CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 2

Karabacak, Kerem
e2644417@ceng.metu.edu.tr

I worked individually.
exxxxxxx@ceng.metu.edu.tr

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1. (a) y'(t) + 5y(t) = x(t)

   (b) We can use the fact $y(t) = y_p(t) + y_h(t)$, and we can ignore u(t) while doing the calculations. We can assume we are working for non-negative t's and add the step function in the end. Also, we know $y_h(t)$ is in form $C.e^{st}$.

$$(C.e^{st})' + 5Ce^{st} = 0$$
$$C.se^{st} + 5Ce^{st} = 0$$
$$Ce^{st}(s + 5) = 0$$

We don't want C = 0, and we can't get 0 from exponential part. So, s = -5. We will compute C with initial conditions at the end.

$$y_h(t) = C.e^{-5t}$$

We know $y_p(t)$ is in form $K.x(t)$, in this question we can convert it to $A.e^{-t} + Be^{-3t}$. New equation is:

$$(A.e^{-t} + Be^{-3t})' + 5(A.e^{-t} + Be^{-3t}) = e^{-t} + e^{-3t}$$
$$-A.e^{-t} - 3Be^{-3t} + 5Ae^{-t} + 5Be^{-3t} = e^{-t} + e^{-3t}$$
$$4A.e^{-t} + 2Be^{-3t} = e^{-t} + e^{-3t}$$
$$A = \frac{1}{4} \qquad B = \frac{1}{2}$$

Therefore general expression for y(t):

$$y(t) = y_p(t) + y_h(t) = \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t} + C.e^{-5t}$$

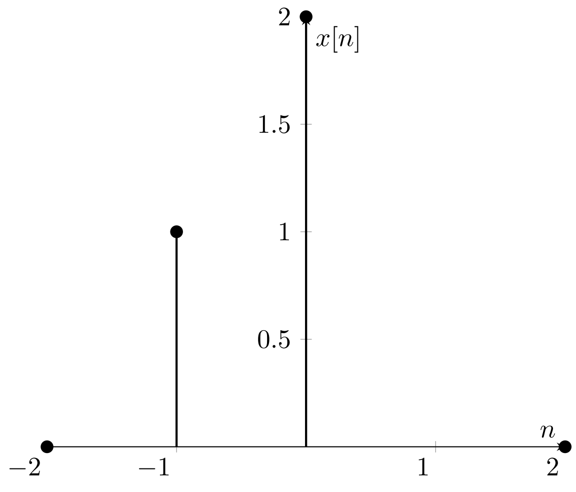
Question states that system is in initial rest, it means system is causal i.e. y(0) = 0, y'(0) = 0, ... We can use this fact to compute C.

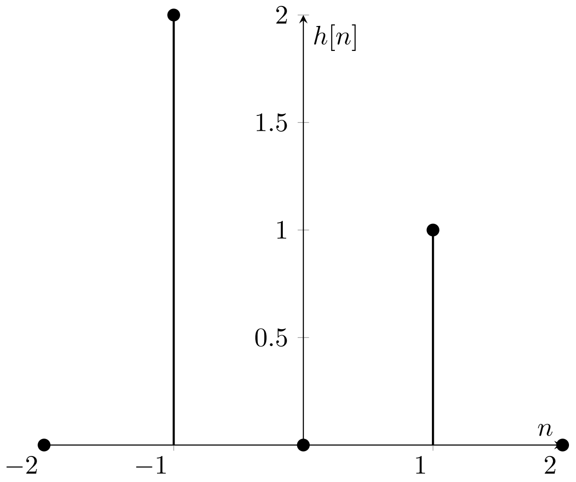
$$y(0) = \frac{1}{4} + \frac{1}{2} + C = 0$$
$$C = \frac{-3}{4}$$

Finally, we can add the step function and the result is:

$$y(t) = (\frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t} - \frac{3}{4}e^{-5t})u(t)$$

