From Partial Equilibrium to General Equilibrium

- In many cases, the motivation for a policy comes from its partial equilibrium effects
- However, the general equilibrium effects, due to the interrelation between markets, may offset the effectiveness of the policy in addressing the original problem and/or cause other problems - the unintended consequences
- General equilibrium effect can be important for comprehensive evaluation of the welfare/efficiency consequence of a shock or policy change

Dutch Disease

natural resource booms
$\Rightarrow$ more input demand by the mining sector
$\Rightarrow$ wage $\uparrow$ due to larger induced demand for workers
$\Rightarrow$ profits in manufacturing sector $\downarrow$ due to higher labour cost

Energy Act and Wheat Price

promotion of renewal fuels in the 2005 Energy Policy Act
$\Rightarrow$ corn price $\uparrow$
$\Rightarrow$ production of wheat, soybean, etc $\downarrow$ due to the switch of farmers to corn
$\Rightarrow$ prices for these other agriculture produce $\uparrow$ due to declining supply

Roadmap

Key questions

- How do interrelated markets achieve equilibrium at the same time?
- Is the equilibrium outcome efficient?

Outline

- Three activities - consumption, production and transaction
- Pure exchange economy
  - Consumption and transaction only, no production
  - With or without money - barter economy or competitive economy
  - Outcome - core and competitive equilibria
- Competition and efficiency - the Fundamental Welfare Theorems
- Introduce production
- Introduce time and uncertainty

Elements in a Pure Exchange Economy

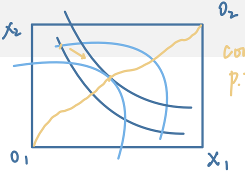
- Two goods 1 and 2
- Consumers A and B
  consumption $\mathbf{x}^A = (x_1^A, x_2^A)$ and $\mathbf{x}^B = (x_1^B, x_2^B)$

  上标是消费者.
- Endowment of two commodities
  $\mathbf{e}^A = (e_1^A, e_2^A)$ and $\mathbf{e}^B = (e_1^B, e_2^B)$
  $\Rightarrow$ total endowment $\mathbf{e} = \mathbf{e}^A + \mathbf{e}^B = (e_1^A + e_1^B, e_2^A + e_2^B)$
- No production
- Voluntary exchange/barter

  就是交换.

Edgeworth Box

原点也P.E.



contract curve
P.E.连起来.

- Graphical presentation: Edgeworth box
- Two origins and four coordinates

中间有开口就可以改进.

- Feasible allocation $F(\mathbf{e}) \equiv \{\mathbf{x} | \sum_{i \in I} \mathbf{x}^i = \sum_{i \in I} \mathbf{e}^i\}$ $\Rightarrow MRS^1_{12} = MRS^2_{12}$

相切!

- Preferences (ICs) for consumer 1 and 2: $\succsim_1$ and $\succsim_2$
- Exchange economy: $\{\succsim_i, \mathbf{e}^i\}_{i \in I}$
- **Pareto efficient set**: A feasible allocation, $\mathbf{x} \in F(\mathbf{e})$, is Pareto efficient if there is no other feasible allocation, $\mathbf{y} \in F(\mathbf{e})$, such that $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all consumers i, with at least one preference strict.
  - Tangency points of the two sets of ICs and the two origins
  - What if the two goods are perfect complements for both consumers?
  - What if the two goods are perfect substitutes for both consumers?

Pure Exchange Economy - Barter Exchange

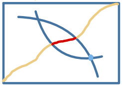
Barter Exchange Equilibrium

feasible allocation. 可行集 box里的所有点.

Voluntary barter trade ⇒ mutually beneficial trade

- Pareto efficient
- Blocking coalitions and core
  - "Coalition S **blocks** allocation $\mathbf{x}$" means there is an allocation $\mathbf{y}$ such that $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$ and $\mathbf{y}^i \succsim \mathbf{x}^i$ for all $i \in S$, with at least one preference strict
  - Unblocked allocation ⇒ Pareto efficient
  - **Core** $C(\mathbf{e})$: the set of all unblocked feasible allocations

可以改进的这些.



core

- The exact outcome depends on the bargaining power of the two parties

禀赋.

这对禀赋点对说是P.I.

Example of Blocking Coalition - Two Agents 1 & 2

- Start with two agents 1 and 2 , with utility functions
  - Agent 1 : $u^1(x_1, x_2) = 2x_1 + x_2$
  - Agent 2 : $u^2(x_1, x_2) = x_1 + 2x_2$
- The initial endowment allocation is $\mathbf{e}^1 = (1, 2)$ and $\mathbf{e}^2 = (2, 1)$.
- $\mathbf{x}^1 = (3, 1)$ and $\mathbf{x}^2 = (0, 2)$ is one possible equilibrium with VT
  - It is PE for 1 and 2
  - Agent 1 gets all the benefits from trade

Example of Blocking Coalition - Add Agent 3

- Introduce Agent 3 with the following endowment and preference
  - $\mathbf{e}^3 = (1,1)$ and $u^3(x_1, x_2) = x_1 + x_2$
- Will Agent 2 further trade with Agent 3 ?
  - No - allocation $\mathbf{x}^2 = (0,2)$ and $\mathbf{x}^3 = (1,1)$ is PE for 2 and 3
- Will Agent 1 further trade with Agent 3?
  - Yes - allocation $\mathbf{x}^1 = (3,1)$ and $\mathbf{x}^3 = (1,1)$ is NOT PE for 1 and 3
  - $\mathbf{x}^1 = (4,0)$ and $\mathbf{x}^3 = (0,2)$ is one possible equilibrium with VT
  - With $\mathbf{x}^1 = (4,0)$ and $\mathbf{x}^3 = (0,2)$, Agent 1 gets all the benefits

