CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

Karabacak, Kerem
e2644417@ceng.metu.edu.tr

I worked individually.
exxxxxxx@ceng.metu.edu.tr

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1. We can start with synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

We can take the integral and calculate b_k according to that.

$$\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 s} ds$$

Now we can use analysis equation on this:

$$b_k = \frac{1}{T} \int_0^T \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 s} e^{-jkw_0 t} ds dt$$

$$b_k = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k \int_0^T \int_{-\infty}^{t} e^{jkw_0 s} e^{-jkw_0 t} ds dt$$

$$b_k = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k \int_0^T \frac{1}{jkw_0} e^{jkw_0 t} e^{-jkw_0 t} dt$$

Exponential terms are cancelled.

$$b_k = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k \int_0^T \frac{1}{jkw_0} dt$$

$$b_k = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k \frac{T}{jkw_0}$$

$$b_k = \sum_{k=-\infty}^{\infty} a_k \frac{1}{jkw_0}$$

Finally, we know $w_0 = \frac{2\pi}{T}$.

$$b_k = \sum_{k=-\infty}^{\infty} a_k \frac{1}{jk\frac{2\pi}{T}}$$

2. (a) Let's denote x(t)x(t)'s coefficients as b_k.
Using multiplication property of continious signals:

$$b_k = \sum_{l=-\infty}^{\infty} a_{l-k} a_l = a_k * a_k$$

(b) We can rewrite Ev{x(t)} as

$$\frac{x(t) + x(-t)}{2}$$

Let's denote this representation's coefficients as b_k.

$$b_k = \frac{a_k + a_{-k}}{2}$$

(c) Let's denote x(t+t_0)'s coefficents as b_k, x(t - t_0)'s coefficients as c_k, all equation's coefficients as d_k. Using time shifting property of continious signals:

$$b_k = e^{jkw_0t}a_k$$
$$c_k = e^{-jkw_0t}a_k$$
$$d_k = b_k + c_k = e^{jkw_0t}a_k + e^{-jkw_0t}a_k$$

3. Let's denote period of system as T_1. Then -2 is $-T_1$ and 2 is T_1. According to that x(t) is:

$$\begin{cases} 2 & 0 < t < \frac{T_1}{2} \\ -2 & -T_1 < t < \frac{-T_1}{2} \\ 0 & otherwise \end{cases}$$

Using analysis equation for CT signals:

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jkw_0t}dt = \frac{1}{T}\int_{0}^{T_1/2} 2e^{-jkw_0t}dt + \frac{1}{T}\int_{-T_1}^{-T_1/2} -2e^{-jkw_0t}dt$$

$$a_k = \frac{-2}{T}(\frac{e^{-jkw_0T_1/2} - 1}{jkw_0}) + \frac{2}{T}(\frac{e^{jkw_0T_1/2} + e^{jkw_0T_1}}{jkw_0})$$

$$a_k = \frac{2}{jkw_0T}(e^{jkw_0T_1/2} + e^{jkw_0T_1} - e^{-jkw_0T_1/2} + 1)$$

Arranging the equation to use Euler's formula:

$$a_k = \frac{4}{kw_0t}(\frac{e^{jkw_0T_1/2} - e^{-jkw_oT_1/2}}{2j} + \frac{e^{jkw_0T_1} + 1}{2j})$$

Also, we can use $w_0 = \frac{2\pi}{T}$:

$$a_k = \frac{2}{k\pi}(sin(kw_0T_1/2) + \frac{cos(kw_0T_1) + jsin(kw_oT_1) + 1}{2j})$$

4. (a) We can rewrite x(t) with the help of Euler's formula as follows:

$$x(t) = 1 + \frac{1}{2j}e^{jw_0t} - \frac{1}{2j}e^{-jw_0t} + e^{jw_0t} + e^{-jw_0t} + \frac{1}{2}e^{j(2w_0t+\frac{\pi}{4})} + \frac{1}{2}e^{-j(2w_0t+\frac{\pi}{4})}$$

If we group them by k (for instance e^{jw_0t} correscponds to a_1 since k is 1 in forumla e^{jkw_0t}), coefficients are:

$$a_0 = 1 \quad a_1 = 1 - \frac{j}{2} \quad a_{-1} = 1 + \frac{j}{2} \quad a_2 = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}j \quad a_{-2} = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}j$$

All other coefficients are 0.

