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Answer 1

a) Since $f(x) \geq 0$ for every element in domain,there is not any negative value in codomain. It means f is not surjective. (Counter example)

$$f(-1) = f(1) = 1$$

We have same result for two different element from domain. It means f is not injective.(Counter example)

b) Since $f(x) \geq 0$ for every element in domain,there is not any negative value in codomain. It means f is not surjective. (Counter example)

For this function if $f(x) = f(y)$, where $x, y \in$ A, x and y must be equal. By definition this function is injective.

$$(f(x) = f(y)) \implies (x = y)) \equiv True$$

c) Let $y \in$ B and $x \in$ A and $f(x) = y$. So $\overset{-1}{f}(y) = x$ and $f(\overset{-1}{f}(y)) = y$. It means that for all $y \in$ B there exists a $x \in$ A. In other words all elements in B is the image of some element in A. By definiton this function is surjective.

$$f(-2) = f(2) = 4$$

We have same result for two different element from domain. It means f is not injective.(Counter example)

d) Let $y \in$ B and $x \in$ A and $f(x) = y$. So $\overset{-1}{f}(y) = x$ and $f(\overset{-1}{f}(y)) = y$. It means that for all $y \in$ B there exists a $x \in$ A. By definiton this function is surjective.

For this function if $f(x) = f(y)$, where $x, y \in$ A, x and y must be equal. By definition this function is injective.

$$(f(x) = f(y)) \implies (x = y)) \equiv True$$

Answer 2

a) We can take any δ. Let's say $0 < \delta < 1$. Since we are working on integers, only x that makes this inequality true is x_0. So $||x_0 - x_0|| = 0$. Therefore $||f(x_0) - f(x_0)|| = 0$, and it shows that $||f(x) - f(x_0)|| < \epsilon$. Since x_0 was random f is continous everywhere on its domain.

b) Let's assume constant function is not the only way. Our domain is R which is a uncountable infinite set. Since this set is mapping to Z which is a countable set, our x values are going to be countable as well. So our δ will not go to the ∞ i.e. we can find a $\delta > 0$.

Let's consider ϵ's situation. We have uncountable domain. These values can map to uncountably many value which different than $f(x_0)$ (They can be same between them.). It means that our $||f(x) - fx_0||$ value can go to the ∞. Therefore we cannot find a ϵ value, but if our function is constant then our $||f(x) - fx_0||$ is going to be 0 and we can find a ϵ value. To sum up, we can give a counter example function which have these properties, and our starting claim is wrong. Proof by contradiction a constant function is the only way.

Answer 3

a) Suppose that a,b,c,... are finite countable sets. If we can show that natural numbers set can map to every single element in the cartesian product set, then it's countable as well, because natural number set is countable.

Sum = n	Sum = n+1	Sum = n+2	Sum = n+3 .....
$(a_1, b_1, c_1....)$	$(a_2, b_1, c_1....)$	$(a_3, b_1, c_1....)$	$(a_4, b_1, c_1....)$
.	$(a_1, b_2, c_1....)$	$(a_1, b_3, c_1....)$	$(a_1, b_4, c_1....)$
.	$(a_1, b_1, c_2....)$	$(a_1, b_1, c_3....)$	$(a_1, b_1, c_4....)$
.	.	$(a_2, b_2, c_1....)$	$(a_3, b_2, c_1...)$
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.

We can arrange the elements of cartesian products by using summation of element number in their own set. Since all sets are countable natural number set can map to this arrangement. Therefore this set is countable.

b) Let's assume that this set is countable. We can use Cantor's theorem to dissprove that. We can take elements from our cartesian product set. $A_1, A_2, ...$
$A_1 = (a_{11}, a_{12}, ...)$
$A_2 = (a_{21}, a_{22}, ...)$

A $= (a_1, a_2, a_3....), a_i$ is equal to 0 or 1.

$$a_i = \begin{cases} 0, & \text{if } a_{ii} = 1 \\ 1, & \text{if } a_{ii} = 0 \end{cases}$$

If we choose elements of A like this, we will see there is not such a element in cartesian product set, i.e. A is not equal to any $A_1, A_2, ...$. Therefore our claim is wrong. Proof by contradiction this set is uncountable.

