EC 504
Spring, 2022
HW 5

Due Saturday, April 2, 8PM on Gradescope.

1. (15 points) Assume you have a scheduling problem with 10 customers. Customer i has value V_i, and must be processed before deadline T_i. The processing time of each customer is exactly one unit of time. The values of V_i and T_i are listed below as arrays:

```
V = {3,4,7,5,2,3,5,8,9,6}
T = {2,3,4,3,1,5,7,3,4,6}
```

Find the optimal sequence of jobs that can be scheduled in order to complete as much value as possible.

Solution:

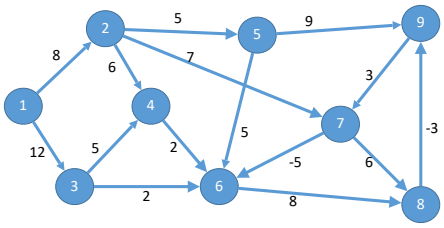
This algorithm admits a greedy solution. Rank stuff by decreasing value, and schedule it if it fits in partial schedule, otherwise skip it. To check it fits into partial schedule, order tasks in increasing deadline and see if it is executable.

Sorting by decreasing value, the task order is: 9,8,3,10,7,4,2,1,6,5. Thus, we first add task 9, then 8, then 3, then 10, then 7, then 4, all of which fit into a schedule. To check this, schedule this group in order of earliest deadline first. This yields a schedule of 4, 8, 9, 3, 10, 7, which meets all the deadlines.

Now try to add task 2, with deadline 3. It would have to be added in the first 3 slots, which would then violate the deadline for task 3, which would slip to slot 5. Thus, task 2 is skipped. Similarly, task 1 is skipped. Task 6 has deadline 5, so it can be added to the schedule to form: 4,8,9,3,6,10,7. Task 5 cannot be added because it has deadline 1 and some other task would have to be dropped.

The final value is 43.

2. (15 points) Consider the directed weighted graph on the right. Show the distance estimates computed by each step of the Bellman-Ford algorithm for finding a shortest path tree in the graph, starting from vertex 1 to all other vertices. When scanning edges out of a vertex, consider them in clockwise order starting from straight up. Show the algorithm using a work queue to determine which vertices to scan, in order to avoid unnecessary iterations.

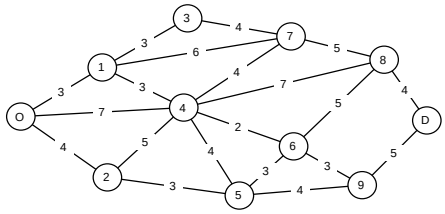


Solution: Algorithm is shown below. Distances in brackets indicate vertices with new distances.

```
Initial Distances:  D(1) = 0; D(2) = inf; D(3) = inf; D(4) = inf; D(5) = inf;
                    D(6) = inf; D(7) = inf; D(8) = inf; D(9) = inf;
Q = {1}
Scan vertex 1:          D(1) = 0; D(2) = 8; D(3) = 12; D(4) = inf; D(5) = inf;
                    D(6) = inf; D(7) = inf; D(8) = inf; D(9) = inf;
Q = {2,3}
Scan vertex 2:          D(1) = 0; D(2) = 8; D(3) = 12; D(4) = [14]; D(5) = [13];
                    D(6) = inf; D(7) = [15]; D(8) = inf; D(9) = inf;
Q = {3,5,7,4}
Scan vertex 3:          D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = [14]; D(7) = 15; D(8) = inf; D(9) = inf;
Q = {5,7,4,6}
Scan vertex 5:          D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 14; D(7) = 16; D(8) = [21]; D(9) = [22];
Q = {7,4,6,9}
Scan vertex 7:        D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
```

```
                    D(6) = [10]; D(7) = 16; D(8) = [23]; D(9) = 22;
Q = {4,6,9,8}
Scan vertex 4:         D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 10; D(7) = 16; D(8) = 23; D(9) = 22;
Q = {6,9,8}
Scan vertex 6:         D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 10; D(7) = 16; D(8) = [18]; D(9) = 22;
Q = {9,8}
Scan vertex 9:         D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 10; D(7) = 16; D(8) = 18; D(9) = 22;
Q = {8}
Scan vertex 8:         D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 10; D(7) = 16; D(8) = 18; D(9) = [15];
Q = {9}
Scan vertex 9:         D(1) = 0; D(2) = 8; D(3) = 12; D(4) = 14; D(5) = 13;
                    D(6) = 10; D(7) = 16; D(8) = 18; D(9) = 15;

Done.  No more vertices in Queue means no other vertices can be reached.
Shortest path tree

    7--6--8--9
   /
  2--5
 / \
1   4
 \
  3
```

3. (15 points) Consider the weighted, undirected graph shown on the right. Assume that the edges can be traveled in both directions with the same distance. Illustrate the steps of Dijkstra's algorithm for finding a shortest path from vertex O to vertex D.

