Continued Fractions

Emil Kerimov

1 History

Continued fractions were often times used off-handedly in the last 2000 years, but the foundations of continued fractions were not established until late 1600's, early 1700's by John Wallis, see a fuller account in the Mathematics Archives [?]. Below is an except from the Mathematics Archives [?].

> In his book Opera Mathematica (1695) Wallis laid some basic groundwork for continued fractions. He explained how to compute the nth convergent and discovered some now familiar properties of convergents. It was also in this work that the term "continued fraction" was first used.
>
> The Dutch mathematician and astronomer Christiaan Huygens (1629-1695) was the first to demonstrate a practical application of continued fractions. He wrote a paper explaining how to use the convergents of a continued fraction to find the best rational approximations for gear ratios. These approximations enabled him to pick the gears with the correct number of teeth. His work was motivated impart by his desire to build a mechanical planetarium.
>
> While the work of Wallis and Huygens began the work on continued fractions, the field of continued fractions began to flourish when Leonard Euler (1707-1783), Johan Heinrich Lambert (1728-1777), and Joseph Louis Lagrange (1736-1813) embraced the topic. Euler laid down much of the modern theory in his work De Fractionlous Continious (1737). He showed that every rational can be expressed as a terminating simple continued fraction. He also provided an expression for e in continued fraction form. He used this expression to show that e and e^2 are irrational. He also demonstrated how to go from a series to a continued fraction representation of the series, and conversely.
>
> Lambert generalized Euler's work on e to show that both e^x and $\tan(x)$ are irrational if x is rational. Lagrange used continued fractions to find the value of irrational roots. He also proved that a real root of a quadratic irrational is a periodic continued fraction.
>
> The nineteenth century can probably be described as the golden age of continued fractions. As Claude Brezinski writes in History of Continued Fractions and Padre Approximations, "the nineteenth

century can be said to be popular period for continued fractions." It was a time in which "the subject was known to every mathematician." As a result, there was an explosion of growth within this field. The theory concerning continued fractions was significantly developed, especially that concerning the convergents. Also studied were continued fractions with complex variables as terms. Some of the more prominent mathematicians to make contributions to this field include Jacobi, Perron, Hermite, Gauss, Cauchy, and Stieljes. By the beginning of the 20th century, the discipline had greatly advanced from the initial work of Wallis.

2 Definition

Definition 2.1 (Finite General Continued Fraction)**.** *A* ***finite general*** *continued fraction is defined as:*

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots + \cfrac{b_n}{a_n}}}}$$

If we allow the number of levels to go to infinity we get a general continued fraction.

Definition 2.2 (General Continued Fraction)**.** *A* ***general*** *continued fraction is defined as:*

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}}$$

Definition 2.3 (Finite Standard Continued Fraction)**.** *A* ***finite standard*** *continued fraction is defined as:*

$$[a_0; a_1, a_2, \cdots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

If we allow the number of levels to go to infinity we get a standard continued fraction.

Definition 2.4 (Standard Continued Fraction)**.** *A* ***standard*** *continued fraction is defined as:*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

This form is also referred to as the simple continued fraction or regular continued fraction, or the canonical form.

2.1 Rational Examples

Every rational number has a unique representation in the standard continued fraction form. Moreover, every rational number can be represented using only the finite continued fraction form. Below are a few examples.

$$\frac{1}{8} = [0; 8] = \mathbf{0} + \frac{1}{\mathbf{8}}$$

$$\frac{2}{5} = [0; 2, 2] = \mathbf{0} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2}}}$$

$$\frac{163}{7} = [23; 3, 2] = \mathbf{23} + \cfrac{1}{\mathbf{3} + \cfrac{1}{\mathbf{2}}}$$

$$\frac{-13}{5} = [-3; 2, 2] = \textbf{-3} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2}}}$$

2.2 Generation of continued fractions

The simple continued fraction representation of a number is both easy to verify and easy to generate. Here we go through two common ways of generating the simple continued fraction representation of a number.

2.2.1 Rational numbers

Given a rational number p/q we can use Euclid's algorithm for determining the greatest common divisor (GCD) between the numbers p and q.
Using the example of $\frac{163}{7}$ as listed above we get the following steps:

See how many times 7 goes into 163 evenly

$$163 = \mathbf{23} \cdot 7 + \mathbf{2}$$

See how many times 2 goes into 7 evenly

$$7 = \mathbf{3} \cdot 2 + \mathbf{1}$$

See how many times 1 goes into 2 evenly

$$2 = \mathbf{2} \cdot 1 + \mathbf{0}$$

Notice that the resulting divisor values exactly match the corresponding simple continued fraction representation of $\frac{163}{7}$ as shown above. In particular

$$\frac{163}{7} = [23; 3, 2] = \mathbf{23} + \cfrac{1}{\mathbf{3} + \cfrac{1}{\mathbf{2}}}$$

2.2.2 General real numbers

We can apply the same reasoning as for the rational numbers to general real numbers.
Using $\sqrt{2} = 1.4142135623730951\ldots$ as an example.

