Variables, Functions and Equations.

Economists are interested in examining types of relationships. For example, an economist may look at the amount of money a person earns and the amount that person chooses to spend. This is a consumption relationship or function. As another example an economist may look at the amount of money a business firm has and the amount it chooses to spend on new equipment. This is an investment relationship or investment function.

A **function** tries to define these relationships. It tries to give the relationship a mathematical form. An equation is a mathematical way of looking at the relationship between concepts or items. These concepts or items are represented by what are called variables.

A **variable** represents a concept or an item whose magnitude can be represented by a number, i.e. measured quantitatively. Variables are called variables because they vary, i.e. they can have a variety of values. Thus, a variable can be considered as a quantity which assumes a variety of values in a particular problem. Many items in economics can take on different values. Mathematics usually uses letters from the end of the alphabet to represent variables. Economics however often uses the first letter of the item which varies to represent variables. Thus, p is used for the variable price and q is used for the variable quantity.

An expression such as $4x^3$ is a variable. It can assume different values because x can assume different values. In this expression x is the variable and 4 is the coefficient of x. Coefficient means 4 works together with x. Expressions such as $4x^3$ which consists of a coefficient times a variable raised to a power are called monomials.

A **monomial** is an algebraic expression that is either a numeral, a variable, or the product of numerals and variables. (Monomial comes from the Greek word, monos, which means one.) Real numbers such as 5 which are not multiplied by a variable are also called monomials. Monomials may also have more than one variable. $4x^3y^2$ is such an example. In this expression both x and y are variables and 4 is their coefficient.

The following are examples of monomials:

x, $4x^2$, $-6xy^2z$, 7

One or more monomials can be combined by addition or subtraction to form what are called **polynomials**. (Polynomial comes from the Greek word, poly, which means many.) A polynomial has two or more terms i.e. two or more monomials. If there are only two terms in the polynomial, the polynomial is called a binomial.

The expression $4x^3y^2 - 2xy^2 + 3$ is a polynomial with three terms.
These terms are $4x^3y^2$, $- 2xy^2$, and 3. The coefficients of the terms are 4, -2, and 3.
The degree of a term or monomial is the sum of the exponents of the variables. The degree of a polynomial is the degree of the term of highest degree. In the above example the degrees of the terms are 5, 3, and 0. The degree of the polynomial is 5.
Remember that variables are items which can assume different values. A function tries to explain one variable in terms of another.
Consider the above example where the amount you choose to spend depends on your salary. Here there are two variables: your salary and the amount you spend.
Independent variables are those which do not depend on other variables. Dependent variables are those which are changed by the independent variables. The change is caused by the independent variable. In our example salary is the independent variable and the amount you spend is the dependent variable.
To continue with the same example what if the amount you choose to spend depends not only on your salary but also on the income you receive from investments in the stock market. Now there are three variables: your salary and your investment income are independent variables and the amount you spend is the dependent variable.

Definition: A function is a mathematical relationship in which the values of a single dependent variable are determined by the values of one or more independent variables. Function means the dependent variable is determined by the independent variable(s).

A goal of economic analysis is to determine the independent variable(s) which explain certain dependent variables. For example, what explains changes in employment, in consumer spending, in business investment etc.?
Functions with a single independent variable are called univariate functions. There is a one to one correspondence. Functions with more than one independent variable are called multivariate functions.
The independent variable is often designated by x. The dependent variable is often designated by y.
We say y is a function of x. This means y depends on or is determined by x.
Mathematically we write $y = f(x)$
It means that mathematically y depends on x. If we know the value of x, then we can find the value of y.

In pronunciation we say " y is f of x." This does not mean that y is the product of two separate quantities, f and x but rather that f is used to indicate the idea of a function. In other words, the parenthesis does not mean that f is multiplied by x.

It is not necessary to use the letter f. For example, we could say

y = g(x) which also means that y is a function of x or we could say y = h(x) which too means that y is a function of x.

We may look at functions algebraically or graphically. If we use algebra, we look at equations. If we use geometry, we use graphs.

<u>A simple example of functional notation</u>

Q_d = the number of pizzas (quantity) demanded

P_p = the price of a pizza

P_t = the price of tomato sauce

P_c = the price of cheese

P_d = the price of pizza dough

N = the number of potential pizza eaters

$P_p = f\ (P_t, P_c, P_d)$

This is an example of a function that says the price of pizza depends on the prices of tomato sauce, cheese, and pizza dough. There is one dependent variable, the price of pizza and there are three independent variables, the prices of tomato sauce, cheese, and pizza dough.

$Q_d = f\ (P_p, N)$

This is another example of a function. It says that the quantity of pizza demanded depends on the price of pizza and the number of potential pizza eaters. There is one dependent variable, the quantity of pizza demanded, and there are two independent variables, the price of pizza and the number of potential pizza eaters.

<u>A common economic example of functional notation</u>

C = consumption, the amount spent on goods and services

Y = income, the amount available to spend

C = C(Y)

This is an example of a function that says the amount spent on consumption depends on income. This is a very general form of the consumption function. In order to use it economists must put it into a more precise mathematical form. For example

$C = 25 + .75Y$

This is a function which says that consumption is 25 regardless of the level of income and that for every extra dollar of income 75 cents are spent on consumption.

The use of functional notation: some examples

Example 1

$y = f(x) = 3x + 4$

This is a function that says that, y, a dependent variable, depends on x, an independent variable. The independent variable, x, can have different values. When x changes y also changes.

Find f (0). This means find the value of y when x equals 0.

f (0) = 3 times 0 plus 4

$f(0) = 3(0) + 4 = 4$

Find f (1). This means find the value of y when x equals 1.

f (1) = 3 times 1 plus 4

$f(1) = 3(1) + 4 = 7$

Find f (-1). This means find the value of y when x equals -1.

f (-1) = 3 times (-1) plus 4

$f(1) = 3(-1) + 4 = 1$

Example 2

$d(p) = p^2 - 20p + 125$

This is a function that describes the demand for an item where p is the dollar price per item. It says that demand depends on price.

Find the demand when one item costs $2.

$d(2) = 2^2 - 20(2) + 125 = 89$

Find the demand when one item costs $5.

$d(5) = 5^2 - 20(5) + 125 = 50$

Notice that, as we might expect, the demand declines as the price rises.

Example 3

Two or more functions can be added, subtracted, multiplied or divided.

$g(x) = x - 3$ $\qquad$ $h(x) = x^2 + 2$

Find g (0) + h (0)

$g(0) = 0 - 3 = -3$

$h(0) = 0^2 + 2 = 2$

$g(0) + h(0) = -3 + 2 = -1$

Find g (1) h (2)

$g(1) = 1 - 3 = -2$

h (2) = $2^2 + 2 = 6$

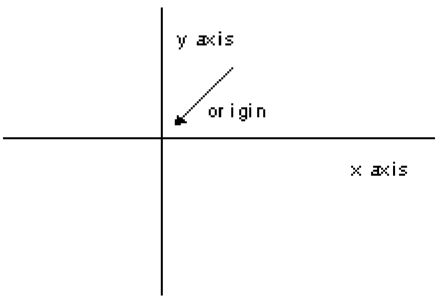
g (1) h (2) = (-2) (6) = -12

Graphs of Two Variable Functions

Many types of economic problems require that we consider two variables at the same time. A typical example is the relation between price of a commodity and the demand or supply of that commodity. The relation can be described algebraically by a two variable function or equation. But it is often useful to represent the relation in a two-dimensional graph. Such a graph is known as a scatter diagram. This is a useful device because if there is a simple relationship between the two variables, it is readily observable once the data are plotted. As the proverb says, “a picture is worth a thousand words.”

