A Simple Gamble

Would you like to pay \$200 to play the following game?

A coin is flipped

- If it is a head, you receive \$300.
- If it is a tail, you receive \$100.
- The coin is even, so it is an actuarially fair game.

A Simple Gamble

之前：付钱就确定有东西.
现在：有不确定性.

Would you like to pay \$200 to play the following game?

A coin is flipped

- If it is a head, you receive \$300.
- If it is a tail, you receive \$100.
- The coin is even, so it is an actuarially fair game.

St. Petersburg Paradox

$\frac{1}{2}\times 2^1 + \frac{1}{2}\times\frac{1}{2}\times 2^2 + \cdots$

- Flip a coin until a head shows up. If it is the nth toss, you win $\$2^n$.
- How much would you like to pay to play the game?

$EV = +\infty$

但是为什么不愿意付大价钱.

pay 1,000

$2^{10} = 1024$ 才可以回本

$1 - [\frac{1}{2} + \cdots + (\frac{1}{2})^9] = \frac{1}{512}$

正好等于这个.

St. Petersburg Paradox

- Flip a coin until a head shows up. If it is the nth toss, you win $\$2^n$.
- How much would you like to pay to play the game?
- What is the expected value of the game?
- Paradox: people only want to pay $XX (on average) to play a game with unlimited amount of money as expected payoff! Why?
  - An uncertain prospect of $1,000 is different from $1,000 in hand
  - What is the most likely outcome of the game?
  - Suppose somebody has paid $1,000 to play the game, what is the probability of winning it back?

$$2^9 = 512;\ 2^{10} = 1024$$

Need at least 9 tails in a row. The chance is about 1 out of 500!

Key Message

When there is uncertainty, need to evaluate

1. the outcomes
2. **feelings about uncertainty in outcomes**

Road-map for Decisions under Uncertainty

做决策⇒概率变化.

- Some basic concepts in probability theory
- Lottery (consumption bundle) 彩票、赌局
- Preference over lotteries and expected utility function (EU)
- Attitudes toward risk (risk preference) under the EU framework
- Market for risk - example of insurance

ranking

$\sum_{1}^{\infty} 2^n (\frac{1}{2})^n \Rightarrow \sum_{1}^{\infty} u(2^n)(\frac{1}{2})^n$ 计算效用.

不同人的风险厌恶情况?

Concepts in Probability Theory (1)

Probability distribution

- Finite number of possible states of nature: 1,2...,S, each with probability p_s. A valid probability distribution satisfies

$$(1)\ p_s \geq 0 \text{ for } \forall s; \quad (2)\ \sum_{s=1}^{S} p_s = 1$$

- Continuum of states: probability density function $f(x)$. A valid distribution satisfies

$$(1)\ f(x) \geq 0 \text{ for } \forall x; \quad (2)\ \int_{-\infty}^{+\infty} f(x) d_x = 1$$

The related "cumulative density function" is $F(x) = \int_{-\infty}^{x} f(z) d_z$.

Concepts in Probability Theory (2)

- Expectation (or expected value) of random variable x

$$E(x) = \sum_{s=1}^{S} p_s x_s$$

$$E(x) = \int_{-\infty}^{+\infty} z\, f(z) d_z$$

- Variance (or dispersion) and risk

$$Var(x) = \sum_{s=1}^{S} p_s \left[x_s - E(x)\right]^2$$

$$Var(x) = \int_{-\infty}^{+\infty} \left[z - E(x)\right]^2 f(z) d_z$$

Concepts in Probability Theory (3)

Law of large numbers:

The probability that the mean of an *i.i.d.* sample is close to the population mean can be made as high as wanted by taking a large enough sample.

$n \to \infty$ 样本的和总体的

Lottery (1)

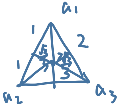
a_1 $p_1=1$ $p_2=0$ | $p_1=0.5$ $p_2=0.5$ | a_2 $p_1=0$ $p_2=1$

离谁近就概率大.

同理

a_1 a_2 a_3

A probability distribution over the space of potential outcomes.



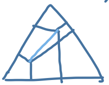
- Set of outcomes $A = \{a_1, ..., a_N\}$
  - a finite set; can be extended to contain infinite outcomes
- A simple lottery on A is

固定a，比较偏好就是比较p

用到边的距离表示.

凸集.

$$L = (p_1 \circ a_1, ..., p_N \circ a_N) \text{ with } \sum_{n=1}^{N} p_n = 1$$



- The set of (simple) lotteries over the set of outcomes A

$\alpha \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} + (1-\alpha) \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$

$$\Delta(A) = \left\{ (p_1, ..., p_N) : p_n \geq 0 \text{ and } \sum_{n=1}^{N} p_n = 1 \right\}$$

  - $\Delta(A)$ is convex, i.e, contains all possible linear combinations of its elements.
  - $\Delta(A)$ can be represented by a simplex.

Lottery (2)

consumer 只关心最终结果.

Compound lottery: a lottery with lotteries as outcomes

- K simple lotteries

k simple.

$$L^k = (p_1^k \circ a_1, ..., p_N^k \circ a_N)$$

- A compound lottery with the K simple lotteries as the outcome set

混合

outcome 是不确定的.

$$L = (\alpha^1 \circ L^1, ..., \alpha^K \circ L^K) \text{ with } \sum_{k=1}^{K} \alpha^k = 1$$

- The corresponding reduced simple lottery

不同的L结合起来.

$$L' = (p_1 \circ a_1, ..., p_N \circ a_N) \text{ where } p_n = \sum_{k=1}^{K} \alpha^k p_n^k$$

Reduction (to simple gamble) axiom of preference

Consequentialists only care about the probability distribution of the final outcomes and it does not matter whether the distribution comes about as a simple lottery or a compound lottery.

Rational and Continuous

偏好

2个组合可比.　　$a>b, b>c \Rightarrow a>c$

- **Axiom 1 & 2**: completeness and transitivity
- **Axiom 3**: continuity (in probabilities)　　还是不突变
  - **Intuition** - if L is preferred to L', a small enough deviation from either of the two lotteries does not change the ranking.
  - The preference is continuous if for any three lotteries

$$L^1,\ L^2,\ L^3 \in \Delta(A) \text{ with } L^1 \succ L^2 \succ L^3$$

there exits $\overline{\alpha} \in (0,1)$ such that

$$\text{for } \forall \alpha < \overline{\alpha} \text{ we have } \alpha L^3 + (1-\alpha)L^1 \succ L^2$$

  - The following two sets are open

$$\{\alpha \in [0,1] : \alpha L^3 + (1-\alpha)L^1 \succ L^2\}$$

$$\{\alpha \in [0,1] : L^2 \succ \alpha L^1 + (1-\alpha)L^3\}$$

  - The following two sets are closed

$$\{\alpha \in [0,1] : \alpha L^3 + (1-\alpha)L^1 \succsim L^2\}$$

$$\{\alpha \in [0,1] : L^2 \succsim \alpha L^1 + (1-\alpha)L^3\}$$

Monotonicity

$L^B = (\beta \circ L, (1-\beta)\circ L')$ $L^A = (\alpha \circ L, (1-\alpha)\circ L')$

