Darboux's formula

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1 Darboux's formula

Let $\phi(t)$ be a polynomial of degree n and $f(x)$ is an analytic function then,

Theorem 1.1.

$$\sum_{m=0}^{n}(-1)^m(z-a)^m\Big[\phi^{(n-m)}(1)f^{(m)}(z)-\phi^{(n-m)}(0)f^{(m)}(a)\Big]=$$
$$(-1)^n(z-a)^{n+1}\int_0^1\phi(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt \quad (1)$$

Proof

For $n=0$ we have $\phi(t)=c$ then

$$LHS_0=c\Big[f(z)-f(a)\Big]$$
$$RHS_0=(z-a)\int_0^1 cf^{(1)}\Big[a+t(z-a)\Big]dt$$

After a change of variables $u=a+t(z-a)$, $du=dt(z-a)$ we get

$$RHS_0=c\int_a^z f^{(1)}(u)du=c\Big[f(u)\Big]\Bigg|_{u=a}^{u=z}=c\Big[f(z)-f(a)\Big]=LHS_0$$

For $n>0$

$$RHS_n=(-1)^n(z-a)^{n+1}\int_0^1\phi(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

Using integration by parts $\int_{t=0}^1 uv'=uv\big|_{t=0}^1-\int_{t=0}^1 u'v$, with $u=\phi(t)$, and $v'=f^{(n+1)}(a+t(z-a))$

$$RHS_n=(-1)^n(z-a)^{n+1}\left[\phi(t)\frac{f^{(n)}\Big[a+t(z-a)\Big]}{(z-a)}\Bigg|_{t=0}^{t=1}-\int_0^1\phi'(t)\frac{f^{(n)}\Big[a+t(z-a)\Big]}{(z-a)}dt\right]$$

Evaluating the first definite integral and multipying by $(z-a)$

$$RHS_n=(-1)^n(z-a)^n\left[\phi(1)f^{(n)}(z)-\phi(0)f^{(n)}(a)-\int_0^1\phi'(t)f^{(n)}\Big[a+t(z-a)\Big]dt\right]$$

Moving the second integral out to it's own term

$$RHS_n = (-1)^n(z-a)^n\Bigg[\phi(1)f^{(n)}(z) - \phi(0)f^{(n)}(a)\Bigg] + (-1)^{n-1}(z-a)^n\int_0^1 \phi'(t)f^{(n)}\Big[a+t(z-a)\Big]dt$$

Repeating the integration by parts process we get

$$RHS_n = (-1)^n(z-a)^n\Bigg[\phi(1)f^{(n)}(z) - \phi(0)f^{(n)}(a)\Bigg]$$

$$+(-1)^{n-1}(z-a)^{n-1}\Bigg[\phi^{(1)}(1)f^{(n-1)}(z) - \phi^{(1)}(0)f^{(n-1)}(a)\Bigg]$$

$$+(-1)^{n-2}(z-a)^{n-1}\int_0^1 \phi^{(2)}(t)f^{(n-1)}\Big[a+t(z-a)\Big]dt$$

Repeating the integration by parts n-times until $\phi^{(n+1)}(t) = 0$ *we get*

$$RHS_n = \sum_{k=0}^{n}(-1)^{n-k}(z-a)^{n-k}\Bigg[\phi^{(k)}(1)f^{(n-k)}(z) - \phi^{(k)}(0)f^{(n-k)}(a)\Bigg] + 0$$

Changing the index of summation $m = n - k$

$$RHS_n = \sum_{m=0}^{n}(-1)^{m}(z-a)^{m}\Bigg[\phi^{(n-m)}(1)f^{(m)}(z) - \phi^{(n-m)}(0)f^{(m)}(a)\Bigg]$$

Therefore we get

$$\sum_{m=0}^{n}(-1)^{m}(z-a)^{m}\Big[\phi^{(n-m)}(1)f^{(m)}(z) - \phi^{(n-m)}(0)f^{(m)}(a)\Big] =$$

$$(-1)^n(z-a)^{n+1}\int_0^1 \phi(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

2 Taylor's formula

Set $\phi(t) = (t-1)^n$ into Darnoux's formula to obtain Taylor's formula.

