On the Playability of Violins.
Part II: Minimum Bow Force and Transients

J. Woodhouse
Cambridge University Engineering Department

Dedicated to the memory of Lothar Cremer (1905–1990)

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Summary

Two issues are addressed concerning the possible variations between violins, or between notes on one violin, of the perceived "playability". A theoretical framework was established in a companion paper, and some general issues were reviewed. The first problem examined here concerns the bow-force limits for the playing of a steady note, and it is found that the minimum bow force may be a source of playability variations. An extension of Schelleng's calculation is used to provide a measurement technique for the note-by-note variation of minimum bow force. The connection between minimum bow force and wolf notes is explored. The second topic is a study by computer simulation of which regime of self-sustained oscillation arises from a particular starting transient, and how this varies with the waveform of applied bow force. This is only a preliminary investigation of a very difficult problem, but the results are very encouraging since they reveal a pattern of behaviour which is at least qualitatively in accordance with playing experience.

Über die Spielbarkeit von Violinen.
Teil II: Mindestbogendruck und transiente Vorgänge

Zusammenfassung

Es werden zwei Aspekte möglicher Unterschiede zwischen Violinen oder zwischen Tönen einer Violine hinsichtlich der subjektiven „Spielbarkeit" angesprochen. In einer begleitenden Arbeit wurde ein theoretischer Rahmen hierfür erstellt und einige allgemeine Gesichtspunkte wurden erneut überprüft. Das erste der hier untersuchten Probleme betrifft die Grenzen des Bogendruckes für das Spielen einer langen Note, und es wurde gefunden, daß der minimale Bogendruck für Unterschiede der Spielbarkeit verantwortlich sein kann. Eine Erweiterung von Schellengs Berechnung wird benutzt, um eine Meßtechnik für die tonweise Veränderung des minimalen Bogendruckes zu erstellen. Des weiteren wird die Verbindung zwischen dem minimalen Bogendruck und dem Wolfton untersucht. Im zweiten Punkt wird durch Computersimulation untersucht, welcher Bereich einer selbsterregten Schwingung von einem bestimmten Anfangstransienten herrührt und wie dieser sich mit dem Verlauf der aufgewandten Bogenkraft verändert. Obwohl es sich hier nur um die vorläufige Untersuchung eines sehr schwierigen Problems handelt, sind die Ergebnisse sehr ermutigend, da sie ein Verhaltensmuster aufzeigen, das zumindest qualitativ in Übereinstimmung mit der Erfahrung des Spielers steht.

L'aisance de jeu des violons.
Partie II: pression d'archet minimale et transitoires

Sommaire

Cet article examine deux des éléments qui peuvent rendre compte des différences perçues entre violons, ou entre notes d'un même violon, en termes de «jouabilité». Dans un article précédent [1], on a établi un cadre théorique et passé en revue quelques problèmes généraux. Ici nous analysons tout d'abord le problème des limites de pression d'archet pour le jeu d'une note entretenue, et l'on constate que ce paramètre peut être une source de variation de la jouabilité. Une extension de la méthode de Schelleng fournit une technique de mesure des variations note par note de la pression minimale d'archet. On explore les liens entre la pression minimale d'archet et les «notes de loup». En second lieu on étudie, par simulation numérique, le régime vibratoire entretenu résultant d'un transitoire d'attaque particulier, et les variations de ce régime avec la forme d'onde de la force appliquée par l'archet. Ce n'est qu'une recherche préliminaire sur un problème très difficile, mais les résultats sont encourageants car ils révèlent des types de comportements qui sont en accord, au moins qualitativement, avec l'expérience du jeu.

1. Introduction

In the first part of this study [1] it was pointed out that violins differ from one another not only in their sounds, but also in what may be called their "playability". This term is meant to cover the range of reactions a player may have which contribute to such judgements as that one instrument is easier to play than another, or that a particular note on an instrument is hard to control. The current understanding of the mechanics of the bowing process was reviewed, and a detailed study made of how, within the simplest models of that process, the response characteristics of the

Received 26 March 1992,
accepted 15 October 1992.

Dr. J. Woodhouse, Cambridge University, Engineering Department, Trumpington Street, Cambridge CB2 1PZ, U.K.

instrument body can influence behaviour under the bow.

Within the theoretical framework thus established, two specific problems related to playability are addressed in this paper. The first, in Section 2, relates to the response of single sustained notes: we consider the detailed characterisation of the minimum bow force for the usual Helmholtz motion. This leads naturally to a short discussion of the "wolf note", the most striking manifestation of note-by-note variation in minimum bow force. After that we enter the more difficult territory of transient behaviour, and consider in Section 3 how computer simulation can be used to explore the "robustness" or "ease of playing" of a given note on a given instrument. Throughout, efforts are made to relate the theoretical ideas to things which are measurable on a real instrument, so that there is a realistic possibility of testing the conclusions. In the first case, relevant measurements are relatively straightforward, while for the second case things are much more difficult. Complete answers certainly cannot be given yet for either case, but the issues and the present state of knowledge are reviewed.

2. Minimum bow force

2.1. Schelleng's method

In the existing literature of the bowed string, only one issue which is obviously related to playability has been discussed in any detail. This concerns the limits on the various playing parameters within which the usual Helmholtz motion is possible with steady bowing. Discussion of the allowed ranges of these parameters has usually been couched in terms of limits on the bow force, for given values of the bow speed and position. One may thus speak of the *minimum bow force*, below which the Helmholtz motion degenerates to a "surface sound" (usually an oscillation regime of the string involving more than one slip per period length), and a *maximum bow force* above which the Helmholtz motion again degenerates into something musically unacceptable, in extreme cases a raucous "crunch". Approximate formulae for both bow force limits have been calculated by Schelleng, who then presented the results in a graphical form which has become very well known [2].

If these bow force limits were to vary strongly from instrument to instrument, or from note to note on a given instrument, one might reasonably expect a player to experience this as a variation in "playability" of some kind. They thus provide a natural first candidate for investigation. We will examine Schelleng's calculations, and find that in the case of minimum bow force an extension of his argument suggests a measurement which may quite easily be made on a real instrument, and which has some promise of relating to playability judgements.

Schelleng's maximum bow force criterion was calculated from the condition that at maximum force, the arrival of the Helmholtz "corner" at the sticking bow is no longer sufficient to trigger a transition to slipping. The accurate time-keeping role of the corner ceases to operate, and the result was described by Schelleng as raucous motion. This state of affairs depends only on conditions near the bowed point: frictional behaviour and the magnitude of the velocity jump in the Helmholtz corner. It does not involve in any obvious way the small changes in the string motion as a consequence of interaction with the instrument body, so does not seem a good candidate for being strongly involved in judgements of playability. For later reference, Schelleng's formula is

$$f_{\max} = \frac{2 v_b}{Y_0 \beta (\mu_s - \mu_d)} \tag{1}$$

where μ_s and μ_d are respectively the coefficients of sticking friction and sliding friction (at the Helmholtz slip speed, $-v_b[1/\beta - 1]$ in the ideal case). Other notation follows ref. [1]: v_b is the bow speed, Y_0 is the characteristic admittance of the string, and β specifies the bowing position as a fraction of the string length.

As has been discussed elsewhere, the maximum bow force used in practice by an experienced player is not governed by the onset of raucous motion. Two other phenomena have been described which can limit maximum bow force under certain circumstances. These are the growth to unacceptable levels of a non-periodic "noise" component accompanying the note [3], and the flattening of the pich of the note as bow force is increased [4]. The physical mechanism leading to noise growth depends on the finite width of bow hair in contact with the string, but is otherwise governed by similar considerations to Schelleng's maximum force. It does not seem likely to vary between instruments or from note to note, and so probably does not contribute strong playability effects. The flattening effects seems rather more likely to be of relevance to this discussion. It depends on the extent, and perhaps the details, of the rounding of the Helmholtz corner. Susceptibility to flattening certainly varies between the strings on an instrument, being greater on the lower strings. This might make an interesting subject for future investigation, but will not be considered further in this paper.

Instead, we turn to the much more promising area of minimum bow force. According to Schelleng's analysis, minimum bow force is intimately connected with the response of the instrument body, to the extent that an ideal text-book stretched string with rigid terminations would not have a non-zero minimum bow force. The analysis is based on three assumptions. Two of these are quite robust: that the string motion is well approximated by a Helmholtz motion, and that the bowed point is very close to the bridge (i.e. that β is small) so that the short section of string between bow and bridge can be treated quasi-statically since it will be approximately straight throughout most of the oscillation period. The final assumption used by Schelleng to derive a simple formula is less realistic. He modelled the response of the body by Raman's model, assuming the string to be terminated in a simple dashpot. In section 2.2 we will improve on this assumption, on lines sketched out by Schelleng himself.

From the geometry of the ideal Helmholtz motion the waveform of transverse force acting on the bridge is a sawtooth, with a ramp of slope $Tv_b/\beta L$ interrupted by jumps of magnitude $-Tv_b/\beta L f_0 = -2v_b/\beta Y_0$ to produce a periodic wave at frequency f_0, equal to the natural frequency of an ideal string of length L. (The string tension is T.) The force will have a non-zero mean value, since the mean value of the friction force at the bow produces a shift of the mean position of the bowed point in the direction of bowing. This has no dynamical significance in our calculation, and can be ignored when calculating the body response to the force.

