1.40 RELATIONAL MODEL FOR DATA BASES

A relation database is a collection of data belonging to a system spread over a number of relations, i.e., to find relations between discrete objects. To establish such relation, we are giving an introduction to a relational model for database. A data base is a collection of records, which are n-tuples, made up of fields. The fields are the entries of the n-tuples. For example a data base of student records of college may be made of fields containing the, *Name, Identification* number, *Branch* and *Percentage* of the student. The rational data base model invented by E.F. Codd in 1970 consists of n-ary relation (i.e., the table has n-columns). Thus, the student records are represented as 5-tuples.

The following table represents a 5-ary relation.

Name (N)	Age (A)	ID Number (I)	Branch (B)	Percentage% (P)
Nitin	23	11021	Civil	62
Anuj	20	21052	Mechanical	65
Rakesh	21	31061	Electronics	72
Ravi	24	41150	Computer Science	81
Kishore	22	51162	Bio-Tech.	65

The given table may be named 'record' on the sets N, A, I, B, and P.

The sets N, A, I, B and P are called *domains* of the table, the number of sets (i.e, 5 here) is called the *degree* of the table and the column of the table (i.e., here columns of 5-ary) relation are called *attributes*.

Thus above table can be expressed as the set Record of student

= {(Nitin, 23, 11021, Civil, 62), (Anuj, 20, 21052, Mechanical, 65), ..., (Kishore, 22, 51162, Bio-Tech, 65)}.

1.41 *PARTIAL ORDER RELATION OR PARTIALLY ORDERED SETS*

[R.G.P.V. June, 2004]

A relation R on a set P is called *partial order relation* or a *partial ordering* in A, if

(i) R is reflexive : $aRa \quad \forall \; a \in P$

(ii) R is anti-symmetric : aRb and $bRa \Rightarrow a = b, \quad \forall \; a, b \in P$

(iii) R is transitive : aRb and $bRc \Rightarrow aRc, \quad \forall \; a, b, c \in P$

It is denoted by symbol '$\leq$', then ordered pair $(P, \leq)$ is called a *partially ordered set* or a *Po-set*.

For example :

Let R be the set of all real numbers. The relation "less than or equal to" or "greater then or equal to" is a partial order on R.

Example : *If P(S) is the set of all subsets of a given set S, the set inclusion relation ($A \subseteq B$), then ($P(S), \leq$) is a po-set.*

Or

Prove that "being a subset of" is a partial order on the power set of a non-empty set of S.

Solution. Let S be a non-empty set and P(S) = {A, B, C, ...} be the power set of S.

Define the given relation: $A \leq B \Leftrightarrow A \subseteq B$

We shall prove that ($P(S), \leq$) is a po-set.

(i) *Reflexive* : Since $A \subseteq A \Rightarrow A \leq A, \ \forall \ A \in P(S)$

Hence relation '$\leq$' is reflexive.

(ii) *Antisymmetric* : Let $A \leq B$ and $B \leq A$

$\Rightarrow$ $A \subseteq B$ and $B \subseteq A$

$\Rightarrow$ $A = B$.

Hence '$\leq$' is antisymmetric.

(iii) *Transitive* :

Let $A \leq B$ and $B \leq C$ $\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow$ $A \leq B \Rightarrow A \subseteq B$.

Hence '$\leq$' is transitive.

Then ($P(S), \leq$) is a partial order set. **Proved**

Example : *Prove that the relation "x divides y", or y = k x, for some integer k, on the set of positive integer N is a partial order relation.*

Solution Given $x \leq y \Leftrightarrow x$ divides y i.e., $x|y$, for all $x, y \in N$

(i) *Reflexive* : since x divides x i.e., $x|x$, $\forall x \in N$

$\Rightarrow$ relation '|' is reflexive.

(ii) *Anti-Symmetric* : For $x, y \in N$

Let $x \leq y$ and $y \leq x \Rightarrow x|y$ and $y | x$

$\Rightarrow$ $y = k_1 x$ and $x = k_2 y$, for some $k_1, k_2 \in N$

$\Rightarrow$ $x = k_2 (k_1 x)$

$\Rightarrow$ $x = k_1 k_2 x$ ....(1)

$\Rightarrow$ $k_1 k_2 = 1$

$\Rightarrow$ $k_1 = k_2 = 1$

Hence $x \leq y$ and $y \leq x \Rightarrow x = y$ [by 1]

$\therefore$ Relation '|' is antisymmetric.

(iii) Transitive : Let x, y, z $\in$ N, we have

$x \leq y$ and $y \leq z \Rightarrow x|y$ and $y|z$

$\Rightarrow$ $y = k_1 x$ and $z = k_2 y$, for $k_1, k_2 \in N$

$\Rightarrow$ $z = k_2 (k_1 x)$

$\Rightarrow$ $z = k x$, where $k = k_1 k_2 \in N$

$\Rightarrow$ $x|z$

$\Rightarrow$ $x \leq z$

Hence relation '|' is transitive.

Thus (N, |) is a po-set. **Proved.**

Example : *Let I be the set of integers. Define a relation*

$x \leq y \Leftrightarrow y = x^r$, for some positive integer r.

Prove that (Z, ≤) is a partial ordered set. ***[R.G.P.V. Dec. 2003]***

Solution Given $x \leq y \Leftrightarrow y = x^r$, for some $r \in N$ and $x, y \in I$

(i) Reflexive : Since $x = x^1 \Rightarrow x \leq x$ for all $x \in I$

(ii) Antisymmetric :

Let $x \leq y$ and $y \leq x \Rightarrow y = x^r$ and $x = y^s$, for some $r, s \in N$ ...(1)

$\Rightarrow$ $y = (y^s)^r \Rightarrow y = y^{rs}$

$\Rightarrow$ $1 = rs$

$\Rightarrow$ $r = 1$ and $s = 1$. [$\because$ $r, s \in N$]

Hence (1) becomes : $x \leq y$ and $y \leq x \Rightarrow x = y$, for all $x, y \in I$

(iii) Transitive :

Let $x \leq y$ and $y \leq z \Rightarrow y = x^r$ and $z = y^s$, for some $r, s \in N$

$\Rightarrow$ $z = (x^r)^s$ [Remove y]

$\Rightarrow$ $z = x^{rs}$

$\Rightarrow$ $z = x^t$ [$\because$ $rs = t \in N$]

$\Rightarrow$ $x \leq z$.

Thus (Z, ≤) is a po-set. **Proved.**

Example : *If R be a partial order on a set A and let R^{-1} be the inverse relation of R then prove that R^{-1} is also partial order relation.*

Solution : (i) *Reflexivity : Since R is reflexive* $\Rightarrow a R^{-1} a$, $\forall a \in A$

$\Rightarrow$ R^{-1} is reflexive.

(ii) *Anti-symmetry* : Let $a R^{-1} b$ and $b R^{-1} a$

[by definition of R^{-1}]

$\Rightarrow$ $b\,R\,a$ and $a\,R\,b$

Since R is partial order relation $\Rightarrow a = b$

Therefore, R^{-1} is antisymmetric.