Remove the largest integer from value

$$\sqrt{2} = \mathbf{1} + 0.4142135623730951\ldots$$

Take the reciprocal of the result and remove the largest integer from that

$$1/0.4142135623730951\cdots = 2.414213562373095 = \mathbf{2} + .414213562373095$$

Repeat the process

$$1/0.414213562373095\cdots = 2.41421356237309 = \mathbf{2} + .414213562373095$$

These calculations seem to imply that

$$\sqrt{2} = [1; 2, 2, 2, \ldots] = \mathbf{1} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \ddots}}}$$

We will see in a later section that this is the exact continued fraction representation of $\sqrt{2}$.

Moreover, we will show that in general if $\sqrt{n}$ is not an integer then it will always have a periodic repeating infinite continued fraction representation.

The algorithm presented here will work for every number as long as a decimal representation is available for said number.

3 Useful lemmas

Definition 3.1 (Convergents). *A convergent* $\frac{p_n}{q_n}$ *of a simple continued fraction* $[a_0; a_1, a_2, a_3, \ldots]$ *is defined as:*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n]$$

For reasons that will become apparent later we define $p_{-1} = 1, p_{-2} = 0$ *and* $q_{-1} = 0, q_{-2} = 1$.

Lemma 3.2 (Recurrence Relation). *For* $\frac{p_n}{q_n}$ *the* n^{th} *convergent of* $[a_0; a_1, a_2, a_3, \ldots]$.

We can find p_n, q_n *using the previous convergents through the following recurrence relation:*

$$\frac{p_n}{q_n} = \frac{a_n \cdot p_{n-1} + p_{n-2}}{a_n \cdot q_{n-1} + q_{n-2}}$$

Proof.

$$\frac{p_0}{q_0} = a_0 = \frac{a_0 \cdot 1 + 0}{a_0 \cdot 0 + 1} = \frac{a_0 \cdot p_{-1} + p_{-2}}{a_0 \cdot q_{-1} + q_{-2}}$$
$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{a_1 \cdot a_0 + 1}{a_1 \cdot 1 + 0}$$
$$\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{a_2}{a_1 a_2 + 1} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} = \frac{a_2 \cdot (a_0 a_1 + 1) + a_0}{a_2 \cdot a_1 + 1}$$

Assume true for k and prove for $k+1$:

$$\frac{p_{k+1}}{q_{k+1}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \cfrac{1}{a_{k+1}}}}}$$
$$\frac{p_{k+1}}{q_{k+1}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{\cfrac{a_k a_{k+1} + 1}{a_{k+1}}}}}$$

Let $a_k^{'} = \frac{a_k a_{k+1} + 1}{a_{k+1}}$ and use the assumption for $\frac{p_k}{q_k}$:

$$\frac{p_{k+1}}{q_{k+1}} = \frac{\frac{a_k a_{k+1} + 1}{a_{k+1}} \cdot p_{k-1} + p_{k-2}}{\frac{a_k a_{k+1} + 1}{a_{k+1}} \cdot q_{k-1} + q_{k-2}}$$
$$\frac{p_{k+1}}{q_{k+1}} = \frac{(a_k a_{k+1} + 1) \cdot p_{k-1} + a_{k+1} p_{k-2}}{(a_k a_{k+1} + 1) \cdot q_{k-1} + a_{k+1} q_{k-2}}$$
$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

Again using the recursive assumption for p_k and q_k we finish the inductive proof:

$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}$$

□

Lemma 3.3 (Determinant Identity)**.** *For* $\frac{p_n}{q_n}$*, the* n^{th} *convergent of* $[a_0; a_1, a_2, a_3, \ldots]$*:*

$$p_{n-1} \cdot q_n - p_n \cdot q_{n-1} = (-1)^n$$

Proof. We proceed by verifying the base cases and then using mathematical induction on n.

