UNIT - 2 LOGIC AND ALGEBRA OF PROPOSITIONS

2.0 *INTRODUCTION*

One of the main aims of mathematical logic or propositional logic is to provide rules. The rules of logic give precise meaning to mathematical statements and distinguish between valid and invalid mathematical arguments. In addition, logic has numerous applications in computer science. These rules are used in the design of computer circuits, construction of computer programs, verification of the correctness of programs, and in many other ways. The kind of Mathematical logic which we shall use here is bi-valued i.e., 'True' or 'False'.

2.1 *PROPOSITIONS OR STATEMENTS*

The Proposition or statement is a declarative sentence which is either 'Universally True' or 'Universally False' but not both. It is denoted by p, q, r etc.

The value of a proposition is true is denoted by 'T' or '1' and false is denoted by 'F' or '0'.

For examples :

The following are *propositions* or *statements* :

Declarative sentence	***Truth Value***
(i) $4+2=6$	T
(ii) 24 is an even integer and 25 is not	T
(iii) 5 is a prime number	T
(iv) New Delhi is the capital of India	T
(v) $4 \in \{1, 3, 5, 7\}$	F
(vi) $142 \geq 151$	F
(vii) Bhopal is the capital of M.P.	T

The following are *not propositions* or *statements* :

(i) where are you going	Interrogative
(ii) $x+3=6$	Depends on x
(iii) $x+y>0$	Depends on x and y.
(iv) please wait	Request

Remarks :

Statements are usually denoted by the letters : p, q, r or P, Q, R,.........

For example :

P = New Delhi is the capital of India.

2.2 LOGICAL OPERATORS OF LOGICAL CONNECTIVE

The are several ways in which we commonly combine simple statement into compound or The words *or* and, not, if.... then and if and only if we can be added to one or more proposi to create a new proposition. Such new propositions are called *compound propositions*.

Connective English word	Name of Connective	Symbol
Not	Negation or Denial	~ or ¬ or – or '
And	Conjunction	∧
Or	Disjunction	∨
If.......Then	Implies (Conditional)	→ or ⇒
If and only if	Bi-Conditional	↔ or ⇔

Remarks :

(i) Proposition such as P, Q, R........or p, q, r.........etc. are known as *logical variables*.

For example :

p : Bhopal is capital of M.P.

(ii) Above propositions are known as *simple propositions* and also ca *atomic propositions*.

(iii) Compound propositions is known as *molecular propositions*.

For example :

p : The sun is not shining and it is raining today.

2.3 TRUTH TABLE

The truth table of a logical operator specifies how the truth value of a proposition using operator is determined by the truth values of the propositions. A truth table lists all pos combination of truth values of the propositions in the left most columns and the truth v of the resulting propositions in the right most column.

Hence, truth table is a special technique to define the operational behaviour of lo connectives. We will discuss in detail about the following logical connectives with the of truth table.

(1) Negation or NOT :

If *p* is a proposition, then its negation *p* i.e. ~p is another proposition called the *negation*

Truth Table for Negation :

p.	~ p
T	F
F	T

Remarks :

$\sim \sim p = p$.

Example : ***Find the negation of the following propositions :***

(i) It is cold

(ii) To day is Monday

(iii) Raju is poor.

Solution The negation of the propositions are :

(i) It is not cold.

(ii) To day is not Monday.

(iii) Raju is not poor. **Ans.**

(2) Conjunction (AND) :

If p and q are propositions, then the propositions "p and q", denoted by $p \wedge q$, is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the *conjunction* of p and q.

Truth Table for Conjunction :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For example :

p : To day is Sunday

q : It is holiday

Then $p \wedge q$: To day is Sunday and it is holiday.

Example : ***Let p be "Raju is rich" and let q be "Raju is happy". Write each of the following in symbolic from.***

(i) Raju is poor but happy.

(ii) Raju is neither rich nor happy.

(iii) Raju is rich and unhappy.

Solution : Given p : Raju is rich and q : Raju is happy, then

(i) $\sim p \wedge q$

(ii) $\sim p \wedge \sim q$

(iii) $p \wedge \sim q$. **Ans.**

Remark :

$p \wedge p \equiv p$.

(3) Disjunction (OR) :

If p and q are propositions, then disjunction p or q, denoted $p \vee q$, is false when p and q are both false and true otherwise. The proposition "$p \vee q$" is called the disjunction of p and

Truth Table for Disjunction :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

For example :

p : Computer science students read C++.

q : Computer science students read discrete structures.

Then $p \vee q$: Computer science students read C++ or discrete structures.

Example : ***Let p be "Raju is tall" and let q be "Raju is handsome". Write each of the followi statements in symbolic form :***

(i) Raju is short or not handsome.

(ii) Raju is tall or handsome.

(iii) It is not true that Raju is short or not handsome.

Solution : Let p : Raju is tall and q : Raju is handsome, then

(i) $\sim p \vee \sim q$

(ii) $p \vee q$

(iii) $\sim(\sim p \vee \sim q)$.

Example : ***Let p : Saturn is the second biggest planet in the Universe.***

q : George Boole was born in Lincoln.

Write the sentence which describe the following $p \wedge \sim q$.

Solution. $\sim q$: George Boole was not born in Lincoln.

Then $p \wedge \sim q$: Saturn is the second biggest planet in the Universe and George Boole w not born in Lincoln.

Remark :

$p \vee p \equiv p$

(4) Conditional (If...Then) :

Let p and q be propositions. The implication $p \to q$ is false when p is true and q is false and true otherwise. In the implication p is called the *premise* or *hypothesis or antecedent* and q is called the *consequence* (or conclusion).

Truth Table for Implication i.e., (If.....Then) :

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

For example :

If p : Raju is a student of Information Technology

q : Raju is a good programmer.

Then $p \to q$: *If* Raju is a student of Information Technology *then* Raju is a good programmer.

Example : *Let p denote "It is below freezing" and let q denote "It is showing". Write the following statement in a symbolic form.*

(i) If it is below freezing, it is also snowing.

(ii) It is not snowing if it is below freezing.

(iii) It is below freezing is a necessary condition for it to be snowing.

Solution : (i) $p \to q$

(ii) $p \to \sim q$

(iii) $q \to p$. **Ans.**

Remarks :

(i) If $p \to q$ is an implication proposition or conditional statement, then **[R.G.P.V. Dec 2002 and 03]**

(a) $q \to p$ is called *converse* of $p \to q$.

(b) $\sim q \to \sim p$ is called *contrapositive* of $p \to q$.

(c) $\sim p \to \sim q$ is called *inverse of* $p \to q$.

Above statements are called kind of conditional statement.

(ii) $\sim p \rightarrow \sim q \equiv q \rightarrow p$ i.e., inverse is equivalent to converse.

(iii) $\sim q \rightarrow \sim p \equiv p \rightarrow q$ i.e., *contra positive is equivalent to implication.*

(5) Bi-conditional (if and only if) :

Let p and q be propositions. The bi-conditional $p \leftrightarrow q$ is true when p and q have the same tru

values and is false otherwise.

The bi-conditional statement is also known as an equivalence, and denoted by $p \leftrightarrow q$ or $p \equiv$

Truth Table for Bi-Conditional (or if and only if) :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

For examples :

(i) Gwalior is in India if and only if $3+3 = 6$

(ii) Two lines are parallel if and only if they have the same slope.