(iii) *Transitive* : Let $a\,R^{-1}\,b$ and $b\,R^{-1}\,c \Rightarrow b\,R\,a$ and $c\,R\,b$

$\Rightarrow$ $c\,R\,b$ and $b\,R\,a$

Since R is transitive

$\Rightarrow$ $c\,R\,a$

$\Rightarrow$ $a\,R^{-1}\,c$.

$\Rightarrow$ R^{-1} is transitive.

Therefore R^{-1} is a partial order relation. **Proved.**

Remark :

The poset (A, R^{-1}) is called the *dual of the poset* (A, R) and the partial order relation R^{-1} is called the *dual partial order relation of* R.

For example : Relations '≥' and '≤' both are partial order relations and are *dual of each other*.

1.42 BOOLEAN ALGEBRA

A non-empty set B with two binary operations '+' and '.' with one unitary operation ' is called *Boolean Algebra* [B, +, ., '], if

(i) *Commutative laws :*

$a + b = b + a$, and $\forall a, b \in B$

$a.b = b . a$,

(ii) *Associative laws :*

$a + (b + c) = (a + b) + c$, and

$a. (b.c) = (a. b).c, \ \forall a, b, c, \in B$

(iii) *Identity laws :*

$a + 0 = 0 + a = a$ and $a.1 = 1.a = a, \ \forall a \in B$.

Here '0' is identity for addition and '1' is multiplication identity.

(iv) *Distributive laws :*

$a. (b+c) = a.b + a.c$

$a + (b.c) = (a+b). (a+c), \ \forall a, b, c \in B$

(v) *Complement laws :*

$a + a' = 1$ and $a.a' = 0, \ \forall a \in B$, where a' is complement of a.

emarks :

(i) Idempotent laws : $a+a = a$ and $a.a = a, \forall \in a \in B$

(ii) Involution laws : $(a')' = a, \forall a \in B$

(iii) Absorption laws : $a + (a.b) = a$ and $a. (a+b) = a$, $\forall a \in B$

(iv) DeMorgan's's laws : $(a+b)' = a'.b'$ and $(a.b)' = a' + b'$, $\forall a, b \in B$.

1.43 ORDER RELATION OR INCLUSION RELATION IN BOOLEAN ALGEBRA

Let (B, +, ., ') be Boolean algebra, then order relation define as

for $a, b \in B$ then $a \le b \Leftrightarrow a.\,b' = 0$.

Theorem : *In Boolean algebra (B, +, ., '), prove that the order relation $\le$ is partial order relation.*

Proof. We known that order relation is Boolean algebra B.

$a \le b \Leftrightarrow a.b' = 0, \quad \forall a, b \in B$

(i) *Reflexive* : Since $a.\,a' = 0 \Rightarrow a \le a, \forall a \in B$

Hence $\le$ is reflexive.

(ii) *Antisymmetric* :

Let $a \le b \Rightarrow a.b' = 0$ ......(1)

and $b \le a \Rightarrow b.a' = 0$ ......(2)

Now, we have

$$
\begin{aligned}
a &= a.\,1 \\
&= a.\,(b+b') && [b+b' = 1] \\
&= a.b + a.b' && \text{[By distributive law]} \\
&= a.b + 0 && \text{[using (1)]}
\end{aligned}
$$

Hence $a = a.b$ ....(3)

Also

$$
\begin{aligned}
b &= b.1 \\
&= b.\,(a+a') && [\because a+a^1 = 1] \\
&= b.a + b.a' && \text{[By distributive law]} \\
&= b.a + 0 && \text{[Using (2)]}
\end{aligned}
$$

Hence $b = b.a$ ....(4)

Thus $a \le b$ and $b \le a \Rightarrow a = a.b$ and $b = b.a$

$\Rightarrow a = b.$

Therefore '$\le$' is antisymmetric.

(iii) *Transitive* : Let $a \le b \Rightarrow a.b' = 0$ ....(5)

and $b \le c \Rightarrow b.c' = 0$ ....(6)

We have

$$
\begin{aligned}
a.c' &= (a.\,1).\,c' && [\because x.1 = x] \\
&= [a.\,(b+b')].\,c' && [\because b+b^1 = 1] \\
&= (a.\,b + a.\,b')\,.\,c' && \text{[by distributive law]} \\
&= (a.b + 0).c' && [\because a.b' = 0] \\
&= (a.b).\,c' && [\because x+0 = x]
\end{aligned}
$$

[by associative law]

[$b.c' = 0$]

$= a.(b.c')$

$= a.0$

Hence $a \leq b$ and $b \leq c \Rightarrow a.c' = 0$

$\Rightarrow \quad a \leq c.$

Therefore '≤' is transitive.

Hence the order relation '≤' is a partial order relation.

1.44 COMPARABILITY

The element a and b of a poset $(P, \leq)$ are called *comparable* if either $a \leq b$ or $b \leq a$. Otherwise a and b are called *non-comparable* if neither $a \leq b$ nor $b \leq a$.

For example :

Let N be the set of natural numbers and relation on N is divisibility. Then 24 and 3 are comparable since 3/24 but 3 and 11 are non-comparable, because neither 3/11 nor 11/3.

1.45 TOTALLY ORDERED SET OR LINEARLY ORDERED SET

Let $(P, \leq)$ be a partial order set. If for all $a, b \in P$ we have either $a \leq b$ or $b \leq a$ (i.e. comparable) then $(P, \leq)$ is called a *totally ordered* set or *linearly ordered*.

For example :

D_{27} i.e., divisors of 27 is a totally ordered set.

1.46 CHAINS AND ANTI-CHAINS [R.G.P.V. June, 2003]

Let $(P, \leq)$ be a partially ordered set or poset. A subset of a poset set such that every element of this subset are comparable is called-*chain*.

A subset of a poset is called *antichain* if every two element of this subset are incomparable.

For example :

(i) D_9, i.e., poset $(\{1, 3, 9\}, |)$ is a chain.

(ii) Let N be the set of positive integer define a relation $x \leq y \Leftrightarrow x$ divides y. Then $(N, \leq)$ is a po-set but $(N, \leq)$ is not a totally ordered set. However in N the sets $\{3, 3^2, 3^3, \ldots\}$, $\{4, 4^2, 4^3, \ldots\}$ and so on, are totally ordered sets or chains.

47 HASSE DIAGRAM

Hasse diagram is a diagrammatic representation of a finite partial order on a set. In this diagram, the elements are shown as vertices (or dots).

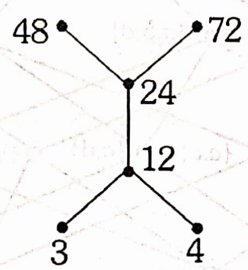
Two related vertices in the Hasse diagram of a partial order are connected by a line if and only if they are related.

