Bernoulli Numbers

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1 Bernoulli Numbers

Examining the Taylor expansion of $\frac{x}{e^x-1}$, this will be required for the evaluation of the even integer values of ζ.

Definition 1.1. *The Bernoulli numbers B_n are defined as the coefficients of the Taylor expansion of the following function*

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n}$$

Theorem 1.1.

$$\sum_{k}^{n} \binom{n+1}{k} B_k = 0 \quad \forall n > 0, \ \textit{with } B_0 = 1 \tag{1}$$

Proof

Starting from definition 1.1

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Using the Taylor expansion of e^x

$$\frac{x}{\sum_{k=1}^{\infty} \frac{x^k}{k!}} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Rearranging

$$x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

$$1 = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x$$

Adjusting the indices

$$1 = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Using Cauchy's product formula for infinite sums [1]

$$1 = \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{n} \underbrace{\left(\frac{B_k}{k!} x^k\right)}_{a_k} \underbrace{\left(\frac{x^{n-k}}{(n-k+1)!}\right)}_{b_{n-k}}}_{c_n}$$

Simplifying

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} B_k x^k x^{n-k}$$
$$1 = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} B_k$$

Comparing coefficients of powers of x of both sides we get

$$n = 0, \quad 1 = B_0$$
$$n \neq 0, \quad 0 = \sum_{k=0}^{n} \binom{n+1}{k} B_k$$

1.1 First few Bernoulli numbers

Using equation 1 with $n = 1$ implies $\binom{2}{0} B_0 + \binom{2}{1} B_1 = 0 = 1 + 2B_1$ which implies

$$B_1 = -\frac{1}{2} \tag{2}$$

Using equation 1 with $n = 2$ implies $\binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 = 0 = 1 - \frac{3}{2} + 3B_2$ which implies

$$B_2 = \frac{1}{6} \tag{3}$$

Similarly when we apply this equation for increasing values of n

$$B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \ldots \tag{4}$$

Theorem 1.2. *In fact we can show that all odd Bernoulli numbers after n=1 are 0*

$$B_n = 0 \quad \forall \text{ odd } n > 1 \tag{5}$$

Proof

Starting from definition 1.1

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

[1] $\left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. See proof: TODO ADD

Removing the first two terms of the sum

$$\frac{x}{e^x - 1} = B_0 + B_1 x + \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$

Rearranging

$$\frac{x}{e^x - 1} - 1 + \frac{x}{2} = \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$
$$\frac{x + xe^x}{2e^x - 2} - 1 = \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$

If we can show that the RHS is an even function then we have established the proof

$$y(x) = \frac{x + xe^x}{2e^x - 2} - 1$$
$$y(-x) = \frac{-x + -xe^{-x}}{2e^{-x} - 2} - 1$$

Simplifying

$$y(-x) = \frac{-x - xe^{-x}}{2e^{-x} - 2} \cdot \frac{e^x}{e^x} - 1$$
$$y(-x) = \frac{-xe^x - x}{2 - 2e^x} - 1$$
$$y(-x) = \frac{x + xe^x}{2e^x - 2} - 1$$
$$y(-x) = y(x)$$
$$B_n = 0 \quad \forall \; odd \; n > 1$$

2 Bernoulli Polynomials

Definition 2.1. *The Bernoulli polynomials $B_n(x)$ are defined as*

$$\boxed{B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}}$$

2.1 Examples

$$
\begin{aligned}
B_0(x) &= B_0 = 1 \\
B_1(x) &= B_0 x + B_1 = x - \frac{1}{2} \\
B_2(x) &= B_0 x^2 + \binom{2}{1} B_1 x + B_2 = x^2 - x + \frac{1}{6} \\
B_3(x) &= B_0 x^3 + \binom{3}{1} B_1 x^2 + \binom{3}{2} B_2 x + B_3 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x
\end{aligned}
$$

Note that $B_n(x)$ is a polynomial of degree n.

