Gamma Function

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1 Definition

Definition 1.1 *The Gamma Function is defined as:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

The first few evaluation of the Gamma function are:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-t}\ dt = -e^{-t}\Big|_0^\infty = 0 - (-1) = 1 \\
\Gamma(2) &= \int_0^\infty e^{-t} t\ dt = -e^{-t} t\Big|_0^\infty + \int_0^\infty e^{-t}\ dt = 1 \\
\Gamma(3) &= \int_0^\infty e^{-t} t^2\ dt = -e^{-t} t^2\Big|_0^\infty + 2\int_0^\infty e^{-t} t\ dt = 2 \\
\Gamma(4) &= \int_0^\infty e^{-t} t^3\ dt = -e^{-t} t^3\Big|_0^\infty + 3\int_0^\infty e^{-t} t\ dt = 6
\end{aligned} \tag{1}$$

2 Relation to the Factorial Function

Expanding on the pattern we found by evaluating the first few integer values of the Gamma function through integration by parts we obtain:

Theorem 2.1 $\Gamma(z+1) = z\Gamma(z)$

Proof by induction

Assume true for k

$$\Gamma(k+1) = k\Gamma(k)$$

Proving for k+1

$$\begin{aligned}
\Gamma(k+1) &= \int_0^\infty e^{-t} t^k\ dt \\
&= -e^{-t} t^k\Big|_0^\infty + k\int_0^\infty e^{-t} t^{k-1}\ dt \\
&= 0 - 0 + k\Gamma(k-1) \\
&= k\Gamma(k-1)
\end{aligned}$$

2.1 Integer Values

Therefore since $\Gamma(1) = 1$, then for positive integer values

$$\Gamma(n) = (n-1)! \tag{2}$$

3 Equivalent Integral Forms

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{3}$$

Theorem 3.1 *Starting from Equation 9, we can use the substitution $t = -ln(u)$ to obtain*

$$\Gamma(z) = \int_0^1 (-lnu)^{z-1} dt \tag{4}$$

Proof

$$t = -ln(u)$$
$$dt = -\frac{du}{u}$$
$$\Gamma(z) = \int_1^0 e^{ln(u)} (-ln(u))^{z-1} \frac{-du}{u}$$
$$\Gamma(z) = -\int_1^0 u(-ln(u))^{z-1} \frac{du}{u}$$
$$\Gamma(z) = -\int_1^0 (-ln(u))^{z-1} du$$
$$\Gamma(z) = \int_0^1 (-ln(u))^{z-1} du$$

4 Finite Gamma Expression

Define the finite gamma function (find out its real name)

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)(x+2)...(x+p)} = \frac{p^x}{x(\frac{x}{1}+1)(\frac{x}{2}+1)...(\frac{x}{p}+1)} \quad x > 0 \tag{5}$$

Note that this function has the following properties:

$$\Gamma_p(1) = \frac{p!p}{1(1+1)(1+2)...(1+p)} = \frac{p!p}{p!(1+p)} = \frac{p}{1+p} \tag{6}$$

$$\begin{aligned} \Gamma_p(x+1) &= \frac{p!pp^x}{(x+1)(x+2)...(x+p)(x+p+1)} \\ &= \frac{xp!pp^x}{x(x+1)(x+2)...(x+p)(x+p+1)} \\ &= \frac{xp}{x+p+1}\Gamma_p(x) \end{aligned} \tag{7}$$

4.1 Taking the limit to obtain Gamma

Using Equation 7 and assuming $\Gamma_\infty(x)$ converges we get:

$$\lim_{p\to\infty} \Gamma_p(x+1) = \lim_{p\to\infty} \frac{xp}{x+p+1}\Gamma_p(x) = x \lim_{p\to\infty} \Gamma_p(x) \tag{8}$$

Therefore using the above properties we see that if $\Gamma_\infty(x)$ converges then it must be equal to $\Gamma(x)$.

5 Euler Mascheroni constant

The Euler Mascheroni constant is defined as:

$$\gamma = \lim_{p\to\infty} \sum_{i=1}^{p} 1/i - ln(p) \approx 0.5772 \tag{9}$$

This constant can be interpreted as the difference between the sum of $1/x$ and the integral of $1/x$ from 1 to ∞. It is unknown whether this number is transcendental or even irrational! One can show numerically that if it is rational then the denominator must be greater than 10^{242080}.

6 Weierstrass Formula

Theorem 6.1 *Let γ be the Euler Mascheroni constant as defined in section 5, and Γ be the Gamma function, then*

$$\frac{1}{\Gamma(x)} = x \cdot e^{x\gamma} \cdot \prod_{k=1}^{\infty} (1 + x/k) \cdot e^{-x/k} \tag{10}$$