To represent a function graphically we use two perpendicular lines called axes. Their point of intersection is called the origin. This method of representation is called the Cartesian coordinate system or plane. The numerical value of one variable is measured along the bottom or horizontal axis. The horizontal axis is called the x axis. The numerical value of the other variable is measured along the side or vertical axis. The vertical axis is called the y axis. The four sections into which the graph is divided are called quadrants. Units of length are indicated along the two axes.

Note that there are four quadrants. If x is positive, we move to the right. If x is negative, we move to the left. If y is positive, we move up. If y is negative, we move down.



Coordinates

Coordinates allow us to look at the relationship between pairs of numbers and points in the plane. Coordinates give the location of a point, P, in relation to the origin.

We have an x coordinate and a y coordinate.

Let (x, y) represent the point whose coordinates are the numbers x and y. Note that the x coordinate comes first. The two coordinates tell us how far we must go first along the x axis and then along the y axis until the point is reached.

Plotting coordinates: some examples

Example 1

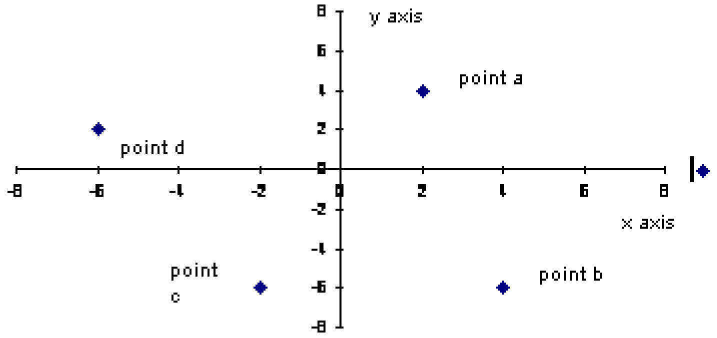
Find the following points

point a (2, 4)

point b (4, -6)

point c (-2, -6)

point d (-6, 2)



Example 2

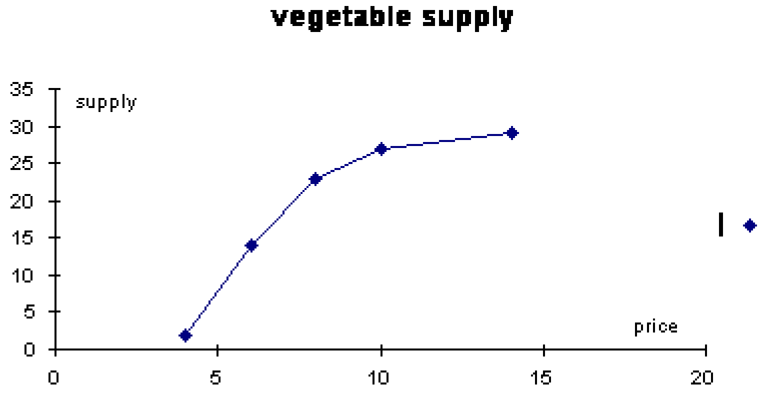
p = the price per dollar of a crate of vegetables

f(p) = the supply in thousands of crates

A store manager has the following data which relate the price of a crate of vegetables to the supply:

price	4	6	8	10	14
supply	2	14	23	27	29

What do the data tell us? The graph shows that as we might expect an increase in price is associated with an increase in supply.



Linear Functions

The linear function is popular in economics. It is attractive because it is simple and easy to handle mathematically. It has many important applications.

Linear functions are those whose graph is a straight line.

A linear function has the following form

$y = f(x) = a + bx$

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y.

a is the constant term or the y intercept. It is the value of the dependent variable when $x = 0$.

b is the coefficient of the independent variable. It is also known as the slope and gives the rate of change of the dependent variable.

Graphing a linear function

To graph a linear function:

1. Find 2 points which satisfy the equation
2. Plot them
3. Connect the points with a straight line

Example:

$y = 25 + 5x$

let $x = 1$

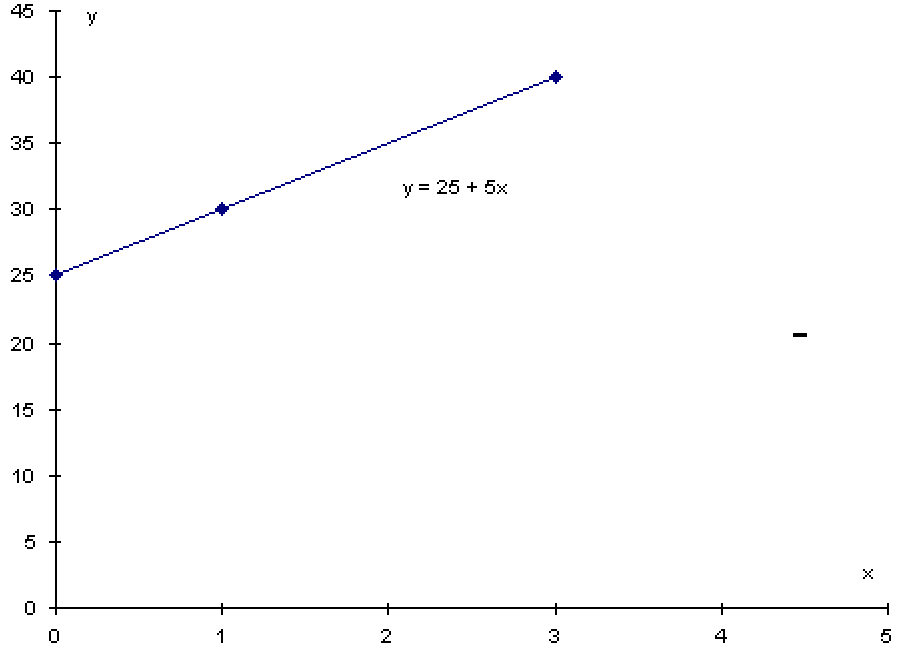
then

$y = 25 + 5(1) = 30$

let x = 3

then

y = 25 + 5(3) = 40



A simple example of a linear equation

A company has fixed costs of $7,000 for plant and equipment and variable costs of $600 for each unit of output.

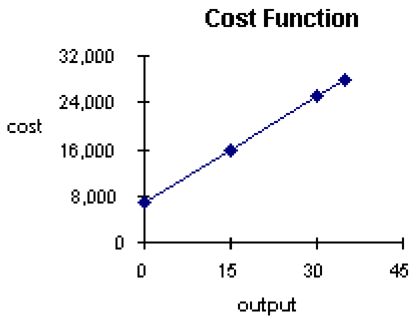
What is total cost at varying levels of output?

let x = units of output

let C = total cost

C = fixed cost-plus variable cost = 7,000 + 600 x

output	**total cost**
15 units	C = 7,000 + 15(600) = 16,000
30 units	C = 7,000 + 30(600) = 25,000



Combinations of linear equations

Linear equations can be added together, multiplied or divided.

A simple example of addition of linear equations

C(x) is a cost function

C(x) = fixed cost + variable cost

R(x) is a revenue function

R(x) = selling price (number of items sold)

profit equals revenue less cost

P(x) is a profit function

P(x) = R(x) - C(x)

x = the number of items produced and sold

Data:

A company receives $45 for each unit of output sold. It has a variable cost of $25 per item and a fixed cost of $1600.

What is its profit if it sells (a) 75 items, (b)150 items, and (c) 200 items?

R(x) = 45x	C(x) = 1600 + 25x
P(x) = 45x -(1600 + 25x)	
= 20x - 1600	
let x = 75	P (75) = 20(75) - 1600 = -100 a loss
let x = 150	P (150) = 20(150) - 1600 = 1400

let $x = 200$ $\quad$ $P(200) = 20(200) - 1600 = 2400$

Slope of Linear Functions

The concept of slope is important in economics because it is used to measure the rate at which changes are taking place. Economists often look at how things change and about how one item changes in response to a change in another item.

