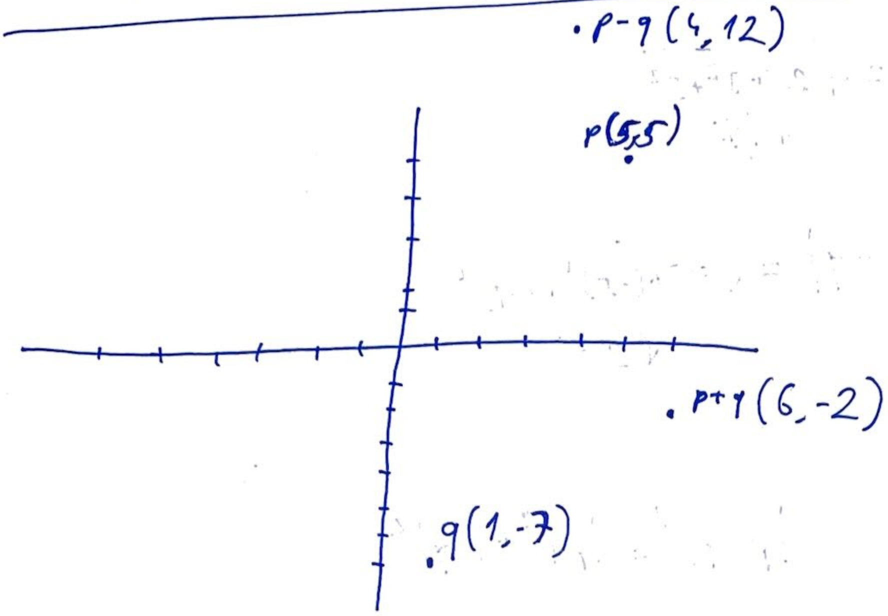
1) $p=(5,5)$ $q=(1,-7)$

$p+q=(6,-2)$ $p-q=(4,12)$



$|p|=\sqrt{5^2+5^2}=5\sqrt{2}$ $|q|=\sqrt{1^2+(-7)^2}=5\sqrt{2}$

$|p+q|=\sqrt{6^2+(-2)^2}=2\sqrt{10}$ $|p-q|=\sqrt{4^2+12^2}=4\sqrt{10}$

$|p+q|^2=(2\sqrt{10})^2=40$

$|p|^2=(5\sqrt{2})^2=50$

$|q|^2=(5\sqrt{2})^2=50$

2) $p=(2,-2,1)$ $q=(2,3,2)$

$p+q=(4,1,3)$ $p-q=(0,-5,-1)$

$|p| = \sqrt{2^2+(-2)^2+1^2}$
$\sqrt{9}$

$|q| = \sqrt{2^2+3^2+2^2}$
$\sqrt{17}$

$|p+q| = \sqrt{4^2+1^2+3^2}$
$\sqrt{26}$

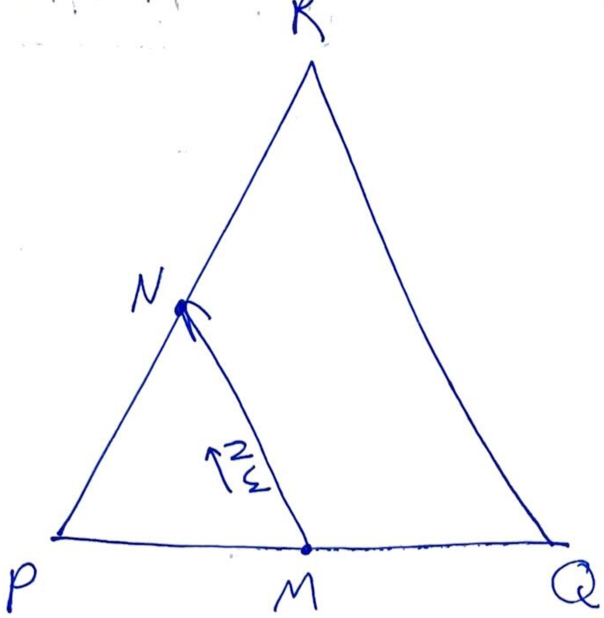
$|p-q| = \sqrt{0^2+(-5)^2+(-1)^2}$
$\sqrt{26}$