Let (P, ≤) be a po-set. An element $b \in P$ is said to be *cover* $a \in P$ if $a < b$ and if there does not exist any element $c \in P$ such that $a \leq c$ and $a \leq b$. If '*b covers a*' then a line is drawn between the elements *a* and b in the Hasse diagram.

For examples :

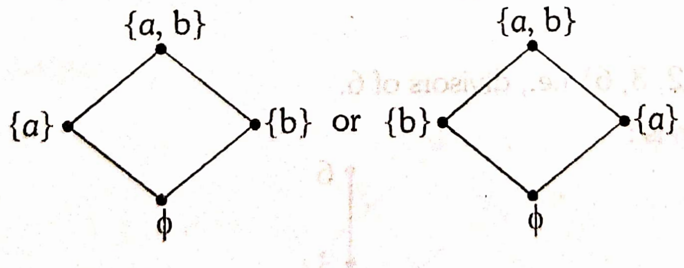
(i) Let $A = \{1, 2, 3\}$, and ≤ be relation 'less than or equal to" on A. Then the Hasse diagram of po-set (A, ≤) is :

(ii) Let A = {3, 4, 12, 24, 48, 72} and the relation ≤ be defined as $a \leq b \Leftrightarrow a$ divides b i.e., $a \mid b$. Then the Hasse diagram of po-set (A, ≤) is :



Example : *If (P, ≤) is a partial ordered set, then prove that the Hasse diagram of (P, ≤) is not unique.*

Solution : Consider S = {*a*, *b*}. The relation of inclusion '⊆' on power set P(S) {ϕ, {*a*}, {*b*}, {*a*,*b*}}*and* then (P(S), ⊆) is a poset. Then the Hasse diagrams of (P (S), ⊆) are given as :

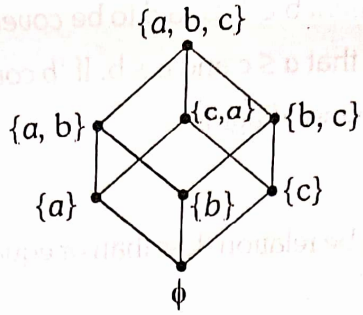


Ans.

Note : Hasse diagram is not unique.

Example : *Let S = {a, b, c} Then P(S) = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a} {a, b, c}}. Consider the poset (P (S), ⊆). Draw Hasse diagram.*

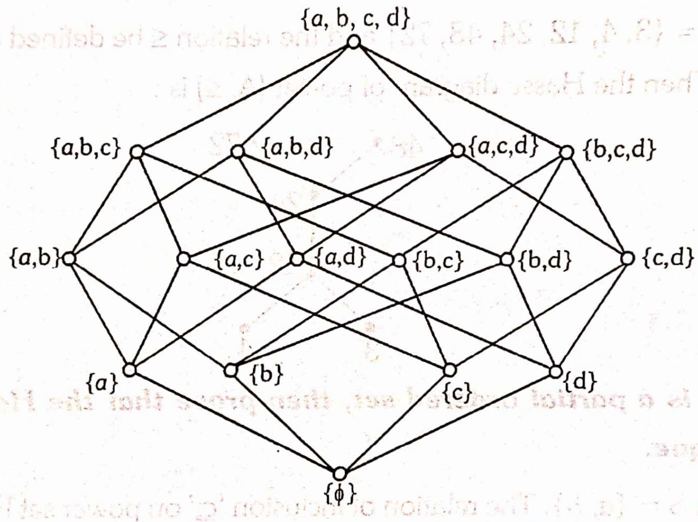
Solution : The Hasse diagram is



Example : Let S = {a, b, c, d} and P(S) be its power set. Let Í be the inclusion relatio P(S). Draw Hasse diagram. **[RGPV June**

Solution : P(S) = {ϕ {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d} {c, d}, {a, b, c}, {a, b, d}, {b, c, d} {a, c, d}, {a b, c, d}}

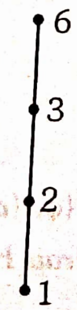
Then (P (S) ≤) is a po-set. The Hasse diagram is :



Example : If N be a positive integers and D_N denote the set of all divisors of N. Conside partial order 'divides' in D_N. Then draw the Hasse diagrams for D_6, D_{24}, D_{30} and D_{18}.

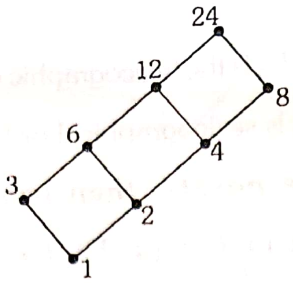
Solution. (i) $D_6 = \{1, 2, 3, 6\}$ i.e., divisors of 6.

Hasse diagram is :



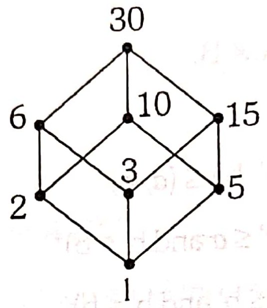
(ii) $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

The Hasse diagram is :



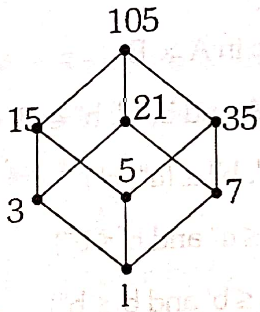
(iii) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

The Hasse diagram is :



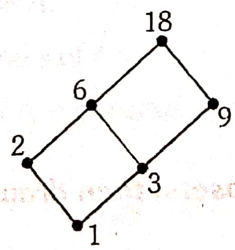
(iv) $D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$

The Hasse diagram is :



(v) $D_{18} = \{1, 2, 3, 6, 9, 18\}$

The Hasse diagram is :



1.48 LEXICOGRAPHICAL ORDER

Let $(A, \leq_1)$ and $(B, \leq_2)$ are posets, then the Cartesian product $A \times B$ defined by

$\{a, b) < (a', b')$ if either $a \leq_1 a'$ or if both $a = a'$ and $b \leq_2 b'$ where $a, a' \in A$ and $b, b' \in B$, is called *lexicographical* order.

For example :

Let $(Z \times Z, \leq)$ be a poset, and $\leq$ is the lexicographic ordering relation on Z, Then $Z \times Z$ defined by $(1, 2, 3, 6) < (1, 2, 5, 4)$ is sexicographical order, because $1 = 1, 2 = 2$, and $3 < 5$.

Example : ***If $(A, \leq)$ and $(B, \leq)$ are posets, then prove that $(A \times B, \leq)$ is a poset, with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.***

Solution. Let $a, a' \in A$ and $b, b' \in B$, then $(a, b), (a', b') \in A \times B$.

(i) *Reflexive :* Since $a \leq a$ and $b \leq b$, $\forall\ a \in A, b \in B$

$\Rightarrow (a, b) \leq (a, b)$

Hence, $\leq$ is reflexive in $A \times B$.