It may show for example how demand changes when price changes or how consumption changes when income changes or how quickly sales are growing.

Slope measures the rate of change in the dependent variable as the independent variable changes. The greater the slope the steeper the line.

Consider the linear function:

$y = a + bx$

b is the slope of the line. Slope means that a unit change in x, the independent variable will result in a change in y by the amount of b.

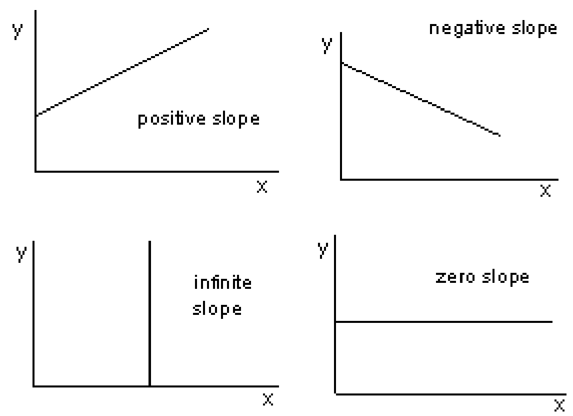
slope = change in y/change in x = rise/run

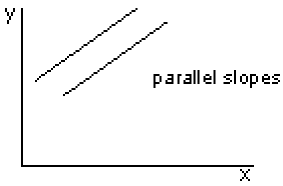
Slope shows both steepness and direction. With **positive** slope the line moves upward when going from left to right. With **negative** slope the line moves down when going from left to right.

If two linear functions have the same slope, they are parallel.

Slopes of linear functions

The slope of a linear function is the same no matter where on the line it is measured. (This is not true for non-linear functions.)





An example of the use of slope in economics

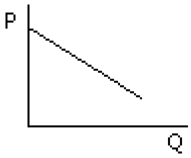
Demand might be represented by a linear demand function such as

Q(d) = a - bP

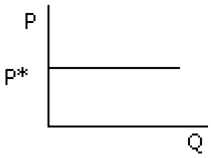
Q(d) represents the demand for a good

P represents the price of that good.

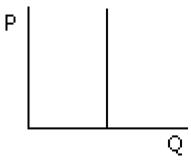
Economists might consider how sensitive demand is to a change in price.



This is a typical downward sloping demand curve which says that demand declines as price rises.



This is a special case of a horizontal demand curve which says at any price above P* demand drops to zero. An example might be a competitor's product which is considered just as good.



This is a special case of a vertical demand curve which says that regardless of the price quantity demanded is the same. An example might be medicine as long as the price does not exceed what the consumer can afford.

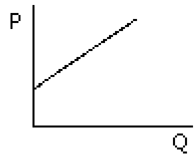
Supply might be represented by a linear supply function such as

Q(s) = a + bP

Q(s) represents the supply for a good

P represents the price of that good.

Economists might consider how sensitive supply is to a change in price.



This is a typical upward sloping supply curve which says that supply rises as price rises.

An example of the use of slope in economics

The demand for a breakfast cereal can be represented by the following equation where p is the price per box in dollars:

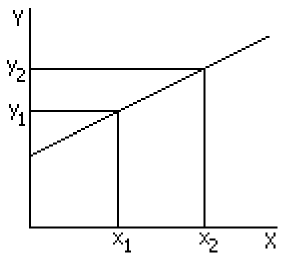
d = 12,000 - 1,500 p

This means that for every increase of $1 in the price per box, demand decreases by 1,500 boxes.

Calculating the slope of a linear function

Slope measures the rate of change in the dependent variable as the independent variable changes. Mathematicians and economists often use the Greek capital letter D or D as the symbol for change. Slope shows the change in y or the change on the vertical axis versus the change in x or the change on the horizontal axis. It can be measured as the ratio of any two values of y versus any two values of x.

$$\text{slope} = \frac{\text{changein y}}{\text{changein x}} = \frac{\Delta Y}{\Delta X} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$



Example 1

Find the slope of the line segment connecting the following points:

(1,1) and (2,4)

$x_1 = 1$ $\quad$ $y_1 = 1$

$x_2 = 2$ $\quad$ $y_2 = 4$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{1-4}{1-2} = \frac{-3}{-1} = 3$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

Example 2

Find the slope of the line segment connecting the following points:

(-1, -2) and (1,6)

$x_1 = -1$ $y_1 = -2$

$x_2 = 1$ $y_2 = 6$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-2-6}{-1-1} = \frac{-8}{-2} = 4$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-2)}{1-(-1)} = \frac{8}{2} = 4$$

Example 3

Find the slope of the line segment connecting the following points:

(-1,3) and (8,0)

$x_1 = -1$ $y_1 = 3$

$x_2 = 8$ $y_2 = 0$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{3-0}{-1-8} = \frac{3}{-9} = -\frac{1}{3}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{8-(-1)} = \frac{-3}{9} = -\frac{1}{3}$$

Slope Intercept Form of a Linear Equation

The slope intercept form of a linear equation has the following form where the equation is solved for y in terms of x:

y = a + bx

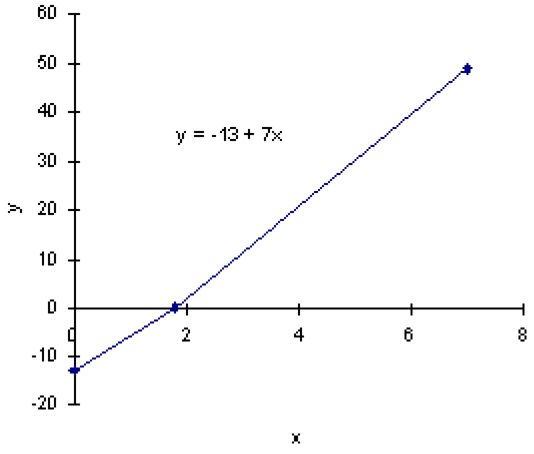
b is the slope

a is a constant term. It is the y intercept, the place where the line crosses the y axis.

Example 1

y = -13 + 7x

This equation is in slope intercept form. The y intercept is (0, -13) and the slope is 7.



Example 2

4x + 3y = 12

Rewrite this equation in slope intercept form.

3y = 12 - 4x

$y = 4 - \frac{4}{3}x$

The equation is now in slope intercept form. The y intercept is (0,4) and the slope is -4/3.

Example 3

5x - 3y - 15 = 0

Rewrite this equation in slope intercept form.

3y = -15 + 5x

$y = -5 + \frac{5}{3}x$

The equation is now in slope intercept form. The y intercept is (0, -5) and the slope is 5/3.

Example 4

$x = 5 + \frac{2}{3}y$

$\frac{2}{3}y = -5 + x$

$\frac{2}{3}y\left(\frac{3}{2}\right) = -5\left(\frac{3}{2}\right) + \frac{3}{2}x$

$y = -\frac{15}{2} + \frac{3}{2}x$

$y = -7.5 + 1.5x$

The equation is now in slope intercept form. The y intercept is (0, -7.5) and the slope is 1.5.

Applications of Linear Functions

Simple interest

With simple interest, interest is earned (charged) only on the amount lent (borrowed).

P	=	principal
		the amount lent or borrowed
i	=	the interest rate
n	=	the number of years
A	=	the future value
		the amount received or due at the end of n years
i(n)(P)	=	interest on lending or borrowing

A = P + P (i) (n)

A = P [1+(i)(n)] The P is factored out.

Example 1

$12,000 is borrowed at a rate of 9 %. How much is owed after n years?