Base Cases:

$$n = 0: \quad p_{-1} \cdot q_0 - p_0 \cdot q_{-1} = 1 \cdot 1 - a_0 \cdot 0 = 1 = (-1)^0,$$
$$n = 1: \quad p_0 \cdot q_1 - p_1 \cdot q_0 = a_0 \cdot a_1 - (a_0 a_1 + 1) \cdot 1 = -1 = (-1)^1,$$

Induction Hypothesis: Assume that for k, the statement holds:

$$p_{k-1} \cdot q_k - p_k \cdot q_{k-1} = (-1)^k.$$

Inductive Step: We need to prove the statement for $k+1$:

$$\begin{aligned}
p_k \cdot q_{k+1} - p_{k+1} \cdot q_k &= p_k \cdot (a_{k+1}q_k + q_{k-1}) - (a_{k+1}p_k + p_{k-1}) \cdot q_k \\
&= (p_k \cdot q_{k-1} - p_{k-1} \cdot q_k) \\
&= -(p_{k-1} \cdot q_k - p_k \cdot q_{k-1}) = -(-1)^k = (-1)^{k+1}.
\end{aligned}$$

Therefore, $p_k \cdot q_{k+1} - p_{k+1} \cdot q_k = (-1)^{k+1}$, completing the induction step. □

Lemma 3.4 (Difference Identity)**.** *For* $\frac{p_n}{q_n}$ *the* n^{th} *convergent of* $[a_0; a_1, a_2, a_3, \dots]$.
We can define a closed form for the difference of two sequential convergents:

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n \cdot q_{n-1}}$$

Proof.

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1} \cdot q_n - p_n \cdot q_{n-1}}{q_n q_{n-1}}$$

Using Lemma **??** we have

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}$$

□

4 Relation to alternating Series

Theorem 4.1 (Series to Continued Fraction)**.**

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 \cfrac{a_3^2}{\cdots + \cfrac{a_{n-1}^2}{a_n - a_{n-1}}}}}}$$

Proof. **Proof by induction**
Base Case n=1:

$$\sum_{k=1}^{1} \frac{(-1)^{k-1}}{a_k} = \frac{1}{a_1} = \frac{1}{a_1}$$

Base Case n=2:

$$\begin{aligned}\sum_{k=1}^{2}\frac{(-1)^{k-1}}{a_k} &= \frac{1}{a_1}-\frac{1}{a_2}\\ &= \frac{a_2-a_1}{a_1\cdot a_2}\\ &= \frac{1}{\frac{a_1\cdot a_2}{a_2-a_1}}\\ &= \cfrac{1}{a_1+\cfrac{a_1^2}{a_2-a_1}}\end{aligned}$$

Inductive Hypothesis: Assume the statement is true for n:

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1+\cfrac{a_1^2}{a_2-a_1+\cfrac{a_2^2}{a_3-a_2\cfrac{a_3^2}{\ddots+\cfrac{a_{n-1}^2}{a_n-a_{n-1}}}}}}$$

Inductive Step: Prove for $n+1$.

The idea of the proof is to re-write the summation for $n+1$ as a summation over n terms by obtaining a new $\widehat{a_n}$ term.

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{a_k} &= \frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\cdots+\frac{(-1)^{n-1}}{a_n}+\frac{(-1)^n}{a_{n+1}}\\ &= \frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\cdots+(-1)^{n-1}\cdot\left(\frac{1}{a_n}-\frac{1}{a_{n+1}}\right)\\ &= \frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\cdots+(-1)^{n-1}\cdot\left(\frac{a_{n+1}-a_n}{a_n\cdot a_{n+1}}\right)\\ &= \frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\cdots+(-1)^{n-1}\cdot\left(\frac{1}{\frac{a_n\cdot a_{n+1}}{a_{n+1}-a_n}}\right)\\ &= \frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\cdots+(-1)^{n-1}\frac{1}{\widehat{a_n}}\end{aligned}$$

Note that

$$\widehat{a_n} = \frac{a_n\cdot a_{n+1}}{a_{n+1}-a_n} = \frac{(a_{n+1}-a_n)\cdot a_n + a_n^2}{a_{n+1}-a_n} = a_n + \frac{a_n^2}{a_{n+1}-a_n}$$

Hence, plugging in this new $\widehat{a_n}$ into our inductive hypothesis we get:

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cfrac{a_3^2}{\ddots + \cfrac{a_{n-1}^2}{\widehat{a_n} - a_{n-1}}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cfrac{a_3^2}{\ddots + \cfrac{a_{n-1}^2}{\left(a_n + \frac{a_n^2}{a_{n+1} - a_n}\right) - a_{n-1}}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cfrac{a_3^2}{\ddots + \cfrac{a_{n-1}^2}{a_n - a_{n-1} + \frac{a_n^2}{a_{n+1} - a_n}}}}}}$$

Which proves the theorem for $n+1$ since this matches the form we were expecting for $n+1$. $\square$

Theorem 4.2 (Euler's Continued Fraction Formula)**.** *https://en.wikipedia.org/wiki/Euler%27s_continued_fraction_formula*

$$\sum_{k=0}^{n} a_0 a_1 \cdots a_k = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{\ddots + 1 + a_{n-1} - \cfrac{a_n}{1 + a_n}}}}}$$

Proof. We proceed by induction on n.