$L \succsim L'$ $\alpha \geq \beta$ A比B好

- Monotonicity (in probability): if $a_1 \succsim a_N$, for any $\beta,\ \gamma \in [0,1]$

好结果概率高的赌局更好

$$(\beta \circ a_1, (1-\beta)\circ a_N) \succsim (\gamma \circ a_1, (1-\gamma)\circ a_N) \Leftrightarrow \overset{\beta}{\cancel{\alpha}} \geq \cancel{\beta}\ \gamma$$

- Monotonicity also means that if $L \succsim L'$, then for any $\beta,\ \gamma \in [0,1]$

$$(\beta \circ L, (1-\beta)\circ L') \succsim (\gamma \circ L, (1-\gamma)\circ L') \Leftrightarrow \beta \geq \gamma$$

- Monotonicity and continuity together means there exists α such that

$$L^2 \sim \alpha L^3 + (1-\alpha)L^1$$

  - Rank the outcomes such that $z_1 \succsim z_2 \succsim \ldots \succsim z_N$. Then for $\forall L \in \Delta(A)$ there exists $\alpha \in [0,1]$ such that

$$L \sim \alpha z_1 + (1-\alpha) z_N$$

  - Can always construct an equivalence to any lottery with just the two extreme outcomes z_1 and z_N - an analogy of the intermediate value theorem.

Utility Function

- Completeness, transitivity and continuity guarantee the existence of a utility function $U : S_L \to R$ such that $U(L) \geq U(L') \Leftrightarrow L \succsim L'$.
- Need to impose more structure on preference to derive utility functions of a particular form -
  The utility from a lottery is just **the expected value of the utilities from different outcomes**.
  Therefore, what matters for the choice among lotteries is NOT the expected value of the outcomes, BUT the expected value of the utilities from the outcomes.

Substitution Axiom

L₁、L₂的结果无差异+分布相同⇒L₁、L₂无差异

Substitution axiom: for two lotteries

$$L = (p_1 \circ a_1, \ldots, p_N \circ a_N) \text{ and } L' = (p_1 \circ a_1', \ldots, p_N \circ a_N')$$

if $a_n \sim a_n'$ **for** $\forall n = 1, \ldots, N$**, then** $L \sim L'$

- **Intuition** - fixing the probability distribution, indifference over outcomes implies indifference over lotteries
- Replace outcomes with lotteries

$$L = (\alpha \circ L_1, (1-\alpha) \circ L_2) \text{ and } L' = (\alpha \circ L_1', (1-\alpha) \circ L_2')$$

  Then $L_i \sim L_i', \ i = 1, 2 \ \Rightarrow L \sim L'$

- If $L \sim L'$, the agent must be indifferent between all linear combinations of L and L'

$$(\alpha \circ L, (1-\alpha) \circ L') \sim (\alpha \circ L, (1-\alpha) \circ L) \Leftrightarrow L \sim L'$$

  What is the implication for indifference curves in the lottery space?

Independence Axiom

Independence Axiom: for any lotteries L^1, L^2 and L, the independence axiom is satisfied if for $\forall \alpha \in (0,1)$

维持偏好关系.

$$L^1 \succsim L^2 \Leftrightarrow (1-\alpha)L^1 + \alpha L \succsim (1-\alpha)L^2 + \alpha L$$

- **Intuition** - shifting the same probability (α in this case) to L does not change the ranking between L^1 and L^2.
- Indifference curves are parallel lines - there is no counterpart in the preference-based consumer theory without uncertainty
  - Suppose $(2,0) \succsim (0,2)$; mix with $(2,2)$ by the same weights

    $$(2,1) = 0.5 \times (2,0) + 0.5 \times (2,2) \text{ and } (1,2) = 0.5 \times (0,2) + 0.5 \times (2,2)$$

  - It is not necessarily true that $(2,1) \succsim (1,2)$
- Focus on the difference between L^1 and L^2 - the preference ranking of these two lotteries is independent of L and α, this is also called **"Independence of Irrelevant Alternatives"**.

Independence Implies Monotonicity

Suppose $L \succ L'$ **and** $0 < \beta < \alpha < 0$. **Given independence, want to show**

$$\alpha L + (1-\alpha) L' \succ \beta L + (1-\beta) L'$$

- Independence $\Rightarrow L \succ \beta L + (1-\beta) L'$
- For any $0 < \gamma < 1$, independence $\Rightarrow$

$$\gamma L + (1-\gamma) \left(\beta L + (1-\beta) L'\right) \succ \beta L + (1-\beta) L'$$

- It is sufficient to show that there exists γ such that

$$\alpha L + (1-\alpha) L' = \gamma L + (1-\gamma) \left(\beta L + (1-\beta) L'\right)$$

- Just let $\gamma = \frac{\alpha-\beta}{1-\beta}$, which satisfies $\gamma \in (0,1)$

Allais Paradox (1)

How reasonable is the IIA Axiom?

- Three possible outcomes $A = \{0, 1000, 1100\}$
- How would you rank the two lotteries below?

$$\begin{aligned} L^1 &= (1\% \circ 0, 66\% \circ 1000, 33\% \circ 1100) \\ L^2 &= (0\% \circ 0, 100\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

- How would you rank the two lotteries below?

$$\begin{aligned} \widehat{L}^1 &= (67\% \circ 0, 0\% \circ 1000, 33\% \circ 1100) \\ \widehat{L}^2 &= (66\% \circ 0, 34\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

Allais Paradox (1)

How reasonable is the IIA Axiom?

- Three possible outcomes $A = \{0, 1000, 1100\}$
- How would you rank the two lotteries below?

$$\begin{aligned} L^1 &= (1\% \circ 0, 66\% \circ 1000, 33\% \circ 1100) \\ L^2 &= (0\% \circ 0, 100\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

- How would you rank the two lotteries below?

$$\begin{aligned} \widehat{L}^1 &= (67\% \circ 0, 0\% \circ 1000, 33\% \circ 1100) \\ \widehat{L}^2 &= (66\% \circ 0, 34\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

- L^1 vs. L^2: (\$0 with 1% and \$1100 with 33%) vs. 1000 with 34%

Allais Paradox (1)

How reasonable is the IIA Axiom?