(iii) Gwalior is in Pakisthan if and only if $3+3 = 7$

(iv) It p : It is raining

q : It is cold,

Then $p \leftrightarrow q$: It is raining if and only if it is cold.

2.4 CONSTRUCTION OF TRUTH TABLES

Truth tables have already been introduce in the definitions of the logical connectives. (basic concern is to determine the truth table of a proposition for each possible combinat of the truth values of the compound propositions. A table showing all such truth val (T or F) is called *truth table* of the proposition or formula. In general, if there are n disti components or variables in a proposition or formula, we need to consider 2^n, possi combinations of truth values in order to obtain the truth table.

Example : Construct the truth table for the proposition

$\sim p \wedge q.$

Solution Here two variable p and q, then number of possible combination is $2^2 = 4$ of truth values.

Hence truth table is given below :

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Example : *Construct the truth table for*

(i) $\sim(\sim p \vee q)$ (ii) $\sim(\sim p \wedge \sim q)$

(iii) $p \wedge (p \vee q)$.

Solution. (i) *The truth table is :*

p	q	$\sim p$	$(\sim p \wedge q)$	$\sim(\sim p \wedge q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

(ii) Let $P \equiv \sim(\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p \wedge q)$	$\sim(\sim p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

(iii) Let $P \equiv p \wedge (p \vee q)$

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Example : *Construct the truth table for the proposition $(p \rightarrow q \wedge r) \vee (\sim p \wedge q)$.*

Solution Here p, q, r are three variables so that number of possible combination, of truth values are $2^3 = 8$.

Let $P \equiv (p \rightarrow q \wedge r) \vee (\sim p \vee q)$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$\sim p$	$\sim p \wedge q$	$(p \to q \wedge r) \vee (\sim p \wedge q)$
T	T	T	T	T	F	F	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Example : ***Construct the truth table for the following propositions :***

(i) $(p \leftrightarrow q) \wedge (r \vee q)$.

(ii) $(p \vee q) \wedge (\sim r) \to q$.

(iii) $(\sim q \to \sim p) \wedge (q \to p) \to (p \leftrightarrow q)$

Solution. (i) Truth table of $p \leftrightarrow q \wedge (r \vee q)$:

p	q	r	$p \leftrightarrow q$	$r \vee q$	$(p \leftrightarrow q) \wedge (r \vee q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	F	F

(ii) Truth table of $(p \vee q) \wedge (\sim r) \to q$:

p	q	r	$\sim r$	$p \vee q$	$(p \vee q) \wedge (\sim r)$	$(p \vee q) \wedge (\sim r) \to q$
T	T	T	F	T	T	T
T	T	F	T	T	F	T
T	F	T	F	T	T	F
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	T	F	T	T	F	T
F	F	T	F	F	F	T
F	F	F	T	F	F	T

(iii) Truth Table for $P \equiv (\sim q \rightarrow \sim p) \wedge (q \rightarrow p) \rightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$A \equiv \sim q \rightarrow \sim p$	$B \equiv q \rightarrow p$	$A \wedge B$	$C \equiv p \leftrightarrow q$	$P \equiv (A \wedge B) \rightarrow C$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T	T

2.5 WELL FORMED FORMULAS (WFF)

A statement formula is a string consisting of variables i.e., p, q, r, parentheses i.e., ( ) and connective symbols i.e., ($\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$).

A statement formula is called a *well formed formula* (w f f) if it can be generated by the following rules :

1. A statement variable p standing alone is a *well formed formula.*
2. If p is a well formed formula, then $\sim p$ is a *well formed formula.*
3. If p and q are *well-formed formulas,* then $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(p \rightarrow q)$ are *well formed formulas.*
4. A string of symbols is a well formed if and only if it is obtained by finitely many application of the rules 1, 2, and 3.

For examples :

(i) The well formed formula are :

$\sim (p \vee q)$, $\sim (p \wedge q)$, $(p \rightarrow (p \wedge q))$, $(p \rightarrow (p \vee q))$, $((p \rightarrow q) \leftrightarrow r)$, etc.

(ii) But the following are not well formed formula :

$(p \wedge q) \rightarrow (\wedge p,$

$(p \vee q) \rightarrow (q \rightarrow r) \rightarrow (r.$ etc.

2.6 TAUTOLOGY, CONTRADICTION AND CONTINGENCY

We know how to construct the truth table of a given formula. In general, the final column of a truth table of given formula contains both T and F. There are some formula whose truth values are always T or always regardless of the truth value assignments to the variables.

(1) *Tautology :*

A statement formula that is true for all possible values of its propositional variables is called a *tautology* or *logically valid* or *logically true.*

For example :

Proposition $p \wedge q \rightarrow p$ is tautology

Proof by Truth Table :

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Clearly all entries in the last columns of $(p \wedge q) \to p$ are T.

(2) Contradiction :

A statement formula that is always false is called *Contradiction* or *absurdity* or *logically fa*

For example :

Proposition $(p \wedge q) \wedge \sim p$ is a contradiction.

Proof by Truth Table :

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \wedge \sim p$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

Clearly all entries in the last column of $(p \wedge q) \wedge \sim p$ are F.

(3) Contingency :

A proposition or statement formula that is neither a tautology nor a contradiction is calle *contingency.*

For example :

Proposition $\sim p \to \sim q$ is a contingency.

Proof by Truth Table :

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Clearly all entries in the last column of $\sim p \to \sim q$ are neither all T nor all F.

Example : ***Prove that the following propositions are tautologies.***

(i) $q \to p \vee q$

(ii) $(p \wedge q) \to (p \vee q)$

(iii) $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \to (p \leftrightarrow r)$.

Solution. (i) Truth table for $P \equiv q \to p \vee q$:

p	q	$p \vee q$	$q \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Clearly all entries in the last column of proposition are true. Hence P is a tautology. **Proved.**

(ii) Truth Table for $P \equiv (p \wedge q) \to (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$P \equiv p \wedge q \to p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Clearly all entries in the last column of P are true. Hence given proposition P is a tautology.

Proved.

(iii) Truth Table for $P \equiv (p \leftrightarrow q) \wedge (q \leftrightarrow r) \to (p \leftrightarrow r)$

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow q$	$p \leftrightarrow r$ $\equiv A$ say	$(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ $\equiv B$ say	$B \to A$ i.e., P
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	T	F	F	T

F	T	T	F	T	F	F	T
F	T	F	F	F	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Clearly all entries in the last column of P are true. Hence given proposition P is a tautol

Example : *Prove that the proposition.*

$p \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$ **is a tautology.**

Solution. Let $P \equiv p \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$

Truth Table for Proposition P :

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \wedge r$	$(p \wedge q) \rightarrow r$	$p \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Clearly all entries in the last column are true. Hence given proposition P is a tautology.

Example : *Show that the following propositions are tautologies.*

(i) $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

(ii) $\{(p \wedge \sim q) \wedge (\sim p \vee \sim q)\} \vee q.$

[R.G.P.V. Dec.

Solution. The truth table for given proposition (i) :

p	q	$\sim p$	$\sim q$	$p \wedge q \equiv A$	$p \wedge \sim q \equiv B$	$\sim p \wedge q \equiv C$	$\sim p \vee \sim q \equiv D$	$A \vee B$	$A \vee B \vee C$	$A \vee B \vee C$
T	T	F	F	T	F	F	F	T	T	T
T	F	F	T	F	T	F	F	T	T	T
F	T	T	F	F	F	T	F	T	T	T
F	F	T	T	F	F	F	T	F	F	T

Clearly all entries in the last column of given proposition are true. Hence given proposition (i) is a tautology.