2. (a) At first let's look x[n] and h[n]'s graphs:





Since we are working on discrete time we should use convolution sum formula. All sums (variable k) are going $-\infty$ to ∞.

$$y[n] = \sum x[k]h[n-k]$$

Now we can consider all values:

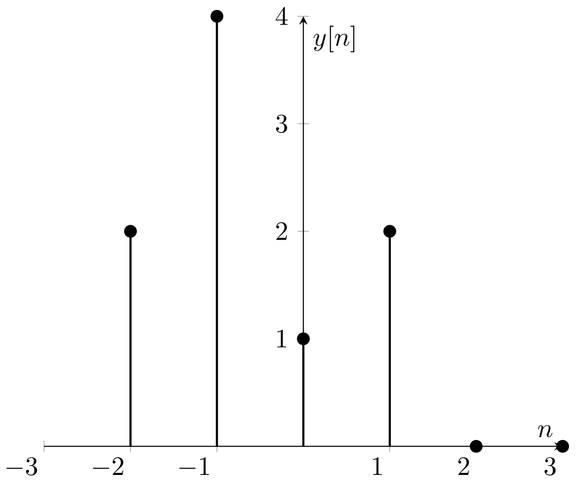
$$y[-2] = \sum x[k]h[-2-k] = x[-1]h[-1] = 2$$

$$y[-1] = \sum x[k]h[-1-k] = x[0]h[-1] = 4$$

$$y[0] = \sum x[k]h[-k] = x[-1]h[1] = 1$$

$$y[1] = \sum x[k]h[1-k] = x[0]h[1] = 2$$

All values except these are 0. Finally, y[n]'s graph is following according to values above:



(b) We know step function's derivative is impulse function,

$$x'(t) = \delta(t-1) + \delta(t+1)$$

We can ignore u(t) and we can assume we are working on non-negative numbers.
We can use distributive property of convolution,

$$y(t) = x'(t) * h(t) = \delta(t-1) * h(t) + \delta(t+1) * h(t)$$

From the definition of convolution, if we take convolution of some function with impulse function the result will equal to function itself, bu will have the shift of impulse function. Therefore,

$$y(t) = x'(t) * h(t) = h(t-1) + h(t+1)$$

$$y(t) = [e^{-t+1} sin(t-1) + e^{-t-1} sin(t+1)]u(t)$$

3. (a) We can use standard form:

$$\int_{-\infty}^{\infty} x(t)h(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

Expression is non-zero when $\tau > 0$ and $t - \tau > 0$ because of step functions. So, $0 < \tau < t$. This is the new boundaries of integral, we can get rid of step functions, since we are working on non-zero values only from now on.

$$\int_0^t e^{-\tau}e^{-2t+2\tau}d\tau = e^{-2t}\int_0^t e^{\tau} = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}$$

Finally, we can put the step function back.

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

(b) Again with the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} (u(\tau) - u(\tau - 1))e^{3t-3\tau}u(t-\tau)d\tau$$

This expression is non-zero only at $0 < \tau < 1$ and $t > \tau$ and x(t) = 1. So we should consider to different cases seperately. Let's start with (0,1):

$$\int_0^1 e^{3t-3\tau}d\tau = e^{3t}\int_0^1 e^{-3\tau}d\tau = \frac{1}{3}e^{3t}(e^{-3}+1)$$

For (1,t) case:

$$\int_1^t e^{3t-3\tau}d\tau = \frac{(e^{3t-3}-1)}{3}$$

Finally, we can add the step functions:

$$y(t) = (\frac{1}{3}e^{3t}(e^{-3}+1))(u(t) - u(t-1)) + \frac{(e^{3t-3}-1)}{3}(u(t-1))$$

4. (a) We have a recursive equation. At first, let's look at the first values of y[n], and try to see its solution trivally.

