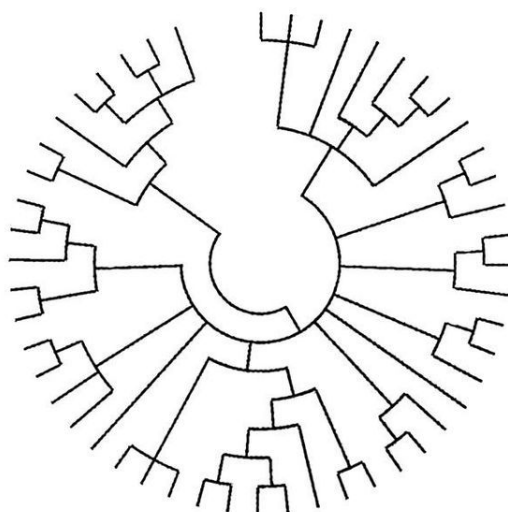


DOCUMENTATION FOR THE PYTHON LIBRARY PEDIGRAD.PY

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VERSION 19.04.a



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Introduction

1.1. About pedigrad and Pedigrad.py

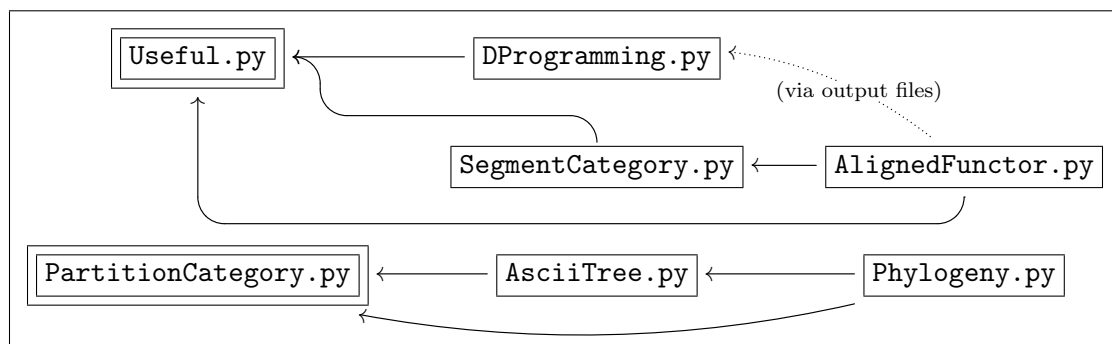
Pedigrad is a mathematical tool that was initially created to model genetic mechanisms (see [1, 2, 3]). Mathematically, pedigrad is a cone-preserving functor going from a certain class of limit sketches to a given category of values. Different aspects of biology can be encoded depending on the considered category of value. The present python library – `Pedigrad.py` – provides tools modelling the content described in [2]. Because earlier versions focused on pedigrad in the category of partitions and their use in phylogenetics, the library also comes with modules on partitions and evolutionary trees.

1.2. About this documentation

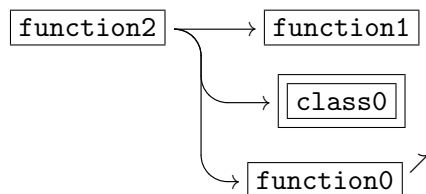
The present book contains a tutorial (see Chapter 2) explaining how to use the various methods and classes contained in `Pedigrad.py` as well as a description of the (importable and non-importable) functions and classes of its sub-modules

- `Useful.py` (see Chapter 3)
- `DProgramming.py` (see Chapter 4)
- `SegmentCategory.py` (see Chapter 5)
- `AlignedFunctor.py` (see Chapter 6)
- `PartitionCategory.py` (see Chapter 7)
- `AsciiTree.py` (see Chapter 8)
- `Phylogeny.py` (see Chapter 9)

The dependencies between the different chapters is displayed below.



Similarly, descriptions of functions given in this documentation will usually start with dependency flow charts showing the modules that they use. Below, we give the example of such a flow chart.



As can be seen above, these flow charts will use three different types of boxes:

- 1) boxes with no specific decoration will usually frame items (such as classes and functions) that belong to the module of the described function;
- 2) double boxes will frame the name of intermediate items that do not have dependencies with other items (final item);
- 3) boxes with arrows in the top-right corner will frame items that are defined in modules external to that of the described function;

We will also give examples and demonstrations within editor mode windows or console mode windows. The editor mode will be used to describe the code of functions and will look as follows.

```

1 class MyClass:
2     """
3     This is a comment
4     """
5     def __init__(self, arg0, arg1): #This is the constructor of the class
6         """
7         Another comment about the code
8         """

```

The console mode will be used for examples and will look as follows.

```

>>> P.obj
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13]

```

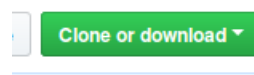
1.3. Acknowledgments

I would like to thank Maxim Wolf and Carles Boix for interesting discussions about DNA and genetics. I would also like to thank Maxim Wolf for answering many of my questions and giving me some of his time.

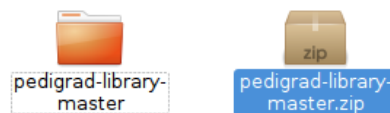
Tutorial

2.1. Installation and preparation

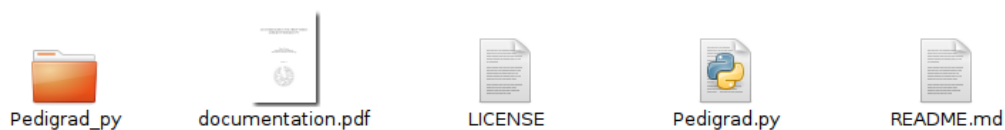
To install the library, first download the package by clicking on the green button on the right of the screen at <https://github.com/remytuyeras/pedigrad-library>.



The downloaded package should be a compressed file named `pedigrad-library-master.zip`. Create a new directory in which you can copy and extract the compressed file using your favorite extraction application.



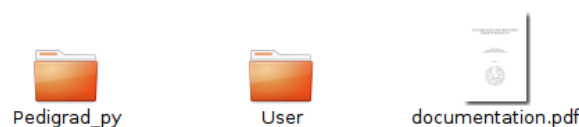
Enter the extracted directory `pedigrad-library-master`. Its inside should contain the following files.



In this tutorial, we will create four files:

- ▷ two multiple sequence alignment files (usually with an extension `.fa` or `.fasta`);
- ▷ a file in which a pre-ordered set will be specified (preferably with an extension `.yaml`);
- ▷ a python file `main.py` in which we will write python functions and use the library.

To do this properly, create a new directory in `pedigrad-library-master`, call it `User`, and copy the file `Pedigrad.py` in `User`.



Now, open the file `Pedigrad.py` that you copied in `User`: you will be able to see several instances of the function `sys.path.insert` in which paths appear in the second argument.

```
1 import sys
2 sys.path.insert(0, 'Pedigrad_py/Useful/')
3 from Useful import *
```

Add the text `../` at the beginning of every path passed to the function `sys.path.insert`, as shown below.

```
1 import sys
2 sys.path.insert(0, '../Pedigrad_py/Useful/')
3 from Useful import *
```

Once the paths are all updated, create four new files with the names `align.fa`; `dprog.fa`; `main.py`; and `omega.yml` in the directory `User`. The inside of the directory `User` should look as follows, where the file `Pedigrad.pyc` will appear later, after compilation.



Open the file `main.py` and insert the following piece of code.

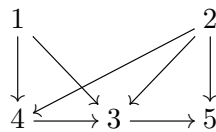
```
1 from Pedigrad import *
```

We are now ready to use the library – proceed to section 2.2.

2.2. Pre-ordered sets

This section demonstrates the use of the class `PreOrder` (section 5.1) belonging to the module `SegmentCategory.py` (section 5). The class `PreOrder` allow the user to store in the random-access memory a pre-ordered structure whose specification is given in a text file. For the sake of making this tutorial as exhaustive as possible, we will consider the pre-ordered set pictured in (2.1), in which every arrow $x \rightarrow y$ represents relation $x > y$ in the pre-order structure.

(2.1)



`omega.yml`

```
1 !obj:
2   - 1;
3   - 2;
4   - 3;
5   - 4;
6   - 5;
7 rel:
8   - 1 > 4, 3;
9   - 2 > 5, 3, 4;
10  - 3 > 5;
11  - 4 > 3;
```

The class has three objects that allows: `.relations`, `.mask` and `.cartesian`.

```
3 Omega = PreOrder("omega.yml")
4 print(Omega.relations)
5 print(Omega.mask)
6 print(Omega.cartesian)
```

2.3. Segments and their morphisms

SegmentCategory

2.4. Categories of segments

SegmentCategory

2.5. Environment functors

SequenceAlignment

2.6. Aligned functors

AlignedFunctor

2.7. Sequence alignments and the dynamic programming

DProgramming

2.8. Useful functions

usf, Useful

Useful

3.1. Description of `usf` (class item)

3.1.1. Description of `.fasta` (method).

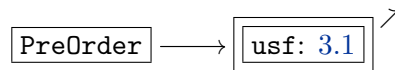
3.2. Description of `CategoryItem` (class)

DProgramming

Presentation of the module SegmentCategory.py

5.1. Description of PreOrder (class)

5.1.1. Introduction. The code of `PreOrder` uses external modules.



The class `PreOrder` models the features of a pre-ordered set. The pre-order relations are specified through either a file or another `PreOrder` item passed to the constructor. The method `.closure` makes sure that the pre-order axioms are satisfied and will (re-)compute the transitive closure of the pre-order relations stored in the object `.relations` if needed; the method `.geq` returns a Boolean value specifying whether there is a pre-order relation between two given elements of the pre-ordered set; the method `.inf` returns the infimum of two elements of the pre-ordered set; and the method `.presence` returns a Boolean value specifying whether an element belongs to the pre-ordered set.

5.1.2. Structure. The following tables give a preview of the class `PreOrder`. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<code>object</code>	N/A
Statistics	
▷ Importable objects: 4	
▷ Non-importable objects: 0	
▷ Importable methods: 5	
▷ Non-importable methods: 2	

The following table gives a description of the 4 importable objects of the class. The type α shown in the table is a polymorphic type.

Objects		
Name	Type	Related sections
<code>.relations</code>	<code>list(list(α))</code>	▷ section 5.1.3 ▷ section 5.1.4
<code>.transitive</code>	<code>bool</code>	▷ section 5.1.3 ▷ section 5.1.4
<code>.mask</code>	<code>bool</code>	▷ section 5.1.3
<code>.cartesian</code>	<code>int</code>	▷ section 5.1.3 ▷ section 5.1.5 ▷ section 5.1.6

Finally, the following table gives a description of the 5 importable methods of the class:

Methods			
Name	Input types	Output types	Related sections
<code>__init__</code>	- <code>string</code> - <code>int</code> - <code>list</code>	- <code>self</code>	▷ section 5.1.3
<code>.closure</code>	- <code>self</code>	- <code>self</code>	▷ section 5.1.4
<code>.geq</code>	- α - α - <code>list</code>	- <code>bool</code>	▷ section 5.1.5
<code>.inf</code>	- α - α - <code>list</code>	- α	▷ section 5.1.6
<code>.presence</code>	- α	- <code>bool</code>	▷ section 5.1.7

5.1.3. Description of `__init__` (method). The code of the function `__init__` is equipped with the following inputs, among which two are optional.

<code>__init__</code>		
Inputs	Types	Specifications
<code>name_of_file</code>	<code>string</code>	necessary
<code>cartesian = 0</code>	<code>int</code>	optional
<code>*args</code>	<code>list</code>	optional

The method possesses two actions, which we describe below through examples.

Action 1	
Case	If <code>cartesian > 0</code> and <code>len(args)==1</code>
Description	In this case, the list <code>args</code> is assumed to contain a <code>PreOrder</code> item that the method uses to initialize the three objects <code>self.relations</code> , <code>self.transitive</code> , <code>self.mask</code> through componentwise attributions. The object <code>self.cartesian</code> is initialized with the value stored in <code>cartesian</code> .
Action 2	
Case	If <code>cartesian <= 0</code> or <code>len(args) != 1</code>
Description	In this case, the method uses the file referenced through the variable <code>name_of_file</code> to fill out the objects of the class. In section 5.1.3.1 (see below), we describe the syntax of the file and how it is parsed.

5.1.3.1. `name_of_file` (variable). The code contained in `name_of_file` should follow a specific syntax that can be described through the following grammar.

```

file      → file | !obj: O rel: R | obj: O rel: R | ε
O         → Separator Word Comment O | ε
R         → Text > Text ; Comment R | Comment | ε
Separator → not(Word) Separator | ε
Word      → [0-9]Word | @Word | [A-Z]Word | _Word | [a-z]Word | ε
Comment   → ε | #Text
Text      → ε | Separator Word Text

```

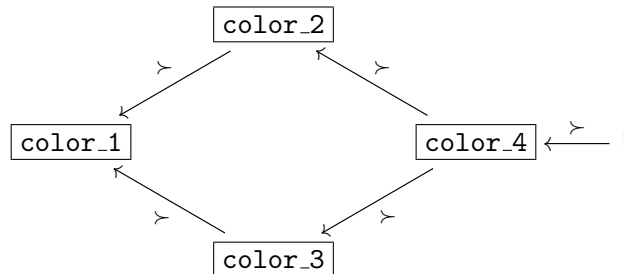
Key words	
!obj: , obj:	Specify that the objects of the pre-ordered set are about to be listed. Each object is defined by a <code>Word</code> variable in the previous grammar.
!obj:	Will add a formal minimal object to the pre-order structure
rel:	Specifies that the relations of the pre-ordered set are about to be listed. The relations should only contain variables that were listed after !obj: or obj: .

Because this syntax can be compatible with the yaml grammar, the file can ideally be specified with the extension `.yaml`. Here is an example of such a file.

preorder.yaml
<pre> 1 #This is an example of a pre-ordered set. 2 3 !obj:#a formal minimal object is added to the pre-order 4 - color_1 #1st object 5 - color_2 #2st object 6 - color_3 #3rd object 7 - color_4 #4th object 8 9 rel:#these are generating relations for the preorder. 10 - color_1 > color_2, color_3; 11 - color_1 > color_3, color_1; 12 - color_3, color_2 > color_4; </pre>

The previous pre-ordered set can be described by atomic relations shown in diagram (5.1), where the additional object `!` is the formal minimal object added to the pre-order structure.

(5.1)



We now give examples showing how the pre-ordered set `preorder.yaml` is used by the constructor of the class `PreOrder`.

After calling the constructor, the object `.relations` will contain the pre-order relations specified in `preorder.yaml` as a list of lists (see below). Every internal list starts with a predecessor of all the elements in that list. There is an internal list for every element specified as an object of the pre-ordered set in `preorder.yaml`.

```
>>> Omega = PreOrder("preorder.yml")
```

```
>>> Omega.relations
```

```
[[‘color_1’, ‘color_2’, ‘color_3’], [‘color_2’, ‘color_4’], [‘color_3’,  
‘color_4’], [‘color_4’]]
```

The object `.mask` is set to `True` if a formal minimal object has been added to the structure via the use of `!obj:` and is set to `False` otherwise.

```
>>> Omega.mask
```

```
True
```

Since we have not specified the value of the parameter `cartesian`, its value is still 0.

```
>>> Omega.cartesian
```

```
0
```

Also, since the method `.closure()` has not been called yet, the object `.transitive` is still equal to `False`.

```
>>> Omega.transitive
```

```
False
```

Calling the method `.closure()` will complete the lists inside `.relations` into down-closures for the first element of every list.

```
>>> Omega.closure()
```

```
>>> Omega.relations
```

```
[[‘color_1’, ‘color_2’, ‘color_3’, ‘color_4’], [‘color_2’, ‘color_4’],  
[‘color_3’, ‘color_4’], [‘color_4’]]
```

We can now verify that the call of the method `.closure()` set the value of the object `.transitive` to `True`.

```
>>> Omega.transitive
```

```
True
```

5.1.4. Description of `.closure` (method). The code of the function `.closure` has no input. The method possesses two actions, which we describe below through examples.

Action 1	
Case	If <code>self.transitive == False</code>
Description	In this case, the value of <code>self.transitive</code> is set to <code>True</code> and the method completes every lists contained in <code>self.relations</code> into down-closures for the first element of each list.

Action 2	
Case	If <code>self.transitive == True</code>
Description	No action is taken.

