problem. Figure 4 illustrates this.

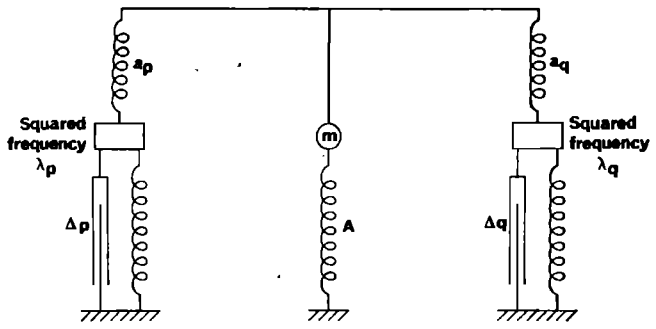


FIG. 4. Notation for the problem of power flow between two oscillators coupled through a third.

The first case is the one discussed in the previous section: A very large compared with the other springs in the system. Gathering up leading-order terms in A from numerator and denominator of the expression (12) for α_{pq} then yields

$$\frac{A^2}{a_p^2 a_q^2}\alpha_{pq} \simeq \frac{\Delta_p + \Delta_q}{(\lambda_p - \lambda_q)^2 + 4(\Delta_p + \Delta_q)(\Delta_p\lambda_q + \Delta_q\lambda_p)} + \frac{m\Delta_p\Delta_q}{\Delta_p a_p^2 + \Delta_q a_q^2}. \tag{53}$$

Notice that if we set $m=0$ in this expression, it agrees with the large-A limit of Eq. (38) which we calculated by hand for the case $m=0$: this gives useful confirmation of the correctness of the computer algebra. Now we note an interesting difference from the usual SEA proportionality constant.[5] In addition to the first term, which is dominated by resonant coupling ($\lambda_p \simeq \lambda_q$), we have a nonresonant term which is proportional to the mass m of the coupling degree of freedom but independent of frequency. Since only the first term has been considered previously, conventional SEA assumes all power flow in weakly coupled systems to be dominated by resonant coupling. We now see that this is not necessarily the case.

The second type of weak coupling which is easy to discuss arises when the mass m becomes very large compared with other masses in the system. Collecting dominant-order terms in m yields

$$1/\alpha_{pq} \simeq m[(\lambda_p^2/a_p^2\Delta_p) + (\lambda_q^2/a_q^2\Delta_q)]. \tag{54}$$

In this case, we find that power flow does not depend to leading order on any resonant terms. There is simply more power transfer, the lower the individual frequencies of the blocked modes. Thus for systems in which coupling is weak because of large mass, standard SEA could give quite the wrong answer by assuming resonant effects to dominate.

The final case of weak coupling which we can usefully consider is when the coupling springs a_p and a_q are both weak compared with A, or equivalently that A and m are both large compared with other springs and masses in the system but are comparable with each other. This enables us to see the effect of coupling resonance on the power flow. Collecting lowest-order terms in a_p and a_q, or equivalently highest-order terms in A and m taken together, yields

$$\alpha_{pq} \simeq \frac{a_p^2 a_q^2 \Delta_p \Delta_q}{m[\Delta_p a_p^2(\Omega - \lambda_q)^2 + \Delta_q a_q^2(\Omega - \lambda_p)^2 + 4\Omega\Delta_p\Delta_q(\Delta_p a_q^2 + \Delta_q a_p^2)]}, \tag{55}$$

where

$$\Omega = A/m \tag{56}$$

is the squared frequency of the coupling oscillator. We now see that power flow is not dominated by resonance of the two subsystem degrees of freedom with each other, but rather by resonance of each separately with the coupling degree of freedom. This is not especially surprising, and it perhaps sheds light on the nonresonant terms found in the preceding two cases, in which Ω tended to infinity and zero, respectively. The fact that conventional SEA needs modification to cope with resonances in the coupling has been mentioned previously,[8] but little appears to have been said about how such modifications should be made. Equation (55) provides a preliminary answer to this question.