The truth table for given proposition (ii) :

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \vee \sim p$	$(p\vee \sim q)\wedge(\sim p\vee\sim q)\equiv A$ *say*	$A \vee q$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	T	T	T

Hence proposition $((p\vee \sim q)\wedge(\sim p\vee\sim q)) \vee q$, is a tautology. **Proved.**

Example : *Prove that $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ is a contradiction.*

Solution Truth table for given proposition $P \equiv (p \vee q) \wedge (\sim p) \wedge (\sim q)$:

p	q	$p \vee q$	$\sim p$	$\sim q$	$(p \vee q) \wedge (\sim p)$	$(p \vee q) \wedge (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	T	F
F	F	F	T	T	F	F

Clearly all entries in the last column are false. Hence given proposition P is a contradiction.

2.7 LOGICAL EQUIVALENCE

Two statements or propositions are called *logically equivalent or simply equivalent,* if truth value of both the statements are always identical.

In other words, the propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

The equivalence of p and q is denoted by $p \equiv q$ or $p \Leftrightarrow q$ or $p \leftrightarrow q$.

Example : *Prove that $q \rightarrow p$ is converse of $p \rightarrow q$ and its inverse $(\sim p) \rightarrow (\sim q)$ are logically equivalent.*

Solution. *Truth table is :*

p	q	$q \rightarrow p$	$\sim p$	$\sim q$	$(\sim p) \rightarrow (\sim q)$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Clearly all entries of third and sixth columns are identical.

Hence $q \to p$ and $(\sim p) \to (\sim q)$ are logically equivalent.

i.e., $$q \to p \equiv (\sim p) \to (\sim q)$$

or $$(q \to p) \leftrightarrow ((\sim p) \to (\sim q))$$ is a tautology.

Example : *Prove that the following propositions are logically equivalent :*

(i) $p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$

(ii) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

(iii) $(p \to q) \vee r \equiv (p \vee r) \to (q \vee r)$. [R.G.P.V. Dec. 200

Solution : (i) *Truth table for the given proposition is :*

p	q	r	$(q \wedge r)$	$p \to (q \wedge r)$	$(p \to q)$	$(p \to r)$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Clearly all entries of fifth and eighth columns are identical. Hence $p \to (q \wedge r)$ is logica equivalent to $(p \to q) \wedge (p \to r)$.

i.e. $p \to (q \wedge r) \equiv (p \to q) \wedge (q \to r)$.

(ii) *The truth table for the given proposition is :*

p	q	r	$(q \vee r)$	$p \to (q \vee r)$	$(p \to q)$	$(p \to r)$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Clearly all entries of fifth and eighth columns are identical.

Hence, $p \Rightarrow (q \vee r) \equiv (p \to q) \vee (p \to r)$. **Proved.**

(iii) The truth table for the given proposition is :

p	q	r	$(p \to q)$	$(p \to q) \vee r)$	$(p \vee r)$	$(q \vee r)$	$(p \vee r) \to (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	T

Clearly all entries of fifth and eighth columns are identical. Hence $(p \to q) \vee r$ is logically equivalent to $(p \vee r) \Rightarrow (q \vee r)$.

i.e., $(p \to q) \vee r \equiv (p \vee r) \to (q \vee r)$. **Proved.**

Example : Prove that

$p \vee r \sim (q \wedge r)$ is equivalent to $(p \vee \sim q) \vee \sim r$. ***[R.G.P.V. Dec. 2002 and 03]***

Solution. *The truth table for the given proposition is :*

p	q	r	$(q \wedge r)$	$\sim (q \wedge r)$	$p \vee \sim (q \wedge r)$	$\sim q$	$\sim r$	$(p \vee \sim q)$	$(p \vee \sim q) \vee \sim r$
T	T	T	T	F	T	F	F	T	T
T	T	F	F	T	T	F	T	T	T
T	F	T	F	T	T	T	F	T	T
T	F	F	F	T	T	T	T	T	T
F	T	T	T	F	F	F	F	F	F
F	T	F	F	T	T	F	T	T	T
F	F	T	F	T	T	T	F	F	T
F	F	F	F	T	T	T	T	T	T

Clearly all entries of fifth and eighth columns are identical.

Hence, $p \vee \sim (q \wedge r) \equiv (p \vee \sim q) \vee \sim r$. **Proved.**

Example : Show that the following is equivalent formula : [R.G.P.V., June 200

$p \vee (p \wedge q) \Leftrightarrow p.$

Solution.

p	q	$p \vee q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Clearly all entries of first and fourth column are identical.

Hence $p \vee (p \wedge q) \equiv p.$ Prove

Example : Show that [R.G.P.V., June 200

$\urcorner(p \wedge q) \Rightarrow (\urcorner p \vee (\urcorner p \vee q) \Leftrightarrow (\urcorner p \vee q).$

Solution. The truth table for the given proposition is :

p	q	$\urcorner p$	$\urcorner q \vee p$	$\urcorner(p \wedge q) \equiv A$ say	$\urcorner p \vee (\urcorner p \vee q) \equiv B$ say	$A \Rightarrow B$
T	T	F	T	F	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	F	T	T	T	T	T

Clearly all entries of fourth and seventh columns are identical

Hence $\urcorner(p \wedge p) \Rightarrow (\urcorner p \vee (\urcorner p \vee (\urcorner p \vee q) \Leftrightarrow (\urcorner p \vee q).$ Prov

Example : Show that $(p \vee q) \wedge (\urcorner p \wedge q))$ is equivalent to $\urcorner p \vee q$. [R.G.P.V., Dec. 200

Solution. The truth table for the given proposition is :

p	q	$p \vee q$	$\urcorner p \wedge q$	$\urcorner p \wedge (\urcorner p \wedge q)$	$(p \vee q) \wedge (\urcorner p \wedge (\urcorner p \wedge q))$	$\urcorner p \vee q$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	T	T	T	T
F	F	F	F	F	F	F

Clearly all entries of sixth and seventh columns are identical.

Hence $(p \vee q) \wedge (\urcorner p \wedge (\urcorner p \wedge q)) \equiv \urcorner p \vee p.$ Prove

Example : Show that the value of the following formula is independent of its component

$$(P \wedge (P \to Q)) \to Q.$$

[R.G.P.V., June 2006]

Solution. Truth table for the given proposition is :

P	Q	$P \to Q$	$P \wedge (P \to Q)$	$(P \wedge (P \to Q)) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Since all entries in the last column are true. Hence the truth value of given formula is independent of its component.

Proved.

Example : Prove that the two propositions.