$|p+q|^2 = |p|^2 + |q|^2$ $|p-q|^2 = |p|^2 + |q|^2$

$\sqrt{26}^2 = \sqrt{9}^2 + \sqrt{17}^2$ $\sqrt{26}^2 = \sqrt{9}^2 + \sqrt{17}^2$

PERPENDICULAR

3)



$$M = \frac{P+Q}{2} \qquad N = \frac{P+R}{2} \qquad \overrightarrow{MN} = N - M$$

$$\overrightarrow{MN} = \frac{P+R}{2} - \frac{P+Q}{2}$$

$$\overrightarrow{MN} = \frac{R-Q}{2}$$

$$\overrightarrow{QR} = R - Q$$

$$\frac{1}{2} \cdot \overrightarrow{QR} = \overrightarrow{MN}$$

4) $p=(-2,4) \quad q=(3,-5)$ $\quad \boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$

$$p \cdot q = -6 - 20 = -26$$

$$|p| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$$

$$|q| = \sqrt{3^2 + (-5)^2} = \sqrt{34}$$

$$-26 = \sqrt{20} \cdot \sqrt{34} \cdot \cos\theta$$

$$-26 = 2\sqrt{170} \cdot \cos\theta$$

$$-13 = \sqrt{170} \cdot \cos\theta$$

$$\frac{-13}{\sqrt{170}} = \cos\theta$$

$$-0,997 = \cos\theta \longrightarrow \underline{a\cos}$$

$$175.56° = \theta$$

5) $p = (1, -2, 4)$ $\quad q = (3, 5, 2)$

$\boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$

$$p \cdot q = 3 - 10 + 8 = 1$$

$$|p| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$$

$$|q| = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38}$$

$$1 = \sqrt{21} \cdot \sqrt{38} \cdot \cos\theta$$

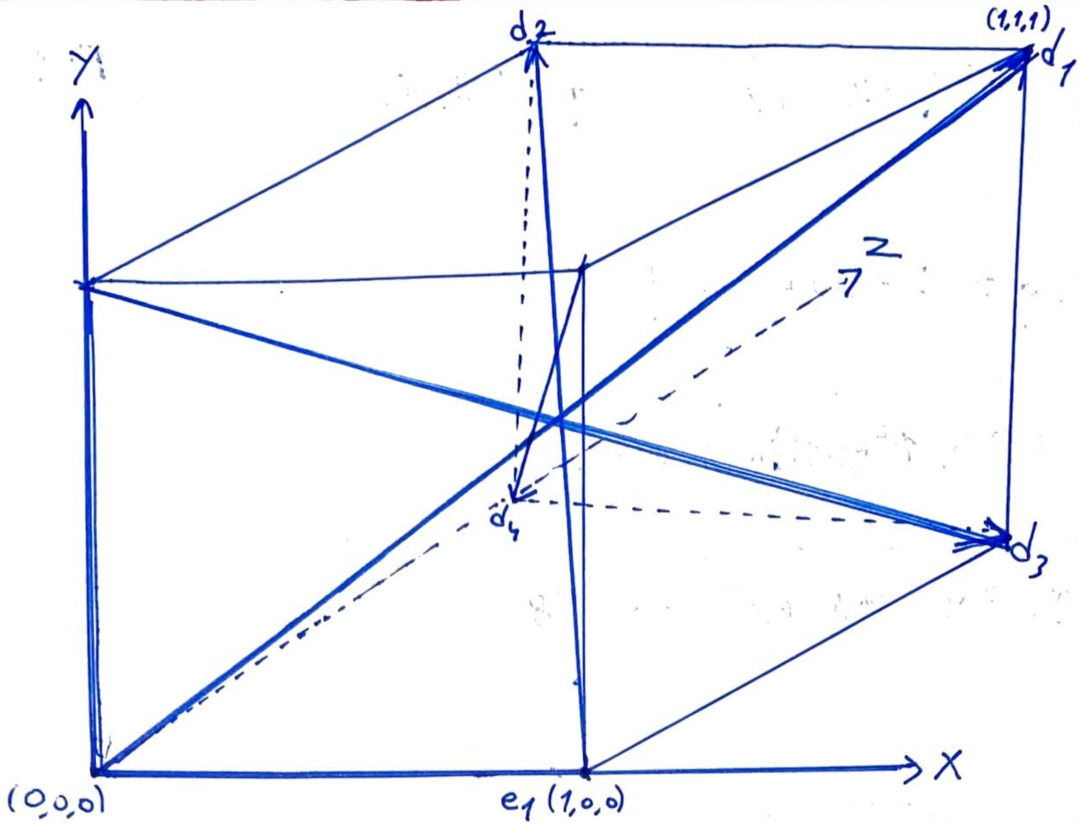
$$1 = \sqrt{798} \cdot \cos\theta$$

$$\frac{1}{\sqrt{798}} = \cos\theta \longrightarrow \text{acos}$$

$$87.97^\circ = \theta$$

6)

$\boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$



$d_1 = (0,0,0) \longrightarrow (1,1,1) = (1,1,1)$ $\quad |d_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$d_2 = (1,0,0) \longrightarrow (0,1,1) = (-1,1,1)$ $\quad |d_2| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{3}$

$d_3 = (0,1,0) \longrightarrow (1,0,1) = (1,-1,1)$ $\quad |d_3| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

$d_4 = (1,1,0) \longrightarrow (0,0,1) = (-1,-1,1)$ $\quad |d_4| = \sqrt{(-1)^2+(-1)^2+1^2} = \sqrt{3}$

$e_1 = (0,0,0) \longrightarrow (1,0,0) = (1,0,0)$ $\quad |e_1| = \sqrt{1^2+0^2+0^2} = \sqrt{1}$