Here is an example that uses the file `preorder.yml` of section 5.1.3.1 as an input.

```
>>> Omega = PreOrder("preorder.yml")
```

```
>>> Omega.relations
```

```
[[‘color_1’, ‘color_2’, ‘color_3’], [‘color_2’, ‘color_4’], [‘color_3’,  
‘color_4’], [‘color_4’]]
```

```
>>> Omega.closure()
```

```
>>> Omega.relations
```

```
[[‘color_1’, ‘color_2’, ‘color_3’, ‘color_4’], [‘color_2’, ‘color_4’],  
[‘color_3’, ‘color_4’], [‘color_4’]]
```

5.1.5. Description of `.geq` (method). The code of the function `.geq` is equipped with the following inputs.

.geq		
Inputs	Types	Specifications
element1	α	necessary
element2	α	necessary

The method possesses two actions, which we describe below through examples. Before starting each of these actions, the method calls the method `.closure()` to make sure that the relations contained in `self.relations` define transitive down-closures.

Action 1	
Case	If <code>self.cartesian == 0</code>
Description	The function considers <code>element1</code> and <code>element2</code> to be elements belonging to the lists contained in <code>self.relations</code> . The method first looks for <code>element1</code> at the beginning of every list in <code>self.relations</code> and if a list is found, the method looks for <code>element2</code> in that list.

The following example shows how the method compares element that belong or not to the pre-ordered set.

```
>>> Omega = PreOrder("preorder.yml")
>>> Omega.geq("?", "color_1")
False
>>> Omega.geq("color_1", "?")
False
>>> Omega.geq("color_1", "color_4")
True
>>> Omega.geq("color_4", "color_1")
False
>>> Omega.geq("color_2", "color_3")
False
```

The second action allows us to simulate any Cartesian product structure on the pre-ordered set.

Action 2	
Case	If <code>self.cartesian > 0</code>
Description	The function considers <code>element1</code> and <code>element2</code> to be lists of elements belonging to the lists contained in <code>self.relations</code> . For every element <code>i</code> in <code>range(self.cartesian)</code> , the method looks for <code>element1[i]</code> at the beginning of every list in <code>self.relations</code> and if a list is found, it looks for <code>element2[i]</code> in that list. Exceptionally for the present case, any component <code>element2[i]</code> is allowed to take Boolean values. If <code>element2[i]</code> happens to be equal to <code>True</code> , then the method skip the search for <code>element1[i]</code> .

This second example shows how the method `.geq` extends to the Cartesian product of the pre-order used in the previous example.

```
>>> Cartesian = PreOrder("", 2, Omega)
>>> Cartesian.geq(["color_1", "color_2"], [Cartesian.mask]*2)
True
>>> Cartesian.geq([Cartesian.mask]*2, ["color_1", "color_2"])
False
>>> Cartesian.geq(["color_1", "color_2"], ["color_4", "color_4"])
True
>>> Cartesian.geq(["color_1", "color_2"], ["color_4", "color_3"])
False
```

5.1.6. Description of .inf (method). The code of the function `.inf` is equipped with the following inputs.

<code>.inf</code>		
Inputs	Types	Specifications
<code>element1</code>	α	necessary
<code>element2</code>	α	necessary

The method possesses two actions, which we describe below through examples. Before starting each of these actions, the method calls the method `.closure()` to make sure that the relations contained in `self.relations` define transitive down-closures.

Action 1	
Case	If <code>self.cartesian == 0</code>
Description	The function considers <code>element1</code> and <code>element2</code> to be elements belonging to the lists contained in <code>self.relations</code> . The method first looks for <code>element1</code> and <code>element2</code> . If one of them is not found, then <code>self.mask</code> is returned. If both elements are present, then the intersection of their down-closures is computed and the first local maxima of the intersection is returned. If the intersection is empty, then <code>self.mask</code> is returned instead.

The following example shows how the method compares element that belong or not to the pre-ordered set.

```
>>> Omega = PreOrder("preorder.yml")
>>> Omega.inf("?", "color_1")
True
>>> Omega.inf("color_2", "color_3")
color_4
```

The second action allows us to simulate any Cartesian product structure on the pre-ordered set.

Action 2	
Case	If <code>self.cartesian > 0</code>
Description	The function considers <code>element1</code> and <code>element2</code> to be lists of elements belonging to the lists contained in <code>self.relations</code> . For every element <code>i</code> in <code>range(self.cartesian)</code> , the method applies Action 1 to the pair <code>(element1[i], element2[i])</code> . The output is then appended to a list, which is eventually returned.

This second example shows how the method `.geq` extends to the Cartesian product of the pre-order used in the previous example.

```
>>> Cartesian = PreOrder("", 2, Omega)
>>> Cartesian.inf(["color_1", "color_2"], ["?"]*2)
[True, True]
>>> Cartesian.inf(["color_1", "color_2"], ["color_4", "color_3"])
['color_4', 'color_4']
```

5.1.7. Description of .presence (method). The code of the function `.presence` is equipped with the following inputs.

<code>.inf</code>		
Inputs	Types	Specifications
<code>element</code>	α	necessary

The method possesses one action, which we describe below through an example.

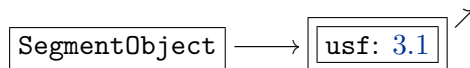
Action 1	
Case	Always
Description	The function looks for <code>element</code> at the beginning of every list contained in <code>self.relations</code> . If <code>element</code> is found, then <code>True</code> is returned, otherwise the method returns <code>False</code> .

The following examples shows how the method `.presence` reacts when given various inputs.

```
>>> Omega = PreOrder("preorder.yml")
>>> Omega.presence("?")
False
>>> Omega.presence("color_2")
True
```

5.2. Description of *SegmentObject* (subclass)

5.2.1. Introduction. The code of *SegmentObject* uses external modules.



The (sub)class *SegmentObject* models the features of segments (as defined in [2, section 2]). A *SegmentObject* item can be seen as a tape equipped with a read head, whose position is stored in the object `.parse` and is displayed through the method `.display` as a red patch of nodes.

(oooo|oooooooo|ooo|o|oooooooo)

The method `.patch` returns the index of a patch (an area in brackets) containing a node whose position is given as an input. Note that the method `.patch` starts searching the index associated with the node from where the read head is and only goes in a direction (left or right) specified in the second argument; the method `.merge` takes a tiling of the segment and merges groups of patches that share the same tiles. The tiling patterns are specified in a list of triples, where each triple contains a start index, a tile length, and an end index; the method `.remove` removes either a node or a patch (from the segment) at the index given in the first argument: whether a node or a patch is removed is decided upon whether the second argument is equal to "nodes-given" or is not specified.

5.2.2. Structure. The following tables give a preview of the class *SegmentObject*. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<i>CategoryItem</i>	section 3
Statistics	
▷ Importable objects: 4	
▷ Non-importable objects: 0	
▷ Importable methods: 5	
▷ Non-importable methods: 1	

The following table gives a description of the 4 importable objects of the class. The type α shown in the table is a polymorphic type.

Objects		
Name	Type	Related sections
<code>.domain</code>	<code>int</code>	▷ section 5.2.3
<code>.topology</code>	<code>list(int * int)</code>	▷ section 5.2.3
<code>.colors</code>	<code>list(list(α))</code>	▷ section 5.2.3
<code>.parse</code>	<code>int</code>	▷ section 5.2.3

Finally, the following table gives a description of the 5 importable methods of the class:

Methods			
Name	Input types	Output types	Related sections
<code>__init__</code>	- <code>int</code> - <code>list(int * int)</code> - <code>list(list(α))</code>	- <code>self</code>	▷ section 5.2.3
<code>.patch</code>	- <code>int</code> - <code>string</code>	- <code>int</code>	▷ section 5.2.5
<code>.display</code>	- <code>self</code>	- <code>self</code> - <code>stdout</code>	▷ section 5.2.4
<code>.merge</code>	- <code>list(int * int * int)</code> - <code>fun: $\alpha * \alpha \rightarrow \alpha$</code>	- <code>SegmentObject(α)</code>	▷ section 5.2.6
<code>.remove</code>	- <code>list</code> - <code>string</code>	- <code>SegmentObject</code>	▷ section 5.2.7

5.2.3. Description of `__init__` (method). The code of the function `__init__` is equipped with the following inputs.

<code>__init__</code>		
Inputs	Types	Specifications
<code>domain</code>	<code>int</code>	necessary
<code>topology</code>	<code>list(int * int)</code>	necessary
<code>colors</code>	<code>list(α)</code>	necessary

The method possesses one action, which we describe below through examples.

Action 1	
Case	Always
Description	The method uses the inputs <code>domain</code> , <code>topology</code> and <code>colors</code> to initialize the objects <code>self.domain</code> , <code>self.topology</code> and <code>self.colors</code> . The object <code>self.parse</code> is given the value 0.

The following examples show how the constructor can be used.

```

>>> t = list()
>>> c = list()
>>> for i in range(5):
...     t = t + [(3*i,3*i+2)]
...     c = c + ["color_1"]
...
>>> s = SegmentObject(20,t,c)
>>> s.domain
20
>>> s.topology
[(0, 2), (3, 5), (6, 8), (9, 11), (12, 14)]
>>> s.colors
['color_1', 'color_1', 'color_1', 'color_1', 'color_1']
>>> s.parse
0

```

5.2.4. Description of .display (method). The code of the function `.display` has no input. The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	Displays on the standard output a segment equipped with a red 'read head' positioned at the patch whose index is <code>self.parse</code> . The intervals defined by the pairs contained in <code>self.topology</code> will be displayed in bold while the nodes that are not comprised within pairs in <code>self.topology</code> will be displayed using a normal font.

Below, we show the action of the method `.display` on the segment `s` defined at the end of section 5.2.3.

```

>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s.parse = 3
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)

```

5.2.5. Description of .patch (method). The code of the function `.patch` is equipped with the following inputs, among which one is optional.

.patch		
Inputs	Types	Specifications
<code>position</code>	<code>int</code>	necessary
<code>search = ">1"</code>	<code>string</code>	optional

The method possesses two actions, which we describe below through examples.

Action 1	
Case	If <code>search == ">"+str(n)</code> where <code>n</code> is of the type <code>int</code>
Description	In this case, the list <code>self.topology</code> is parsed from the pair at index <code>self.parse</code> to the pair at index <code>len(self.topology)-1</code> with an incrementation equal to <code>n</code> . If the value <code>position</code> is within the range of one of the parsed pairs in <code>self.topology</code> , then the index of the pair (within <code>self.topology</code>) is returned. In any other case, the value -1 is returned.

Action 2	
Case	If <code>search == "<" + str(n)</code> where <code>n</code> is of the type <code>int</code>
Description	In this case, the list <code>self.topology</code> is parsed from the pair at index <code>self.parse</code> to the pair at index 0 with an incrementation equal to <code>n</code> . If the value <code>position</code> is within the range of one of the parsed pairs in <code>self.topology</code> , then the index of the pair (within <code>self.topology</code>) is returned. In any other case, the value -1 is returned.

Below, we show the action of the method `.patch` on the segment `s` considered at the end of section 5.2.4.

```
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s.patch(5,"<1")
1
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s.patch(7,">1")
2
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s.patch(11,">2")
-1
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s.patch(4,"<2")
-1
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
```

5.2.6. Description of `.merge` (method). The code of the function `.merge` is equipped with the following inputs.

<code>.merge</code>		
Inputs	Types	Specifications
<code>folding_format</code>	<code>list(int * int * int)</code>	necessary
<code>infimum</code>	<code>fun: $\alpha * \alpha \rightarrow \alpha$</code>	necessary

The method possesses one action.

Action	
Case	Always
Description	The method returns a new segment that is a merged version of <code>self</code> . Specifically, the patches of <code>self</code> are merged according to the tiling patterns specified in <code>folding_format</code> , in which a tiling pattern is given by a pair (x, l, y) . For every pair (x, l, y) in <code>folding_format</code> , the method will transitively merge any subset of patches whose indices in <code>self.topology</code> belong to a same interval of the form $(x + k * l, \min(x + (k + 1) * l, y))$. For each merging, the resulting pair that is included in the new topology consists of the first component of the first merged pair and the last component of the last merged pair. The color associated with that pair is the output of the recursive two-by-two application $f(_, \dots(_, f(_, f(_, _)))$ of the function $f := \text{infimum}$.

The following examples will probably make the previous description more explicit. We use the `preOrder` constructed from the file `preorder.yml` shown in section 5.1.3.1.

```
>>> Omega = PreOrder("preorder.yml")
>>> t = list()
>>> c = list()
>>> for i in range(5):
...     t = t + [(3*i,3*i+2)]
...     if i % 2 == 0:
...         c = c + ["color_2"]
...     if i % 2 == 1:
...         c = c + ["color_3"]
...
>>> s = SegmentObject(20,t,c)
>>> s.display()
(ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.merge([(0,1,3)],Omega.inf)
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1.colors
['color_2', 'color_3', 'color_2', 'color_3', 'color_2']
>>> s1 = s.merge([(0,2,3)],Omega.inf)
>>> s1.display()
(oooooo|oooooo|ooo|ooooo)
>>> s1.colors
['color_4', 'color_4', 'color_2']
>>> s1 = s.merge([(0,3,3)],Omega.inf)
>>> s1.display()
(ooooooooo|ooo|ooo|ooooo)
>>> s1.colors
['color_4', 'color_3', 'color_2']
>>> s1 = s.merge([(0,4,3)],Omega.inf)
>>> s1.display()
(ooooooooooooo|ooo|ooooo)
>>> s1.colors
['color_4', 'color_2']
>>> s1 = s.merge([(0,5,3)],Omega.inf)
>>> s1.display()
(ooooooooooooo|ooo|ooooo)
>>> s1.colors
['color_4', 'color_2']
```

5.2.7. Description of `.remove` (method). The code of the function `...init...` is equipped with the following inputs, among which one is optional.

...init...		
Inputs	Types	Specifications
<code>a_list</code>	<code>list</code>	necessary
<code>option = "patches-given"</code>	<code>string</code>	optional

The method possesses two actions, which we describe below through examples.

Action 1	
Case	If <code>option != "nodes-given"</code>
Description	In this case, the method removes every patch whose index is an integer <code>i</code> in <code>a.list</code> .

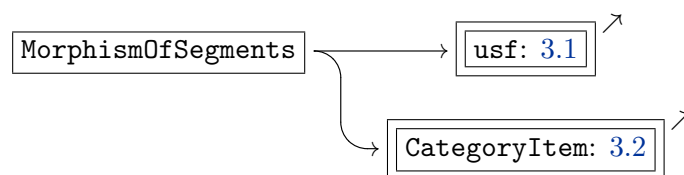
Action 2	
Case	If <code>option == "nodes-given"</code>
Description	In this case, the method removes every patch whose index is an integer <code>i</code> for which there exists an index <code>j</code> in <code>a.list</code> such that <code>i = self.patch(j)</code> .

The following examples shows the effects of choosing either two options. We use the segment `s` defined at the end of section 5.2.6.

```
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([0], "patches-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([0], "nodes-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([1], "patches-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([1], "nodes-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([2], "patches-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([2], "nodes-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([0,1,2,6], "patches-given")
>>> s1.display()
(ooooo|ooo|ooo|ooo|ooo|ooooo)
>>> s1 = s.remove([0,1,2,6], "nodes-given")
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
```

5.3. Description of `MorphismOfSegments` (subclass)

5.3.1. Introduction. The code of `MorphismOfSegments` uses external modules.



The (sub)class `MorphismOfSegments` models the features of a morphism of segments (as defined in [2]). As such, its constructor takes a source segment, a target segment, a list `f1` of increasing integers describing the image of an injection $i \rightarrow f1[i]$ relating the domain of

the source segment to the domain of the target segment, and a pre-order relation compatible with the pre-ordered sets of two the segments.

5.3.2. Structure. The following tables give a preview of the class *MorphismOfSegments*. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
CategoryItem	section 3
Statistics	
▷ Importable objects: 3 ▷ Non-importable objects: 0 ▷ Importable methods: 1 ▷ Non-importable methods: 1	

The following table gives a description of the 3 importable objects of the class:

Objects		
Name	Type	Related sections
.defined	bool	▷ section 5.3.3
.f0	list(int)	▷ section 5.3.3
.f1	list(int) or string	▷ section 5.3.3

Finally, the following table gives a description of the only importable method of the class. The type α shown in the table is a polymorphic type.

Methods			
Name	Input types	Output types	Related sections
__init__	- SegmentObject(α) - SegmentObject(α) - list(int) - fun $\alpha * \alpha \rightarrow \text{bool}$	- self	▷ section 5.3.3

5.3.3. Description of __init__ (method). The code of the function __init__ is equipped with the following inputs.

__init__		
Inputs	Types	Specifications
source	SegmentObject(α)	necessary
target	SegmentObject(α)	necessary
f1	list(int) or string	necessary
geq	fun $\alpha * \alpha \rightarrow \text{bool}$	necessary

The method possesses two actions, which we describe below through examples.