- Three possible outcomes $A = \{0, 1000, 1100\}$
- How would you rank the two lotteries below?

$$\begin{aligned} L^1 &= (1\% \circ 0, 66\% \circ 1000, 33\% \circ 1100) \\ L^2 &= (0\% \circ 0, 100\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

- How would you rank the two lotteries below?

$$\begin{aligned} \widehat{L}^1 &= (67\% \circ 0, 0\% \circ 1000, 33\% \circ 1100) \\ \widehat{L}^2 &= (66\% \circ 0, 34\% \circ 1000, 0\% \circ 1100) \end{aligned}$$

- L^1 vs. L^2: (\$0 with 1% and \$1100 with 33%) vs. 1000 with 34%
- $\widehat{L}^1$ vs. $\widehat{L}^2$: (\$0 with 1% and \$1100 with 33%) vs. 1000 with 34%

Allais Paradox (2)

Let $L = (\frac{0.01}{0.34} \circ 0, \frac{0.33}{0.34} \circ 1100)$ and $L' = (1 \circ 1000)$

$$\begin{aligned} L^1 &= (1\% \circ 0, 66\% \circ 1000, 33\% \circ 1100) \sim (34\% \circ L, 66\% \circ 1000) \\ L^2 &= (0\% \circ 0, 100\% \circ 1000, 0\% \circ 1100) \sim (34\% \circ L', 66\% \circ 1000) \end{aligned}$$

IIA $\Rightarrow$ L^1 vs. L^2 ranking the same as L vs. L' ranking

$$\begin{aligned} \widehat{L}^1 = (67\% \circ 0, 0\% \circ 1000, 33\% \circ 1100) &\sim (66\% \circ 0, 34\% \circ L) \\ \widehat{L}^2 = (66\% \circ 0, 34\% \circ 1000, 0\% \circ 1100) &\sim (66\% \circ 0, 34\% \circ L') \end{aligned}$$

IIA $\Rightarrow$ $\widehat{L}^1$ vs. $\widehat{L}^2$ ranking the same as L vs. L' ranking

L^1 vs. L^2 ranking should be the same as $\widehat{L}^1$ vs. $\widehat{L}^2$ ranking

Expected Utility Function Form

- The preference is completeness, transitive and continuous
  $\Rightarrow$ there exist continuous real-valued utility functions to represent the preference.

伯努利：用钱衡量

- Expected Utility Function Form

衡量lottery的效用

$$U(L) = \sum_{n=1}^{N} p_n u_n$$

where $L = (p_1 \circ a_1, ..., p_N \circ a_N)$ and u_n is the utility number assigned to the n*th* outcome.

- $U : S_L \to R$ is a function from the set of lotteries to real numbers.
- The function form is linear in the probabilities over the outcomes.
- vNM form - John von Neumann & Oskar Morgenstern.

EU or vNM Form $\Leftrightarrow$ Linear

$E(u(\pi)) = u(E(\pi))$

The vNM expected utility form $\Leftrightarrow$ the function is linear, i.e., preserving the operation of adding and multiplying by a constant. For any K lotteries L_k, $k = 1, 2, ..., K$ and probability distribution $(\gamma^1, \gamma^2, ..., \gamma^K) \geq 0$ with $\sum_{k=1}^{K} \gamma^k = 1$

可以把概率提到外面来

$$U(\sum_{k=1}^{K} \gamma^k L^k) = \sum_{k=1}^{K} \gamma^k U(L^k)$$

$U(g) = E[u(\pi)]$

- Linear $\Rightarrow$ vNM form
  Take each outcome as a degenerate lottery
- vNM form $\Rightarrow$ Linear

$$U(\sum_{k=1}^{K} \gamma^k L^k) = \sum_{n=1}^{N} \left(\sum_{k=1}^{K} \gamma^k p_n^k \right) u_n = \sum_{k=1}^{K} \gamma^k \left(\sum_{n=1}^{N} p_n^k u_n \right) = \sum_{k=1}^{K} \gamma^k U(L^k$$

vNM Form and Affine Transformation

- An affine transformation transforms parallel lines to parallel lines and preserves ratios of distances along parallel lines.

$$\widehat{U}(L) = \beta_0 + \beta_1 U(L)$$

- The vNM form can be preserved **by and only by** positive affine transformation with $\beta_1 \geq 0$.

Affine Transformation ⇒ vNM Form

If $U(L)$ has the vNM form and $\widehat{U}(L)$ is the outcome of a positive affine transformation, i.e.,

$$\widehat{U}(L) = \beta_0 + \beta_1 U(L) \quad with \quad \beta_1 \geq 0,$$

$\widehat{U}(.)$ is linear thus of the vNM form. That is, for any $(\alpha^1, \alpha^2, ..., \alpha^K) \geq 0$ and $\sum_{k=1}^{K} \alpha^k = 1$, it must be

$$\widehat{U}(\sum_{k=1}^{K} \alpha^k L^K) = \sum_{k=1}^{K} \alpha^k \widehat{U}(L^k)$$

Proof:

$$\begin{aligned}
\widehat{U}(\sum_{k=1}^{K} \alpha^k L^k) &= \beta_0 + \beta_1 U(\sum_{k=1}^{K} \alpha^k L^k) = \beta_0 + \beta_1 \sum_{k=1}^{K} \alpha^k U(L^k) \\
&= \sum_{k=1}^{K} \alpha^k \left(\beta_0 + \beta_1 U(L^k)\right) = \sum_{k=1}^{K} \alpha^k \widehat{U}(L^k)
\end{aligned}$$

vNM Form $\Rightarrow$ Affine Transformation

If both $U(L)$ and its transformation $\widehat{U}(L)$ are of the vNM form, the transformation must be a positive affine transformation, i.e., we can find β_0 and β_1 such that

$$\widehat{U}(L) = \beta_0 + \beta_1 U(L)$$

Proof:
There exists α such that $L \sim \alpha \overline{L} + (1-\alpha)\underline{L}$. Thus,

$$U(L) = \alpha U(\overline{L}) + (1-\alpha)U(\underline{L}),\ \widehat{U}(L) = \alpha \widehat{U}(\overline{L}) + (1-\alpha)\widehat{U}(\underline{L})$$

Solve for α as a function of $(U(L), U(\overline{L}), U(\underline{L}))$ and substitute in the expression of $\widehat{U}(L)$

$$\begin{aligned}
\widehat{U}(L) &= \frac{U(L) - U(\underline{L})}{U(\overline{L}) - U(\underline{L})}\widehat{U}(\overline{L}) + \frac{U(\overline{L}) - U(L)}{U(\overline{L}) - U(\underline{L})}\widehat{U}(\underline{L}) \\
&= \frac{U(\overline{L})\widehat{U}(\underline{L}) - U(\underline{L})\widehat{U}(\overline{L})}{U(\overline{L}) - U(\underline{L})} + \frac{\widehat{U}(\overline{L}) - \widehat{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})}U(L)
\end{aligned}$$

Linear in Probability - a Cardinal Feature

- Positive affine transformation is one special type of monotone transformation
- Since the property of being linear in probabilities only survives positive affine transformation but not monotone transformation in general, this feature is not ordinal but cardinal.
- Cardinal (expected) utility function means the differences in utilities have meanings
- Positive affine transformations preserve the ranking of utility differences

Preserve Ranking in Utility Difference

Consider two lotteries of 4 potential outcomes.

$$L^a = (\frac{1}{2} \circ a_1, 0 \circ a_2, 0 \circ a_3, \frac{1}{2} \circ a_4) \text{ and } L^b = (0 \circ a_1, \frac{1}{2} \circ a_2, \frac{1}{2} \circ a_3, 0 \circ a_4)$$

If $a_1 \succ a_2 \succ a_3 \succ a_4$ with $u_1 - u_2 > u_3 - u_4$, we have

$$U(L^a) = 0.5u_1 + 0.5u_4 > U(L^b) = 0.5u_2 + 0.5u_3 \text{ and } L^a \succsim L^b$$

With a positive affine transformation of $U(.)$ we still have

$$\widehat{U}(L^a) = \beta_0 + \beta_1 (0.5u_1 + 0.5u_4) > \beta_0 + \beta_1 (0.5u_2 + 0.5u_3) = \widehat{U}(L^b)$$

Thus the ranking in the difference of utilities is preserved, so is the ranking between L^a and L^b.