Proof

Re-writing p^x

$$p^x = e^{xlnp} = e^{x[lnp-1-1/2-1/3-...-1/p]}e^{x+x/2+x/3+...+x/p}$$

Plugging p^x *into equation 9*

$$\Gamma_p(x) = \frac{e^{x[lnp-1-1/2-1/3-...-1/p]}e^{x+x/2+x/3+...+x/p}}{x(\frac{x}{1}+1)(\frac{x}{2}+1)...(\frac{x}{p}+1)}$$

$$\Gamma_p(x) = e^{x[lnp-1-1/2-1/3-...-1/p]} \cdot \frac{1}{x} \cdot \frac{e^x}{x+1} \cdot \frac{e^{x/2}}{x/2+1} \cdot \frac{e^{x/3}}{x/3+1} \cdot ... \cdot \frac{e^{x/p}}{x/p+1}$$

$$\Gamma_p(x) = e^{x[lnp-1-1/2-1/3-...-1/p]} \cdot \frac{1}{x} \cdot \prod_{k=1}^{p} \frac{e^{x/k}}{1+x/k}$$

Taking the limit as $p \to \infty$

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x) = \lim_{p\to\infty} e^{x[lnp-1-1/2-1/3-...-1/p]} \cdot \frac{1}{x} \cdot \prod_{k=1}^{p} \frac{e^{x/k}}{1+x/k}$$

$$\Gamma(x) = e^{-x\gamma} \cdot \frac{1}{x} \cdot \prod_{k=1}^{\infty} \frac{e^{x/k}}{1+x/k}$$

Equivalently

$$\frac{1}{\Gamma(x)} = x \cdot e^{x\gamma} \cdot \prod_{k=1}^{\infty} (1+x/k) \cdot e^{-x/k}$$

6.1 Symmetry around 1/2 relation

Using 6 we derive the following functional form.

Theorem 6.2

$$\boxed{\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(1-x)} = \frac{sin(\pi x)}{\pi}} \tag{11}$$

Proof
Starting from 6 and considering $\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(-x)}$ *we obtain*

$$\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(-x)} = x \cdot e^{x\gamma} \cdot \prod_{k=1}^{\infty} (1+x/k) \cdot e^{-x/k} \cdot -x \cdot e^{-x\gamma} \cdot \prod_{k=1}^{\infty} (1-x/k) \cdot e^{x/k}$$

$$\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(-x)} = -x^2 \cdot \prod_{k=1}^{\infty} (1-x^2/k^2)$$

Using the functional form of Γ*, we get* $\Gamma(1-x) = \Gamma(-x+1) = -x\Gamma(-x)$

$$\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(1-x)} = x \cdot \prod_{k=1}^{\infty} (1-x^2/k^2)$$

Using the product form of $\frac{sin(\pi x)}{\pi x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)$ *, See Euler's Sine Product Form document*

$$\frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(1-x)} = \frac{sin(\pi x)}{\pi}$$

6.2 Special Examples

Plugging in various values for x we obtain certain values and relations for $\Gamma(x)$

$x = \frac{1}{2}$

$$\Gamma(\frac{1}{2}) = \sqrt{\frac{\pi}{sin(\frac{\pi}{2})}} = \sqrt{\pi} \tag{12}$$

$x = \frac{1}{3}$

$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}} \tag{13}$$

$x = \frac{1}{4}$

$$\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \frac{\pi}{sin(\frac{\pi}{4})} = \sqrt{2}\pi \tag{14}$$

Theorem 6.3 $x = -\frac{1}{2}$ *with equation 12*

$$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi} \tag{15}$$

Proof

$$\begin{aligned} \Gamma(-\frac{1}{2}) \cdot \Gamma(1 - (-\frac{1}{2})) &= \frac{\pi}{sin(-\frac{\pi}{2})} \\ \Gamma(-\frac{1}{2}) &= -\frac{\pi}{\Gamma(\frac{3}{2})} \\ \Gamma(-\frac{1}{2}) &= -\frac{\pi}{\frac{1}{2}\Gamma(\frac{1}{2})} \\ \Gamma(-\frac{1}{2}) &= -\frac{\pi}{\frac{1}{2}\sqrt{\pi}} \\ \Gamma(-\frac{1}{2}) &= -2\sqrt{\pi} \end{aligned}$$

Note that equation 12 implies that

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{e^t\sqrt{t}}dt = \sqrt{\pi} \tag{16}$$

Since $\Gamma(x) = (x-1)!$ for integer values of x, this is where the idea of $(-\frac{1}{2})! = \sqrt{\pi}$ comes from.

7 Other relations

Using equation 9 we derive the following.

Theorem 7.1 *This is the more general multiplication equation referred to as the multiplication formula or Gauss's multiplication formula*

$$\boxed{\Gamma(x) \cdot \Gamma(x + \frac{1}{n}) \cdot \Gamma(x + \frac{2}{n}) \cdots \Gamma(x + \frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}} n^{1/2-nx}\Gamma(nx)} \tag{17}$$

Proof
TODO.

Plugging in $x = 0$ into Equation 17 we get

$$\Gamma(\frac{1}{n}) \cdot \Gamma(\frac{2}{n}) \cdots \Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \tag{18}$$

Theorem 7.2 *There is a multiplicative relation to the Zeta function. Refer to the Zeta function document for the definition and properties of the Zeta function.*

$$\boxed{\Gamma(x)\zeta(x) = \int_0^\infty \frac{u^{s-1}}{e^u - 1} du} \tag{19}$$

Proof
See the Zeta function document.

