Technique Recursive sum of powers

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1 Technique Recursive sum of powers

To solve for $\sum_{i=1}^{n} i^p$ consider $\sum_{i=1}^{n} (1+i)^{p+1} - i^{p+1} = (1+n)^{p+1} - 1$

1.1 Sum of Squares

Let p=2

$$\sum_{i=1}^{n}(1+i)^3 - i^3 = \sum_{i=1}^{n} 1 + 3i + 3i^2 + i^3 - i^3 = \sum_{i=1}^{n} 1 + 3i + 3i^2$$

and

$$\sum_{i=1}^{n}(1+i)^3 - i^3 = (n+1)^3 - 1$$

Hence

$$\sum_{i=1}^{n} 1 + 3\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} i^2 = (n+1)^3 - 1$$

$$n + 3\frac{n(n-1)}{2} + 3\sum_{i=1}^{n} i^2 = (n+1)^3 - 1$$

$$3\sum_{i=1}^{n} i^2 = n^3 + 3n^2 + 3n + 1 - n - 3\frac{n(n-1)}{2}$$

$$3\sum_{i=1}^{n} i^2 = n(n+1)(n+\frac{1}{2})$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n}{3}\cdot(n+1)\cdot(n+\frac{1}{2})}$$

1.2 Sum of Cubes

Let p=3

$$\sum_{i=1}^{n}(1+i)^4 - i^4 = \sum_{i=1}^{n} 1 + 4i + 6i^2 + 4i^3 + i^4 - i^4 = \sum_{i=1}^{n} 1 + 4i + 6i^2 + 4i^3$$

and

$$\sum_{i=1}^{n}(1+i)^4 - i^4 = (n+1)^4 - 1$$

Hence

$$\sum_{i=1}^{n} 1 + 4\sum_{i=1}^{n} i + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i^3 = (n+1)^4 - 1$$

$$n + 4\frac{n(n-1)}{2} + 6\frac{n}{3} \cdot (n+1) \cdot (n + \frac{1}{2}) + 4\sum_{i=1}^{n} i^3 = (n+1)^4 - 1$$

$$4\sum_{i=1}^{n} i^3 = n^4 + 4n^3 + 6n^2 + 4n - n - 2n(n-1) - n \cdot (n+1) \cdot (2n+1)$$

Simplifying

$$4\sum_{i=1}^{n} i^3 = n^4 + 2n^3 + n^2$$

$$4\sum_{i=1}^{n} i^3 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$$

$$\boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$$