Finding slope of tangent line without the Derivative

Emil Kerimov

June 17, 2017

1 A Quadratic Function

Given a quadratic function $y = ax^2 + bx + c$, to find the slope of the tangent line for all x, we need to solve for m such that $y_t = mx + b$ intersects the quadratic function.

$$ax^2 + bx + c = mx + k \tag{1}$$

Since we know the quadratic function is strictly convex, we know that the tangent line will only intersect the quadratic curve at one and only one point.

$$ax^2 + bx + c = mx + k$$

Rearranging

$$ax^2 + (b - m)x + (c - k) = 0$$

Using the quadratic equation

$$x = \frac{m - b \pm \sqrt{(b - m)^2 - 4a(c - k)}}{2a}$$

Since we know that there is only one intersection between these two curve, we know that the term inside the square root needs to be 0, hence making both solution of x the same.

$$x = \frac{m - b}{2a}$$

Solving for m

$$m = 2ax + b$$

Notice that this is the same as the derivative of $ax^2 + bx + c$.

2 A reciprocal function

Given a reciprocal function $y = \frac{a}{x}$, to find the slope of the tangent line for all x, we need to solve for m such that $y_t = mx + b$ intersects the quadratic function.

$$\frac{a}{x} = mx + k \tag{2}$$

Since we know the reciprocal function is strictly concave, we know that the tangent line will only intersect the reciprocal curve at one and only one point.

$$\frac{a}{x} = mx + k$$

Rearranging

$$mx^2 + kx - a = 0$$

Using the quadratic equation

$$x = \frac{-k \pm \sqrt{k^2 + 4ma}}{2m}$$

Since we know that there is only one intersection between these two curve, we know that the term inside the square root needs to be 0, hence making both solution of x the same.

$$x = \frac{-k}{2m}$$

Substituting for k

$$x = \frac{-(\frac{a}{x} - mx)}{2m}$$

Simplifing

$$2mx = mx - \frac{a}{x}$$

Solving for m

$$m = -\frac{a}{x^2}$$

Notice that this is the same as the derivative of $\frac{a}{x}$.