Schelleng's argument now runs as follows. The bridge force evokes a velocity response of the "body" which is simply the force waveform divided by the value R of the assumed dashpot rate. This (small) motion of the string termination then induces a fluctuating component of force at the bow. If that force causes limiting friction to be reached at some time during the nominal sticking phase of the Helmholtz motion, then the assumption of Helmholtz motion is not self-consistent, and some other regime of oscillation must occur. The threshold for this then defines the minimum bow force for the Helmholtz motion. If bow force is reduced beyond this point once a Helmholtz motion is established, a second slip will indeed occur at the moment when limiting friction is first exceeded. This usually leads to the establishment of a periodic oscillation with two or more slips per cycle, the result being commonly described by players as "surface sound". A computer-simulated example of such motion is given later, in Fig. 12(b).

The perturbing force at the bow is simply proportional to the displacement waveform at the bridge (by the assumption that the short section of string in between is straight). Integrating the velocity response just obtained and multiplying by the appropriate constant, one cycle of this force waveform may be written

$$f_{\text{pert}} = \frac{T^2 v_b t^2}{2R\beta^2 L^2} + K, \quad -\frac{1}{2f_0} < t < \frac{1}{2f_0} \tag{2}$$

where the origin of time has been chosen half-way through the Helmholtz sticking phase (neglecting the time delays on the short section of string, since it is being treated quasi-statically). The constant of integration K is determined by the condition that the perturbing force must be zero during slipping. This is because the total force then is assumed to be uniquely determined by the friction curve. (The mechanism by which the perturbing force acquires a non-zero mean value is that the mean position of the bowed point on the string is free to shift sideways by the appropriate amount, taking advantage of the indeterminacy of the friction force during sticking.) The result is

$$K = -\frac{v_b}{2R Y_0^2 \beta^2}. \tag{3}$$

For the Helmholtz motion to be possible, this force waveform must evoke an equal and opposite component of the frictional force, which from eq. (2) will be a parabolic "arch" with a peak value $-K$ at time $t=0$. The condition for minimum bow force arises when this value, added to the steady friction force $\mu_d f_b$, just reaches the limiting value $\mu_s f_b$. Thus

$$f_{\min} = \frac{v_b}{2R Y_0^2 \beta^2 (\mu_s - \mu_d)} \tag{4}$$

which is Schelleng's result [2]. If Raman's model is used in a less approximate way to calculate the perturbing force at the bow, essentially the same result is obtained [5], lending credence to Schelleng's approximate method. The main difference is that the "arch" ceases to be a continuous curve but becomes a stepped arch, as is illustrated for example in Fig. 4.3 of Cremer [6].

2.2. Measuring minimum bow force

Schelleng's discussion of bow-force limits highlighted two features of eqs. (1) and (4). First, the minimum force does indeed tend to be much less than the maximum force, from the appearance of the extra factor $(R Y_0)^{-1}$ in the minimum force. This is the string-to-"body" impedance ratio, and will always be small for any remotely realistic body model. Second, maximum bow force varies as β^{-1}, while minimum bow force varies as β^{-2}. Thus both limits increase as the bow is brought nearer to the bridge, but the minimum force increases more rapidly so that the range of usable bow force becomes narrower. This is all in accordance with the general experience of players.

It is not immediately obvious how to go beyond such qualitative inferences, however, since it is far from clear what value should be used for R. Is the energy dissipation rate in the resistance the important thing, or is the magnitude of the complex body impedance a more appropriate comparison? To answer this, we must repeat the calculation without the restrictive assumption of Raman's model. The other aspects of Schelleng's method seem quite robust, and can be retained. The incident force at the bridge notch in a Helmholtz motion is the same as before. The perturbation force at the bow is proportional to the displacement response of the bridge to this forcing, with the same proportionality constant as before. We must simply improve the step in between, to obtain a better approximation to this displacement response. This can be done by using a more realistic theoretical model for the body, but it can also be done using a measured admittance function at the bridge notch on a real instrument. We examine this possibility first.

The sawtooth bridge force may be expressed as a Fourier series:

$$f_{\mathrm{br}}(t) = \sum_{n=1}^{\infty} a_n \sin 2\pi n f_0 t$$

with

$$a_n = (-1)^{n+1} \frac{T v_{\mathrm{b}}}{n\pi f_0 \beta L}. \tag{5}$$

If the admittance at the bridge notch is $Y_1(\omega)$, the displacement response to this force is

$$\mathrm{Re} \sum_{n=1}^{\infty} \frac{(-1)^n v_{\mathrm{b}}}{n^2 \pi^2 \beta Y_0 f_0} Y_1(2n\pi f_0)\, \mathrm{e}^{2n\pi \mathrm{i} f_0 t}$$

and the perturbation force at the bow is

$$\begin{aligned} f_{\mathrm{pert}}(t) &= \frac{T}{\beta L} \mathrm{Re} \sum_{n=1}^{\infty} \frac{(-1)^n v_{\mathrm{b}}}{n^2 \pi^2 \beta Y_0 f_0} Y_1(2n\pi f_0)\, \mathrm{e}^{2n\pi \mathrm{i} f_0 t} + K \\ &= \frac{2 v_{\mathrm{b}}}{\pi^2 \beta^2 Y_0^2} \mathrm{Re} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} Y_1(2n\pi f_0)\, \mathrm{e}^{2n\pi \mathrm{i} f_0 t} + K. \end{aligned} \tag{6}$$

As before, the value of K is determined by the condition $f_{\mathrm{pert}}(\pm 1/2f_0) = 0$, so that

$$K = -\frac{2 v_{\mathrm{b}}}{\pi^2 \beta^2 Y_0^2} \mathrm{Re} \sum_{n=1}^{\infty} \frac{Y_1(2n\pi f_0)}{n^2}. \tag{7}$$

So the minimum bow force is

$$\begin{aligned} f_{\min} = {} & \frac{2 v_{\mathrm{b}}}{\pi^2 \beta^2 Y_0^2 (\mu_{\mathrm{s}} - \mu_{\mathrm{d}})} \\ & \cdot \left[\max_t \left\{ \mathrm{Re} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} Y_1(2n\pi f_0)\, \mathrm{e}^{2n\pi \mathrm{i} f_0 t} \right\} \right. \\ & \left. + \mathrm{Re} \sum_{n=1}^{\infty} \frac{Y_1(2n\pi f_0)}{n^2} \right], \end{aligned} \tag{8}$$

since the condition we are seeking is that the perturbation force at some stage during the cycle just takes the friction force up to the limiting value. (It is also possible to imagine a force perturbation which went sufficiently negative as to reach limiting friction in the opposite sense, leading to slipping *forwards* relative to the bow, but this is certainly not the normal condition at minimum bow force and we ignore the possibility here.)

As a simple check on this result, consider again Schelleng's case in which $Y_1(\omega) = 1/R$ for all frequencies. The factor $1/n^2$ in eq. (8) suggests that if $Y_1(\omega)$ has no strong variation, a reasonable approximation may be given by taking just the $n=1$ term in the Fourier series (as Schelleng himself noted in Appendix A of ref. [2]). Equation (8) then yields a result which has the identical functional form to eq. (4), but multiplied by a factor $8/\pi^2$. The maximum perturbation force again occurs at $t=0$, mid-way through the sticking phase of the Helmholtz motion, as in the original analysis. For such a crude comparison, this seems entirely satisfactory.

Although eq. (8) appears rather complicated, the qualitative features of Schelleng's formula can all be seen in it. The dependence on β^{-2} is the same, and the whole expression [...] (multiplied by the factor $4/\pi^2$) can be seen as the appropriate combination of body admittances which replaces Schelleng's factor $1/R$. The Fourier series will be rapidly convergent, so that higher harmonics will have a rapidly-decreasing influence on the minimum bow force, but they certainly will have an influence, and this gives quantitative form to the intuitive idea that particularly strong effects might be expected for frequencies such that either peaks or troughs of the body admittance fall in harmonic relations. Conversely, adverse effects of a strong peak in $Y_1(\omega)$ at the fundamental might perhaps be compensated to some extent by suitable behaviour an octave higher. Such inter-relations between behaviour at different frequencies might well be a contributory factor in making one instrument "easier to play" than another.

It is straightforward to apply eq. (8) to data from real instruments. Fig. 1(a) shows a measured admittance function at the G-string bridge notch of a violin, and Fig. 1(b) shows the corresponding admittance for a cello (measured at the C-string notch). Both measurements are made in the bowing direction for the string in question, as the analysis here requires. Fig. 1(a) is directly comparable with the violin admittances shown by Cremer [6, Fig. 10.1]. From such admittance functions, the waveform of bridge displacement (i.e. the expression within the braces in eq. (8)) may then be calculated for any given note. A typical set of results is shown in Fig. 2, from the violin admit-

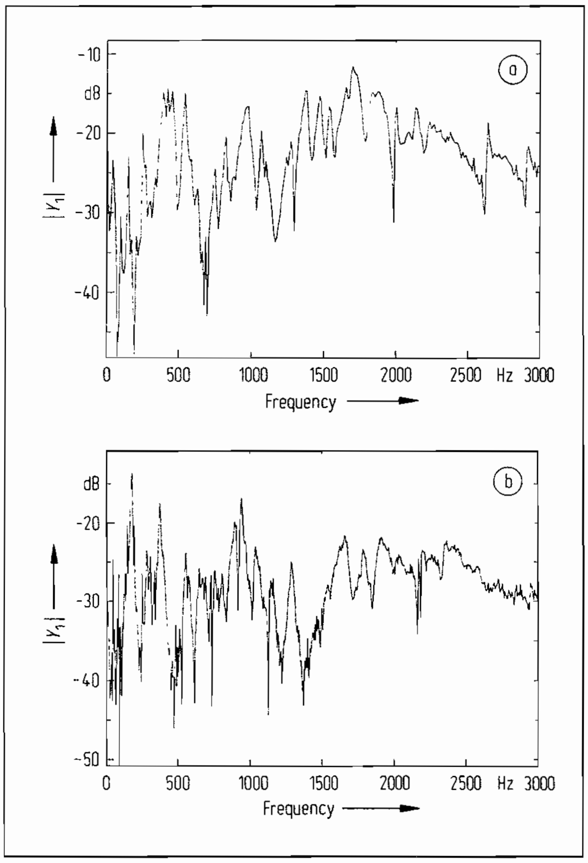

Fig. 1. Input admittance of (a) a violin; (b) a cello, measured at the lowest string-notch on the bridge in the bowing direction. Both instruments were made by the author. The frequency scales are linear, the admittances are plotted in dB relative to 1 m/Ns.

