Gaussian Integral

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1 Gaussian Integral

Theorem 1.1

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Proof

Defining I

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

Squaring I

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2} dx \cdot \int_{-\infty}^{\infty} e^{-ay^2} dy$$

Combining the integrals

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ax^2 - ay^2} dxdy$$

Converting to Polar coordinates

$$I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-ar^2} rdrd\theta$$

Solving

$$I^2 = \int_{0}^{2\pi} \tfrac{-1}{2a} e^{-ar^2} \Big|_0^{\infty} d\theta$$

$$I^2 = \int_{0}^{2\pi} \tfrac{1}{2a} d\theta$$

$$I^2 = 2\pi \tfrac{1}{2a}$$

$$I^2 = \frac{\pi}{a}$$

$$I = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Since e^{-ax^2} is an even function we also get the integral

$$\int_{0}^{\infty} e^{-ax^2} dx = \frac{\sqrt{\frac{\pi}{a}}}{2} = \sqrt{\frac{\pi}{4a}} \tag{1.0.1}$$

2 Gaussian Type Integrals

Using $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ we can take derivatives with respect to (w.r.t.) a

$$\frac{\partial}{\partial a} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}}$$
$$\int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx = \sqrt{\pi} \frac{-1}{2} a^{-3/2}$$
$$\boxed{\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}}$$

This process can be repeated to obtain the general identity

$$\boxed{\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{\pi}{|\Gamma(\frac{1}{2} - n)| a^n \sqrt{a}}} \tag{2.0.1}$$