(ii) *Antisymmetric :*

Let $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$

$\Rightarrow (a \leq a'$ and $b \leq b')$ and $(a' \leq a$ and $b' \leq b)$

$\Rightarrow (a \leq a'$ and $a' \leq a)$ and $(b \leq b'$ and $b' \leq b)$ [Since A and B are poset

$\Rightarrow a = a'$ and $b = b'$

$\Rightarrow (a, b) = (a', b')$.

Hence $\leq$ is antisymmetric in $A \times B$.

(iii) *Transitive :* For $a, a' a'' \in A$ and $b, b', b'' \in B$.

Let $(a, b) \leq (a', b')$ and $(a', b') \leq (a'', b'')$.

$\Rightarrow (a \leq a'$ and $b \leq b')$ and $(a' \leq a''$ and $b' \leq b'')$

$\Rightarrow (a \leq a'$ and $a' \leq a'')$ and $(b \leq b'$ and $b' \leq b'')$

$\Rightarrow a \leq a''$ and $b \leq b''$ [Since A and B are transitiv

$\Rightarrow (a, b) \leq (a'', b'')$.

Hence, $\leq$ is transitive.

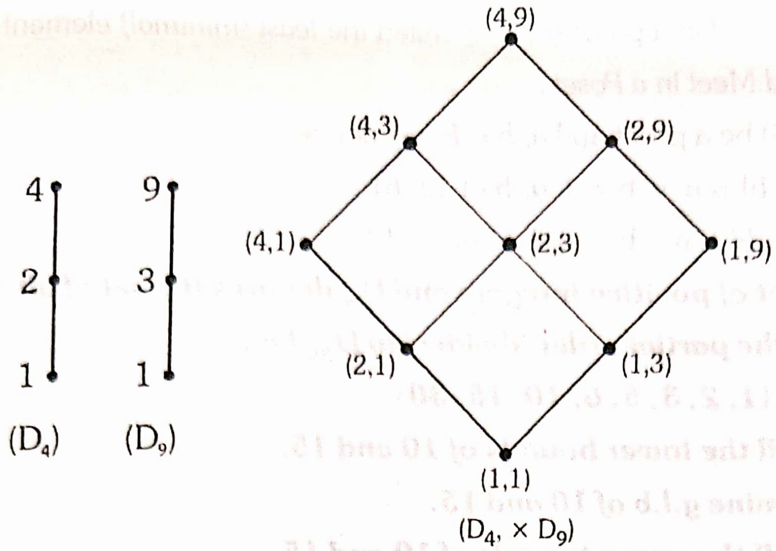
Thus, $(A \times B, \leq)$ is a poset. **Proved**

xample : ***Let $(D_4, \leq)$ and $(D_9 \leq)$ be posets, then draw Hasse diagram for $D_4 \times D_9$, under the product partial order.***

lution The Hasse diagram of poset $(D_4 \times D_9, \leq)$ is as follows ;

$D_4 = \{1, 2, 4\}$ and $D_9 = \{1, 3, 9\}$

Then $D_4 \times D_9 = \{(1,1), (1,3), (1, 9), (2,1), (2,3), (2,9), (4,1), (4,3)\ (4,9)\}$



1.49 BOUNDS OF ELEMENTS OF A SET

1. **Upper Bound :** Let $A \subseteq S$, then an element $s \in S$ is called *upper bound* of *A*, if and only if $a \leq s$, $\forall\ a \in S$.
2. **Lower Bound :** Let $A \subseteq S$, then an element $s \in S$ is called the *lower bound* of *A*, if and only if $s \leq a$, $\forall\ a \in A$.
3. **Least Upper Bound :** If $A \subseteq S$ and $g \in S$, then g is called the *least upper bound* of *A* if and only if the following two conditions hold :

   (a) g is upper bound of the set *A*.

   (b) $g \leq s$ for every upper bound s of *A*.

   The least upper bound of *A* is denoted by *l.u.b.A or sup A.*
4. **Greatest Lower Bound :** If $A \subseteq S$, and $l \in S$, then l is called the *greatest lower bound* of *A* if and only if the following two conditions hold :

   (a) l is lower bound of the set *A*.

   (b) $s \leq l$ for every lower bound s of *A*.

   The greatest lower bound of *A* is denoted by *g.l.b. A or inf A.*

Remarks :

(i) In a Boolean algebra [B, +, . , '], then

$$a + b = l.\,u.\,b\ \{a, b\} \text{ and}$$
$$a \,.\, b = g.l.b\ \{a, b\}, \text{ for all } a, b \in B.$$

(ii) *Greatest element :*

An element m in a poset $(P, \leq)$ is called the *greatest (maximal)* element, if for all $a \in P$, $a \leq m$

(iii) *Least element :*

An element l in a poset $(P, \leq)$ is called the *least (minimal)* element, if for all $a \in P, l \leq a$

(iv) *Join and Meet in a Poset :*

Let $(P, \leq)$ be a poset and $a, b \in P$, we define

Join (a, b) is $a \vee b = l.\ u.\ b$ of $\{a, b\}$

Meet (a, b) is $a \wedge b = g.\ l.\ b.$ of $\{a, b\}$.

Example : ***If N be a set of positive integers and D_N denotes the set of all divisors of N. Consider the partial order 'divides' in D_{30} i.e.,***

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

(i) Find all the lower bounds of 10 and 15.

(ii) Determine g.l.b of 10 and 15.

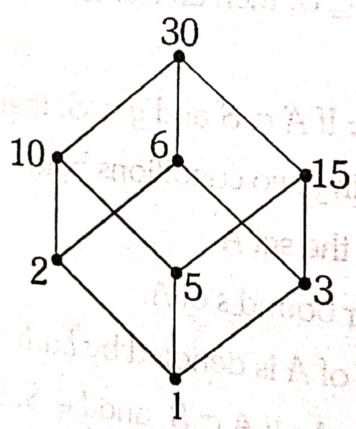
(iii) Find all the upper bounds of 10 and 15.

(iv) Determine l.u.b. of 10 and 15.

(v) Find greatest element of D_{30}.

(vi) Find least element of D_{30}.

Solution Since D_{30} is a poset, then Hasse diagram is



(i) The lower bounds of 10 are 1, 2, 5, 10 and lower bounds of 15 are 1, 3, 5, 15.
Hence lower bounds of 10 and 15 are 1 and 5 [i.e., common divisors of 10 and 15]

(ii) The greatest lower bound (*g. l. b.*) of 10 and 15 is $5 \in D_{30}$.

(iii) The upper bounds of 10 are 10, 30 and upper bounds of 15 are 15, 30.
Hence upper bounds of 10 and 15 is 30.

(iv) The *l. u. b.* of 10 and 15 is $30 \in D_{30}$.

(v) The greatest element of D_{30} is 30.

(vi) The least element of D_{30} is 1.