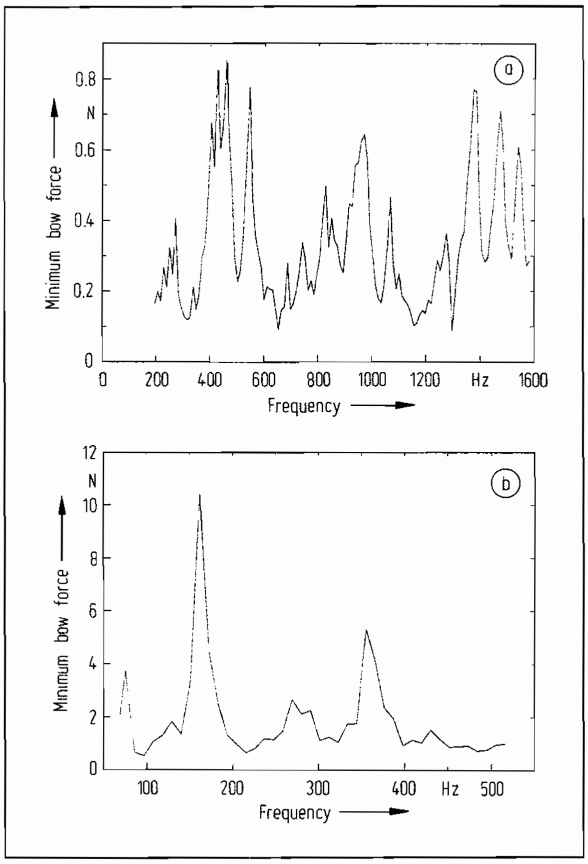

Fig. 3. The Schelleng minimum bow force deduced, as described in the text, from the measured admittances of Fig. 1 for (a) a violin and (b) a cello.

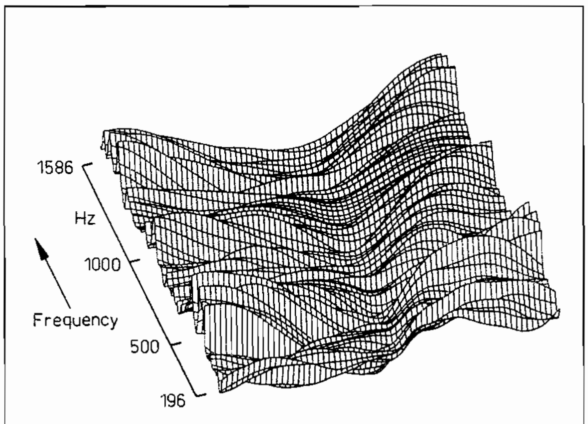

Fig. 2. Bridge displacement waveforms calculated from Fig. 1(a) assuming an incident Helmholtz motion. For each frequency in a three-octave range (upwards from the lowest note of the violin) one cycle of the waveform is plotted across the picture. The separate waveforms are joined by line segments (running from front to back of the diagram) to form an impression of a surface.

tance of Fig. 1(a). It shows one cycle of the displacement waveform as it varies over a three-octave range of frequencies from that of the open G string (196 Hz). The phase is chosen so that the Helmholtz slip occurs in the centre of the range plotted. Notice that the maximum displacement usually occurs near the ends of the plotted range, so that (as with Schelleng's calculation based on Raman's model) a second slip will tend to happen near the middle of the Helmholtz sticking phase. However, this is not the case at all frequencies, from the effect of the phase of the admittance function and the influence of higher terms in the Fourier series.

Finally, Fig. 3(a) shows the minimum bow from eq. (8) deduced from these waveforms over this three-octave range. The following values have been used in this calculation: $\mu_s = 0.8$, $\mu_d = 0.3$, $Y_0 = 3.1\ \mathrm{m\ N^{-1}\ s^{-1}}$, $\beta = 0.1$, $v_b = 0.5\ \mathrm{m\ s^{-1}}$. Notice that the value of β has been held constant, so that it is envisaged that the bow is moved nearer to the bridge as higher notes are played on the G string. (Of course, one cannot play three octaves on a single string in practice without running off the end of the fingerboard, but the pattern revealed here is appropriate to the other strings as well if suitable scaling factors are applied: for frequencies low enough that the top portion of the bridge moves as a rigid body, all four strings see essentially the same admittance function.) Fig. 3(b) shows the corresponding pattern of minimum bow force determined from the cello admittance of Fig. 1(b), using the same parameters except $Y_0 = 1.0\ \mathrm{m\,N^{-1}\,s^{-1}}$.

To test the results of Fig. 3 quantitatively would require a reliable bowing machine, and would make an interesting future project. One would not perhaps expect agreement to be very precise, since there are many factors missing from the analysis, such as the effect of the finger reflection and of losses at the bow. Nevertheless one might hope that eq. (8) captures the essential note-to-note variation in minimum force since these other factors probably do not have strong frequency dependence.

An equally interesting test would be to investigate whether the fluctuations revealed by a measurement like Fig. 3 correlate with perceived note-by-note variations in playability. Such correlation might take the form of notes with high values of $f_{\min}$ seeming hard to control in situations where one is close to minimum force, or it might turn out that the level of $f_{\min}$ matters less than the degree of variation from note to note. It is encouraging to notice that at least one feature of Fig. 3(b) certainly does indicate a familiar anomaly in the playing behaviour of cellos: the large peak around 175 Hz is the "main body resonance", and here is the lair of the "wolf note". The intimate connection between wolf notes and minimum bow force will be discussed in section 2.4. The second large peak, around 370 Hz, also corresponds to a note which is conspicuously awkward to play on this particular cello.

2.3. Minimum bow force with narrow reflection functions

The analysis just developed for minimum bow force can be applied equally well to theoretical models of the instrument body behaviour. As a first example, it is natural to turn to the case of narrow reflection functions discussed previously [1]. If the bridge reflection function $h_1(t)$ is narrow compared to the period of the note in question, the corresponding admittance function $Y_1(\omega)$ will be approximately constant for the first few harmonics of the note. Furthermore, if the constraint

$$\int_{-\infty}^{\infty} h_1(t)\,\mathrm{d}t = -1 \tag{9}$$

is satisfied, this constant value will be *zero*. If follows from eq. (8) that the predicted minimum bow force will be very small, depending on the values of $Y_1(\omega)$ at higher harmonics. If eq. (9) is not satisfied, any narrow reflection function will behave approximately the same as a Raman-model delta function, and Schelleng's original formula eq. (4) will apply.

To see what is going on in a little more detail, this problem can be treated directly in the time domain. The bridge velocity response is simply the difference between the incident sawtooth and the reflected wave, given by one convolution with $h_1(t)$. These two waveforms are shown in Figs. 4(a) and (b), where Cremer's model (discussed in detail in section 4 of ref. [1]) has been used for the bridge reflection function. The force perturbation is as usual proportional to the displacement response, the integral of the velocity difference, plotted in Fig. 4(c). If eq. (9) is satisfied, as it is for Cremer's model, then the slope of the ramp in the reflected wave is unchanged by convolution with $h_1(t)$.

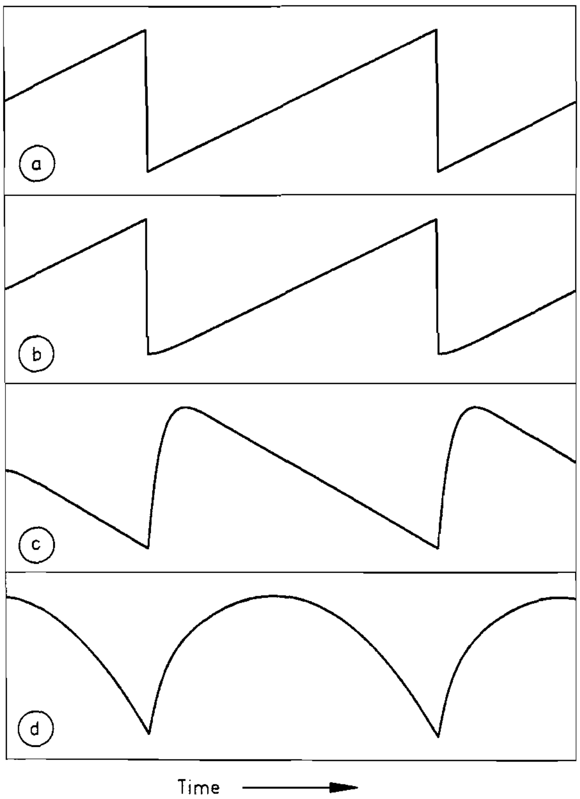

Fig. 4. Ideal time histories illustrating the perturbing force at the bow which may be deduced by Schelleng's argument assuming a narrow reflection function at the bridge: (a) the incident force for a Helmholtz motion; (b) the result of one convolution with the Cremer-model reflection function (shown inverted for ease of comparison with (a)); (c) the waveform of force perturbation at the bridge; (d) as (c), but with the reflection function artificially scaled by a factor 0.97 to violate eq. (9).

