Tribonacci Numbers

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1 Definition

The Tribonacci sequence is defined as:

$$\begin{aligned} t_0 &= 0 \\ t_1 &= 1 \\ t_2 &= 1 \\ t_n &= t_{n-1} + t_{n-2} + t_{n-3} \quad \forall \quad n \in \mathbb{Z}^+ \end{aligned} \tag{1}$$

The first few Tribonacci numbers then would be:

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$$

2 Ratios

Taking the ratios of consecutive Tribonacci numbers we find this pattern:

$$\begin{aligned} \frac{t_2}{t_1} &= 1 \\ \frac{t_3}{t_2} &= \frac{2}{1} \\ \frac{t_4}{t_3} &= \frac{2}{1} \\ \frac{t_5}{t_4} &= \frac{7}{4} \\ \frac{t_6}{t_5} &= \frac{13}{7} \\ \frac{t_7}{t_6} &= \frac{26}{13} \end{aligned}$$

Theorem 2.1 $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \frac{1}{3} \cdot \left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right)$

Proof TODO

3 Closed Form Expression

Using the characteristic equation of this sequence definition (see Characteristic Equation of recurrence relations), we have $x^3 - x^2 - x - 1 = 0$, with distinct roots of α, β, γ.

Theorem 3.1

$$t_n = c_1 \cdot \alpha^n + c_2 \cdot \beta^n + c_3 \cdot \gamma^n \quad \forall n \in \mathbb{Z}^+$$

Where the constants c_1, c_2, c_3 are derived using the initial conditions.

Theorem 3.2

$$t_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \forall n \in \mathbb{Z}^+$$

Proof

Using the initial conditions we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 && (n = 0) \\ \alpha c_1 + \beta c_2 + \gamma c_3 &= 1 && (n = 1) \\ \alpha^2 c_1 + \beta^2 c_2 + \gamma^2 c_3 &= 1 && (n = 2) \end{aligned}$$

Re-writing this as a matrix we get a classic vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Using Gaussian elimination we can solve for the coefficients

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ \alpha & \beta & \gamma & 1 \\ \alpha^2 & \beta^2 & \gamma^2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \beta - \alpha & \gamma - \alpha & 1 \\ 0 & \beta^2 - \alpha^2 & \gamma^2 - \alpha^2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{\gamma - \alpha}{\beta - \alpha} & \frac{1}{\beta - \alpha} \\ 0 & 1 & \frac{\gamma^2 - \alpha^2}{\beta^2 - \alpha^2} & \frac{1}{\beta^2 - \alpha^2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{\gamma - \alpha}{\beta - \alpha} & \frac{1}{\beta - \alpha} \\ 0 & 0 & \frac{\gamma^2 - \alpha^2}{\beta^2 - \alpha^2} - \frac{\gamma - \alpha}{\beta - \alpha} & \frac{1}{\beta^2 - \alpha^2} - \frac{1}{\beta - \alpha} \end{array}\right)$$

Therefore we can write c_3:

$$c_3 = \frac{\frac{1}{\beta^2 - \alpha^2} - \frac{1}{\beta - \alpha}}{\frac{\gamma^2 - \alpha^2}{\beta^2 - \alpha^2} - \frac{\gamma - \alpha}{\beta - \alpha}}$$

$$c_3 = \frac{(\beta - \alpha) - (\beta^2 - \alpha^2)}{(\gamma^2 - \alpha^2)(\beta - \alpha) - (\gamma - \alpha)(\beta^2 - \alpha^2)}$$

$$c_3 = \frac{\beta - \alpha}{(\beta - \alpha)(\gamma - \alpha)} \cdot \frac{1 - (\beta + \alpha)}{(\gamma + \alpha) - (\beta + \alpha)}$$

$$c_3 = \frac{1 - (\beta + \alpha)}{(\gamma - \beta)(\gamma - \alpha)}$$

$$c_3 = \frac{1 - (\beta + \alpha + \gamma) + \gamma}{(\gamma - \beta)(\gamma - \alpha)}$$

Since α, β and γ satisfy $x^3 - x^2 - x - 1 = 0$ we get the following identities

$$\alpha + \beta + \gamma = 1$$

$$\alpha \cdot \beta \cdot \gamma = 1$$

Using these identities we can simplify the expression for c_3:

$$c_3 = \frac{\gamma}{\gamma^2 - (\alpha + \beta)\gamma + \alpha\beta}$$

$$c_3 = \frac{\gamma}{\gamma^2 - (\alpha + \beta + \gamma)\gamma + \gamma^2 + \frac{\alpha\beta\gamma}{\gamma}}$$

$$c_3 = \frac{\gamma}{2\gamma^2 - \gamma + \frac{1}{\gamma}}$$

$$c_3 = \frac{1}{2\gamma - 1 + \frac{1}{\gamma^2}}$$

Re-writing $\frac{1}{\gamma^2}$ as $2\gamma - \gamma^2$:

$$1 = \gamma^3 - \gamma^2 - \gamma$$

$$\frac{1}{\gamma} = \gamma^2 - \gamma - 1$$

$$\frac{1}{\gamma^2} = \gamma - 1 - \frac{1}{\gamma}$$

$$\frac{1}{\gamma^2} = \gamma - 1 - \gamma^2 + \gamma + 1$$

$$\frac{1}{\gamma^2} = 2\gamma - \gamma^2$$

Therefore:

$$c_3 = \frac{1}{-\gamma^2 + 4\gamma - 1}$$

By symmetry we can see that:

$$c_1 = \frac{1}{-\alpha^2 + 4\alpha - 1}$$

$$c_2 = \frac{1}{-\beta^2 + 4\beta - 1}$$

Hence:

$$t_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \forall n \in \mathbb{Z}^+$$

4 Negative Indices

If we ignore the condition of $n > 0$ we can can use the functional form definition of Tribonacci numbers to define negative indices:

$$t_{n-3} = t_n - t_{n-1} - t_{n-2} \tag{2}$$

Therefore the first few will be:

$$\begin{aligned}
t_{-1} &= t_2 - t_1 - t_0 = 1 - 1 - 0 = 0 \\
t_{-2} &= t_1 - t_0 - t_{-1} = 1 - 0 - 0 = 1 \\
t_{-3} &= t_0 - t_{-1} - t_{-2} = 0 - 0 - 1 = -1 \\
t_{-4} &= t_{-1} - t_{-2} - t_{-3} = 0 - 1 - (-1) = 0 \\
t_{-5} &= t_{-2} - t_{-3} - t_{-4} = 1 - (-1) - 0 = 2 \\
t_{-6} &= t_{-3} - t_{-4} - t_{-5} = -1 - 0 - 2 = -3
\end{aligned}$$

Theorem 4.1

$$t_{-n} = t_{n-1}^2 - t_{n-2} \cdot t_n$$

Proof by induction

Assume true for k, k+1, and k+2

$$\begin{aligned}
t_{-k} &= t_{k-1}^2 - t_{k-2} \cdot t_k \\
t_{-k-1} &= t_{k-2}^2 - t_{k-3} \cdot t_{k-1} \\
t_{-k-2} &= t_{k-3}^2 - t_{k-4} \cdot t_{k-2}
\end{aligned}$$

Proving for k+3

$$TODO$$