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Answer 1

1) BASIS:

For n = 1:

$$6^{2.1} - 1 = 35$$
$$5|35 \quad and \quad 7|35,$$

is true. | stands for divisibility. Therefore our basis holds.

2) INDUCTIVE STEP:
Assume $6^{2k} - 1$ is divisible by both 5 and 7. If we can show that that statement is true for k+1 which is arbitary then this statement is true by mathemtical induction. ($k \in N^{+}$) From our assumption:

$$6^{2k} \equiv 1(mod5)$$
$$6^{2k} \equiv 1(mod7)$$

I am going to use following principle from number theory:

$$a \equiv d(modc)$$

$$b \equiv f(modc)$$
$$a.b \equiv (f.d(modc))(modc)$$

Then:

$$6^{2k+2} = 36.6^{2k}$$
$$36 \equiv 1(mod5)$$
$$36.6^{2k} \equiv 1(mod5)$$

and

$$36 \equiv 1(mod7)$$
$$36.6^{2k} \equiv 1(mod7)$$

Therefore $5|6^{2k+2} - 1$ and $7|6^{2k+2} - 1$ and our claim is true which we proved by mathemtical induction.

Answer 2

1) BASIS:

$$H_3 = 8H_2 + 8H_1 + 9H_0$$
$$H_3 = 8.7 + 8.5 + 9.1 = 105 \leq 9^3$$

Basis holds.

2) INDUCTIVE STEP:

Assume that $H_3, H_4, H_5, ..., H_k$ is less than 9^i, $(3 \leq i \leq k)$ Since k is arbitary if we can show that this expression is correct for $k+1$ our proof will be done.

$$H_{k+1} = 8H_k + 8H_{k-1} + 9H_{k-2}$$

From our inductive step:

$$H_k \leq 9^k$$
$$H_{k-1} \leq 9^{k-1}, 9^{k-1} = \frac{1}{9}9^k$$
$$H_{k-2} \leq 9^{k-2}, 9^{k-2} = \frac{1}{81}9^k$$

Now, we can apply these into the H_{k+1}'s equation:

$$8H_k \leq 8.9^k$$
$$8H_{k-1} \leq \frac{8}{9}9^k$$
$$9H_{k-2} \leq \frac{1}{9}9^k$$

If we sum these inequations:

$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9.9^k$$
$$H_{k+1} \leq 9.9^k$$
$$H_{k+1} \leq 9^{k+1}$$

We were able to show that $H_{k+1} \leq 9^{k+1}$. Then our claim is correct. We proved it by mathematical induction.

Answer 3

We can seperate the problem into two subproblems. First is consecutive 0's and second is consecutive 1's.

$$\underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}$$

$$\underline{.}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}$$

$$.$$

$$.$$

We can take '0000' as a group and change the other elements. As you can see '0000' can be in 5 different places, and we can choose other elements with $2^4 = 16$ (product rule) different ways. Therefore, for '0000' we have $5.16 = 80$ different ways, but we count some strings twice for instance "00000111". These repeated elements coming from cases that have 5,6,7,8 consecutive 0's. We can apply same operations to this cases(product rule).

5 consecutive 0's:

$$4.2^3 = 32$$

6 consecutive 0's:

$$3.2^2 = 12$$

7 consecutive 0's:

$$2.2^1 = 4$$

8 consecutive 0's:

$$1.2^0 = 1$$

From Principle of Inclusion/Exclusion we should exclude all these cases, $80 - 32 - 12 - 4 - 1 = 31$

Same logic is valid for 1's as well:

$$\underline{1}\ \ \underline{1}\ \ \underline{1}\ \ \underline{1}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}$$

$$\underline{.}\ \ \underline{1}\ \ \underline{1}\ \ \underline{1}\ \ \underline{1}\ \ \underline{.}\ \ \underline{.}\ \ \underline{.}$$

$$.$$

$$.$$

So, we have 31 different ways for '1111'. Now, we can apply Principle of Inclusion/Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Since we count '00001111' and '11110000' strings twice the $|A \cap B|$ is 2.
Therefore final answer for this problem is $31 + 31 - 2 = \underline{60}$.

Answer 4

We can seperate this problem to subproblems. Selection of a star, selection and arrangement of non-habitable and habitable planets, selection of number of non-habitable planets between habitable planets.

1) Selection of a star:

$$C(10, 1) = 10$$

2) Selection and arrangement of non-habitable planents:

$$P(80, 8)$$

3) Selection and arrangemnet of habitable planets:

$$P(20, 2)$$

Since we are using permutation, arrangement issue is done. Now, we should check number of non-habitable planets between habitable planets:

H, 6xNH, H, 2xNH

NH, H, 6xNH, H, NH

2xNH, H, 6xNH, H

H, 7xNH, H, NH

NH, H, 7xNH, H

H, 8xNH, H

(H : habitable planet NH: non-habitable planet)

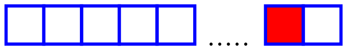
As you can see there are 6 different cases for selection of number of non-habitable planets between habitable planets.

Now we can apply product rule.
Therefore the final answer is:

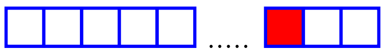
$$\underline{10.P(80, 8).P(20, 2).6}$$

Answer 5

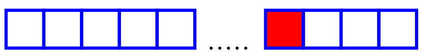
a) $W_n \rightarrow$ number of ways robot move to nth box

 $\rightarrow n$ boxes

Our robot is at the box which is filled with red. ($n-1th$ box) The robot can move to $n-1th$ box W_{n-1} different ways, and last step will not change the number of different ways.

 $\rightarrow n$ boxes

Our robot is at the box which is filled with red. ($n-2th$ box) The robot can move to $n-2th$ box W_{n-2} different ways, and last step, which is 2, will not change the number of different ways.

 $\rightarrow n$ boxes

Our robot is at the box which is filled with red. ($n-3th$ box) The robot can move to $n-3th$ box W_{n-3} different ways, and last step, which is 3, will not change the number of different ways.

Therefore, with these 3 different situtations we consider all the cases possibly occur, and the recursion expression is the following:

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} \quad , n \geq 4$$

b) The robot must jump every time so there is no initial condition as W_0. Since our recursion expression has a term with $n-3$ we should have 3 different initial conditions:

$$W_1 \rightarrow (1) = 1$$

$$W_2 \rightarrow [(1,1),(2)] = 2$$

$$W_3 \rightarrow [(1,1,1),(2,1),(1,2),(3)] = 4$$

c) We can use our initial conditions and recursion expression to calculate this:

$$W_4 = W_1 + W_2 + W_3 = 1 + 2 + 4 = 7$$

$$W_5 = W_2 + W_3 + W_4 = 2 + 4 + 7 = 13$$

$$W_6 = W_3 + W_4 + W_5 = 4 + 7 + 13 = 24$$

$$W_7 = W_4 + W_5 + W_6 = 7 + 13 + 24 = 44$$

$$W_8 = W_5 + W_6 + W_7 = 13 + 24 + 44 = 81$$

$$W_9 = W_6 + W_7 + W_8 = 24 + 44 + 81 = \underline{149}$$