

# Basic Trig Functions

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## 1 Basic Inverse Trig Functions

If we assume we have this as a given.

**Definition 1.1** (Euler's Formula).

$$e^{ix} = \cos(x) + i \cdot \sin(x) \quad (1)$$

**Theorem 1.2** (Sine Definition).

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = -i \frac{e^{ix} - e^{-ix}}{2} \quad (2)$$

*Proof.*

$$\begin{aligned} e^{-ix} &= \cos(-x) + i \cdot \sin(-x) = \cos(x) - i \cdot \sin(x) \\ e^{ix} - e^{-ix} &= 2i \cdot \sin(x) \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} = -i \frac{e^{ix} - e^{-ix}}{2} \end{aligned}$$

□

**Theorem 1.3** (Cosine Definition).

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (3)$$

*Proof.* Same as above.

□

**Theorem 1.4** (Tangent Definition).

$$\tan(x) = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \quad (4)$$

*Proof.* Same as above.

□

**Theorem 1.5** (Cotangent Definition).

$$\cot(x) = i + \frac{2i}{e^{2ix} - 1} \quad (5)$$

*Proof.*

$$\begin{aligned}\tan(x) &= -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \\ \cot(x) &= i \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} \\ \cot(x) &= i \frac{(e^{ix} - e^{-ix}) + (2e^{-ix})}{e^{ix} - e^{-ix}} \\ \cot(x) &= i + 2i \frac{e^{-ix}}{e^{ix} - e^{-ix}} \\ \cot(x) &= i + \frac{2i}{e^{2ix} - 1}\end{aligned}$$

□

If we for now ignore the non-uniqueness of the square root and logarithm of complex numbers, we can obtain

**Theorem 1.6** (Arcsine Function).

$$\arcsin(x) = -i \cdot \ln(ix + \sqrt{1 - x^2}) \quad (6)$$

*Proof.* Let  $y = \arcsin(x)$

$$\begin{aligned}\sin(y) &= \frac{e^{iy} - e^{-iy}}{2i} \\ \sin(\arcsin(x)) &= x = \frac{e^{iy} - e^{-iy}}{2i} \\ 2ix &= e^{iy} - e^{-iy} \\ e^{2iy} - 2ixe^{iy} - 1 &= 0 \\ e^{iy} &= \frac{2ix \pm \sqrt{-4x^2 + 4}}{2} \\ e^{iy} &= ix \pm \sqrt{1 - x^2} \\ iy &= \ln(ix + \sqrt{1 - x^2}) \\ y &= \frac{\ln(ix + \sqrt{1 - x^2})}{i} = -i \cdot \ln(ix + \sqrt{1 - x^2}) \\ \arcsin(x) &= -i \cdot \ln(ix + \sqrt{1 - x^2})\end{aligned}$$

□

**Theorem 1.7** (Arccosine Function).

$$\arccos(x) = -i \cdot \ln(x + \sqrt{x^2 - 1}) \quad (7)$$

*Proof.* Same as above. □

**Theorem 1.8** (Arctangent Function).

$$\arctan(x) = \frac{i}{2} \cdot \ln\left(\frac{i+x}{i-x}\right) \quad (8)$$

*Proof.* Same as above. □

**Theorem 1.9** (Arcsecant Function).

$$\text{arcsec}(x) = -i \ln \left( \frac{1 \pm \sqrt{1-x^2}}{x} \right) \quad (9)$$

*Proof.* Same as above. □

**Theorem 1.10** (Arccosecant Function).

$$\text{arccsc}(x) = -i \ln \left( \frac{i}{x} + \sqrt{1 - \frac{1}{x^2}} \right) \quad (10)$$

*Proof.* Same as above. □

**Theorem 1.11** (Arccotangent Function).

$$\text{arccot}(x) = \frac{i}{2} \cdot \ln \left( \frac{x-i}{x+i} \right) \quad (11)$$

*Proof.* Same as above. □

The same can be applied to hyperbolic functions, using

**Definition 1.12** (Hyperbolic Functions).

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (12)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (13)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \quad (14)$$

For each normal trigonometry identities there is a similar hyperbolic trigonometry identity.

The relation of the two are

**Theorem 1.13** (Relation to Circular Functions).

$\cosh(ix) = \cos(x)$	$\sinh(ix) = i \cdot \sin(x)$
$\tanh(ix) = i \cdot \tan(x)$	$\cos(ix) = \cosh(x)$
$\sin(ix) = i \cdot \sinh(x)$	$\tan(ix) = i \cdot \tanh(x)$