Having obtained approximate proportionality constants for a variety of weak-coupling configurations and found that these depart significantly from the conventional SEA expressions, we can ask whether these differences are directly observable, and how they affect SEA modeling of real systems. We first try the usual SEA procedure, and apply to Eqs. (53)–(55) a set of sweeping assumptions which enable us to generate simple statistical predictions. We suppose that the driving forces on the various coordinates are statistically independent, and that all coordinates in each subsystem have the same energy, E_1 for subsystem 1 and E_2 for subsystem 2. We then suppose for simplicity that only subsystem 1 is directly excited, with band-limited white noise having a flat spectrum between frequencies ω and $\omega + B$. This means that the total power flow into coordinate q of subsystem 2 is approximately

$$(E_1 - E_2)\sum_p \alpha_{pq}, \tag{57}$$

where the sum is over coordinates p of subsystem 1 whose resonant frequencies lie within the band of excitation. Thus the total power flow Π_{12} from subsystem 1 to subsystem 2 within this frequency band is approximately

$$\Pi_{12} \simeq (E_1 - E_2)\sum_p \sum_q \alpha_{pq}, \tag{58}$$

summing over modes q of subsystem 2 lying within the excitation band. Assuming that couplings a_i and dissipation factors Δ_i are constant for each separate subsystem in the band of interest makes these summations easier, since it leaves only the frequency dependence of α_{pq} to consider. If we finally assume that the frequencies of resonance of the two subsystems are independent random variables, uniformly probable (in an ensemble-average sense) over the band in question, with densities

ρ_1 and ρ_2, respectively, we can approximate the summations in Eq. (58) by integrals to give

$$\Pi_{12} \simeq \rho_1\rho_2(E_1 - E_2) \int_{\omega}^{\omega+B} \int_{\omega}^{\omega+B} \alpha(\omega_p, \omega_q) d\omega_p d\omega_q , \quad (59)$$

where

$$\omega_i^2 = \lambda_i \quad (60)$$

and we now regard α as a continuous function of ω_p and ω_q.

Equation (59) represents the normal SEA approach, and it is valid for the normal SEA type of proportionality constant α, exemplified by the first term of Eq. (53). Under our approximations we can replace all occurences of ω_p and ω_q in that term by ω, except in the difference $\omega_p - \omega_q$. We can then do the ω_q integral straightforwardly to obtain

$$\int \alpha(\omega_p, \omega_q) d\omega_q \simeq \frac{a_1^2 a_2^2}{A^2} \frac{\pi}{4\omega^2} , \quad (61)$$

where a_1 and a_2 are the values of a_p and a_q appropriate to subsystem 1 and 2, respectively (assumed constant for each subsystem). Now the integrand of the ω_p integral is approximately independent of ω_p, so that the final approximation to Π_{12} is

$$\Pi_{12} \simeq \rho_1\rho_2(\pi/4\omega^2)(a_1^2 a_2^2/A^2)(E_1 - E_2) , \quad (62)$$

which agrees with Eq. (3.2.9) of Lyon.[5]

However, if we were to carry out similar integrations on the other term of Eq. (53) we would obtain an erroneous answer, a contribution to Π_{12} proportional to B^2, since this second term is independent of frequency. The answer would be

$$\Pi_{12} \simeq \left(B \frac{\pi}{4\omega^2} \frac{a_1^2 a_2^2}{A^2} + B^2 \frac{m\Delta_p\Delta_q}{\Delta_p a_p^2 + \Delta_q a_q^2} \right) \rho_1\rho_2(E_1 - E_2) . \quad (63)$$

We can see that this makes no sense: in the approximation $B \ll \omega$ which we have made, we are only entitled to retain terms linear in B since we have already made linearizing assumptions. This does not mean that the second term is necessarily negligible, but indicates that we have made inconsistent approximations. The problem is that in deriving Eq. (60) we have taken the values of α_{pq} calculated by integrating from zero to infinity over frequency, and have used these values to approximate the band-limited integrals from ω to $\omega + B$. This is an acceptable approximation for a normal SEA form of α_{pq} dominated by resonance of ω_p with ω_q, but is wrong for nonresonant forms of α_{pq} like the second term of Eq. (53).

Similarly, the expression for α_{pq} in Eq. (54), if integrated in the same way, yields an inconsistent value of Π_{12} proportional to the square of bandwidth since the integrand is only slowly varying over the range of integration. The expression (55) for α_{pq}, however, yields a correct approximation to Π_{12} proportional to $\log(B)$ provided the resonant frequency of the coupling lies in the band over which we integrate: both integrals are approximated by resonant terms in this case, so that power flow is not strongly dependent on bandwidth.