$d_1 \cdot d_2 = |d_1| \cdot |d_2| \cdot \cos\theta$

$-1+1+1 = \sqrt{3} \cdot \sqrt{3} \cdot \cos\theta$

$1 = 3 \cdot \cos\theta$

$\frac{1}{3} = \cos\theta \longrightarrow a\cos$

$70{,}52° = \theta$

$d_1 \cdot e_1 = |d_1| \cdot |e_1| \cdot \cos\theta$

$1+0+0 = \sqrt{3} \cdot \sqrt{1} \cdot \cos\theta$

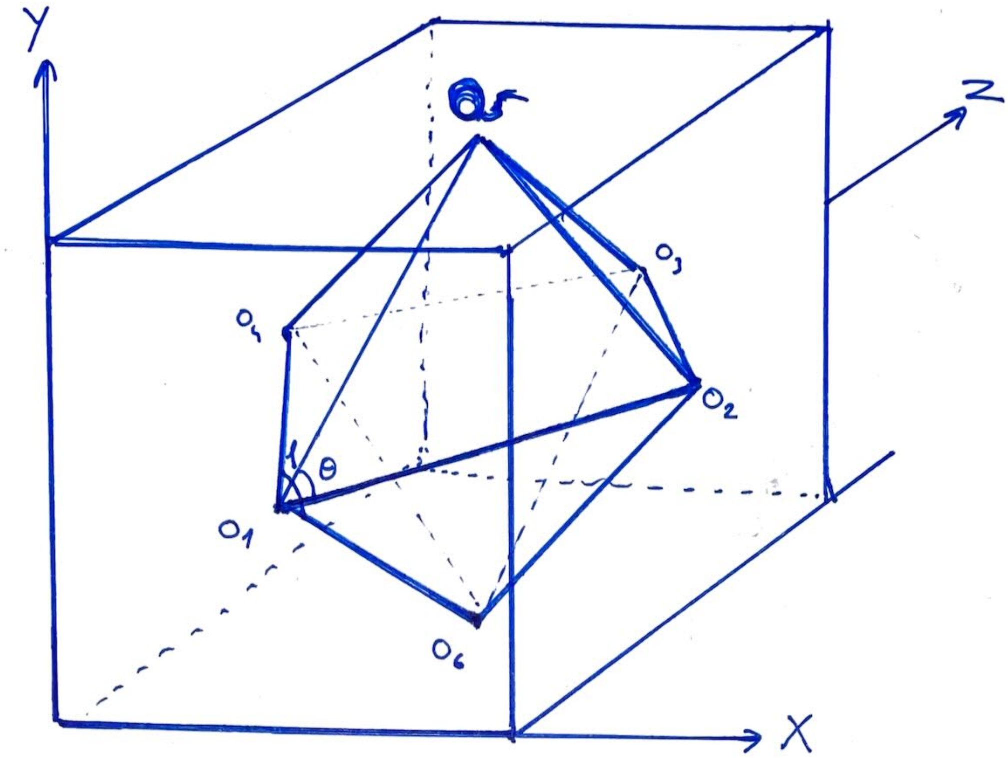
$1 = \sqrt{3} \cdot \cos\theta$

$\frac{1}{\sqrt{3}} = \cos\theta \longrightarrow a\cos$

$54{,}73° = \theta$

7)

$$\boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$$



$$O_{12} = \overset{O_1}{(0.5,\ 0.5,\ 0)} \longrightarrow \overset{O_2}{(1,\ 0.5,\ 0.5)} = (0.5,\ 0,\ 0.5)$$

$$O_{14} = \overset{O_1}{(0.5,\ 0.5,\ 0)} \longrightarrow \overset{O_4}{(0,\ 0.5,\ 0.5)} = (-0.5,\ 0,\ 0.5)$$

$$O_{15} = (0.5,\ 0.5,\ 0) \longrightarrow (0.5,\ 1,\ 0.5) = (0,\ 0.5,\ 0.5)$$