Example of Blocking Coalition - Coalition by 2 & 3

- $\mathbf{x}^1 = (4, 0)$, $\mathbf{x}^2 = (0, 2)$ and $\mathbf{x}^3 = (0, 2)$ is a PEA for all three
- Suppose 1 and 2 haven't traded before they meet 3, all three start VT together
    - Will Agent 2 and 3 accept the previous terms of trade they had with Agent 1?
    - Will the PE above be an equilibrium under VT?

Pure Exchange Economy - Perfect Competition

Perfect Competition

Instead of barter trades and bargaining over the terms of trade, all individuals take prices as given and respond to prices only

...Imagine there exists an auctioneer ...

How would the equilibrium differ?

- ⋄ Individual optimization: utility maximization with endowment
- ⋄ Market clears

Finding WE: Example 1

- Preferences:

$$
\begin{aligned}
U^A(x_1^A, x_2^A) &= (x_1^A)^a (x_2^A)^{1-a} \\
U^B(x_1^B, x_2^B) &= (x_1^B)^b (x_2^B)^{1-b}
\end{aligned}
$$

- Endowment: $\mathbf{e}^A = (1,0)$ and $\mathbf{e}^B = (0,1)$
- Walrasian equilibrium $\frac{p_2}{p_1}^*$?

Finding WE: Example 2

- Preferences:

$$\begin{aligned} U^A(x_1^A, x_2^A) &= x_1^A - \frac{1}{8}(x_2^A)^{-8} \\ U^B(x_1^B, x_2^B) &= -\frac{1}{8}(x_1^B)^{-8} + x_2^B \end{aligned}$$

- Endowment: $\mathbf{e}^A = (2, r)$, $\mathbf{e}^B = (r, 2)$ and $r = 2^{8/9} - 2^{1/9}$
- Walrasian equilibrium $\frac{p_2}{p_1}^*$?

Finding WE: Example 2

- Preferences:

$$
\begin{aligned}
U^A(x_1^A, x_2^A) &= x_1^A - \frac{1}{8}(x_2^A)^{-8} \\
U^B(x_1^B, x_2^B) &= -\frac{1}{8}(x_1^B)^{-8} + x_2^B
\end{aligned}
$$

- Endowment: $\mathbf{e}^A = (2, r)$, $\mathbf{e}^B = (r, 2)$ and $r = 2^{8/9} - 2^{1/9}$
- Walrasian equilibrium $\frac{p_2}{p_1}^*$?
- With normalization $p_1 = 1$

$$r(p_2^* - 1) = p_2^{*8/9} - p_2^{*1/9}$$

Finding WE: Example 3

- Preferences:

$$
\begin{aligned}
U^A(x_1^A, x_2^A) &= lnx_1^A + x_2^A \\
U^B(x_1^B, x_2^B) &= \sqrt{x_1^B}
\end{aligned}
$$

- Endowment: $\mathbf{e}^A = (0, a)$ and $\mathbf{e}^B = (b, 0)$ with $a > 0$ and $b > 0$
- Walrasian equilibrium $\frac{p_2}{p_1}^*$?

Design a Question

- Preferences: U^A Cobb-Douglas, U^B quasi-linear in x_2
- Endowment: $\mathbf{e}^i \gg 0$ for $i = A, B$
- Find examples of $U^A(\cdot)$ and $U^B(\cdot)$ such that under the Walrasian equilibrium $\frac{p_2}{p_1}^*$ consumer B optimizes at a corner solution.

A Formal Presentation (1)

- Consumer's problem: $u^i(.)$ continuous, strongly increasing (or locally non-satiated preference) and strictly quasi-concave

$$max_{\mathbf{x}^i \in \Re^n_+} \ u^i(\mathbf{x}^i) \ s.t. \ \mathbf{p}'\mathbf{x}^i \leq \mathbf{p}'\mathbf{e}^i$$

  $\Rightarrow$ unique solution $\mathbf{x}^i(\mathbf{p}, \mathbf{p}'\mathbf{e}^i)$ with $\mathbf{p}'\mathbf{x}^i(\mathbf{p}, \mathbf{p}'\mathbf{e}^i) = \mathbf{p}'\mathbf{e}^i$ (Walras' Law)

- Excess demand

$$\begin{aligned} z_k(\mathbf{p}) &\equiv \sum_{i \in I} x_k^i(\mathbf{p}, \mathbf{p}'\mathbf{e}^i) - \sum_{i \in I} e_k^i \\ \mathbf{z}(\mathbf{p}) &= (z_1(\mathbf{p}), ..., z_n(\mathbf{p})) \end{aligned}$$

  Properties:

  - Continuous in $\mathbf{p}$
  - Homogeneous of degree 0 (HOD 0) in $\mathbf{p}$, i.e, $\mathbf{z}(c\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for $\forall c > 0$
  - Walras' Law $\mathbf{p}'\mathbf{z}(\mathbf{p}) = 0$
  - If $p_l \to 0$ for some l, then $z_l \to +\infty$ (under strongly increasing utility function)

A Formal Presentation (2)

- Walrasian equilibrium: $\mathbf{p}^* \gg 0$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$
  (or $\mathbf{p}^* \geq 0$ such that $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$)
- Homogeneity $\Rightarrow$ if $\mathbf{p}^*$ is an equilibrium price vector, $c\mathbf{p}^*$ is also an equilibrium price vector for $\forall c > 0$.
- Walras' Law $\Rightarrow$ if all $n-1$ markets are in equilibrium for some $\mathbf{p}$, then the nth market must be in equilibrium as well.