The only significant differences between incident and reflected waves occur near the "flyback", the jump in the sawtooth, which will be smoothed in some way by the convolution. If on the other hand eq. (9) is violated, as in Raman's model, the convolution will also reduce the ramp slope. It is the difference between these ramp slopes which, when integrated, produces the "arch" in f_{pert}, the peak of which is responsible for Schelleng's value of minimum bow force. This is illustrated in Fig. 4(d), which shows the integrated velocity difference waveform following one convolution with the Cremer-model reflection function scaled by a factor 0.97 in order to violate eq. (9). It is scarcely distinguishable from the Raman-model result, eq. (2), and a minimum bow force could be deduced from it by the argument already presented.

If eq. (9) is satisfied the peak in f_{pert} tends to occur near the Helmholtz corner, as Fig. 4(c) shows. It is then quite wrong to use the value of this peak perturbing force in Schelleng's analysis to give a minimum bow force. If the assumption of a perfect Helmholtz motion leads to the prediction that limiting friction is reached for times very close to the slip phase of the Helmholtz cycle, this is *not* suggesting that the motion is tending to give way to a multiple-slipping regime. Instead, it is saying that the effect of rounding of the Helmholtz corner is to lengthen the slipping time as a proportion of the period length. This is precisely the effect of corner-rounding on the form of the Helmholtz motion which has been extensively discussed in the past (e.g. [6], § 5.2). The conclusion is that, so far as this simple analysis can tell us, a model with narrow reflection functions which satisfy eq. (9) will have a very small minimum bow force, but one which cannot reliably be calculated using Schelleng's argument.

This is borne out by simulations. If the bow force is slowly reduced once a stable Helmholtz motion is established, the Helmholtz corner becomes more and more rounded, and the slipping time occupies a larger and larger proportion of the period length. Eventually, slipping occupies the entire period length. Without a sticking phase to feed energy back into the system, the motion may then decay slowly to a state of steady sliding without oscillation. Fig. 5 shows three stages in this process, from a simulation based again on Cremer's model. Such behaviour is not seen in real bowed strings, and this points to a significant element of unreality in any model based on narrow reflection functions. However, this may not be quite as serious as it appears at first sight. We will return to this issue in Section 3, when a more thorough discussion will be given of what can be learned from simulations.

Simulations using narrow reflection functions do not always behave like this. If the bow force is reduced more rapidly, one may indeed see a transition from Helmholtz motion to double-slipping motion at some critical bow force. The explanation lies in another source of perturbing force at the bow, which has been neglected up to now. These are Cremer's "secondary waves", perturbations to the ideal Helmholtz motion generated in the interaction of a rounded Helmholtz corner with the bow, which subsequently reflect back and forth on sections of the string, gradually being damped out by successive convolutions with the reflection functions [6, § 5.3]. These are not easy to incorporate in a simple analytical treatment in the spirit of Schelleng's, but they are fully allowed for in simulations which use the method described in previous papers [4, 7]. Forces at the bow due to these secondary waves may contribute to limiting friction being reached. This can produce a transition to double-slipping when the simple analysis does not predict any such transition. The effect of secondary waves in influencing the details of a transition to double-slip motion is often apparent in the timing of the second slip relative to the original Helmholtz slip. Schelleng's analysis usually suggests a second slip which is roughly central within the Helmholtz sticking interval, whereas in practice the second slip occurs first on a secondary wave peak, usually before the centre of the sticking interval.

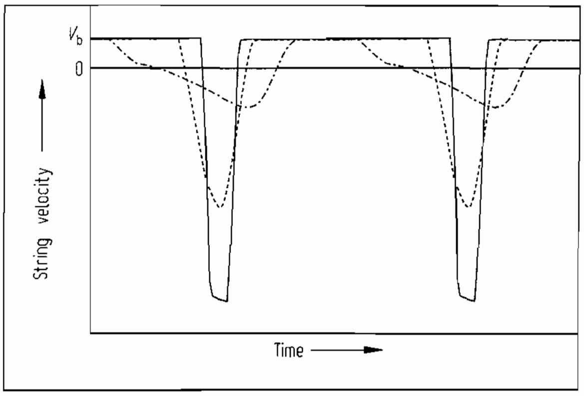

Fig. 5. Waveforms of string velocity at the bowed point for simulated Helmholtz motions using Cremer's model, showing the evolution as bow force is slowly reduced: solid line – bow force 0.5 (using units with $v_b = 1$, $Y_0 = 2$); dashed line – bow force 0.15; dashdot line – bow force 0.05.

2.4. The wolf note

As a first step away from narrow reflection functions, we can consider the effect of a single body resonance. This will allow an investigation of the phenomenon of wolf notes, and it is convenient to couch the discussion in terms of the cello rather than the violin, since cellos suffer most conspicuously from wolf notes. In the measured admittance function of Fig. 1(b), the so-called main body resonance of this cello appears as the large peak at 175 Hz. The simplest model which can embody such behaviour is a mass-spring-dashpot arrangement as illustrated in Fig. 6. This gives an admittance

$$Y_1(\omega) = \frac{\mathrm{i}\omega}{S + \mathrm{i}\omega R - M\omega^2} \tag{10}$$

and a reflection function, using the approximation of eq. (6) of ref. [1] and a small-damping approximation,

$$h_1(t) \approx -\delta(t-2\beta L/c) + \frac{2}{Y_0 M} \cos\sqrt{\frac{S}{M}}(t-2\beta L/c)\, \mathrm{e}^{-R(t-2\beta L/c)/2M} \quad (t \geq 2\beta L/c). \tag{11}$$

It is easy to fit parameters to this model to match the measured peak height, frequency and bandwidth of the main body resonance of the test cello, with the results $M = 0.087$ kg, $R = 3.8$ N s m^{-1}, $S = 0.11$ MN m^{-1}. Using these, the minimum bow force eq. (8) can be calculated in the same way that it was for the measured admittance function. The result is Fig. 7, which is directly comparable with Fig. 3(b). The value of minimum bow force right at the peak is particularly easy to write down. The Fourier series in eq. (8) is then strongly dominated by the $n = 1$ term, and we obtain a result very similar to that of Raman's model:

$$f_{\min} \approx \frac{4 v_b}{\pi^2 R Y_0^2 \beta^2 (\mu_s - \mu_d)}, \tag{12}$$

where it should noted that $1/R$ is simply the peak value of $Y_1(\omega)$.

If the main body resonance is sufficiently strongly coupled to the string, an attempt to play the corresponding note may lead to the undesirable "warbling" effect known as a wolf note. It has been shown [4] that the relatively simple theoretical model discussed in ref. [1], together with a bridge reflection function as in eq. (11) (modified by a little rounding of the initial delta function), can reproduce the detailed behaviour of a wolf note quite well. Wolf notes are closely related to minimum bow force behaviour [8]. It may be possible to start a Helmholtz motion with a bow force below the minimum, because the resonant response of the body takes several period-lengths to build up. As the body displacement amplitude grows, the effective minimum bow force increases, by Schelling's argument, until it passes the actual bow force and a transition to a double-slip regime takes place. This does not excite the body resonance so strongly, since it applies less force to the bridge at the fundamental frequency. Thus it may be possible for the Helmholtz motion to re-establish as the body vibration dies down. An alternation of regimes then ensues, which is the wolf.

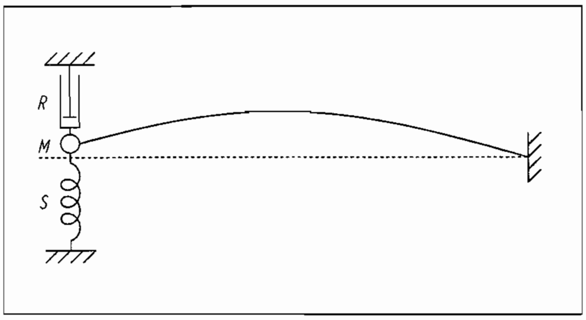

Fig. 6. Sketch of the wolf-note model, in which an ideal string is terminated rigidly at one end and in a mass-spring-dashpot combination at the other.

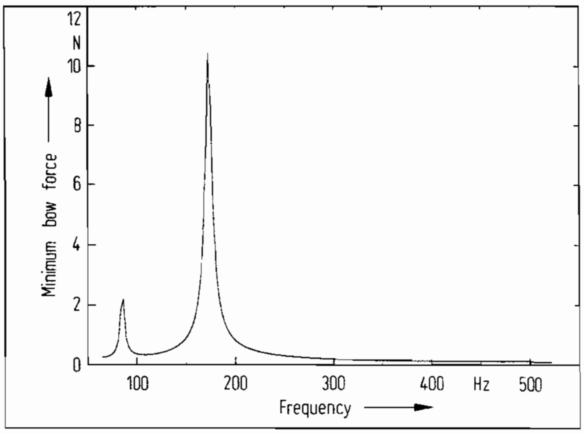

Fig. 7. Schelleng's minimum bow force for a wolf note model, with parameters matched to the main body resonance of the cello whose admittance was shown in Fig. 1(b). The diagram is directly comparable with Fig. 3(b).