A = 12,000 (1+.09n)

The independent variable is n, the length of the loan. The dependent variable is A, the amount which must be repaid.

Example 2

You borrow $3,200 and want to pay it back in a lump sum after 4 years. How much will you have to pay if you are being charged 8% simple interest?

A = 3,200 [1+.08(4)]

A = 4,224

How much will you have to pay if you are being charged 10 % simple interest?

A = 3,200 [1+.10(4)]

A = 4,480

Systems of Linear Equations

Often it is necessary to look at several functions of the same independent variable. Consider the prior example where x, the number of items produced and sold, was the independent variable in three functions, the cost function, the revenue function, and the profit function.

In general, there may be:

n equations

v variables

Solving systems of equations

There are four methods for solving systems of linear equations:

a. graphical solution
b. algebraic solution
c. elimination method
d. substitution method

Graphical solution

Example 1

given are the two following linear equations:

$f(x) = y = 1 + .5x$

$f(x) = y = 11 - 2x$

Graph the first equation by finding two data points. By setting first x and then y equal to zero it is possible to find the y intercept on the vertical axis and the x intercept on the horizontal axis.

If $x = 0$, then $f(0) = 1 + .5(0) = 1$

If $y = 0$, then $f(x) = 0 = 1 + .5x$

$-.5x = 1$

$x = -2$

The resulting data points are (0,1) and (-2,0)

Graph the second equation by finding two data points. By setting first x and then y equal to zero it is possible to find the y intercept on the vertical axis and the x intercept on the horizontal axis.

If $x = 0$, then $f(0) = 11 - 2(0) = 11$

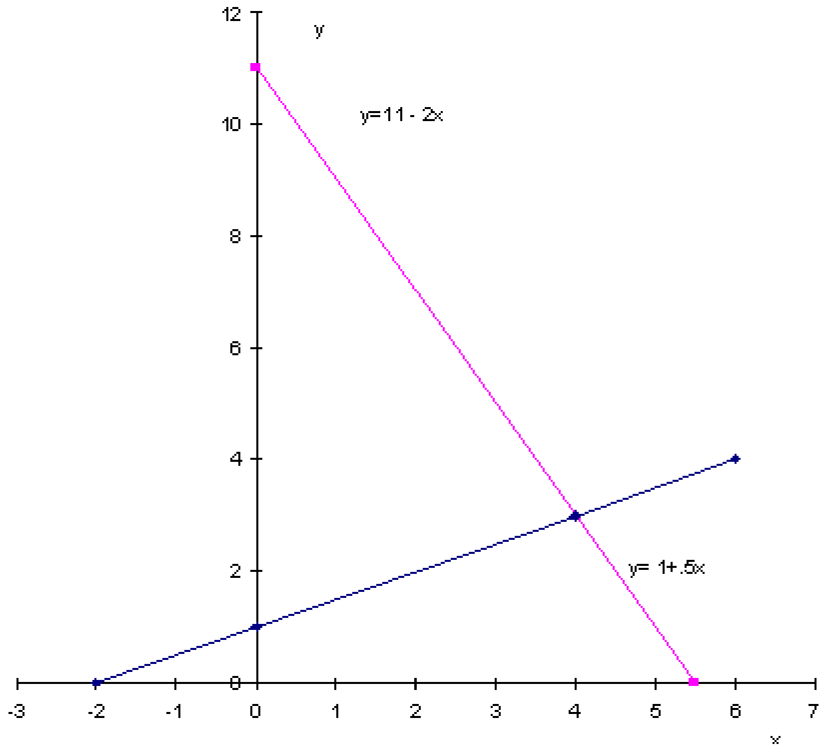
If $y = 0$, then $f(x) = 0 = 11 - 2x$

$2x = 11$

$x = 5.5$

The resulting data points are (0,11) and (5.5,0)

At the point of intersection of the two equations x and y have the same values. From the graph these values can be read as $x = 4$ and $y = 3$.



Example 2

given are the two following linear equations:

$f(x) = y = 15 - 5x$

$f(x) = y = 25 - 5x$

Graph the <u>first</u> equation by finding two data points. By setting first x and then y equal to zero it is possible to find the y intercept on the vertical axis and the x intercept on the horizontal axis.

If $x = 0$, then $f(0) = 15 - 5(0) = 15$

If $y = 0$, then $f(x) = 0 = 15 - 5x$

$5x = 15$

$x = 3$

The resulting data points are (0,15) and (3,0)

Graph the <u>second</u> equation by finding two data points. By setting first x and then y equal to zero it is possible to find the y intercept on the vertical axis and the x intercept on the horizontal axis.

If $x = 0$, then $f(0) = 25 - 5(0) = 25$

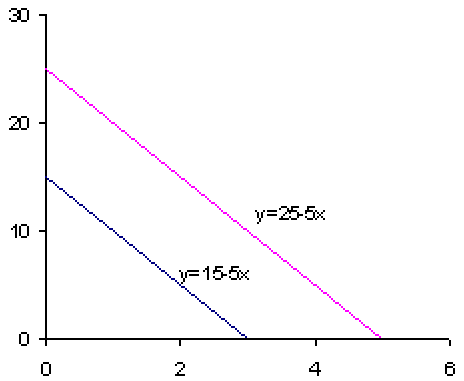
If $y = 0$, then $f(x) = 0 = 25 - 5x$

$5x = 25$

x = 5

The resulting data points are (0,25) and (5,0)

From the graph it can be seen that these lines do not intersect. They are parallel. They have the same slope. There is no unique solution.



Example 3

given are the two following linear equations:

21x - 7y = 14

-15x + 5y = -10

Rewrite the equations by putting them into slope intercept form.

The first equation becomes

7y = -14 + 21x

y = -2 + 3x

The second equation becomes

5y = -10 + 15x

y = -2 + 3x

Graph either equation by finding two data points. By setting first x and then y equal to zero it is possible to find the y intercept on the vertical axis and the x intercept on the horizontal axis.

If x = 0, then f (0) = -2 +3(0) = -2

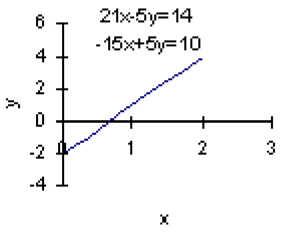
If y = 0, then f(x) = 0 = -2 + 3x

3x = 2

x = 2/3

The resulting data points are (0, -2) and (2/3,0)

From the graph it can be seen that these equations are equivalent. There are an infinite number of solutions.



Algebraic solution

This method will be illustrated using supply and demand analysis. This type of analysis is derived from the work of the great English economist Alfred Marshall.

Q = quantity and P = price

P (s)= the supply function and P (d) = the demand function

When graphing price is placed on the vertical axis. Thus, price is the dependent variable. It might be more logical to think of quantity as the dependent variable and this was the approach used by the great French economist, Leon Walras. However, by convention economists continue to graph using Marshall's analysis which is referred to as the Marshallian cross.

The objective is to find an equilibrium price and quantity, i.e. a solution where price and quantity will have the same values in both the supply function and the price function.

Q_E = the equilibrium quantity P_E = the equilibrium price

For equilibrium

supply = demand

or P (s) = P (d)

Given the following functions

$P(s) = 3Q + 10$ and $P(d) = -1/2Q + 80$

Set the equations equal to each other and solve for Q.

$P(s) = 3Q + 10 = -1/2Q + 80 = P(d)$

$3.5Q = 70$

$Q = 20$ The equilibrium quantity is 20.

