1 Integral Representation of Zeta(2)

$$\int_0^\infty \frac{t}{e^{\alpha t}-1}dt = \int_0^\infty \frac{te^{-\alpha t}}{1-e^{-\alpha t}}dt = \int_0^\infty t\sum_{n=1}^{\infty} e^{-n\alpha t}dt$$

By the Dominant Convergence Theorem: (Include Reference)

$$\int_0^\infty t\sum_{n=1}^{\infty} e^{-n\alpha t}dt = \sum_{n=1}^{\infty}\int_0^\infty te^{-n\alpha t}dt$$

Using $n \cdot \alpha \cdot t = x \to n \cdot \alpha \cdot dt = dx$

$$\sum_{n=1}^{\infty}\int_0^\infty \frac{x}{n\alpha}e^{-x}\frac{dx}{n\alpha} = \sum_{n=1}^{\infty}\frac{1}{n^2\alpha^2}\int_0^\infty xe^{-x}dx = \frac{1}{\alpha^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Gamma(2)$$

Recall $\Gamma(2) = 1! = 1$ (Include Reference)

$$\frac{1}{\alpha^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Gamma(2) = \frac{1}{\alpha^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\alpha^2}\zeta(2)$$

Recall $\zeta(2) = \frac{\pi^2}{6}$ (Include Reference)

$$=> \int_0^\infty \frac{t}{e^{\alpha t}-1}dt = \frac{\pi^2}{6\alpha^2}$$