One of Ramanujan Identity

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1 Ramanujan Identity

Prove that:

$$\alpha\sum_{k=1}^{\infty}\frac{k}{e^{2\alpha k}-1}+\beta\sum_{k=1}^{\infty}\frac{k}{e^{2\beta k}-1}=\frac{\alpha+\beta}{24}-\frac{1}{4}$$

$$\alpha>0,\beta>0,\alpha\cdot\beta=\pi^2$$

$$f(t)=\begin{cases}\frac{|t|}{e^{2\alpha|t|}-1} & t\neq 0\\ \frac{1}{2\alpha} & t=0\end{cases}$$

$$\hat{f}(s)=\int_{-\infty}^{\infty}f(t)e^{-i2\pi st}dt=\int_{-\infty}^{\infty}f(t)\cos(2\pi st)dt+i\int_{-\infty}^{\infty}f(t)\sin(2\pi st)dt=2\int_{0}^{\infty}\frac{t\cos(2\pi st)}{e^{2\alpha t}-1}dt \text{ and } s\neq 0$$

$$\hat{f}(0)=2\int_{0}^{\infty}\frac{t}{e^{2\alpha t}-1}dt=\frac{2}{(2\alpha)^2}\zeta(2)=\frac{\pi^2}{12\alpha^2} \text{ (Insert Reference)}$$

Notice both the function and it's Fourier Transform are even.
By Poisson Summation formula (Include Reference):

$$\hat{f}(0)+2\sum_{m=1}^{\infty}\hat{f}(m)=f(0)+2\sum_{n=1}^{\infty}f(n)$$

Hence applying this:

$$\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\pi^2}{12\alpha^2}+4\sum_{m=1}^{\infty}\int_{0}^{\infty}\frac{t\cos(2\pi mt)}{e^{2\alpha t}-1}dt$$

Examining

$$\int_{0}^{\infty}\frac{t\cos(2\pi mt)}{e^{2\alpha t}-1}dt$$

By considering

$$\int_{0}^{\infty}\frac{\sin(cx)}{e^{x}-1}dx=\int_{0}^{\infty}\frac{e^{-x}\sin(cx)}{1-e^{-x}}dx=\int_{0}^{\infty}\sin cx\sum_{k=1}^{\infty}e^{-kx}dx$$

By the Dominant Convergence Theorem: (Include Reference)

$$\int_0^\infty \sin(cx) \sum_{k=1}^{\infty} e^{-kx} dx = \sum_{k=1}^{\infty} \int_0^\infty \sin(cx) e^{-kx} dx$$

By the LaPlace Transform of $\sin at$:

$$\int_0^\infty \sin(ct) e^{-sx} dt = \frac{c}{c^2 + s^2} \text{ (Insert Reference)}$$

$$\sum_{k=1}^{\infty} \int_0^\infty \sin(cx) e^{-kx} dx = \sum_{k=1}^{\infty} \frac{c}{c^2 + k^2}$$

By the Tangent Identity (Insert Reference)

$$\sum_{n=-\infty}^{\infty} \frac{1}{\alpha^2 + n^2} = \frac{\pi}{\alpha} \coth(\pi\alpha)$$

Where

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}$$

Hence:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\alpha^2 + n^2} = \frac{\pi}{\alpha} \cdot \frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1}$$

Hence:

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2 + n^2} = \frac{1}{2} \cdot \left(\frac{\pi}{\alpha} \cdot \frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1} - \frac{1}{\alpha^2}\right)$$

$$=> \sum_{k=1}^{\infty} \frac{c}{c^2 + k^2} = \frac{\pi}{2} \cdot \frac{e^{2\pi c} + 1}{e^{2\pi c} - 1} - \frac{1}{2c}$$

$$=> \int_0^\infty \frac{\sin(cx)}{e^x - 1} dx = \frac{\pi}{2} \cdot \frac{e^{2\pi c} + 1}{e^{2\pi c} - 1} - \frac{1}{2c}$$

Taking the derivative of both sides using Leibniz Rule

$$\frac{d}{dc} \int_0^\infty \frac{\sin(cx)}{e^x - 1} dx = \int_0^\infty \frac{\partial}{\partial c} \frac{\sin(cx)}{e^x - 1} dx = \int_0^\infty \frac{x\cos(cx)}{e^x - 1} dx$$

$$= \frac{d}{dc}\left(\frac{\pi}{2} \cdot \frac{e^{2\pi c} + 1}{e^{2\pi c} - 1} - \frac{1}{2c}\right) = \frac{\pi}{2} \cdot \frac{2\pi e^{2\pi c}}{e^{2\pi c} - 1} - \frac{\pi}{2} \cdot \frac{2\pi e^{4\pi c} + 2\pi e^{2\pi c}}{(e^{2\pi c} - 1)^2} + \frac{1}{2c^2} = \frac{-2\pi^2 e^{2\pi c}}{(e^{2\pi c} - 1)^2} + \frac{1}{2c^2}$$