Theorem 2.1.

$$\boxed{f(z) = \sum_{m=0}^{n} \frac{(z-a)^m f^{(m)}(a)}{m!} + (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)} \Big[a + t(z-a)\Big] dt} \tag{2}$$

Proof

Starting from Darboux's formula **??**

$$\sum_{m=0}^{n} (-1)^m (z-a)^m \Big[\phi^{(n-m)}(1) f^{(m)}(z) - \phi^{(n-m)}(0) f^{(m)}(a)\Big] =$$
$$(-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)} \Big[a + t(z-a)\Big] dt$$

Setting $\phi(t) = (t-1)^n$, which is an n-degree polynomial

Note that $\phi^{(k)}(t) = \frac{n!}{(n-k)!}(t-1)^{n-k}$ for $k \leq n$, therefore

$$\phi^{(k)}(0) = \frac{n!}{(n-k)!}(-1)^{n-k} \quad \textit{for } k \leq n$$
$$\phi^{(k)}(1) = 0 \quad \textit{for } k < n$$
$$\phi^{(n)}(1) = n!$$

$$\sum_{m=1}^{n} (-1)^m (z-a)^m \Big[0 - \frac{n!(-1)^m}{(m)!} f^{(m)}(a)\Big] + (-1)^0 (z-a)^0 \Big[n! f^{(0)}(z) - n! f^{(0)}(a)\Big]$$
$$= (-1)^n (z-a)^{n+1} \int_0^1 (t-1)^n f^{(n+1)} \Big[a + t(z-a)\Big] dt$$

Simplfiying and dividing by $n!$

$$-\sum_{m=1}^{n} \frac{(z-a)^m}{m!} f^{(m)}(a) + \Big[f(z) - f(a)\Big]$$
$$= (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)} \Big[a + t(z-a)\Big] dt$$

Rearranging

$$f(z) = f(a) + \sum_{m=1}^{n} \frac{(z-a)^m}{m!} f^{(m)}(a) + (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)} \Big[a + t(z-a)\Big] dt$$

Adding the extra term into the summation to finish the proof

$$f(z) = \sum_{m=0}^{n} \frac{(z-a)^m}{m!} f^{(m)}(a) + (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)} \Big[a + t(z-a)\Big] dt$$

2.1 Infinite Series

Theorem 2.2. *If* $|f^{(k)}(x)| \leq M$ *and* $|z-a| \leq R \quad \forall x$ *then*

$$\boxed{f(z) = \sum_{m=0}^{\infty} \frac{(z-a)^m f^{(m)}(a)}{m!}} \tag{3}$$

Proof

Starting from Taylor's formula 2 and taking the limit as n approaches infinity

$$f(z) = \sum_{m=0}^{\infty} \frac{(z-a)^m f^{(m)}(a)}{m!} + \lim_{n\to\infty} (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)}\Big[a + t(z-a)\Big] dt$$

Define the remainder term, and then we will show that it converges to 0

$$R_n = (-1)^n \frac{(z-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n+1)}\Big[a + t(z-a)\Big] dt$$

$$|R_n| = \Big|(-1)^n\Big| \cdot \Big|\frac{(z-a)^{n+1}}{n!}\Big| \cdot \Big|\int_0^1 (t-1)^n f^{(n+1)}\Big[a + t(z-a)\Big] dt\Big|$$

$$|R_n| \leq \frac{|z-a|^{n+1}}{n!} \cdot \int_0^1 \Big|(t-1)^n\Big| \cdot \Big|f^{(n+1)}\Big[a + t(z-a)\Big]\Big| dt$$