$$O_{16} = \overset{O_1}{(0.5,\ 0.5,\ 0)} \longrightarrow \overset{O_6}{(0.5,\ 0,\ 0.5)} = (0,\ -0.5,\ 0.5)$$

$$|O_{12}| = \sqrt{\left(\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

$$|O_{16}| = \sqrt{0^2 + \left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

$$|O_{15}| = \sqrt{0^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

$$|O_{14}| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

$$O_{12} \cdot O_{15} = |O_{12}| \cdot |O_{15}| \cdot \cos\theta$$

$$\frac{1}{4} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \cos\theta$$

$$\frac{1}{4} = \frac{1}{2} \cdot \cos\theta$$

$$\frac{1}{2} = \cos\theta \longrightarrow \text{acos}$$

$$60^\circ = \theta$$

$$O_{14} \cdot O_{16} = |O_{14}| \cdot |O_{16}| \cdot \cos\ell$$

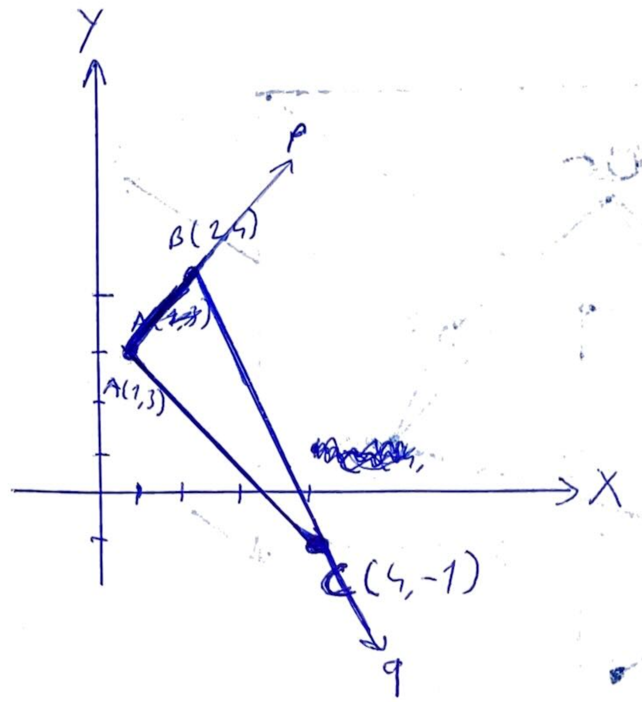
$$\frac{1}{4} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \cos\ell$$