Expected Utility Theorem (1)

If a preference on the lottery set $\Delta(A)$ satisfies all the above axioms - completeness, transitivity, continuity, independence of irrelevant alternatives （substitution and monotonicity） and reduction - then there exists a utility function with the vNM form that represents the preference.

Part I: Define the Function

- Continuity and completeness $\Rightarrow$
  For $\forall L \in \Delta(A)$, there exists α_L such that $L \sim \alpha_L \overline{L} + (1-\alpha_L)\underline{L}$.
- Monotonicity $\Rightarrow$

$$\forall \alpha, \beta \in [0,1],\ \beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L} \Leftrightarrow \beta > \alpha$$

  Therefore α_L is unique.
- Define $U(L) = \alpha_L$, it represents $\succsim$ because

$$U(L) \geq U(L') \Leftrightarrow \alpha_L \geq \alpha_{L'}$$

Expected Utility Theorem (2)

Part II: Verify the vNM Form (simple case - convex combination of two)
$\forall L_1, L_2 \in \Delta(A), t \in (0,1)$, for $L_3 = tL_1 + (1-t)L_2$,
$U(L_3) = tU(L_1) + (1-t)U(L_2)$?

- Based on previous discussion, we can find $\alpha_1, \alpha_2 \in [0,1]$ such that

$$L_1 \sim \alpha_1 \overline{L} + (1-\alpha_1)\underline{L}, \quad \text{thus } U(L_1) = \alpha_1$$

$$L_2 \sim \alpha_2 \overline{L} + (1-\alpha_2)\underline{L}, \quad \text{thus } U(L_2) = \alpha_2$$

  IIA and Reduction $\Rightarrow$

$$\begin{aligned} L_3 &\sim t(\alpha_1 \overline{L} + (1-\alpha_1)\underline{L}) + (1-t)\Phi\alpha_2 \overline{L} + (1-\alpha_2)\underline{L}\Psi \\ &\sim (t\alpha_1 + (1-t)\alpha_2)\,\overline{L} + (t(1-\alpha_1) + (1-t)(1-\alpha_2))\,\underline{L} \end{aligned}$$

  By definition, $U(L_3) = t\alpha_1 + (1-t)\alpha_2 = tU(L1) + (1-t)U(L_2)$.

- $U(L)$ is of the vNM form

Expected Utility Theorem (3)

General case - convex combination of many

$$L^k \sim U(L^k)\overline{L} + \left(1 - U(L^k)\right)\underline{L} \text{ and } \gamma^k > 0,\ \forall k = 1, \ldots K \text{ with } \sum_{k=1}^{K} \gamma^k = 1$$

$$\Longrightarrow\ U\left(\sum_{k=1}^{K} \gamma^k L^k\right) = \sum_{k=1}^{K} \gamma^k U\left(L^k\right)$$

Proof:

$$\begin{aligned} & L^k \sim U(L^k)\overline{L} + \left(1 - U(L^k)\right)\underline{L}\ \forall k \\ \Rightarrow\ & \sum_{k=1}^{K} \gamma^k L^k \sim \left(\sum_{k=1}^{K} \gamma^k U(L^k)\right)\overline{L} + \left(\sum_{k=1}^{K} \gamma^k \left(1 - U(L^k)\right)\right)\underline{L} \\ \Rightarrow\ & U\left(\sum_{k=1}^{K} \gamma^k L^k\right) = \sum_{k=1}^{K} \gamma^k U(L^k) \end{aligned}$$

Risk Attitudes

- The goal of a decision maker is to maximize the expected utility
- Solutions by Daniel Bernoulli and Gabriel Cramer to the St. Petersburg Paradox - use a log or square root function to evaluate the outcomes
- Lotteries with non-negative wealth levels as outcomes
- Two utility functions
  - vNM expected utility function $U(L)$: evaluate lottery
  - Bernoulli utility function $u(w)$: evaluate wealth
  - $u(\cdot)$ is effectively $U(\cdot)$ taking w as a degenerate lottery
- "Manipulate" $u(\cdot)$ to capture different risk attitudes

Definition

$u[p_1w_1+\cdots+p_nw_n]$ 求和

- $L = (p_1 \circ w_1, ..., p_N \circ w_N)$, $E(w) = \sum_{n=1}^{N} p_n w_n$, $U(L) = E(u(w))$
  - Risk-averse $\Leftrightarrow$ $\underline{u(E(w)) > U(L) = E(u(w))}$ $p_1u_1+\cdots+p_nu_n$ 凹函数
  - Risk-neutral $\Leftrightarrow$ $u(E(w)) = U(L) = E(u(w))$
  - Risk-loving $\Leftrightarrow$ $u(E(w)) < U(L) = E(u(w))$

- It all depends on the marginal utility of wealth $MU_w = u'(w)$
  - 2阶导<0 Concave $u(w)$ $\Leftrightarrow$ decreasing MU_w $\Leftrightarrow$ risk-averse
  - Linear $u(w)$ $\Leftrightarrow$ constant MU_w $\Leftrightarrow$ risk-neutral
  - Convex $u(w)$ $\Leftrightarrow$ increasing MU_w $\Leftrightarrow$ risk-loving

组合小于确切
纵轴是效用

Measure Risk Aversion (1)

Arrow-Pratt measure of absolute risk aversion

$R_a(w) > 0 \Rightarrow$ R.A.

主要还是二阶导的符号.

$$\boxed{R_a(w) \equiv -\frac{u''(w)}{u'(w)}}$$

- The sign: positive for risk aversion
- The magnitude: u'' describes how fast the slope changes
- Division by u': $R_a(w)$ not affected by any affine transformation of $U(\cdot)$

  仿射.
- The value depends on wealth level

  DARA: decreasing absolute risk aversion

  $w\uparrow$, 风险厌恶$\downarrow$ $R_a'(w) < 0$
  - Increasing willingness to accept small gambles at a higher wealth level
  - Risky assets are "normal goods": demand for risky assets $\Uparrow$ in w

有钱了就不那么厌恶.