The nonresonant terms in some of the expressions for α_{pq} are telling us something, but we have just seen that the usual SEA procedure is not appropriate for revealing the message. It is not hard, however, to see what that message is, at least in qualitative terms. One particular problem with expressions like Eq. (63) is that they appear to treat ω and B as separate variables. However, in fact all information about the dependence of total power flow on both these parameters is contained in a single function of frequency, a convenient representation of which is the cumulative curve of power transfer between the two subsystems as a function of frequency; that is, the indefinite integral with respect to frequency of the power transfer function. This curve is monotonically increasing since energy always flows from the directly excited subsystem to the indirectly excited one, at each individual frequency, so we are integrating a strictly positive function. From such a curve, we can read off the power transfer for any given band of noise driving one subsystem, as the difference between the values of the function at the extremities of the band in question. Thus the question we should ask about the three forms of weak coupling discussed above is what shape this curve should have in each case. We would expect to obtain curves of the general form indicated in Fig. 5. For spring coupling [Fig. 5(a)], the curve is more or less straight, so that power flow is roughly the same for a band of a given width with any center frequency. For mass coupling [Fig. 5(b)], on the other hand, the power flow for a given bandwidth increases as the center frequency of the band is reduced. For resonant coupling [Fig. 5(c)], as might be expected, the power flow is greatest in the vicinity of the resonant frequency of the coupling.

This description can be put on a quantitative basis, but we do not go into it further here since it requires a slightly different approach to the one we are discussing: we were only just able to do the integrals to infinity, so

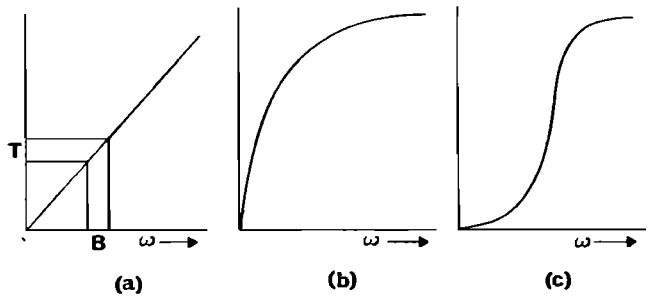


FIG. 5. Cumulative curves of power transfer between two subsystems through various sorts of weak coupling, as a function of frequency ω. Thus total power flow between the two subsystems in case (a) when driven with the band B of noise as indicated is T. Case (a) refers to weak coupling through a strong spring. Case (b) refers to weak coupling through a large mass, and has a form approximately ω^{-3}. Case (c) refers to coupling through a resonator to which the subsystems are only weakly linked, and power flow is greatest in the vicinity of the resonant frequency of the coupling. This way of representing the trends revealed by a statistical analysis in cumulative curves seems particularly useful, since such curves can be measured directly in some experimental situations, obtaining reasonably smooth curves which can readily be compared with the predictions.

cannot use our method to treat finite ranges properly. The proportionality constants α_{pq} calculated above are not the right starting point. It is interesting to note, though, that the approach offers the possibility of making useful SEA estimates (embodied in such cumulative curves) for the case when the coupling has a few resonances only, too few to treat statistically but few enough that they can be predicted with reasonable accuracy deterministically. From a knowledge of the character and frequency of each coupling resonance of interest, one could piece together a curve like Fig. 6, made up of sections of the various kinds shown in Fig. 5. Such a combination of statistical and deterministic analysis appears to be new in the area of structural vibration, although known to room acousticians. It deserves to be further investigated.

The analysis of power flow between weakly coupled subsystems presented in the previous section enables us to link the present approach to another area of existing work on SEA. If we combine the expression (62) for power flow in the case $m=0$ with expression (45) for the shift in eigenvalue caused by the coupling, we obtain

$$\rho_1\rho_2 B\pi\Delta\omega_1\Delta\omega_2(E_1-E_2), \tag{64}$$

where $\Delta\omega_1$ is the shift in angular frequency of a typical mode of subsystem 1 as the blocking constraint is relaxed, and $\Delta\omega_2$ is the same thing for subsystem 2. This is identical in form to an expression due to Newland,[12] which has been further discussed subsequently.[11] Equation (64) differs from Newland's formula by a factor 2, but the two results do not really refer to the same situation: we are discussing eigenvalue shift between a system blocked in our manner and the coupled system, whereas Newland treats eigenvalue shift between *decoupled* (i.e., separated) subsystems and subsystems blocked in the Lyon manner.

This qualitative agreement between two quite different approaches suggests that one could characterize power flow in a similar way using any suitable definition of eigenvalue shift ("suitable" meaning one which goes to zero as the coupling becomes progressively weaker). However, this surely does not indicate a direct connection between eigenvalue shift and power flow, but rather a link between eigenvalue shift and an appropriate measure of coupling strength, and a further link between coupling strength and power flow. It is surely clear that any sort of weak coupling produces an eigenvalue shift which increases continuously as the measure of coupling strength increases, so that by Taylor's theorem when this measure of coupling is sufficiently small, the eigenvalue shift is simply proportional to it.