Since susceptibility to wolf behaviour is obviously an issue of playability, it is appropriate to explore wolf note criteria in a little detail. The well-known discussion of this subject by Schelleng [9] will be re-examined and given a rather different slant, and then an alternative criterion based on very simple arguments about bow-force limits will be derived and compared with Schelleng's result. Schelleng's discussion focussed on the driving-point response at the bowed point near the fundamental frequency of the string. He argued that if the coupling between string and body is sufficiently strong that $\mathrm{Im}\{G(\omega)\}$ exhibits three zeros rather than a single zero near this frequency, then the frequencies of the outer two zeros might be excited simultaneously by the frictional nonlinearity, leading to beating behaviour which he identified with the wolf note "warble". He thus derived a criterion which successfully predicted the increasing susceptibility to wolves of the viola and the cello (relative to the violin).

The worst case will occur when the nominal fundamental frequency of the string (assuming fixed end conditions) coincides precisely with the resonant frequency of the "body" alone (decoupled from the string). The behaviour of $G(\omega)$ depends on the frequency separation of the two coupled string/body modes which then result. The form of these two modes is sketched in Fig. 8. If the damping is temporarily neglected and the string-to-body impedance ratio $\gamma=(Y_0\sqrt{SM})^{-1}$ is assumed small, it is readily shown that the end corrections b for both modes have the same magnitude:

$$\frac{b}{L}\approx\pm\sqrt{\frac{\gamma}{2\pi}}. \tag{13}$$

The relative separation of the two mode frequencies is thus

$$\frac{\Delta\omega}{\omega}\approx 2\sqrt{\frac{\gamma}{2\pi}}. \tag{14}$$

As is usual in a problem of coupled oscillators with matching uncoupled frequencies, both modes exhibit equipartition of energy between the string and "body". It follows that when (small) damping in the body is allowed for, while assuming the string itself to be lossless, both coupled modes have approximately the same Q factor given by

$$Q\approx 2Q_{\text{body}}=2\sqrt{SM}/R. \tag{15}$$

When Schelleng calculated his wolf-note threshold (Fig. 9 of ref. [9]) he chose a fixed value of bow position β [10]. This suppressed the β-dependence implicit in these mode shapes, and if we reinstate it we discover a weakness in Schelleng's detailed argument. It is straightforward to use the standard expression for the driving-point admittance as a sum over normal modes to investigate the behaviour of $\mathrm{Im}\{G(\omega)\}$ near these two coupled resonances. One immediate conclusion is that if the bow is applied at the nodal point of the higher of the two modes there will only be a single zero of this function, since the other mode is "invisible" at that point. According to Schelleng's argument, then, the wolf should disappear at this particular value of β.

This prediction of the behaviour does not seem to be correct. The critical distance, from eq. (13) and Schelleng's data, is rather close to the bridge for a cello C string with a typical wolf note around F. The author could not detect any disappearance of the wolf note around this particular bow position on his own cello, but this should not be taken too seriously as an experimental finding. More significantly, the description of the physics of the wolf note summarised earlier offers no obvious possibility of such sensitive dependence on β. This suggests that Schelleng's argument is not correct but it does not necessarily mean that Schelleng's *criterion* is essentially wrong. There is an alternative interpretation of this criterion which does not depend critically on β. The condition that $\mathrm{Im}\{G(\omega)\}$ has three zeros for at least some value of β can be viewed as a condition on the *modal overlap* of the two coupled modes. If the coupling is strong enough that this reaches a sufficiently low value, three zeros will certainly appear.

However something else depends on this modal overlap factor. The strongly nonlinear nature of the bowed string self-excitation process means that many harmonics are involved, and there is no reason to expect to be able to explain everything about a wolf note by arguments based just on what happens around the fundamental frequency. A condition for low modal overlap of the coupled pair of modes is also the condition for significant inharmonicity, since both are shifted away from the expected frequency by an amount at least comparable to the modal bandwidth. The higher modes, though, will still form an approximately harmonic series based on that expected fundamental. The arguments of Benade in connection with the influence of inharmonicity on playability of wind instruments [11] lead one to expect some serious impairment of playability under those conditions, and perhaps that is the origin of wolf-note behaviour. Modal overlap and inharmonicity are, of course, properties of the system which have no dependence on the particular bow position β chosen. Using eqs. (14) and (15), the condition for modal overlap to be less than unity (i.e. frequency separation to be greater than

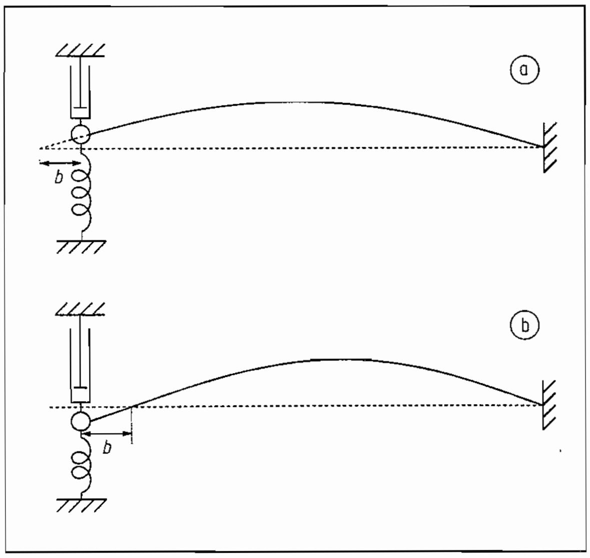

Fig. 8. The two modes of vibration of the model shown in Fig. 6, when the fundamental string resonance matches the frequency of the "body" oscillator.

the half-power bandwidth of each mode) is

$$\frac{1}{2\,Q_{\text{body}}} < 2\sqrt{\frac{\gamma}{2\pi}}. \tag{16}$$

This turns out to be precisely the condition for Schelleng's argument to apply for all values of β.

It is instructive to compare this condition with one which we can derive by an entirely different argument, based on very simple use of the bow-force limits discussed earlier. For a fully-developed wolf, it is necessary that the bow force is below the minimum, eq. (12), but this is not sufficient: if the force is too low, the motion may simply turn into a double-slip regime and stay like that, as usually happens below minimum bow force. Another condition is needed to say that the double-slip regime does not persist, but turns back into a Helmholtz motion when the body motion has died away sufficiently. This will be some kind of maximum force condition for the double-slip regime, based on the relative stability of the double-slip and Helmholtz regimes. It is not possible to give an accurate condition for this without a more careful study (which would also need to take into account the pre-existing level of body vibration induced by the Helmholtz phase of the wolf's warbling cycle before double-slipping commences), but we can give a crude estimate which may capture at least the flavour of the true condition if we require that the force should be above the Schelleng maximum for the double-slip regime. A sufficiently good approximation for this follows from eq. (1), noting that a symmetric double-slip regime has approximately half the Helmholtz slipping velocity:

$$f_{\text{max, d.slip}} \approx \frac{v_{\text{b}}}{Y_0\,\beta\,(\mu_{\text{s}}-\mu_{\text{d}})}. \tag{17}$$

So we might consider that some kind of wolf behaviour is inevitable if the bow force can lie above this but below the Helmholtz minimum, eq. (12). This requires

$$\frac{v_{\text{b}}}{Y_0\,\beta\,(\mu_{\text{s}}-\mu_{\text{d}})} < \frac{4\,v_{\text{b}}}{\pi^2 R\,Y_0^2\,\beta^2(\mu_{\text{s}}-\mu_{\text{d}})},$$

i.e.

$$\beta < \frac{4}{\pi^2 R\,Y_0}.$$

This suggests that some kind of undesirable behaviour will occur for sufficiently small β. No doubt a player would call this a wolf, whether or not it produced the classical warbling behaviour. It is obviously not the same condition as the modal-overlap criterion discussed above, since it involves β. It does not suffer from the objection which was advanced against the β-dependence of Schelleng's argument, since the predicted behaviour is quite different. The idea that a wolf can be made to warble better by bowing closer to the bridge does not conflict with playing experience. On the author's own cello, for which data was shown earlier, the wolf can be made to warble for any bowing position when playing F on the C string, but on the G string it only begins to warble for β less than about 1/7.

It is obviously of some interest to compare numerical values of these two quite different criteria of wolf susceptibility. To make the second criterion look recognisably similar to the first, we might substitute a value for β which is larger than the usual range of normal playing, say 1/4. Then we can say that there is a range of bow forces within which something "wolfish" must happen for all reasonable bowing positions if

$$\frac{1}{2\,Q_{\text{body}}} < \frac{8\gamma}{\pi^2}. \tag{18}$$

This differs from eq. (16) mainly in the occurrence of γ rather than $\sqrt{\gamma}$, since the numerical factors are rather arbitrary in both cases.

The two criteria (16) and (18) are plotted in Fig. 9, in the same format as Schelleng's original diagram (Fig. 9 of ref. [9]) although showing wider ranges of both parameters. Schelleng's original curve is also included, together with some of the experimental data he showed (corrected for his erratum [12]) and also some data on the test cello mentioned above. The condition (16) gives lower numerical values than Schelleng's line (based on a particular value of β), which in turn is rather low for the experimental data on cello strings. If the argument about modal overlap has any validity, this implies that a much lower level than unity is needed as a threshold of safety from wolves – Fig. 9 suggests a value around 1/20 to make the cello D string just safe.

The condition (18) is much closer to the measured data, but a little on the high side. This is no surprise – using the Schelleng maximum force of eq. (17) certainly produces an over-estimate of the wolf threshold, and a lower force limit taking account of stability of the double-slip oscillation would be better. However the precise position of the plotted line in Fig. 9 should not be taken too seriously, since the use of the particular value $\beta = 1/4$ was rather arbitrary. The experimental data given by Schelleng is not really adequate to tell whether either criterion really conforms well to actual behaviour. An interesting approach to resolving this question might be via computer simulations, in the manner to be explored in section 3.