Measure Risk Aversion (2)

- Certainty equivalent of a lottery - $CE(L)$
  - Certainty: an amount of money for sure
  - Equivalent: to the lottery in terms of utility
  - Risk premium $EV(L) - CE(L)$
- Probability premium $\pi(w, \epsilon, u)$

$U(L) = u(CE)$
固定的钱产生的效用
$E(L) > CE \Rightarrow R.A.$

消除风险需要的钱

$$L = ((0.5 + \pi) \circ (w + \epsilon), (0.5 - \pi) \circ (w - \epsilon)) \sim w$$

Find the π to make $CE(L) = w$, that is

$$U(L) = (0.5 + \pi)u(w + \epsilon) + (0.5 - \pi)u(w - \epsilon) = u(w)$$

Solve for $\pi(w, \epsilon; u(\cdot))$

$\pi \geq 0 \Rightarrow$ 风险厌恶.

Compare Risk Aversion across Agents: R_a and CE (1)

1更R.A. $\Leftrightarrow R_a^1 > R_a^2 \Leftrightarrow CE^1 < CE^2 \Leftrightarrow \pi^1 > \pi^2$

L带来多少utility

For two individuals $i = 1, 2$,

$R_a^1(w) \geq R_a^2(w)$ **for** $\forall w \geq 0 \;\; \Leftrightarrow \;\; CE_1(L) \leq CE_2(L)$ **for** $\forall L$**?**

- Outcomes: $w_1, ..., w_N$
- Evaluation by agent 1: $u_1, ..., u_N$
- Evaluation by agent 2: $v_1, ..., v_N$
- Construct function $h(.)$ such that $u_n = h(v_n)$
  - Thus function $h(.)$ captures the connection between u_n and v_n
    - $h(s)$ represents the utility agent 1 gets from the amount of money that gives agent 2 utility s.

$$h(s) = u\left(v^{-1}(s)\right)$$

  - $CE_1(L) \leq CE_2(L)$ for $\forall L \;\; \Leftrightarrow \;\; h(.)$ is concave $\;\; \Leftrightarrow \;\; R_a^1(w) \geq R_a^2(w)$
    - Concavity of $h(.)$ means $u(.)$ is the outcome of a concave transformation of $v(.)$ thus "more concave".

Compare Risk Aversion across Agents: R_a and CE (2)

- Let $h(s) = u\left[f(s)\right]$ and $x = f(s) = v^{-1}(s)$
  - Second order derivative of inverse function $\frac{d^2x}{dy^2} = -\frac{y''}{y'^3}$

$$f''(s) = -\frac{v''(x)}{[v'(x)]^3}$$

  - Second order derivative of composite function

$$h''(s) = u''(x)\left[f'(s)\right]^2 + u'(x)f''(s) = \frac{u'(x)}{v'(x)^2}\left(\frac{u''(x)}{u'(x)} - \frac{v''(x)}{v'(x)}\right)$$

- $R_a^1(w) \geq R_a^2(w) \quad \Leftrightarrow \quad \frac{u''(x)}{u'(x)} - \frac{v''(x)}{v'(x)} \leq 0 \quad \Leftrightarrow \quad h(.)$ is concave
- By Jensen's inequality

$$u(CE_1) = \sum_{n=1}^{N} p_n u_n = \sum_{n=1}^{N} p_n h(v_n) \leq h\left(\sum_{n=1}^{N} p_n v_n\right) = u(CE_2)$$

Compare Risk Aversion across Agents: R_a and $\pi(\epsilon, w; u)$

For two individuals $i = 1, 2$,
$R_a^1(w) > R_a^2(w)$ **for** $\forall w \geq 0 \;\; \Leftrightarrow \;\; \pi(w, \epsilon, u(\cdot) > \pi(w, \epsilon, v(\cdot)$ **for** $\forall w$ **and** ϵ**? where** $u(\cdot)$ **and** $v(\cdot)$ **are the Bernoulli utility functions for agent 1 and 2 respectively.**

- Use the definition identity, take derivative w.r.t. ϵ twice, then show that
$$R_a(w) = 4\frac{\partial \pi(\epsilon, w; .)}{\partial \epsilon}|_{\epsilon=0}$$
- $R_a^1(w) > R_a^2(w) \Leftrightarrow \frac{\partial \pi(\epsilon; w, u(\cdot))}{\partial \epsilon}|_{\epsilon=0} > \frac{\partial \pi(\epsilon; w, v)}{\partial \epsilon}|_{\epsilon=0}$
- At wealth level w, the more risk-averse agent as measured by $R_a(w)$ requires a larger probability premium for a small disturbance ϵ to w.

Investment in Risky Asset (1)

风投

- Wealth: w
- Investment in risky asset: β
- Possible (net) returns: r_i, $i = 1, 2, ..., N$ with probability p_i
- The agent is risk averse, i.e, has concave Bernoulli utility function $u(\cdot)$

Investment in Risky Asset (2)

- List of possible outcomes: $\underline{w + \beta r_i}$ 最大化.
- Expected utility: $EU = \sum_{i=1}^{N} p_i u(w + r_i \beta)$
- Constrained optimization: $0 \leq \beta \leq w$
- Corner solution: $\beta^* = 0$ or $\beta^* = w$
- Interior solution: $0 < \beta^* < w$
- Wealth effect $\frac{d\beta^*}{dw}$

Investment in Risky Asset (3)

- Corner solution $\beta^* = 0$ if expected return is negative, i.e

$$\sum_{i=1}^{N} p_i r_i < 0$$ 期望收益 < 0

- Corner solution $\beta^* = w$ if expected marginal utility of investment is always positive, i.e.

$$\sum_{i=1}^{N} p_i r_i u'(w + r_i w) > 0$$

- Interior solution is achieved where the expected marginal utility of investment is 0 (so it is the optimal investment), i.e,

$$\sum_{i=1}^{N} p_i r_i u'(w + r_i \beta^*) = 0$$

Investment in Risky Asset (4)

Comparative statics - wealth effect under *DARA* **($R_a(w) \Downarrow$ in w)**

- Start with an interior solution, i.e, β^* that satisfies

$$\sum_{i=1}^{N} p_i r_i u'(w + r_i\beta^*) = 0$$

- Take total derivative w.r.t w and β^* and derive $\frac{d\beta^*}{dw}$

正常品

$$\begin{aligned}
\frac{d\beta^*}{dw} &= -\frac{\sum_{i=1}^{N} p_i r_i u''(w + r_i\beta^*)}{\sum_{i=1}^{N} p_i r_i^2 u''(w + r_i\beta^*)} \\
&= \frac{\sum_{i=1}^{N} p_i r_i R_a(w + r_i\beta^*) u'(w + r_i\beta^*)}{\sum_{i=1}^{N} p_i r_i^2 u''(w + r_i\beta^*)} \\
&> \frac{\sum_{i=1}^{N} p_i r_i R_a(w) u'(w + r_i\beta^*)}{\sum_{i=1}^{N} p_i r_i^2 u''(w + r_i\beta^*)} = 0
\end{aligned}$$