8 Digamma function

8.1 Definition

Define the Digamma function

$$\Psi(x) = \frac{\partial}{\partial x} ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{20}$$

8.2 Series Representation

Theorem 8.1 *Let γ be the Euler Mascheroni constant as defined in section 5 then,*

$$\boxed{\Psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \Big(\frac{1}{k} - \frac{1}{x+k}\Big)} \tag{21}$$

Proof

$$\Psi(x) = \frac{\partial}{\partial x} ln(\Gamma(x))$$

Using equation 10

$$\Psi(x) = -\frac{\partial}{\partial x} ln(x \cdot e^{x\gamma} \cdot \prod_{k=1}^{\infty} (1 + x/k) \cdot e^{-x/k})$$

$$\Psi(x) = -\frac{\partial}{\partial x} \Big[ln(x) + x\gamma + \sum_{k=1}^{\infty} ln(1 + x/k) - \frac{x}{k} \Big]$$

$$\Psi(x) = -\frac{1}{x} - \gamma - \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{1 + x/k} - \frac{1}{k}$$

$$\Psi(x) = -\frac{1}{x} - \gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{x+k}$$

8.3 Derivatives

Theorem 8.2

$$\boxed{\Psi'(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x-1)^2}} \tag{22}$$

Proof

$$\textit{Using equation 21}$$

$$\Psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k}\right)$$

$$\Psi'(x) = \frac{1}{x^2} + \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}$$

$$\textit{Including the first term in the sum}$$

$$\Psi'(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k-1)^2}$$

Theorem 8.3

$$\boxed{\Psi^n(x) = \sum_{k=1}^{\infty} \frac{(-1)^{n+1} n!}{(k+x-1)^{n+1}}} \tag{23}$$

Proof

Take repeated deriviates of equation 22

8.4 Special values

Theorem 8.4

$$\Psi^n(x+1) = \Psi^n(x) + \frac{(-1)^n n!}{x^{n+1}} \tag{24}$$

Proof

$$\Psi^n(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{n+1} n!}{(k+x)^{n+1}}$$

$$\Psi^n(x+1) = \frac{(-1)^n n!}{x^{n+1}} + \sum_{k=2}^{\infty} \frac{(-1)^{n+1} n!}{(k+x)^{n+1}}$$

$$\Psi^n(x+1) = \frac{(-1)^n n!}{x^{n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^{n+1} n!}{(k+x-1)^{n+1}}$$

$$\Psi^n(x+1) = \frac{(-1)^n n!}{x^{n+1}} + \Psi^n(x)$$

Theorem 8.5 *Reflection formula*

$$\Psi(1-x) = \Psi(x) + \pi cot(\pi x) \tag{25}$$

Proof

$$\Psi(1-x) = -\gamma - \frac{1}{1-x} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+1-x}\Big)$$

Adding and subtracting $\Psi(x)$

$$\Psi(1-x) = -\gamma - \frac{1}{1-x} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+1-x}\Big) + \Psi(x) - \Psi(x)$$

$$\Psi(1-x) = -\gamma - \frac{1}{1-x} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+1-x}\Big) + \Psi(x) - \Bigg(-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{x+k}\Big)\Bigg)$$

Simplifying

$$\Psi(1-x) = \Psi(x) + \frac{1}{x} - \frac{1}{1-x} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+1-x} - \frac{1}{k} + \frac{1}{x+k}\Big)$$

$$\Psi(1-x) = \Psi(x) + \frac{1}{x} - \frac{1}{1-x} + \sum_{k=1}^{\infty}\Big(\frac{1}{x+k} - \frac{1}{k+1-x}\Big)$$

$$\Psi(1-x) = \Psi(x) + \sum_{k=0}^{\infty}\Big(\frac{1}{x+k} + \frac{1}{x-k-1}\Big)$$

$$\Psi(1-x) = \Psi(x) + \sum_{k=-\infty}^{\infty}\Big(\frac{1}{x+k}\Big)$$

Using the series [1]*for* $\pi cot(\pi x) = \sum_{k=-\infty}^{\infty} \frac{1}{x+k}$

$$\Psi(1-x) = \Psi(x) + \pi cot(\pi x)$$

Theorem 8.6

$$\Psi(2x) = \frac{1}{2}\cdot\Psi(x) + \frac{1}{2}\cdot\Psi(x+\frac{1}{2}) + ln(2) \tag{26}$$

Proof
TODO

$$\Psi(1) = \Gamma'(1) = -\gamma \tag{27}$$

$$\Psi^{(n)}(1) = (-1)^{n+1}n!\zeta(n+1) \tag{28}$$

Where $\zeta(z)$ is the Riemann Zeta function defined as $\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z}$. CONFIRM THIS?

$$\Psi(n) = \frac{\Gamma'(n)}{\Gamma(n)} = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} = -\gamma + H_{n-1} \tag{29}$$

Where H_n is the sum of the n first harmonic numbers. Note that this implies that the derivative of the Γ function is diverges to ∞.

[1]See the Euler's Sine Product formula document.

