1 PascalRowSum

Prove that:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n}=2^n$$

Draw Pascal's Triangle.

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

Notice, the sum of the nth row seems to be 2^n.

1.0.1 Sidenote 1

Prove that

$$\binom{p}{0}=\binom{p+1}{0}$$

Proof:

$$\binom{p}{0}=\frac{p!}{0!\cdot(p-0)!}=\frac{p!}{p!}=1=\frac{(p+1)!}{0!\cdot(p+1-0)!}=\binom{p+1}{0}$$

1.0.2 Sidenote 2

Prove that

$$\binom{p}{p}=\binom{p+1}{p+1}$$

Proof:

$$\binom{p}{p}=\frac{p!}{0!\cdot(p-0)!}=\frac{p!}{p!}=1=\frac{(p+1)!}{0!\cdot(p+1-0)!}=\binom{p+1}{p+1}$$

1.0.3 Sidenote 3

Prove that

$$\binom{p}{q}+\binom{p}{q+1}=\binom{p+1}{q+1}$$

Proof:

$$\binom{p}{q}+\binom{p}{q+1}=\frac{p!}{q!\cdot(p-q)!}+\frac{p!}{(q+1)!\cdot(p-q-1)!}$$

$$=\frac{p!\cdot(q+1)}{(q+1)!\cdot(p-q)!}+\frac{p!\cdot(p-q)}{(q+1)!\cdot(p-q)!}$$

$$= \frac{p! \cdot (q+1+p-q)}{(q+1)! \cdot (p-q)!} = \frac{p! \cdot (p+1)}{(q+1)! \cdot (p-q)!}$$
$$= \frac{(p+1)!}{(q+1)! \cdot (p-q)!} = \binom{p+1}{q+1}$$

Using induction:
for n=0:

$$\binom{0}{0} = \frac{0!}{0! \cdot 0!} = 1 = 2^0$$

Assume n=k works:

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k} = 2^k$$

Prove for n=k+1

$$\binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + ... + \binom{k+1}{k-1} + \binom{k+1}{k} + \binom{k+1}{k+1}$$

Using sidenote 1, replace $\binom{k+1}{0}$ with $\binom{k}{0}$

Using sidenote 2, replace $\binom{k+1}{k+1}$ with $\binom{k}{k}$

Using sidenote 2, replace $\binom{k+1}{q+1}$ with $\binom{k}{q} + \binom{k}{q+1}$

$$= \binom{k}{0}+\binom{k}{0}+\binom{k}{1}+\binom{k}{1}+\binom{k}{2}+...+\binom{k}{k-2}+\binom{k}{k-1}+\binom{k}{k-1}+\binom{k}{k}+\binom{k}{k}$$

$$= 2 \cdot (\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k}) = 2 \cdot (2^k) = 2^{k+1}$$

Therefore, proven by induction.