It seems a futile exercise to seek circumstances under which the eigenvalue-shift formula remains a valid approximation for strong coupling, as has been suggested. Whether this is ever the case will depend strongly on details of the model in use, in a way which will be hard to assess in any practical problem. In any case, it is extremely doubtful whether eigenvalue shift can ever form the basis for a useful method of determining the average power-flow proportionality constant in SEA modeling. SEA is only of use in frequency regions where the modes of the structure are quite dense. Now, if one changes the system in some way and tries to identify the average eigenvalue shift, one has first to trace which mode of the changed system corresponds to each mode one of the original system. Since high modes are notoriously sensitive to small changes in structure, this is a nontrivial exercise. Even then the eigenvalue shift will be very small and thus hard to measure with useful accuracy. Thus while it may be barely possible to make the method work in a controlled laboratory experiment, it is unlikely to be feasible in the field.

We close this section by noting the one important type of weak coupling between structural subsystems which does *not* fit naturally into our general model. This is coupling via a large impedance discontinuity. Such coupling can, of course, be included in finite-degree-of-freedom models and thus discussed in terms of potential, kinetic, and dissipation matrices as we have done. The reason it does not fit into our model is that it needs a different type of blocking to separate the subsystems

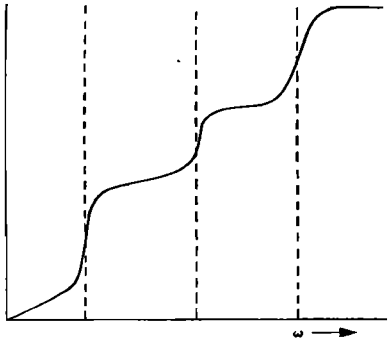


FIG. 6. Qualitative indication of the sort of cumulative curve of power transfer to be expected when the coupling network has a few (three in this case) resonances in the region of interest. The coupling resonances are indicated by dotted lines. Such curves could be predicted by a combination of statistical analysis of the subsystems and deterministic analysis of the coupling, and could represent a useful extension of normal SEA to cope with such cases of coupling with insufficient resonances to be treated statistically.

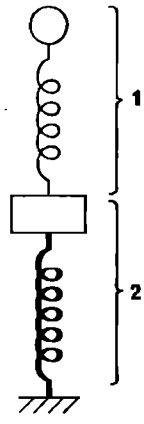


FIG. 7. The simplest example of coupling of subsystems through and impedance jump. The impedance of oscillator "2" is much higher than that of oscillator "1."

in a natural way, and thus provide the appropriate coordinates for a discussion of power flow due to coupling. The essential thing about the method of blocking which we have discussed is that by performing one action (immobilizing one or more coupling degrees of freedom) we block all subsystems simultaneously. By contrast, for impedance-jump coupling the most natural way of "blocking" the subsystems on either side of the coupling involves doing two different things to the two subsystems. We can best illustrate by an example. Figure 7 shows the simplest mass-and-spring example of this type of coupling: a high-impedance oscillator with a lower-impedance one sitting on top of it. Now the natural way to describe oscillator 2 alone is to *remove* oscillator 1, while the natural way to describe oscillator 1 alone is to *immobilize* oscillator 2. This separation describes, of course, what happens to the normal modes of the coupled system as the impedance difference tends to infinity. This in turn is precisely the limit in which the coupling we consider goes to zero, and it is in this sense that we can claim to have described the most natural form of "blocking."

This asymmetric blocking appropriate to impedance-jump coupling is essentially the blocking used exclusively by previous workers in the field (e.g., Lyon[5]). Since impedance-jump coupling was especially important for the early applications of SEA, this may have been a sensible direction for the early theory to take. However, it leads to difficulties in presenting a general theory which is entirely convincing, and it is precisely by getting away from it to the simpler, symmetric blocking that we have been able to give a reasonably simple account of the fundamentals of SEA for situations where such blocking is appropriate. Our account is thus in some ways complementary to the existing literature, since problems best suited to the traditional treatment are not included here, while precisely those questions which present difficulties to the traditional approach (e.g., resonances in the coupling) are easily dealt with here. The question of whether an analysis similar to the one presented above can be usefully performed for weak coupling via impedance jumps is not pursued further at present.

VI. CONCLUSIONS AND DIRECTIONS OF FURTHER WORK

We have given a self-contained, systematic account of some of the circumstances under which the rate of power flow between two subsystems of a complicated structure is proportional to the difference of "temperature," or mean kinetic energy per vibration mode, between those subsystems. This proportionality result forms the basis of the SEA approach to vibration analysis of the structure. We have found that the most natural context in which to study the proportionality result is between pairs of coordinates which correspond to normal modes of the subsystems when blocked (isolated) in an appropriate way. Our approach differs somewhat from that taken by previous workers in the field, particularly in the treatment of coupling between subsystems, and hence in the appropriate method of blocking this coupling to define precisely what is meant by the separate subsystems.