Example : ***Draw the Hasse diagram for $(P, \leq)$, where $P = \{2, 3, 6, 12, 24, 36\}$ and $x \leq y$ if and only if i.e., x divides y'.***

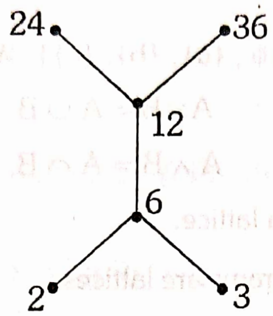
(i) The l. u. b. and the g.l.b. of $A = \{2, 3, 6\}$

(ii) The l. u. b. and the g. l. b. of $B = \{2, 3\}$

(iii) The l. u. b. and the g. l. b. of $C = \{6, 12\}$.

Solution.

Draw the Hass diagram :

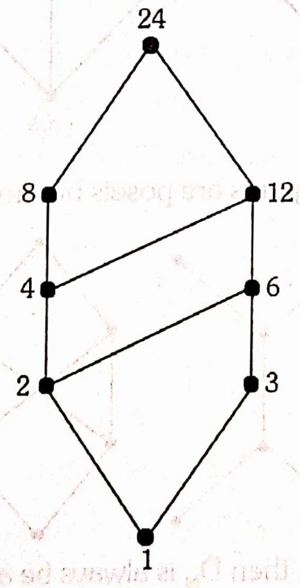


(i) *l. u. b.* of $A = 6$; *and g. l. b* of *A* does not exist.

(ii) *l. u. b.* of $B = 6$; but there is no *g. l. b.*

(iii) *l. u. b* of $C = 12$; *and g. l. b* of $C = 6$.

Example : ***In the Hasse diagram***



(i) Find all lower bounds 8 and 12.

(ii) Find all upper bounds 8 and 12.

(iii) Find the g. l. b. of 8 and 12.

(iv) Find the l. u. b of 8 and 12.

Solution : (i) Lower bounds of 8 and 12 are 1, 2, 4.

(ii) Upper bound of 8 and 12 is 24.

(iii) *g. l. b.* of 8 and 12 is 4.

(iv) *l. u. b.* of 8 and 12 is 24.

[R.G.P.V. June, 2003]

1.50 LATTICES

Lattice is a particular type of poset. A *lattice* is a poset $(L, \leq)$ in which every pair of element has *l. u. b.* and *g. l. b.*

Let $a, b \in L$, then
$$a \vee b = l.\ u.\ b\ \{a, b\}$$
$$a \wedge b = g.\ l.\ b\ \{a, b\}.$$

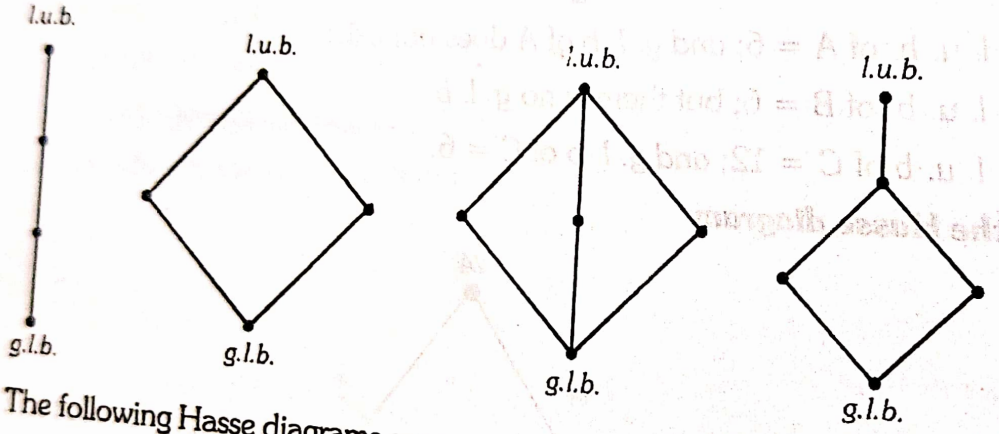
It is denoted by $(L, \leq, \wedge, \vee)$.

For examples :

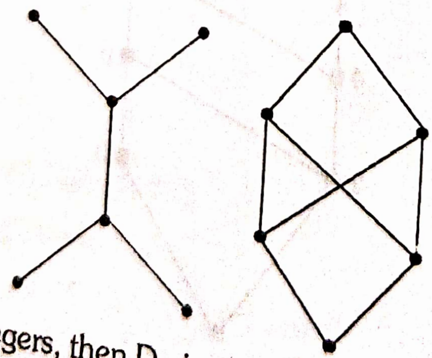
(i) $S = \{a, b\}$ and $P(S) = \{\phi, \{a\}, \{b\}, \{c\}\}$. Where join and meet defined as
$$A \vee B = A \cup B.$$
$$A \wedge B = A \cap B.$$

Then $(P(S), \subseteq, \cup, \cap)$ is a lattice.

(ii) The following Hasse diagrams are lattices :



(iii) The following Hasse diagrams are posets but not lattices :



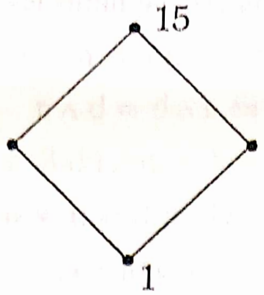
(iv) If N is a positive integers, then D_N is always be a lattice under the relation divisibility.

ırk :

Other notations of meet are *or. or $\cap$ and join are , $\oplus$, or + or $\cup$.

ıle : *Let $D_{15} = \{1, 3, 5, 15\}$. Prove that partially order set D_{15} under the relation 'divides' is a lattice.*

n The Hasse diagram is



Here *g. l. b* of D_5 is 1

and *l. u. b.* of D_5 is 15

Hence $(D_{15}, |)$ is a lattice. **Ans.**

Example : *Determine whether or not each of the following sets is a lattice with respect to divisibility :*

$A = \{2, 3, 4, 12\}$ and $B = \{1, 2, 3, 9, 18\}$.

Solution (i) $(A, |)$ is not lattice, because every pair of element does not Contain *g. l. b*, i.e., the *g. l. b* of 2 and 3 is $1 \notin A$.

(ii) $(B, |)$ is also not a lattice, because every pair of element does not contain *l. u. b.* i.e., the *l. u. b.* of 2 and 3 is $6 \notin B$.

Remarks :

(i) *Let $S = \{a, b, c\}$ and $P(S)$ be power set of S, then $(P(S), \subseteq)$ is a lattice.*

(ii) *Every linearly ordered or chain is a lattice.*

Theorem : Let $(L, \leq)$ be a lattice and for $a, b \in L$ then prove that :

(i) *Idempotent laws :*

(a) $a \vee a = a$

(a') $a \wedge a = a$

(ii) *Commutative laws :*

(b) $a \vee b = b \vee a$

(b') $a \wedge b = b \wedge a$

(iii) *Associative laws :*

(c) $a \vee (b \vee c) = (a \vee b) \vee c$

(c') $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(iv) *Absorption laws :*

(d) $a \vee (a \wedge b) = a$

(d') $a \wedge (a \vee b) = a.$

Proof. (i) Since set L has one element a, then *g. l. b.* of L = *l. u. b* of $L = a \in L$.

