

# Zeta Function

Emil Kerimov

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## 1 Definition

**Definition 1.1** *The Riemann Zeta function is defined as*

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

We have already shown that <sup>1</sup>

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad (1)$$

## 2 Bernoulli Numbers

Examining the Taylor expansion of  $\frac{x}{e^x-1}$ , this will be required for the evaluation of the even integer values of  $\zeta$ .

**Definition 2.1** *The Bernoulli numbers  $B_n$  are defined as the coefficients of the Taylor expansion of the following function*

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n}$$

**Theorem 2.1**

$$\sum_k^n \binom{n+1}{k} B_k = 0 \quad \forall n > 0, \text{ with } B_0 = 1 \quad (2)$$

*Proof*

*Starting from definition 2.1*

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

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<sup>1</sup>See Gamma function document

Using the Taylor expansion of  $e^x$

$$\sum_{k=1}^{\infty} \frac{x^k}{k!} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Rearranging

$$\begin{aligned} x &= \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \\ 1 &= \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x \end{aligned}$$

Adjusting the indicies

$$1 = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Using Cauchy's product formula for infinite sums <sup>2</sup>

$$1 = \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^n \left( \underbrace{\frac{B_k}{k!} x^k}_{a_k} \right) \left( \underbrace{\frac{x^{n-k}}{(n-k+1)!}}_{b_{n-k}} \right)}_{c_n}$$

Simplifying

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^n \binom{n+1}{k} B_k x^k x^{n-k} \\ 1 &= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \sum_{k=0}^n \binom{n+1}{k} B_k \end{aligned}$$

Comparing coefficients of powers of  $x$  of both sides we get

$$\begin{aligned} n = 0, \quad 1 &= B_0 \\ n \neq 0, \quad 0 &= \sum_{k=0}^n \binom{n+1}{k} B_k \end{aligned}$$

## 2.1 First few Bernoulli numbers

Using equation 2 with  $n = 1$  implies  $\binom{2}{0}B_0 + \binom{2}{1}B_1 = 0 = 1 + 2B_1$  which implies

$$B_1 = -\frac{1}{2} \tag{3}$$

Using equation 2 with  $n = 2$  implies  $\binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 = 0 = 1 - \frac{3}{2} + 3B_2$  which implies

$$B_2 = \frac{1}{6} \tag{4}$$

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<sup>2</sup>  $\left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n$  where  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . See proof: TODO ADD

Similarly when we apply this equation for increasing values of  $n$

$$B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \dots \quad (5)$$

**Theorem 2.2** *In fact we can show that all odd Bernoulli numbers after  $n=1$  are 0*

$$B_n = 0 \quad \forall \text{ odd } n > 1 \quad (6)$$

*Proof*

*Starting from definition 2.1*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

*Removing the first two terms of the sum*

$$\frac{x}{e^x - 1} = B_0 + B_1 x + \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n$$

*Rearranging*

$$\begin{aligned} \frac{x}{e^x - 1} - 1 + \frac{x}{2} &= \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n \\ \frac{x + xe^x}{2e^x - 2} - 1 &= \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n \end{aligned}$$

*If we can show that the RHS is an even function then we have established the proof*

$$\begin{aligned} y(x) &= \frac{x + xe^x}{2e^x - 2} - 1 \\ y(-x) &= \frac{-x + -xe^{-x}}{2e^{-x} - 2} - 1 \end{aligned}$$

*Simplifying*

$$\begin{aligned} y(-x) &= \frac{-x - xe^{-x}}{2e^{-x} - 2} \cdot \frac{e^x}{e^x} - 1 \\ y(-x) &= \frac{-xe^x - x}{2 - 2e^x} - 1 \\ y(-x) &= \frac{x + xe^x}{2e^x - 2} - 1 \\ y(-x) &= y(x) \\ B_n &= 0 \quad \forall \text{ odd } n > 1 \end{aligned}$$

### 3 Even integer values $\zeta(2n)$

While the odd integer values of  $\zeta(n)$  do not have a closed form expression, the even values do.