$$y[2] - y[1] - y[0] = 0, \quad y[2] = 2$$

$$y[3] - y[2] - y[1] = 0, \quad y[3] = 3$$

$$y[4] - y[3] - y[2] = 0, \quad y[4] = 5$$

And so on. We have 1,1,2,3,5,8.. It's famous Fibonacci recurrence relation, but its solution is not trival, so we need to use some recursion solving techniques. Let's start with characteristic polynomial.

$$r^2 - r - 1 = 0, \quad r_1 = \frac{1-\sqrt{5}}{2}, \quad r_2 = \frac{1+\sqrt{5}}{2}$$

Our y[n] is $A(\frac{1-\sqrt{5}}{2})^n + B(\frac{1+\sqrt{5}}{2})^n$. We can find A and B constants with using initial conditions.

$$y[0] = 1 \rightarrow 1 = A + B$$

$$y[1] = \rightarrow 1 = A(\frac{1-\sqrt{5}}{2}) + B(\frac{1+\sqrt{5}}{2})$$

$$A = \frac{\sqrt{5}-1}{2\sqrt{5}} \quad B = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

Therefore general solution is:

$$y[n] = \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n + \frac{\sqrt{5}+1}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n, \quad y[0] = 1, y[1] = 1, \quad n \geq 2$$

It also holds for initial conditions.

(b) We can look for characteristic equation of system.

$$y^3 - 6y^2 + 13y - 10$$

Since we have 3^{rd} degree equation, let's try to find a trival root which is 2.

$$8 - 6.4 + 13.2 - 10 = 0$$

Since we know the root, now we can do polynomial division.

$$\frac{y^3 - 6y^2 + 13y - 10}{y-2} = y^2 - 4y + 5$$

Let's consider this polynom's roots with help of discriminant formula, and roots are:

$$y_1 = 2 - 2j, \quad y_2 = 2 + 2j$$

If we think in form of $a \pm bj$, we know the roots are in form of $c_1.e^{at}cos(bt) + c_2.e^{at}sin(bt)$. Therefore general solution:

$$y(t) = y_h(t) = c_1e^{2t} + c_2e^{2t}cos(2t) + c_3e^{2t}sin(2t)$$

Now we can find the coefficents with initial conditions.

$$y''(t) = 4c_1e^{2t} - 8c_2e^{2t}sin(2t) + 8c_3e^{2t}cos(2t)$$

$$y''(0) = 4c_1 + 8c_3 = 3$$

$$y'(t) = 2c_1e^{2t} + 2c_2e^{2t}(cos(2t) - sin(2t)) + 2c_3e^{2t}(cos(2t) + sin(2t))$$

$$y'(0) = 2c_1 + 2c_2 + 2c_3 = \frac{3}{2}$$

$$y(0) = c_1 + c_2 = 1$$

$$c_1 = \frac{5}{4} \quad c_2 = -\frac{1}{4} \quad c_3 = -\frac{1}{4}$$

Therefore final general solution to differential equation:

$$y(t) = y_h(t) = \frac{5}{4}e^{2t} - \frac{1}{4}e^{2t}cos(2t) - \frac{1}{4}e^{2t}sin(2t)$$

5. (a) $y_p(t)$ will be in form Acos(5t) + Bsin(5t). According to this:

$$(Acos(5t) + Bsin(5t))'' + 5(Acos(5t) + Bsin(5t))' + 6(Acos(5t) + Bsin(5t)) = cos(5t)$$

$$(-25Acos(5t) - 25Bsin(5t)) + (-25Asin(5t) + 25Bcos(5t)) + (6Acos(5t)6Bsin(5t)) = cos(5t)$$

$$(-19B - 25A)sin(5t) + (25B - 19A)cos(5t) = cos(5t)$$

$$-19B - 25A = 0$$

$$25B - 19A = 1$$

$$A = \frac{-19}{986} \qquad B = \frac{25}{986}$$

Therefore,

$$y_p(t) = \frac{-19}{986}cos(5t) + \frac{25}{986}sin(5t)$$

(b) We can use directly use characteristic equation:

$$r^2 + 5r + 6 = 0$$

$$r = -3 \qquad r = -2$$

And we know the form of $x_h(t) : Ce^{st}$ Therefore,

$$x_h(t) = C_1e^{-3t} + C_2e^{-2t}$$

(c) Initially rest means x(0) = 0, x'(0) = 0 and so on.