$$\frac{1}{4} = \frac{1}{2} \cdot \cos\ell$$

$$\frac{1}{2} = \cos\ell \rightarrow \text{acos}$$

$$60^\circ = \ell$$

8)



$$proj_q(p) = \frac{p \cdot q}{|q|^2} \cdot q$$

Distance $= \vec{BA} - proj_{BC}(\vec{BA})$

$p = \vec{AB} = B - A = (1,1)$ $\quad |p| = \sqrt{2}$

$q = \vec{BC} = C - B = (2,-5)$ $\quad |q| = \sqrt{29}$

Projection of p onto q

$$\frac{p \cdot q}{|q|^2} \cdot q = \frac{(2-5)}{\sqrt{29}^2} \cdot (2,-5) = \frac{-3}{29} \cdot (2,-5) = \left(\frac{-6}{29}, \frac{15}{29}\right)$$

$$proj_q(p) = \left(\frac{-6}{29}, \frac{15}{29}\right)$$

Distance of A to $q(\vec{BC})$

$\vec{BA} = A - B = -p = (-1,-1)$

$$\vec{BA} - proj_q(\vec{BA}) = \vec{BA} - proj_q(-p) = \vec{BA} - \left(\frac{+6}{29}, \frac{-15}{29}\right) = (-1,-1) - \left(\frac{+6}{29}, \frac{-15}{29}\right)$$

$$= \left(\frac{-35}{29}, \frac{-14}{29}\right)$$

$$\text{Distance} = \sqrt{\left(\frac{-35}{29}\right)^2 + \left(\frac{-14}{29}\right)^2} = \sqrt{\frac{1421}{(29)^2}} = \frac{7\sqrt{29}}{29}$$

Area: Distance $\times |\vec{BC}| \times \frac{1}{2}$

$$\frac{7\sqrt{29}}{29} \times \sqrt{29} \times \frac{1}{2} = \frac{7}{2} = \boxed{3,5}$$

9) $p = (2, -3, 1)$ $\quad P_{paralel} = Proj_q(P)$

$q = (12, 3, 4)$ $\quad P_{perpendicular} = P - Proj_q(P)$

$$\boxed{Proj_q(P) = \frac{P \cdot q}{|q|^2} \cdot q}$$

$\boxed{P_{paralel}}$

$$\frac{P \cdot q}{|q|^2} \cdot q = \frac{24 + (-9) + 4}{(12^2 + 3^2 + 4^2)} \cdot (12, 3, 4) = \frac{19}{169} \cdot (12, 3, 4)$$

$$= \left(\frac{228}{169}, \frac{57}{169}, \frac{76}{169}\right)$$

$\boxed{P_{perpendicular}}$

$$P - Proj_q(P) = (2, -3, 1) - \left(\frac{228}{169}, \frac{57}{169}, \frac{76}{169}\right)$$

$$= \left(\frac{110}{169}, \frac{-564}{169}, \frac{93}{169}\right)$$

$\boxed{P = P_{paralel} + P_{perpendicular}}$ ✓

$$\left(\frac{228}{169}, \frac{57}{169}, \frac{76}{169}\right) + \left(\frac{110}{169}, \frac{-564}{169}, \frac{93}{169}\right) = (2, -3, 1)$$

$\boxed{P_{perpendicular} \cdot q = 0}$ ✓

$$\left(\frac{110}{169}, \frac{-564}{169}, \frac{93}{169}\right) \cdot (12, 3, 4) = \frac{1320 - 1692 + 372}{169} = \frac{0}{169} = 0$$