$[(p \wedge q) \vee (p \wedge r)) \to s$ **and** $((\sim p \vee (\sim q \wedge \sim r)) \vee s$ **are equivalent.**

[R.G.P.V. Dec. 2003 EI]

Solution Let $A = (p \wedge q) \vee (p \wedge r)$ and $B = \sim p \vee (\sim q \wedge \sim r)$

Then we shall prove that $A \to s$ are equivalent to $B \vee s$

The truth table is :

p	q	r	s	$(p \wedge q)$	$(p \wedge r)$	A	$A \to s$	$\sim p$	$\sim q$	$\sim r$	$\sim q \wedge \sim r$	B	$B \vee s$
T	T	T	T	T	T	T	T	F	F	F	F	F	T
T	T	T	F	T	T	T	F	F	F	F	F	F	F
T	T	F	T	T	F	T	T	F	F	T	F	F	T
T	T	F	F	T	F	T	F	F	F	T	F	F	F
T	F	T	T	F	T	T	T	F	T	F	F	F	T
T	F	T	F	F	T	T	F	F	T	F	F	F	F
T	F	F	T	F	F	F	T	F	T	T	T	T	T
T	F	F	F	F	F	F	T	F	T	T	T	T	T
F	T	T	T	F	F	F	T	T	F	F	F	T	T
F	T	T	F	F	F	F	T	T	F	F	F	T	T

F	T	F	T	F	F	F	T	T	F	T	F	T	T
F	T	F	F	F	F	F	T	T	F	T	F	T	T
F	F	T	T	F	F	F	T	T	T	F	F	T	T
F	F	T	F	F	F	F	T	T	T	F	F	T	T
F	F	F	T	F	F	F	T	T	T	T	T	T	T
F	F	F	F	F	F	F	T	T	T	T	T	T	T

Clearly the entries of eighth and fourteenth columns are identical. Hence $(p \wedge q) \vee (p \wedge r) \equiv$ and $((\sim p \vee (\sim q \wedge \sim r)) \vee s$ are equivalent.

Example : Construct the truth table for the following :

(i) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(ii) $p \leftrightarrow (\overline{p} \vee \overline{q})$

(Here $\overline{p}$ means $\sim p$) ***[R.G.P.V., June 2004, and Dec., 2003 (E***

Solution Let $P \equiv (p \to (q \to r)) \to ((p \to q) \to (p \to r))$

The truth table is :

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$ $\equiv A$ say	$(p \to q) \to (p \to r)$ $\equiv B$ say	$P \equiv A \to B$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Clearly all entries in the last column are true.

Hence given proposition P is tautology.

Ans.

(ii) *The truth table is :*

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$	$p \leftrightarrow (\overline{p} \vee \overline{q})$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

Example : ***Let p denote the statement "The weather is nice" and q denote the statement "we have a picnic". Translate the following in English and simplify it.***

(i) $p \wedge \overline{q}$

(ii) $p \leftrightarrow q$

(iii) $\overline{q} \to \overline{p}$ ***[R.G.P.V., June 2004 (EI)]***

Solution. Given : p = the weather is nice and q = we have a picnic.

(i) $p \wedge \overline{q}$: The weather is nice and we don't have a picnic.

(ii) $p \leftrightarrow q$: The weather is nice if any only if we have a picnic.

(iii) $\overline{q} \leftrightarrow \overline{p}$ If we don't have a picnic, then the weather is not nice.

Truth table for $p \wedge \overline{q}$ is :

p	q	$\overline{q}$	$p \wedge \overline{q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(ii) The truth table for $p \leftrightarrow q$ is :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(iii) The truth table for $\overline{q} \to \overline{p}$ is :

p	q	$\overline{p}$	$\overline{q}$	$\overline{q} \to \overline{p}$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

2.8 LAW OF ALGEBRA OF PROPOSITIONS

Equivalence / Proposition	Name
1. (a) $p \Leftrightarrow (p \vee p)$	Idempotents of $\vee$
(b) $p \Leftrightarrow (p \wedge p)$	Idempotents of $\wedge$
2. (a) $(p \vee q) \Leftrightarrow (q \vee p)$	Communativity of $\vee$
(b) $(p \wedge q) \Leftrightarrow (q \wedge p)$	Communativity of $\wedge$
3. (a) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associativity of $\vee$
(b) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associativity of $\wedge$
4. (a) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$	De Morgan's law
(b) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$	
5. (a) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive of $\wedge$ over $\vee$
(b) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive of $\vee$ over $\wedge$
6. (a) $p \vee 1 \Leftrightarrow 1$	(Null) or domination laws
(b) $p \wedge 0 \Leftrightarrow 0$	
7. (a) $p \wedge 1 \Leftrightarrow p$	Identity laws
(b) $p \vee 0 \Leftrightarrow p$	
8. (a) $p \vee \sim p \Leftrightarrow 1$	Negation laws
(b) $p \wedge \sim p \Leftrightarrow 0$	
9. (a) $p \vee (p \wedge q) \Leftrightarrow p$	Absorption laws
(b) $p \wedge (p \vee q) \Leftrightarrow p$	
10. $\sim(\sim p) \Leftrightarrow p$	Double negation law (involution)
11. $p \rightarrow q \Leftrightarrow \sim p \vee q$	Implication law
12. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$	Equivalence law
13. $(p \wedge q) \rightarrow r \Leftrightarrow p \rightarrow (q \rightarrow r)$	Exportation law
14. $(p \rightarrow q) \wedge (p \rightarrow \sim q) \Leftrightarrow \sim p$	Absurdity law
15. $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$	Contrapositive law
16. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$	Bi-conditional law

All the laws can be proved, easily by truth table.

Note : Notation $\Leftrightarrow$ Can be replaced by '$\leftrightarrow$' or $\equiv$ 'or' '='.

[A] : Commutative Laws :

(a) $p \vee q \Leftrightarrow q \vee p$

(b) $p \wedge q \Leftrightarrow q \wedge p$.

Proof (a) :

p	q	$p \vee q$	$q \vee p$	$p \vee q \Leftrightarrow q \vee p$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

Similarly (b) can be proved easily.

[B] : Associative Laws :

(a) : $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

(b) $(p \vee q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$.

Proof (b) :

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

Similarly (a) can be proved easily.

[C] : Distributive Laws :

(a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

Proof (a) :

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

Clearly seventh and eighth column are identical.

Hence $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Similarly (b) can be proved easily.

[D] : Demorgan's Laws :

(a) $\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$,

(b) $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$.

Proof (a) : $\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$	$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

(b) : $\sim (p \vee q) \Leftrightarrow (\sim p) \vee (\sim q)$

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

[E] : Identity Laws :

(a) $p \wedge t \Leftrightarrow p$ or $t \wedge p \Leftrightarrow p$

(b) $p \vee f \Leftrightarrow p$ or $f \vee p \Leftrightarrow p$

Proof (a) :

p	t	$p \wedge t$	$t \wedge p$	$p \wedge t \Leftrightarrow p$	$t \wedge p \Leftrightarrow p$
T	T	T	T	T	T
F	T	F	F	T	T

(b) :

p	f	$p \vee f$	$f \vee p$	$p \vee f \Leftrightarrow p$	$f \vee p \Leftrightarrow p$
T	F	T	T	T	T
F	F	F	F	T	T

[F] : Complement Laws :

(a) $p \vee (\sim p) \Leftrightarrow t$.

(b) $p \wedge (\sim p) \Leftrightarrow f$.

Truth Table for (a)

p	$(\sim p)$	$p \vee (\sim p)$
T	F	T
F	T	T

Truth Table for (b)

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

Example : ***Show that without truth table :***

$$\sim (p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r.$$

Solution :

$(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow (\sim p \wedge (\sim q \wedge r)) \vee ((q \vee p) \wedge r)$ [∵ by distributive la

$\Leftrightarrow ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)$ [∵ by associative la

$\Leftrightarrow ((\sim p \wedge \sim q) \vee (q \vee p) \wedge r$ [∵ by distributive la

$\Leftrightarrow (\sim (p \vee q) \vee (p \vee q) \wedge r$ [∵ by De Morgan's law and commutative la

$\Leftrightarrow T \wedge r$ [∵ by negation la

$\Leftrightarrow r$ [∵ by Identity la

This shows that $\sim (p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$.