$$y(t) = y_h(t) + y_p(t) = \frac{-19}{986}cos(5t) + \frac{25}{986}sin(5t) + C_1e^{-3t} + C_2e^{-2t}$$

$$y(0) = \frac{-19}{986} + C_1 + C_2 = 0$$

$$y'(0) = \frac{125}{986} - 3C_1 - 2C_2 = 0$$

$$C_1 + C_2 = \frac{19}{986}$$

$$3C_1 + 2C_2 = \frac{125}{986}$$

$$C_1 = \frac{87}{986} \qquad C_2 = \frac{68}{986}$$

Final form:

$$y(t) = \frac{-19}{986}cos(5t) + \frac{25}{986}sin(5t) + \frac{87}{986}e^{-3t} + \frac{68}{986}e^{-2t}$$

6. (a) In the first system input is x and output is w. If we replace x with $\delta[n]$ output will be $h_0[n]$.So,

$$h_0[n] - \frac{1}{2}h_0[n-1] = \delta[n]$$

We know system is initially at rest so when n <0 we have 0 as output. Let's look some values with this info to compute $h_0[n]$:

$$h_0[0] - 1/2h_0[-1] = 1 \quad h_0[0] = 1$$

$$h_0[1] - 1/2h_0[0] = 0 \quad h_0[1] = \frac{1}{2}$$

$$h_0[2] - 1/2h_0[1] = 0 \quad h_0[2] = \frac{1}{4}$$

And so on. We can trivally see $h_0[n]$ is $(\frac{1}{2})^n$.

(b) If we cascade the system (implication property of convolution sum):

$$x[n] * (h_0[n] * h_0[n]) = y[n]$$

Also,

$$x[n] * h[n] = y[n]$$

Therefore

$$h_0[n] * h_0[n] = h[n]$$

We know $h_0[n]$ is $(\frac{1}{2})^n$,

$$h[n] = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^k (\frac{1}{2})^{n-k}$$

$$h[n] = (\frac{1}{2})^n$$

(c) We can arrange y[n] as $h[n] * x[n]$ (commutative). Also we know system is initally at rest so when n is negative output is 0. According to that we can change the borders of sum as 0 to ∞.

$$y[n] = \sum_{k=0}^{\infty}(\frac{1}{2})^k x[n-k]$$

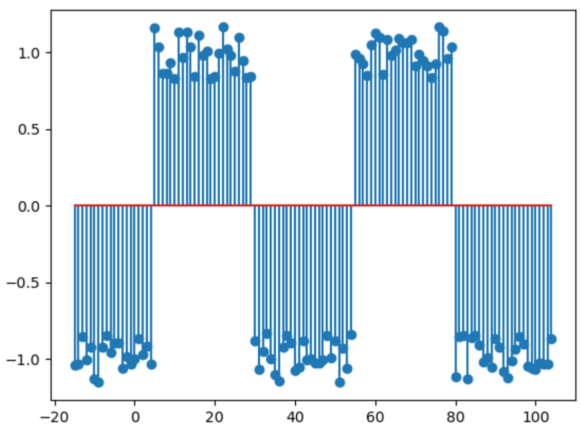
We can take out y[0],

$$y[n] = x[n] + \sum_{k=1}^{\infty}(\frac{1}{2})^k x[n-k]$$

We can clearly see $\sum_{k=1}^{\infty}(\frac{1}{2})^k x[n-k]$ is actually $\frac{1}{2}y[n-1]$. Therefore difference equation is,

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$



7. (a)

Behaviour of signal is shifting for $\delta(n-5)$. It shifted the signal to the right 5 unit.

(b)