1 Geometric Series

Prove that:

$$\sum_{i=1}^{n} a \cdot r^{i-1} = \frac{a \cdot (1 - r^{n+1})}{1 - r}$$

Proof:
Define S_n such that:

$$S_n = a + ar + ar^2 + ... + ar^{n-1} + ar^n$$

Note that

$$r \cdot S_n = ar + ar^2 + ... + ar^{n-1} + ar^n + ar^{n+1}$$

Such that

$$S_n - r \cdot S_n = a - ar^{n+1}$$

$$=> S_n = \frac{a \cdot (1 - r^{n+1})}{1 - r}$$

If $|r| < 1$:
Then

$$\sum_{i=1}^{\infty} a \cdot r^{i-1} = \lim_{n \to +\infty} \frac{a \cdot (1 - r^{n+1})}{1 - r} = \frac{a}{1 - r}$$

Note that by taking derivatives of both sides and a bit of reindexing:

$$\sum_{i=0}^{\infty} \frac{d}{dr} a \cdot r^i = \sum_{i=0}^{\infty} i \cdot a \cdot r^{i-1} = 0 + \sum_{i=1}^{\infty} i \cdot a \cdot r^{i-1} = \frac{a}{(1 - r)^2}$$

$$=> \sum_{i=1}^{\infty} i \cdot r^i = \frac{r}{(1 - r)^2} \text{ when } |r| < 1$$