We have considered coupling such that an appropriate method of blocking consists of immobilizing one or more parts of the structure: coupling of plates through substantial ribs would represent a typical example. An important effect of such blocking is that it removes some degrees of freedom from the system, and in order to describe properly the vibration of the complete, coupled system we must restore these degrees of freedom. This enables us to take account readily of a variety of different types of coupling, including coupling through a structure (such as the rib separating the plates mentioned above) which exhibits a small number of resonant frequencies itself in the frequency range of interest. This appears to represent an advance of the conventional SEA approach. Degrees of freedom in the coupling are not explicitly taken into account in that conventional approach, so that if a coupling structure does not have enough resonances to be treated as a statistical subsystem in its own right, it cannot usefully be treated at all.

We first sought conditions under which the power-flow proportionality relation is exactly satisfied. We were able to reproduce the well-known results of Lyon and Maidanik[2] for two coupled oscillators and of Lyon and Scharton[3] for many identical oscillators, in both cases much more simply than the original derivations. We were also able to provide a significant extension of Lyon and Maidanik's result to the general case of two oscillators coupled through any conservative network. The proportionality result remains exactly true in that case. We were then able to show that the proportionality result is not exactly true when we allow for more than two oscillators, so that to make further progress one has to seek approximate results.

The most obvious approximations one can make are of weak coupling in some sense, and of small damping compared to the potential and kinetic energies of the subsystems. A discussion of the various possible types of weak coupling within our formulation revealed that the case of two oscillators coupled through a third represents a useful approximation to several different types of weak coupling, while one important type (coupling via an impedance jump) is not readily allowed for within our general model. For the cases which we could treat, we obtained expressions for the expected proportionality constant, and these expressions yield interesting predictions which differ in some ways from traditional SEA predictions. These predictions concern the expected total power flow between two subsystems when excited with random noise with a flat spectrum within a limited bandwidth. Three types of weak coupling were described, which can be characterized as spring coupling, mass coupling, and resonant coupling. The effect on power flow of the three types was illustrated qualitatively in Fig. 5. Further study of the implications for SEA modeling strategy of the forms of power-flow proportionality constants arising from these forms of coupling is desirable. Of the new effects, the "doubly-resonant" behavior arising from resonances in the coupling structure is fairly readily allowed for, but

the "long-range" effects of nonresonant terms are much more awkward. (By "long-range" we mean that a given mode in one subsystem communicates energy at a significant rate not only to modes of the other subsystem which are close to it in frequency, but also to more distant ones.) Possibly these effects can be included in a SEA model of a structure in a similar way to that used by Crocker *et al.*[13] who modeled transmission loss through a partition separating two rooms with some success using SEA. They incorporated the nonresonant "mass law" contribution to the coupling of the rooms in their model as a separate pathway from the resonant "coincident" transmission. Our nonresonant terms in Eqs. (51) and (52) also represent "mass law" coupling in a sense, although the argument justifying the normal mass law for sound transmission through partitions is not at all relevant to us.

If the new effects can be thus successfully included in SEA models of structures, the next question to address is how they influence the statistical behavior of the SEA estimates. After all, an estimate, to be useful, needs to carry some error bounds. Such bounds have been discussed by Lyon[5] for conventional SEA modeling, and this discussion would have to be extended to include the new effects. One qualitative result which we can immediately anticipate is that coupling resonances, while they can be treated as described above, will tend to increase the variance with respect to the SEA estimate of the mean, since the statistical fluctuations of the real system compared with the simple assumptions which we have made will have more effect in the doubly resonant case than in the usual SEA model.

As well as studying the results of the analysis presented here, there is more work to do on refining that analysis. We have made a number of sweeping assumptions in the course of the study (although no more so than previous workers in the field), and it is important to know how much effect these assumptions have. Probably the most important of these is the very special form assumed for the dissipation function. While it may be acceptable when damping is small to treat it in a cavalier way, damping is not always small in structures to which we would like to apply SEA. Indeed, the prediction of the effect of high-damping treatment applied to certain parts of a structure is an entirely appropriate question to ask of a SEA model. Thus more work needs to be done on how the analysis is modified by a more thorough treatment of damping, and by a relaxation of other special assumptions made to simplify the initial study. It is hoped that such further studies will extend the analysis given here to a complete and theoretically satisfactory base for all SEA modeling of structural vibration.

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