Existence of Walrasian Equilibrium

不会供小于求.

WE: 消费者 max u + Z(p*) = 0.
+ P.E.
市场出清
n-1个市场出清 ⇒ 第n个市场出清.

Suppose $\sum_{i=1}^{I} \mathbf{e}^i \gg 0$,

- If each consumer's utility function is
  - Continuous
  - Strongly increasing
  - Strictly quasi-concave

  there exists $\mathbf{p}^* \gg 0$ such that there is no excess demand for any product, i.e, $\mathbf{z}(\mathbf{p}^*) = 0$ $\Sigma x_j^i - \Sigma e_j^i$ j的超额需求.
- Alternatively, if each consumer's preference is
  - Continuous
  - Locally non-satiated
  - Strictly convex

  there exists $\mathbf{p}^* \geq 0$ such that there is no excess demand for any product, i.e, $\mathbf{z}(\mathbf{p}^*) \leq 0$. Since $\mathbf{p}'\mathbf{z}(\mathbf{p}) = 0$, this implies
  - $p_l^* > 0$ and $z_l(\mathbf{p}^*) = 0$ or
  - $p_l^* = 0$ and $z_l(\mathbf{p}^*) \leq 0$

Existence - Price Normalization

- $\mathbf{z}(\mathbf{p})$ being HOD 0
  $\Rightarrow$ Normalization of price does not affect excess demands
- Define relative prices

$$\tilde{p}_l = \frac{p_l}{\sum_l p_l} \quad \text{and} \quad \sum_l \tilde{p}_l = 1$$

- Price vectors belong to $L-1$ dimensional unit simplex (with L vertices)

$$S^{L-1} = \{\tilde{\mathbf{p}} \in \mathcal{R}^n_+ : \sum_l \tilde{p}_l = 1\}$$

- Can we find $\tilde{\mathbf{p}} \in S^{L-1}$ that would **result in no excess demand for any product**?
- *Intuition: imagine a function of $\tilde{\mathbf{p}}$ which captures the adjustment process (upward adjustment to reduce excess demand) and returns a new price vector as the value of the function.*
  Then an equilibrium price would be a **fixed point** *of such a function.*

Existence - Brouwer Fixed-point Theorem (FPT)

If $f : S^{L-1} \to S^{L-1}$ is a continuous function from the unit simplex to itself, there is some X in S^{L-1} such that $X = f(X)$

Proof: Scarf(1973: the computation of Economic Equation).

- A simple case of $L = 2$, the existence of a fixed point in the unit 1-dimensional simplex S^1 of function $f : [0,1] \to [0,1]$
- Consider $g(x) = f(x) - x$, a fixed point is an x^* such that $g(x^*) = 0$
  - $g(0) = f(0) - 0 \geq 0$ since $f(0) \in [0,1]$
  - $g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0,1]$
  - g is continuous $\Rightarrow \exists\ x^* \in [0,1]$ such that $g(x^*) = 0$, and thus $f(x^*) = x^*$ (Intermediate Value Theorem)

Existence - Price Adjustment Function

Define function $\mathbf{z}^+(.)$ on S^{L-1} as

$$z_l^+(\tilde{\mathbf{p}}) = max\{z_l(\tilde{\mathbf{p}}), 0\} \quad for \quad l = 1, ..., L$$

Define function $f : S^{L-1} \to S^{L-1}$ as

$$f_l(\tilde{\mathbf{p}}) = \frac{\tilde{p}_l + z_l^+(\tilde{\mathbf{p}})}{\alpha(\tilde{\mathbf{p}})} \quad \text{for} \quad l = 1, ..., L$$

where

$$\alpha(\tilde{\mathbf{p}}) = \sum_l (\tilde{p}_l + z_l^+(\tilde{\mathbf{p}}))$$

$f_l(\mathbf{p}) \geq 0$ and $\sum_l f_l(\mathbf{p}) = 1$, thus $f(\tilde{\mathbf{p}}) \in S^{L-1}$

Existence of WE - Fixed Point of f Function

By Brouwers' FPT, there exists $\tilde{\mathbf{p}}^* \in S^{L-1}$ such that $\tilde{\mathbf{p}}^* = f(\tilde{\mathbf{p}}^*)$

$$\tilde{p}_l^* = \frac{\tilde{p}_l^* + z_l^+(\tilde{\mathbf{p}}^*)}{1 + \sum_j z_j^+(\tilde{\mathbf{p}}^*)}$$

Thus

$$\begin{aligned}
z_l^+(\tilde{\mathbf{p}}^*) &= \tilde{p}_l^* \sum_j z_j^+(\tilde{\mathbf{p}}^*) \\
\sum_l z_l(\tilde{\mathbf{p}}^*) z_l^+(\tilde{\mathbf{p}}^*) &= \sum_l z_l(\tilde{\mathbf{p}}^*) \tilde{p}_l^* \sum_j z_j^+(\tilde{\mathbf{p}}^*) \\
\sum_l z_l(\tilde{\mathbf{p}}^*) z_l^+(\tilde{\mathbf{p}}^*) &= 0 \quad \text{(Walras' Law)}
\end{aligned}$$

Since $z_l(\tilde{\mathbf{p}}^*) z_l^+(\tilde{\mathbf{p}}^*) \geq 0$, we must have $z_l(\tilde{\mathbf{p}}^*) \leq 0$ for $l = 1, ..., L$.
Thus all markets clear.