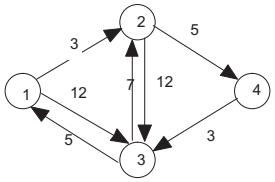


Solution: As before, keep track of distances which change using brackets, and delete vertices already scanned from distance list

```
Initial Distances: D(O)=0; D(1)=inf; D(2) = inf; D(3)=inf; D(4)=inf; D(5) = inf;
                    D(6) = inf; D(7) = inf; D(8) = inf; D(9) = inf; D(D) = inf
Scan vertex O: D(1)=[3]; D(2) = [4]; D(3)=inf; D(4)=[7]; D(5) = inf;
                    D(6) = inf; D(7) = inf; D(8) = inf; D(9) = inf; D(D) = inf

Scan vertex 1: D(2) = 4; D(3)=[6]; D(4)=[6]; D(5) = inf;
                    D(6) = inf; D(7) = [9]; D(8) = inf; D(9) = inf; D(D) = inf
Scan vertex 2: D(3)=6; D(4)=6; D(5) = [7];
                    D(6) = inf; D(7) = 9; D(8) = inf; D(9) = inf; D(D) = inf
Scan vertex 3: D(4)=6; D(5) = 7;
                    D(6) = inf; D(7) = 9; D(8) = inf; D(9) = inf; D(D) = inf
Scan vertex 4: D(5) = 7;
                    D(6) = [8]; D(7) = 9; D(8) = [14]; D(9) = inf; D(D) = inf
Scan vertex 5: D(6) = 8; D(7) = 9; D(8) = 14; D(9) = [11]; D(D) = inf
Scan vertex 6: D(7) = 9; D(8) = [13]; D(9) = 11; D(D) = inf
Scan vertex 7: D(8) = 13; D(9) = 11; D(D) = inf
```

```
Scan vertex 8: D(9) = 11; D(D) = [17]
Scan vertex 9: D(D) = [16]

Shortest path in reverse:  D--9--5--2--0, length 16.
```

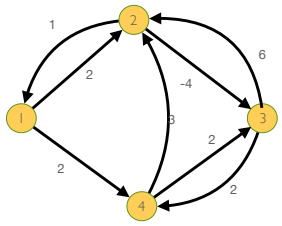
4. (15 points) Consider the graph indicated on the right. This is a directed graph. Use the Floyd-Warshall algorithm to find the shortest distance for all pairs of vertices. Show your work as a sequence of 4 by 4 tables.



Solution: Here is the initial table, obtained from the edges in the graph. the following tables expand the algorithm.

```
D(i,j)     j=1    j=2     j=3    j=4
i=1         0       3       12    inf
i=2        inf      0       12     5
i=3         5       7        0    inf
i=4        inf     inf       3     0

iteration: k = 1 as intermediate vertex
D(i,j)     j=1    j=2     j=3    j=4
i=1         0       3       12    inf
i=2        inf      0       12     5
i=3         5       7        0    inf
i=4        inf     inf       3     0

iteration: k = 2 as intermediate vertex
D(i,j)     j=1    j=2     j=3    j=4
i=1         0       3       12   [8]
i=2        inf      0       12     5
i=3         5       7        0   [12]
i=4        inf     inf       3     0

iteration: k = 3 as intermediate vertex
D(i,j)     j=1    j=2     j=3    j=4
i=1         0       3       12     8
i=2        [17]     0       12     5
i=3         5       7        0    12
i=4        [8]     [10]      3     0

iteration: k = 4 as intermediate vertex
D(i,j)     j=1    j=2     j=3    j=4
i=1         0       3      [11]    8
i=2       [13]      0       [8]    5
i=3         5       7        0    12
i=4         8      10        3     0
```

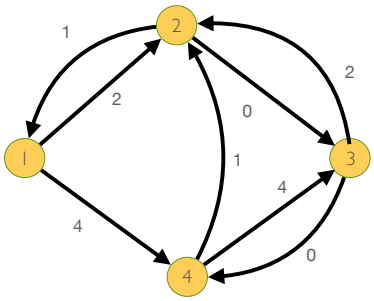
5. (15 points) Consider the graph indicated on the right. This is a directed graph. Note the negative weight edge from 2 to 3. Use Johnson's algorithm, followed by Dijkstra's algorithm, to find the shortest distance for all pairs of vertices.