9 Using The Gamma Function for Euler Integrals

Theorem 9.1 *Let $p = x + yi$ and $\alpha = tan^{-1}(\frac{y}{x})$ then*

$$\boxed{\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n) du} \tag{30}$$

$$\boxed{\frac{\Gamma(s)}{n|p|^s} cos(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} cos(yu^n) du} \tag{31}$$

Proof

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

Using substitution $t = p \cdot u^n$ hence $dt = p \cdot n \cdot u^{n-1} \cdot du$

$$\Gamma(s) = \int_0^\infty p^{n-1} u^{ns-n} e^{-pu} npu^{n-1} du = n \int_0^\infty p^s u^{ns-1} e^{-pu^n} du$$

$$\frac{\Gamma(s)}{np^s} = \int_0^\infty u^{ns-1} e^{-pu^n} du$$

Assuming p is a complex number with real part x, imaginary part y

Taking the conjugate of both sides, we obtain a second equation.

$$\frac{\Gamma(s)}{n\bar{p}^s} = \int_0^\infty u^{ns-1} e^{-\bar{p}u^n} du$$

Note that $p = x + yi = |p|e^{i\alpha}$ and $\bar{p} = x - yi = |p|e^{-i\alpha}$ where $\alpha = tan^{-1}(\frac{y}{x})$

Adding and subtracting both equations we get

$$\frac{\Gamma(s)}{n}\Big[\frac{1}{\bar{p}^s} \pm \frac{1}{p^s}\Big] = \int_0^\infty u^{ns-1}\Big[e^{-\bar{p}u^n} \pm e^{-pu^n}\Big] du$$

$$\frac{\Gamma(s)}{n|p|^s}\Big[\frac{1}{e^{-i\alpha s}} \pm \frac{1}{e^{i\alpha s}}\Big] = \int_0^\infty u^{ns-1}\Big[e^{-xu^n+yu^n i} \pm e^{-xu^n-yu^n i}\Big] du$$

$$\frac{\Gamma(s)}{n|p|^s}\Big[e^{i\alpha s} \pm e^{-i\alpha s}\Big] = \int_0^\infty u^{ns-1} e^{-xu^n}\Big[e^{yu^n i} \pm e^{-yu^n i}\Big] du$$

With addition we get

$$\frac{\Gamma(s)}{n|p|^s}\Big[2cos(\alpha s)\Big] = \int_0^\infty u^{ns-1} e^{-xu^n}\Big[2cos(yu^n)\Big] du$$

$$\frac{\Gamma(s)}{n|p|^s} cos(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} cos(yu^n)\Big] du$$

With subtraction we get

$$\frac{\Gamma(s)}{n|p|^s}\Big[2isin(\alpha s)\Big] = \int_0^\infty u^{ns-1} e^{-xu^n}\Big[2isin(yu^n)\Big] du$$

$$\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n)\Big] du$$

9.1 Examples

9.1.1 Sinc Integral $\int \frac{\sin(x)}{x}$

Theorem 9.2

$$\boxed{\int_0^\infty \frac{\sin(x)}{x} = \frac{\pi}{2}} \tag{32}$$

Proof
Starting from equation 30

$$\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n) du$$

$$\textit{setting } y = 1, n = 1, x = 0 \textit{ and taking the limit as } s \to 0 \textit{ we obtain}$$

$$|p| = 1, \alpha = tan^{-1}(1/0) = \pi/2$$

$$\lim_{s\to 0} \Gamma(s) sin(s\frac{\pi}{2}) = \lim_{s\to 0} \int_0^\infty u^{s-1} sin(u) du = \int_0^\infty \frac{sin(u)}{u} du$$

$$\textit{To evaluate the RHS limit we use equation 11 } \Gamma(s) = \frac{\pi}{\Gamma(1-s) sin(\pi s)}$$

$$\lim_{s\to 0} \Gamma(s) sin(s\frac{\pi}{2}) = \lim_{s\to 0} \frac{\pi}{\Gamma(1-s) sin(\pi s)} sin(s\frac{\pi}{2})$$

We can split this limit into the product of 3 limits since all of these limits exist

$$\lim_{s\to 0} \Gamma(s) sin(s\frac{\pi}{2}) = \lim_{s\to 0} \frac{\frac{\pi}{2}}{\Gamma(1-s)} \cdot \lim_{s\to 0} \frac{\pi}{sin(\pi s)} \cdot \lim_{s\to 0} \frac{sin(s\frac{\pi}{2})}{\frac{\pi}{2}}$$

$$\lim_{s\to 0} \Gamma(s) sin(s\frac{\pi}{2}) = \frac{\pi}{2}$$

$$\int_0^\infty \frac{sin(u)}{u} du = \frac{\pi}{2}$$

$$\textit{since } \frac{sin(u)}{u} \textit{ is an even function}$$

$$\int_{-\infty}^\infty \frac{sin(u)}{u} du = \pi$$

9.1.2 $\int \sin(x^2)$

Theorem 9.3

$$\boxed{\int_0^\infty \sin(x^2) = \frac{\sqrt{2\pi}}{4}} \tag{33}$$

Proof

Starting from equation 30

$$\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n) du$$
$$setting\ y = 1, n = 2, x = 0\ and\ ns - 1 = 0\ we\ obtain$$
$$|p| = 1, \alpha = tan^{-1}(1/0) = \pi/2, s = 1/2$$
$$\frac{\Gamma(\frac{1}{2}) sin(\frac{\pi}{4})}{2} = \int_0^\infty sin(u^2) du$$
$$using\ equation\ 12\text{:}\ \Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$\int_0^\infty sin(u^2) du = \frac{\sqrt{\pi}\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2\pi}}{4}$$