Answer 4

I am going to use limits to show which function's growth rate is bigger, and I am going to use L'hospital rule to determine results.

a)

$$\lim_{n \to +\infty} \frac{2^n}{n^{50}} = \lim_{n \to +\infty} \frac{(2^n)log2}{50(n^{49})} = ... = \lim_{n \to +\infty} \frac{(2^n)(log2)^{50}}{50!} = \infty$$

Therefore $n^{50} = O(2^n)$

b)

$$\lim_{n \to +\infty} \frac{n^{50}}{(logn)^2} = \lim_{n \to +\infty} \frac{50n^{49}}{\frac{2logn}{n}} = \lim_{n \to +\infty} \frac{50 * 49n^{48}}{\frac{2}{n^2}} = \lim_{n \to +\infty} 25 * 49n^{50} = \infty$$

$(logn)^2 = O(n^{50})$

c)

$$\lim_{n \to +\infty} \frac{(logn)^2}{\sqrt{n}logn} = \lim_{n \to +\infty} \frac{\frac{2logn}{n}}{\frac{logn + 2}{2\sqrt{n}}} = 0$$

$(logn)^2 = O(\sqrt{n}logn)$

d)

$$\lim_{n \to +\infty} \frac{\sqrt{n}logn}{5^n} = \lim_{n \to +\infty} \frac{\frac{logn + 2}{n}}{5^n log5} = 0$$

$\sqrt{n}logn = O(5^n)$

e)

$$\lim_{n \to +\infty} \frac{5^n}{(n!)^2} = \frac{5 * 5 * 5 * 5 * 5 * 5 * 5...}{1 * 1 * 2 * 2 * 3 * 3 * 4 * 4 * 5 * 5 * 6 * 6 * 7 * 7..} = 0$$

$5^n = O((n!)^2)$

f)

$$\lim_{n\to+\infty} \frac{(n!)^2}{n^{51}+n^{49}} = \infty$$

$n^{51}+n^{49} = O((n!)^2)$

We can see $2^n = O(5^n)$ and $n^{50} = O(n^{51}+n^{49})$ clearly. So the final arrangement is:

$$(n!)^2, 5^n, 2^n, n^{51}+n^{49}, n^{50}, \sqrt{n}logn, (logn)^2$$

Answer 5

a) Euclid's algorithm states that gcd(a,b) = gcd(b,a mod b).

$$134 \equiv 40(mod94)$$

$$gcd(134,94) = gcd(94,40)$$
$$94 \equiv 14(mod40)$$
$$gcd(94,40) = gcd(40,14)$$
$$40 \equiv 12(mod14)$$
$$gcd(40,14) = gcd(14,12)$$
$$14 \equiv 2(mod12)$$
$$gcd(14,12) = gcd(12,2)$$
$$12 \equiv 0(mod12)$$

We have 0. Then the last number is the result of gcd(134,94) which is 2.

b) PROOF OF SUM OF THREE PRIMES BY USING GOLDBACH'S CONJECTURE:

Let $a > 5$ be an integer.

CASE 1: a is even.

$$a = 2n, n \geq 3 \quad and \quad a - 2 = 2n - 2$$

is even as well. By using Goldbach's conjecture:

$$2n - 2 = p_1 + p_2$$

where p_1 and p_2 are prime numbers.

$$a = 2n = p_1 + p_2 + 2$$

Therefore a is sum of three prime numbers.

CASE 2: a is odd.

$$a = 2n + 1, n \geq 3 \quad and \quad a - 3 = 2n - 2$$

is even. By using Goldbach's conjecture:

$$2n - 2 = p_1 + p_2$$

where p_1 and p_2 are prime numbers.

$$a = 2n + 1 = p_1 + p_2 + 3$$

Therefore a is sum of three prime numbers.

PROOF OF GOLDBACH'S CONJECTURE BY USING SUM OF THREE PRIMES:

CASE 1: a is even.

We can express a even number as sum of three numbers as follows:

$$even + even + even \quad or \quad even + odd + odd$$

So we need at least 1 even number. This number must be 2 according to sum of three primess rule.

$$a = 2n, n \geq 3$$

$$2n = p_1 + p_2 + 2$$

where p_1 and p_2 are prime numbers.

$$2n - 2 = p_1 + p_2$$

CASE 2: a is odd.

We can express a odd number as sum of three numbers as follows:

$$even + even + odd \quad or \quad odd + odd + odd$$

Even numbers must be 2 according to sum of three primes rule.

$$a = 2n + 1, n \geq 3$$

$$2n + 1 = p_1 + 2 + 2$$

where p_1 is odd prime number.

$$2n - 1 = p_1 + 2$$

Left hand side is still odd and right and side is sum of two prime numbers.

<u>Now let's consider other case:</u>

$$a = p_1 + p_2 + p_3$$

where all numbers are odd prime numbers. Sum of two of them is even.

$$a - p_1 - p_2 = p_3$$

a is still odd. If we sum both sides with 2.

$$a - p_1 - p_2 + 2 = p_3 + 2$$

Left hand side is still odd and right and side is sum of two prime numbers.