Hence $a \vee a = a$ and $a \wedge a = a$.

(ii) Since $g.\ l.\ b$ and $l.\ u.\ b$ of pair of elements is a lattice L must be unique.

Hence $g.l.b$ of a & b = $g.l.b.$ of b & a

$$\Rightarrow a \wedge b = b \wedge a$$

Similarly $l.\ u.\ b.\ a$ & b = $l.\ u.\ b.$ of b & a

$$\Rightarrow a \vee b = b \vee a.$$

(iii) Let $x = a \vee (b \vee c)$ and $y = (a \vee b) \vee c$.

We have $x = a \vee (b \vee c) \Rightarrow a \leq x, (b \vee c) \leq x$

$$\Rightarrow a \leq x, b \leq x, c \leq x$$

$$\Rightarrow a \vee b \leq x, \quad c \leq x \qquad [a \leq x, b \leq x \Rightarrow a \vee$$

$$\Rightarrow (a \vee b) \vee c \leq a$$

$$\Rightarrow y \leq x.$$

Similarly, we can easily shown that $x \leq y$.

From (1) and (2) $s \Rightarrow x = y$.

$$\Rightarrow a \vee (b \vee c) = (a \vee b) \vee c.$$

Similarly we can proved (c') of Associative law.

(iv) Let $a \in L$. We have

$a \leq a$ and $a \leq a \vee b \Rightarrow a \leq a \wedge (a \vee b)$

By definition of '$\vee$', $a \leq a \vee (a \wedge b)$.

Using antisymmetric property of '$\leq$', we get $a \vee (a \wedge b) = a$

Similarly we can proved (d') of absorption law.

Theorem : Let (L, ≤) be a lattice, for a, b ∈ L, then prove that

(i) $a \leq b \Leftrightarrow a \wedge b = a$ ***(ii)*** $a \leq b \Leftrightarrow a \vee b = b$

Solution Let $a, b \in L$ and we assume $a \leq b$, and since $a \leq a$

$\Rightarrow a$ is lower bound of a and b

$$\Rightarrow a \leq a \wedge b.$$

Also by definition of $a \wedge b$, we have $a \wedge b \leq a$.

From (1) and (2), we get $a \wedge b = a$

Conversely, if we assume $a \wedge b = a$.

Then by definition of $a \wedge b$, $a \wedge b \leq b$

$$\Rightarrow a \leq b. \qquad [\because a \wedge b$$

In a similar manner, it can also be shown that

$$a \leq b \Leftrightarrow a \vee b = b.$$

Remark : *Dual of a Lattice :*

let (L, ≤) be a poset and let (L, ≥) be its dual. If (L, ≤) is a lattice, then (L, ≥) is also a lat

For example:

If $(P(S), \subseteq, \cap, \cup)$ is a lattice, then its dual i.e., $(P(S), \supseteq, \cap, \cup)$ is also a lattice.

1.51 SUBLATTICE

Let $(L, \leq)$ be a lattice and consider a non-empty subset M of L. Then $(M, \leq)$ is a sublattice of $(L, \leq)$, if $(M, \leq)$ it self is a lattice with respect to the operation of L.

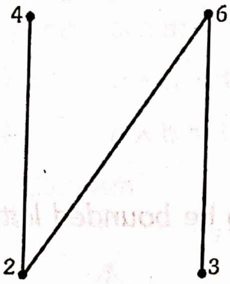
i.e., $a \wedge b \in M$ and $a \vee b \in M$, $\forall\, a, b \in M$.

For examples:

(i) Let $(N, |)$ is a lattice and D_N of all divisors of N is a sublattice of the lattice N under the relation divisibility.

(ii) Let $(N, |)$ be a lattice and $S = \{2, 3, 4, 6\}$ under the same relation is not a sublattice, because *l. u. b* of 4 and 6 is $12 \notin S$.

i.e., By Hasse diagram :



Example : *Prove that intersection of two sub-lattices is a sub-lattice.*

Solution Let $(L, \leq)$ be a lattice.

Suppose $(L_1, \leq)$ and $(L_2, \leq)$ are sublattice of L.

Then we shall prove that $L_1 \cap L_2$ is a sub-lattice of L.

Let $a, b \in L_1 \wedge L_2 \Rightarrow a, b \in L_1$ and $a, b \in L_2$

Since L_1 and L_2 are sub-lattices.

$\Rightarrow$ $(a \wedge b \in L_1$ and $a \vee b \in L_1)$ and $(a \wedge b \in L_2$ and $a \vee b \in L_2)$.

$\Rightarrow$ $(a \wedge b \in L_1$ and $a \wedge b \in L_2)$ and $(a \vee b \in L_1$ and $a \vee b \in L_2)$.

$\Rightarrow$ $a \wedge b \in L_1 \wedge L_2$ and $a \vee b \in L_1 \wedge L_2$.

Thus $a, b \in L_1 \wedge L_2 \Rightarrow a \wedge b \in L_1 \wedge L_2$ and $a \vee b \in L_1 \wedge L_2$.

Hence $L_1 \wedge L_2$ is a sublattice. **Proved.**

Example : *Show with an example that the union of two sub-lattices may not be a sub-lattice.*

Solution. Let $L = (D_{12}, |)$ i.e., $L = \{1, 2, 3, 4, 6, 12\}$ under divisibility of 12 be a lattice.

Let $L_1 = \{1, 2\}$ and $L_2 = \{1, 3\}$ be sub-latices of L.

Then $L_1 \cup L_2 = \{1, 2, 3\}$ is not a sub-lattice, because

l. u. b. of 2 and 3 is 6 i.e., $2 \vee 3 = 6 \notin L_1 \cup L_2$. **Proved.**

1.52 COMPLETE LATTICE

Let L be a lattice, then L is said to be *complete* lattice if its non-empty subsets of L possesses a *l.u.b.* and *g.l.b.*

For example :

Let S = {*a*, *b*, *c*} be a set. and P(S) be a power set, then (P(S), ⊆, ∩, ∪) be a *complete lattice*, because every non empty subset of P(S) has a *g.l.b* and *l.u.b.*

Remarks :

(i) Every finite lattice must be complete lattice which must have a least element 0 and greatest element 1.

(ii) Element 0 and 1 are called *universal lower bound* and *universal upper bound,* respectively.

For example :

If (L, ≤, ∧, ∨) be the lattice and L = $\{a_1, a_2, a_3, \ldots a_n\}$, then

$$\overset{n}{\underset{i=i}{\wedge}} a_i = 0 \quad \text{and} \quad \overset{n}{\underset{i=i}{\vee}} a_i = 1.$$

1.53 BOUNDED LATTICE

A lattice (L, ≤, ∧, ∨) is said to be *bounded lattice* if there exist 0 and 1 in L such that for all $a \in L$, $0 \le a \le 1$.

