Zeta Function

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1 Definition

Definition 1.1 *The Riemann Zeta function is defined as*

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

We have already shown that [1]

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1}$$

2 Bernoulli Numbers

Examining the Taylor expansion of $\frac{x}{e^x-1}$, this will be required for the evaluation of the even integer values of ζ.

Definition 2.1 *The Bernoulli numbers B_n are defined as the coefficients of the Taylor expansion of the following function*

$$\boxed{\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n}$$

Theorem 2.1

$$\sum_{k}^{n} \binom{n+1}{k} B_k = 0 \quad \forall n > 0,\ \textit{with } B_0 = 1 \tag{2}$$

Proof

Starting from definition 2.1

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

[1] See Gamma function document

Using the Taylor expansion of e^x

$$\frac{x}{\sum_{k=1}^{\infty} \frac{x^k}{k!}} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Rearranging

$$x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$
$$1 = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x$$

Adjusting the indicies

$$1 = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Using Cauchy's product formula for infinite sums [2]

$$1 = \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{n} \underbrace{\left(\frac{B_k}{k!} x^k\right)}_{a_k} \underbrace{\left(\frac{x^{n-k}}{(n-k+1)!}\right)}_{b_{n-k}}}_{c_n}$$

Simplifying

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} B_k x^k x^{n-k}$$
$$1 = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} B_k$$

Comparing coefficients of powers of x of both sides we get

$$n = 0, \quad 1 = B_0$$
$$n \neq 0, \quad 0 = \sum_{k=0}^{n} \binom{n+1}{k} B_k$$

2.1 First few Bernoulli numbers

Using equation 2 with $n = 1$ implies $\binom{2}{0} B_0 + \binom{2}{1} B_1 = 0 = 1 + 2B_1$ which implies

$$B_1 = -\frac{1}{2} \tag{3}$$

Using equation 2 with $n = 2$ implies $\binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 = 0 = 1 - \frac{3}{2} + 3B_2$ which implies

$$B_2 = \frac{1}{6} \tag{4}$$

[2] $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. See proof: TODO ADD

Similarly when we apply this equation for increasing values of n

$$B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, ... \tag{5}$$

Theorem 2.2 *In fact we can show that all odd Bernoulli numbers after n=1 are 0*

$$B_n = 0 \quad \forall \text{ odd } n > 1 \tag{6}$$

Proof

Starting from definition 2.1

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Removing the first two terms of the sum

$$\frac{x}{e^x - 1} = B_0 + B_1 x + \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$

Rearranging

$$\frac{x}{e^x - 1} - 1 + \frac{x}{2} = \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$
$$\frac{x + xe^x}{2e^x - 2} - 1 = \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$

If we can show that the RHS is an even function then we have established the proof

$$y(x) = \frac{x + xe^x}{2e^x - 2} - 1$$
$$y(-x) = \frac{-x + -xe^{-x}}{2e^{-x} - 2} - 1$$

Simplfying

$$y(-x) = \frac{-x - xe^{-x}}{2e^{-x} - 2} \cdot \frac{e^x}{e^x} - 1$$
$$y(-x) = \frac{-xe^x - x}{2 - 2e^x} - 1$$
$$y(-x) = \frac{x + xe^x}{2e^x - 2} - 1$$
$$y(-x) = y(x)$$
$$B_n = 0 \quad \forall \text{ odd } n > 1$$

3 Even integer values $\zeta(2n)$

While the odd integer values of $\zeta(n)$ do not have a closed form expression, the even values do.