2.2 Useful Properties

Theorem 2.1. $B_n(0)$

$$\boxed{B_n(0) = B_n} \tag{6}$$

Proof

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

$$B_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_k x^{n-k} + B_n$$

$$B_n(0) = \sum_{k=0}^{n-1} \binom{n}{k} B_k \cdot 0 + B_n = B_n$$

Theorem 2.2. $B_n(1)$

$$\boxed{B_n(1) = B_n \quad n \neq 1 \quad B_1(1) = -B_1} \tag{7}$$

Proof

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

$$B_n(1) = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Using equation 1, $\sum_k^n \binom{n+1}{k} B_k = 0$

$$\sum_k^{n-1} \binom{n}{k} B_k = 0$$

Adding B_n *to both sides*

$$\sum_k^{n} \binom{n}{k} B_k = B_n$$

Theorem 2.3. *Derivative*

$$\boxed{B_n^{'}(x) = nB_{n-1}(x)} \tag{8}$$

Proof

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

Taking derivatives of both sides

$$B_n^{'}(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_k (n-k) x^{n-k-1}$$

$$B_n^{'}(x) = \sum_{k=0}^{n-1} \frac{n!(n-k)}{k!(n-k)!} B_k x^{n-1-k}$$

$$B_n^{'}(x) = \sum_{k=0}^{n-1} \frac{(n-1)! \cdot n}{k!(n-k-1)!} B_k x^{n-1-k}$$

$$B_n^{'}(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} B_k x^{n-1-k}$$

$$B_n^{'}(x) = nB_{n-1}(x)$$

Theorem 2.4. *Integral*

$$\boxed{\int_{x=0}^{1} B_n(x) dx = 0} \tag{9}$$

Proof

Starting from the derivative equation, using $m = n - 1$

$$B_{m+1}^{'}(x) = (m+1) \cdot B_m(x)$$

Integrating both sides

$$\int_{x=0}^{1} B_{m+1}^{'}(x) = B_{m+1}(1) - B_{m+1}(0) = (m+1) \cdot \int_{x=0}^{1} B_m(x)$$

Using the Bernoulli polynomial properties

$$B_{m+1}(1) - B_{m+1}(0) = B_{m+1} - B_{m+1} = 0$$

Therefore for $m \neq -1$

$$\int_{x=0}^{1} B_m(x) = 0$$

3 Euler-Maclaurin formula

Let $\{x\} = x - \lfloor x \rfloor$ then

Theorem 3.1.

$$\sum_{k=1}^{n-1} f(k) = \int_{x=1}^{n} f(x)dx - \sum_{k=1}^{m} \frac{B_k}{k!} \cdot \big(f^{(k-1)}(n) - f^{(k-1)}(1)\big) + R_{mn} \tag{10a}$$

$$R_{mn} = \frac{(-1)^{m+1}}{m!} \cdot \int_{x=1}^{n} B_m(\{x\}) f^{(m)}(x)dx \tag{10b}$$

Proof

See the Darboux Formula paper for proof

3.1 Stirling's Approximation

Theorem 3.2.

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot exp\left(\sum_{k=1}^{\lfloor m/2 \rfloor} \frac{B_{2k}}{2k \cdot (2k-1)} \frac{1}{n^{2k-1}} + O(\frac{1}{n^m})\right) \tag{11}$$

Proof

See the Gamma function paper for proof

4 Connection to Zeta(2n)

See the Zeta function document.

Theorem 4.1. *We can express all even integer values of ζ using Bernoulli numbers*

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2\cdot(2n)!} \tag{12}$$

Proof: See the Zeta function document.

4.1 The first few even integer values of Zeta

$$\zeta(2)=\frac{\pi^2}{6},\zeta(4)=\frac{\pi^4}{90},\zeta(6)=\frac{\pi^6}{945},\zeta(8)=\frac{\pi^8}{9450},\zeta(10)=\frac{\pi^{10}}{93555},$$
$$\zeta(12)=\frac{691}{638512875}\pi^{12},\zeta(14)=\frac{2}{18243225}\pi^{14},\zeta(16)=\frac{3617}{325641566250}\pi^{16},$$
$$\zeta(18)=\frac{43867}{38979295480125}\pi^{18},\zeta(20)=\frac{174611}{1531329465290625}\pi^{20},\zeta(22)=\frac{155366}{13447856940643125}\pi^{22},$$
$$\zeta(24)=\frac{236364091}{201919571963756521875}\pi^{24},\ldots$$