**Theorem 3.1** *We can express all even integer values of  $\zeta$  using Bernoulli numbers*

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2 \cdot (2n)!} \quad (7)$$

*Proof*

*Starting from the series representation <sup>3</sup> of cotangent*

$$\pi x \cot(\pi x) = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k}$$

*Using the complex exponential representation <sup>4</sup> of cotangent*

$$i\pi x + \frac{2i\pi x}{e^{2i\pi x} - 1} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k}$$

*Using the Bernoulli numbers definition 2.1*

$$i\pi x + \sum_{n=0}^{\infty} \frac{B_n}{n!} (2i\pi x)^n = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k}$$

*Rearranging and using the first two values of the Bernoulli numbers*

$$\begin{aligned} i\pi x + B_0 + B_1(2i\pi x) + \sum_{n=2}^{\infty} \frac{B_n}{n!} (2i\pi x)^n &= 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k} \\ i\pi x + 1 + -i\pi x + \sum_{n=2}^{\infty} \frac{B_n}{n!} (2i\pi x)^n &= 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k} \\ \sum_{n=2}^{\infty} \frac{B_n}{n!} (2i\pi)^n x^n &= -2 \sum_{k=1}^{\infty} \zeta(2k) x^{2k} \end{aligned}$$

*Using the fact the the odd Bernoulli numbers after  $B_1$  are 0*

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (-1)^k (2\pi)^{2k} x^{2k} = \sum_{k=1}^{\infty} -2\zeta(2k) x^{2k}$$

*Comparing coefficients*

$$\zeta(2k) = \frac{B_{2k}}{2 \cdot (2k)!} (-1)^{k+1} (2\pi)^{2k} \quad \forall k \in \mathbb{N}_{>0}$$

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<sup>3</sup>See Euler's Sine Product Formula document

<sup>4</sup> $\cot(x) = i + \frac{2i}{e^{2ix} - 1}$ . See Basic trig functions document

### 3.1 The first few even integer values of $\zeta$

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \zeta(10) = \frac{\pi^{10}}{93555}, \\ \zeta(12) &= \frac{691}{638512875}\pi^{12}, \zeta(14) = \frac{2}{18243225}\pi^{14}, \zeta(16) = \frac{3617}{325641566250}\pi^{16}, \\ \zeta(18) &= \frac{43867}{38979295480125}\pi^{18}, \zeta(20) = \frac{174611}{1531329465290625}\pi^{20}, \zeta(22) = \frac{155366}{13447856940643125}\pi^{22}, \\ \zeta(24) &= \frac{236364091}{201919571963756521875}\pi^{24}, \dots\end{aligned}$$

## 4 Related functions

### 4.1 Dirichlet Eta Function

**Definition 4.1** *The Dirichlet Eta function is defined as*

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \dots$$

**Theorem 4.1**

$$\boxed{\eta(z) = (1 - 2^{1-s})\zeta(z)} \quad (8)$$

*Proof*

*Starting from the definition of  $\eta(z)$*

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = \sum_{k=1}^{\infty} \frac{1}{k^z} - 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^z}$$

*Using the definition of  $\zeta(z)$*

$$\eta(z) = \zeta(z) - \frac{2}{2^z} \zeta(z)$$

$$\eta(z) = (1 - 2^{1-z})\zeta(z)$$

#### 4.1.1 Values

$$\eta(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2) \quad ^5 \quad (9)$$

$$\eta(2) = (1 - 2^{1-2}) \cdot \zeta(2) = \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12} \quad (10)$$

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<sup>5</sup>This can be seen by taking the Taylor-Expansion of  $\ln(x+1)$  and evaluating at  $x=1$

## 4.2 Dirichlet Lambda Function

**Definition 4.2** *The Dirichlet Lambda function is defined as*

$$\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^z} = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \dots$$