Although the wolf note is not of very great significance in itself, further study would be justified because it makes a useful probe of our understanding of the

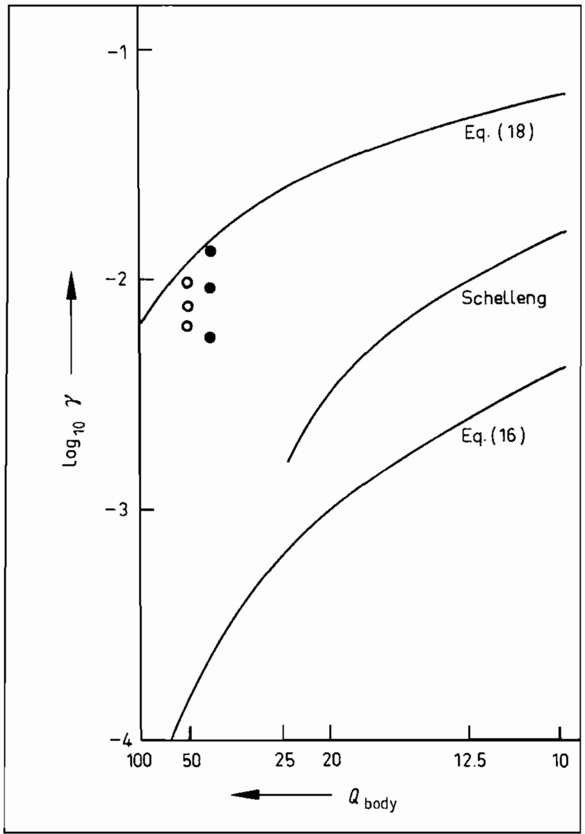

Fig. 9. Thresholds for wolf-note activity, according to the various arguments discussed in the text. Solid circles show data cited by Schelleng for cello C, G and D strings (from the top down), while open circles show the equivalent data for the test cello whose admittance was shown in Fig. 1(b).

detailed physics of the bowed string. One final point about wolves which may be noted before we leave the subject relates to the relative susceptibility of cellos and violins to wolf notes. The criteria discussed above relate the greater susceptibility of cellos to the closer impedance match between strings and body (resulting from the small size of the cello compared with a scaled-up violin [9]). The admittance measurements shown in Fig. 1 highlight another difference, though. The main body resonance of the cello (around 370 Hz in Fig. 1(b)) is a clear, isolated peak. On the violin, by contrast, the corresponding region shows a cluster of peaks rather than a single strong peak, and this behaviour seems typical of many violins. This might have a bearing on the player's perception: if one note stands out from its neighbours as being grossly different in playing properties, that may present more of a problem than if a broader range of notes (from the cluster of peaks in a violin) is affected.

3. Pattern selection in starting transients

With the discussion of wolf notes, we perhaps reach the limit of what can usefully be said about playability by studying conditions for steady bowing. It seems safe to assume that most phenomena which are significant to players involve transient response of one kind or another. The highly nonlinear nature of the bowing process means that very little progress can be made in understanding transient behaviour without resort to computer simulation. An efficient algorithm for simulating the bowed-string model discussed in ref. [1] was described some years ago [7], and we now enquire what use might be made of this to further the study of playability.

Early implementations of the simulation algorithm took the form of interactive programs, in which the playing parameters could be varied during a run so that the program could be "played" roughly like the real string. This yielded many valuable insights into the bowing process and the strengths and weaknesses of the particular model used. However, the parameter space which one is exploring in this way is so large that it is extremely difficult to discern any structure in the overall behaviour from watching individual interactive runs of the program. A more organised use of simulation is needed.

Some inspiration may be drawn from the comments of players about instruments. Recall the discussion in ref. [1] of the common player reaction that a particular instrument is found "easy to play". Two types of phenomenon spring to mind to explain this reaction. First, the player might be gaining an impression of having to work less hard than usual to produce good tone. That impression might possibly have to do with the issues discussed earlier – in particular, a low value of minimum bow force might mean that a lower force can be used to produce a given effect, so that less work really is being done by the player. The second type of phenomenon is rather different, and concerns the *reliability* of starting transients. There will always be some uncertainty in the player's control of bowing parameters. Perhaps an "easy to play" instrument is one which is tolerant of variations in transient form, and does not readily switch to an undesirable oscillation regime when a perturbation is made by the player.

Again, we can subdivide perturbations into two categories which could be studied separately. A perturbation might take the form of a well-controlled transient with a slightly different value of one or more parameters (such as the eventual steady bow force). On the other hand, one can imagine a perturbation which consists of small random fluctuations superimposed on the intended smoothly-varying transient (for example, force fluctuations from a slightly shaking bow-hand). We might characterise these as "deterministic" and "stochastic" perturbations respectively. Attempts could be made to quantify the sensitivity to perturbations of both types using simulation. In this initial study, we concentrate on deterministic perturbations.

The aim is to use simulations to map out some part of the player's parameter space, and then represent the results in diagrammatic form so that any interesting structure may be readily discerned. This strongly suggests that a two-dimensional subspace be studied, since it is hard to convey results in more than two dimensions. To choose a suitable subspace for a preliminary investigation requires some care, since the computational cost will be quite high and there are many possibilities to consider. In the light of the previous discussion of bow-force limits, it seems natural that the steady bow force at the end of the transient should be one variable to consider. A second variable is then wanted which can be used to specify a range of different starting transients, all of which share the same eventual bow force.

It was decided to use a second force-like variable. Starting transients can be simulated in which the initial force is different from the asymptotic value, with the offset decaying exponentially with time. This allows a range of somewhat plausible bowing transients to be simulated. If the force starts from zero and increases to the final level, this gives a simple representation of a string-crossing transient in which the bow alights on the string and the force takes a finite time to build up. If the initial force is the same as the final value, a switch-on transient is produced. This probably does not represent anything done in normal playing, but it is a favourite condition for previous simulations and makes a useful comparison. Finally, if the force starts high and decreases to the final value, then we have at least a crude representation of a martelé transient. This will be less accurate than the string-crossing transient, since the bow speed will also vary significantly during a martelé. Despite this reservation, the two-dimensional space of transients in which the initial and final forces are varied (keeping the exponential time scale and the rest of the model constant) seems quite a promising candidate for a first study.

The calculation to be done is now analogous to the computation of the famous Mandelbrot set: for each point in the parameter plane, a nonlinear process is simulated using the coordinates of the point as input data. The simulation is continued long enough to indicate the eventual outcome (in our case, whether a periodic Helmholtz motion is or is not produced), then that point can be coloured in some way to represent this outcome, and the calculation moves on to the next point. When a reasonable area has been covered in this way, a picture will have been built up of the region of the parameter subspace in which Helmholtz motion actually occurs from a starting transient. This might turn out to be quite different from the region within which Helmholtz motion is *possible*, since several different nonlinear oscillation regimes are often possible at a given point in parameter space. The problem being addressed here is of *pattern selection*: which one of the multiple possibilities is chosen by the system under given conditions.

The analogy with the Mandelbrot set should not be taken too far. It is highly likely that the results for bowed-string behaviour will exhibit fascinating fractal structure which could be studied at higher and higher resolution to give pretty pictures, but such behaviour would be entirely irrelevant to the problem in hand. We are interested in rather coarse-grained structure in the results. The player only has limited control over the values of initial and final bow force. The question of interest relates to behaviour at this error scale and above. If the Helmholtz region turns out to be large and empty, with reasonably smooth boundaries, then the model in question could presumably be said to be "easy to play". If, on the other hand, the Helmholtz region has many islands within it of non-Helmholtz behaviour, or a boundary which is highly convoluted, then the model is in some sense "hard to play". The implication would be that a small change in the player's parameters might cause the Helmholtz motion to give way unpredictably to some other, undesirable, behaviour. Such transitions are certainly not unknown to string players.

Having specified what is to be varied, it remains to specify the rest of the model parameters. In the light of the earlier discussion, by far the most obvious candidate for a first study is Cremer's model. So an open A string is considered, terminated rigidly at the nut end and with a spring and dashpot at the bridge as specified in section 4 of ref. [1]. A simulated digitisation rate was used corresponding approximately to compact disc recordings: 44 kHz sampling of 440 Hz yields a very convenient 100 samples per nominal period length. The chosen value of β was 11/100, fairly typical of normal playing. The assumed friction curve is shown in Fig. 10, consisting of a rectangular hyperbola for the slipping portion, and a sloping line representing sticking with some loss into torsional waves. The bow force offset from the specified asymptotic value decays according to $\exp(-t/\tau)$, where τ is eight nominal period-lengths, approximately 18 ms. This seems a plausible order of magnitude.

