EC 504
Spring, 2021
HW 8

Due Monday, April 26, 8PM on Gradescope.

1. (16 pts) Answer True or False to each of the questions below, **AND YOU MUST PROVIDE SOME FORM OF EXPLANATION (brief is OK)**. Each question is worth 2 points. Answer true only if it is true as stated, with no additional assumptions.

   (a) There are instances of the integer knapsack problem which can be solved in polynomial time by a greedy algorithm.

   (b) Every problem in class P is also in class NP.

   (c) Every problem in class P is can be reduced in polynomial time to a problem that is NP-complete.

   (d) The problem of finding a shortest path in a graph can be polynomially reduced to an instance of the integer knapsack problem.

   (e) Consider the class of integer knapsack problems where the maximum knapsack capacity is bounded by 1000. The decision problems associated with this class belong to class P.

   (f) Assume that there is a decision problem which can be described in an input with n bits. Assume that the worst-case instance of this decision problem of size n bits requires $O(2^n)$ operations to solve. Then, the decision problem belongs to class NP.

   (g) The class NP consists of all decision problems which cannot be solved in time $O(n^k)$ for some positive integer k, where n is the size of the problem instance description in bits.

   (h) Finding the minimum-distance tour in a traveling salesperson problem is an NP-Hard problem.

2. (20 pts) A wide range of questions can be asked about social networks: graphs whose nodes represent people, with edges joining pairs who know one another.

   Consider a small unnamed college (BC), where the Office of Student Life decides to make use of some social network algorithms. BC, they reason, is one of the best places for making friendships that last a lifetime; they want to make sure the students are taking as much advantage of this opportunity as possible. So to facilitate new interactions, they decide to invite selected groups of students to a series of"ice-breaking events."

   To avoid the situation where people talk only to their friends, they would like to invite a set of people to the event with the property that no two know each other. Naturally these restrictions impose some limits on how large an ice-breaking event one can have. Thus, the Ice-breaking Event problem is the following. Given a set of students, and a list of all pairs of students who know one another, is there a set S of k students such that no two students in S know each other?

   (a) Show that this problem is in **NP**.

   (b) Suppose the social network graph is a line (lots of strangers here, and no student knows more than two other students!) Can you describe a polynomial time algorithm for solving the ice-breaking event problem for a line of n students?

   (c) Suppose the social network graph is a tree. Note following observation: if person k is a leaf in the tree, then there exists a maximum cardinality ice-breaking event that includes person k. The reason is that, if it were not, then a maximum cardinality set must include the parent vertex of that leaf (or else adding that leaf increases cardinality). However, one can exchange that leaf for its parent and still have a valid ice-breaking event of the same cardinality. Use this fact to develop a polynomial algorithm for finding the maximum size of an ice-breaking event when the social network is a tree.

   (d) Show that, for general undirected graphs, the ice-breaking problem above is NP-complete.

3. (20 pts) You are consulting for a trucking company that does a large amount of business shipping packages between New York and Boston. The volume is high enough that they have to send a number of trucks each day between the two locations. Trucks have a fixed limit W on the maximum amount of weight they are allowed to carry.

   Boxes arrive to the New York station one-by-one, and each package i has a weight w_i. At the moment the company is using a simple greedy algorithm for packing: they pack boxes in the order they arrive, and whenever the next box does not fit, they send the truck on its way.

   Assume that we have a set of n packages such that the weight of package i is w_i, and each truck has a limit W on the maximum amount of weight. You can assume each w_i and W are integers. The goal is to minimize the number of trucks that are needed in order to carry all the packages.

   (a) Let us consider the decision version of the problem as follows: given a set of packages with weights w_1, ..., w_n, and a weight limit for each truck W, determine whether or not we can fit all the packages into k trucks. Show that this problem is NP-complete.

   (b) Show that the number of trucks used by the greedy algorithm is within a factor of 2 of the minimum possible number, for any set of weights and any value of W.

4. (20 pts) Consider the weighted, undirected graph in Figure 1 below. Assume the real graph is dense, and all the missing edges have length 2, so Figure 1 is only showing the length 1 edges. Thus, the edges between nodes 2 and 3, and between nodes 1 and 4, have length 2.

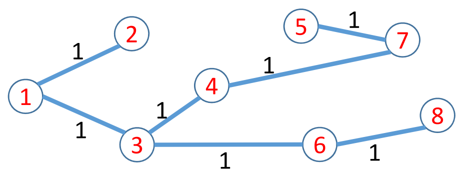

Figure 1: Figure for Problem 1

   (a) What is the weight of the minimum spanning tree for this graph?

   (b) Show that the distances in the arcs in this graph satisfy the metric property: $d(i, j) \leq d(i, k) + d(k, j)$ for any i, j, k nodes.

   (c) Given the unique minimum spanning tree in this graph, double the edges in the spanning tree and construct an Euler tour that visits all the vertices, albeit more than once, starting in vertex 1 and selecting the edges at branches that goes to the lowest numbered node.

   (d) From this Euler tour, form a Traveling Salesperson Tour by skipping vertices that have already been visited. What is the length of this tour?

   (e) Is the above tour optimal? Explain why or why not.