10) $|p+q|^2 + |p-q|^2 = 2|p|^2 + 2|q|^2$

$|p+q|^2 = |p+q| \cdot |p+q| = |p|^2 + 2pq + |q|^2$

$|p-q|^2 = |p-q| \cdot |p-q| = |p|^2 - 2pq + |q|^2$

$(|p|^2 + 2pq + |q|^2) + (|p|^2 - 2pq + |q|^2) = 2|p|^2 + 2|q|^2$

$\boxed{\text{☆ Perpendicular}}$

$p = (1,0) \quad q = (0,1)$

$|p| = \sqrt{1^2+0^2} \quad |q| = \sqrt{0^2+1^2}$

$p+q = (1,1) \quad p-q = (1,-1)$

$|p+q| = \sqrt{1^2+1^2} \quad |p-q| = \sqrt{1^2+(-1)^2}$

$(\sqrt{2})^2 + (\sqrt{2})^2 = 2.(\sqrt{1})^2 + 2.(\sqrt{1})^2$

$p = (1,0) \quad q = (1,1)$

$\boxed{\text{☆ Non-Perpendicular}}$

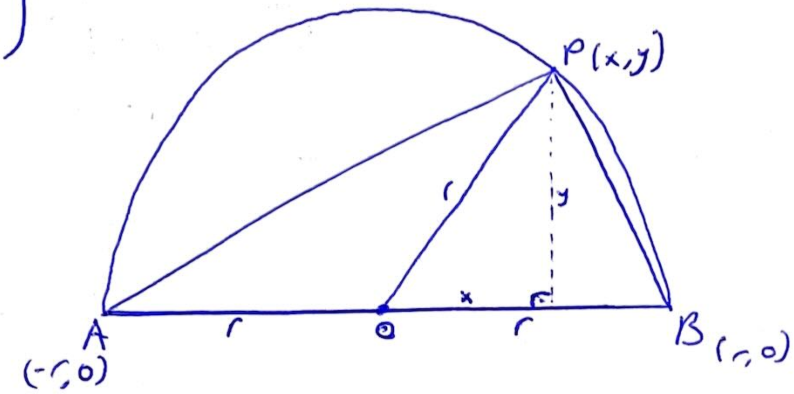
$|p| = \sqrt{1^2+0^2} \quad |q| = \sqrt{1^2+1^2}$

$p+q = (2,1) \quad p-q = (0,-1)$

$|p+q| = \sqrt{5} \quad |p-q| = \sqrt{1}$

$(\sqrt{5})^2 + (\sqrt{1})^2 = 2.(\sqrt{1})^2 + 2.(\sqrt{2})^2$

11)



$\boxed{x^2+y^2=r^2}$

$\boxed{p.q=|p|.|q|.\cos\theta}$

$\overrightarrow{AP} = \vec{P} - \vec{A} = (x,y)-(-r,0) = (x+r, y)$

$\overrightarrow{BP} = \vec{P} - \vec{B} = (x,y)-(r,0) = (x-r, y)$

$\overrightarrow{AP} . \overrightarrow{BP} = (x+r, y).(x-r, y) = (x^2-r^2+y^2)$

$= \underbrace{x^2+y^2}-r^2$

$= r^2 - r^2$

$= 0$

$\cos(0) = 90°$

PERPENDICULAR

12) $(p - \lambda q) \cdot (p - \lambda q) \geqslant 0$

$$\lambda = \frac{p \cdot q}{|q|^2}$$

$$\cancel{p^2 - 2\lambda(p \cdot q) + \lambda^2 (q \cdot q) \geqslant 0}$$

$$|p|^2 - 2\lambda(p \cdot q) + \lambda^2 |q|^2 \geqslant 0$$

$$|p|^2 - 2\left(\frac{p \cdot q}{|q|^2}\right)(p \cdot q) + \left(\frac{p \cdot q}{|q|^2}\right)^2 \cdot |q|^2 \geqslant 0$$

$$|p|^2 - 2 \cdot \frac{(p \cdot q)^2}{|q|^2} + \frac{(p \cdot q)^2}{|q|^2} \geqslant 0$$

$$|p|^2 - \frac{(p \cdot q)^2}{|q|^2} \geqslant 0$$

$$|p|^2 \cdot |q|^2 \geqslant (p \cdot q)^2$$

$|p| \cdot |q| \geqslant (p \cdot q)$ → Cauchy-Schwarz Inequality

Two vectors can never be more "aligned" than the length of each one multiplied together