Action 1	
Case	If f1 == 'id'
Description	In this case, the super objects <code>self.source</code> and <code>self.target</code> are given the value of the variables <code>source</code> and <code>target</code> , respectively. The object <code>self.f1</code> is given the value 'id' and the object <code>self.f0</code> will given the list containing the integers <code>self.target.patch(i)</code> , for every index <code>i</code> in <code>range(self.source.topology)</code> if <code>self.f0</code> defines a function. In this case, the object <code>self.defined</code> is set to True.

Action 2	
Case	If <code>f1 != 'id'</code> and <code>f1</code> is of the type <code>list(int)</code>
Description	In this case, the super objects <code>self.source</code> and <code>self.target</code> are given the value of the variables <code>source</code> and <code>target</code> , respectively. The object <code>self.f1</code> is given the list stored in <code>f1</code> and the object <code>self.f0</code> will be given the list containing the integers <code>self.target.patch(f1[i])</code> , for every index <code>i</code> in <code>range(self.source.topology)</code> if <code>self.f0</code> defines a function. In this case, the object <code>self.defined</code> is set to <code>True</code> .

The following examples illustrate the previous actions. We consider the lists `t` and `c` defined in section 5.2.6.

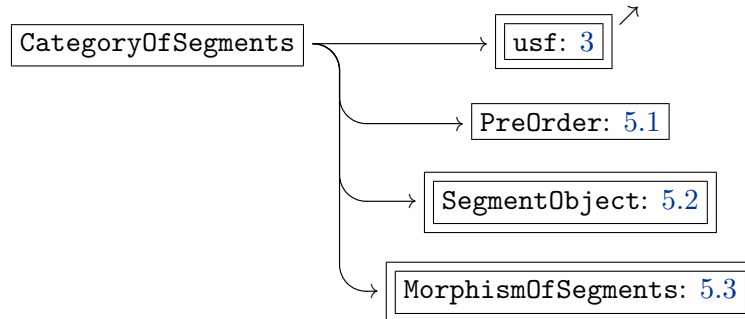
```
>>> s = SegmentObject(20,t,c)
>>> t = SegmentObject(25,t,c)
>>> s.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> t.display()
(ooo|ooo|ooo|ooo|ooo|ooooooooo)
>>> s1 = s.remove([2])
>>> s1.display()
(ooo|ooo|ooo|ooo|ooo|ooooo)
>>> t1 = t.remove([0,2,3])
>>> t1.display()
(ooo|ooo|ooooo|ooo|ooooo)
>>> f1 = range(20)
>>> m = MorphismOfSegments(s1,t1,f1,Omega.geq)
>>> m.defined
True
>>> m.f0
[-1, 0, -1, 1]
>>> m.f1
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]
```

Not all morphisms are well-defined. For instance, the following examples contained non-well-defined morphisms of segments. We obtained these examples by changing the topology or the mapping `f1`. As can be seen the resulting object `self.f0` is an empty list.

```
>>> s2 = s1.merge([(1,2,2)],Omega.inf)
>>> s2.display()
(ooo|ooooooooo|ooo|ooooo)
>>> m = MorphismOfSegments(s2,t1,f1,Omega.geq)
>>> m.defined
False
>>> m.f0
[]
>>> f1 = f1[1:]+[20]
>>> f1
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]
>>> m = MorphismOfSegments(s1,t1,f1,Omega.geq)
>>> m.defined
False
>>> m.f0
[]
```

5.4. Description of *CategoryOfSegments* (class)

5.4.1. Introduction. The code of *CategoryOfSegments* uses other functions of the module *SegmentCategory.py* as well as external modules.



The class *CategoryOfSegments* models the features of a category of segments. The class is initialized by passing a *PreOrder* item to its constructor and allows one to know or compute information related to a category structure. The method *.identity* returns a Boolean value that specifies whether there may exist an identity morphism between two *SegmentObject* items (these may be saved in different places in the memory); the method *.initial* returns an local initial object in the category, where the local aspect is determined by the colors of the segment; the method *.homset* computes the hom-set of a pair of *SegmentObject* items.

5.4.2. Structure. The following tables give a preview of the class *CategoryOfSegments*. The table given below describe the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<i>object</i>	N/A
Statistics	
▷ Importable objects: 1	
▷ Non-importable objects: 0	
▷ Importable methods: 4	
▷ Non-importable methods: 0	

The following table gives a description of the only importable object of the class:

Objects		
Name	Type	Related sections
<i>.preorder</i>	<i>PreOrder</i>	▷ section 5.4.3

Finally, the following table gives a description of the 4 importable methods of the class. The type α shown in the table is a polymorphic type.

Methods			
Name	Input types	Output types	Related sections
<i>__init__</i>	- <i>PreOrder</i>	- <i>object</i> - <i>SegmentObject</i>	▷ section 5.4.3
<i>.identity</i>	- <i>SegmentObject</i> - <i>SegmentObject</i>	- <i>bool</i>	▷ section 5.4.4
<i>.initial</i>	- <i>int</i> - <i>list(list(α))</i>	- <i>SegmentObject</i>	▷ section 5.4.5
<i>.homset</i>	- <i>SegmentObject</i> - <i>SegmentObject</i>	- <i>list(MorphismOfSegments)</i>	▷ section 5.4.6

5.4.3. Description of `__init__` (method). The code of the function `__init__` is equipped with the following inputs.

<code>__init__</code>		
Inputs	Types	Specifications
<code>preorder</code>	<code>PreOrder</code>	necessary

The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	The method attributes the value of the variable <code>preorder</code> to the object <code>self.preorder</code>

The following example shows how the constructor of the class `CategoryOfSegments` can be used with the class `PreOrder` (see section 5.1). We use the file `preorder.yml` displayed in section 5.1.3.1.

```
>>> Omega = PreOrder("preorder.yml")
>>> Seg = CategoryOfSegments(Omega)
>>> Seg.preorder.relations
[['color_1', 'color_2', 'color_3'], ['color_2', 'color_4'], ['color_3',
'color_4'], ['color_4']]
>>> Seg.preorder.mask
True
```

5.4.4. Description of `identity` (method). The code of the function `identity` is equipped with the following inputs.

<code>identity</code>		
Inputs	Types	Specifications
<code>segment1</code>	<code>SegmentObject</code>	necessary
<code>segment2</code>	<code>SegmentObject</code>	necessary

The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	The method tests whether each type of objects <code>.domain</code> , <code>topology</code> and <code>colors</code> are equal for both inputs <code>segment1</code> and <code>segment2</code> .

In the following example, we use the file `preorder.yml` displayed in section 5.1.3.1 and the method `.initial` detailed in section 5.4.5.

```
>>> Omega = PreOrder("preorder.yml")
>>> Seg = CategoryOfSegments(Omega)
>>> s = Seg.initial(20,"color_1")
>>> s.display()
(●○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○)
>>> t = Seg.initial(20,"color_1")
(●○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○)
>>> t.display()
>>> Seg.identity(s,t)
True
```

If we now use the method `.remove`, detailed in section 5.2.7, to change the colors of `t`, then the identity does not hold anymore, as shown in the following example.

5.4.5, the method `.merge` detailed in section 5.2.6 and the constructor of the class `PreOrder` (see section 5.1).

```
>>> Omega = PreOrder("preorder.yml")
>>> Seg = CategoryOfSegments(Omega)
>>> s = Seg.initial(8,"color_1")
>>> s = s.merge([(0,2,7)],Omega.inf)
>>> t = Seg.initial(12,"color_1")
>>> t = t.merge([(0,3,10)],Omega.inf)
>>> s.display()
(oo|oo|oo|oo)
>>> t.display()
(ooo|ooo|ooo|oo|o)
>>> for m in Seg.homset(s,t):
...     print m.f1
...
[0, 1, 3, 4, 6, 7, 9, 10]
[0, 1, 3, 4, 6, 8, 9, 10]
[0, 1, 3, 4, 7, 8, 9, 10]
[0, 1, 3, 5, 6, 7, 9, 10]
[0, 1, 3, 5, 6, 8, 9, 10]
[0, 1, 3, 5, 7, 8, 9, 10]
[0, 1, 4, 5, 6, 7, 9, 10]
[0, 1, 4, 5, 6, 8, 9, 10]
[0, 1, 4, 5, 7, 8, 9, 10]
[0, 2, 3, 4, 6, 7, 9, 10]
[0, 2, 3, 4, 6, 8, 9, 10]
[0, 2, 3, 4, 7, 8, 9, 10]
[0, 2, 3, 5, 6, 7, 9, 10]
[0, 2, 3, 5, 6, 8, 9, 10]
[0, 2, 3, 5, 7, 8, 9, 10]
[0, 2, 4, 5, 6, 7, 9, 10]
[0, 2, 4, 5, 6, 8, 9, 10]
[0, 2, 4, 5, 7, 8, 9, 10]
[1, 2, 3, 4, 6, 7, 9, 10]
[1, 2, 3, 4, 6, 8, 9, 10]
[1, 2, 3, 4, 7, 8, 9, 10]
[1, 2, 3, 5, 6, 7, 9, 10]
[1, 2, 3, 5, 6, 8, 9, 10]
[1, 2, 3, 5, 7, 8, 9, 10]
[1, 2, 4, 5, 6, 7, 9, 10]
[1, 2, 4, 5, 6, 8, 9, 10]
[1, 2, 4, 5, 7, 8, 9, 10]
```

Presentation of the module

AlignedFunctor.py

6.1. Description of PointedSet (class)

6.1.1. Introduction. The class `PointedSet` models the features of a pointed set. It is made of a list `.symbols`, representing the underlying set, and an integer `.index` indicating where the point is located within `.symbols`.

6.1.2. Structure. The following tables give a preview of the class `PreOrder`. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<code>object</code>	N/A
Statistics	
▷ Importable objects: 2 ▷ Non-importable objects: 0 ▷ Importable methods: 2 ▷ Non-importable methods: 0	

The following table gives a description of the 2 importable objects of the class. The type α shown in the table is a polymorphic type.

Objects		
Name	Type	Related sections
<code>.symbols</code>	<code>list(α)</code>	▷ section 6.1.3
<code>.index</code>	<code>int</code>	▷ section 6.1.3

Finally, the following table gives a description of the 2 importable methods of the class:

Methods			
Name	Input types	Output types	Related sections
<code>.__init__</code>	- <code>list(α)</code> - <code>int</code>	- <code>self</code>	▷ section 6.1.3
<code>.point</code>	- <code>self</code>	- <code>α</code>	▷ section 6.1.3

6.1.3. Description of `.__init__` (method). The code of the function `.__init__` is equipped with the following inputs.

...init...		
Inputs	Types	Specifications
symbols	string	necessary
index	int	necessary

The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	The method attributes the value stored in the input variable symbols to the object self.symbols . If the index is in the range of the list self.symbols , then the value of the input variable index is passed to the object self.index . Otherwise, the object self.index is set to 0.

In the following example, we see how the object **.index** can be used with the object **.symbols** to return the 'point' of the **PointedSet** item.

```
>>> E = PointedSet(["-", "A", "C", "G", "T"], 0)
>>> E.symbols
['-', 'A', 'C', 'G', 'T']
>>> E.index
0
>>> E.symbols[E.index]
-
```

6.1.4. Description of .point (method). The code of the function **.point** has no input. The method possesses one action, which we describe below through an example.

Action	
Case	Always
Description	The method returns the point self.symbols[self.index] of the structure.

The following example illustrates the use of the method **.point**.

```
>>> E = PointedSet(["A", "C", "G", "T", "epsilon"], 4)
>>> E.point()
epsilon
```

6.2. Description of Sequence (class)

6.2.1. Introduction. The class **Sequence** models the features of a sequence in a set of multiple sequence alignments meant to be displayed in a fasta file. Its objects **.name** and **.color** are used to form the fasta reference associated with a sequence of characters contained in its object third object **.seq**.

6.2.2. Structure. The following tables give a preview of the class **PreOrder**. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
object	N/A
Statistics	
▷ Importable objects: 3	
▷ Non-importable objects: 0	
▷ Importable methods: 1	
▷ Non-importable methods: 0	

The following table gives a description of the 3 importable objects of the class:

Objects		
Name	Type	Related sections
<code>.name</code>	<code>string</code>	▷ section 6.2.3
<code>.seq</code>	<code>list(string)</code>	▷ section 6.2.3
<code>.color</code>	<code>string</code>	▷ section 6.2.3

Finally, the following table gives a description of the only importable method of the class:

Methods			
Name	Input types	Output types	Related sections
<code>.__init__</code>	- <code>name</code> - <code>seq</code> - <code>color</code>	- <code>self</code>	▷ section 6.2.3

6.2.3. Description of `.__init__` (method). The code of the function `.__init__` is equipped with the following inputs.

<code>.__init__</code>		
Inputs	Types	Specifications
<code>name</code>	<code>string</code>	necessary
<code>sequence</code>	<code>list(int)</code>	necessary
<code>color</code>	<code>string</code>	necessary

The method possesses one action, which we describe below through an example.

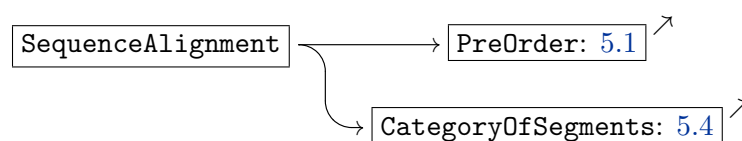
Action	
Case	Always
Description	The method attributes the values of the input variables <code>name</code> , <code>sequence</code> and <code>color</code> to the objects <code>self.name</code> , <code>self.seq</code> , and <code>self.color</code> , respectively.

The following example shows a canonical use of the constructor of the class `Sequence`.

```
>>> sam = Sequence("Samuel", ["A", "C", "T", "T", "C", "g", "g", "a", "T"], "blue")
>>> sam.name
Samuel
>>> sam.seq
['A', 'C', 'T', 'T', 'C', 'g', 'g', 'a', 'T']
>>> sam.color
blue
```

6.3. Description of *SequenceAlignment* (class)

6.3.1. Introduction. The code of `SequenceAlignment` uses external modules.



The class `SequenceAlignment` models the features of a sequence alignment functor, as defined in [2]. The images of the sequence alignment functor are stored in the object `.database` and can be queried through the method `.eval`. The method also computes the images of the right Kan extension of this functor through the method `.ran` (TO BE CODED). The extending

diagram (see [2]) used to compute this right Kan extension can be computed through the method `.extending_diagram`.

6.3.2. Structure. The following tables give a preview of the class `SequenceAlignment`. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<code>object</code>	N/A
Statistics	
▷ Importable objects: 5 ▷ Non-importable objects: 0 ▷ Importable methods: 4 ▷ Non-importable methods: 0	

The following table gives a description of the 5 importable objects of the class. The type α shown in the table is a polymorphic type.

Objects		
Name	Type	Related sections
<code>.env</code>	<code>Environment</code>	▷ section 6.3.3
<code>.Seg</code>	<code>list</code>	▷ section 6.3.3
<code>.indiv</code>	<code>list</code>	▷ section 6.3.3
<code>.base</code>	<code>list(SegmentObject)</code>	▷ section 6.3.3
<code>.database</code>	<code>list(α)</code>	▷ section 6.3.3

Finally, the following table gives a description of the 4 importable methods of the class:

Methods			
Name	Input types	Output types	Related sections
<code>.__init__</code>	- <code>Environment</code> - <code>CategoryOfSegments</code> - <code>list(SegmentObject)</code> - <code>list(α)</code>	- <code>self</code>	▷ section 6.3.3
<code>.eval</code>	- <code>SegmentObject</code>	- <code>α</code>	▷ section 6.3.4
<code>.extending_diagram</code>	- <code>SegmentObject</code>	- <code>list(int* MorphismOfSegments)</code>	▷ section 6.3.5
<code>.ran</code>	- <code>CategoryItem</code>	- <code>α</code>	▷ section 6.3.6

6.3.3. Description of `.__init__` (method). The code of the function `.__init__` is equipped with the following inputs.

.__init__		
Inputs	Types	Specifications
<code>env</code>	<code>Environment</code>	necessary
<code>indiv</code>	<code>list</code>	necessary
<code>base</code>	<code>list(SegmentObject)</code>	necessary
<code>database</code>	<code>list(α)</code>	necessary

The method possesses one action, which we describe below through an example.

Action	
Case	Always
Description	The method attributes the values stored in the input variables <code>env</code> , <code>indiv</code> , <code>base</code> and <code>database</code> to the objects <code>self.env</code> , <code>self.indiv</code> , <code>self.base</code> and <code>self.database</code> . It also uses the object <code>self.env.spec</code> to compute the <code>(self.env.spec)</code> -fold Cartesian product of the pre-ordered set encoded by <code>self.env.Seg.preorder</code> and attributes to the object <code>self.Seg</code> the <code>CategoryOfSegments</code> item that can be constructed from it.