Existence - Strongly Increasing Utility Function

So far, we have assumed the preference is locally non-satiated; thus it is possible that

- $p_l = 0$
- $z_l < 0$

If we assume each consumer's preference to be strongly increasing, then

- $p_l \to 0 \Rightarrow z_l \to \infty$

that is, the market of l won't clear if $p_l = 0$. In equilibrium, we must have $\mathbf{p}^* \gg 0$ and $\mathbf{z}(\mathbf{p}) = 0$

Efficiency

- Walrasian equilibrium allocation (WEA)

$$\mathbf{W}(\mathbf{e}) = \mathbf{x}(\mathbf{p}^*) = \left(\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), ..., \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1)\right) \subset F(\mathbf{e})$$

- $\mathbf{W}(\mathbf{e}) \subset C(\mathbf{e})$, thus Pareto efficient
  Otherwise there exists a coalition S to block WEA $\mathbf{x}$
  - There exists $\{\mathbf{y}^i\}_{i \in S}$ such that for $\forall i \in S$

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i) \quad \text{strict for some } i \text{ and } \sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

  - $\mathbf{x}^i$ maximizes i's utility under $\mathbf{p}^*$,
    thus $\mathbf{p}^{*\prime}\mathbf{y}^i \geq \mathbf{p}^{*\prime}\mathbf{x}^i = \mathbf{p}^{*\prime}\mathbf{e}^i$ with inequality strict for some i
  - But this implies $\mathbf{p}^{*\prime} \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^{*\prime} \sum_{i \in S} \mathbf{e}^i$
    which is in contradiction with $\mathbf{p}^{*\prime} \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^{*\prime} \sum_{i \in S} \mathbf{e}^i$

Welfare Theorems

福一

- First welfare theorem
  WEA $\Rightarrow$ Pareto efficiency
  - No distortion of price
  - No externalities
  - No asymmetric information
  - No market power

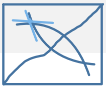
福二.

- Second welfare theorem

  Any Pareto efficient allocation can result from a WE (given that endowments can be redistributed in a lump sum way)

Pure Exchange Economy - Core vs. WEA

WEA and the Core



$MRS^1 > MRS^2$，在左上方.

$MRS^1 = \frac{P_1}{P_2}$, $MRS^2 = \frac{P_1}{P_2}$

$MRS^1 = MRS^2$

$x_1^1 + x_1^2 = e_1^1 + e_1^2$

$x_2^1 + x_2^2 = e_2^1 + e_2^2$.

WEA和价格之比相关.

$F(e) \supset P(e) \supset C(e) \supset W(e)$

包括原点!

C.C.有边界解的可能.

The set of WEAs is a subset of core allocations: $W(\mathbf{e}) \subset C(\mathbf{e})$

- When there are a large number of consumers, a competitive equilibrium is achieved only if no subgroup of consumers can make themselves better off by forming a coalition.
- In other words, increasing competition from a larger group of consumers reduces bargaining power by each individual.

Core Allocation and WEA

Question: what is the relationship between the outcome of a barter economy - allocations in the core - and the outcome of a competitive economy - the WEAs?

- ◇ In a pure exchange economy, the set of WEAs is a subset of the set of core allocations
- ◇ As the size of the economy grows, the set of core allocations will shrink and converge to the set of WEAs

Model Setup

- There are I types of consumers - consumers of the same type have the same endowment and preference
- Start with an economy in which there is only one consumer of each type, then duplicate the economy
- An r-fold replica economy ε_r: r consumers of each of the I types, which gives a total of rI consumers.
- Denote consumers of type i by iq where $q = 1, ..., r$

Edgeworth-Debreu-Scarf Limit Theorem on the Core:

If x $\in C_r$ for any $r = 1, 2, ...$, then x is a WEA for ε_1

Proof Outline

- Equal treatment of the same type in the core
  - Otherwise the "badly" treated agents can form a coalition to block the "unequal" allocation
  - We focus on allocations in an r-fold replica economy that are r-fold replica of the allocations in the core of the basic economy
- Some core allocations in the basic economy can be blocked in the replica economy
  - Thus the core shrinks as the size of the economy grows
- A WEA in an r-fold replica economy $\Leftrightarrow$ a replica of a WEA in the basic economy
  - Thus a WE remains a WE as the size of the economy grows
  - Thus the set of core allocations converge to the set of WEAs

Equal Treatment (of the same type) in Core

If $\mathbf{x}$ is an allocation in the core, then $\mathbf{x}^{iq} = \mathbf{x}^{iq'}$ for $\forall i$ and $\forall q, q' = 1, ..., r$.

- $I = 2$ and $r = 2$
- Allocation $(\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22})$ is in the core, so it is feasible and $\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$
- Suppose $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$ and $\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}$
- Coalition by consumer 12 and 22: allocation $\overline{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2}$ and $\overline{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2}$ is feasible and preferable.

