Symmetric Positive-Definite matrix bounds

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1 Motivating Example

Suppose there are three assets, a, b, c , by knowing the volatility of each of the assets, and the correlation between assets a and b, as well as the correlation between assets b and c, what is the interval of valid correlations between and assets a and c?

This document presents one general method of addressing this problem, and solves the case of three assets analytically.

2 Definition

Definition 2.1 *A positive-definite matrix is defined as TODO.*

Definition 2.2 *A semi-positive-definite matrix is defined as TODO.*

Definition 2.3 *A principle minor is defined as TODO.*

3 Background Theory

We make use of some known properties of positive-definite (PD) and positive-semi-definite (PSD) matrices.

Theorem 3.1 *A PD matrix must have all positive eigenvalues.*
Proof:

From the definition of PD

$$TODO$$

Theorem 3.2 *A PD matrix must have all positive principle minors.*
Proof:

From the definition of PD

$$TODO$$

4 Main Idea

All principle majors of a PD matrix, must be positive. Therefore compute them all, and figure out bounds on the unknown quantities.

4.1 Two Asset

The case of $n = 2$ is trivial, due to the definition of correlation being bounded by -1 and 1.

$$|\rho_{a,b}| \leq 1$$
$$|\frac{\sigma_{a,b}}{\sigma_a \cdot \sigma_b}| \leq 1$$

Since σ_a and σ_b are positive by definition

$$|\sigma_{a,b}| \leq \sigma_a \cdot \sigma_b$$

We show that the positive principle minors method arrives at the same bound.

Let matrix C be a 2x2 matrix,

$$C = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \tag{1}$$

4.2 3x3

We can

$$C = \begin{pmatrix} 1 & \rho_{ab} & \rho_{\mathbf{ac}} \\ \rho_{ab} & 1 & \rho_{bc} \\ \rho_{\mathbf{ac}} & \rho_{bc} & 1 \end{pmatrix} \tag{2}$$

Theorem 4.1

$$\boxed{|\rho_{\mathbf{ac}} - \rho_{ab} \cdot \rho_{bc}| \leq \sqrt{(1 - \rho_{ab}^2) \cdot (1 - \rho_{bc}^2)}} \tag{3}$$

Proof:

Det(C) ¿= 0

TODO

4.3 4x4

We can

$$C = \begin{pmatrix} 1 & \rho_{ab} & \rho_{ac} & x \\ \rho_{ab} & 1 & \rho_{bc} & \rho_{bd} \\ \rho_{ac} & \rho_{bc} & 1 & \rho_{cd} \\ x & \rho_{bd} & \rho_{cd} & 1 \end{pmatrix} \tag{4}$$

Theorem 4.2 *Bounds on $x = \rho_{ad}$*

$$\boxed{\begin{gathered} |\rho_{\mathbf{ad}} - \frac{(\rho_{a,b} \cdot \rho_{b,d} + \rho_{a,c} \cdot \rho_{c,d}) - \rho_{b,c} \cdot (\rho_{a,b} \cdot \rho_{c,d} + \rho_{a,c} \cdot \rho_{b,d})}{1 - \rho_{b,c}^2}| \leq (1 - \rho_{b,c}^2)\sqrt{D} \\ D = (1 - \rho_{a,b}^2 - \rho_{a,c}^2 - \rho_{b,c}^2 + 2 \cdot \rho_{a,b} \cdot \rho_{a,c} \cdot \rho_{b,c}) \cdot (1 - \rho_{b,c}^2 - \rho_{b,d}^2 - \rho_{c,d}^2 + 2 \cdot \rho_{b,c} \cdot \rho_{b,d} \cdot \rho_{c,d}) \end{gathered}}$$