Base case $(n = 0)$**.**

$$\sum_{k=0}^{0} a_0 a_1 \cdots a_k = a_0, \qquad \frac{a_0}{1} = a_0.$$

Base case $(n = 1)$**.**

$$\text{LHS} = a_0 + a_0 a_1 = a_0(1 + a_1),$$

$$\text{RHS} = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1}} = \frac{a_0(1 + a_1)}{1 + a_1 - a_1} = a_0(1 + a_1).$$

Inductive hypothesis. Assume for some $k \geq 1$ that

$$\sum_{j=0}^{k} a_0 a_1 \cdots a_j = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{\ddots - \cfrac{a_k}{1 + a_k}}}}.$$

Inductive step. We show the formula holds for $k+1$.
Let

$$S_k := \sum_{j=0}^{k} a_0 a_1 \cdots a_j.$$

Then

$$S_{k+1} = S_k + a_0 a_1 \cdots a_{k+1} = S_k \left(1 + a_{k+1} \frac{a_0 a_1 \cdots a_k}{S_k}\right).$$

From the inductive hypothesis, the ratio

$$\frac{a_0 a_1 \cdots a_k}{S_k}$$

is exactly the value of the *last convergent* of the continued fraction, namely

$$\frac{a_k}{1 + a_k}.$$

Hence

$$S_{k+1} = S_k \left(1 + \frac{a_k a_{k+1}}{1 + a_k}\right).$$

Now observe that appending one more level to the continued fraction replaces the final denominator

$$1 + a_k \quad \text{by} \quad 1 + a_k - \frac{a_{k+1}}{1 + a_{k+1}}.$$

Indeed,

$$1 + a_k - \frac{a_{k+1}}{1 + a_{k+1}} = \frac{(1 + a_k)(1 + a_{k+1}) - a_{k+1}}{1 + a_{k+1}} = \frac{1 + a_k + a_{k+1} + a_k a_{k+1}}{1 + a_{k+1}}.$$

This produces exactly the multiplicative factor

$$1 + \frac{a_k a_{k+1}}{1 + a_k}$$

in the numerator, while preserving the continued fraction structure above it.
Therefore,

$$S_{k+1} = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{\ddots - \cfrac{a_{k+1}}{1 + a_{k+1}}}}}.$$

This completes the induction. □

Theorem 4.3 (Infinite Product Alternatives)**.**

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} = \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \cfrac{a_3}{\ddots + a_{n-1} - 1 + \cfrac{a_{n-1}}{a_n - 1}}}}}$$

Proof. **Proof by induction**

We proceed by induction on n.

Base Case: For $n = 1$, the left-hand side of the equation is:

$$\frac{(-1)^{1-1}}{a_1} = \frac{1}{a_1}$$

And the right-hand side for $n = 1$ is:

$$\frac{1}{a_1}$$

Thus, the equation holds for $n = 1$.

Base Case: For $n = 2$, the left-hand side of the equation is:

$$\begin{aligned} \frac{1}{a_1} - \frac{1}{a_1 a_2} &= \frac{a_2 - 1}{a_1 a_2} = \frac{1}{\frac{a_1 a_2}{a_2 - 1}} \\ &= \frac{1}{\frac{a_1(a_2-1)+a_1}{a_2-1}} \\ &= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1}} \end{aligned}$$

And the right-hand side for $n = 2$ is:

$$\cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1}}$$

Thus, the equation holds for $n = 2$.