The following examples show how to use the constructor of the class in combination with the classes `PreOrder` (see section 5.1), `CategoryOfSegments` (see section 5.4), `PointedSet` (see section 6.1) and `Environment` (see section 6.4). We use the file `preorder.yml` displayed in section 5.1.3.1.

```
>>> Omega = PreOrder("preorder.yml")
>>> Seg = CategoryOfSegments(Omega)
>>> E = PointedSet(["-", "A", "C", "G", "T"], 0)
>>> Env = Environment(Seg, E, 2, ["color_2"]*2)
>>> base = [Env.Seg.initial(8, ["color_1"]*2), Env.Seg.initial(8, ["color_2"]*2),
Env.Seg.initial(5, ["color_2"]*2), Env.Seg.initial(7, ["color_3"]*2)]
```

In partrice, the following list should contain actual sequence alignments, namely each `string` item should be replaced with a succession of `string` items representing the sequences of a sequence alignment.

```
>>> database = [[["Alignment:1:1"], ["Alignment:1:2"]], [["Alignment:2:1"],
["Alignment:2:2"]], [["Alignment:3:1"], ["Alignment:3:2"],
["Alignment:1:3"]], [["Alignment:4:1"], ["Alignment:4:2"]]]
>>> Seqali = SequenceAlignment(Env, ["A", "B", "C", "D"], base, database)
```

The following code shows how to recover the data passed to the constructor using the various objects of the class.

```
>>> Seqali.indiv
['A', 'B', 'C', 'D']
>>> for i in range(len(Seqali.base)):
...     print str(i)+" color:  "+ str(Seqali.base[i].colors[
Seqali.base[i].parse ])
...     Seqali.base[i].display()
...     for j in range(len(Seqali.database[i])):
...         for k in range(len(Seqali.database[i][j])):
...             print Seqali.database[i][j][k]
...         print ""
...
0) color:  ['color_1', 'color_1']
(●|○|○|○|○|○|○|○|○)
Alignment:1:1

Alignment:1:2

1) color:  ['color_2', 'color_2']
(●|○|○|○|○|○|○|○|○)
Alignment:2:1

Alignment:2:2
```

```
2) color: ['color_2', 'color_2']
(o|o|o|o|o)
Alignment:3:1
```

```
Alignment:3:2
```

```
Alignment:1:3
```

```
3) color: ['color_3', 'color_3']
(o|o|o|o|o|o|o)
Alignment:4:1
```

```
Alignment:4:2
```

6.3.4. Description of .eval (method). The code of the function `.eval` is equipped with the following input.

.eval		
Inputs	Types	Specifications
segment	SegmentObject	necessary

The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	The method returns the image of the sequence alignment functor for the given input SegmentObject item. If the input does not belong to the list stored in self .base, then the empty list is returned.

The following example shows that the images of the SegmentObject items stored in self.base recovers the elements stored in the list self.database. We use the Environment item constructed in the example given in section 6.3.3.

```
>>> for i in range(len(Seqali.base)):
...     print "image for base[" + str(i) + "]"
...     Seqali.base[i].display()
...     sal = Seqali.eval(Seqali.base[i])
...     for j in range(len(sal)):
...         for k in range(len(sal[j])):
...             print sal[j][k]
...             print ""
...
image for base[0]
(o|o|o|o|o|o|o|o|o)
Alignment:1:1

Alignment:1:2
```

```
image for base[1]
(●|○|○|○|○|○|○|○)
Alignment:2:1
```

```
Alignment:2:2
```

```
image for base[2]
(●|○|○|○|○)
Alignment:3:1
```

```
Alignment:3:2
```

```
Alignment:1:3
```

```
image for base[3]
(●|○|○|○|○|○|○|○)
Alignment:4:1
```

```
Alignment:4:2
```

The following example shows that the images of the `SegmentObject` items stored in `self.base` recovers the elements of the list stored in `self.database`.

```
>>> s = Env.Seg.initial(7, ["color_1"]*2)
>>> s.display()
(●|○|○|○|○|○|○)
>>> sal = Seqali.eval(s)
>>> for j in range(len(sal)):
...     for k in range(len(sal[j])):
...         print sal[j][k]
...         print ""
...
>>>
```

6.3.5. Description of `.extending_diagram` (method). The code of the function `.extending_diagram` is equipped with the following input.

<code>.extending_diagram</code>		
Inputs	Types	Specifications
<code>segment</code>	<code>SegmentObject</code>	necessary

The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	This method computes the set of objects of the <i>extending category</i> defined in [2, Definition 3.25]. More specifically, the method returns the list of pairs (i, m) such that m is a <code>MorphismOfSegments</code> item whose source object is <code>segment</code> and whose target object is <code>self.base[i]</code> .

The following example shows an example of calculation of an extending category.

```

>>> diag = Seqali.extending_diagram(Env.Seg.initial(7,["color_1"]*2))
>>> for i,m in diag:
...     print i
...     print m.f1
...     print m.f0
...     m.source.display()
...     m.target.display()
...     for j in range(len(Seqali.database[i])):
...         for k in range(len(Seqali.database[i][j])):
...             print Seqali.database[i][j][k]
...             print ""
...
0
[0, 1, 2, 3, 4, 5, 6]
[0, 1, 2, 3, 4, 5, 6]
(o|o|o|o|o|o|o)
(o|o|o|o|o|o|o)
Alignment:1:1

Alignment:1:2

0
[0, 1, 2, 3, 4, 5, 7]
[0, 1, 2, 3, 4, 5, 7]
(o|o|o|o|o|o|o)
(o|o|o|o|o|o|o)
Alignment:1:1

Alignment:1:2

0
[0, 1, 2, 3, 4, 6, 7]
[0, 1, 2, 3, 4, 6, 7]
(o|o|o|o|o|o|o)
(o|o|o|o|o|o|o)
Alignment:1:1

Alignment:1:2

0
[0, 1, 2, 3, 5, 6, 7]
[0, 1, 2, 3, 5, 6, 7]
(o|o|o|o|o|o|o)
(o|o|o|o|o|o|o)
Alignment:1:1

Alignment:1:2

```

```

0
[0, 1, 2, 4, 5, 6, 7]
[0, 1, 2, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:1:1

```

```

Alignment:1:2

```

```

0
[0, 1, 3, 4, 5, 6, 7]
[0, 1, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:1:1

```

```

Alignment:1:2

```

```

0
[0, 2, 3, 4, 5, 6, 7]
[0, 2, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:1:1

```

```

Alignment:1:2

```

```

0
[1, 2, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:1:1

```

```

Alignment:1:2

```

```

1
[0, 1, 2, 3, 4, 5, 6]
[0, 1, 2, 3, 4, 5, 6]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 1, 2, 3, 4, 5, 7]
[0, 1, 2, 3, 4, 5, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 1, 2, 3, 4, 6, 7]
[0, 1, 2, 3, 4, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 1, 2, 3, 5, 6, 7]
[0, 1, 2, 3, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 1, 2, 4, 5, 6, 7]
[0, 1, 2, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 1, 3, 4, 5, 6, 7]
[0, 1, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[0, 2, 3, 4, 5, 6, 7]
[0, 2, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

1
[1, 2, 3, 4, 5, 6, 7]
[1, 2, 3, 4, 5, 6, 7]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:2:1

```

```

Alignment:2:2

```

```

3
id
[0, 1, 2, 3, 4, 5, 6]
(○|○|○|○|○|○|○)
(○|○|○|○|○|○|○)
Alignment:4:1

```

```

Alignment:4:2

```

6.3.6. Description of .ran (method). The code of the function `.ran` is equipped with the following input.

.ran		
Inputs	Types	Specifications
<code>cat_item</code>	<code>CategoryItem</code>	necessary
<code>B = ""</code>	<code>string</code>	optional

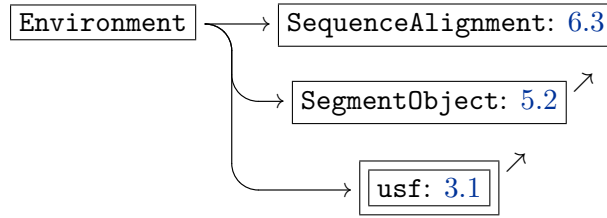
The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	This method computes the images of the right Kan extension of the functor encoded by the method <code>self.eval</code> .

TO BE CODED

6.4. Description of Environment (class)

6.4.1. Introduction. The code of `Environment` uses other functions of the module `AlignedFunctor.py` as well as external modules.



This structure models the features of an aligned environment functors, as defined in [2, section 3]. As in aligned environment functors, this structure is associated with a `PointedSet` item `.pset`. The class is also equipped with a fiber operation (pullback along a point in the image of the aligned environment functor)

$$\begin{array}{ccc}
 \{\tau\} & \xrightarrow{\subseteq} & \mathbf{Seg}(\Omega) \\
 \downarrow \text{!} & & \downarrow \mathbf{AE}_b^\varepsilon \\
 \mathbf{1} & \xrightarrow{\text{input}} & \mathbf{Set}
 \end{array}$$

and a sequence alignment functor constructor call. Specifically, the method `.segment` returns a segment that is the pullback of the aligned environment functor above any input list that represents an element in one of its images. If the input list contains a character that is not in the object `.pset.symbols`, then the node associated with that character is masked in the returned segment. Additionally, the method `.seqali` constructs a sequence alignment functor from a file of sequence alignments, as shown in the discussion of [2, Example 3.22].

6.4.2. Structure. The following tables give a preview of the class `Environment`. The table given below describes the various dependencies of the class.

Dependencies	
Superclass ancestry	Module section
<code>object</code>	N/A
Statistics	
▷ Importable objects: 4 ▷ Non-importable objects: 0 ▷ Importable methods: 3 ▷ Non-importable methods: 0	

The following table gives a description of the 4 importable objects of the class. The type α shown in the table is a polymorphic type.

Objects		
Name	Type	Related sections
<code>.Seg</code>	<code>CategoryOfSegments(α)</code>	▷ section 6.4.3
<code>.pset</code>	<code>PointedSet(α)</code>	▷ section 6.4.3
<code>.spec</code>	<code>int</code>	▷ section 6.4.3
<code>.b</code>	<code>list(α)</code>	▷ section 6.4.3

Finally, the following table gives a description of the 3 importable methods of the class. The type β shown in the table is a polymorphic type.

Methods			
Name	Input types	Output types	Related sections
<code>...int__</code>	- <code>Seg</code> - <code>pset</code> - <code>exponent</code> - <code>threshold</code>	- <code>self</code>	▷ section 6.4.3
<code>.segment</code>	- <code>list(β)</code> - α	- <code>self</code>	▷ section 6.4.4
<code>.seqali</code>	- <code>string</code>	- <code>SequenceAlignment</code>	▷ section 6.4.5

6.4.3. Description of `...int__` (method). The code of the function `...init__` is equipped with the following inputs.

<code>...init__</code>		
Inputs	Types	Specifications
<code>Seg</code>	<code>CategoryOfSegments(α)</code>	necessary
<code>pset</code>	<code>PointedSet(α)</code>	necessary
<code>exponent</code>	<code>int</code>	necessary
<code>threshold</code>	<code>list(α)</code>	necessary

The method possesses two actions, which we describe below through examples.

Action 1	
Case	If $\exists i$: <code>self.Seg.preorder.presence(self.b[i]) == False</code>
Description	In this case, the method replaces the element <code>self.b[i]</code> with the Boolean value <code>self.Seg.preorder.mask</code> . Also, if the length of the list <code>self.b</code> is not equal to <code>self.spec</code> , then <code>self.b</code> is completed into such a list by appending copies of the value <code>self.Seg.preorder.mask</code> to it.

Action 2	
Case	If $\forall i$: <code>self.Seg.preorder.presence(self.b[i]) == True</code>
Description	In this case, the method checks whether the length of the list <code>self.b</code> is equal to <code>self.spec</code> . If not, then <code>self.b</code> is completed into such a list by appending copies of the value <code>self.Seg.preorder.mask</code> to it.

The following examples show how to use the constructor of the class in combination with the classes `PreOrder` (see section 5.1), `CategoryOfSegments` (see section 5.4), and `PointedSet` (see section 6.1). We use the file `preorder.yml` displayed in section 5.1.3.1.

```
>>> Omega = PreOrder("preorder.yml")
>>> Seg = CategoryOfSegments(Omega)
>>> E = PointedSet(["-", "A", "C", "G", "T"], 0)
>>> Env = Environment(Seg, E, 4, ["color_2"] * 4)
>>> Env.Seg.preorder.relations
[['color_1', 'color_2', 'color_3'], ['color_2', 'color_4'], ['color_3',
'color_4'], ['color_4']]
>>> Env.pset.symbols
['-', 'A', 'C', 'G', 'T']
>>> Env.pset.point()
-
>>> Env.spec
4
>>> Env.b
['color_2', 'color_2', 'color_2', 'color_2']
```

6.4.4. Description of `.segment` (method). The code of the function `.segment` is equipped with the following inputs.

...init...		
Inputs	Types	Specifications
<code>a_list</code>	<code>list(β)</code>	necessary
<code>color</code>	α	necessary

The method possesses one action, which we describe below through an example.

Action	
Case	Always
Description	The method takes a list of symbols <code>a_list</code> whose elements are assumed to belong to the object <code>self.Env.pset.symbols</code> and returns the pull-back (a <code>SegmentObject</code> item) of the underlying environment functor defined by <code>self</code> above the element represented by <code>a_list</code> . Every element of <code>a_list</code> that is not a symbol in <code>self.Env.pset.symbols</code> is removed from the topology of the <code>SegmentObject</code> item.

The following example shows what the effect of the method `.segment` on a list of strings looks like. We use the `Environment` item defined at the end of section 6.4.3.

```
>>> r = Env.segment(["A", "T", "G", "ERROR", "-", "-", "G", "?", "?"], "color_1")
>>> r.display()
(○|○|○|○|○|○|○|○)
>>> r.colors
['color_1', 'color_1', 'color_1', 'color_1', 'color_1', 'color_1']
```

6.4.5. Description of `.seqali` (method). The code of the function `.seqali` is equipped with the following input.

.seqali		
Inputs	Types	Specifications
<code>name_of_file</code>	<code>string</code>	necessary








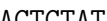

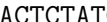




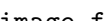
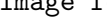




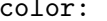


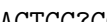

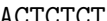


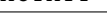
















































The method possesses one action, which we describe below through examples.

Action	
Case	Always
Description	The method constructs a sequence alignment functor (<i>i.e.</i> a <code>SequenceAlignment</code> item – see section 6.3) from a <code>fasta</code> file <code>name_of_file</code> of sequence alignments (as shown in [2, Example 3.22]) and in which every sequence label succeeding the usual key symbol <code>></code> takes the form <code>group_label:individual:color</code> , where <code>group_label</code> is a reference number gathering sequences under the same sequence alignment, where <code>individual</code> is the name of the individual from which the sequence comes, and <code>color</code> is the color that will be used to color the segment associated with the sequence.