The range of asymptotic bow force was chosen after some empirical experiments to find the actual limits for the Helmholtz motion. The program is written in

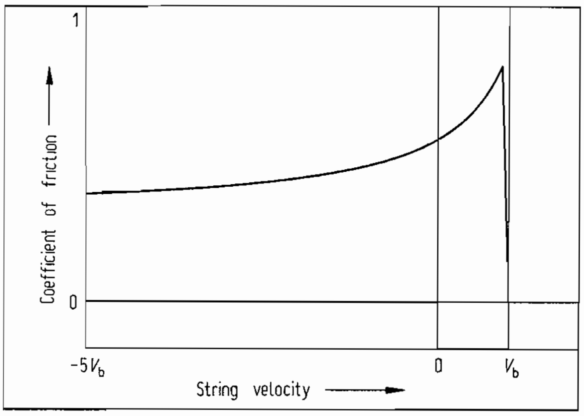

Fig. 10. The friction curve used in the simulations reported in this paper. The slipping portion is a rectangular hyperbola, with a peak friction coefficient 0.8 and an asymptotic coefficient 0.3, which passes through the value mid-way between these two when the string speed is zero. The "sticking" portion is a straight line connecting the nominal bow speed v_b to a peak at 0.9 v_b, to represent some loss into torsional waves.

terms of dimensionless units, based on the convenient values $v_b = 1$, $Y_0 = 2$. In these terms, the Schelleng maximum force, eq. (1), is 18.2. For reasons discussed in section 2.3, it is not easy to calculate the Schelleng minimum bow force for a model based on narrow reflection functions, but it is easy to determine a value by experimenting with an interactive version of the simulation program. If a Helmholtz motion is established and the bow force is reduced very slowly, a transition to a double-slip regime never occurs, but by the time the force reaches 0.04 or so the slip phase grows to occupy the whole period length, as explained earlier and illustrated in Fig. 5. Anywhere above that force, a stable Helmholtz motion is possible in principle, but the results will show that in fact much higher forces are needed to establish a Helmholtz motion from any starting transient of the type considered here. With these facts in mind, the limits of asymptotic bow force for the study were set at 0.5 and 20. The range of transients studied was such that the initial force was varied from zero to four times the value of the asymptotic force.

This computation lends itself very easily to implementation on a parallel computer. Different processors are simply assigned their own values of the asymptotic force and the initial force, and they can get on with simulating the string motion independently of one another. The results to be shown shortly were calculated using 16384 processors of a Connection Machine [13], so that a single run could cover a grid of points of dimensions 128×128 in the parameter plane. The advantages of such an approach are obvious, and indeed this study would scarcely have been possible on a serial computer. (The results to be shown actually occupy a grid which is 256 points square, created from several runs by piecing together the output files.)

The most difficult aspect of modifying the simulation program to scan parameter space automatically was provide a reliable algorithm to recognise the nature of the eventual oscillation regime. Here the skill of the human eye and brain in processing the results of an interactive simulation is very difficult to reproduce. Further effort is undoubtedly needed in this area of the study, and eventually an "expert system" capable of being trained to emulate the discrimination of a human may be necessary. For this preliminary study, we have used a simple approach which seems to discriminate the main features of interest reasonably well. The simulation is run for 100 nominal period lengths, longer than any musically-acceptable transient. The last five period-lengths are then subjected to three different analyses, the results of which are used to classify the various possible regimes of oscillation.

The first two analyses are very straightforward. First, the correlation coefficient is calculated between neighbouring pairs of nominal period-lengths, to see whether the motion has settled to an acceptably periodic form. This does not take account of the flattening effect at high bow force, but with the particular model used here this does not cause serious problems. (The reason has to do with the specific reflection functions of Cremer's model: they induce very little "corner-rounding", and hence allow very little flattening to occur even when significant frictional hysteresis is present.) Second, the number of episodes of slipping during the last five period-lengths is counted, to produce an average number of slips per cycle. This is an unambiguous quantity, but its interpretation is not always as easy as might appear because there are sometimes very small slips of little dynamical consequence.

The final quantity which it seemed desirable to know is the number of travelling "corners" on the string, which for a periodic motion corresponds to the "type number" in Raman's classification. The problem which arises in trying to compute this is to distinguish reliably between corners and the effects of secondary waves. The present algorithm is based on calculating a smoothed derivative of the outgoing velocity wave from the bow and counting how often this rises above a suitable threshold. The details were evolved iteratively using the interactive simulation program. This method seems to work adequately with the particular model in use here, but a more satisfactory and robust algorithm could probably be developed by modelling the expected roundedness of a travelling corner at a given bow force and using this information in the processing.

Putting all three of these measures together, the program decides that a Helmholtz motion has been achieved if the correlation exceeds a threshold value (very close to unity), and the number of slips per cycle and the number of travelling corners are both equal to one. If anything, this measure errs on the conservative side. It is possible, for example, for the motion to be "almost exactly Helmholtz", but with an insignificant second slip or some irregularity or subharmonic activity [3] which reduces the correlation coefficient.

The result is shown in Fig. 11(a). Asymptotic bow force is plotted logarithmically on the horizontal axis, and the ratio of initial to final force is plotted linearly on the vertical axis. Points plotted in white correspond to motion which passed the "Helmholtz" tests just described, while black ones failed one or more of these tests. Fig. 11(b) shows an outline sketch of the results, with some annotations to show what is happening in regions where Helmholtz motion was not found. The vertical dashed line plotted on this figure shows the Schelleng maximum bow force given earlier. (The effective minimum bow force, around 0.04, is far off the left-hand side of the diagram.) Fig. 12 shows some time histories of the final motion at certain particular points in Fig. 11, selected to illustrate the main oscillation regimes of interest. In each case, the velocity waveform at the bow and the bridge-force waveform are both plotted.

If asymptotic bow force had been the only significant factor in determining the nature of motion, Fig. 11(a) would show only vertical stripes. It is immediately apparent that this is not the case, and we discuss in turn the various structures which the figure reveals. First, we see that there is indeed an upper force limit, and that the Schelleng maximum bow force predicts at least the right order of magnitude for it. At forces of this order and above, the motion does not settle into a periodic oscillation of any kind. This is precisely the behaviour which led Schelleng to call such motion "raucous". In strict numerical terms, the simulated results show periodic Helmholtz motion failing to become established at forces above 9 or so, about half the Schelleng value.

A more interesting situation is revealed when we look at low bow force. For asymptotic forces below about 1.5 in these units, periodic double-slipping motion is usually found. When the asymptotic force becomes lower still, this gives way in turn to motion with three and more slips per cycle. Fig. 12(a) shows typical Helmholtz motion, and Fig. 12(b) shows typical double-slipping motion on the same time-scale. This effective "minimum bow force" of around 1.5 is almost two orders of magnitude greater than the value discussed

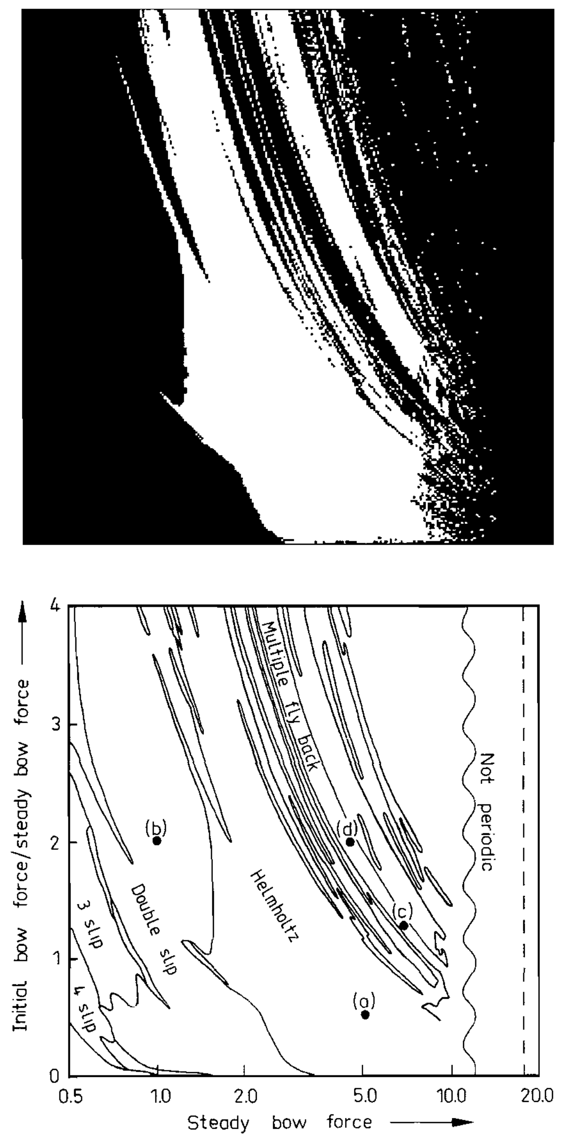

Fig. 11. (Top) The result of the simulation experiment described in the text, white points representing periodic Helmholtz motion and black points any other outcome; (Bottom) a sketch of the features of Fig. 11 (Top), showing the axis scales and the regions in which various important regimes were found. The four labelled solid circles correspond to the waveforms plotted in Fig. 12. The vertical dashed line shows the Schelleng maximum bow force.

earlier, below which a Helmholtz motion is not possible. The fact, discussed in section 2.3, that Schelleng's argument does not yield a satisfactory prediction of minimum bow force for narrow reflection functions is thus seen to be of little significance. The practical lower limit on bow force is set by the entirely different physical mechanism which governs pattern selection