- With *DARA*, $r_i R_a(w + r_i\beta^*) < r_i R_a(w)$ holds for $r_i > 0$ & $r_i < 0$

Insurance

- "Wealth" at risk: health/property/unemployment insurance
- Actuarial cost: expected payment by insurance company
- Fairly priced insurance policy - zero expected profit
  revenue(insurance premium) = actuarial cost
- Perfectly competitive insurance market

$rI = p \cdot I$

How Much Insurance to Buy

$W^0 \begin{cases} p: W^0 - W^r \\ 1-p: W^0 \end{cases}$

- An agent has total asset of w_0
- w_r is under risk
  – if the accident happens, w_r is lost and the agent is left with $w_0 - w_r$
- The probability of the accident is p.
- The insurance premium is r
  – the agent needs to pay r to claim one dollar loss from the insurance company when the accident happens.
- The agent is risk averse with Bernoulli utility function $u(\cdot)$

  How much of wealth under risk w_r should be insured?

Diagram for Insurance Purchase Problem

Expected utility with I insured

$$EU(I; p, r, w_0, w_r) = (1-p)u(w_0 - rI) + pu(w_0 - rI - w_r + I)$$

The agent's problem

$$Max_{I} EU(I; p, r, w_0, w_r, w_b)$$

$$F.O.C \quad (1-p)ru'(w_0 - rI^*) = p(1-r)u'(w_0 - rI^* - w_r + I^*)$$

- "sq" - status quo/no accident; "b" - the bad event has happened

现状.
$$w^{sq} = w_0 - rI; \quad w^b = w_0 - rI - w_r + I$$
保险金. 补偿.

- What is the fair price r^F of the insurance?
- What is I^* under fair insurance price r^F?
- What is the consumer's willingness to pay for full insurance?
- What if the insurance price is higher than the fair one, i.e, $r > r^F$?

Budget Lines

fair : r = p ⇒ 2种情况的wealth相同.

$W_0 - Ir$ $W_0 - Ir + I - W$ $= 0$

- Point representing no insurance - $(w_0, w_0 - w_r)$
- Points representing no uncertainty - 45 degree line

相同的金额.

- Point representing full insurance $(I = w_r)$ - $(w_0 - rw_r, w_0 - rw_r)$
- Budget line - connecting $(w_0, w_0 - w_r)$ and $(w_0 - rw_r, w_0 - rw_r)$
- Slope of budget line - $\frac{1-r}{r}$

$$\begin{aligned} & w^{sq} \downarrow \text{ by r to have 1 more dollar insured} \\ \Rightarrow \quad & w^{b} \uparrow \text{ by } (-r+1) \end{aligned}$$

体现在这.

-r is for paying the premium; 1 is the insurance claim

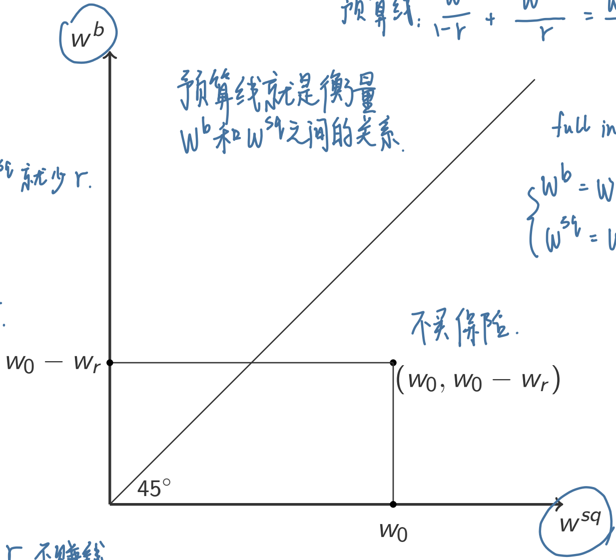
- If an insurance policy leaves consumer on a budget line flatter than the one implied by the fair price, the insurance company earns positive expected profits.

$$r > p \Rightarrow E\pi = rI - pI \text{ for } \forall I > 0$$

BL Graph - No Insurance vs. Full Insurance

联立 w^b、w^{sq}，消掉 I.

预算线: $\frac{w^b}{1-r} + \frac{w^{sq}}{r} = \frac{w^0 - w^r}{1-r} + \frac{w^0}{r}$



预算线就是衡量 w^b 和 w^{sq} 之间的关系.

full insurance.

$$\begin{cases} w^b = w^0 - rI + I - w^r \\ w^{sq} = w^0 - rI \end{cases}$$

不买保险.

人(r, I)

w^b 多保1, w^{sq} 就少 r.

⇒ 斜率 $-\frac{1-r}{r}$

fair: $-\frac{1-p}{p}$.

这不是全保.

$pI - rI$

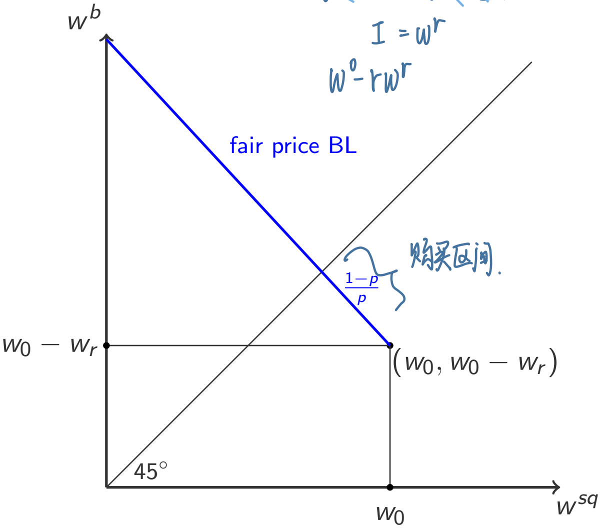
$p = r$ 不赚钱.