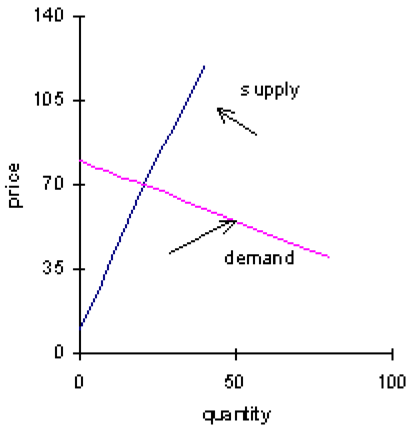
Substitute this value for Q in either equation and solve for P.

$P(s) = 3(20) + 10$

$P(s) = 70$

P (d) = -1/2(20) + 80

P (d) = 70 The equilibrium price is 70.



Elimination method

This method involves removing variables from the equations. Variables are removed successively until only a single last variable is left, i.e. until there is one equation with one unknown. This equation is then solved for the one unknown. The solution is then used in finding the second to last variable. The procedure is repeated by adding back variables as their solutions are found.

Example 1

$2x + 3y = 5$

$-5x - 2y = 4$

Procedure: eliminate y. The coefficients of y are not the same in the two equations but if they were it would possible to add the two equations and the y terms would cancel out. However, it is possible through multiplication of each equation to force the y terms to have the same coefficients in each equation.

Step 1: Multiply the first equation by 2 and multiply the second equation by 3. This gives

$4x + 6y = 10$

$-15x - 6y = 12$

Step 2: Add the two equations. This gives

$-11x = 22$

$x = -2$

Step 3: Solve for y in either of the original equations

$2(-2) + 3y = 5$

$3y = 9$

$y = 3$ or

$-5(-2) - 2y = 4$

$10 - 2y = 4$

$2y = 6$

$y = 3$

Alternate Procedure: eliminate x. The coefficients of x are not the same in the two equations but if they were it would possible to add the two equations and the y terms would cancel out. However, it is possible through multiplication of each equation to force the x terms to have the same coefficients in each equation.

Step 1: Multiply the first equation by 5 and multiply the second equation by 2. This gives

$$10x + 15y = 25$$
$$-10x - 4y = 8$$

Step 2: Add the two equations. This gives

$$11y = 33$$
$$y = 3$$

Step 3: Solve for x in either of the original equations

$2x + 3(3) = 5$

$2x = -4$

$x = -2$ or

$-5x - 2(3) = 4$

$- 5x = 10$

$x = -2$

Example 2

$$2x_1 + 5x_2 + 7x_3 = 2$$
$$4x_1 - 4x_2 - 3x_3 = 7$$
$$3x_1 - 3x_2 - 2x_3 = 5$$

In this example there are three variables: x_1, x_2, and x_3. One possible procedure is to eliminate first x_1, to eliminate next x_2, and then to solve for x_3. The value obtained for x_3 is used to solve for x_2 and finally the values obtained for x_3 and x_2 are used to solve for x_1.

Procedure Part A First eliminate x_1.

Step 1 Multiply the first equation by 2 and subtract the second equation from the first equation. This gives

$4x_1 + 10x_2 + 14x_3 = 4$ first equation

$4x_1 - 4x_2 - 3x_3 = 7$ second equation

$14x_2 + 17x_3 = -3$ second equation subtracted from the first

Step 2 Multiply the first equation by 3, multiply the third equation by 2, and subtract the third equation from the first equation. This gives

$6x_1 + 15x_2 + 21x_3 = 6$ first equation

$6x_1 - 6x_2 - 4x_3 = 10$ third equation

$21x_2 + 25x_3 = -4$ third equation subtracted from the first

Procedure Part B Second eliminate x_2. From Part A there are two equations left. From these two equations eliminate x_2.

$14x_2 + 17x_3 = -3$ first equation

$21x_2 + 25x_3 = -4$ second equation

Step 1 Multiply the first equation by 21, multiply the second equation by 14. and subtract the second equation from the first equation. This gives

$294x_2 + 357x_3 = -63$ first equation

$294x_2 + 350x_3 = -56$ second equation

$7x_3 = -7$ second equation subtracted from first

$x_3 = -1$

Part C Solve for x_2 by inserting the value obtained for x_3 in either equation from Part B.

$14x_2 + 17(-1) = -3$

$1 4x_2 = 14$

$x_2 = 1$ or

$21x_2 + 25(-1) = -4$

$21x_2 = 21$

$x_2 = 1$

Part D Solve for x_1 by inserting the values obtained x_2 andx_3 in any of the three original equations.

$2x_1 + 5x_2 + 7x_3 = 2$ first original equation

$2x_1 + 5(1) + 7(-1) = 2$

$2x_1 = 4$

$x_1 = 2$ or

$4x_1 - 4x_2 - 3x_3 = 7$ second original equation

$4x_1 - 4(1) - 3(-1) = 7$

$4x_1 \quad = 8$

$x_1 \quad = 2$ or

$3x_1 - 3x_2 - 2x_3 = 5$ third original equation

$3x_1 - 3(1) -2(-1) = 5$

$3x_1 \quad = 6$

$x_1 \quad = 2$

Substitution method

This involves expressing one variable in terms of another until there is a single equation in one unknown. This equation is then solved for that one unknown. The result is then used to solve for the variable which was expressed in terms of the variable whose solution has just been found.

Example

$12x - 7y = 106$ first equation

$8x \quad + y \quad = 82$ second equation

Solve the second equation for y and then substitute the value obtained for y in the first equation.

$y = 82 - 8x$ second equation solved for y

$12x - 7(82 – 8x) = 106$ first equation rewritten in terms of x

$12x - 574 \ +56x = 106$

$68x = 680$

$x = 10$

Substitute the value obtained for x in either of the original equations.

$12x - 7y = 106$ first equation

$12(10) - 7y = 106$

$7y = 14$

$y = 2$

$8(10) \quad + y \quad = 82$ second equation

$y \quad = 2$

Non-Linear Functions

Often in economics a linear function cannot explain the relationship between variables. In such cases a non-linear function must be used. Non-linear means the graph is not a straight line. The graph of a non-linear function is a curved line. A curved line is a line whose direction constantly changes.

A cautionary note: Economists are accustomed to designate all lines in graphs as curves - both straight lines and lines which are actually curved.

Although the slope of a linear function is the same no matter where on the line it is measured, the slope of a non-linear function is different at each point on the line. Thus, there is no single slope for a non-linear function. However, the slope can be determined at any point on the line. The techniques of differential calculus are used to determine the slopes of non-linear functions.

Three non-linear functions commonly used in economics are

- the exponential function
- the quadratic function
- the logarithmic function

Exponents

What are exponents? Exponential notation is a form of mathematical shorthand which allows us to write complicated expressions more succinctly. An exponent is a number or letter written above and to the right of a mathematical expression called the base. It indicates that the base is to be raised to a certain power. x is the base and n is the exponent or power.

Definition: If x is a positive number and n is its exponent, then x^n means x is multiplied by itself n times.