**Theorem 4.2**

$$\boxed{\lambda(z) = (1 - 2^{-s})\zeta(z)} \quad (11)$$

*Proof*

*Starting from the definition of  $\lambda(z)$*

$$\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^z} = \sum_{k=1}^{\infty} \frac{1}{k^z} - \sum_{k=1}^{\infty} \frac{1}{(2k)^z}$$

*Using the definition of  $\zeta(z)$*

$$\lambda(z) = \zeta(z) - \frac{1}{2^z}\zeta(z)$$

$$\lambda(z) = (1 - 2^{-z})\zeta(z)$$

### 4.2.1 Values

$$\lambda(2) = (1 - 2^{-2}) \cdot \zeta(2) = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8} \quad (12)$$

## 5 Integral Forms

### 5.1 Gamma with Zeta functions

**Theorem 5.1** *See the Gamma function document for the definition and properties of the Gamma function.*

$$\boxed{\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du} \quad (13)$$

*Proof*

Using  $\Gamma(z) = n^z \int_0^\infty e^{-nu} u^{z-1} du$  from the Gamma Function document and rearranging

$$\Gamma(z) \frac{1}{n^z} = \int_0^\infty e^{-nu} u^{z-1} du$$

Let  $n$  be all integers and take the sum of all equalities

$$\sum_{n=1}^\infty \Gamma(z) \frac{1}{n^z} = \Gamma(z)\zeta(z) = \sum_{n=1}^\infty \int_0^\infty e^{-nu} u^{z-1} du$$

Bringing the summation into the integral since this integral converges uniformly

$$\Gamma(z)\zeta(z) = \int_0^\infty \left( \sum_{n=1}^\infty e^{-nu} \right) u^{z-1} du$$

Notice that  $\sum_{n=1}^\infty e^{-nu}$  is a geometric series if  $u > 0$

$$\Gamma(z)\zeta(z) = \int_0^\infty \left( \frac{e^{-u}}{1 - e^{-u}} \right) u^{z-1} du$$

Simplifying by multiplying top and bottom by  $e^u$

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du$$



## 6 Jacobi Theta Function

### 6.1 Definition

**Definition 6.1** The Jacobi Theta function is defined as

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

**Theorem 6.1** The Jacobi Theta function has the following functional form

$$\Theta(x) = \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right) \quad (14)$$

*Proof*

Using the Poisson Summation Formula <sup>6</sup>

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x - 2\pi i k y} dy$$

Re-writting the exponent

$$-\pi y^2 x - 2\pi i k y = -\pi x \left(y^2 - 2\pi i \frac{k}{x} y\right) = -\pi x \left(y + i \frac{k}{x}\right)^2 - \pi \frac{k^2}{x}$$

Plugging in the factored exponent, we get

$$\Theta(x) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x \left(y + i \frac{k}{x}\right)^2 - \pi \frac{k^2}{x}} dy = \sum_{k=-\infty}^{\infty} e^{-\pi \frac{k^2}{x}} \int_{-\infty}^{\infty} e^{-\pi x \left(y + i \frac{k}{x}\right)^2} dy$$

After a change of variables  $z = y + i \frac{k}{x}$  we can use the Guassian integral identity as proved before, refer to the Guassian integral document,  $\int_{-\infty}^{\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}}$ , we get:

$$\Theta(x) = \sum_{k=-\infty}^{\infty} e^{-\pi \frac{k^2}{x}} \sqrt{\frac{\pi}{x\pi}} = \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right)$$

## 7 Functional Form

### 7.1 First Form

**Theorem 7.1** The symmetric functional equation of  $\zeta$

$$\boxed{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)} \quad (15)$$

*Proof*

TODO

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<sup>6</sup>  $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$ . See proof: TODO ADD

## 7.2 Second Form

**Theorem 7.2**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (16)$$

*Proof*

*TODO*

## 8 Riemann Hypothesis Statement