Proof:

$$
\begin{gathered}
Det(C) \geq 0 \\
1-(\rho_{c,d}^2+\rho_{b,c}^2+\rho_{b,d}^2+\rho_{a,b}^2+\rho_{a,c}^2) \\
+2\cdot(\rho_{b,c}\rho_{b,d}\rho_{c,d}+\rho_{a,b}\rho_{b,c}\rho_{a,c}) \\
-2(\rho_{a,b}\rho_{b,d}\rho_{a,c}\rho_{c,d}) \\
+(\rho_{a,b}^2\rho_{c,d}^2+\rho_{a,c}^2\rho_{b,d}^2) \\
+2x(\rho_{a,b}\rho_{b,d}+\rho_{a,c}\rho_{c,d}) \\
-2x(\rho_{a,b}\rho_{b,c}\rho_{c,d}+\rho_{a,c}\rho_{b,d}\rho_{b,c}) \\
+x^2(\rho_{b,c}^2-1) \geq 0
\end{gathered}
$$

This is just a quadradic equation

$$
\begin{gathered}
a=\rho_{b,c}^2-1 \\
b=2(\rho_{a,b}\rho_{b,d}+\rho_{a,c}\rho_{c,d})-2(\rho_{a,b}\rho_{b,c}\rho_{c,d}+\rho_{a,c}\rho_{b,d}\rho_{b,c}) \\
c=1-(\rho_{c,d}^2+\rho_{b,c}^2+\rho_{b,d}^2+\rho_{a,b}^2+\rho_{a,c}^2) \\
+2\cdot(\rho_{b,c}\rho_{b,d}\rho_{c,d}+\rho_{a,b}\rho_{b,c}\rho_{a,c}) \\
-2(\rho_{a,b}\rho_{b,d}\rho_{a,c}\rho_{c,d}) \\
+(\rho_{a,b}^2\rho_{c,d}^2+\rho_{a,c}^2\rho_{b,d}^2) \\
ax^2+bx+c \geq 0 \\
|2ax+b| \leq \sqrt{b^2-4ac}
\end{gathered}
$$

Dividing all sides by 2

$$
\begin{gathered}
a=\rho_{b,c}^2-1 \\
b=(\rho_{a,b}\rho_{b,d}+\rho_{a,c}\rho_{c,d})-(\rho_{a,b}\rho_{b,c}\rho_{c,d}+\rho_{a,c}\rho_{b,d}\rho_{b,c}) \\
c=1-(\rho_{c,d}^2+\rho_{b,c}^2+\rho_{b,d}^2+\rho_{a,b}^2+\rho_{a,c}^2) \\
+2\cdot(\rho_{b,c}\rho_{b,d}\rho_{c,d}+\rho_{a,b}\rho_{b,c}\rho_{a,c}) \\
-2(\rho_{a,b}\rho_{b,d}\rho_{a,c}\rho_{c,d}) \\
+(\rho_{a,b}^2\rho_{c,d}^2+\rho_{a,c}^2\rho_{b,d}^2) \\
|ax+b| \leq \sqrt{b^2-ac}
\end{gathered}
$$

Simplifying the square root

$$
\begin{gathered}
b^2-ac=(a,b)^2(b,d)^2+(a,c)^2(c,d)^2+(b,c)^2(c,d)^2+(b,c)^2(b,d)^2 \\
+(b,c)^2(a,b)^2+(b,c)^2(a,c)^2-(a,b)^2(c,d)^2-(a,c)^2(b,d)^2 \\
+(2*(b,c)^2+(c,d)^2+(b,d)^2+(a,b)^2+(a,c)^2)-1 \\
+4(a,b)(b,c)^2(c,d)(a,c)(b,c)-2((b,c)(b,d)(c,d)+(a,b)(b,c)(a,c))-(b,c)^4 \\
-2((a,b)^2(b,d)(b,c)(c,d)+(a,b)(b,d)^2(b,c)(a,c)+(a,c)(c,d)^2(a,b)(b,c) \\
+(a,c)^2(c,d)(b,c)(b,d))-2((b,c)^3(b,d)(c,d)+(b,c)^3(a,b)(a,c))
\end{gathered}
$$