Shrinking Core - $C_1 \supseteq C_2 \supseteq C_3 \ldots \supseteq C_r \ldots$ (1)

- Starting from a basic economy, the border point in the core $\tilde{\mathbf{x}}$ will not be in the core of the two-fold replica economy ε_2
- $\tilde{\mathbf{x}}$ will be blocked by coalition of two consumers of type 1 and one consumer of type 2 in ε_2
- Construct an alternative allocation among the three to make Pareto improvement
  - Type-1: linear combination of their endowment point $\mathbf{e}^1$ and $\tilde{\mathbf{x}}^1$
  - Quasi-concave preference: type-1 consumers prefer the linear combination to $\tilde{\mathbf{x}}^1$, the core allocation in ε_1
  - Type-2: still $\tilde{\mathbf{x}}^2$
  - The alternative allocation is feasible

$$\begin{aligned} 2\left(\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1\right) + \tilde{\mathbf{x}}^2 &= \mathbf{e}^1 + (\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2) \\ &= \mathbf{e}^1 + (\mathbf{e}^1 + \mathbf{e}^2) \end{aligned}$$

Shrinking Core - $C_1 \supseteq C_2 \supseteq C_3 ... \supseteq C_r ...$ (2)

Similarly, any non-WEA allocation in C_1 can be blocked by some coalition when r is large enough.

- Line connecting endowment E and $\tilde{\mathbf{x}}$, a non-WEA allocation in C_1, cuts through the indifference curve of some type, say i. Otherwise, it is a WEA.
- Some linear combination of the intersection point A and $\tilde{\mathbf{x}}$ is strictly better than $\tilde{\mathbf{x}}$ for i due to strict convexity of preference.
- With large enough r, $\frac{1}{r}\mathbf{e}^i + \frac{r-1}{r}\tilde{\mathbf{x}}^i$ will be on the segment between A and $\tilde{\mathbf{x}}$, thus strictly better than $\tilde{\mathbf{x}}^i$ for type i.
- Allocation of having r type i consuming $\frac{1}{r}\mathbf{e}^i + \frac{r-1}{r}\tilde{\mathbf{x}}^i$ and $r-1$ type j consuming $\tilde{\mathbf{x}}^j$ is feasible as total consumption sums to $r\mathbf{e}^i + (r-1)\mathbf{e}^j$
- Thus there exists a coalition of r individuals of type i and $r-1$ individuals of type j

WEA in ε_r vs. r-fold Replica of WEA in ε_1

- A WEA in an r-fold replica economy $\Rightarrow$ a replica of a WEA in the basic economy
  WEA in ε_r $\Rightarrow$ it is in the core $\Rightarrow$ equal treatment and thus a replica of an allocation in the basic economy $\Rightarrow$ it is a WEA in the basic economy as MRS is the same across consumers
- A replica of a WEA in the basic economy $\Rightarrow$ WEA allocation in an r-fold replica economy
  Keep the same set of prices in the basic economy - utility maximization and market clearing conditions are satisfied in the r-fold replica economy

GE with Production - Robinson Crusoe

Robinson Crusoe - Social Planner's Problem

RC经济.

- One final product: banana

自产自销.

- One production input: labour

$$b = f(l) = Al^{\beta} \text{ where } \beta \in (0, 1]$$

劳动力.

- Consume: banana and leisure

$$u(b, l) = b^{\alpha} l^{1-\alpha} \text{ where } \alpha \in (0, 1)$$

- Production frontier plays the role of budget line
- What is the social planner's problem?

$$max_{b,l}\ b^{\alpha}(L - l)^{1-\alpha}\ s.t.\ b = Al^{\beta}$$

Solution?

Robinson Crusoe - Social Planner's Problem

- One final product: banana
- One production input: labour

$$b = f(l) = Al^{\beta} \text{ where } \beta \in (0, 1]$$

- Consume: banana and leisure

$$u(b, l) = b^{\alpha} l^{1-\alpha} \text{ where } \alpha \in (0, 1)$$

- Production frontier plays the role of budget line
- What is the social planner's problem?

$$max_{b,l}\ b^{\alpha}(L - l)^{1-\alpha}\ s.t.\ b = Al^{\beta}$$

Solution?

$$l^* = \frac{\alpha\beta}{1 - \alpha + \alpha\beta} L$$

$$b^* = A\left[\frac{\alpha\beta}{1 - \alpha + \alpha\beta} L\right]^{\beta}$$

Robinson Crusoe - Competitive Equilibrium

- Now introduce an imaginary auctioneer announcing prices, and assume Mr. Crusoe the producer and Mr. Crusoe the consumer are both price takers
- Where is the competitive equilibrium?

$$
\begin{aligned}
producer : max_{b,l} \quad & pb - wl \ s.t. \ b = f(l) = Al^{\beta} \\
\Leftrightarrow max_{l} \quad & pAl^{\beta} - wl \\
consumer : max_{b,l} \quad & b^{\alpha}(L-l)^{1-\alpha} \ s.t. \ pb = wl + \pi(w,p)
\end{aligned}
$$

$$pb = wl + \pi(w,p) \Leftrightarrow pb + w(L-l) = wL + \pi(w,p)$$

Mr. Crusoe the Producer

$\underline{A^{\frac{\beta}{1-\beta}} \cdot A} = A^{\frac{\beta+1-\beta}{1-\beta}} = A^{\frac{1}{1-\beta}}.$

FOC for profit maximization $\underline{pA\beta l^{\beta-1} = w}$ 反解出I.