Solution: The first step in Johnson's algorithm is to add a vertex s with outgoing edges to vertices 1, 2, 3, 4, with weight 0. Then, we find the shortest distance to each vertex from s using the Bellman-Ford algorithm. Let's do that first.

```
Initial Distances:  D(s) = 0; D(1) = inf; D(2) = inf; D(3) = 0; D(4) = inf
Q = {s}
Scan vertex s:        D(1) = 0; D(2) = 0; D(3) = 0; D(4) = 0;
Q = {1,2,3,4}
Scan vertex 1:        D(1) = 0; D(2) = 0; D(3) = 0; D(4) = 0;
Q = {2,3,4}
Scan vertex 2:        D(1) = 0; D(2) = 0; D(3) = -4; D(4) = 0;
Q = {3,4}
Scan vertex 3:        D(1) = 0; D(2) = 0; D(3) = -4; D(4) = -2;
Q = {4}
Scan vertex 4:        D(1) = 0; D(2) = 0; D(3) = -4; D(4) = -2;
```

Done. These distances now form the weights for the second part of the algorithm, where $h(1) = h(2) = 0, h(3) = -4, h(4) = -2$. The revised weights on edge (u, v) are $w_h(u, v) = w(u, v) + h(u) - h(v)$. Thus, the revised weights have $w_h(2, 3) = 0, w_h(3, 2) = 2$, and $w_h(3, 4) = 2 + -4 + 2 = 0, w_h(4, 3) = 2 + -2 + 4 = 4$, and $w_h(4, 2) = 3 - 2 = 1, w_h(1, 4) = 2 + 0 + 2 = 4$. The revised weights are shown below:



We now use Dijkstra's algorithm from each of the 4 starting vertices.

```
Starting at 1,    D[1] = 0, D[2] = inf, D[3] = inf, D[4] = inf.
PQ = {1, 2, 3, 4}
Scan 1:   D[2] =2, D[4] = 4, D[3] = inf
PQ = {2,4,3}
Scan 2: D[3] = 2, D[4] = 4;
PQ = {3,4}
Scan 3:   D[4] = 2
PQ = {4}
Pop 4 and we are done.  Shortest paths are 1 --> 2 --> 3 --> 4, Distances 0, 2, -2, 0.

Starting at 2, D[1]= inf, D[2] = 0, D[3] = inf, D[4] = inf;
PQ = {2, 1, 3, 4}
Scan 2: D[1] = 1, D[3] = 0, D[4] = inf;
PQ = {3,1,4]
```

```
Scan 3: D[1] = 1, D[4] = 0;
PQ = {4,1}
Scan 4: D[1] = 1;
PQ = {1}
Pop 1, we are done.   Shortest paths are 2 --> 3 --> 4,
                                           \
                                            1
True distances D[2] = 0, D[1] = 1, D[3] = -4, D[4] = -2;

Starting at 3, D[1]= inf, D[2] = inf, D[3] = 0, D[4] = inf;
PQ = {3, 1, 2, 4}
Scan 3: D[1] = inf, D[2] = 2, D[4] = 0;
PQ = {4,2,1]
Scan 4: D[2] = 1, D[1] = inf;
PQ = {2,1}
Scan 2: D[1] = 2;
PQ = {1}
Pop 1, we are done.   Shortest paths are 3 --> 4 --> 2 --> 1,
True distances D[3] = 0, D[1] = 6, D[2] = 5, D[4] = 2;

Starting at 4, D[1]= inf, D[2] = inf, D[3] = inf, D[4] = 0;
PQ = {4, 1, 2, 3}
Scan 3: D[1] = inf, D[2] = 1, D[3] = 4;
PQ = {2,3,1]
Scan 2: D[3] = 1, D[1] = 2;
PQ = {3,1}
Scan 3: D[1] = 2;
PQ = {1}
Pop 1, we are done.   Shortest paths are 4 --> 2 --> 3 ,
                                                \
                                                 1
True distances D[3] = -1, D[1] = 4, D[2] = 3, D[4] = 0;
```

6. (15 points) In city streets, the length of an edge often depends on the time of day. Suppose you have a directed graph of streets connecting vertices that represent intermediate destinations, and you are given the travel time on the edge as a function of the time at which you start to travel that edge. Thus, for edge e, you are given $d_e(t)$, the time it takes to travel edge e if you start at time t. These travel times must satisfy an interesting ordering property: You can't arrive earlier if you started later. That is, if $s < t$, then $s + d_e(s) <= t + d_e(t)$. Suppose that you start at time 0 at the origin vertex 1. Describe an algorithm for computing the minimum time path to all vertices when travel times on edges are time dependent and satisfy the ordering property.

   Solution: Turns out Dijkstra's algorithm is exactly right for this problem. Dijkstra's algorithm scans vertices in the order of arrival time. The travel time on the edges for the vertex that are being scanned can be computed because we know the time at which that vertex is reached. The only difference is that the travel times on edges are computed as a function of the travel time to the origin vertex of the edge, when that vertex is permanently labeled by the algorithm.