Useful Imaginary Identities

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January 19, 2019

1 Definition

Define a number i such that

Definition 1.1

$$\boxed{i^2 = -1}$$

2 Euler's equation

This formula is the underlying connection between imaginary numbers and the trigonometric functions.

Theorem 2.1 *Euler's equation*

$$\boxed{e^{ix} = cos(x) + i \cdot sin(x)} \tag{1}$$

Proof 1 through Taylor series expansion

Taking the taylor series of $cos(x), sin(x), e^{ix}$ *we obtain*

$$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - ...$$
$$sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - ...$$
$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}$$

Note that we can split the taylor series of e^{ix} *into a real part and an imaginary part*

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\Big) + i\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\Big)$$
$$e^{ix} = cos(x) + i \cdot sin(x)$$

Proof 2 through partial fraction decomposition

Evaluating the integral of $\int \frac{1}{x^2+1}$ *two different ways*

First way, change of variables

$$Let\ tan(\theta) = x,\ then\ dx = \frac{1}{cos^2(\theta)}d\theta$$

$$\int \frac{1}{x^2+1}dx = \int \frac{1}{\frac{sin^2(\theta)}{cos^2(\theta)}+1}\frac{1}{cos^2(\theta)}d\theta = \int \frac{1}{sin^2(\theta)+cos^2(\theta)}d\theta = \int d\theta = \theta + C_1$$

$$\int \frac{1}{x^2+1}dx = arctan(x) + C_1$$

Second way, partial fractions decomposition

$$\int \frac{1}{x^2+1}dx = \int \frac{1}{(x+i)(x-i)}dx = \frac{i}{2}\cdot\int \frac{1}{x+i}dx - \frac{i}{2}\cdot\int \frac{1}{(x-i)}dx$$

$$\int \frac{1}{x^2+1}dx = \frac{i}{2}\cdot ln(x+i) - \frac{i}{2}ln(x-i) + C_2$$

$$\int \frac{1}{x^2+1}dx = \frac{i}{2}ln\left(\frac{x+i}{x-i}\right) + C_2$$

Equating the two expressions

$$\frac{i}{2}ln\left(\frac{x+i}{x-i}\right) + C_2 = arctan(x) + C_1$$

simplifying

$$-ln\left(\frac{x+i}{x-i}\right) + C = 2i\cdot arctan(x)$$

$$ln\left(\frac{x-i}{x+i}\right) + C = 2i\cdot arctan(x)$$

$$K\frac{x-i}{x+i} = e^{2i\cdot arctan(x)}$$

Using a change of variables of $tan(\theta) = x$

$$K\frac{tan(\theta)-i}{tan(\theta)+i} = e^{2i\cdot\theta}$$

simplifying

$$e^{2i\cdot\theta} = K\frac{(tan(\theta)-i)^2}{tan^2(\theta)+1} = K\frac{tan^2(\theta)-2i\cdot tan(\theta)-1}{\frac{1}{cos^2(\theta)}} = K(sin^2(\theta) - i\cdot 2sin(\theta)cos(\theta) - cos^2(\theta)$$

Using the double angle sin and cos formulas

$$e^{2i\cdot\theta} = K(-cos(2\theta) - isin(2\theta)$$

Plugging in $\theta = 0$ *we get*

$$K = -1$$

Replacing 2θ with x

$$e^{i \cdot x} = cos(x) + isin(x)$$

Proof 3 through differential equations

Consider the soluions of the differential equation $y' = i \cdot y$, with $y(0) = 1$

Notice that both $y(x) = e^{ix}$ and $y(x) = cos(x) + isin(x)$ both satisfy the equation

Prove that the solution is unique

$$\text{let } y_1(x), y_2(x) \text{ be two solutions to the above equation, then define}$$
$$f(x) = y_1(x) - y_2(x)$$
$$\text{Notice that } f(0) = 1 - 1 = 0$$
$$f'(x) = y_1'(x) - y_2'(x) = i \cdot f(x)$$
$$\text{To prove that f' is 0, hence f is constant, we define}$$
$$g(x) = e^{-ix} f(x)$$
$$\text{Notice that } g(0) = 1 \cdot f(0) = 0$$
$$g'(x) = -i \cdot g(x) + e^{-ix} f'(x)$$
$$g'(x) = -i \cdot g(x) + i \cdot e^{-ix} f(x)$$
$$g'(x) = -i \cdot g(x) + i \cdot g(x) = 0$$
$$g(x) = 0 \quad \forall x$$
$$e^{-ix} f(x) = 0 \quad \forall x$$

Assuming e^{ix} is not 0 for any x

$$f(x) = 0 \quad \forall x$$
$$y_1(x) - y_2(x) = 0 \quad \forall x$$
$$e^{i \cdot x} = cos(x) + isin(x) \quad \forall x$$

3 Properties

Starting from Euler's equation

Definition 3.1

$$e^{ix} = cos(x) + i \cdot sin(x)$$

Using this equation we can plug in the value of $x = \frac{\pi}{2}$ and obtain

$$e^{i\frac{\pi}{2}} = cos(\frac{\pi}{2}) + i \cdot sin(\frac{\pi}{2}) = 0 + i \cdot 1 = i \tag{2}$$

Therefore we can claim

Definition 3.2

$$i = e^{i\frac{\pi}{2}}$$

3.1 Reciprocal

$$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = -i \tag{3}$$

3.2 Square Root

$$\sqrt{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = cos(\frac{\pi}{4}) + i \cdot sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \tag{4}$$

3.3 i to the power of i

$$i^i = (e^{i\frac{\pi}{2}})^i = e^{-\frac{\pi}{2}} \approx 0.208 \tag{5}$$