Example : ***Show the following equivalences, without truth table :***

(i) $p \to (q \to p) \Leftrightarrow \sim p \to (p \to q)$

(ii) $p \to (q \vee p) \Leftrightarrow (p \to q) \vee (p \to r)$

(iii) $(p \to q) \wedge (r \to q) \Leftrightarrow (p \vee r) \to q.$

Solution.

(i) L.H.S. $= p \to (q \to p) \equiv \sim p \vee (\sim q \vee p)$ [∵ by implication l

$\equiv (\sim p \vee p) \vee \sim q$ [∵ by associative and commutative la

$\equiv T \vee \sim q$ [∵ by negation l

R.H.S.$= \sim p \to (p \to q) \equiv \sim(\sim p) \vee (\sim p \vee q)$ [∵ by implication l

$\equiv p \vee (\sim p \vee q)$ [∵ by involution l

$\equiv (p \vee \sim p) \vee q$ [∵ by associative l

$\equiv T \vee q$ [∵ by negation l

$\equiv T.$

Hence $p \to (q \to p) \Leftrightarrow \sim p \to (p \to q)$. [by implication law]

(ii) $p \to (q \vee r) \Leftrightarrow \sim p \vee (q \vee r)$

$\Leftrightarrow \sim p \vee p \vee (q \vee r)$ $[\sim p \Leftrightarrow \sim p \vee$

$\Leftrightarrow (\sim p \vee q) \vee (\sim p \vee r)$ [∵ by associative and commutative l

$\Leftrightarrow (p \to q) \vee (p \to r).$ [∵ by implication l

This shows that $p \to (q \vee r) \leftrightarrow (p \to q) \vee (p \to r)$. ***Proved.***

(iii) $(p \to q) \wedge (r \to q) \Leftrightarrow (\sim p \wedge q)\ (\sim r \vee q)$ [$\because$ by implication law]

$\Leftrightarrow\ (\sim p \wedge \sim r) \vee q$ [$\because$ by distributive law]

$\Leftrightarrow\ \sim (p \vee r) \vee q$ [$\because$ by De morgan's law]

$\Leftrightarrow\ (p \vee q) \to q$ [$\because$ by implication law]

This shows that $(p \to q) \wedge (r \to q) \leftrightarrow (p \vee q) \to q$. ***Proved.***

Example : Simplify the following propositions :

(i) $(p \wedge q) \vee p$

(ii) $(p \wedge q) \wedge \sim p$

(iii) $\sim (\sim p \wedge q) \wedge (\sim p \vee q) \wedge (p \vee q)$.

Solution. (i) We have

$(p \wedge q) \vee p \equiv p \vee (p \wedge q)$ [$\because$ By commutative law]

$\equiv (p \wedge \pi) \vee (p \wedge q)$ [$\because p \wedge \pi \equiv p$]

$\equiv p \wedge (\pi \vee q)$ [$\because$ By distributive law]

$\equiv p \wedge \pi$ [$\because \pi \vee q \equiv \pi$]

$\equiv p.$ [$\because p \wedge p \equiv p$] ***Ans.***

(ii) We have

$(p \wedge q) \wedge \sim p \equiv \sim p \wedge (p \wedge q)$ [$\because$ by commutative law]

$\equiv (\sim p \wedge p) \wedge (\sim p \wedge q)$

$\equiv \phi \wedge (\sim p \wedge q),$ where ϕ is contradiction

$\equiv \phi.$ [$\because \phi \wedge a = \phi$] ***Ans.***

(iii) We have

$\sim (\sim p \wedge q) \wedge (\sim p \vee q) \wedge (p \vee q)$

$\equiv (\sim\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q)$ [$\because$ By Demorgan's law]

$\equiv (p \vee \sim q) \wedge [(\sim p \wedge p) \vee q]$ [$\because$ By distributive law]

$\equiv (p \vee \sim q) \wedge (\phi \wedge q)$ [$\because \sim p \vee p \equiv \phi$]

$\equiv (p \vee \sim q) \wedge q$ [$\because \phi \vee q = q$]

$\equiv (p \wedge q) \vee (\sim q \wedge q)$ [$\because$ By distributive law]

$\equiv (p \wedge q) \vee \phi$ [$\because \sim q \wedge q \equiv \phi$]

$\equiv p \wedge q.$ ***Ans.***

Example : *Simplify the following propositions :*

(i) $(P \vee Q) \wedge \sim P$,

(ii) $P \vee (P \vee Q)$,

(iii) $\sim (P \vee Q) \vee (\sim P \wedge Q)$.

Solution. (i) $(P \vee Q) \wedge \sim P$

$\equiv \sim P \wedge (P \vee Q)$ [Commutative la

$\equiv (\sim P \wedge P) \vee (\sim P \wedge Q)$ [Distributive la

$\equiv F \vee (\sim P \wedge Q)$ [Complement la

$\equiv (\sim P \wedge Q)$ [Identity law] A

(ii) $P \vee (P \wedge Q)$

$\equiv (P \wedge T) \vee (P \wedge Q)$ [$\because P \wedge T \equiv$

$\equiv P \wedge (T \vee Q)$ [$\because P \wedge (T \vee Q) \equiv (P \wedge T) \vee (P \wedge$

$\equiv P \wedge T$ [$\because P \vee T \equiv$

$\equiv P$. [$\because P \wedge T \equiv P$] A

(iii) $\sim (P \vee Q) \vee (\sim P \wedge Q)$

$\equiv (\sim P \wedge \sim Q) \vee (\sim P \wedge Q)$ [by De-Morgan's la

$\equiv \sim P \wedge (\sim Q \vee Q)$ [by Distributive la

$\equiv \sim P \wedge T$ [$\because \sim Q \vee Q \equiv$

$\equiv \sim P$. [$\because P \wedge T = P$] A

Example : *Let Apq denote p ∧ q and Np denote ~p. Rewrite the following propositio using A and N instead of ∧ and ~.*

(i) $p \wedge \sim q$,

(ii) $\sim (p \wedge \sim q) \wedge (\sim q \wedge \sim r)$.