Solve for supply functions (labour demand and banana supply)

$b = AI^{\beta}$

$$l^d(w,p) = (A\beta)^{1/(1-\beta)} \left(\frac{p}{w}\right)^{1/(1-\beta)}$$
$$b^s(w,p) = (A)^{1/(1-\beta)} (\beta)^{\beta/(1-\beta)} \left(\frac{p}{w}\right)^{\beta/(1-\beta)}$$
$$\pi(w,p) = A^{\frac{1}{1-\beta}} \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}\right) w^{\frac{-\beta}{1-\beta}} p^{\frac{1}{1-\beta}}$$

Mr. Crusoe the Consumer

- Budget $m(w, p) = wL + \pi(w, p)$
- Solution to utility maximization with Cobb-Douglas utility function

$$
\begin{aligned}
b^d(w, p) &= \frac{\alpha(wL + \pi(w, p))}{p} \\
l^s(w, p) &= L - \frac{(1-\alpha)(wL + \pi(w, p))}{w}
\end{aligned}
$$

Market Clearing Conditions

人的结果 market clear.

$$
\begin{aligned}
b^s(w,p) &= b^d(w,p) \\
l^s(w,p) &= l^d(w,p)
\end{aligned}
$$

producer: k^*, y^*, π^*
consumer: $\hat{y}, \hat{l}$

$$\Rightarrow \begin{cases} h^* + \hat{l} = T \\ y^* = \hat{y} \end{cases}$$

Thus

$$
\begin{aligned}
(A)^{1/(1-\beta)} (\beta)^{\beta/(1-\beta)} \left(\frac{p}{w}\right)^{\beta/(1-\beta)} &= \frac{\alpha(wL + \pi(w,p))}{p} \\
(A\beta)^{1/(1-\beta)} \left(\frac{p}{w}\right)^{1/(1-\beta)} &= L - \frac{(1-\alpha)(wL + \pi(w,p))}{w} \\
\textit{where } \pi(w,p) &= A^{\frac{1}{1-\beta}} \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}\right) w^{\frac{-\beta}{1-\beta}} p^{\frac{1}{1-\beta}}
\end{aligned}
$$

Solve for $\frac{w}{p}$.

Market Clearing Conditions

$$
\begin{aligned}
b^s(w,p) &= b^d(w,p) \\
l^s(w,p) &= l^d(w,p)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(A)^{1/(1-\beta)}(\beta)^{\beta/(1-\beta)}\left(\frac{p}{w}\right)^{\beta/(1-\beta)} &= \frac{\alpha(wL+\pi(w,p))}{p} \\
(A\beta)^{1/(1-\beta)}\left(\frac{p}{w}\right)^{1/(1-\beta)} &= L-\frac{(1-\alpha)(wL+\pi(w,p))}{w} \\
\textit{where } \pi(w,p) &= A^{\frac{1}{1-\beta}}\left(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}}\right)w^{\frac{-\beta}{1-\beta}}p^{\frac{1}{1-\beta}}
\end{aligned}
$$

Solve for $\frac{w}{p}$.

$$
\begin{aligned}
\left(\frac{w}{p}\right)^* &= A\beta^\beta\left(\frac{1-\alpha+\alpha\beta}{\alpha L}\right)^{1-\beta} \\
l^{d*}=l^{s*} &= \frac{\alpha\beta}{1-\alpha+\alpha\beta}L
\end{aligned}
$$

Social Planner's Solution vs. WE

Both involve the tangency condition

$$\frac{\Delta b}{\Delta l} = MP_l^b = \frac{MU_l}{MU_b}$$

Interpretation:

- Social planner:
  Utility from the last unit of leisure consumption MU_l the same as the utility from consuming the amount of banana that can be produced with the same amount of time $MP_l^b * MU_b$
- Competitive equilibrium:
  - Producer profit maximization $MP_l^b = \frac{w}{p}$
  - Consumer utility maximization $\frac{MU_l}{MU_b} = \frac{w}{p}$

Equilibrium Analysis - Change in A Productivity

Comparative statics

$$\left(\frac{w}{p}\right)^* \Uparrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \text{ unaffected}$$

Equilibrium Analysis - Change in A Productivity

Comparative statics

$$\left(\frac{w}{p}\right)^* \Uparrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \text{ unaffected}$$

Adjustment process

1. Direct impact on firms - productivity effect
   - l^d shifts out, $w \Uparrow$
   - b^s shifts out, $p \Downarrow$

Equilibrium Analysis - Change in A Productivity

Comparative statics

$$\left(\frac{w}{p}\right)^* \Uparrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \text{ unaffected}$$

Adjustment process

1. Direct impact on firms - productivity effect
   - l^d shifts out, $w \Uparrow$
   - b^s shifts out, $p \Downarrow$
2. Consumers respond to the price and profit changes
   - Profits $\Uparrow$
   - b^d shifts out
     - with both demand and supply shifting out, $b^* \Uparrow$
   - l^s shifts up
     - completely offset the quantity change due to the shift of l^d
     - with demand out and supply up, $w^* \Uparrow$