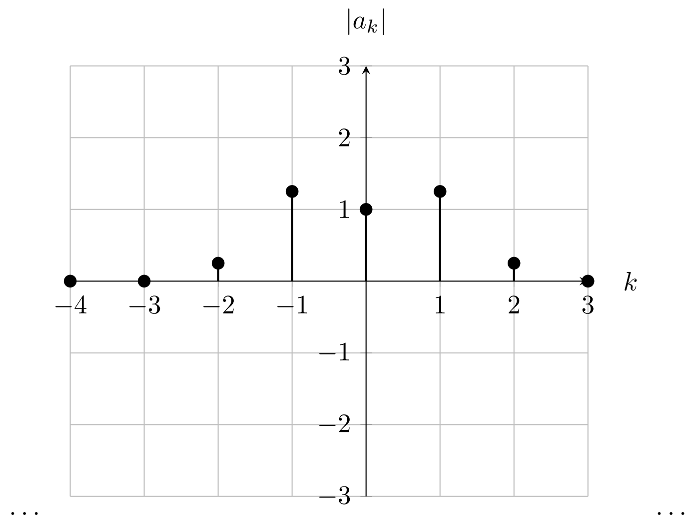

Figure 1: k vs. $|a_k|$.

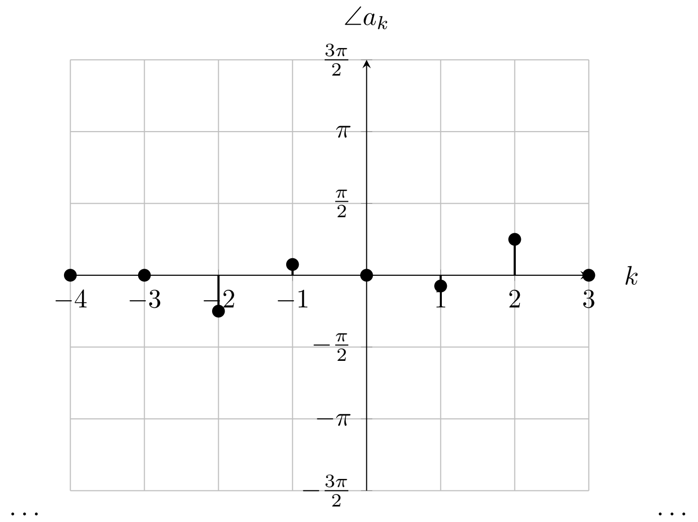

Figure 2: k vs. $\angle a_k$.

(b) We can set x(t) $= e^{jw_0t}$ for simplicity. Also, we know $y(t) = H(jw_0)x(t)$. Let's put those onto the formula given:

$$H(jw_0)x'(t) + H(jw_0)x(t) = x(t)$$

$$H(jw_0)e^{jw_0t}jw_0 + H(jw_0)e^{jw_0t} = e^{jw_0t}$$

e^{jw_0t}'s are cancelled.

$$H(jw_0) = \frac{1}{1+jw_0}$$

(c) Let's denote y(t)'s coefficients as b_k. Also we know:

$$b_k = a_k H(jw_0)$$

I took T as 2π since x(t) only contains sine and cosine terms i.e. $w_0 = 1$. According to that:

$$b_{-2} = (\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}j)(\frac{1}{1+j}) = \frac{1-j}{2\sqrt{2}+2\sqrt{2}j}$$

$$b_{-1} = (1+\frac{j}{2})(\frac{1}{1+j}) = \frac{2+j}{2+2j}$$

$$b_0 = \frac{1}{1+j}$$

$$b_{-1} = (1-\frac{j}{2})(\frac{1}{1+j}) = \frac{2-j}{2+2j}$$

$$b_{-2} = (\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}j)(\frac{1}{1+j}) = \frac{1+j}{2\sqrt{2}+2\sqrt{2}j} = \frac{1}{2\sqrt{2}}$$

For all other values of k, b_k is 0.

