EC 504
Spring, 2021
Quiz 4

Solutions

1. Which of the following statements are true?

   (a) An undirected graph with no cycles is a tree.

   (b) Consider an undirected graph where, for every pair of nodes i, j, there exists a unique simple path between them. This graph must be a tree.

   Solution: The first statement is false, as there is no guarantee the graph is connected. Hence, it could be a forest.

   The second statement is true. The graph cannot have cycles, and it must be connected, hence it is a tree.

2. Which of the following statements are true?

   (a) Kruskal's algorithm for finding a minimum spanning tree grows a single tree into a minimum spanning tree.

   (b) The worst case complexity of the fastest minimum spanning tree algorithm is $O(n \log n)$, where n is the number of vertices in the graph.

   Solution: Both statements are false. Kruskal's algorithm grows a forest, and the fastest MST algorithm is $O(m + n\log(n))$, where m is the number of edges.

3. Which of the following statements are true?

   (a) Consider a minimum spanning tree T in a connected undirected graph (V, E). Assume that the edge weights in the graph are all distinct (e.g. no two edges have the same weight). Then, an edge i, j in T must have the property that it is either the minimum weight edge connected to vertex i or the minimum weight edge connected to vertex j.

   Solution: This is false. Consider a simple graph with four nodes arranged in a line, where the weight from 1 to 2 is 1, the weight from 2 to 3 is 3, and the weight from 3 to 4 is 1. This graph is already a minimum spanning tree. Note that the edge from 2 to 3 is in the MST, but is neither the smallest weight edge connected to 2 or the smallest weight edge connected to 3.

   (b) Vertices v_1 and v_2 are neighbors in an undirected graph if and only if there is a path between them.

   Solution: This is false; there must be an edge between them.

4. Which of the following statements are true?

   (a) Prim's algorithm and Kruskal's algorithm can also be used to find the maximum weight spanning tree in a graph.

   Solution: This is true. To find the maximum weight, simply compute the negative value of the weights and find the minimum weight. Note that you can always add a constant to every arc, and get the same minimum spanning tree, since the number of arcs is the same for every spanning tree. Thus, you can solve the problem with negative weight arcs.

   (b) Consider a minimum spanning tree in an undirected graph $G = (V, E)$ where all the edge weights are distinct. Then, if an edge e has the largest weight of all the edges in a cycle, that edge e cannot be part of a minimum spanning tree.

   Solution: This is true. If e is in a spanning tree T, then there is an edge e' in the cycle that is not in T (because a tree can have no cycles), and has weight less than e. Then, replacing e with e' results in a new spanning tree that has less weight than T, hence T cannot be a minimum spanning tree.

5. Which of the following are true:

   (a) If a weighted, undirected graph has unique weights on its edges, the smallest weight edge on every cycle must be part of the minimum spanning tree.

   Solution: This is False. Consider a fully connected graph with four vertices, and thus 6 edges. Assign edge weights $w_{12} = 1, w_{13} = 2, w_{14} = 3, w_{23} = 4, w_{24} = 5, w_{34} = 6$. Then, the MST has the edges $w_{12} = 1, w_{13} = 2, w_{14} = 3$, and none of the edges in the cycle 2-3-4 are in the MST.

   (b) If you add an extra edge to a minimum spanning tree, you form a graph with exactly one cycle.

   Solution: This is true. If the edge is $\{i, j\}$ and is not in the MST, then the one cycle formed has the edges in the MST in the path connecting i and j, plus the edge $\{i, j\}$.