Equilibrium Analysis - Change in L Endowment

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Downarrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow \text{ but not as much as } L \Uparrow$$

Equilibrium Analysis - Change in L Endowment

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Downarrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow \text{ but not as much as } L \Uparrow$$

Adjustment process

1. Direct impact on consumers - endowment effect
   - l^s shifts out, $w \Downarrow$
   - b^d shifts out, $p \Uparrow$

Equilibrium Analysis - Change in L Endowment

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Downarrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow \text{ but not as much as } L \Uparrow$$

Adjustment process

1. Direct impact on consumers - endowment effect
   - l^s shifts out, $w \Downarrow$
   - b^d shifts out, $p \Uparrow$
2. Firms respond to price changes
   - b^s shifts out
     - with both demand and supply shifting out, $b^* \Uparrow$
   - l^d shifts out
     - with both demand and supply shifting out, $l^* \Uparrow$

Equilibrium Analysis - Change in α Preference

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow$$

Equilibrium Analysis - Change in α Preference

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow$$

Adjustment process

1. Direct impact on consumers
   - l^s shifts out, $w \Downarrow$
   - b^d shifts out, $p \Uparrow$

Equilibrium Analysis - Change in α Preference

Comparative statics

$$\left(\frac{w}{p}\right)^* \Downarrow, \frac{b^*}{L - l^{s*}} \Uparrow, b^* \Uparrow, l^{s*} = l^{d*} \Uparrow$$

Adjustment process

1. Direct impact on consumers
   - l^s shifts out, $w \Downarrow$
   - b^d shifts out, $p \Uparrow$
2. Firms respond to price changes
   - b^s shifts out
     - with both demand and supply shifting out, $b^* \Uparrow$
   - l^d shifts out
     - with both demand and supply shifting out, $l^* \Uparrow$

Impact of a Distorting Income Tax

- The firm pays w and the worker receives $(1 - t)w$ per hour
- A government takes the difference as tax revenue
- The same government gives lump sum transfer to consumers
- The government keeps fiscal balance

Questions

1. Where is the competitive equilibrium?
2. What is socially optimal t^*?

GE with Production - Small Open Economy

An Introduction to the Small Open Economy Setting

- Open:
  - Free trade in products:
    – consumers buy from/producers sell to domestic and foreign markets
  - Balance of payments:
    – but no need to each domestic good market
  - Nontradable/immobile primary production factors:
    – clear primary input markets in equilibrium
- Small:
  - consumers and producers take international product prices as given
- We are effectively solving an efficient production problem.
- Study the $2 \times 2 \times 2$ model in detail.

GE Model with Production and Multiple Inputs

- There are J firms
- There are L inputs
- The total endowment of the economy is $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L)$
- Firm j produces q^j with production function $f^j(\mathbf{z}^j)$ where
  - $\mathbf{z}^j = (z_1^j, ..., z_L^j)$
  - $\mathbf{f}(.)$ is concave, strictly increasing and differentiable
- Product prices are $\mathbf{p} = (p_1, ..., p_J)$
- Factor prices are denoted by $\mathbf{w} = (w_1, ..., w_L)$
- Small open economy with immobile inputs - taking output prices as given; input prices will be endogenously determined

Firm j's Problem

$$max_{\mathbf{z}^j \geq 0} \; p^j f^j(\mathbf{z}^j) - \mathbf{w} \cdot \mathbf{z}^j$$

- FOC

$$p^j \frac{\partial f^j(\mathbf{z}^j)}{\partial z_l^j} = w_l \; for \; \forall j \text{ and } \forall l$$

- Factor supply: $\bar{\mathbf{z}} = (\bar{z}_1, ..., \bar{z}_L)$
- Market clearing condition

$$\sum_{j \in J} \mathbf{z}^j(\mathbf{w}, \mathbf{p}) = \bar{\mathbf{z}}$$

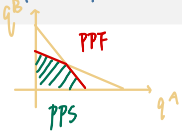
- All together $L(J+1)$ conditions
- Efficiency of the outcome allocation

A 2X2 example: $J = L = 2$

生产一单位需要的要素

$$\begin{cases} q^A a_1^A + q^B a_1^B \le \bar{a}_1 \\ q^A a_2^A + q^B a_2^B \le \bar{a}_2 \end{cases}$$

要素供给量



产品 q^A q^B 要素 a_1^A a_1^B a_2^A a_2^B

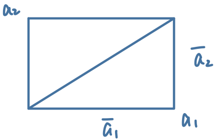
production possiblity set PPS

...... frontier PPF

上标公司.

- Suppose the technology is of constant return to scale - $q^i = f^i(K^i, L^i)$, $i = 1, 2$, is homogeneous of degree one
- Edgeworth box with iso-quants - Pareto efficient points?
- Homothetic production function $\Rightarrow$
  - The Pareto set and the diagonal - no cuts
  - Factor intensities by the two firms along the Pareto set
  - Single intersection of a ray from the origin and the Pareto set
  - Monotone change of factor intensity and factor price ratio along the Pareto set

$\frac{a_1^A}{a_2^A} > \frac{a_1^B}{a_2^B}$ A 偏向于 a_1，是a_1-intensive.