13) $|p| + |q| \geq |p+q|$ ⟶ Triangle Inequality

$|p| \cdot |q| \geq (p \cdot q)$ ⟶ Cauch - Schwarz Inequality

---

$$(|p| + |q|)^2 \geq (|p+q|)^2$$

$$|p|^2 + 2\underbrace{|p||q|}_{(p \cdot q)} + |q|^2 \geq |p+q|^2$$

$$|p|^2 + 2p \cdot q + |q|^2 \geq |p+q|^2$$

$$(|p| + |q|)^2 \geq |p+q|^2$$

$$\underline{|p| + |q| \geq |p+q|}$$

14) $\big|\,|p| - |q|\,\big| \leq |p-q|$

$$\big(\big|\,|p| - |q|\,\big|\big)^2 \leq \big(|p-q|\big)^2$$

$$|p|^2 - 2|p||q| + |q|^2 \leq |p|^2 - 2p\cdot q + |q|^2$$

$$-2|p||q| \leq -2p\cdot q$$

$$\underline{|p||q| \geq p\cdot q} \longrightarrow \text{Cauch-Schwarz Inequality}$$

15) $p = |p|(\cos\phi, \sin\phi)$ $\quad$ R^2

$\mathbf{p} = (p_1, p_2)$

$|p| = \sqrt{p_1^2 + p_2^2}$

$$\boxed{p_1 = |p| \cdot \cos\phi \qquad p_2 = |p| \cdot \sin\phi}$$

$\mathbf{p} = (p_1, p_2) = (|p| \cdot \cos\phi, |p| \cdot \sin\phi) = \underline{|p| \cdot (\cos\phi, \sin\phi)}$

$$\boxed{u_p = \frac{p}{|p|}}$$

$p = |p| \cdot (\cos\phi, \sin\phi)$

$$u_p = \frac{|p| \cdot (\cos\phi, \sin\phi)}{|p|}$$

$\underline{U_p = (\cos\phi, \sin\phi)}$

$\sin^2\phi + \cos^2\phi = 1$

$$\boxed{p_1 = |p| \cdot \cos\phi \qquad p_2 = |p| \cdot \sin\phi}$$

$\cos\phi = \dfrac{p_1}{|p|} \qquad \sin\phi = \dfrac{p_2}{|p|}$

$$\frac{p_1^2 + p_2^2}{|p|^2} = 1 \longrightarrow \frac{p_1^2 + p_2^2}{\left(\sqrt{p_1^2 + p_2^2}\right)^2} = 1$$

16) $p = (p_1, p_2, p_3)$ $\quad$ R^3

$$\boxed{U_p = \frac{p}{|p|}}$$

standard unit vectors

$i(1,0,0) \quad j(0,1,0) \quad k(0,0,1)$

$$|p| = \sqrt{p_1^2 + p_2^2 + p_3^2}$$

$a = (a_1, a_2, a_3) \quad b = (b_1, b_2, b_3) \quad a.b = (a_1 b_1 + a_2 b_2 + a_3 b_3)$

$$U_p . i = \left(\frac{p_1}{|p|}, \frac{p_2}{|p|}, \frac{p_3}{|p|}\right) . (1,0,0) = \frac{p_1}{|p|}$$

$$U_p . j = \left(\frac{p_1}{|p|}, \frac{p_2}{|p|}, \frac{p_3}{|p|}\right) . (0,1,0) = \frac{p_2}{|p|}$$

$$U_p . k = \left(\frac{p_1}{|p|}, \frac{p_2}{|p|}, \frac{p_3}{|p|}\right) . (0,0,1) = \frac{p_3}{|p|}$$

$$\boxed{p = |p| . \cos \alpha}$$

$$\cos a_1 = \frac{p_1}{|p|} \qquad \cos a_2 = \frac{p_2}{|p|} \qquad \cos a_3 = \frac{p_3}{|p|}$$