It is denoted by (L, ≤, ∧, ∨, 0, 1) and

l. u. b. of L is $\overset{n}{\underset{i=i}{\vee}} a_i = 1$

and *g. l. b.* of L is $\overset{n}{\underset{i=i}{\wedge}} a_i = 0$.

For examples :

(i) S = {*a*, *b*, *c*} and P(S) be a power set of S, then the lattice (P(S) ⊆, ∩, ∪) is a bounded lattice, because its least element is ϕ and its greatest element is {*a*, *b*, *c*}.

(ii) Lattice $\{D_N, \le)$ is a bounded lattice, because its least element is 1 and its greatest element is N.

Remarks :

(i) $a \vee 0 = a$, $a \wedge 0 = 0$, $a \wedge 1 = a \vee 1 = 1$.

i.e., 0 and 1 are the identity of the ∨ and ∧ respectively.

(ii) Every finite lattice (L, ≤ ∨, ∧) is bounded.

1.54 COMPLEMENTED LATTICE

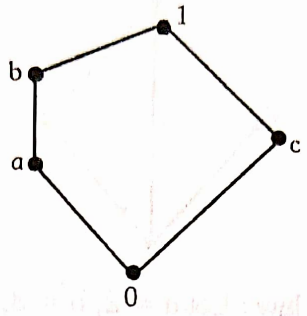
A Lattice (L, ≤, ∧, ∨, 0,1) is called complemented lattice, if every element of L has at least one complement.

i.e., for $a, b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$, then b is known as *complement* of a or a is the complement of b.

For examples :

(i) If $S = \{1, 2\}$ and P(S) is power set, then $(P(S), \subseteq, \cap, \cup)$ is the complemented lattice.

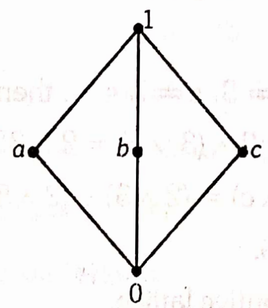
(ii) Consider the following Hasse diagram :



Here the element c has two complements a and b, because

$$c \vee a = 1, \qquad c \wedge a = 0$$
$$c \vee b = 1, \qquad c \wedge b = 0.$$

(iii) Consider the following Hasse diagram :



Here the complement of b are a and c, because

$$b \vee a = 1, \qquad b \wedge a = 0$$
$$b \vee c = 1, \qquad b \wedge c = 0.$$

Remarks :

(i) For the complemented lattice $(L, \leq, \wedge, \vee)$, the complement of any element of L need not be unique.

(ii) Two bounded lattices A and B are complemented if and only if $A \times B$ is also complemented.

(iii) Dual of a complemented lattice is complemented.

1.55 DISTRIBUTIVE LATTICE

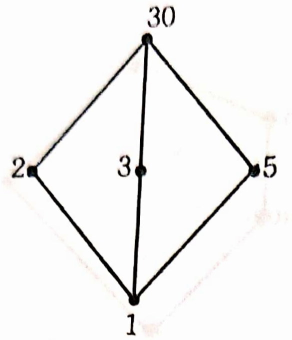
A lattice $(L, \leq, \wedge, \vee)$ is said to be *distributive lattice,* if for any $a, b, c \in L$ it satisfies the following distributive properties :

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

For examples :

(i) Let poset $L = \{1, 2, 3, 5, 30\}$ under the relation, 'divides', then lattice $(L, \leq, \wedge, \vee)$ is n
a distributive lattice. The Hasse diagram is



For satisfies distributive law : Let $a = 2, b = 3, c = 30 \in L$

$\therefore$ L. H.S. $= a \wedge (b \vee c) = 2 \wedge (3 \vee 30)$

$= 2 \wedge (30)$ $\quad$ [$3 \vee 30$ = L.C.M of 3 and 30 = 3

$= 2$

R. H.S. $= (a \wedge b) \vee (a \wedge c) = (2 \wedge 3) \vee (2 \wedge 30)$

$= 1 \vee 2 = 2.$

Thus L.H.S. = R.H.S.

Also if we take $a = 2, b = 3, c = 5 \in L$, then

L.H.S. $= a \wedge (b \vee c) = 2 \wedge (3 \vee 5) = 2 \wedge 30 = 2$

R.H.S. $= (a \wedge b) \vee (a \wedge c) = (2 \wedge 3) \vee (2 \wedge 5) = 1 \vee 1 = 1$

Hence L.H.S. $\neq$ R.H.S.

$\therefore$ $(L, \leq, \wedge, \vee)$ is not a distributive lattice.

(ii) Let S be any finite set and P(S) is the power set of S.

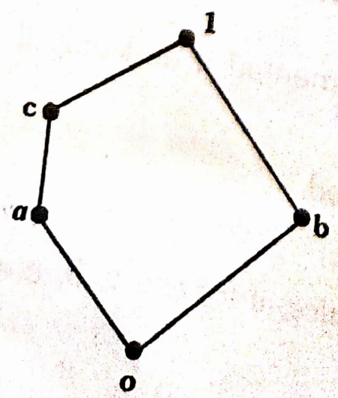
Then $(P(S), \subseteq, \cap, \cup)$ is the distributive lattice, because

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

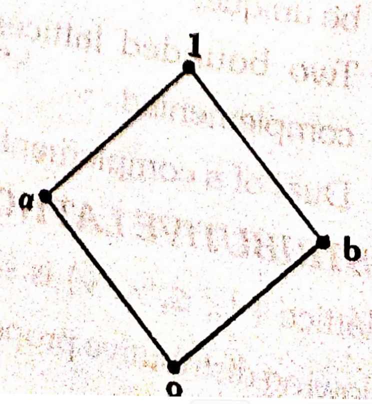
and $$A \cup (B \cap C) = (A \cup B) \cap (A \cap C), \text{ for all } A, B, C \in P(S).$$

mple : Check whether the lattices given by Hasse diagrams are distributed ?

(i)



(ii)



Solution (i) By distributive law

L.H.S. $= a \wedge (b \vee c) = a \wedge 1 = a$ $[\because b \vee c = 1]$

R.H.S. $= (a \wedge b) \vee (a \wedge c) = 0 \vee b = b$ $[\because a \vee b = a$ and $a \wedge c = b]$

Hence $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

Thus (i) is non-distributive lattice and (ii) is distributive lattice. **Ans.**

Example : A lattice L is distributive then prove that

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a), \ \forall \ a, b, c \in L.$$

[R.G.P.V. June, 2005]

Solution Let L be a distributive lattice. Then,

L.H.S. $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= [a \wedge ((b \vee c) \wedge (c \vee a))] \vee [b \wedge [(b \vee c) \wedge (c \vee a))]$

$= [(a \wedge (c \vee a)) \wedge (b \vee c)] \vee [(b \wedge (b \wedge c)] \wedge (c \vee a)]$ [By associativity]

$= [a \wedge (b \vee c)] \vee (b \wedge (c \vee a)]$ [By Idempotent law]

$= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \vee (b \wedge a)$

$= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) =$ R.H.S. **Proved.**

Example : Let D(6) be the set of elements which are divisors of 6, then determine :

(i) [0 (6), |] is a lattice, where | denotes divides.