9.1.3 $\int \cos(x^2)$

Theorem 9.4

$$\boxed{\int_0^\infty \cos(x^2) = \frac{\sqrt{2\pi}}{4}} \tag{34}$$

Proof
Starting from equation 31

$$\frac{\Gamma(s)}{n|p|^s} cos(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} cos(yu^n) du$$
$$setting\ y = 1, n = 2, x = 0\ and\ ns - 1 = 0\ we\ obtain$$
$$|p| = 1, \alpha = tan^{-1}(1/0) = \pi/2, s = 1/2$$
$$\frac{\Gamma(\frac{1}{2}) cos(\frac{\pi}{4})}{2} = \int_0^\infty cos(u^2) du$$
$$using\ equation\ 12\text{:}\ \Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$\int_0^\infty cos(u^2) du = \frac{\sqrt{\pi}\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2\pi}}{4}$$

9.1.4 $\int \frac{\sin(x^2)}{x^2}$

Theorem 9.5

$$\boxed{\int_0^\infty \frac{\sin(x^2)}{x^2} = \sqrt{\frac{\pi}{2}}} \tag{35}$$

Proof

Starting from equation 30

$$\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n) du$$

$$setting\ y = 1, n = 2, x = 0\ and\ ns - 1 = -2\ we\ obtain$$

$$|p| = 1, \alpha = tan^{-1}(1/0) = \pi/2, s = -1/2$$

$$\frac{\Gamma(\frac{-1}{2}) sin(-\frac{\pi}{4})}{2} = \int_0^\infty \frac{sin(u^2)}{u^2} du$$

$$using\ equation\ 15\text{:}\ \Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$$

$$\int_0^\infty \frac{sin(u^2)}{u^2} du = -\frac{\Gamma(\frac{-1}{2})}{2\sqrt{2}} = \sqrt{\frac{\pi}{2}}$$

9.1.5 $\int \frac{\sin(x^n)}{x^n}$

Theorem 9.6

$$\boxed{\int_0^\infty \frac{\sin(x^n)}{x^n} = \frac{\Gamma(\frac{1}{n})}{n-1} \cdot cos(\frac{\pi}{2n}) \quad , n > 1} \tag{36}$$

Proof
Starting from equation 30

$$\frac{\Gamma(s)}{n|p|^s} sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-xu^n} sin(yu^n) du$$

$$setting\ y = 1, x = 0\ and\ ns - 1 = -n\ we\ obtain$$

$$|p| = 1, \alpha = tan^{-1}(1/0) = \pi/2, s = 1/n - 1$$

$$\frac{\Gamma(\frac{1}{n} - 1) sin(\frac{\pi}{2n} - \frac{\pi}{2})}{n} = \int_0^\infty \frac{sin(u^n)}{u^n} du$$

$$using\ equation\ 2.1\ and\ assuming\ n \neq 1\text{:}\ \Gamma(k+1) = k\Gamma(k)$$

$$\int_0^\infty \frac{sin(u^n)}{u^n} du = \frac{\Gamma(\frac{1}{n})}{n(\frac{1}{n} - 1)} sin(\frac{\pi}{2n} - \frac{\pi}{2})$$

$$\int_0^\infty \frac{sin(u^n)}{u^n} du = \frac{\Gamma(\frac{1}{n})}{1-n)} sin(\frac{\pi}{2n} - \frac{\pi}{2})$$

$$using\ sin(x - \frac{\pi}{2}) = -cos(x)$$

$$\int_0^\infty \frac{sin(u^n)}{u^n} du = \frac{\Gamma(\frac{1}{n})}{n-1} \cdot cos(\frac{\pi}{2n})$$

10 Relation to the Beta Function

10.1 Definition

The Beta Function is defined as:

Definition 10.1

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

10.2 Symmetry

Theorem 10.1

$$\beta(x, y) = \beta(y, x) \tag{37}$$

Proof
Starting from the definition 10.1

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

setting $u = 1 - t$, hence $du = -dt$ we obtain

$$\beta(x, y) = \int_1^0 (1-u)^{x-1} u^{y-1} - du$$

Switching the bounds of integration we get

$$\beta(x, y) = \int_0^1 (1-u)^{x-1} u^{y-1} du = \beta(y, x)$$

10.3 Relation to Gamma

Theorem 10.2

$$\boxed{\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}} \tag{38}$$

Proof

Starting from the definition 10.1

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

$$\textit{setting } t=\frac{1}{1+u}, \textit{ hence } dt=-\frac{1}{(1+u)^2}du \textit{ we obtain}$$

$$\beta(x,y)=\int_\infty^0 (\frac{1}{1+u})^{x-1}(1-\frac{1}{1+u})^{y-1}\frac{1}{(1+u)^2}(-du)$$