Let us consider the following `fasta` file of sequence alignments whose sequence labels takes the form `group_label:individual:color`.

alignment.fa	
1	<code>>1:A:color_1</code>
2	<code>ACTCGATCTCTG?TCGATCGATCG</code>
3	<code>CCTATCGGATCGATC</code>
4	<code>>1:B:color_1</code>
5	<code>ACTCTAT?TCTATCC-ACTCATCA</code>
6	<code>CCTACTATCTCGAAA</code>
7	<code>>1:C:color_1</code>
8	<code>ACTCTATC----TCCGACT?ATCA</code>
9	<code>CCTACTATCTCGAAA</code>
10	<code>>1:D:color_1</code>
11	<code>ACTATTTA?TTTC----TTTTCTA</code>
12	<code>CCGGGCTGGGGGGGG</code>
13	<code>>2:A:color_1</code>
14	<code>ACTCGATCGGATC-AT??CGATCG</code>
15	<code>CCTATCGGATCGATCG</code>
16	<code>>2:B:color_1</code>
17	<code>ACTCC?CCTA--CTGGGCTGCTCA</code>
18	<code>CCTACTATCTCGAAAG</code>
19	<code>>2:C:color_1</code>
20	<code>ACTCTCTATC?CTCTATC---TCA</code>
21	<code>CCTACTATC?CGAAAG</code>
22	<code>>2:D:color_1</code>
23	<code>ACTATT-??T--AC--CTTTTCTA</code>
24	<code>CCGGGCTTTTCGAAAA</code>

The following example shows what the effect of the applying the method `.seqali` on the previous file returns. In particular, we display the outputs of the method `.eval` on the `SegmentObject` items contained in the object `.base`.

```
>>> Seqali = Env.seqali("alignment.fa")
>>> for i in range(len(Seqali.base)):
...     print "image for base[" + str(i) + "]"
...     Seqali.base[i].display()
...     print "color:  "+str(Seqali.base[i].colors[Seqali.base[i].parse])
...     sal = Seqali.eval(Seqali.base[i])
...     for j in range(len(sal)):
...         for k in range(len(sal[j])):
...             print sal[j][k]
...     print ""
...
image for base[0]
(||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||
```


Presentation of the module PartitionCategory.py

7.1. Description of `_image_of_partition`

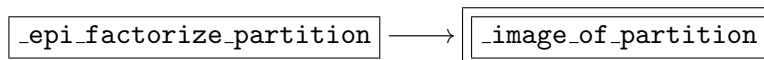
The function `_image_of_partition` takes a list of elements and returns the list of its elements without repetition in the order in which they can be accessed from the left to the right.

```
1 def _image_of_partition(partition):
2     """ the source code of this function can be found in iop.py """
3     return the_image
```

This corresponds to returning the image object of the underlying partition of the list.

```
>>> print(_image_of_partition([3,3,2,1,1,2,4,5,6,5,2,6]))
[3, 2, 1, 4, 5, 6]
>>> print(_image_of_partition(['A',4,'C','C','G',4,0,0,1,'a','A']))
['A', 4, 'C', 'G', 0, 1, 'a']
```

7.2. Description of `_epi_factorize_partition`



The function `_epi_factorize_partition` relabels the elements of a list with non-negative integers. It starts with the integer 0 and attributes a new label by increasing the previously attributed label by 1. The first element of the list always receives the label 0 and the highest integer used in the relabeling equals the length of the image (section 7.1) of the list decreased by 1.

```
1 def _epi_factorize_partition(partition):
2     """ the source code of this function can be found in efp.py """
3     return epimorphism
```

Even though a list already encodes an epimorphism, the goal of the function

`_epi_factorize_partition`

is to return a canonical *choice* of epimorphism.

$$S \xrightarrow[e(f)]{\quad} \text{Im}(f) \xrightarrow[\cong]{\quad} K$$

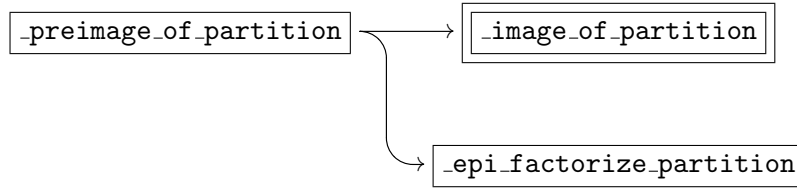
f

This choice ensures that two partitions characterized by the same set of universal properties are *equal* as python lists. In [4, Appendix A], this type of construction is formulated in terms of a factorization system in the category of sets (see the diagram above).

```
>>> p = [3,3,2,1,1,2,4,5,6,5,2,6]
>>> print(_epi_factorize_partition(p))
[0, 0, 1, 2, 2, 1, 3, 4, 5, 4, 1, 5]
>>> im = _image_of_partition(p)
>>> print(_epi_factorize_partition(im))
[0, 1, 2, 3, 4, 5]
>>> print(_epi_factorize_partition(['A',4,'C','C','G',4,0,0,1,'a','A']))
[0, 1, 2, 2, 3, 1, 4, 4, 5, 6, 0]
```

Note that the function does not literally relabel the input list, but allocates a new space in the memory to store the relabeled list.

7.3. Description of `_preimage_of_partition`



The function `_preimage_of_partition` takes a list and returns the list of the lists of indices that index the same element.

```
1 def _preimage_of_partition(partition):
2     """ the source code of this function can be found in piop.py """
3     return the_preimage
```

From the point of view of partitions [4, Appendix A], the returned list `the_preimage` (see the code above) is the preimage of the underlying epimorphism $f : S \rightarrow K$ of the input `partition`, where the preimage of f is defined as the K -indexed set of the fibers of the epimorphism.

$$\text{Prelm}(f) = \{f^{-1}(k)\}_{k \in K}$$

Note that, from an implementation viewpoint, the set K might not be equipped with an obvious order relation, which makes it difficult to define the preimage of $f : S \rightarrow K$ as a python list. To rectify this flaw, the preimage is computed with respect to the canonical epimorphism $e(f) : S \rightarrow \text{Im}(f)$ whose codomain is equipped with the natural order on integers (see section 7.2).

$$\text{Prelm}(f) := \{e(f)^{-1}(k)\}_{k \in \text{Im}(f)}$$

```
>>> p = ['a','a',2,2,3,3,'a']
>>> print(_preimage_of_partition(p))
[[0, 1, 6], [2, 3], [4, 5]]
>>> print(_epi_factorize_partition(p))
[0, 0, 1, 1, 2, 2, 0]
```

```
>>> p = [2,1,0,6,5,4,2,1,0]
>>> print(_preimage_of_partition(p))
[[0, 6], [1, 7], [2, 8], [3], [4], [5]]
>>> print(_epi_factorize_partition(p))
[0, 1, 2, 3, 4, 5, 0, 1, 2]
```

In the first example given above:

- the list `[0,1,6]` is the fiber of the element 'a' and its index in the preimage is 0;
- the list `[2,3]` is the fiber of the element 2 and its index in the preimage is 1;
- the list `[4,5]` is the fiber of the element 3 and its index in the preimage is 2.

The preimage will always order its fibers with respect to the order in which the elements of the input list appear.

7.4. Description of `print_partition`

`print_partition` \longrightarrow `_preimage_of_partition`

The function `print_partition` is a debug function that takes a list of elements and prints its preimage on the standard output.

```
1 def print_partition(partition):
2     print(_preimage_of_partition(partition))
```

See section 7.3 for examples.

7.5. Description of `_join_preimages_of_partitions`

`_join_preimages_of_partitions` \longrightarrow `_image_of_partition`

The function `_join_preimages_of_partitions` takes two lists of lists of indices (the indices can be repeated and should only be non-negative integers) as well as a Boolean value and returns the list of the maximal unions of internal lists that intersect within the concatenation of the two input lists (see the examples below).

```
1 def _join_preimages_of_partitions(preimage1,preimage2,speed_mode):
2     """ the source code of this function can be found in jpop.py """
3     return the_join
```

While the two input lists `preimage1` and `preimage2` could be two outputs of the procedure

`_preimage_of_partition(-)`

for two input lists of the same length, the Boolean value `speed_mode` would indicate whether one of the two input lists may contain at least two different sublists with the same index, as shown below.

`[[0, 1], [0, 4], [2, 3, 4, 5]]`

Note that the global variable `FAST` is reserved to this use.

```
>>> print(FAST)
True
```

From the point of view of partitions, the composition of `_preimage_of_partition` with `_join_preimages_of_partitions` would amount to computing the coproduct of two partitions. Since a category of partitions is also a partially ordered set, this coproduct is also the *join* of the two partitions, which explains the name of the procedure.

For illustration, if we consider the following two lists of lists of indices

```
>>> p1 = [[0, 3], [1, 4], [2]]
>>> p2 = [[0, 1], [2], [3], [4]]
```

we can notice that

- the internal list [0,3] of **p1** intersects with the internal lists [0,1] and [3] in **p2**;
- the internal list [0,1] of **p1** intersects with the internal lists [1,4] and [1] in **p2**;
- the internal list [1,4] of **p1** intersects with the internal list [4] in **p2**;

and

- the internal list [2] of **p1** only intersects with the internal list [2] in **p2**,

so that we have

```
>>> print(_join_preimages_of_partitions(p1,p2,FAST))
[[1, 4, 0, 3], [2]]
```

In terms of implementation, the program

(7.1) `_join_preimages_of_partitions(p1,p2,FAST)`

considers each internal list of **p1** and searches for the lists of **p2** that intersect it. If an intersection is found between two internal lists, it merges the two internal lists in **p1** and empties that of **p2** (the list is emptied and *not* removed in order to preserve a coherent indexing of the elements of **p2**). The function continues until all the possible intersections have been checked.

Here is a detail of what program (7.1) does with respect to the earlier example:

The element 0 of [0,3] is searched in the list [0,1] of **p2**;

The element 0 is found;

The lists [0,3] and [0,1] are merged in **p1** and [0,1] is emptied from **p2** as follows:

```
p1 = [[0, 3, 1], [1, 4], [2]]
p2 = [[], [2], [3], [4]]
```

Because the element 0 has now been found in **p2** and the third input was set to **FAST**, no other sublist of **p2** is supposed to contain the element 0 and the search of the element 0 stops here. Note that if **not(FAST)** were given in the third argument, then the earlier union operation would also be operated on the remaining sublists of **p2**.

The element 3 of [0, 3] is searched in the list [] of **p2**;

The element 3 is not found (continues);

The element 3 of [0, 3] is searched in the list [2] of **p2**;

The element 3 is not found (continues);

The element 3 of [0, 3] is searched in the list [3] of **p2**;

The element 3 is found;

The lists [0,3] and [3] are merged in **p1** and [3] is emptied from **p2** as follows:

```
p1 = [[0, 3, 1], [1, 4], [2]]
p2 = [[], [2], [], [4]]
```

The element 3 has now been found in **p2** and does not need to be searched again.

All elements of the initial list [0, 3] have been searched.

The first lists of **p1** is appended to **p2** in order to ensure the transitive computation of the maximal unions through the next iterations.

The list [0, 3, 1] of **p1** is emptied as follows:

```
p1 = [[], [1, 4], [2]]
p2 = [[], [2], [], [4], [0, 3, 1]]
```

Repeat the previous procedure with respect to the list [1, 4] of p1. We obtain the following pair:

```
p1 = [[], [], [2]]
p2 = [[], [2], [], [], [], [1, 4, 0, 3]]
```

Repeat the previous procedure with respect to the remaining list [2] of p1. We obtain the following pair:

```
p1 = [[], [], []]
p2 = [[], [], [], [], [], [1, 4, 0, 3], [2]]
```

The function stops because there is no more list to process in p1. The output is all the non-empty lists of p2; i.e. [[1, 4, 0, 3], [2]]

Note that, because of the iterative nature of the previous algorithm, the procedure

```
_join_preimages_of_partitions(-,-,-)
```

does not necessarily presents its output in the same way as the procedure

```
_preimage_of_partition(-)
```

does. For instance, while the index 0 will always be contained in the first list of the output of `_preimage_of_partition`, it might not be contained in the first list of the output of `_join_preimages_of_partitions` as illustrated below.

```
>>> l = _preimage_of_partition([1,2,4,5,1,2,3,2,2,1,3])
>>> print(l)
[[0, 4, 9], [1, 5, 7, 8], [2], [3], [6, 10]]
>>> m = _preimage_of_partition([1,2,5,4,2,2,5,6,5,7,8])
>>> print(m)
[[0], [1, 4, 5], [2, 6, 8], [3], [7], [9], [10]]
>>> print(_join_preimages_of_partitions(l,m))
[[3], [6, 10, 2, 1, 5, 7, 8, 0, 4, 9]]
```

Interestingly, the procedure `_join_preimages_of_partitions` can also be used to compute the intersection-free closure of a set of sets, as shown below.

```
>>> a = [[1,0,2,1,3,2,5,4,6],[15,15,18,0,13],[7,11,12,22],[23,12]]
>>> closure_of_a = _join_preimages_of_partitions(a,a,not(FAST))
>>> print(closure_of_a)
[[15, 18, 0, 13, 1, 2, 3, 5, 4, 6], [23, 12, 7, 11, 22]]
```

7.6. Description of *EquivalenceRelation* (class)

`EquivalenceRelation` \longrightarrow `_join_preimages_of_partitions`

The class *EquivalenceRelation* possesses two objects, namely

- `.classes` (list of lists);
- `.range` (integer);

and three methods, namely

- `__init__` (constructor);
- `.closure`;
- `.quotient`.

The constructor `__init__` takes between 1 and 2 arguments: the first argument should either be an empty list or a list of lists of indices (i.e. non-negative integers) and the second argument, which is optional when the first argument is not an empty list, should be an integer that is greater than or equal to the maximum index contained in the first input, if it exists.

```

1 class EquivalenceRelation:
2     #The objects of the class are:
3     #.classes (list of lists);
4     #.range (integer);
5     #The following constructor takes between 1 and 2 arguments,
6     #the first one being a list and the second being an integer.
7     def __init__(self,*args):
8         """ the source code of this constructor can be found in cl_er.py """
9     def closure(self):
10         self.classes = _join_preimages_of_partitions(self.classes,
11 self.classes,not(FAST))
12     def quotient(self):
13         """ the source code of this function can be found in cl_er.py """
14         return the_quotient

```

If the first input is not empty, then it is stored in the object `.classes` while the object `.range` receives:

- either the second input, when this second input is given;
- or the maximum index contained in the first input when no second input is given.

```

>>> eq1 = EquivalenceRelation([[0,1,2,9],[7,3,8,6],[4,9,5]])
>>> print(eq1.classes)
[[0, 1, 2, 9], [7, 3, 8, 6], [4, 9, 5]]
>>> print(eq1.range)
9
>>> eq2 = EquivalenceRelation([[0,1,2,9],[7,3,8,7],[9,15]],18)
>>> print(eq2.classes)
[[0, 1, 2, 9], [7, 3, 8, 7], [9, 15]]
>>> print(eq2.range)
18

```

If the first input is empty, then the second argument is required. In this case, the object `.classes` receive the lists containing all the singleton lists containing the integers from 0 to the integer given in the second argument, which is stored in the object `.range`.

```

>>> eq3 = EquivalenceRelation([],5)
>>> print(eq3.classes)
[[0], [1], [2], [3], [4], [5]]

```

The method `.closure()` replaces the content of the object `.classes` with the transitive closure of its classes. After this procedure, the object `.classes` describes an actual equivalence relation (modulo the singleton equivalence classes, which do not need to be specified for obvious reasons).

```

>>> eq1.closure()
>>> print(eq1.classes)
[[7, 3, 8, 6], [4, 9, 5, 0, 1, 2]]

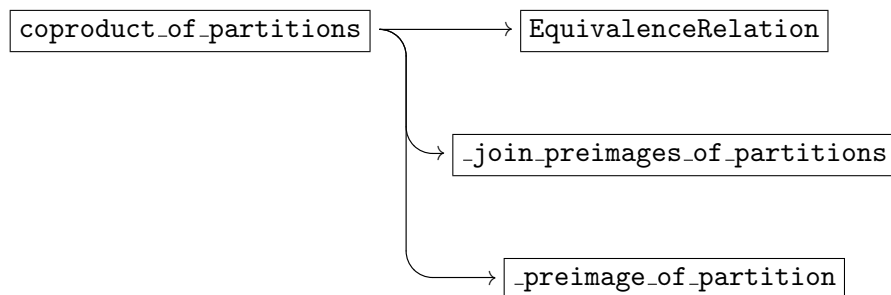
```

```
>>> eq2.closure()
>>> print(eq2.classes)
[[7, 3, 8], [9, 15, 0, 1, 2]]
```

The method `.quotient()` returns a list of integers whose length is equal to the integer contained in the object `.range` decreased by 1 and whose non-trivial fibers are those contained in the object `.classes` after a call of the method `.closure()`.

```
>>> print(eq1.quotient())
[1, 1, 1, 0, 1, 1, 0, 0, 0, 1]
>>> print(eq2.quotient())
[1, 1, 1, 0, 2, 3, 4, 0, 0, 1, 5, 6, 7, 8, 9, 1, 10, 11, 12]
```

7.7. Description of coproduct_of_partitions



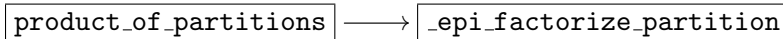
The function `coproduct_of_partitions` takes two lists of the same length and returns their coproduct (or join) as partitions. Specifically, the procedure outputs the quotient of the join of their preimages (see the piece of code given below). If the two input lists do not have the same length, then an error message is outputted and the program is aborted.

```
1 def coproduct_of_partitions(partition1,partition2):
2     if len(partition1) == len(partition2):
3         #Returns the coproduct of two partitions as the quotient of the
4         #equivalence relation induced by the join of the preimages
5         #of the two partitions.
6         the_join = EquivalenceRelation(_join_preimages_of_partitions(
7             _preimage_of_partition(partition1),_preimage_of_partition(partition2),
8             FAST))
9         return the_join.quotient()
10    else:
11        print("Error:  in coproduct_of_partitions:  lengths do not match.")
12    exit()
```

Note that the outputs of the procedure `coproduct_of_partitions` do not necessarily belong to the set of outputs of the procedure `_epi_factorize_partition`. The reason comes from the way in which the procedure `_join_preimages_of_partitions` is implemented (see the end of section 7.5).

```
>>> l = [1,2,4,5,1,2,3,2,2,1,3]
>>> m = [1,2,5,4,2,2,5,6,5,7,8]
>>> c = coproduct_of_partitions(l,m)
>>> print(c)
[1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1]
>>> print(_epi_factorize_partition(c))
[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
```

7.8. Description of `product_of_partitions`



The function `product_of_partitions` takes two lists and returns a list that is the relabeling of the zipping of the two lists (i.e. the list of pairs of elements with corresponding indices in each of the input lists) via the procedure `_epi_factorize_partition`.

```

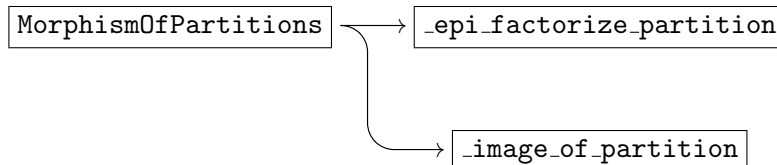
1 def product_of_partitions(partition1,partition2):
2     #The following line checks if the product of the two lists is possible.
3     if len(partition1) == len(partition2):
4         #Constructs the list of pairs of element with the
5         #same index in the two lists, and then relabels
6         #the pairs using _epi_factorize_partition.
7         return _epi_factorize_partition(zip(partition1,partition2))
8     else:
9         print("Error:  in product_of_partitions:  lengths do not match.")
10    exit()
  
```

The function outputs an error if the two input lists do not have the same length.

```

>>> product_of_partitions([1,1,1,1,2,3],[‘a’,‘b’,‘c’,‘c’,‘c’,‘c’])
[0, 1, 2, 2, 3, 4]
>>> product_of_partitions([1,1,1,1,2],[‘a’,‘b’,‘c’,‘c’,‘c’,‘c’])
Error:  in product_of_partitions:  lengths do not match.
  
