1 Properties of Fourier Series

A function $f(x)$ can be written as the series shown in eq. (1.0.1), known as a Fourier Series. The following provides derivation of the properties of Fourier Series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) \tag{1.0.1}$$

1.0.1 Side Note: Trigonometric Identities of addition

Recall equations eqs. (1.0.2) to (1.0.5)

$$cos(a+b) = cos(a)cos(b) - sin(a)sin(b) \tag{1.0.2}$$
$$cos(a-b) = cos(a)cos(b) + sin(a)sin(b) \tag{1.0.3}$$
$$sin(a+b) = sin(a)cos(b) + sin(b)cos(a) \tag{1.0.4}$$
$$sin(a-b) = sin(a)cos(b) - sin(b)cos(b) \tag{1.0.5}$$

From equations eqs. (1.0.2) to (1.0.5) we obtain eqs. (1.0.6) to (1.0.8)

$$cos(a)cos(b) = \frac{1}{2}(cos(a+b) + cos(a-b)) \tag{1.0.6}$$
$$sin(a)cos(b) = \frac{1}{2}(sin(a+b) + sin(a-b)) \tag{1.0.7}$$
$$sin(a)sin(b) = \frac{1}{2}(cos(a-b) - cos(a+b)) \tag{1.0.8}$$

1.0.2 Integrals

Since $sin(nx)$ is an odd function, its integral over a symmetric region is zero.

$$\int_{-\pi}^{\pi} sin(nx)dx = 0 \tag{1.0.9}$$

Since n is an integer, and assuming $n > 0$, we obtain the following.

$$\int_{-\pi}^{\pi} cos(nx)dx = \frac{2sin(\pi n)}{n} = 0 \tag{1.0.10}$$

Since $sin(mx)cos(nx)$ is an odd function, its integral over a symmetric region is zero.

$$\int_{-\pi}^{\pi} sin(mx)cos(nx)dx = 0 \tag{1.0.11}$$

Use eq. (1.0.6) to write the following.

$$\int_{-\pi}^{\pi} cos(mx)cos(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} cos((n+m)x)dx + \frac{1}{2}\int_{-\pi}^{\pi} cos((n-m)x)dx$$
$$= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \tag{1.0.12}$$

Using eq. (1.0.8) we obtain the following.

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} cos((n-m)x)dx - \frac{1}{2}\int_{-\pi}^{\pi} cos((n+m)x)dx$$
$$= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \tag{1.0.13}$$

1.0.3 a_0 coefficient

Integrate both sides of eq. (1.0.1) to obtain:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cos(nx)dx + b_n \int_{-\pi}^{\pi} sin(nx)dx)$$

Using eqs. (1.0.9) and (1.0.10) the sum terms become zero.

$$= \int_{-\pi}^{\pi} a_0 + 0 + 0 = a_0 x|_{-\pi}^{\pi} = a_0\pi + a_0\pi = 2\pi a_0$$

And thus we obtain

$$a_0 = \frac{\int_{-\pi}^{\pi} f(x)dx}{2\pi} \tag{1.0.14}$$

1.0.4 a_n coefficients

Multiply both sides of eq. (1.0.1) by $cos(mx)$ to obtain:

$$f(x)cos(mx) = a_0 cos(mx) + \sum_{n=1}^{\infty} a_n cos(nx)cos(mx) + b_n sin(nx)cos(mx)$$

Integrate both sides.

$$\int_{-\pi}^{\pi} f(x)cos(mx)dx = a_0\int_{-\pi}^{\pi} cos(mx)dx + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cos(nx)cos(mx)dx + b_n \int_{-\pi}^{\pi} sin(nx)cos(mx)dx)$$

$\rightarrow \int_{-\pi}^{\pi} cos(mx)dx = 0$ (from eq. (1.0.10))

$\rightarrow \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \int_{-\pi}^{\pi} a_m cos(mx)cos(mx) + \sum_{n=1, n\neq m}^{\infty} \int_{-\pi}^{\pi} cos(nx)cos(mx),$

which (from eq. (1.0.12)) is equal to $a_m\pi + \sum\limits_{n=1,n\neq m}^{\infty} 0 = a_m\pi$.
$\rightarrow \int\limits_{-\pi}^{\pi} sin(nx)cos(mx)dx = 0$ (from eq. (1.0.13)). Thus we obtain:

$$\int\limits_{-\pi}^{\pi} f(x)cos(mx)dx = 0 + a_m\pi + 0$$

And thus:

$$a_m = \frac{\int\limits_{-\pi}^{\pi} f(x)cos(mx)dx}{\pi}$$

Which, by changing the index from m to n gives:

$$a_n = \frac{\int\limits_{-\pi}^{\pi} f(x)cos(nx)dx}{\pi} \tag{1.0.15}$$

1.0.5 b_n coefficients

Using similar arguments to section 1.0.4 we obtain:

$$b_n = \frac{\int\limits_{-\pi}^{\pi} f(x)sin(nx)dx}{\pi} \tag{1.0.16}$$