Switching the bounds of integration and simplifying we get

$$\beta(x,y)=\int_0^\infty (\frac{1}{1+u})^{x-1}(\frac{u}{1+u})^{y-1}\frac{1}{(1+u)^2}du$$

collecting powers of $(1+u)$

$$\beta(x,y)=\int_0^\infty \frac{u^{y-1}}{(1+u)^{x+y}}du$$

Using the Gamma definition 9 and applying a change of variables $\alpha w=t$*, hence* $dt=\alpha dw$

$$\Gamma(z)=\alpha^z\int_0^\infty e^{-\alpha w}w^{z-1}dw$$

setting $z=x+y$ *and* $\alpha=1+u$*, we obtain*

$$\Gamma(x+y)=(1+u)^{x+y}\int_0^\infty e^{-(1+u)w}w^{x+y-1}dw$$

Combining with the Beta function integral we obtained

$$\beta(x,y)\Gamma(x+y)=\int_0^\infty \Gamma(x+y)\frac{u^{y-1}}{(1+u)^{x+y}}du$$

$$\beta(x,y)\Gamma(x+y)=\int_0^\infty \Big((1+u)^{x+y}\int_0^\infty e^{-(1+u)w}w^{x+y-1}dw\Big)\frac{u^{y-1}}{(1+u)^{x+y}}du$$

Simplfying

$$\beta(x,y)\Gamma(x+y)=\int_0^\infty\int_0^\infty e^{-(1+u)w}w^{x+y-1}u^{y-1}dwdu$$

Rearranging

$$\beta(x,y)\Gamma(x+y)=\int_0^\infty \Big(\int_0^\infty e^{-uw}u^{y-1}du\Big)e^{-w}w^{x+y-1}dw$$

$$\textit{Using } \Gamma(z)=\alpha^z\int_0^\infty e^{-\alpha u}u^{z-1}du$$

$$\beta(x,y)\Gamma(x+y)=\int_0^\infty \frac{\Gamma(y)}{w^y}e^{-w}w^{x+y-1}dw$$

Simplfying and moving $\Gamma(y)$ *outside the integral*

$$\beta(x,y)\Gamma(x+y)=\Gamma(y)\int_0^\infty e^{-w}w^{x-1}dw$$

Using the definition of the $\Gamma(z)$

$$\beta(x,y)\Gamma(x+y)=\Gamma(y)\Gamma(x)$$

$$\beta(x,y)=\frac{\Gamma(y)\Gamma(x)}{\Gamma(x+y)}$$

10.4 Relation to integral of powers of sin(x) and cos(x)

Theorem 10.3

$$\boxed{\int_0^{\pi/2} sin^n(t)cos^m(t)dt = \frac{1}{2}\beta\Big(\frac{n+1}{2}, \frac{m+1}{2}\Big) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n+m}{2}+1)}} \tag{39}$$

Proof

$$A(m,n) = \int_0^{\pi/2} sin^n(t)cos^m(t)dt$$
$$setting\ sin^2(t) = u,\ hence\ 2sin(t)cos(t)dt = du\ we\ obtain$$
$$note\ cos^2(t) = 1-u,\ and\ sin^2(0) = 0\ and\ sin^2(\frac{\pi}{2}) = 1$$
$$A(m,n) = \int_0^1 u^{\frac{n}{2}}(1-u)^{\frac{m}{2}}\frac{du}{2\sqrt{u}\sqrt{1-u}}$$
$$Simplifying\ the\ powers\ we\ get$$
$$A(m,n) = \frac{1}{2}\int_0^1 u^{\frac{n}{2}-\frac{1}{2}}(1-u)^{\frac{m}{2}-\frac{1}{2}}du$$
$$Writing\ it\ in\ the\ Beta\ function\ form\ we\ get$$
$$A(m,n) = \frac{1}{2}\int_0^1 u^{\frac{n+1}{2}-1}(1-u)^{\frac{m+1}{2}-1}du = \frac{1}{2}\beta(\frac{n+1}{2}, \frac{m+1}{2})$$

10.5 Proof of the Legendre duplication formula

Theorem 10.4

$$\boxed{\Gamma(x)\cdot\Gamma(x+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)} \tag{40}$$

Proof

Using the relation to the Beta function, and the definition of the Beta function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

let $x = y = s$

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_0^1 t^{s-1}(1-t)^{s-1}dt$$

let $t = \frac{1+x}{2}$, hence $dt = \frac{dx}{2}$

note when $t = 0$ then $x = -1$, and when $t = 1$ then $x = 1$

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_{-1}^1 (\frac{1+x}{2})^{s-1}(\frac{1-x}{2})^{s-1}\frac{dx}{2}$$

Simplifying the powers of 2 and rearranging

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\int_{-1}^1 (1+x)^{s-1}(1-x)^{s-1}dx$$

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\int_{-1}^1 (1-x^2)^{s-1}dx$$

Notice that the integral is even around 0, hence

$$2^{2s-1}\Gamma(s)\Gamma(s) = 2\Gamma(2s)\int_0^1 (1-x^2)^{s-1}dx$$

To evaluate the integral, notice that after a substitution of $u = x^2$ the integral is just $\beta(1/2, s)$

$$\beta(1/2, s) = \int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}dt$$

Let $t = x^2$, hence $dt = 2xdx$

$$\beta(1/2, s) = \int_0^1 \frac{1}{x}(1-x^2)^{s-1}2xdx = 2\int_0^1 (1-x^2)^{s-1}dx$$