Solution (i) $p \wedge \sim q \quad = p \wedge N q$

$= A\, p\, N\, q$. A

(ii) $\sim (p \wedge \sim q) \wedge (\sim q \wedge \sim r)$

$= \sim (A\, p\, N\, q) \wedge (A\, N\, q\, N\, r)$

$= (N\, A\, p\, N\, q) \wedge (A\, N\, q\, N\, r)$

$= A\, N\, A\, p\, N\, q\, A\, N\, q\, N\, r$. A

Example : *Rewrite the following propositions using ∧ and ~ instead of A and N.*

(i) **A A p q r.**

(ii) **A p A q r.**

(iii) **A N A p A q p A A N q r p.**

Solution (i) $A\,A\,p\,q\,r = A\,(p \wedge q)\,r$

$= (p \wedge q) \wedge r.$ **Ans.**

(ii) $A\,p\,A\,q\,r = A\,p\,(q \wedge r)$

$= p \wedge (q \wedge r).$ **Ans.**

(iii) $A\,N\,A\,p\,A\,q\,p\,A\,A\,N\,q\,r\,p$

$= A\,N\,A\,p\,A\,q\,p\,A\,A\,(\sim q)\,r\,p$

$= A\,N\,A\,p\,A\,q\,p\,A\,(\sim q \wedge r)\,p$

$= ANApAqp\,[(\sim q \wedge r) \wedge p]$

$= ANAp\,(q \wedge p)\,[(\sim q \wedge r) \wedge p\}$

$= AN\,[p \wedge (q \wedge p)\,]\,[(\sim q \wedge r) \wedge p]$

$= \sim[p \wedge (q \wedge p)] \wedge [\sim q \wedge r) \wedge p].$ **Ans.**

2.9 DUALITY PRINCIPLE

The principle of duality states that any established result involving statement formulae and connectives $\wedge$ and $\vee$ and special variables *t* and *f* are *dual* to each other. On replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$ in a formula we obtain the *dual* of given formula. Similarly, if the formula contains the variable *t* and *f* then replace *t* by *f* and *f* by *t* respectively to obtain the dual formula.

For examples :

(i) Dual of $(p \wedge q) \vee r$ is $(p \vee q) \wedge r$

(ii) Dual of $\sim (p \vee q) \wedge (p \vee \sim q \wedge \sim r)$ is $\sim(p \wedge q) \vee (p \wedge \sim q \vee \sim r)$.

2.10 OTHER CONNECTIVES

[A] : Exclusive OR :

If *p* and *q* are two statements or propositions then *exclusive or* if *p* and *q* is true when either of *p* and *q* is true otherwise it in false. It is denoted by symbol $\oplus$

Truth Table for exclusive OR :

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Remarks :

(i) $p \oplus q \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

(ii) $p \oplus q \equiv \sim (p \leftrightarrow q)$

[B] NAND :

The connective NAND is a combination of 'NOT' and 'AND' where 'NOT' stands for negation and 'AND' stands for conjunction. It is denoted by $\uparrow$ i.e. $p \uparrow q$ is true, when both p and q are not true.

The truth table for NAND :

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Remark : $p \uparrow q \equiv \sim (p \wedge q)$.

[C] NOR :

The connective NOR is a combination of 'NOT' and 'OR', where 'NOT' stands for negation and 'OR' stands for disjunction. It is denoted by $\downarrow$ i.e., $p \downarrow q$ is true only when p and q both false.

The truth table for NOR :

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Remark : $p \downarrow q \equiv \sim (p \vee q)$

2.11 ARGUMENTS

Arguments are at the heart of logic or least at the heart of using logic. An *argument* is denoted by Symbol $\vdash$ which is called *turnstile.*

Argument is a sequence of propositions that perport to imply another proposition. The sequence of such propositions serving as evidence will be called *premises* and the proposition Inferred will be called *conclusion.*

A premise is a list of Boolean expressions and assumptions. If the conclusion follows from premises, then the argument is valid. If not, it is not valid.

Let premises be $p_1, p_2 ...., p_n$ and let argument yield the conclusion q, then such an argument is denoted by

$p_1, p_2, ..., p_n \vdash q$ and written as

$$\left.\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$
$$\therefore\ q \text{ (conclusion)}$$

For example of an argument :

$$\left.\begin{array}{l} p \to q \\ p \end{array}\right\} \textit{premises}$$
$$\therefore q \text{ (conclusion)}$$

The expressions above lines are the premises, and those below are the conclusions. Actually, the conclusion is always a single Boolean expression.

2.12 VALID ARGUMENT AND FALLACY ARGUMENT (INVALID)

An argument $p_1, p_2, ..., p_n \vdash q$ is called *valid argument* if q is true, whenever all its premises $p_1, p_2, ..., p_n$ are true.

In other words, the argument $p_1, p_2, p_n \vdash q$ is said to be *valid* if and only if the proposition $(p_1 \wedge p_2 \wedge p_3 ..... \wedge p_n) \to q$ is tautology.

An argument which is not valid is said to be *fallacy* or an *invalid argument.*

Example : *Show that the logical validity of the following :*

$$\begin{array}{r} p \vee q \\ \sim p \\ \hline q \end{array}$$

Solution Here in the given argument $p \vee q$ and $\sim p$ are premises and q is the conclusion.

We know that argument is valid if the proposition $((p \vee q) \wedge \sim p) \to q$ is tautology.

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	T	T
F	F	T	F	F	T

Clearly all entries in the last columns are true.

Hence given argument is valid. **Proved.**

Example : *Check the logical validity of the following :*

If Sumit learns programming, he will become a programmer.

Sumit learns programming.

Thereforè, he will become a programmer.

Solution Let $p \equiv$ Sumit learns programming

$q \equiv$ He will become a programmer

Then $p \to q \equiv$ If sumit learns programming, he will become a programmer

Then argument is
$$\left.\begin{array}{l} p \to q \\ p \end{array}\right\} \text{premises}$$
$$\therefore q \text{ (conclusion)}$$

For the validity of argument, to construct truth table of the proposition :

$((p \to q) \wedge p) \to q$ is tautology.

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Clearly all entries in the last column are true. Hence argument is valid.

Remark : The valid argument is called law of *direct inference* or *modus ponen*.

Example : *Show that the argument $p, p \to q, q \to r \vdash r$ is valid.*

Solution We shall prove that the given proposition.

$(p \wedge (p \to q) \wedge (q \to r)) \to r$ is a tautology.

p	q	r	$p \to q$	$q \to r$	$p \wedge (p \to q)$	$p \wedge (p \to q) \wedge (q \to r) \equiv$ A (say)	A →
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

Clearly all entries in the last column are true.

Hence given argument is valid.

...mple : *Check the validity of the argument :*

If Dr. Bajaj buys a car, then he can go home in time.

If he goes home in time, then his family will be happy.

Solution Let

p : Dr. Bajaj buys a car.

q : He can go home in time.

r : His family will be happy.

The argument is

$$\left.\begin{array}{l} p \to q \\ q \to r \end{array}\right\} \text{ premises}$$
$$\therefore\ p \to r \text{ (conclusion)}$$

To construct truth table for the proposition say, $A \equiv ((p \to q) \wedge (q \to r)) \to (p \to r)$.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$(p \to r)$	A
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given argument is valid. **Ans**

mple : *Test the validity of the argument :*

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

$\therefore$ ***The opposite angles are not equal.***

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tion. Let p : Two sides of a triangle are equal

q : The opposite angles of a triangle are equal.

The given argument is

$$\left.\begin{array}{l} p \to q \\ \sim p \end{array}\right\} \text{ premises}$$
$$\therefore\ \sim q \quad \text{(conclusion)}$$

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We shall construct the truth table for proposition : $((p \to q) \wedge \sim p) \to \sim q$.

p	q	$p \to q$	$\sim p$	$\sim q$	$(p \to q) \wedge \sim p$	$((p \to q) \wedge \sim p) \to \sim q$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

Clearly all entries in the last column in not true.

Hence $((p \to q) \wedge \sim p) \to \sim q$ is not valid.

Example : ***State whether the argument given below is valid. If it is valid, identify tautology or tautologies used :***

If I drive to work then I will arrive tired

I drive to work. [R.G.P.V. June 2

$\therefore$ ***I will arrive tired.***

Solution Let p : I drive to work

q : I will arrive tired.