```

7.9. Description of `MorphismOfPartitions` (class)



The class `MorphismOfPartitions` possesses three objects, namely

- `.arrow` (list)
- `.source` (list)
- `.target` (list)

and a constructor `__init__`. The constructor `__init__` takes two lists as well as an optional argument and stores, in the object `.arrow`, the list that describes, if it exists, the (unique) morphism of partitions from the first input list (seen as a partition) to the second input list (seen as a partition). If the morphism does not exist, then the method returns an error message unless the value `False` was given as a third input.

The canonical epimorphisms associated with the partitions of the first and second input lists (see section 7.2) are stored in the objects `.source` and `.target`, respectively.

```

1 class MorphismOfPartitions:
2     #The objects of the class are:
3     #.arrow (list);
4     #.source (list);
5     #.target (list).
6     def __init__(self,source,target,*args):
7         """ the source code of this constructor can be found in cl.mop.py """
  
```

The list that is contained in the object `.arrow` is the image of the function $\text{source} \mapsto \text{target}(\text{source})$ that can be constructed from the parametrization $t \mapsto \text{source}[t]$ and $t \mapsto \text{target}[t]$, which are given by the list structure of the two input lists `target` and `source`. This is illustrated below in more detail.

For illustration, let us consider the following pair of lists.

```
>>> p1 = [0,1,2,3,3,4,5]
>>> p2 = [0,1,2,3,3,3,1]
```

To construct the function $\text{source} \mapsto \text{target}(\text{source})$, we can first try to construct the graph $(\text{source}[t], \text{target}[t])$, for which we use the procedure `zip`.

```
>>> p3 = zip(p1,p2)
>>> print(p3)
[(0, 0), (1, 1), (2, 2), (3, 3), (3, 3), (4, 3), (5, 1)]
```

The image of the zipping is then as follows:

```
>>> p4 = _image_of_partition(p3)
>>> print(p4)
[(0, 0), (1, 1), (2, 2), (3, 3), (4, 3), (5, 1)]
```

We can see that for each pair (x,y) in `p4`, every component x is mapped to a unique image y so that `p4` defines the *graph* of the morphism of partitions between `p1` and `p2`. The constructor `__init__` then records, in the object `.arrow`, the second projections of the pairs contained in `p4`, which also corresponds to the image of the underlying graph encoded by `p4`.

```
>>> m = MorphismOfPartitions(p1,p2)
>>> print(m.arrow)
[0, 1, 2, 3, 3, 1]
```

If there exists no morphism from the first input list to the second input list, then the function outputs an error message.

For example, if we modify `p2` as follows

```
>>> p2 = [0,1,2,3,6,3,1]
```

then the image of the zipping of `p1` and `p2` is as follows:

```
>>> p4 = _image_of_partition(zip(p1,p2))
>>> print(p4)
[(0, 0), (1, 1), (2, 2), (3, 3), (3, 6), (4, 3), (5, 1)]
```

As can be seen, the argument 3 is ‘mapped’ to two different images, namely 3 and 6. In this case, the constructor `__init__` exits the program with an error message.

Here is a complete example summarizing the previous explanation.

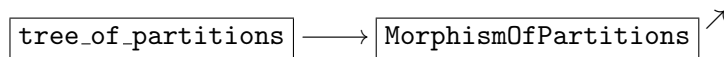
```
>>> m = MorphismOfPartitions([1,6,5,3,3,4,2],[1,2,5,4,4,4,2])
>>> print(m.source)
[0, 1, 2, 3, 3, 4, 5]
>>> print(m.target)
[0, 1, 2, 3, 3, 3, 1]
>>> print(m.arrow)
[0, 1, 2, 3, 3, 1]
>>> m = MorphismOfPartitions([1,6,5,3,3,4,2],[1,2,5,4,6,4,2])
Error: in MorphismOfPartitions.__init__: source and target are not
compatible.
>>> m = MorphismOfPartitions([1,6,5,3,3,4,2],[1,2,5,4,6,4,2],False)
```

Finally, note that the constructor of `MorphismOfPartitions` can be used to test whether there is a morphism between two partitions by using the key words `try` and `except` and setting the third argument to `False`, as illustrated below.

```
>>> try:
>>>     m = MorphismOfPartitions([1,6,5,3,3,4,2],[1,2,5,4,4,4,2],False)
>>>     print("True")
>>> except:
>>>     print("False")
True
>>> try:
>>>     m = MorphismOfPartitions([1,6,5,3,3,4,2],[1,2,5,4,6,4,2],False)
>>>     print("True")
>>> except:
>>>     print("False")
False
```

Presentation of the module AsciiTree.py

8.1. Description of tree_of_partitions



The function `tree_of_partitions` takes a list of lists whose pairs of successive lists can be related via `MorphismOfPartitions` items (see section 7.9)

$$\begin{array}{c}
 \text{pair of succ. lists} \\
 \underbrace{[l_1, l_2, l_3, \dots, l_n]} \\
 \text{pair of succ. lists}
 \end{array}$$

and returns the actual lists of `MorphismOfPartitions` items between these.

```

1 def tree_of_partitions(partitions):
2     """ the source code of this function can be found in top.py """
3     return the_tree
  
```

The input list should always start with the target of the first arrow, then present its source, which should also be the target of the next arrow, etc.

```

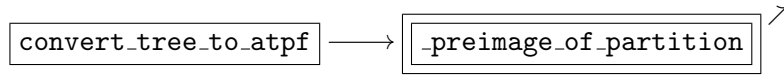
>>> l = [[0,0,0,0,0,0], [0,0,0,0,0,1], [0,0,2,2,2,1], [0,0,3,1,1,2],
         [0,1,2,3,4,5]]
>>> tree = tree_of_partitions(l)
>>> for i in range(len(tree)):
...     print(tree[len(tree)-1-i].source)
...     print("  |\n  | "+"arrow num "+str(len(tree)-i) + ":  " + str(
        tree[len(tree)-1-i].arrow) + "\n  |\n  V")
...     print(tree[0].target)
[0, 1, 2, 3, 4, 5]
|
| arrow num 4:  [0, 0, 1, 2, 2, 3]
|
V
  
```

```

[0, 0, 1, 2, 2, 3]
|
| arrow num 3:  [0, 1, 1, 2]
|
V
[0, 0, 1, 1, 1, 2]
|
| arrow num 2:  [0, 0, 1]
|
V
[0, 0, 0, 0, 0, 1]
|
| arrow num 1:  [0, 0]
|
V
[0, 0, 0, 0, 0, 0]

```

8.2. Description of `convert_tree_to_atpf`

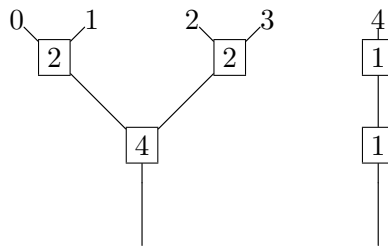


The function `convert_tree_to_atpf` takes a list of `MorphismOfPartitions` item (as returned by the procedure `tree_of_partitions`) and converts it into its associated *ascii tree pre-format* (abbrev. *atpf*), which is a regular expression of type ATPF given by the following double grammar rules.

The grammar terms	and their associated weights
$\text{ATPF} := [\text{Tree}_1, \text{Tree}_2, \dots, \text{Tree}_k];$	
$\text{Tree} := (\text{weight}(\text{Tree}), [\text{Tree}_1, \text{Tree}_2, \dots, \text{Tree}_k]),$	$\text{weight}(\text{Tree}) := \sum_i \text{weight}(\text{Tree}_i);$
$\text{Tree} := (\text{weight}(\text{Tree}), [\text{Leaf}_1, \text{Leaf}_2, \dots, \text{Leaf}_k]),$	$\text{weight}(\text{Tree}) := \sum_i \text{weight}(\text{Leaf}_i);$
$\text{Leaf} := (\text{weight}(\text{Leaf}), l), \text{ where } l \text{ is a list}$	$\text{weight}(\text{Leaf}) := \text{len}(l);$

The idea of such a construction is to give access to the number of leaves contained in each fork of a tree. This type of information will later be used to display trees with *ascii* characters on the console. For illustration, the following line gives the example of an *atpf* that describes the forest displayed below it.

```
atpf = [(4, [(2, [0, 1]), (2, [2, 3])]), (1, [(1, [4])])]
```



Note that the number of levels in the trees can be linked to what could be called the *depth* of the bracketing structure of the *atpf*. Specifically, we define the *depth* of an *atpf* according

to the following recursive equations, relative to the definition of the terms given above.

$$\begin{aligned} \text{depth(ATPF)} &:= \max\{\text{depth}(\text{Tree}_i) \mid i = 1, \dots, k\}; \\ \text{depth(Tree)} &:= \max\{\text{depth}(\text{Tree}_i) \mid i = 1, \dots, k\} + 1; \\ \text{depth(Tree)} &:= \max\{\text{depth}(\text{Leaf}_i) \mid i = 1, \dots, k\} + 1; \\ \text{depth(Leaf)} &:= 1; \end{aligned}$$

The procedure `convert_tree_to_atpf` then returns a list and an integer, where the list is the atpf of the input (i.e. of the tree) and the integer is equal to the depth of the atpf, which is equal to `len(tree)+1`.

```
1 def _convert_tree_to_atpf(tree):
2     """ the source code of this function can be found in ctt.py """
3     return (the_atpf, len(tree)+1)
```

Since the depth is only returned for parsing purposes (see section 8.3 and section 8.4), the main task of the procedure `convert_tree_to_atpf` is to compute the atpf associated with the input list of composable `MorphismOfPartitions` items by considering the successive preimages of the objects `.arrow` of each `MorphismOfPartitions` item contained in the list.

For illustration, consider the following list of partitions:

```
>>> a = [0,1,0,0,0,0]
>>> b = [0,2,0,0,0,1]
>>> c = [0,4,2,3,3,5]
```

This list can be associated with an obvious sequence of `MorphismOfPartitions` item that is constructed by the procedure `tree_of_partitions` (see section 8.1). Below, we make the steps of this construction explicit for the list `[a, b, c]`.

First, one wants to consider the preimage of the last partition `c`. This preimage is outputted by the function `_preimage_of_partition` (see section 7.3) as shown below.

```
>>> _preimage_of_partition(c)
[[0], [1], [2], [3, 4], [5]]
```

The internal lists appearing in the previous list are important for our next step. More specifically, we want to follow the recursive definition of the grammar of atpfs (given above) and take the lists `[0]`, `[1, 2]`, `[3, 4]`, and `[5]` to be the initial values of the recursion so that the first level of the atpf is as shown below, where the red numbers are the weight of the `Leaf` terms (see grammar rules above).

```
the_atpf = [(1, [0]), (1, [1]), (1, [2]), (2, [3, 4]), (1, [5])]
```

For the next level, we need to compute the preimage of the object `.arrow` encoding the morphism $c \rightarrow b$. Recall that the list contained in this object encodes the mapping rules associated with the morphism $c \rightarrow b$. Below, we display this mapping up to relabeling of the lists `c` and `b` as `[0,1,2,3,3,4]` and `[0,1,0,0,0,2]`, respectively.

```
>>> f = MorphismsOfPartitions(c,b)
>>> for i in range(len(f.arrow)):
...     print("f: "+str(i)+" |--> "+str(f.arrow[i]))
f: 0 |-> 0
f: 1 |-> 1
f: 2 |-> 0
f: 3 |-> 0
f: 4 |-> 2
```

Now, since `b` has three elements in its image, the `MorphismOfPartitions` item `f` possesses three fibers, which are given by the following list of lists.

```
>>> fiber = _preimage_of_partition(f.arrow)
>>> print(fiber)
[[0, 2, 3], [1], [4]]
```

Following the atpf grammar and, more specifically, the syntax for the **Tree** terms, we now want to use the list `fiber` computed above to create the next level of the atpf. The idea is to replace the intergers living in `fiber` with the elements of the list `the_atpf`. Precisely, we want to replace:

```
fiber[0][0] = 0 with the_atpf[0]
fiber[0][1] = 2 with the_atpf[2]
fiber[0][2] = 3 with the_atpf[3]
fiber[1][0] = 1 with the_atpf[1]
fiber[2][0] = 4 with the_atpf[4]
```

Doing so, the list `fiber` is turned into the following list.

```
fiber = [[(1, [0]), (1, [2])], (2, [3, 4]), [(1, [1])], [(1, [5])]]
```

To complete the construction of the next level of the atpf, there remains to compute the weight for each internal list. Precisely, we can see that

- the weight of $[(1, [0]), (1, [2])]$ is $1+1+2 = 4$;
- the weight of $[(1, [1])]$ is 1 ;
- the weight of $[(1, [5])]$ is 1 .

We then equip each list with its weight by using tuples, as shown below.

```
the_atpf = [(4, [(1, [0]), (1, [2])], (2, [3, 4])), (1, [(1, [1])]), (1, [(1, [5])])]
```

We then repeat the previous procedure with, this time, the fiber of the morphism $b \rightarrow a$ and the earlier list `the_atpf` so that the final atpf is of the following form.

```
the_atpf = [(5, [(4, [(1, [0]), (1, [2])], (2, [3, 4])], (1, [(1, [5])])]), (1, [(1, [(1, [1])])])]
```

The user that verify that the previous list corresponds to the output of the following command.

```
>>> print(convert_tree_to_atpf(tree_of_partitions([a,b,c]))[0])
```

8.3. Description of `convert_atpf_to_atf`

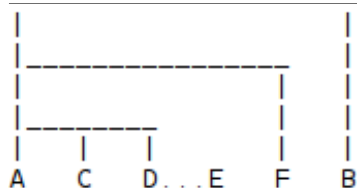
The function `convert_atpf_to_atf` takes an atpf and its depth (see section 8.2) and returns the associated *ascii tree format* (abbrev. *atf*), which is a modified version of an atpf in which one substracts all the weights by the rightmost weight of the next level, as shown by the following grammar rules

```
ATPF := [Tree1, Tree2, ..., Treek];
Tree := ((weight(Tree), weight(Tree) - weight(Treek)), [Tree1, Tree2, ..., Treek]);
Tree := ((weight(Tree), weight(Tree) - weight(Treek)), [Leaf1, Leaf2, ..., Leafk]);
Leaf := ((weight(Leaf), 0), l), where l is a list;
```

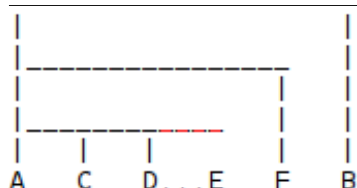
Note that this function uses the depth of the atpf in order to differentiate between the leaves and the intermediate levels of the tree, which require two different types of treatment.

```
1 def convert_atpf_to_atf(atpf, depth):
2     """ the source code of this function can be found in cata.py """
3     return the_atf
```

The reason for this is that the procedure `print_atf` (see section 8.4) is to display ascii trees whose trunks are on the left of the screen, as show below.