$$\boxed{\cos^2_{a_1} + \cos^2_{a_2} + \cos^2_{a_3} = 1}$$

$$\left(\frac{p_1}{|p|}\right)^2 + \left(\frac{p_2}{|p|}\right)^2 + \left(\frac{p_3}{|p|}\right)^2 = 1 \qquad \frac{p_1^2 + p_2^2 + p_3^2}{\left(\sqrt{p_1^2 + p_2^2 + p_3^2}\right)^2} = 1$$

$$\boxed{p = |p| (\cos a_1, \cos a_2, \cos a_3)}$$

$$p_1 = |p| . \cos a_1 \qquad p_2 = |p| . \cos a_2 \qquad p_3 = |p| . \cos a_3$$

$$p = (p_1, p_2, p_3) = |p| (\cos_{a_1}, \cos_{a_2}, \cos_{a_3})$$

17) $p = (3, -4, 12)$

$$\boxed{\cos_{\alpha_1} = \frac{p_1}{|p|}}$$

$$\boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$$

$$|p| = \sqrt{3^2 + (-4)^2 + 12^2}$$

$$|p| = 13$$

$$\cos_{\alpha_1} = \frac{3}{13} = 76,66°$$

$$\cos_{\alpha_2} = \frac{-4}{13} = 107.92°$$

$$\cos_{\alpha_3} = \frac{12}{13} = 22.62°$$

18)

$$\boxed{p \cdot q = |p| \cdot |q| \cdot \cos\theta}$$

$$\cos(a-b) = \cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b)$$

$$e_a = (\cos(a), \sin(a)) \qquad e_b = (\cos(b), \sin(b))$$

$$a = (a_1, a_2) \qquad b = (b_1, b_2)$$

$$e_a \cdot e_b = (\cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b))$$

$$e_a \cdot e_b = |e_a| \cdot |e_b| \cdot \cos\theta$$

Unit Vectors = 1 $\qquad \theta \rightarrow (a-b)$

$$|e_a| \cdot |e_b| \cdot \cos(a-b) = \cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b)$$

$$1 \cdot 1 \cdot \cos(a-b) = \cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b)$$

$$\cos(a-b) = \cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b)$$

19) $p \cdot q = 2 p_1 \cdot q_1 + p_2 \cdot q_2$ $\quad R^2$

$a, b \quad R \qquad\qquad p, r \quad R^2$

$$(a.p + b.r) \cdot q = 2(a.p_1 + b.r_1).q_1 + (a.p_2 + b.r_2).q_2$$

$$= 2.a.p_1.q_1 + 2.b.r_1.q_1 + a.p_2.q_2 + b.r_2.q_2$$

$$= a(2.p_1.q_1 + p_2.q_2) + b(2.r_1.q_1 + r_2.q_2)$$

$$= a(p.q) + b(r.q)$$

$p \cdot p \geqslant 0$

$p \cdot p = 2 p_1 \cdot p_1 + p_2 \cdot p_2 = 2p_1^2 + p_2^2$

$\underline{2p_1^2 + p_2^2 \geqslant 0}$

20)

$$g_{ij} = e_i \cdot e_j$$

$$p \cdot q = |p| \cdot |q| \cdot \cos\theta$$

$$p \cdot q = p_1 \cdot q_1 + p_2 \cdot q_2$$

$$p \cdot q = \sum_{i,j} g_{ij} \cdot p^i \cdot q^j$$

$$p \cdot q = g_{11} \cdot p^1 \cdot q^1 + g_{12} \cdot p^1 \cdot q^2 + g_{21} \cdot p^2 \cdot q^1 + g_{22} \cdot p^2 \cdot q^2$$

$$g_{12} = g_{21}$$

$$\underline{p \cdot q = g_{11} \cdot p^1 \cdot q^1 + 2 g_{12} \cdot p^1 \cdot q^2 + g_{22} \cdot p^2 \cdot q^2}$$