(ii) [D(6), |] is a complemented lattice.

(iii) [D(6), |] is a distributive lattice.

(iv) [D(6), |] is a chain,

(v) [D(6), |] is a bounded lattice. **[R.G.P.V. June, 2002]**

Solution : Let D(6) = {1, 2, 3, 6}.

(i) Now we show that A = [D(6), |] is a lattice.

Reflexive : Since $a = a$, $\forall a \in A. \Rightarrow a/a$, $\forall a \in A$.

Antisymmetric : Let a, b, $\in$ A, we have

a/b and $b/a \Rightarrow b = k_1 a$ and $a = k_2 b$, $k_1, k_2 \in N$

$\Rightarrow \quad b = k_1 k_2 b$

$\Rightarrow \quad k_1 k_2 = 1$

$\Rightarrow \quad k_1 = k_2 = 1.$

Transitivity : Let a, b, c, $\in$ A, we have

$a/b \Rightarrow b = k_1 a$ and $c = k_2 b$, where $k_1, k_2 \in N$.

$\Rightarrow$ $c = k_1 k_2 a$

$\Rightarrow$ $c = ka$, where $k = k_1 k_2 \in N$

$\Rightarrow$ a/c

Also $a \wedge b = \inf. \{a, b\}$ and $a \vee b = \sup. \{a, b\} \in A$.

Consider $3 \wedge 6 = \inf. \{3, 6\} = 3 \in A$.

$3 \vee 6 = \sup. \{3, 6\} = 6 \in A$.

Thus $A = [D(6), |]$ is a lattice.

(ii) For $a \in A$, there exist $a' \in A$ such that

$a \vee a' = 1$, and $a \wedge a' = 0 \notin A$.

Hence A is not complemented lattice.

(iii) *Distributive lattice :*

For distributive lattice D(6) satisfies following laws :

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(ii) $a \wedge (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Let $a = 1, b = 3, c = 6 \in A$, then

$a \wedge (b \vee c) = 1 \wedge (3 \vee 6) = 1 \wedge \sup \{3, 6\} = 1 \wedge 6 = 1,$

and $(a \wedge b) \vee (a \wedge c) = (1 \wedge 3) \vee (1 \wedge 6) = 1 \vee 1 = \sup \{1, 1\} = 1$.

i.e., $1 \wedge (3 \vee 6) = (1 \wedge 3) \vee (1 \vee 6)$

by duality $1 \vee (3 \wedge 6) = (1 \vee 3) \wedge (3 \vee 6)$.

Hence $A = [D(6), |]$ is a distributive lattice.

(iv) *Chain* : $D(6) = \{1, 2, 3, 6\}$

Since not every pair of element is comparable i.e.,

$2 + 3$ or 2 dos not divide 3.

Hence $A = [D(6), |]$ is not a Chain.

(v) *Bounded Lattice;*

Since $A = [D(6), |]$ has a least element 1 and greatest element 6, so that A is bounded lattice

Example : ***Let A be a distributive lattice, for a, b, $\in$ A, prove that $a \vee (\alpha \wedge b) = a$ and $a \wedge (a \vee b) = a$.*** **[R.G.P.V. Dec, 2002]**

Solutio : Let $a, b \in A$ be such that $a \le b$.

$a \wedge b = g.l.b \{a, b\} = a$ and $a \vee b = l.u.b \{a, b\} = b$.

We have $a \vee (a \wedge b) = l.u.b. \{a, a \wedge b\}$

$= l.u.b. (a, a)$ $[\because a \wedge b = a]$

$= l.u.b \{a,\} = a$.

Also, $a \wedge (a \vee b) = g.l.b. \{a, a \vee b\}$

$= g.l.b \{a, b\}$ $[\because a \vee b = b]$

$= a$.

Proved

Example : Prove that is a distributive lattice, if an element has a complement, then this complement is unique. **[R.G.P.V. Dec., 2002]**

Solution : Let $(L, \leq, \wedge, \vee)$ be a bounded distributive lattice.

Let b and c be two complement of $a \in L$, then

$a \vee b = 1, a \wedge b = 0.$

$a \vee c = 1, a \wedge c = 0.$

We have, $b = b \wedge 1 = b \wedge (a \vee c)$ $[\because a \vee c = 1]$

$= (b \wedge a) \vee (b \wedge c)$ [$\because$ L is distributive lattice]

$= 0 \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c)$ $[\because a \wedge c = 0]$

$= (a \vee b) \wedge c$ [L is distributive lattice]

$= 1 \wedge c$ $[\because a \vee b = 1]$

$= c$

Hence complement of $a \in L$ is unique. **Proved.**

Example : ***If L and M are two distributive lattices then prove that L × M is also distributive lattice.***

Solution Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L \times M$, where $a_1, a_2, a_3 \in L$ and $b_1, b_2, b_3 \in M$.

For distributive law

$$\begin{aligned}
\text{L.H.S.} &= (a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] \\
&= (a_1, b_1) \wedge [(a_2 \vee a_3, b_2 \vee b_3)] \\
&= (a_1, b_1) \wedge [(a_2 \vee a_3), (b_2 \vee b_3)] \\
&= [(a_1 \wedge (a_2 \vee a_3), b_1 \wedge (b_2 \vee b_3)] \\
&= [(a_1 \wedge a_2) \vee (a_1 \wedge a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)] \\
&= (a_1 \wedge a_2, b_1 \wedge b_2) \vee (a_1 \wedge a_3, b_1 \wedge b_3) \\
&= [(a_1, b_1) \wedge (a_2, b_2)] \vee [(a_1, b_1) \wedge (a_3, b_3)] = \text{R.H.S.}
\end{aligned}$$

Similarly by duality principle, we get

$$(a_1, b_1) \vee [(a_2, b_2) \wedge (a_3, b_3) = [(a_1, b_1) \vee (a_2, b_2)] \wedge [(a_1, b_1) \vee (a_3, b_3)]$$

Hence, L × M is distributive lattice. **Proved.**

1.56 A JOB-SCHEDULING PROBLEM

We consider the problem of scheduling the execution of a set of tasks on a multiprocessor computing system which has a set of identical processors.

i.e., Job-scheduling problem means to make a schedule for a finite number of workers to complete a given set of tasks.

Let $T = \{T_1, T_2, ..., T_m\}$ denote a set of tasks to be executed on the computing system. Suppose that the execution of a task occupies one and only one processor. Moreover, since the processors are identical, a task can be executed on any one of the processors. Let $\mu(T_i)$ denote the execution time of task T_i, i.e., the amount of time it takes to execute T_i on a processor. There is also a partial ordering relation $\leq$ specified over T such that for $T_i \neq T_j$, T_i