Understanding SVD and Implementing

1) Eigenvalues and Eigenvectors

$$\lambda_i = eigenvalue, x_i = eigenvector$$

$$Ax_1 = \lambda x_1$$

$$Ax_2 = \lambda x_2$$

$$Ax_3 = \lambda x_3$$

$$...$$

$$Ax_n = \lambda x_n$$

2) Diagonizaling a Matrix

We can write the expression about eigenvalues and eigenvectors in a matrix form.

$$A[x_1, x_2, ..., x_n] = [\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_n x_n]$$

$$AX = X\Lambda$$

$$A = X\Lambda X^{-}$$

3) For symmetric matrices the case is special

A is a symmetric matrix

$$A = X\Lambda X^T$$

4) SVD is basically a rotation and a scaling operation which is applied on a matrix

In other words it is a mapping from row space to column space of A:

Let v1 and v2 be orthonormal basis in row space and let u1 and u2 be orthonormal basis in column space of A.

Then we can formulate the following mapping:

$$Av_1 = \sigma_1 u_1$$

We know that symmetric positive definite matrices have positive eigenvalues and has orthonormal eigenvectors. This satisfies the above expression.

Let A be an mxn matrix, we can write the following expression in matrix form for all $Av_i = \sigma_i u_i$

$$AV = U\Sigma$$

$$A = U\Sigma V^{-}$$

Because V is orthonormal we can write like this

$$A = U\Sigma V^T$$

4) SVD for any matrix

What we want now is a symmetric matrix to get to the conclusion

$$AA^T = U\Sigma V^T V \Sigma^T U^T$$

$$AA^T = U\Sigma(V^T V)\Sigma^T U^T$$

$$AA^T = U\Sigma(I)\Sigma^T U^T$$

$$AA^T = U(\Sigma\Sigma^T)U^T$$

$$AA^T = U\Sigma^2 U^T$$

So, we see from the last expression that A and AA^T have Σ and Σ^2 as their diagonal matrices / eigenvalues. Here, Σ is also called singular values. Since AA^T is symmetric positive definite we can compute it's eigenvalues and eigenvectors to solve for Σ and U.

Similarly:

$$A^T A = V\Sigma^T U^T U \Sigma V^T$$

$$A^T A = V\Sigma^T(U^T U)\Sigma V^T$$

$$A^T A = V\Sigma^T(I)\Sigma V^T$$

$$A^T A = V(\Sigma^T\Sigma)V^T$$

$$A^T A = V\Sigma^2 V^T$$

Thank to properties of symmetric positive definite matrices you can go at any direction to find SVD of any given matrix.