Examples:

$3^2 = 3 \times 3$

$3^5 = 3 \times 3 \times 3 \times 3 \times 3$

Rules of exponents

$$x^n x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{-n} = \frac{1}{x^n}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$x^0 = 1$$

The Exponential Function

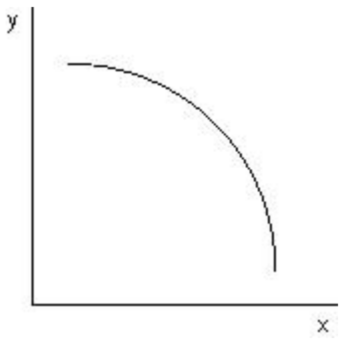
The exponential function has the form:

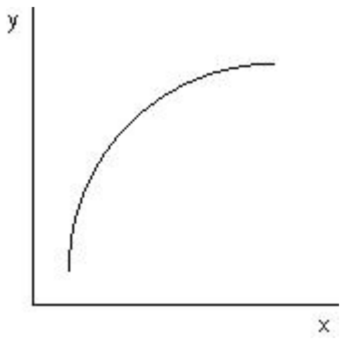
$F(x) = y = ab^x$

where $a \neq 0$ and b is a constant called the base of the exponential function. $b > 0$ and $b \neq 1$

x is the independent variable. It is the exponent of the constant, b. Thus, exponential functions have a constant base raised to a variable exponent

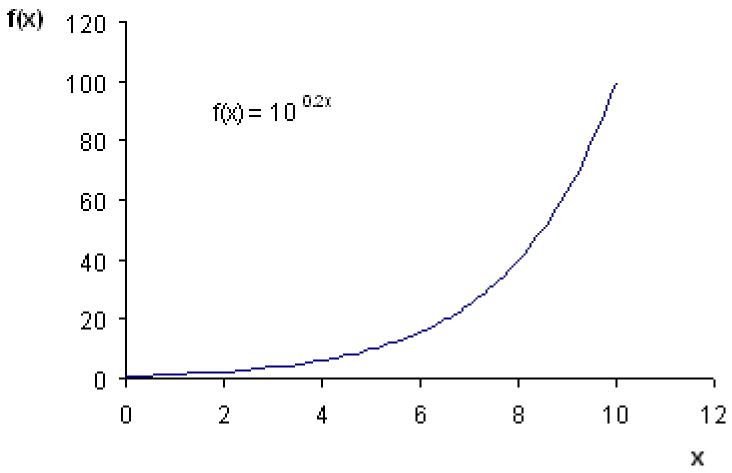
In economics exponential functions are important when looking at growth or decay. Examples are the value of an investment that increases by a constant percentage each period, sales of a company that increase at a constant percentage each period, models of economic growth or models of the spread of an epidemic.

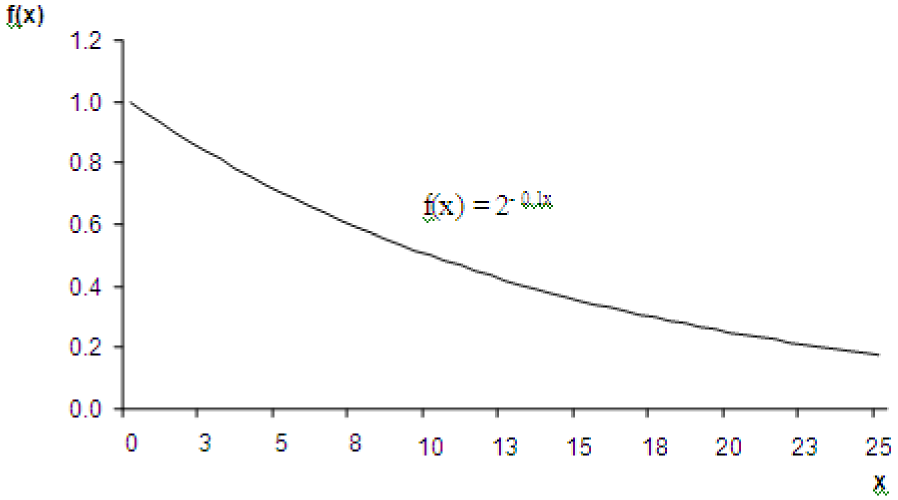




Notice that as the value of x increases, the value of y increases or decreases more and more rapidly.

Examples of the exponential function





Logarithms

A logarithm is an exponent. A logarithm is an exponent which indicates to what power a base must be raised to produce a given number.

$y = b^x$ exponential form

$x = \log_b y$ logarithmic form

x is the logarithm of y to the base b

$\log_b y$ is the power to which we have to raise b to get y

We are expressing x in terms of y

Examples

$\boldsymbol{x = \log_b y}$

$x = \log_2 8$ — This means the logarithm of 8 to the base 2. It is the exponent to which 2 must be raised to get 8. We know that 2(2)(2) = 8. Therefore x = 3.

$x = \log_6 36$ — This means the logarithm of 36 to the base 6. It is the exponent to which 6 must be raised to get 36. We know that 6(6) = 36. Therefore x = 2.

$x = \log_{10} 10{,}000$ — This means the logarithm of 10,000 to the base 10. It is the exponent to which 10 must be raised to get 10,000. We know that 10(10)(10)(10) = 10,000. Therefore x = 4.

$\log_b b = 1$ — The logarithm of any number to the same base equals 1.

$x = \log_{11} 11$ — This means the logarithm of 11 to the base 11. It is the exponent to which 11 must be raised to get 11. We know that 1 (1) = 11. Therefore x = 1.

$\log_b 1 = 0$ — The logarithm of 1 always equals 0.

Any number can serve as b, the base.

Common (Briggsian) logarithms the base is 10.

Logarithms to the base 10 are widely used. Thus, it is common to drop the subscript. If the base does not appear it is understood that the base is 10.

$\log_{10} y = \log y$

Natural (Naperian) logarithms the base is e.

Remember e is the irrational number where e = 2.71828... The symbol "ln" refers to natural logarithms.

$\log_e x = \ln x$ — ln x is the exponent to which e must be raised to get x.

Why do we want to use logarithms? To simplify calculations in many cases.

Rules for logarithms

Product rule $\log_b(MN) = \log_b M + \log_b N$

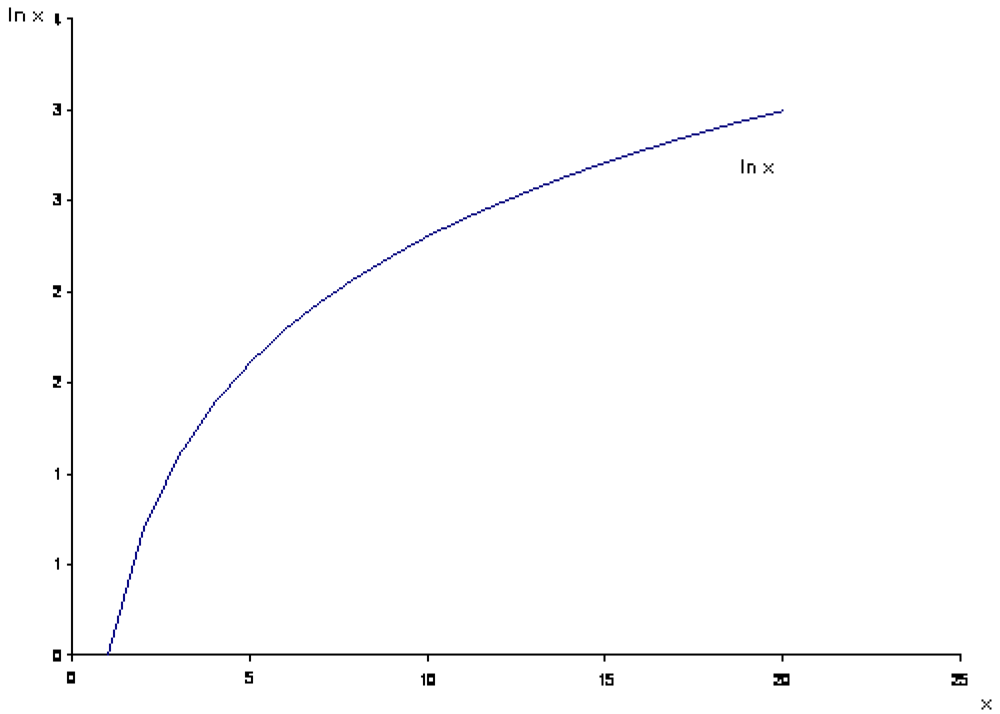
Quotient rule $\log_b \frac{M}{N} = \log_b M - \log_b N$