Inductive Step: Assume that the equation holds for some $n = K$, that is:

$$\sum_{k=1}^{K} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} = \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{a_K - 1}}}}}$$

We want to prove that the equation holds for $n = K + 1$, the left-hand side becomes:

$$\begin{aligned}
\sum_{k=1}^{K+1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} &= \sum_{k=1}^{K-1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} + \frac{(-1)^{K-1}}{a_1 a_2 \cdots a_K} - \frac{(-1)^{K-1}}{a_1 a_2 \cdots a_{K+1}} \\
&= \sum_{k=1}^{K-1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} + \frac{(-1)^{K-1}}{a_1 a_2 \cdots a_{K-1}} \cdot \left(\frac{1}{a_K} - \frac{1}{a_K a_{K+1}} \right) \\
&= \sum_{k=1}^{K-1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} + \frac{(-1)^{K-1}}{a_1 a_2 \cdots a_{K-1}} \cdot \left(\frac{a_{K+1} - 1}{a_K a_{K+1}} \right)
\end{aligned}$$

Define $\widehat{a_K} = \frac{a_K a_{K+1}}{a_{K+1}-1}$

$$= \sum_{k=1}^{K} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_{K-1}\widehat{a_K}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{\widehat{a_K} - 1}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{\frac{a_K a_{K+1}}{a_{K+1}-1} - 1}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{\frac{a_K a_{K+1} - a_{K+1} + 1}{a_{K+1}-1}}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{\frac{(a_K-1)\cdot(a_{K+1}-1)+a_K}{a_{K+1}-1}}}}}}$$

$$= \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \ddots + \cfrac{a_{K-2}}{a_{K-1} - 1 + \cfrac{a_{K-1}}{a_K - 1 + \frac{a_K}{a_{K+1}-1}}}}}}$$

$\square$

4.1 Example $\ln(2)$

Lemma 4.4 (Taylor Series for ln)**.**

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot x^k}{k}$$

Proof. This can be shown using the Taylor Series for $\ln(x+1)$ centred around $a = 0$. See Darboux document for a proof of the Taylor Series.

Note that

$$\frac{\partial^n \big(\ln(x+1)\big)}{\partial x^n}|_{x=0} = (-1)^{n+1} \cdot (n-1)!$$

Thus plugging this into Taylor's equation we get:

$$\begin{aligned}
\ln(1+x) &= \sum_{m=0}^{\infty} \frac{(x-a)^m f^{(m)}(a)}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot x^m (m-1)!}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot x^m}{m}
\end{aligned}$$

□

Lemma 4.5 (Continued Fraction for $\ln(2)$)**.**

$$\ln(2) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \ddots}}}}$$

Proof. Plugging in $x = 1$ to the Taylor expansion of $\ln(x+1)$ and using Theorem **??** we obtain the result. □

4.2 Example $\frac{\pi}{4}$

Lemma 4.6 (Taylor Series for arctan)**.**

$$\arctan(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot x^{2k-1}}{2k-1}$$

Proof. This can be shown using the Taylor Series for $\arctan(x)$ centred around $a = 0$. See Darboux document for a proof of the Taylor Series. □

Lemma 4.7 (Continued Fraction for $\pi/4$)**.**

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

Proof. Plugging in $x = 1$ to the Taylor expansion of $\arctan(x) = \arctan(1) = \frac{\pi}{4}$ and using Theorem **??** we obtain the result. $\square$

5 Square root numbers

Theorem 5.1 (Square Root CF)**.** *For any $x > 0$:*

$$\sqrt{x} = 1 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \ddots}}}$$

Proof. Let y denote the value of the continued fraction:

$$y = 1 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \ddots}}}$$

Since the continued fraction is infinite and periodic, we can observe that the denominator starting from the first 2 is equal to $1 + y$. That is:

$$2 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \ddots}} = 1 + y$$

Therefore, we can rewrite y as:

$$y = 1 + \frac{x-1}{1+y}$$

Multiplying both sides by $(1 + y)$:

$$\begin{aligned} y(1+y) &= (1+y) + (x-1) \\ y + y^2 &= 1 + y + x - 1 \\ y^2 &= x \end{aligned}$$

Since y represents the value of a continued fraction with positive terms and $x > 0$, we have $y > 0$. Therefore:

$$y = \sqrt{x}$$

This proves that the continued fraction equals $\sqrt{x}$. $\square$

6 Approximation of rational numbers

todo

7 Relation to Pell's Equation

todo

Theorem 7.1 (Pell's Equation Relation)**.** *If p, q are integers such that $p^2 - n \cdot q^2 = \pm 1$, Then p, q are convergents of $\sqrt{n}$.*

8 Solving linear Diophantine Equations

todo

9 Gauss's continued fraction

todo
https://en.wikipedia.org/wiki/Gauss%27s_continued_fraction

10 General continued fraction examples

todo

10.1 pi

todo

References

[1] *Mathematics Archives.* http://archives.math.utk.edu/articles/atuyl/confrac/history.html Addison-Wesley, Reading, Massachusetts, 1993.