Theorem 3.1 *We can express all even integer values of ζ using Bernoulli numbers*

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2\cdot(2n)!} \tag{7}$$

Proof

Starting from the series representation [3] *of cotangent*

$$\pi x cot(\pi x) = 1 - \sum_{k=1}^{\infty}\zeta(2k)x^{2k}$$

Using the complex exponential representation [4] *of cotangent*

$$i\pi x + \frac{2i\pi x}{e^{2i\pi x}-1} = 1 - \sum_{k=1}^{\infty}\zeta(2k)x^{2k}$$

Using the Bernoulli numbers definition 2.1

$$i\pi x + \sum_{n=0}^{\infty}\frac{B_n}{n!}(2i\pi x)^n = 1 - \sum_{k=1}^{\infty}\zeta(2k)x^{2k}$$

Rearranging and using the first two values of the Bernoulli numbers

$$\begin{aligned}
i\pi x + B_0 + B_1(2i\pi x) + \sum_{n=2}^{\infty}\frac{B_n}{n!}(2i\pi x)^n &= 1 - \sum_{k=1}^{\infty}\zeta(2k)x^{2k}\\
i\pi x + 1 + -i\pi x + \sum_{n=2}^{\infty}\frac{B_n}{n!}(2i\pi x)^n &= 1 - \sum_{k=1}^{\infty}\zeta(2k)x^{2k}\\
\sum_{n=2}^{\infty}\frac{B_n}{n!}(2i\pi)^n x^n &= -\sum_{k=1}^{\infty}\zeta(2k)x^{2k}
\end{aligned}$$

Using the fact the the odd Bernoulli numbers after B_1 are 0

$$\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(-1)^k(2\pi)^{2k}x^{2k} = \sum_{k=1}^{\infty}-\zeta(2k)x^{2k}$$

Comparing coefficients

$$\zeta(2k) = \frac{B_{2k}}{(2k)!}(-1)^{k+1}(2\pi)^{2k} \quad \forall k\in\mathbb{N}_{>0}$$

[3] See Euler's Sine Product Formula document

[4] $cot(x) = i + \frac{2i}{e^{2ix}-1}$. See Basic trig functions document

3.1 The first few even integer values of ζ

$$\zeta(2)=\frac{\pi^2}{6},\zeta(4)=\frac{\pi^4}{90},\zeta(6)=\frac{\pi^6}{945},\zeta(8)=\frac{\pi^8}{9450},\zeta(10)=\frac{\pi^{10}}{93555},$$
$$\zeta(12)=\frac{691}{638512875}\pi^{12},\zeta(14)=\frac{2}{18243225}\pi^{14},\zeta(16)=\frac{3617}{325641566250}\pi^{16},$$
$$\zeta(18)=\frac{43867}{38979295480125}\pi^{18},\zeta(20)=\frac{174611}{1531329465290625}\pi^{20},\zeta(22)=\frac{155366}{13447856940643125}\pi^{22},$$
$$\zeta(24)=\frac{236364091}{201919571963756521875}\pi^{24},...$$

4 Related functions

4.1 Dirichlet Eta Function

Definition 4.1 *The Dirichlet Eta function is defined as*

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + ...$$

Theorem 4.1

$$\boxed{\eta(z) = (1 - 2^{1-s})\zeta(z)} \tag{8}$$

Proof

Starting from the definition of $\eta(z)$

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = \sum_{k=1}^{\infty} \frac{1}{k^z} - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^z}$$

Using the definition of $\zeta(z)$

$$\eta(z) = \zeta(z) - \frac{2}{2^z}\zeta(z)$$

$$\eta(z) = (1 - 2^{1-z})\zeta(z)$$

4.1.1 Values

$$\eta(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = ln(2) \qquad ^5 \tag{9}$$

$$\eta(2) = (1 - 2^{1-2}) \cdot \zeta(2) = \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12} \tag{10}$$

[5] This can be seen by taking the Taylor-Expansion of $ln(x + 1)$ and evaluating at $x = 1$

4.2 Dirichlet Lambda Function

Definition 4.2 *The Dirichlet Lambda function is defined as*

$$\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1))^z} = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + ...$$