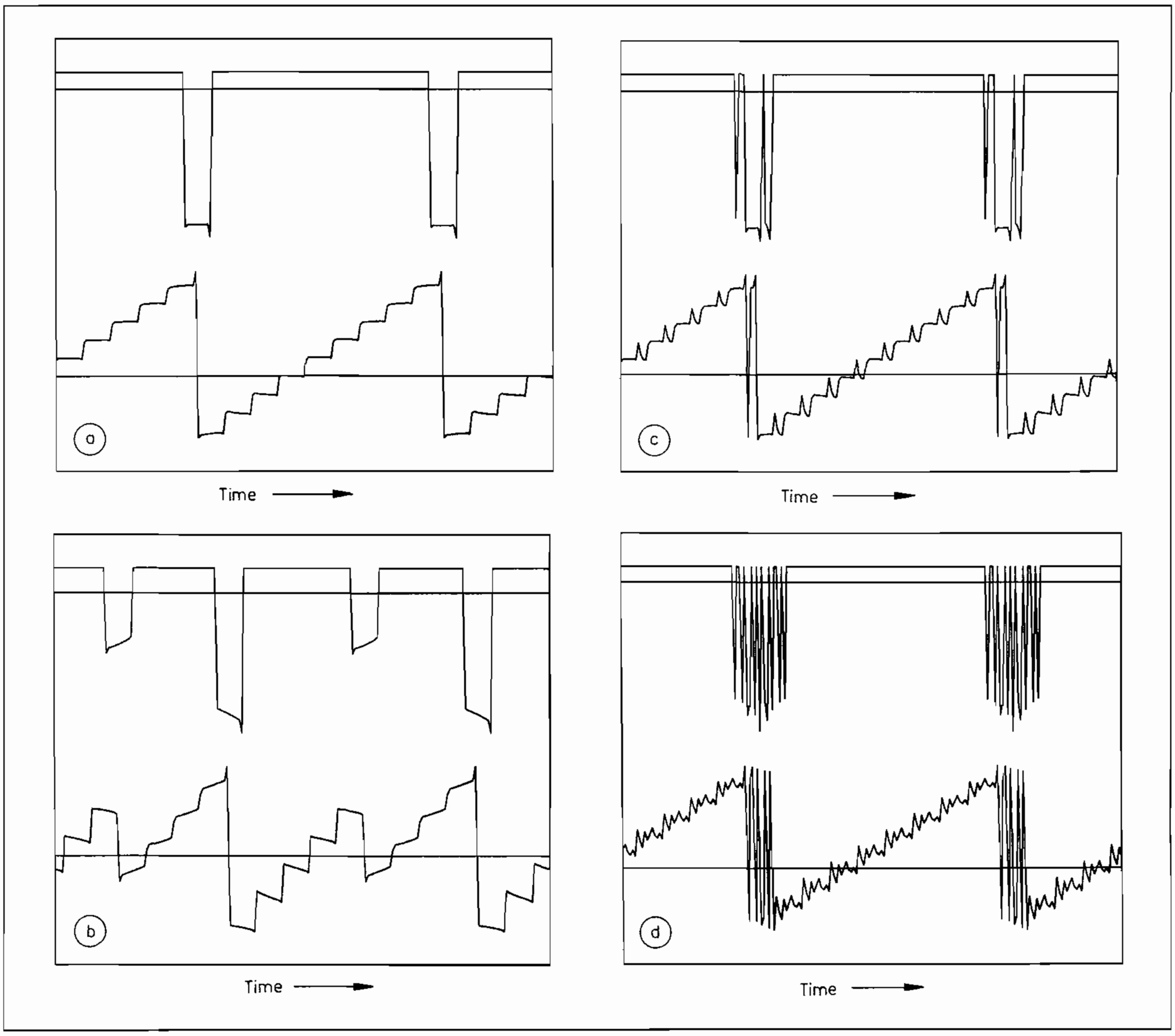

Fig. 12. Waveforms of string velocity at the bow (top) and transverse force on the bridge (bottom) for four particular points in the parameter space explored in the simulations summarised in Fig. 11. Case (a) shows Helmholtz motion, case (b) shows double-slipping motion, and cases (c) and (d) show different varieties of multiple-flyback motion.

in transients. It is not easy to see how analytic studies might be made of this phenomenon, but results like Fig. 11 give a powerful incentive to further investigation. It would be of enormous interest to know whether this limit depends on the behaviour of the instrument body, and whether it can vary between instruments or from note to note on one instrument and thus give rise to strong "playability" effects.

There is a little more to be said about the tendency of the model to produce double-slipping motion well within the range where the Helmholtz motion is possible. If the detailed behaviour is studied for points just inside the white region of Fig. 11(a), very protracted transients are seen in which a long period of double-slipping slowly gives way to Helmholtz motion. This appears to be an instability of the double-slip regime, in which the velocity in one slip gradually grows at the expense of the other until one of them disappears, leaving Helmholtz motion. The question of stability of the various oscillation regimes will be addressed in a later paper. One consequence of these very slow transitions between regimes is that the precise position of the double-slipping boundary as plotted in Fig. 11 may not be of great significance, since it depends on the choice of 100 period-lengths for the running time of the simulations. A longer running time pushes the boundary a little to the left in Fig. 11. This does not seem to be a significant objection to the results – at this stage we are mostly interested in qualitative structure, and exploring the potential of the approach. (It may turn out that these very long transients are characteristic of Cremer's model, which is not very far re-moved from Raman's model and which might thus be expected to produce unrealistically-long transients.)

The final area of structure in Fig. 11 is perhaps the most exciting. The set of curving steaks marked "multiple flyback motion" in Fig. 11(b) give the first definite information about a regime of oscillation which has been mentioned in the past [14] but which has eluded detailed study. They are characterised by motion in which the bridge-force waveform looks roughly like a Helmholtz sawtooth, except that the single sharp flyback is replaced by a closely-spaced multiple set. This motion has been observed quite commonly on real violin strings, using a bridge-force transducer. It gives a characteristic timbre, readily recognised once it has been identified. It was originally christened "double flyback motion", after its most characteristic manifestation. However, some of the simulated results show three or more flybacks, so a more general phrase seems appropriate.

Two examples from different regions of Fig. 11 are shown in Fig. 12(c) and (d). They both differ in detail from the description of this regime published previously [14]. Specifically, they involve three and nine slips per cycle respectively, whereas the previous description (in terms of travelling corners in a space-time diagram) showed a special case with just two slips because a pair of travelling corners cross at the bowed point. The essential nature of the motion shown in Fig. 12(c) is as described previously, the only required change being a shift of the assumed position of the bow. The motion shown in Fig. 12(d) is similar, but with more travelling corners. It should be noted that once the motion becomes as complicated as Fig. 12(d), its details become very sensitive. Different answers may be obtained from the same program run on different machines. This is presumably because, when frictional hysteresis is operating, two similar solutions can diverge strongly if a small deviation causes one to switch to sticking while the other continues to slip. Thereafter they may develop along quite different lines.

The satisfying thing about the appearance of this regime in Fig. 11 is that the circumstances which produce it are at least qualitatively in agreement with the conditions which seem to be necessary in practice. A transient in which a high initial force is held for rather too long before reducing it seems to be the essential factor in producing this regime (based on informal observation only). Fig. 11 makes this notion much more definite, revealing some structure in the transient conditions needed. It is encouraging that the very simple model in use here shows this degree of realism in its response pattern, even if only at a qualitative level. It will be of great interest to see whether this regime is found with all body models, or whether it is sensitive to details. If the latter, it could be an interesting candidate for variations in playability, since this oscillation regime is definitely undesirable in practice.

The final comment about Fig. 11 is a negative one – it is very interesting that no other essentially different regimes of oscillation are found anywhere in it. Raman has described a large gallery of possible bowed string motions [15], and many of them have been observed under special conditions in experiments [16]. Lawergren has developed and unified some of these "higher types" [17]. It is of course possible that some of these would be found under different conditions. Lawergren's "S-motion", in particular, is expected for certain rather particular values of β, near to but not at the $1/n$ points. This can be sought in later trials, but with that proviso, the selection of regimes actually occurring in Fig. 11 agrees precisely with the set described by McIntyre and Woodhouse [14] on the basis of observation of played notes. Again, this shows a perhaps surprising level of realism in the predictions from Cremer's model.

So should we say that the open A string in Cremer's model is easy to play? Surely the answer is no. The low bow-force part of the Helmholtz region looks quite empty, but much of it in fact contains unacceptably long starting transients. (It would be possible to modify the display of Fig. 11(a) to convey information about the length of transient within the Helmholtz region, but a satisfactory way of reproducing such details in print has not yet been found.) The higher bow-force region is cluttered with islands of multiple-flyback motion, which would make it very hard to achieve Helmholtz motion reliably. A goal of further studies might be to find a way to quantify such notions, in terms of some kind of "ease-of-playing index".

This initial attempt at parameter-space mapping by simulation certainly seems very promising. It would be of obvious interest to explore how the picture changes as more realistic features are incorporated into the modelling. Such studies are under way. Of particular interest would be to incorporate reflection functions and other model data measured directly from a real instrument. One could then at least contemplate comparing a computed figure like Fig. 11(a) with the results of a survey on the same instrument using a sophisticated bowing machine which could control force in a transient. The first step of such a project would be to develop an accurate method for measuring the two reflection functions on a real string. This will involve experimental ingenuity, coupled with signal processing algorithms tailored to this particular problem. The approach currently being explored is to measure impulse responses at several points along the string then attempt to unpack the two separate reflection functions from the multiple convolutions of eq. (2) of ref. [11].

Acknowledgements

The computations described in Section 3 would not yet have progressed beyond being an untested idea without the enthusiastic cooperation of Robert T. Schumacher. The calculations were made possible by a starter grant for the Connection Machine from the Pittsburgh Supercomputer Center, which is supported by the National Science Foundation, Carnegie Mellon University, and by the State of Pennsylvania through the Benjamin Franklin Trust. The author expresses profound gratitude to these individuals and bodies, and to many other friends and colleagues for invaluable discussions and comments.

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