Using $|f^{(k)}(x)| \leq M$

$$|R_n| \leq \frac{|z-a|^{n+1}}{n!} \cdot \int_0^1 |t-1|^n \cdot M dt$$

Using $|z-a| \leq R$

$$|R_n| \leq \frac{R^{n+1}M}{n!} \cdot \int_0^1 |t-1|^n dt$$

Note that between $0 \leq t \leq 1$, $|t-1| \leq 1$, *therefore*

$$|R_n| \leq \frac{R^{n+1}M}{n!}$$

Therefore since M *and* R *are constant:*

$$\lim_{n\to\infty} R_n \leq \lim_{n\to\infty} |R_n| \leq 0$$

Therefore

$$f(z) = \sum_{m=0}^{\infty} \frac{(z-a)^m f^{(m)}(a)}{m!}$$

3 Euler-Maclaurin formula

Set $\phi(t) = B_n(x)$ into Darnoux's formula to obtain the Euler-Maclaurin formula, where $B_n(x)$ is the Bernoulli polynomial[1].

Theorem 3.1. *For any analytical $g(x)$*

$$\boxed{\begin{aligned} w\sum_{j=0}^{r-1} g(a+jw) - \int_{x=a}^{a+rw} g(x)dx = \frac{w}{2}\Big(g(a) - g(a+rw)\Big) + \\ \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{w^{2k}B_{2k}}{(2k)!}\Big(g^{(2k-1)}(a+rw) - g^{(2k-1)}(a)\Big) + R_{n,r} \\ R_{n,r} = \frac{(-1)^{n+1}w^{n+1}}{n!}\int_0^r B_n(\{t\})f^{(n+1)}(a+tw)dt \end{aligned}} \tag{4}$$

Proof

Starting from Darboux's formula **??**

$$\sum_{m=0}^{n}(-1)^m(z-a)^m\Big[\phi^{(n-m)}(1)f^{(m)}(z) - \phi^{(n-m)}(0)f^{(m)}(a)\Big] =$$
$$(-1)^n(z-a)^{n+1}\int_0^1 \phi(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

Setting $\phi(t) = B_n(t)$, which is an n-degree polynomial

Note that $\phi^{(k)}(t) = \frac{n!}{(n-k)!}B_{n-k}(t)$ for $k \le n$, therefore

$$\phi^{(k)}(0) = \phi^{(k)}(1) = \frac{n!}{(n-k)!}B_{n-k} \quad \textit{for } k \le n, k \neq 1$$
$$\phi^{(1)}(0) = -\phi^{(1)}(1) = B_1$$

adjusting the indicies we get

$$\phi^{(n-m)}(0) = \phi^{(n-m)}(1) = \frac{n!}{m!}B_m \quad \textit{for } n \ge m \ge 0, m \neq 1$$
$$\phi^{(n-1)}(0) = B_1 = -\phi^{(n-1)}(1)$$

Plugging these values into Darboux's formula, and removing the first two terms we get

$$\sum_{m=2}^{n}(-1)^m(z-a)^m\Big[\frac{n!}{m!}B_m f^{(m)}(z) - \frac{n!}{m!}B_m f^{(m)}(a)\Big] + n!B_0\Big(f(z) - f(a)\Big)$$
$$+(-1)n!(z-a)\Big(-B_1f^{'}(z) - B_1f^{'}(a)\Big) = (-1)^n(z-a)^{n+1}\int_0^1 B_n(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

[1] $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}$. See the Bernoulli number document for the full definition.

simplfying

$$\sum_{m=2}^{n}\frac{(-1)^m(z-a)^m B_m}{m!}(f^{(m)}(z)-f^{(m)}(a))+B_0\Big(f(z)-f(a)\Big)+(z-a)B_1\Big(f^{'}(z)+f^{'}(a)\Big)=$$
$$\frac{(-1)^n(z-a)^{n+1}}{n!}\int_0^1 B_n(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

To simplify notation let us define R_n^a

$$R_n^a=\frac{(-1)^n(z-a)^{n+1}}{n!}\int_0^1 B_n(t)f^{(n+1)}\Big[a+t(z-a)\Big]dt$$

Therefore we get

$$\sum_{m=2}^{n}\frac{(-1)^m(z-a)^m B_m}{m!}(f^{(m)}(z)-f^{(m)}(a))+B_0\Big(f(z)-f(a)\Big)+(z-a)B_1\Big(f^{'}(z)+f^{'}(a)\Big)=R_n^a$$