Therefore making a change of variables $x = 2\alpha t$

$$\int_0^\infty \frac{x\cos(cx)}{e^x - 1} dx = \int_0^\infty \frac{2\alpha t \cos(2\alpha c t)}{e^{2\alpha t} - 1} 2\alpha dt = (2\alpha)^2 \int_0^\infty \frac{t\cos(2\alpha c t)}{e^{2\alpha t} - 1} dt = \frac{-2\pi^2 e^{2\pi c}}{(e^{2\pi c} - 1)^2} + \frac{1}{2c^2}$$

Letting $c = \pi \cdot \frac{m}{\alpha}$

$$(2\alpha)^2 \int_0^\infty \frac{t\cos(2\pi m \cdot t)}{e^{2\alpha t} - 1} dt = \frac{-2\pi^2 e^{2\pi^2 \cdot \frac{m}{\alpha}}}{(e^{2\pi^2 \cdot \frac{m}{\alpha}} - 1)^2} + \frac{1}{2\pi^2 \cdot \frac{m^2}{\alpha^2}} = \frac{-2\pi^2 e^{2\pi^2 \cdot \frac{m}{\alpha}}}{(e^{2\pi^2 \cdot \frac{m}{\alpha}} - 1)^2} + \frac{\alpha^2}{2\pi^2 m^2}$$

Plugging this result back into our original statement:

$$\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\pi^2}{12\alpha^2}+\sum_{m=1}^{\infty}\left(\frac{-2\pi^2e^{2\pi^2\cdot\frac{m}{\alpha}}}{\alpha^2(e^{2\pi^2\cdot\frac{m}{\alpha}}-1)^2}+\frac{1}{2\pi^2m^2}\right)=\frac{\pi^2}{12\alpha^2}+\sum_{m=1}^{\infty}\frac{-2\pi^2e^{2\pi^2\cdot\frac{m}{\alpha}}}{\alpha^2(e^{2\pi^2\cdot\frac{m}{\alpha}}-1)^2}+\frac{1}{2\pi^2}\zeta(2)$$

$$\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\pi^2}{12\alpha^2}+\sum_{m=1}^{\infty}\frac{-2\pi^2e^{2\pi^2\cdot\frac{m}{\alpha}}}{\alpha^2(e^{2\pi^2\cdot\frac{m}{\alpha}}-1)^2}+\frac{1}{12}$$

Let $\beta=\frac{\pi^2}{\alpha}$

$$=>\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\pi^2}{12\alpha^2}+\frac{1}{12}-\frac{2\pi^2}{\alpha^2}\sum_{m=1}^{\infty}\frac{e^{2m\beta}}{(e^{2m\beta}-1)^2}$$

Examining $\sum_{m=1}^{\infty}\frac{e^{2m\beta}}{(e^{2m\beta}-1)^2}$

$$\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}=\sum_{n=1}^{\infty}\frac{ne^{-2\beta n}}{1-e^{-2\beta n}}=\sum_{n=1}^{\infty}(n\sum_{m=1}^{\infty}e^{-2\beta\cdot n\cdot m})$$

$$\sum_{n=1}^{\infty}(n\sum_{m=1}^{\infty}e^{-2\beta\cdot n\cdot m})=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}ne^{-2\beta\cdot n\cdot m}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}ne^{-2\beta\cdot n\cdot m}$$

Insert Reference from Geometric Series Section

$$\sum_{m=1}^{\infty}\frac{e^{-2\beta\cdot m}}{(1-e^{-2\beta\cdot m})^2}=\sum_{m=1}^{\infty}\frac{e^{2\beta\cdot m}}{(e^{2\beta\cdot m}-1)^2}=\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}$$

Hence combining this fact we obtain:

$$=>\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\pi^2}{12\alpha^2}+\frac{1}{12}-\frac{2\pi^2}{\alpha^2}\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}$$

$$\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\beta}{12\alpha}+\frac{1}{12}-\frac{2\beta}{\alpha}\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}$$

$$\frac{1}{2}+2\alpha\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}=\frac{\beta}{12}+\frac{\alpha}{12}-2\beta\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}$$

$$\boxed{\alpha\sum_{n=1}^{\infty}\frac{n}{e^{2\alpha n}-1}+\beta\sum_{n=1}^{\infty}\frac{n}{e^{2\beta n}-1}=\frac{\beta+\alpha}{24}-\frac{1}{4}}$$

Where $\alpha\cdot\beta=\pi^2$
The special case of $\alpha=\beta=\pi$:

$$\boxed{\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1}=\frac{1}{24}-\frac{1}{8\pi}}$$