$\therefore$ The given argument, is

$$\left.\begin{array}{r} p \to q \\ p \end{array}\right\} \text{premises}$$

$$\therefore \quad q \quad \text{(conclusion)}$$

We shall construct the truth table for the proposition : $(p \wedge (p \to q)) \to q$.

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Clearly all entries in the last column are true.

Hence argument is valid.

Example : ***State whether the argument given below is valid or not valid. If it is valid, ide the tautology or tautologies used :***

I will become famous or I will be writer.

I will not be a writer.

Solution Let p : I will become famous

q : I will be writer

The given argument is

$$\left.\begin{array}{r} p \vee q \\ \sim q \end{array}\right\} \text{premises}$$
$$\therefore \quad p \text{ (conclusion)}$$

We shall construct the truth table for the proposition : $((p \vee q) \wedge (\sim q)) \rightarrow p$.

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge (\sim q)$	$((p \vee q) \wedge (\sim q)) \rightarrow q$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since all entries in the last column are true. Hence given argument $((p \vee q) \wedge (\sim q)) \rightarrow p$ is valid.

Ans.

2.13 PREDICATES AND LOGICAL QUANTIFIERS

Predicate Calculus :

Predicate calculus is generalization of propositional calculus. It is important for several reasons this has application in expert system, in data base and also basis for the Prolog language.

Consider the declarative sentence. "He is a good student".

Clearly this sentence is not proposition because it is not clear that whether it is true or false unless we know "Who is she". Such sentences are known as *propositional functions* or *predicates*.

A predicate is symbolised by capital letters, namely P (x) and the names of individuals or objects by small letters i.e., x. The sentence "x is a bachelor" is symbolised as P(x), where x is a *predicate variable* and P(x) is called a *propositional function.*

For example :

Let P (x) : $x^2 - 6x + 9 = 0$

Then P(x) is a propositional function on the set of integers $x \in Z$.

$\Rightarrow$ P (x) is true when $x = 3$ only.

Quantifiers :

Quantifiers are words that refer to quantise such as "some" or "all" and indicate how frequently a certain statement is true. It has been classified into two types :

(i) Universal Quantifiers :

In the propositional function "for all" or "for every" denoted by $\forall$ is called the *universal quantifies.*

For example :

Consider the sentence "All human beings are moral"

Let P(x) : 'x is moral'

Then above sentence can be written as

$\forall x\, P(x)$ or $(\forall x \in S)\, P(x)$, where S denotes the set of all human beings.

(ii) *Existential Quantifiers :*

In the propositional function "there exist" or "for some" denoted by $\exists$ is called *existential quantifier.*

For example :

Consider the sentence there exists *x* such that $x^2 = 11$.

Then above can be written as

$(\exists x \in R)\, P(x)$ or $\exists x\, P(x)$.

Remarks :

(i) The variable concerned with universal and existential quantifiers is called bo *variable.*

(ii) The variable which is not concerned with any quantifier is called *free variable.*

(iii) The *domain* or universe of discourse of a predicate variable is the set of all possible va that may be substituted in place of variable.

Example : *Let I be the set of integer, consider the statement :*

(i) $(\forall x \in I), x^2 = x$

(ii) $(\exists x \in I), x^2 = x$.

Find the truth values of each of the statements.

Solution. Let proposition P (x) : $x^2 = x$

(i) Then $\forall x\, P(x)$ is false, because P(2) is false i.e., $2^2 = 2$ is false.

(ii) $\exists x\, P(x)$ is true, because P(0) and p(1) are true.

Example : *Let P (x) : x is man*

Q (x) : x is moral.

Express the following using quantifiers :

(i) All men are moral.

Solution $\forall x\, P(x) \rightarrow Q(x)$.

Example : *Let P (x) : x is student*

Q (x) : x is clever

R (x) : x is successful.

Express the following using quantifiers :

(i) There exists a student

(ii) Some student are clever

(iii) Some students are not successful

Solution. (i) $(\exists x)\, P(x)$

(ii) $(\exists x)\, (P(x) \wedge Q(x))$.

i.e., There exist x such that x is student and x is clever.

(iii) $(\exists x)\, (P(x) \wedge \sim R(x))$. **Ans.**

Example : Write negation of quantifier :

$\forall\, x \exists\, y\, P(x, y)$.

Solution The negation of $\forall x \exists y\ P(x, y)$ is

$$\sim [\forall x \exists y\, P(x, y)] \equiv \exists x\, (\sim \exists y\, P(x, y))$$

$$\equiv \exists x\, \forall y\, (\sim P(x, y)).$$ **Ans.**

Example : Negate the statement :

$\forall\, x \exists\, y\, (P(x, y) \to Q(y))$.

Solution.

$$\sim [\forall x \exists y\, (P(x, y) \to Q(y)]$$

$$\equiv \exists x\, \forall y\, (\sim(P(x, y) \to Q(y))$$

$$\equiv \exists x\, \forall y\, (\sim (\sim P(x, y) \vee Q(y)) \qquad [\because p \to q = \sim p \vee q]$$

$$\equiv \exists x\, \forall y\, (P(x,y) \wedge \sim Q(y)).$$ **Ans.**

2.14 *CONSISTENCY*

Let $P_1, P_2, \ldots, P_n$ be the set of formulae. If the conjunction of set of formulae $P_1, P_2, \ldots, P_n$ has the true values T for some assigned truth values to the atomic variables in the set of formulae, then $P_1, P_2, \ldots, P_n$ are called '*consistent*'. Otherwise $P_1, P_2, \ldots, P_n$ are called '*in consistent*'.

2.15 *NORMAL FORMS*

Let $a_1, a_2, a_3, \ldots, a_n$ be n atomic or statement variables. Then expression is of the form : $a_1^* \wedge a_2^* \wedge a_3^* \ldots \wedge a_n^*$ where a_i^* is either a_i or $\sim a_i$ is called a *minterm*. There are 2^n such minterms.

Also the expression is of the form : $a_1^* \vee a_2^* \vee a_3^* \ldots \vee a_n^*$ is called a *maxterm*. There are 2^n such maxterms.

For example :

Consider the two variables p and q, then there are 2^2 i.e., 4 minterms and 4 maxterms are as follows :

p	***q***	***minterms*** ($\wedge$)	***maxterms*** ($\vee$)
T	T	$p \wedge q$	$p \vee q$
T	F	$p \wedge \sim q$	$p \vee \sim q$
F	T	$\sim p \wedge q$	$\sim p \vee q$
F	F	$\sim p \wedge \sim q$	$\sim p \vee \sim q$

Remarks :

(i) The type of $p \wedge \sim p$ and $p \vee \sim p$ are not minterm and maxterm.

(ii) Minterms is know as *sum of product (SOP)* and maxterms is known as *product of sums (POS)*.

2.16 DISJUNCTIVE AND CONJUNCTIVE NORMAL FORMS

Normal forms are of two types : (i) Disjunctive form (ii) Conjunctive form :

Disjunctive Normal Form (DNF) :

A formula or statement which contains minterms i.e., sum of elementary products, is called *disjunctive normal form or DNF.*

The disjunction normal form of the given formula is not unique.

Example : Obtain DNF of the following :

(i) $p \wedge (p \to q)$

(ii) $(\sim p \vee \sim q) \to (p \leftrightarrow \sim q)$.