Plugging in $B_0=1$ *and* $B_1=-\frac{1}{2}$

$$\sum_{m=2}^{n}\frac{(-1)^m(z-a)^m B_m}{m!}\Big(f^{(m)}(z)-f^{(m)}(a)\Big)+\Big(f(z)-f(a)\Big)-(z-a)\frac{1}{2}\Big(f^{'}(z)+f^{'}(a)\Big)=R_n^a$$

Adding $(z-a)f^{'}(a)$ *to both sides and rearranging we get*

$$(z-a)f^{'}(a)=f(z)-f(a)-\frac{(z-a)}{2}\Big(f^{'}(z)-f^{'}(a)\Big)+\sum_{m=2}^{n}\frac{(-1)^m(z-a)^m B_m}{m!}\Big(f^{(m)}(z)-f^{(m)}(a)\Big)-R_n^a$$

Define $w=z-a$ *and* $g(x)=f'(x)$ *therefore we get*

$$wg(a)=\int_{x=a}^{a+w}g(x)dx-\frac{w}{2}\Big(g(a+w)-g(a)\Big)+\sum_{m=2}^{n}\frac{(-1)^m w^m B_m}{m!}\Big(g^{(m-1)}(a+w)-g^{(m-1)}(a)\Big)-R_n^a$$

Allowing a *to be multiples of* w, $a'=a+jw$ *and summing over* i *we get*

$$\sum_{j=0}^{r-1}wg(a+jw)=\sum_{j=0}^{r-1}\int_{x=a+jw}^{a+(j+1)w}g(x)dx-\sum_{j=0}^{r-1}\frac{w}{2}\Big(g(a+(j+1)w)-g(a+jw)\Big)$$
$$+\sum_{j=0}^{r-1}\sum_{m=2}^{n}\frac{(-1)^m w^m B_m}{m!}\Big(g^{(m-1)}(a+(j+1)w)-g^{(m-1)}(a+jw)\Big)-\sum_{j=0}^{r-1}R_n^{a+jw}$$

Simplyfying and evaluating the sums we get

$$w\sum_{j=0}^{r-1}g(a+jw)=\int_{x=a}^{a+rw}g(x)dx-\frac{w}{2}\Big(g(a+rw)-g(a)\Big)$$
$$+\sum_{m=2}^{n}\frac{(-1)^m w^m B_m}{m!}\Big(g^{(m-1)}(a+rw)-g^{(m-1)}(a)\Big)-\sum_{j=0}^{r-1}R_n^{a+jw}$$

Rearranging

$$w\sum_{j=0}^{r-1} g(a+jw) - \int_{x=a}^{a+rw} g(x)dx = \frac{w}{2}\Big(g(a) - g(a+rw)\Big)$$
$$+\sum_{m=2}^{n} \frac{(-1)^m w^m B_m}{m!}\Big(g^{(m-1)}(a+rw) - g^{(m-1)}(a)\Big) - \sum_{j=0}^{r-1} R_n^{a+jw}$$

Using the fact that $B_{2n+1} = 0, n \geq 1$ *we can simplfy the sum using* $m = 2k$

$$w\sum_{j=0}^{r-1} g(a+jw) - \int_{x=a}^{a+rw} g(x)dx = \frac{w}{2}\Big(g(a) - g(a+rw)\Big)$$
$$+\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{w^{2k} B_{2k}}{(2k)!}\Big(g^{(2k-1)}(a+rw) - g^{(2k-1)}(a)\Big) - \sum_{j=0}^{r-1} R_n^{a+jw}$$

Recall the remainder term definition

$$R_n^a = \frac{(-1)^n (z-a)^{n+1}}{n!} \int_0^1 B_n(t) f^{(n+1)}\Big[a + t(z-a)\Big]dt$$

Plugging in the definition of $w = (z-a)$ *we get*

$$R_n^a = \frac{(-1)^n w^{n+1}}{n!} \int_0^1 B_n(t) f^{(n+1)}\Big[a + tw\Big]dt$$

