Basic Trig Functions

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1 Basic Inverse Trig Functions

If we assume we have this as a given.

Definition 1.1 (Euler's Formula)**.**

$$e^{ix} = \cos(x) + i \cdot \sin(x) \tag{1}$$

Theorem 1.2 (Sine Definition)**.**

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = -i\frac{e^{ix} - e^{-ix}}{2} \tag{2}$$

Proof.

$$\begin{gathered} e^{-ix} = \cos(-x) + i \cdot \sin(-x) = \cos(x) - i \cdot \sin(x) \\ e^{ix} - e^{-ix} = 2i \cdot \sin(x) \\ \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = -i\frac{e^{ix} - e^{-ix}}{2} \end{gathered}$$

□

Theorem 1.3 (Cosine Definition)**.**

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{3}$$

Proof. Same as above. □

Theorem 1.4 (Tangent Definition)**.**

$$\tan(x) = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \tag{4}$$

Proof. Same as above. □

Theorem 1.5 (Cotangent Definition)**.**

$$\cot(x) = i + \frac{2i}{e^{2ix} - 1} \tag{5}$$

Proof.

$$\tan(x) = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$
$$\cot(x) = i\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$
$$\cot(x) = i\frac{(e^{ix} - e^{-ix}) + (2e^{-ix})}{e^{ix} - e^{-ix}}$$
$$\cot(x) = i + 2i\frac{e^{-ix}}{e^{ix} - e^{-ix}}$$
$$\cot(x) = i + \frac{2i}{e^{2ix} - 1}$$

□

If we for now ignore the non-uniqueness of the square root and logarithm of complex numbers, we can obtain

Theorem 1.6 (Arcsine Function)**.**

$$\arcsin(x) = -i \cdot \ln(ix + \sqrt{1 - x^2}) \tag{6}$$

Proof. Let $y = \arcsin(x)$

$$\sin(y) = \frac{e^{iy} - e^{-iy}}{2i}$$
$$\sin(\arcsin(x)) = x = \frac{e^{iy} - e^{-iy}}{2i}$$
$$2ix = e^{iy} - e^{-iy}$$
$$e^{2iy} - 2ixe^{iy} - 1 = 0$$
$$e^{iy} = \frac{2ix \pm \sqrt{-4x^2 + 4}}{2}$$
$$e^{iy} = ix \pm \sqrt{1 - x^2}$$
$$iy = \ln(ix + \sqrt{1 - x^2})$$
$$y = \frac{\ln(ix + \sqrt{1 - x^2})}{i} = -i \cdot \ln(ix + \sqrt{1 - x^2})$$
$$\arcsin(x) = -i \cdot \ln(ix + \sqrt{1 - x^2})$$

□

Theorem 1.7 (Arccosine Function)**.**

$$\arccos(x) = -i \cdot \ln(x + \sqrt{x^2 - 1}) \tag{7}$$

Proof. Same as above. □

Theorem 1.8 (Arctangent Function)**.**

$$\arctan(x) = \frac{i}{2} \cdot \ln\left(\frac{i + x}{i - x}\right) \tag{8}$$

Proof. Same as above. ☐

Theorem 1.9 (Arcsecant Function)**.**

$$\operatorname{arcsec}(x) = -i \ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right) \tag{9}$$

Proof. Same as above. ☐

Theorem 1.10 (Arccosecant Function)**.**

$$\operatorname{arccsc}(x) = -i \ln\left(\frac{i}{x} + \sqrt{1 - \frac{1}{x^2}}\right) \tag{10}$$

Proof. Same as above. ☐

Theorem 1.11 (Arccotangent Function)**.**

$$\operatorname{arccot}(x) = \frac{i}{2} \cdot \ln\left(\frac{x-i}{x+i}\right) \tag{11}$$

Proof. Same as above. ☐

The same can be applied to hyperbolic functions, using

Definition 1.12 (Hyperbolic Functions)**.**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{12}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{13}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \tag{14}$$

For each normal trigonometry identities there is a similar hyperbolic trigonometry identity.

The relation of the two are

Theorem 1.13 (Relation to Circular Functions)**.**

$$\begin{aligned} \cosh(ix) &= \cos(x) & \sinh(ix) &= i \cdot \sin(x) \\ \tanh(ix) &= i \cdot \tan(x) & \cos(ix) &= \cosh(x) \\ \sin(ix) &= i \cdot \sinh(x) & \tan(ix) &= i \cdot \tanh(x) \end{aligned}$$