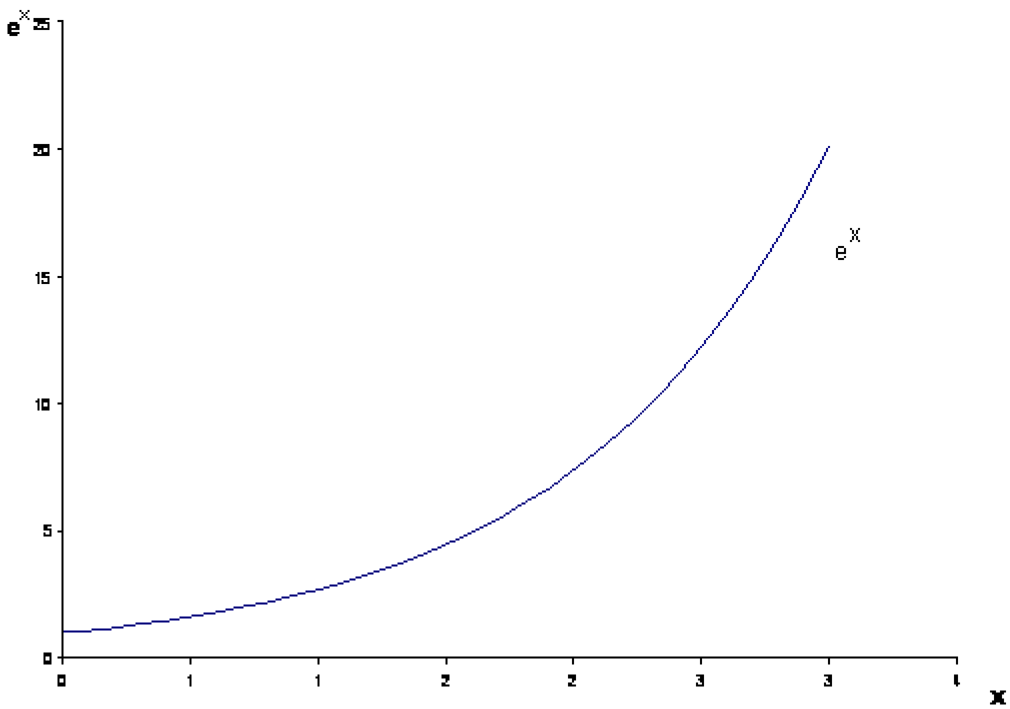
Power rule $\log_b N^k = k \log_b N$ This rule is useful because it allows us to solve equations where the variable is an exponent.

Exponential and Logarithmic Functions are inverse functions

Consider the following tables and the associated graphs:

x	$f(x) = e^x$	x	$f(x) = \ln x$
0	1	1	0
1	2.7	2.7	1
2	7.39	7.39	2
3	20	20	3





The Logarithmic Function

Linear functions are useful in economic models because a solution can easily be found. However non-linear functions can be transformed into linear functions with the use of logarithms.

Given the function

$$y = ax^b$$

Taking logarithms of both sides this function can be rewritten as

$$\log y = \log a + b \log x$$

The resulting function is linear in the log of the variables.

The Quadratic Function

The quadratic function has the form:

F(x) = y = a + bx + cx^2

where a, b, and c are numerical constants and c is not equal to zero.

Note that if c were zero, the function would be linear.

An advantage of this notation is that it can easily be generalized by adding more terms. We could for example write equations such as

$y = a + bx + cx^{2} + dx^3$

$y = a + bx + cx^{2} + dx^3 + ex^4$

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

In many books quadratic equations are written as

$ax^{2} + bx + c = 0$

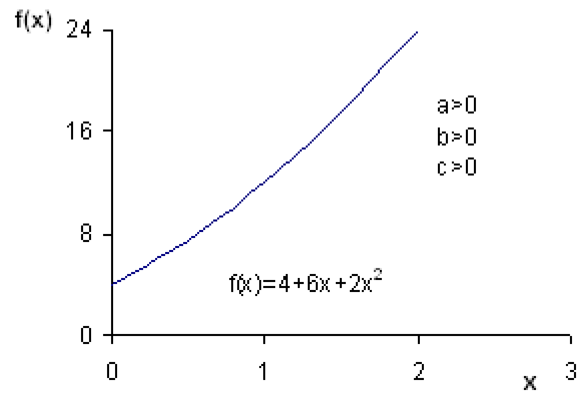
In this case the quadratic formula is given by

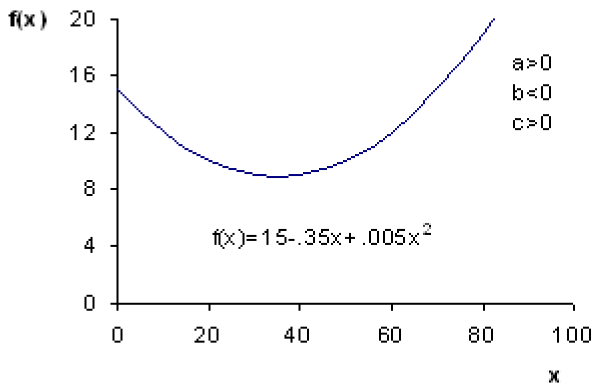
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

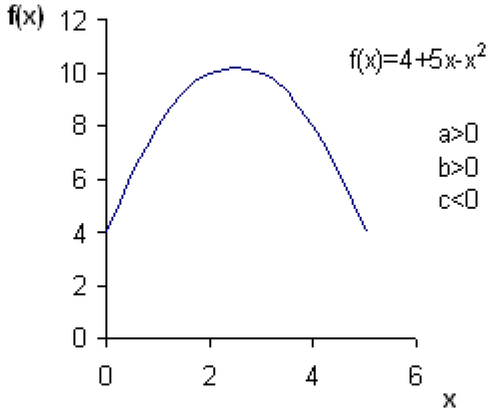
Note that the denominator is then 2a instead of 2c.

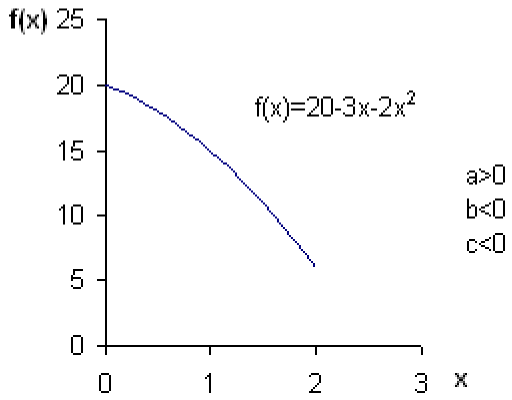
Some common examples of the quadratic function

Notice that the graph of the quadratic function is a parabola. This means it is a curve with a single bump. The graph is symmetric about a line called the axis of symmetry. The point where the axis of symmetry intersects the parabola is known as the vertex.









