1 Basic Inverse Trig Functions

If we assume we have this as a given.

Definition 1.1

$$e^{ix} = cos(x) + i \cdot sin(x)$$

Theorem 1.1

$$sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = -i\frac{e^{ix} - e^{-ix}}{2}$$

Proof

$$e^{-ix} = cos(-x) + i \cdot sin(-x) = cos(x) - i \cdot sin(x)$$

$$e^{ix} - e^{-ix} = 2i \cdot sin(x)$$

$$sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = -i\frac{e^{ix} - e^{-ix}}{2}$$

Theorem 1.2

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Proof: Same as above.

Theorem 1.3

$$tan(x) = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$

Proof: Same as above.

Theorem 1.4

$$cot(x) = i + \frac{2i}{e^{2ix} - 1}$$

Proof

$$tan(x) = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$

$$cot(x) = i\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$

$$cot(x) = i\frac{(e^{ix} - e^{-ix}) + (2e^{-ix})}{e^{ix} - e^{-ix}}$$

$$cot(x) = i + 2i\frac{e^{-ix}}{e^{ix} - e^{-ix}}$$

$$cot(x) = i + \frac{2i}{e^{2ix} - 1}$$

If we for now ignore the non-uniqueness of the square root and logarithm of complex numbers, we can obtain

Theorem 1.5

$$arcsin(x) = -i \cdot ln(ix + \sqrt{1 - x^2})$$

Proof: Let $y = arcsin(x)$

$$sin(y) = \frac{e^{iy} - e^{-iy}}{2i}$$

$$sin(arcsin(x)) = x = \frac{e^{iy} - e^{-iy}}{2i}$$
$$2ix = e^{iy} - e^{-iy}$$
$$e^{2iy} - 2ixe^{iy} - 1 = 0$$
$$e^{iy} = \frac{2ix \pm \sqrt{-4x^2 + 4}}{2}$$
$$e^{iy} = ix \pm \sqrt{1 - x^2}$$
$$iy = ln(ix + \sqrt{1 - x^2})$$
$$y = \frac{ln(ix + \sqrt{1 - x^2})}{i} = -i \cdot ln(ix + \sqrt{1 - x^2})$$
$$arcsin(x) = -i \cdot ln(ix + \sqrt{1 - x^2})$$

Theorem 1.6
$$arccos(x) = -i \cdot ln(x + \sqrt{x^2 - 1})$$

Proof: Same as above.

Theorem 1.7
$$arctan(x) = \frac{i}{2} \cdot ln\Big(\frac{i + x}{i - x}\Big)$$

Proof: Same as above.

Theorem 1.8
$$arcsec(x) = -iln\Big(\frac{1}{x} + \sqrt{1 - \frac{i}{x^2}}\Big)$$

Proof: Same as above.

Theorem 1.9
$$arccsc(x) = -iln\Big(\frac{i}{x} + \sqrt{1 - \frac{1}{x^2}}\Big)$$

Proof: Same as above.

Theorem 1.10
$$arccot(x) = \frac{i}{2} \cdot ln\Big(\frac{x - i}{x + i}\Big)$$

Proof: Same as above.

The same can be applied to hyperbolic functions, using

Definition 1.2
$$sinh(x) = \frac{e^x - e^{-x}}{2}$$

Definition 1.3
$$cosh(x) = \frac{e^x + e^{-x}}{2}$$

Definition 1.4
$$tanh(x) = \frac{sinh(x)}{cosh(x)}$$

For each normal trigonometry identities there is a similar hyperbolic trigonometry identity.

The relation of the two are

Definition 1.5

$$cosh(ix) = cos(x)$$

$$sinh(ix) = i \cdot sin(x)$$

$$tanh(ix) = i \cdot tan(x)$$

$$cos(ix) = cosh(x)$$

$$sin(ix) = i \cdot sinh(x)$$

$$tan(ix) = i \cdot tanh(x)$$