Theorem 4.2

$$\boxed{\lambda(z) = (1 - 2^{-s})\zeta(z)} \tag{11}$$

Proof

$$\textit{Starting from the definition of } \lambda(z)$$
$$\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1))^z} = \sum_{k=1}^{\infty} \frac{1}{k^z} - \sum_{k=1}^{\infty} \frac{1}{(2k)^z}$$
$$\textit{Using the definition of } \zeta(z)$$
$$\lambda(z) = \zeta(z) - \frac{1}{2^z}\zeta(z)$$
$$\lambda(z) = (1 - 2^{-z})\zeta(z)$$

4.2.1 Values

$$\lambda(2) = (1 - 2^{-2}) \cdot \zeta(2) = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8} \tag{12}$$

5 Integral Forms

5.1 Gamma with Zeta functions

Theorem 5.1 *See the Gamma function document for the definition and properties of the Gamma function.*

$$\boxed{\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du} \tag{13}$$

Proof

Using $\Gamma(z) = n^z \displaystyle\int_0^\infty e^{-nu}u^{z-1}du$ *from the Gamma Function document and rearranging*

$$\Gamma(z)\frac{1}{n^z} = \int_0^\infty e^{-nu}u^{z-1}du$$

Let n be all integers and take the sum of all equalities

$$\sum_{n=1}^{\infty}\Gamma(z)\frac{1}{n^z} = \Gamma(z)\zeta(z) = \sum_{n=1}^{\infty}\int_0^\infty e^{-nu}u^{z-1}du$$

Bringing the summation into the integral since this integral converges uniformly

$$\Gamma(z)\zeta(z) = \int_0^\infty \Big(\sum_{n=1}^{\infty} e^{-nu}\Big)u^{z-1}du$$

Notice that $\displaystyle\sum_{n=1}^{\infty} e^{-nu}$ *is a geometric series if* $u > 0$

$$\Gamma(z)\zeta(z) = \int_0^\infty \Big(\frac{e^{-u}}{1 - e^{-u}}\Big)u^{z-1}du$$

Simplfiying by multiplying top and bottom by e^u

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du$$

6 Jacobi Theta Function

6.1 Definition

Definition 6.1 *The Jacobi Theta function is defined as*

$$\Theta(x) = \sum_{k=-\infty}^{\infty} e^{-\pi n^2 x}$$

Theorem 6.1 *The Jacobi Theta function has the following functional form*

$$\Theta(x) = \frac{1}{\sqrt{x}}\Theta(\frac{1}{x}) \tag{14}$$

Proof

Using the Poisson Summation Formula [6]

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi iky} dy = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x - 2\pi iky} dy$$

Re-writting the exponent

$$-\pi y^2 x - 2\pi iky = -\pi x(y^2 - 2\pi i\frac{k}{x}y) = -\pi x(y + i\frac{k}{x})^2 - \pi\frac{k^2}{x}$$

Plugging in the factored exponent, we get

$$\Theta(x) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x(y+i\frac{k}{x})^2 - \pi\frac{k^2}{x}} dy = \sum_{k=-\infty}^{\infty} e^{-\pi\frac{k^2}{x}} \int_{-\infty}^{\infty} e^{-\pi x(y+i\frac{k}{x})^2} dy$$

After a change of variables $z = y + i\frac{k}{x}$ we can use the Guassian integral identity as proved before, refer to the Guassian integral document, $\int_{-\infty}^{\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}}$, we get:

$$\Theta(x) = \sum_{k=-\infty}^{\infty} e^{-\pi\frac{k^2}{x}} \sqrt{\frac{\pi}{x\pi}} = \frac{1}{\sqrt{x}}\Theta(\frac{1}{x})$$

7 Functional Form

7.1 First Form

Theorem 7.1 *The symmetric functional equation of ζ*

$$\boxed{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)} \tag{15}$$

Proof

TODO

[6] $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$. See proof: TODO ADD

7.2 Second Form

Theorem 7.2

$$\boxed{\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)} \tag{16}$$

Proof

TODO

8 Riemann Hypothesis Statement