BL Graph - Fair Price BL

$\cancel{w^0 - rI} = \cancel{w^0 - rI} + I - w^r$

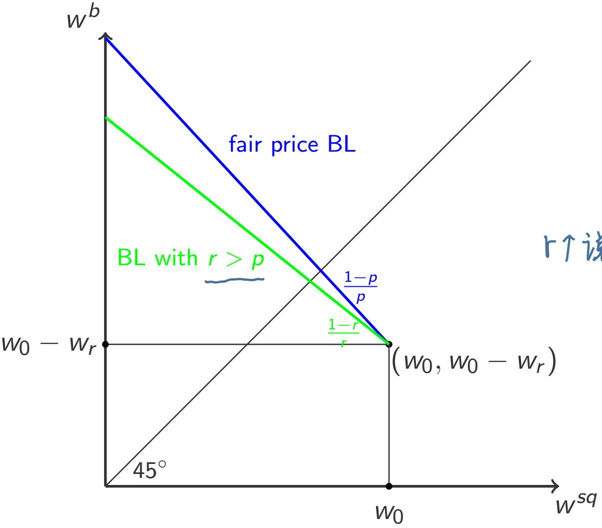
$I = w^r$

$w^0 - r w^r$



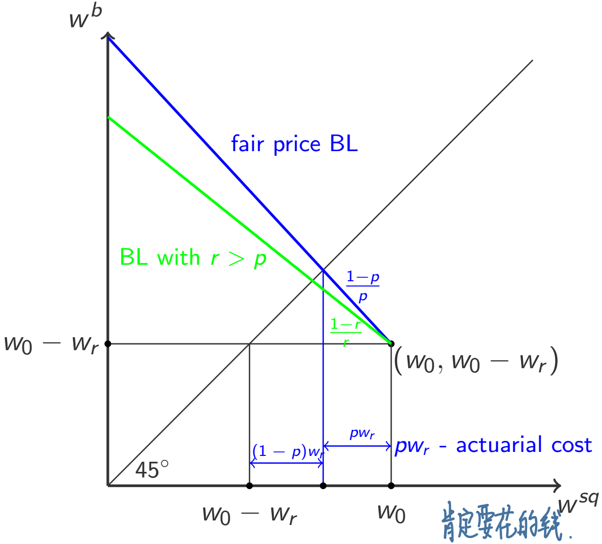
BL Graph - BL with $r > p$

约束收缩.

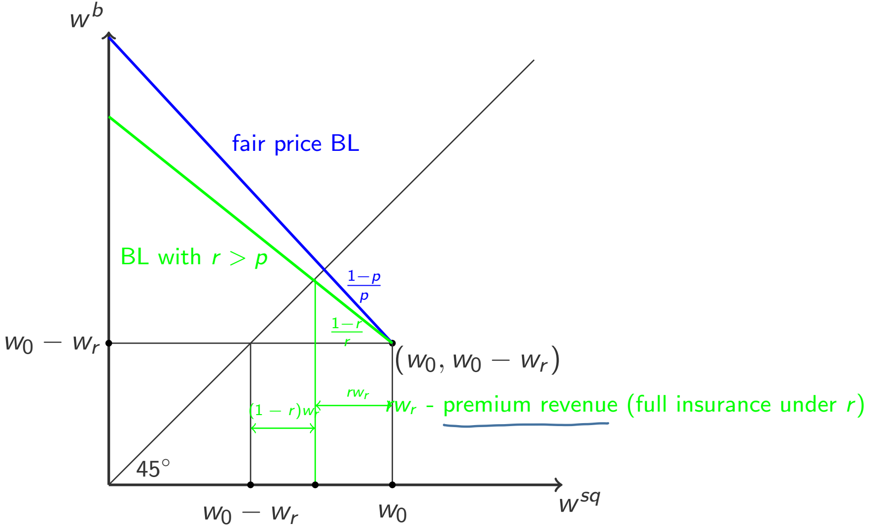


r↑说明保费更贵.

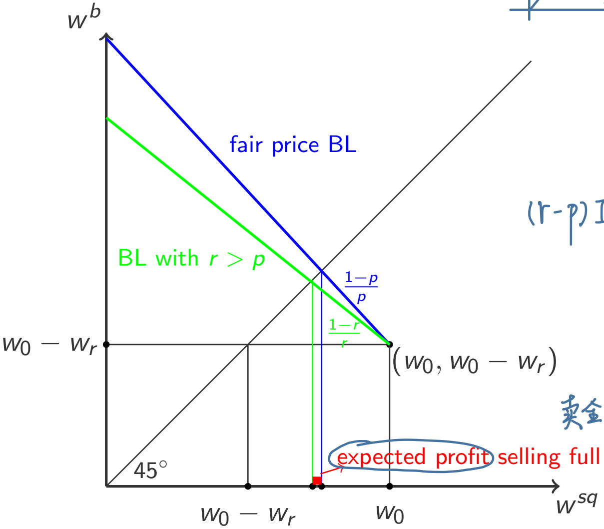
BL Graph - Actuarial Cost for Full Insurance

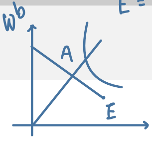


BL Graph - Premium Revenue of Full Insurance under r



BL Graph - Expected Profit $E(\pi)$





$E = (1-p)u(w^{sq}) + p\,u(w^b)$

$\frac{(1-p)u'(w^{sq})}{p\,u'(w^b)}$

在A、E如果不能买保 $\frac{1-r}{r}$，就是角点解

A点解：$p > r$

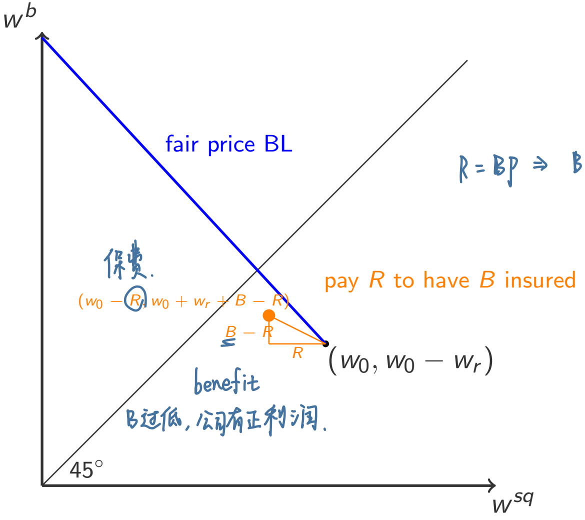
保费低 ⇒ 全保

保险公司利润为负.

$(r-p)I$.

卖全保赚的！$I = w^r$

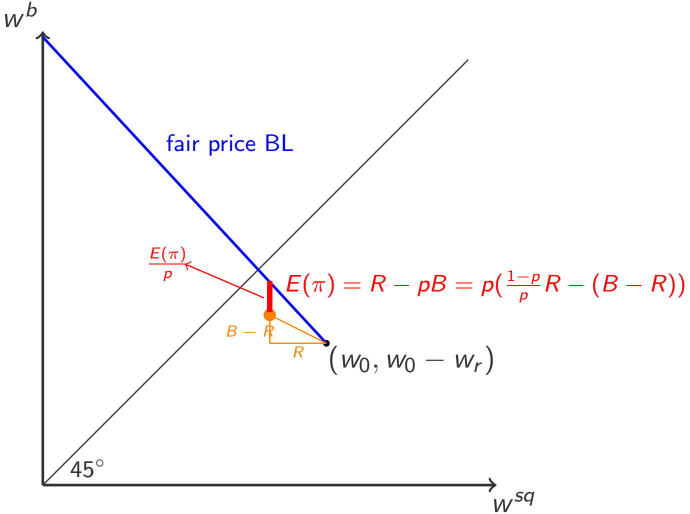
BL Graph - Partial Insurance Product



$R = Bp \Rightarrow B = \frac{R}{p}$ 令公司没利润

BL Graph - $E(\pi)$ for Partial Insurance Product

Proportional to the vertical distance to fair price BL!