Subtracting the rightmost weights of the atpf from the weight placed below it, in the tree, allows `print_atf` to know when it needs to stop printing the horizontal level of the tree. Intuitively, the following pictures shows what atpf would look like without conversion into an atf, where the red underscore symbols sticking out toward the right symbolize the amount of weight subtracted in the atf.



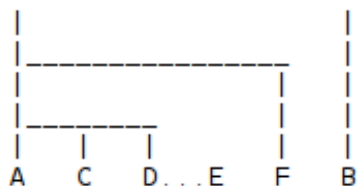
8.4. Description of `print_atf`

The function `print_atf` takes an atf and its depth and prints the ascii tree associated with the atf on the standard output.

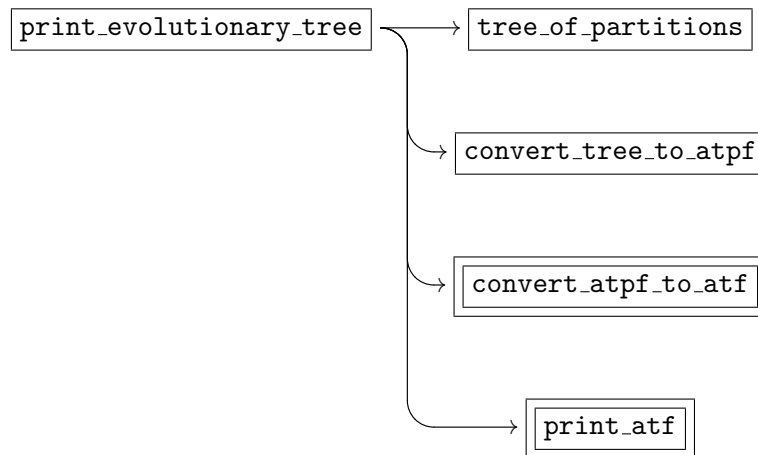
```
1 def print_atf(atf,depth):
2     """ the source code of this function can be found in patf.py """
```

The following example shows how `print_atf` can be combined with the procedures `convert_tree_to_atpf` and `convert_atpf_to_atf`.

```
>>> l = [[0,1,0,0,0,0], [0,2,0,0,0,1], [0,4,2,3,3,5]]
>>> tree = tree_of_partitions(l)
>>> atpf = convert_tree_to_atpf(tree)
>>> atf = convert_atpf_to_atf(*atpf)
>>> print_atf(atf,atpf[1])
```



8.5. Description of `print_evolutionary_tree`



The function `print_evolutionary_tree` takes a list of lists whose pairs of successive lists can be related via `MorphismOfPartitions` items and returns the tree encoded by this sequence of morphisms.

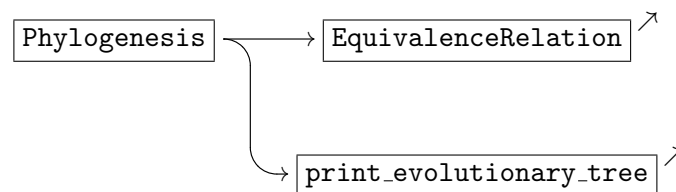
```

1 def print_evolutionary_tree(atf,depth):
2     #Returns a sequence of morphisms of partitions.
3     tree = tree_of_partitions(partitions)
4     #Returns an ascii tree pre-format and its depth.
5     atpf = convert_tree_to_atpf(tree)
6     #Returns the ascii tree format of the atpf.
7     atf = convert_atpf_to_atf(*atpf)
8     #Prints the atf on the standard output.
9     print_atf(atf,atpf[1])
  
```

See the example given in section 8.5 to see what this procedure does.

Presentation of the module Phylogeny.py

9.1. Description of Phylogenesis (class)



The class `Phylogenesis` possesses two objects, namely

- `.taxon` (non-negative integer);
- `.history` (list of lists of indices);

and three methods, namely

- `__init__` (constructor)
- `.partitions`
- `.print_tree`

A `Phylogenesis` item is meant to be part of another structure called a `Phylogeny` (see section 9.2). The first object `.taxon` of the class `Phylogenesis` stores an integer that allows us to identify the `Phylogenesis` item with respect to other `Phylogenesis` items in the `Phylogeny` structure. On the other hand, the object `.history` stores a list of lists encoding a historical record of the coalescence events between the `Phylogenesis` item and the other taxa contained by the `Phylogeny` structure. The list of lists contained in the object `.history` should:

- start with a singleton list containing the integer representing the taxon itself;
- be such that every list should contain its predecessor list;

```

1 class Phylogenesis:
2     #The objects of the class are:
3     #.taxon (non-negative integer);
4     #.history (list of lists of indices);
5     def __init__(self,history):
6         """ the source code of this constructor can be found in cl_pgs.py """
7     def partitions(self):
8         """ the source code of this function can be found in cl_pgs.py """
9         return partitions
10    def print_tree(self):
11        #Returns the evolutionary tree described by the list of lists outputted
12        #by the procedure partitions().
13        return print_evolutionary_tree(self.partitions())

```

The constructor `__init__` takes a non-empty list of lists whose first list is a singleton and allocates

- the index contained in the first internal list of the input list to the object `.taxon`,
- the input list to the object `.history`.

Before terminating, the procedure checks whether the list of lists is made of non-negative integers and whether each list preceding another is contained in the successor list. Below, we give the example of an initialization of a *Phylogenesis* item.

```

>>> history = [[4],[4,6],[5,4,7,8,6],[6,5,4,7,8,20]]
>>> phylogenesis = Phylogenesis(history)
>>> print(phylogenesis.taxon)
4
>>> print(phylogenesis.history[len(phylogenesis.history)-1])
[6, 5, 4, 7, 8, 20]

```

The method `.partitions()` returns the sequence of partitions induced by the list of lists contained in the object `.history` over the set of indices ranging from 0 to the maximum index of the last list of the object `.history`.

```

>>> p = phylogenesis.partitions()
>>> for i in range(len(p)):
>>>     print(p[i])
[1, 2, 3, 4, 0, 0, 0, 0, 0, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0]
[1, 2, 3, 4, 0, 0, 0, 0, 0, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]
[1, 2, 3, 4, 0, 5, 0, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]
[1, 2, 3, 4, 0, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]

```

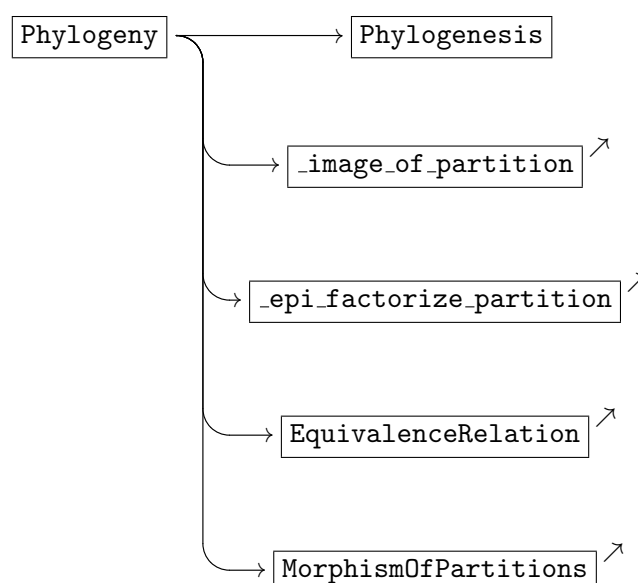
The method `.print_tree()` returns the evolutionary tree associated with the sequence of partitions returned by `.partition()` (for more intuition, see section 8).

```

>>> phylogenesis.print_tree()

```

9.2. Description of Phylogeny (class)



The class `Phylogeny` possesses one object, namely

- `.phylogenesises` (list of `Phylogenesis` items)

and ten methods, namely

- `__init__` (constructor)
- `.coalescent`
- `.extend`
- `.count_uniformity`
- `.boolean_partition`
- `.make_friends`
- `.set_up_friendship`
- `.score`
- `.choose`
- `.set_up_competition`

The object `.phylogenesis` is supposed to contained a list of `Phylogenesis` items. The taxon associated with the i -th phylogenesis should be indexed by the interger i itself and any label appearing in the `Phylogenesis` items of the list should have its own `Phylogenesis` item included in the list.

```

1 class Phylogeny:
2     #The objects of the class are:
3     #.phylogenesises (lists of Phylogenesis items);
4     def __init__(self, phylogenesises):
5         """ the source code of this constructor can be found in cl_pgs.py """
  
```

9.2.1. First generation. Below, we will often use the term *first generation* of the phylogenesis of a certain taxon `t` to refer to the last list contained in the list

`self.phylogenesises[t].history`

relative to the indexing order associated with the list structure.

9.2.2. Constructor. The constructor `__init__` takes a list of lists of lists containing non-negative integers and use every internal list of the input to create a `Phylogenesis` item, which is stored in the object `.phylogeneses`.

The following lines show how a phylogeny can be created via the constructor of the class.

```
>>> p = list()
>>> for i in range(8):
>>>     p.append([i], [i, (i+10) % 8], [i, (i+10) % 8, (i+5*i+13) % 8]))
>>> pg = Phylogeny(p)
>>> for i in range(len(pg.phylogeneses)):
>>>     print("taxon: "+str(pg.phylogeneses[i].taxon))
>>>     print(pg.phylogeneses[i].history)
taxon: 0
[[0], [0, 2], [0, 2, 5]]
taxon: 1
[[1], [1, 3], [1, 3, 3]]
taxon: 2
[[2], [2, 4], [2, 4, 1]]
taxon: 3
[[3], [3, 5], [3, 5, 7]]
taxon: 4
[[4], [4, 6], [4, 6, 5]]
taxon: 5
[[5], [5, 7], [5, 7, 3]]
taxon: 6
[[6], [6, 0], [6, 0, 1]]
taxon: 7
[[7], [7, 1], [7, 1, 7]]
```

9.2.3. Elementary methods. `Phylogeny` items can be updated, modified or analyzed through the methods `.coalescent`; `.extend`; and `.make_friends`, which we present in the following sections.

```
6     def coalescent(self):
7         """ the source code of this constructor can be found in cl_pgs.py """
8         return coalescent
9     def extend(self,extension):
10        """ the source code of this constructor can be found in cl_pgs.py """
11        return False
12    def make_friends(self,taxon):
13        """ the source code of this constructor can be found in cl_pgs.py """
14        return (friends,coalescence_hypothesis)
```

9.2.4. Coalescent. The method `.coalescent()` returns the list of the last lists of the objects `.history` of each `Phylogenesis` item contained in the object `.phylogeneses` (*i.e.* what one would like to understand as the *first generations* of the current phylogeny). The k -th list of the outputted list is therefore the first generation (or last list) of the history of taxon k (see example below).

```
>>> coalescent = pgy.coalescent()
>>> for i in range(len(coalescent)):
>>>     print("1st generation of " + str(i) + "'s history:  " + " " +
            str(coalescent[i]))
1st generation of 0's history:  [0, 2, 5]
1st generation of 1's history:  [1, 3, 3]
1st generation of 2's history:  [2, 4, 1]
1st generation of 3's history:  [3, 5, 7]
1st generation of 4's history:  [4, 6, 5]
1st generation of 5's history:  [5, 7, 3]
1st generation of 6's history:  [6, 0, 1]
1st generation of 7's history:  [7, 1, 7]
```

9.2.5. Extend. The method `.extend` takes a list of pairs of the form (t, l) where t is the label of a taxon of the `Phylogeny` item (accessible through `pgy.phylogenesees[-].taxon`) and l is a list of taxa and it updates the object `.phylogenesees` as follows:

- for all pairs (t, l) contained in the input passed to `.extend`:
 - 1) ▷ if *every* list l contains the last list of `self.phylogenesees[t].history`,
 ▷ if *at least one* of the lists l strictly contains the last list of
 `self.phylogenesees[t].history`,
 → then every list l is appended to the list `self.phylogenesees[t].history`
 and the value `True` is returned;
 - 2) if there is no strict inclusion of the last list of `self.phylogenesees[t].history`
 into l , then the object `.phylogenesees` is not modified and the value `False` is
 returned;
 - 3) otherwise, an error message is returned and the procedure exit the program;
- if an update has happened, then for all other taxa t of the phylogeny that do not
 appear in the input of `.extend`, the last list of `self.phylogenesees[t].history`
 (*i.e.* the first generation of the history of the phylogenesis of t) is again repeated
 (*i.e.* appended again) in the list `self.phylogenesees[t].history`.

The following code lines illustrate these various cases for the example used in the previous sections (starting from section 9.2.2).

```
>>> extension = [(5, [3, 7, 7, 5]), (7, [7, 1])]
>>> flag = pgy.extend(extension)
>>> print(flag)
False
>>> for i in range(len(pgy.phylogenesees)):
>>>     print(pgy.phylogenesees[i].history)
[[0], [0, 2], [0, 2, 5]]
[[1], [1, 3], [1, 3, 3]]
[[2], [2, 4], [2, 4, 1]]
[[3], [3, 5], [3, 5, 7]]
[[4], [4, 6], [4, 6, 5]]
[[5], [5, 7], [5, 7, 3]]
[[6], [6, 0], [6, 0, 1]]
[[7], [7, 1], [7, 1, 7]]
>>> extension = [(5, [3, 7, 7, 5]), (7, [7, 1]), (1, [1, 3, 4, 5])]
>>> flag = pgy.extend(extension)
>>> print(flag)
```

```

True
>>> for i in range(len(pgy.phylogenesees)):
>>>     print(pgy.phylogenesees[i].history)
[[0], [0, 2], [0, 2, 5], [0, 2, 5]]
[[1], [1, 3], [1, 3, 3], [1, 3, 4, 5]]
[[2], [2, 4], [2, 4, 1], [2, 4, 1]]
[[3], [3, 5], [3, 5, 7], [3, 5, 7]]
[[4], [4, 6], [4, 6, 5], [4, 6, 5]]
[[5], [5, 7], [5, 7, 3], [3, 7, 5]]
[[6], [6, 0], [6, 0, 1], [6, 0, 1]]
[[7], [7, 1], [7, 1, 7], [7, 1]]
>>> extension = [(-1,["error"])]
>>> pgy.extend(extension)
>>> for i in range(len(pgy.phylogenesees)):
>>>     print(pgy.phylogenesees[i].history)
[[0], [0, 2], [0, 2, 5], [0, 2, 5]]
[[1], [1, 3], [1, 3, 3], [1, 3, 4, 5]]
[[2], [2, 4], [2, 4, 1], [2, 4, 1]]
[[3], [3, 5], [3, 5, 7], [3, 5, 7]]
[[4], [4, 6], [4, 6, 5], [4, 6, 5]]
[[5], [5, 7], [5, 7, 3], [3, 7, 5]]
[[6], [6, 0], [6, 0, 1], [6, 0, 1]]
[[7], [7, 1], [7, 1, 7], [7, 1]]
>>> extension = [(5,[1,4,80]),(7,["error"])]
>>> pgy.extend(extension)
>>> for i in range(len(pgy.phylogenesees)):
>>>     print(pgy.phylogenesees[i].history)
Error: in Phylogeny.extend: the extension is not compatible with the
phylogenesis of taxon 5

```

9.2.6. Make friends. The method `.make_friends` takes the label of a taxon (i.e. a non-negative integer) and returns a pair of lists (`friends`, `hypothesis`) where

- the list `friends` contains all those taxa that have not coalesced with the input taxon, which means that there are not in the first generation of the phylogenesis of the taxon;
- the list `hypothesis` contains the (sorted) lists obtained by making the union of the first generation of the input taxon with the first generation of one of the taxon in `friends`.