Graphing the quadratic function

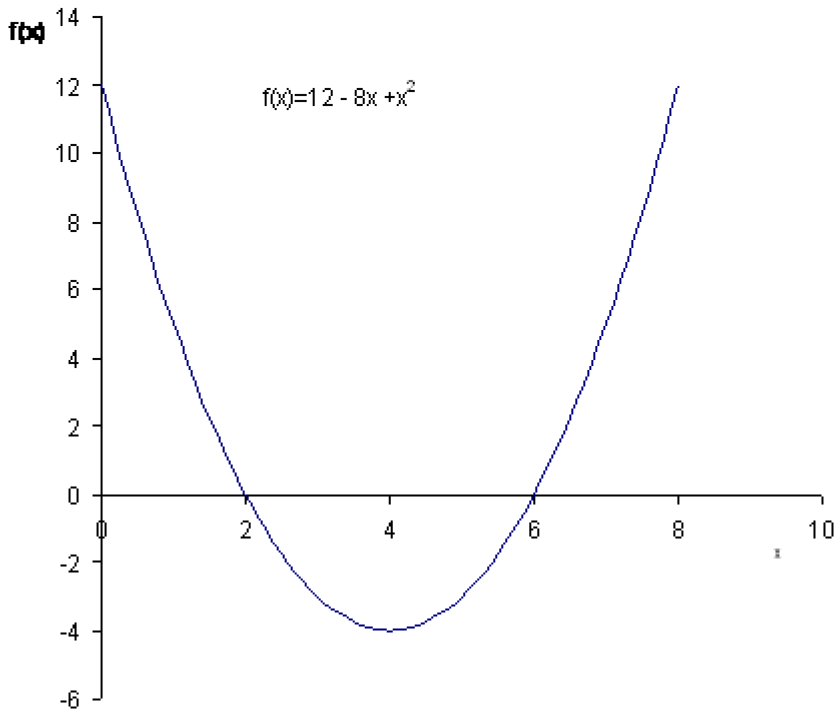
Construct a table with values of x and f(x).

Plot the data points.

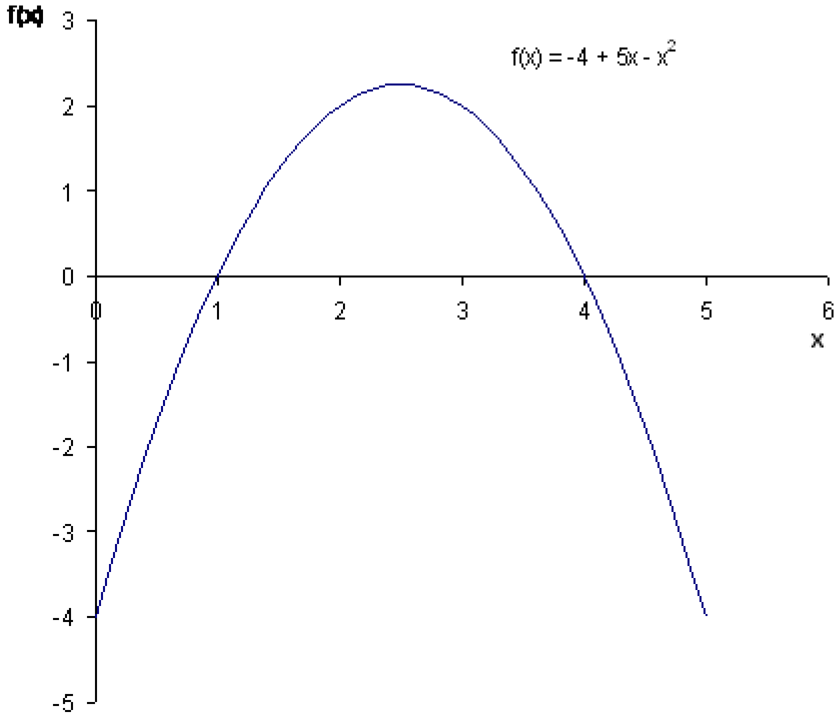
Connect the data points with a smooth line.

This is easily done with Excel.

Example 1 $f(x) = 12 - 8x + x^2$



Example 2 $f(x) = -4 + 5x - x^2$



The quadratic formula, an example.

In general, the supply of a commodity increases with price and the demand decreases. The market for the commodity is in equilibrium when supply equals demand.

In this example we are considering two functions of the same independent variable, price. We want to find the equilibrium price and the corresponding demand.

The supply function is a quadratic equation given by $S(p) = 2p + 4p^2$

The demand function is a linear function given by $D(p) = 231 - 18p$

To find the intersection of the two curves set supply equal to demand and solve for p.

$$S(p) = 2\mathbf{p} + \mathbf{4p^2} = \mathbf{231 - 18p} = D(p)$$

After collecting terms, we obtain the quadratic equation

$$231 - 20p - 4p^2 = 0$$

Note that this has the form of the quadratic equation

$$y = a + bx + cx^2$$

$$0 = 231 - 20p - 4p^2$$

Solve the equation by means of the quadratic formula where a = 231, b = -20, and c = -4.

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(231)(-4)}}{2(-4)}$$

$$x = \frac{20 \pm \sqrt{400 + 3696}}{-8}$$

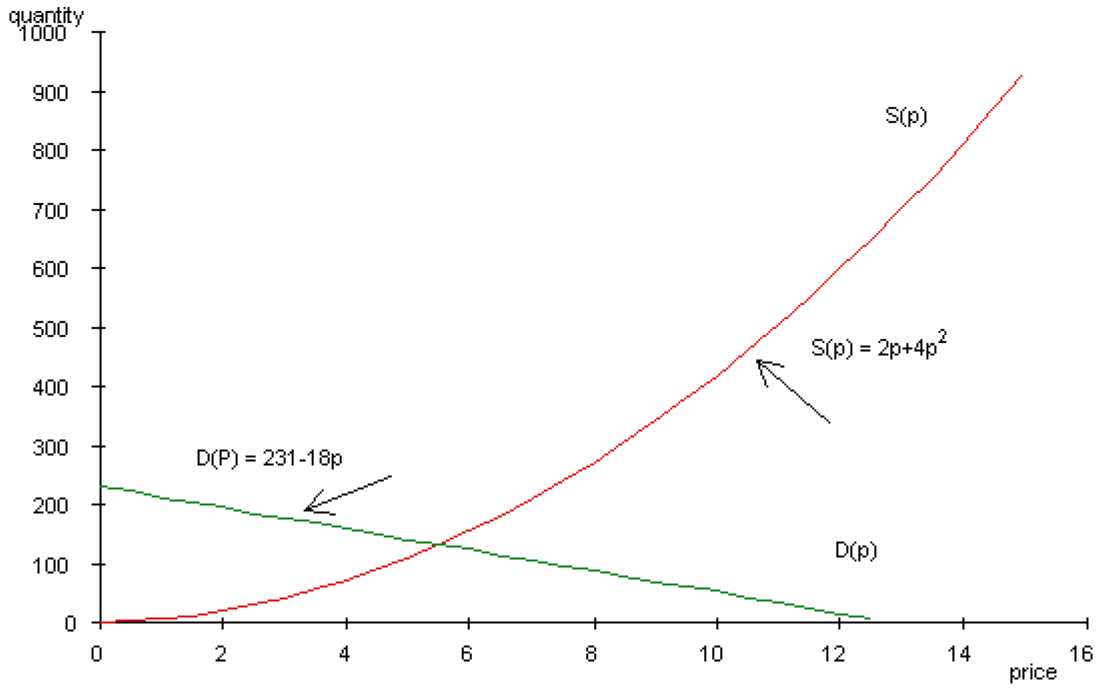
$$x = \frac{20 \pm 64}{-8} \quad x = \frac{20 \pm \sqrt{4096}}{-8}$$

$$x_1 = \frac{84}{-8} = -10.5 \quad \text{and} \quad x_2 = \frac{-44}{-8} = 5.50$$

Since price cannot be negative the value of -10.5 can be eliminated. Supply will equal demand when the price is $5.50. At that price it is possible to find the corresponding demand and supply.

D (5.50) = 231 - 18(5.50) = 132

S (5.50) = 2(5.50) + 4(5.50^{2} = 132


quantity
1000
900
800
700
600
500
400
300
200
100
0
0
2
4
6
8
10
12
14
16
price
S(p)
S(p) = 2p+4p2
D(P) = 231-18p
D(p)