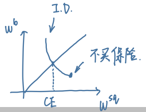


Indifference Curves

$Eu = (1-p)u(w^{sq}) + p\,u(w^b) \Rightarrow (1-p)u'(w^{sq})\Delta w^{sq} + p\,u'(w^b)\cdot\Delta w^b = 0$

- Slope of indifference curves $\frac{(1-p)u'(w^{sq})}{pu'(w^b)}$ Slope of BL: $\frac{1-p}{p}$ 相切 $\Rightarrow$ fully insurance
- At a given point (w^{sq}, w^b) the shape/slope depends on
  - Probability of the accident p
  - Concavity of Bernoulli function $u(\cdot)$
- Holding p constant
  More risk averse $\Rightarrow$ $u(\cdot)$ more concave $\Rightarrow$ more convex ICs
- Holding Bernoulli utility function the same
  Higher $p \Rightarrow$ IC flatter, higher relative value for 1 dollar in w^b
- Slope of indifference curves at full insurance points - $\frac{1-p}{p}$
- Find "CE - certainty equivalent" and "EV - expected value" for the uninsured situation

找出CE、EV



$EV = (1-p)\,w^0 + p(w^0 - w^r)$
$= w^0 - p w^r$

p=r时的全保就是这个值.

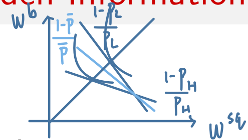
Optimization

- Conditions for full insurance to be optimal $\frac{1-r}{r} = \frac{1-p}{p}$
- Optimal insurance when $r > p$

$$r > p \Rightarrow \frac{1-r}{r} < \frac{1-p}{p}$$

  - Full insurance point is not a tangency point, thus it is not optimal
  - More specifically, the indifference curve through the full insurance point goes beneath the budget line to the right of the full insurance point
  - Optimization achieved with partial insurance
- When not to buy insurance at all?

Hidden Information on Insurance Market (1)

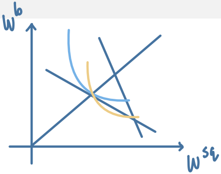


$p_H > p_L$ $\bar{p} = \alpha_H p_H + \alpha_L p_L$

α_H高更偏向p_H.

- An agent owns total wealth w_0 which includes w_r risky assets. p is the probability of losing w_r.
- An insurance contract specifies B, the amount the insurance company will pay if the accident happens, and R, the insurance premium.
  – Full insurance means $B = w_r$.
- There are two types of consumers with different probabilities of having the accident, p_H and p_L, and otherwise identical.
- The proportion of the H-type in the population is α_H and the proportion for the L-type is α_L. $\alpha_H + \alpha_L = 1$.

Hidden Information on Insurance Market (2)



哪个是产品线.

分别均衡

如果按照L，在原来的下方.

$$p_L < p_H \Rightarrow \frac{1-p_L}{p_L} > \frac{1-p_H}{p_H}$$

- At each point in the diagram, the L-type agents have steeper indifference curves.
  This is because the L-type is less likely to have the accident, which makes 1 more dollar in the status quo more valuable.
- The fair price budget line for the L-type is steeper.

Hidden Information on Insurance Market (3)

- With symmetric information, a situation where the insurance company has the same information as the consumer, the insurance company can charge different insurance premium.
- With asymmetric information, the insurance company knows the distribution of accident probabilities α_H and α_L but cannot identify the type of a particular consumer.

Adverse Selection (1)

- How about setting $R = (\alpha_H p_H + \alpha_L p_L)w_r$? This is the fair insurance price for full insurance based on the average probability of the accident in the population.
- The H-types will buy the insurance and be VERY HAPPY!
  - Note that not only uncertainty is gone, expected value is higher!
  - In other words, this insurance point is beyond their fair price BL.
- The L-types may or may not buy the insurance.
  - If the L-types buy, the L-types are subsidizing the H-types in this pooling situation.
  - If the L-types do not buy, the actual actuarial cost would be $p_H w_r$ which is higher than the insurance price they pay, $R = (\alpha_H p_H + \alpha_L p_L)w_r$. The insurance company loses money.

Adverse Selection (2)

If the L-types are willing to buy the full insurance priced at $R = (\pi_H p_H + \pi_L p_L)w_r$, can the pooling result be the equilibrium on a perfectly competitive insurance market?

No. Another insurance company can attract L-type consumers by offering an option of partial insurance with lower premium.

Adverse Selection (3)

If only the H-types are willing to buy the full insurance priced at $R = (\pi_H p_H + \pi_L p_L) w_r$, the insurance company would have to increase R to $p_H w_r$ to break even. All the L-types are excluded from the market.

More generally, for an agent to pay R to get fully insured, it must be

$$
\begin{aligned}
& u(w_0 - R) \geq (1-p)u(w_0) + pu(w_0 - w_r) \\
\Leftrightarrow \quad & pu(w_0) - pu(w_0 - w_r) \geq u(w_0) - u(w_0 - R) \\
\Leftrightarrow \quad & p \geq \frac{u(w_0) - u(w_0 - R)}{u(w_0) - u(w_0 - w_r)}
\end{aligned}
$$

Adverse selection: an increase in insurance premium R will increase the average level of risk of people who buy insurance and thus increase the actuarial cost of the insurance firm.

Solution to Adverse Selection Problem

Separating equilibrium (does not necessarily exist) -
Design a menu with two (B, R) options, one more attractive to the H-type, the other more attractive to the L-type.

- $(B_1 = w_r, R_1 = p_H w_r)$ - a fair-priced full-insurance package for H
- Design (B_2, R_2) that will only attract the L-type
    - Attract the L-type to buy

$$\begin{aligned} & p_L u(w_0 - R_2) + (1 - p_L)u(w_0 - R_2 - w_r + B_2) \\ \geq \quad & max\{p_L u(w_0) + (1 - p_L)u(w_0 - w_r), u(w_0 - p_H w_r)\} \end{aligned}$$

    - Not attract the H-type away from (B_1, R_1)

$$\begin{aligned} & p_H u(w_0 - R_1) + (1 - p_H)u(w_0 - R_1 - w_r + B_1) \\ \geq \quad & p_H u(w_0 - R_2) + (1 - p_H)u(w_0 - R_2 - w_r + B_2) \end{aligned}$$

    - Non-negative profit for the insurance company

$$R_2 \geq p_L B_2$$

Separating Equilibrium Summary

- H-types are fully insured
- L-types are partially insured: lower premium and lower coverage
- The bad type (high risk type in this example) gets what they can achieve under perfect information.
- The good type (low risk type in this case) bears the cost of asymmetric information.
- There are other examples where asymmetric information hurts the good.