(d)

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0t}$$

5. (a) For periodicity $\frac{\pi}{2}n$ must be equal to 2π. So we have period N = 4.
We can rewrite x[n] as follows:

$$x[n] = \frac{1}{2j}e^{j\pi n/2} - \frac{1}{2j}e^{-j\pi n/2}$$

As you can see first part is a_1 and second part is a_{-1}.So

$$a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j}$$

Since x[n] is discrete signal, $a_n = a_{n+N}$. Therefore

$$a_{-3} = a_1 = a_5 = \frac{1}{2j} \qquad a_{-5} = a_{-1} = a_3 = \frac{-1}{2j}$$

(b) For periodicity $\frac{\pi}{2}n$ must be equal to 2π. So we have period N = 4. We can rewrite y[n] as follows:

$$y[n] = 1 + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n}$$

First part is b_0, second part is b_1 and third part is b_{-1}.

$$b_0 = 1 \qquad b_1 = \frac{1}{2} \qquad b_{-1} = \frac{1}{2}$$

Since y[n] is discrete signal $b_n = b_{n+N}$. Therefore

$$b_{-4,0,4..} = 1 \qquad b_{-3,1-5..} = \frac{1}{2} \qquad b_{-5,-1,3..} = \frac{1}{2}$$

(c) We can directly use multiplication property:

$$c_k = \sum_{l=0}^{3} a_l b_{k-l}$$

$$c_0 = 0 + \frac{1}{4j} + 0 - \frac{1}{4j} = 0$$

$$c_1 = 0 + \frac{1}{2j} + 0 + 0 = \frac{1}{2j}$$

$$c_2 = 0 + \frac{1}{4j} + 0 - \frac{1}{4j} = 0$$

$$c_3 = 0 + 0 + 0 - \frac{1}{2j} = \frac{-1}{2j}$$

And also periodicity rule is valid for this one as well.

(d) Multiplying both signals in analysis equation:

$$\frac{1}{4}\sum_{n=0}^{3} sin(\frac{\pi}{2}n) + sin(\frac{\pi}{2}n)cos(\frac{\pi}{2}n)e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\sum_{n=0}^{3} sin(\frac{\pi}{2}n) + \frac{1}{2}sin(\pi n)e^{-jk\frac{\pi}{2}n}$$

$$c_0 = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}(0 + e^{-j\frac{\pi}{2}} + 0 + e^{-j\frac{3\pi}{2}}) = \frac{-j}{2} = \frac{1}{2j}$$

$$c_2 = \frac{1}{4}(0 + e^{-j\pi} + 0 - e^{-j3\pi}) = 0$$

$$c_3 = \frac{1}{4}(0 + e^{-j3\pi/2} + 0 - e^{-j3\pi/2}) = \frac{j}{2} = \frac{-1}{2j}$$

6. (a) x[n]'s period N is 4. We can directly use analysis equation to compute coefficients:

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jkw_0 n} = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\pi n/2}$$

$$a_0 = (0 + 1 + 2 + 1)/4 = 1$$

$$a_1 = (0 + e^{-j\pi/2} + 2e^{-j\pi} + e^{-j3\pi/2})/4 = -\frac{1}{2}$$

$$a_2 = (0 + e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi})/4 = 0$$

$$a_3 = (0 + e^{-3j\pi/2} + 2e^{-j3\pi} + e^{-j9\pi/2})/4 = \frac{-1}{2}$$

Since x[n] is discrete signal, $a_n = a_{n+N}$

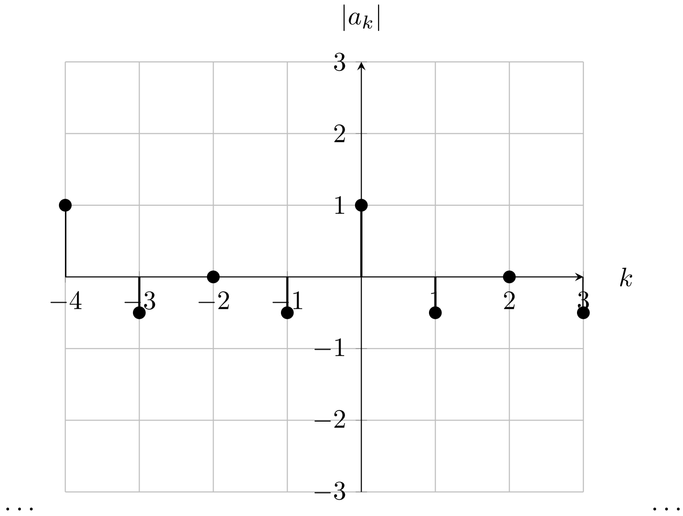

Figure 3: k vs. a_k.

(b) Difference between two signal is difference at the 4k-1 points, -5, -1 ,3. So we can express y[n] as follows:

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} x[n]\delta[n-4k+1]$$

We can use synthesis equation:

$$b_k = \frac{1}{4}\sum_{n=0}^{3}(x[n] - \sum_{k=-\infty}^{\infty} x[n]\delta[n-4k+1])e^{-jkn\pi/2}$$

$$b_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = (0+1+2+1)/4 = 1$$

$$b_1 = \frac{1}{4}sum_{n=0}^{3} x[n]e^{-jn\pi/2} = (0-j-2+j)/4 = \frac{-1}{2}$$

$$b_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n} = (-1+2-1)/4 = 0$$

$$b_3 = \frac{1}{4}\sum_{n=0}^{3} e^{-j3\pi n/2} = (1+j-1-j)/4 = 0$$

Since signal is discrete, $b_n = b_{n+N}$.

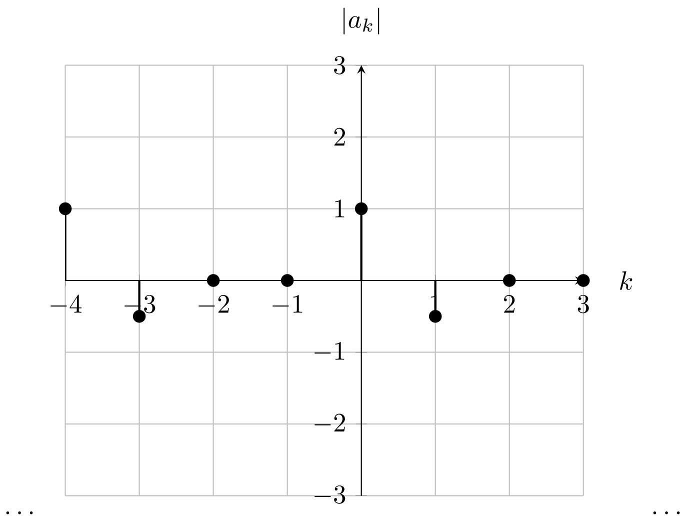

Figure 4: k vs. a_k.

7. (a)

$$b_k = \sum_{k=-\infty}^{\infty} a_k H(jw_0)$$

If x[n] $=$ y[n], then their coefficients must be equal. Since y[n]'s coefficients are non-zero at [-80,80], a_k is zero when $|k|$ >80.

(b) If x(t) $\neq$ y(t), then LTI system has some effect on input and output. Because of this a_k is non zero at least one point on $|k| > 80$.

8.