Solution (i) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ [$\because a \to b \equiv \sim a$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$. [$\because$ By distributive

$\equiv p \wedge q$. [$\because p \wedge \sim p$

Which is DNF.

(ii) $(\sim p \vee \sim q) \to (p \leftrightarrow \sim q)$

$\equiv (\sim p \vee \sim q) \to ((p \to \sim q) \wedge (\sim q \to p))$ [$\because a \leftrightarrow b \equiv (a \to b) \wedge (b-$

$\equiv (\sim p \vee \sim q) \to ((\sim p \vee \sim q) \wedge (q \vee p))$ [$\because a \to b \equiv \sim a$

$\equiv \sim (\sim p \vee \sim q) \vee ((\sim p \vee \sim q) \wedge (q \vee p))$ [$\because a \to b \equiv \sim a$

$\equiv (p \wedge q) \vee ((\sim p \wedge q) \vee (\sim p \wedge p)) \vee (\sim q \wedge q) \vee (q \wedge p)$

$\equiv (p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge q)$

$\equiv (p \wedge q) \vee (\sim p \vee q)$. [$\because a \vee a$

Which is DNF.

Conjunctive Normal Form (CNF) :

A formula or statement which contains maxterms i.e. product of elementary sums called *conjunctive normal* form or CNF.

Example : *Obtain the following into conjunctive normal form : $\sim (p \vee q) \leftrightarrow (p \wedge q)$*

Solution $\sim (p \vee q) \leftrightarrow (p \wedge q)$

$\equiv A \leftrightarrow B$ [Let $A \equiv \sim (p \vee q)$ and $B = (p \wedge$

$\equiv (A \to B) \wedge (B \to A)$

$\equiv (\sim A \vee B) \wedge (\sim B \vee A)$

$\equiv ((p \vee q) \vee (p \wedge q) \wedge (\sim (p \wedge q) \vee \sim (p \vee q)$ [$\because a \to b = \sim a \vee$

$\equiv \quad ((p \vee q) \vee (p \wedge q) \wedge (\sim (p \wedge q) \vee \sim (p \vee q))$

$\equiv \quad ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q)$ [By Demorgan laws]

$\equiv \quad (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$ [by distributive law]

$\equiv \quad (p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\equiv \quad (p \vee q) \wedge (\sim p \vee \sim q).$ $[\because a \wedge a = a]$

Which is CNF. **Ans.**

Example : Obtain the CNF for the following statement :

$$(\mathbf{p} \to (\mathbf{q} \wedge \mathbf{r})) \wedge (\sim \mathbf{p} \to (\sim \mathbf{q} \wedge \sim \mathbf{r})).$$

Solution. $(p \to (q \wedge r)) \wedge (\sim p \to (\sim q \wedge \sim r))$

$\equiv \quad (\sim p \vee (q \wedge r) \wedge (p \vee (\sim q \wedge \sim r))$ $[\because a \to b \equiv \sim a \vee b]$

$\equiv \quad (\sim p \vee q) \wedge (\sim p \vee r) \wedge (p \vee \sim q) \wedge (p \vee \sim r)$ [by distributive law]

$\equiv \quad (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r)$

$\wedge (p \vee \sim q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim q \vee \sim r)$

$\equiv \quad (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$

$\wedge (p \vee q \vee \sim r)$ $[\because a \vee a = a]$

Which is CNF. **Ans.**

Example : Obtain the following into conjunctive normal form of : $(\sim \mathbf{p} \to \mathbf{r}) \wedge (\mathbf{q} \leftrightarrow \mathbf{p})$.

Solution $\quad (\sim p \to r) \wedge (q \leftrightarrow p)$

$\equiv \quad [\sim(\sim p) \vee r] \wedge [(q \to p) \wedge (p \to q)]$ $[\because a \to b \equiv \sim a \vee b]$

$\equiv \quad (p \vee r) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$

$\equiv \quad [(p \vee r) \vee F] \wedge [(\sim q \vee p) \vee F] \wedge [(\sim p \vee q) \vee F]$

$\equiv \quad [(p \vee r) \vee (q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)]$ [By distributive law]

$\equiv \quad (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r) \vee (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\equiv \quad (p \vee q \vee r) \wedge (p \vee r \vee \sim q) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$ $[a \vee a \equiv a]$

Which is CNF. **Ans.**

Example : Obtain the function $f = [(x \wedge y')' \vee z'] \vee (x' \vee z')$ into conjunctive normal form (CNF).

Solution $\quad ((x \wedge y')' \vee z') \wedge (x' \vee z)' \equiv ((x' \vee y) \vee z') \wedge (x \wedge z')$ [By Demorgan's law]

$\equiv \quad (x' \vee y \vee z') \wedge (x \wedge z')$

$\equiv \quad (x \wedge z') \wedge (x' \vee y \vee z')$ [By Commutative law]

$\equiv \quad (x \wedge z' \wedge x') \vee (x \wedge z' \wedge y) \vee (x \wedge z' \wedge z').$ [By distributive law]

Ans

Which is CNF.

Example : Find the CNF of the function $P = [x \wedge (y' \vee z)] \vee z'$ and then find its DNF.

Solution. $P = (x \wedge (y' \vee z)) \vee z'$

$= (x \vee z') \wedge [(y' \vee z) \vee z')]$ [By distributive

$= (x \vee z') \wedge (y' \vee (z \vee z'))$ [By Associative

$= (x \vee z') \wedge (y' \vee T)$ [$\because x \vee x'$

$= (x \vee z') \wedge T$ [$\because a \vee T$

$= x \vee z'$ [$\because a \wedge T$

$= (x \vee z') \vee F$

$= (x \vee z') \vee (y \wedge y')$

$= (x \vee z' \vee y) \wedge (x \vee y' \vee z').$

Which is CNF.

Let $P(x,y,z) = (x \vee z' \vee y)\ (x \vee y' \vee z')'$

Since $P \equiv (P')' = ([(x \vee y \vee z') \wedge (x \vee y' \vee z')]')$

$= [(x \vee y \vee z')' \vee (x \vee y' \vee z')']'$ [By Demorgan'

$= [(x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)]'$ [By Demorgan'

$= (x' \wedge z)'$ [$\because$ By Demorgan'

$= x \vee z'$

$= (x \wedge T) \vee (z' \wedge T)$

$= [x \wedge (y \vee y')] \vee [z' \wedge (y \vee y')]$

$= (x \wedge y) \vee (x \wedge y') \vee (z' \wedge y) \wedge (z' \wedge y').$

Which is DNF.

Example : Show that

$$\neg(p \wedge q) \Rightarrow (\neg p \vee (\neg p \vee q) \Leftrightarrow (\neg p \vee q).$$

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Solution. $\neg(p \wedge q) \Rightarrow (\neg p \vee (\neg p \vee q))$

$\Leftrightarrow (p \wedge q) \vee (\neg p \vee (\neg p \vee q))$

$\Leftrightarrow (p \wedge q) \vee ((\neg p \vee q) \vee q)$

$\Leftrightarrow (p \wedge q) \vee \neg p \vee q$

$\Leftrightarrow (p \vee \neg p) \wedge (q \vee \neg p) \vee q$

$\Leftrightarrow (q \vee \neg p) \vee q$

$\Leftrightarrow q \vee \neg p$

$\Leftrightarrow \neg p \vee q.$

Pro