Define the new remainder term

$$R_n = -\sum_{j=0}^{r-1} R_n^{a+jw}$$

Plug in the definition of R_n^{a+jw} *and simplfy*

$$R_n = -\sum_{j=0}^{r-1} \frac{(-1)^n w^{n+1}}{n!} \int_0^1 B_n(t) f^{(n+1)}(a + jw + tw)dt$$

Simplfy the remainder term using the fractional operator $\{x\}$

$$R_{n,r} = \frac{(-1)^{n+1} w^{n+1}}{n!} \int_0^r B_n(\{t\}) f^{(n+1)}(a + tw)dt$$

3.1 Infinite Series

Note that as n approaches infinity, the remainder term goes to 0.

Theorem 3.2. *If* $|g^{(k)}(x)| \leq M \quad \forall x$ *and* a, w, M *finite then*

$$\boxed{\lim_{n\to\infty} R_{n,r} = \lim_{n\to\infty} \frac{(-1)^{n+1} w^{n+1}}{n!} \int_0^r B_n(\{t\}) f^{(n+1)}(a + tw)dt = 0} \qquad (5)$$

Proof

Starting from the definition of $R_{n,r}$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(-1)^{n+1}w^{n+1}}{n!} \int_0^r B_n(\{t\}) f^{(n+1)}(a+tw)dt = 0$$

$$\lim_{n\to\infty} R_n \leq \lim_{n\to\infty} |R_n|$$

$$|R_n| \leq \frac{|w|^{n+1}}{n!} \int_0^r |B_n(\{t\})| M dt \leq \frac{|w|^{n+1}|B_n(t)|rM}{n!}$$

Using $\lim_{n\to\infty} \frac{B_n(x)}{n!} = 0$ *from BERNOULLI DOCUMENT TODO*

$$\lim_{n\to\infty} R_n \leq \lim_{n\to\infty} |R_n| = 0$$

3.2 Simple version

By specifying $a = 0$ and $w = 1$ we obtain an exact difference between the summation and the integral of any analytic function $g(x)$.

Theorem 3.3. *For any analytical $g(x)$*

$$\boxed{\begin{gathered} \sum_{j=0}^{r} g(j) - \int_{x=0}^{r} g(x)dx = \frac{g(0)+g(r)}{2} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!}\Big(g^{(2k-1)}(r) - g^{(2k-1)}(0)\Big) + R_{n,r} \\ R_{n,r} = \frac{(-1)^{n+1}}{n!} \int_0^r B_n(\{t\}) g^{(n)}(t)dt \end{gathered}} \tag{6}$$

Proof

Starting from the Euler-Maclaurin formula

$$w\sum_{j=0}^{r-1} g(a+jw) - \int_{x=a}^{a+rw} g(x)dx = \frac{w}{2}\Big(g(a) - g(a+rw)\Big) +$$

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{w^{2k}B_{2k}}{(2k)!}\Big(g^{(2k-1)}(a+rw) - g^{(2k-1)}(a)\Big) + R_{n,r}$$

$$R_{n,r} = \frac{(-1)^{n+1}w^{n+1}}{n!} \int_0^r B_n(\{t\}) g^{(n)}(a+tw)dt$$

Plugging in $a = 0$ and $w = 1$ we get

$$\sum_{j=0}^{r-1} g(j) - \int_{x=0}^{r} g(x)dx = \frac{1}{2}\Big(g(0) - g(r)\Big) + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!}\Big(g^{(2k-1)}(r) - g^{(2k-1)}(0)\Big) + R_{n,r}$$

$$R_{n,r} = \frac{(-1)^{n+1}}{n!} \int_0^r B_n(\{t\}) g^{(n)}(t)dt$$

Adding $g(r)$ to both sides

$$\sum_{j=0}^{r} g(j) - \int_{x=0}^{r} g(x)dx = \frac{g(0)+g(r)}{2} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!}\Big(g^{(2k-1)}(r) - g^{(2k-1)}(0)\Big) + R_{n,r}$$