Hence

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\beta(1/2, s)$$

Using theorem 10.2

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\frac{\Gamma(1/2)\Gamma(s)}{\Gamma(s+1/2)}$$

Rearranging and using equation 12

$$\Gamma(s)\Gamma(s+1/2) = \Gamma(2s)\frac{\sqrt{\pi}}{2^{2s-1}}$$

11 Approximations

11.1 Stirling's Approximation

Theorem 11.1 *The basic approximation of the factorial function for large n is*

$$\boxed{n! \approx (\frac{n}{e})^n \sqrt{2\pi n}} \tag{41}$$

Proof
Starting from the definition of the Gamma function 9

$$\Gamma(n+1) = n! = \int_0^\infty e^{-t} t^n dt$$

the idea is to take the log of the integrand and then slightly perturb it

$$ln(e^{-t}t^n) = nln(t) - t$$

$$let\ t = n + \epsilon$$

$$\ln(e^{-t}t^n) = nln(n+\epsilon) - (n+\epsilon) = nln(n) + nln(1+\frac{\epsilon}{n}) - (n+\epsilon)$$

$$for\ large\ n,\ \frac{\epsilon}{n} << 1$$

Using the Taylor-Expansion of $ln(1+x)$

$$\ln(e^{-t}t^n) = n\Big(ln(n) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\frac{\epsilon}{n})^k\Big) - (n+\epsilon)$$

$$\ln(e^{-t}t^n) = nln(n) - n + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\frac{\epsilon^k}{n^{k-1}}) - \epsilon$$

$$\ln(e^{-t}t^n) = nln(n) - n - \frac{\epsilon^2}{2n} + \frac{\epsilon^3}{3n^2} - \frac{\epsilon^4}{4n^3} + ...$$

$$\ln(e^{-t}t^n) \approx nln(n) - n - \frac{\epsilon^2}{2n}$$

$$e^{-t}t^n \approx \frac{n^n}{e^n} e^{-\frac{\epsilon^2}{2n}}$$

$$n! \approx \int_{-n}^{\infty} \frac{n^n}{e^n} e^{-\frac{\epsilon^2}{2n}} d\epsilon$$

$$n! \approx \frac{n^n}{e^n} \int_{-n}^{\infty} e^{\frac{-\epsilon^2}{2n}} d\epsilon$$

Assuming n is large enough

$$n! \approx \frac{n^n}{e^n} \int_{-\infty}^{\infty} e^{-\frac{\epsilon^2}{2n}} d\epsilon$$

$$Using \int_{-\infty}^{\infty} e^{-px^2} dx = \sqrt{\frac{\pi}{p}}$$

$$n! \approx \frac{n^n}{e^n} \sqrt{2n\pi}$$

11.1.1 Limit as n approaches infinity

The approximation in the this result can be stated more correctly as follows

$$\boxed{\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n\sqrt{2\pi n}} = 1} \tag{42}$$

We can state this equivalently by taking logarithms of both sides. We will use this form to help derive the full Stirling approximation.

$$\boxed{\lim_{n\to\infty} ln(n!) - nln(n) + n - \frac{ln(n)}{2} = ln(\sqrt{2\pi})} \tag{43}$$

11.2 Stirling's Full Approximation

Theorem 11.2 *Expanding on Stirling's approximation to include as many terms as needed we need to use the Euler-Maclaurin formula*[2].

$$\boxed{n! = (\frac{n}{e})^n\sqrt{2\pi n}\cdot exp\Big(\sum_{k=1}^{\lfloor m/2\rfloor} \frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{n^{2k-1}} - \frac{1}{m}\int_{n-1}^{\infty}\frac{B_m(\{t\})}{t^m}dt\Big)} \tag{44}$$

Proof
Starting from the definition of the simple version of the Euler-Maclaurin formula

$$\sum_{j=0}^{r} g(j) - \int_{x=0}^{r} g(x)dx = \frac{g(0)+g(r)}{2} + \sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{(2k)!}\Big(g^{(2k-1)}(r) - g^{(2k-1)}(0)\Big) + R_{r,m}$$

$$R_{r,m} = \frac{(-1)^{m+1}}{m!}\int_0^r B_m(\{t\})g^{(m)}(t)dt$$

Define $f(x) = g(x+1)$ *and* $n = r+1$*, so that* $f(n) = f(r+1) = g(r)$*. we get*

$$\sum_{j=1}^{n} f(j) - \int_{x=1}^{n} f(x)dx = \frac{f(1)+f(n)}{2} + \sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(1)\Big) + R_{n,m}$$

$$R_{n,m} = \frac{(-1)^{m+1}}{m!}\int_1^{n-1} B_m(\{t\})f^{(m)}(t)dt$$

[2]See the Darboux Formula document

Let $f(x) = ln(x)$, therefore

$$f^{(m)}(x) = \frac{(-1)^{m-1} \cdot (m-1)!}{(x)^m} \text{ or } f^{(2k-1)}(x) = \frac{(-1)^{2k-2} \cdot (2k-2)!}{(x)^{2k-1}}$$