For illustration, consider the phylogeny constructed in section 9.2.5. Below, we first recall the structure of that phylogeny.

```

>>> for i in range(len(pgy.phylogenesees)):
>>>     print(pgy.phylogenesees[i].history)
[[0], [0, 2], [0, 2, 5], [0, 2, 5]]
[[1], [1, 3], [1, 3, 3], [1, 3, 4, 5]]
[[2], [2, 4], [2, 4, 1], [2, 4, 1]]
[[3], [3, 5], [3, 5, 7], [3, 5, 7]]
[[4], [4, 6], [4, 6, 5], [4, 6, 5]]

```

```
[[5], [5, 7], [5, 7, 3], [3, 7, 5]]
[[6], [6, 0], [6, 0, 1], [6, 0, 1]]
[[7], [7, 1], [7, 1, 7], [7, 1]]
```

The output of the method `.make_friends` for taxa 1, 2 and 7 is as follows.

```
>>> friends_made = pgy.make_friends(1)
>>> friends = friends_made[0]
>>> coalescence_hypothesis = friends_made[1]
>>> for i in range(len(friends)):
>>>     print("taxa 1 and "+ str(friends[i])+" have ancestor
        "+str(coalescence_hypothesis[i]))
taxa 1 and 0 have ancestor [0, 1, 2, 3, 4, 5]
taxa 1 and 2 have ancestor [1, 2, 3, 4, 5]
taxa 1 and 6 have ancestor [0, 1, 3, 4, 5, 6]
taxa 1 and 7 have ancestor [1, 3, 4, 5, 7]
>>> friends_made = pgy.make_friends(2)
>>> friends = friends_made[0]
>>> coalescence_hypothesis = friends_made[1]
>>> for i in range(len(friends)):
>>>     print("taxa 2 and "+ str(friends[i])+" have ancestor
        "+str(coalescence_hypothesis[i]))
taxa 2 and 0 have ancestor [0, 1, 2, 4, 5]
taxa 2 and 3 have ancestor [1, 2, 3, 4, 5, 7]
taxa 2 and 5 have ancestor [1, 2, 3, 4, 5, 7]
taxa 2 and 6 have ancestor [0, 1, 2, 4, 6]
taxa 2 and 7 have ancestor [1, 2, 4, 7]
>>> friends_made = pgy.make_friends(7)
>>> friends = friends_made[0]
>>> coalescence_hypothesis = friends_made[1]
>>> for i in range(len(friends)):
>>>     print("taxa 7 and "+ str(friends[i])+" have ancestor
        "+str(coalescence_hypothesis[i]))
taxa 7 and 0 have ancestor [0, 1, 2, 5, 7]
taxa 7 and 2 have ancestor [1, 2, 4, 7]
taxa 7 and 3 have ancestor [1, 3, 5, 7]
taxa 7 and 4 have ancestor [1, 4, 5, 6, 7]
taxa 7 and 5 have ancestor [1, 3, 5, 7]
taxa 7 and 6 have ancestor [0, 1, 6, 7]
```

9.2.7. Algorithm for constructing phylogenies. The next set of methods are procedures meant to be used to implement the algorithm described in [4], which constructs a phylogeny according to the definition given thereof.

```
15 def set_up_friendships(self):
16     """ the source code of this constructor can be found in cl_pgs.py """
17     return (friendships,coalescence_hypotheses)
15 def score(self,partitions,friendship_network):
16     """ the source code of this constructor can be found in cl_pgs.py """
17     return score_cardinality_adjusted
18 def choose(self,scores):
19     """ the source code of this constructor can be found in cl_pgs.py """
20     return result
```

```

21  def set_up_competition(self,best_fit):
22  """ the source code of this constructor can be found in cl_pgs.py """
23  return coalescence_hypothesis

```

9.2.8. Set up friendships. The method `.set_up_friendships()` returns a pair of lists (`friendships,hypotheses`) containing the lists of the two different outputs of the method `.make_friends` for every taxon of the phylogeny. More specifically,

- `friendships` is the list of lists whose t -th list contains the first output of the procedure `self.make_friends` for taxon t ;
- `hypotheses` is the list of lists whose t -th list contains the second output of the procedure `self.make_friends` for taxon t ;

The following code lines give a description of the output of `.set_up_friendships()` for the phylogeny used in section 9.2.6.

```

>>> friendships_made = pgy.set_up_friendships()
>>> friendships = friendships_made[0]
>>> coalescence_hypotheses = friendships_made[1]
>>> for t in range(len(friendships)):
>>>     for r in range(len(friendships[t])):
>>>         print("taxa " + str(t) + " and " + str(friendships[t][r])+" have
            ancestor "+str(coalescence_hypotheses[t][r]))

```

```

taxa 0 and 1 have ancestor [0, 1, 2, 3, 4, 5]
taxa 0 and 3 have ancestor [0, 2, 3, 5, 7]
taxa 0 and 4 have ancestor [0, 2, 4, 5, 6]
taxa 0 and 6 have ancestor [0, 1, 2, 5, 6]
taxa 0 and 7 have ancestor [0, 1, 2, 5, 7]
taxa 1 and 0 have ancestor [0, 1, 2, 3, 4, 5]
taxa 1 and 2 have ancestor [1, 2, 3, 4, 5]
taxa 1 and 6 have ancestor [0, 1, 3, 4, 5, 6]
taxa 1 and 7 have ancestor [1, 3, 4, 5, 7]
taxa 2 and 0 have ancestor [0, 1, 2, 4, 5]
taxa 2 and 3 have ancestor [1, 2, 3, 4, 5, 7]
taxa 2 and 5 have ancestor [1, 2, 3, 4, 5, 7]
taxa 2 and 6 have ancestor [0, 1, 2, 4, 6]
taxa 2 and 7 have ancestor [1, 2, 4, 7]
taxa 3 and 0 have ancestor [0, 2, 3, 5, 7]
taxa 3 and 1 have ancestor [1, 3, 4, 5, 7]
taxa 3 and 2 have ancestor [1, 2, 3, 4, 5, 7]
taxa 3 and 4 have ancestor [3, 4, 5, 6, 7]
taxa 3 and 6 have ancestor [0, 1, 3, 5, 6, 7]
taxa 4 and 0 have ancestor [0, 2, 4, 5, 6]
taxa 4 and 1 have ancestor [1, 3, 4, 5, 6]
taxa 4 and 2 have ancestor [1, 2, 4, 5, 6]
taxa 4 and 3 have ancestor [3, 4, 5, 6, 7]
taxa 4 and 7 have ancestor [1, 4, 5, 6, 7]
taxa 5 and 0 have ancestor [0, 2, 3, 5, 7]
taxa 5 and 1 have ancestor [1, 3, 4, 5, 7]
taxa 5 and 2 have ancestor [1, 2, 3, 4, 5, 7]
taxa 5 and 4 have ancestor [3, 4, 5, 6, 7]
taxa 5 and 6 have ancestor [0, 1, 3, 5, 6, 7]

```

```

taxa 6 and 2 have ancestor [0, 1, 2, 4, 6]
taxa 6 and 3 have ancestor [0, 1, 3, 5, 6, 7]
taxa 6 and 4 have ancestor [0, 1, 4, 5, 6]
taxa 6 and 5 have ancestor [0, 1, 3, 5, 6, 7]
taxa 6 and 7 have ancestor [0, 1, 6, 7]
taxa 7 and 0 have ancestor [0, 1, 2, 5, 7]
taxa 7 and 2 have ancestor [1, 2, 4, 7]
taxa 7 and 3 have ancestor [1, 3, 5, 7]
taxa 7 and 4 have ancestor [1, 4, 5, 6, 7]
taxa 7 and 5 have ancestor [1, 3, 5, 7]
taxa 7 and 6 have ancestor [0, 1, 6, 7]

```

9.2.9. Scoring system. The method `.score` takes a list of lists of non-negative integers (i.e. partitions), call it `partitions`, and a pair of lists, say `(friendships, hypotheses)`, where

- `friendships` is a list of lists;
- `hypotheses` is a list of length `len(friendships)` whose `t`-th element is a list of length `len(friendships[t])` whose elements are lists of integers ranging from 0 to

`len(self.phylogenesees)-1`

(preferably sorted from smallest to greatest);

and returns a list of length `len(friendships)` whose `t`-th element is a list of triples of the form `(r, large, exact)` where

- `r` runs over the elements of `friendships[t]`,
- `large` is the large score [4] of the hypothetical ancestor `hypotheses[t][r]` within the set of ancestors contained in `hypotheses[t]` for the list of partitions given in the input,
- `exact` is the exact score [4] of the hypothetical ancestor `hypotheses[t][r]` within the set of ancestors contained in `hypotheses[t]` for the list of partitions given in the input.

This means that `large` is the number of partitions belonging to the first input list `partitions` for which there is a morphism of partitions `x.quotient() → partitions[i]` where we take

`x = EquivalenceRelation([hypotheses[t][r]], len(self.phylogenesees)-1)`

and `exact` is the number of partitions that were counted in the large score of `r` such that if these partitions belong to the large score of any other element `s` in `friendships[t]`, then either the equality

$$\text{hypotheses}[t][r] = \text{hypotheses}[t][s]$$

holds or the intersection of `hypotheses[t][r]` with `hypotheses[t][s]` is empty.

The second input of the method `.score` can, for instance, be taken to be the output of the procedure

`self.set_up_friendships()`.

For example, consider the following file containing a sequence alignment.

```

                                Align.fa
1 >Alice
2 ACGCTAGCGCGATCGATCGGATCGATCGATC
3 >Bob
4 ACGACTTAGCGGATCTGATACTCCCTCGATC
5 >Carles
6 ACGACCTAGCGGATCTTATAACTCACCGATC
7 >Doug
8 ACGCTAGCGGCTGATAACGATCGTATCGATC
9 >Eric
10 ACGCATGCGCGATCGACGGATCGTATCTATC
11 >Fred
12 ACGACCTAGCAGATTTCTAATCTCAACGATC
13 >Gary
14 ACGCGAGCATCTGAACACGATTGTAACGATC
15 >Haley
16 ACGCTACGCGACGATCGGCTTTAGATCGATC

```

Let us now construct the list `local` of partitions induced by each non-trivial column of the previous alignment, as shown below.

```

>>> from efp import _epi_factorize_partition
>>> pre_local = usf.fasta("Align1.fa")[1]
>>> local = list()
>>> for j in range(len(pre_local[0])):
...     column = list()
...     for i in range(len(pre_local)):
...         column.append(pre_local[i][j])
...     r = _epi_factorize_partition(column)
...     if r != [0]*len(column):
...         local.append(r)
...

```

Then, we can use the method `.score`, as shown below, with the *Phylogeny* item `pgy` considered in section 9.2.6.

```

>>> l = pgy.score(local,pgy.set_up_friendships())
>>> for t in range(len(l)):
>>>     for (r,large,exact) in l[t]:
>>>         print("taxa " + str(t) + " and " + str(r)+" coalesce with large
              score "+str(large)+" and with exact score "+str(exact))
taxa 0 and 1 coalesce with large score 1 and with exact score 1
taxa 0 and 3 coalesce with large score 2 and with exact score 0
taxa 0 and 4 coalesce with large score 1 and with exact score 0
taxa 0 and 6 coalesce with large score 1 and with exact score 0
taxa 0 and 7 coalesce with large score 1 and with exact score 0
taxa 1 and 0 coalesce with large score 1 and with exact score 0
taxa 1 and 2 coalesce with large score 2 and with exact score 1
taxa 1 and 6 coalesce with large score 0 and with exact score 0
taxa 1 and 7 coalesce with large score 0 and with exact score 0

```

```

taxa 2 and 0 coalesce with large score 2 and with exact score 2
taxa 2 and 3 coalesce with large score 0 and with exact score 0
taxa 2 and 5 coalesce with large score 0 and with exact score 0
taxa 2 and 6 coalesce with large score 0 and with exact score 0
taxa 2 and 7 coalesce with large score 0 and with exact score 0
taxa 3 and 0 coalesce with large score 2 and with exact score 0
taxa 3 and 1 coalesce with large score 0 and with exact score 0
taxa 3 and 2 coalesce with large score 0 and with exact score 0
taxa 3 and 4 coalesce with large score 2 and with exact score 1
taxa 3 and 6 coalesce with large score 1 and with exact score 0
taxa 4 and 0 coalesce with large score 1 and with exact score 0
taxa 4 and 1 coalesce with large score 1 and with exact score 0
taxa 4 and 2 coalesce with large score 1 and with exact score 0
taxa 4 and 3 coalesce with large score 2 and with exact score 1
taxa 4 and 7 coalesce with large score 0 and with exact score 0
taxa 5 and 0 coalesce with large score 2 and with exact score 0
taxa 5 and 1 coalesce with large score 0 and with exact score 0
taxa 5 and 2 coalesce with large score 0 and with exact score 0
taxa 5 and 4 coalesce with large score 2 and with exact score 1
taxa 5 and 6 coalesce with large score 1 and with exact score 0
taxa 6 and 2 coalesce with large score 0 and with exact score 0
taxa 6 and 3 coalesce with large score 1 and with exact score 0
taxa 6 and 4 coalesce with large score 0 and with exact score 0
taxa 6 and 5 coalesce with large score 1 and with exact score 0
taxa 6 and 7 coalesce with large score 2 and with exact score 1
taxa 7 and 0 coalesce with large score 1 and with exact score 0
taxa 7 and 2 coalesce with large score 0 and with exact score 0
taxa 7 and 3 coalesce with large score 1 and with exact score 0
taxa 7 and 4 coalesce with large score 0 and with exact score 0
taxa 7 and 5 coalesce with large score 1 and with exact score 0
taxa 7 and 6 coalesce with large score 2 and with exact score 1

```

9.2.10. Choose. The method `.choose` takes a list of lists of triples (r, l, e) where l and e are non-negative integers and returns a list of lists whose i -th list is the list of those elements r of the i -th internal list of the input list for which the associated pairs (l, e) are equal to the greatest local maxima of the function $\Gamma : (e, l) \mapsto (l, e)$ ordered by the lexicographical order and relative to the pairs of the i -th internal list of the input list (see the example below and [4, Definition 3.14]).

```

>>> k1 = [[("a", 1, 0), ("b", 2, 0), ("c", 1, 1), ("d", 3, 3)]]
>>> choose = pgy.choose(k1)
>>> print(choose)
[['d']]
>>> k2 = [[("a", 1, 0), ("b", 2, 0), ("c", 1, 1), ("d", 3, 3), ("e", 3, 3)]]
>>> choose = pgy.choose(k2)
>>> print(choose)
[['d', 'e']]
>>> k3 = [[("a", 1, 0), ("b", 2, 0), ("c", 8, 1), ("d", 3, 3), ("e", 3, 3)]]
>>> choose = pgy.choose(k3)
>>> print(choose)
[['c']]

```

```
>>> k4 = [(("a",1,0),("b",9,0),("c",8,1),("d",3,3),("e",3,3))]
>>> choose = pgy.choose(k4)
>>> print(choose)
[['b']]
>>> k = k1 + k2 + k3 + k4
>>> choose = pgy.choose(k)
>>> print(choose)
[['d'], ['d', 'e'], ['c'], ['b']]
```

The following example shows the type of output that one gets if one gives the list 1 of the example of section 9.2.9 to the method `.choose`.

```
>>> choose = pgy.choose(1)
>>> for t in range(len(choose)):
>>>     print("Closest relatives of taxon " + str(t) + " are " +
            str(choose[t]))
Closest relatives of taxon 0 are [3]
Closest relatives of taxon 1 are [2]
Closest relatives of taxon 2 are [0]
Closest relatives of taxon 3 are [4]
Closest relatives of taxon 4 are [3]
Closest relatives of taxon 5 are [4]
Closest relatives of taxon 6 are [7]
Closest relatives of taxon 7 are [6]
```

9.2.11. Set up competition. The method `.set_up_competition` takes a list of lists of integers whose length must be equal to the length of `self.phylogenesees` (i.e. the number of taxa of the phylogeny) and returns a list of lists of integers whose length is also equal to the length of `self.phylogenesees` and whose t -th internal list is the union of the t -th list of `self.coalescent()` with the r -th list of `self.coalescent()` for every element r in the t -th internal list of the input list.

The following example shows the output of the method `.set_up_competition` when it is given the list `choose` constructed in the last example of section 9.2.10.

```
>>> competition = pgy.set_up_competition(choose)
>>> for t in range(len(competition)):
>>>     print("Next competing generation representing taxon " + str(t) + " is
            " + str(competition[t]))
Next competing generation representing taxon 0 is [0, 2, 3, 5, 7]
Next competing generation representing taxon 1 is [1, 2, 3, 4, 5]
Next competing generation representing taxon 2 is [0, 1, 2, 4, 5]
Next competing generation representing taxon 3 is [3, 4, 5, 6, 7]
Next competing generation representing taxon 4 is [3, 4, 5, 6, 7]
Next competing generation representing taxon 5 is [3, 4, 5, 6, 7]
Next competing generation representing taxon 6 is [0, 1, 6, 7]
Next competing generation representing taxon 7 is [0, 1, 6, 7]
```

Bibliography

- [1] R. Tuyéras, (2017), *Category theory for genetics*, [arXiv:1708.05255](#)
- [2] R. Tuyéras, (2018), *Category theory for genetics I: mutations and sequence alignments*, [arXiv:1805.07002](#)
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- [4] R. Tuyéras, (2018), *Category theory for genetics III: natural selection, evolution and phylogeny*.