Plugging in $f(x)$ and $f^{(m)}(x)$ into the Euler-Maclaurin formula we get

$$\sum_{j=1}^{n} ln(j) = ln(n!) = \int_{x=1}^{n} ln(x)dx + \frac{ln(1) + ln(n)}{2}$$

$$+ \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{(2k)!} \Big(\frac{(-1)^{2k-2} \cdot (2k-2)!}{n^{2k-1}} - \frac{(-1)^{2k-2} \cdot (2k-2)!}{1^{2k-1}} \Big) + R_{n,m}$$

$$R_{n,m} = \frac{(-1)^{m+1}}{m!} \int_{1}^{n-1} B_m(\{t\}) \frac{(-1)^{m-1} \cdot (m-1)!}{t^m} dt$$

Evalulating the integral and simplifying

$$ln(n!) = \big(n \cdot ln(n) - n\big) - \big(1 \cdot ln(1) - 1\big) + \frac{ln(n)}{2} + \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k} \cdot (2k-2)!}{(2k)!} \Big(\frac{1}{n^{2k-1}} - 1 \Big) + R_{n,m}$$

$$R_{n,m} = \frac{1}{m} \int_{1}^{n-1} \frac{B_m(\{t\})}{t^m} dt$$

Further simplifying

$$ln(n!) = n \cdot ln(n) - n + 1 + \frac{ln(n)}{2} + \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{2k \cdot (2k-1)} \Big(\frac{1}{n^{2k-1}} - 1 \Big) + R_{n,m}$$

To eliminate some terms we can rearranging and take limits of both sides. This would allow us to use the simple Stirling Approximation to skip the evaluation of one of the sums.

$$\lim_{n \to \infty} ln(n!) - n \cdot ln(n) + n - \frac{ln(n)}{2}$$

$$= \lim_{n \to \infty} 1 + \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{2k \cdot (2k-1)} \frac{1}{n^{2k-1}} - \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{2k \cdot (2k-1)} + R_{n,m}$$

$$= 1 - \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{2k \cdot (2k-1)} + \lim_{n \to \infty} R_{n,m}$$

Using equation 43

$$1-\sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{2k\cdot(2k-1)}+\lim_{n\to\infty}R_{n,m}=ln(\sqrt{2\pi})$$

Plugging this back in, we get:

$$ln(n!)-n\cdot ln(n)+n-\frac{ln(n)}{2}=ln(\sqrt{2\pi})+\sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{n^{2k-1}}+R_{n,m}-\lim_{n\to\infty}R_{n,m}$$

Using the integral form of $R_{n,m}$ we can then re-write the difference $D=R_{n,m}-\lim_{n\to\infty}R_{n,m}$

$$D=R_{n,m}-\lim_{n\to\infty}R_{n,m}=\frac{1}{m}\int_1^{n-1}\frac{B_m(\{t\})}{t^m}dt-\frac{1}{m}\int_1^{\infty}\frac{B_m(\{t\})}{t^m}dt=-\frac{1}{m}\int_{n-1}^{\infty}\frac{B_m(\{t\})}{t^m}dt$$

Now raising e to the value of both sides, we get

$$n!=(\frac{n}{e})^n\sqrt{2\pi n}\cdot exp\Big(\sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{n^{2k-1}}-\frac{1}{m}\int_{n-1}^{\infty}\frac{B_m(\{t\})}{t^m}dt\Big)$$

11.2.1 Approximation big-O notation

Theorem 11.3 *The integral in the exponent can be bound by a polynomial power with respect to n.*

$$\boxed{n!=(\frac{n}{e})^n\sqrt{2\pi n}\cdot exp\Big(\sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{n^{2k-1}}+O(\frac{1}{n^m})\Big)} \tag{45}$$

Proof

We need to show that $\frac{1}{m}\int_{n-1}^{\infty}\frac{B_m(\{t\})}{t^m}dt=O\Big(\frac{1}{n^m}\Big)$

$$D=-\frac{1}{m}\int_{n-1}^{\infty}\frac{B_m(\{t\})}{t^m}dt$$

$$\begin{aligned}D\leq|D|\leq\frac{1}{|m|}\int_{n-1}^{\infty}\frac{|B_m(\{t\})|}{t^m}dt&\leq\frac{\max_{x\in[0,1]}|B_m(x)|}{|m|}\int_{n-1}^{\infty}\frac{1}{t^m}dt\\&=\underbrace{\frac{\max_{x\in[0,1]}|B_m(x)|}{m\cdot(m+1)}}_{\neq f(n)}\underbrace{\frac{1}{(n-1)^m}}_{f(n)}=O(\frac{1}{n^m})\end{aligned}$$

11.3 Plugging in values

Generating an approximation with $m=7$ and using $B_2=\frac{1}{6},B_4=-\frac{1}{30},B_6=\frac{1}{42}$ we obtain:

$$n!=(\frac{n}{e})^n\sqrt{2\pi n}\cdot exp\Big(\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}+O(\frac{1}{n^7})\Big) \tag{46}$$

Since the factorial function grows very fast, it's useful to consider the logarithm of the factorial function. This would result in the following approximation.

$$ln(n!)=\frac{ln(2\pi)}{2}+nln(n)-n++\frac{ln(n)}{2}+\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}+O(\frac{1}{n^7}) \tag{47}$$