Firms' Problem

- Production functions are homogeneous of degree one $\Rightarrow$ focus on unit input employment
- Use $c^i(r, w)$ for the unit cost and $\left(k^i(r, w), l^i(r, w)\right)$ for the unit input combination by firm i.
- They are solutions to the following cost minimization problem

$$\begin{aligned} c^i(r, w) &\equiv min_{k,l} \quad rk + wl \quad s.t.\ f^i(k, l) \geq 1 \\ &= r \times k^{i*}(r, w) + w \times l^{i*}(r, w) \end{aligned}$$

Equilibrium

- Equilibrium: w, r, q^1, q^2
- Equilibrium conditions
  - Zero profits

$$\begin{aligned} c^1(r,w) &= p^1 \\ c^2(r,w) &= p^2 \end{aligned}$$

  - Factor markets clear (full employment of factor)

$$\begin{aligned} k^{1*}(r,w)q^1 + k^{2*}(r,w)q^2 &= \overline{K} \\ l^{1*}(r,w)q^1 + l^{2*}(r,w)q^2 &= \overline{L} \end{aligned}$$

  - Four equations and four unknowns (r, w, q^1, q^2)

Finding Equilibrium - Factor Prices

When can we solve for factor prices solely from zero profit conditions?

$$\begin{aligned} c^1(r,w) &= p^1 \\ c^2(r,w) &= p^2 \end{aligned}$$

- Unit-value curve (unit-quant curve)
- **NO** factor intensity reversals (FIR): the production of one product is **always** (for any factor prices) more capital intensive than the other product. We assume product 2 is more capital intensive
- Single intersection of the unit-value curves
- The equilibrium unit factor employment $\left(k^{1*}, l^{1*}\right)$ and $\left(k^{2*}, l^{2*}\right)$ are also determined

Factor Prices Summary

$k^{i'}$ and $l^{i'}$ represent factor requirement for producing output of one dollar in value

$$
\begin{aligned}
f^1(k^{1'}, l^{1'}) &= \frac{1}{p^1} && (1)\\
f^2(k^{2'}, l^{2'}) &= \frac{1}{p^2} && (2)\\
rk^{1'} + wl^{1'} &= 1 && (3)\\
rk^{2'} + wl^{2'} &= 1 && (4)\\
MRTS^1(k^{1'}, l^{1'}) &= \frac{w}{r} && (5)\\
MRTS^2(k^{2'}, l^{2'}) &= \frac{w}{r} && (6)
\end{aligned}
$$

$\Rightarrow k^{1'}, l^{1'}, k^{2'}, l^{2'}, r, w$ as functions of p^1 and p^2

Finding Equilibrium - Full Employment of Factors

Solve for the two unknowns from

$$\begin{aligned} p^1 k^{1'} q^1 + p^2 k^{2'} q^2 &= \overline{K} \\ p^1 l^{1'} q^1 + p^2 l^{2'} q^2 &= \overline{L} \end{aligned}$$

In vector form

$$p^1 q^1 (k^{1'}, l^{1'})^T + p^2 q^2 (k^{2'}, l^{2'})^T = (\overline{K}, \overline{L})^T$$

1. Cone of diversification: factor price equalization
   Free trade in good $\Rightarrow$ factor prices independent of endowment & equalized across trading partners
2. Specialization: pricing factors according to MPVs
3. Factor intensity reversal: multiple cones of diversification

Production Possibility Frontier - Leontief Technology

Suppose there are 3000 units of K and 2000 units of L

- The production technologies are as following

$$Q^m = min(\frac{1}{3}K^m, L^m) \text{ and } Q^t = min(\frac{1}{2}K^t, \frac{1}{2}L^t)$$

Translate into factor requirement

$$a_K^m = 3;\ a_L^m = 1;\ a_K^t = 2;\ a_L^t = 2$$

What is the implied PPF?

$$\begin{aligned} 3Q^m + 2Q^t &\leq 3000 \\ Q^m + 2Q^t &\leq 2000 \end{aligned}$$

The set is convex and PPF is concave to the origin.

- What if the technologies show perfect substitution?

$$Q^m = 2K^m + L^m \text{ and } Q^t = K^t + L^t$$

Production Possibility Frontier - CRS Technology (1)

In general, with intermediate degree of substitutability or complementarity between factors

- Take two points within the PPF, to show that the set is convex we only need to show a combination of the two points is within the set.
- The two points represent two output profiles; behind each output profile is some specific combinations of inputs.
- Let's see what will happen if we reorganize the inputs...

PPF under CRS (2)

- Let $Q^{iA} = f^i(K^{iA}, L^{iA})$ and $Q^{iB} = f^i(K^{iB}, L^{iB})$, $i = t, m$
- Make a linear combination of the factors used in producing (Q^{tA}, Q^{mA}) and (Q^{tB}, Q^{mB})
- With the linear combination of factors, one can produce $f^t(\lambda K^{tA} + (1-\lambda)K^{tB}, \lambda L^{tA} + (1-\lambda)L^{tB})$ and $f^m(\lambda K^{mA} + (1-\lambda)K^{mB}, \lambda L^{mA} + (1-\lambda)L^{mB})$
- Since the production function is concave, this is no less than $\lambda f^i(K^{iA}, L^{iA}) + (1-\lambda)f^i(K^{iB}, L^{iB})$, for $i = t, m$.
- That is, any point in between A and B is in the production possibility set.
- The set is convex and PPF is concave to origin.

Production Possibility Frontier

The production possibility frontier is concave to the origin

$$q^2 = g(q^1, K, L)$$

$f^i(K^i, L^i)$ being strictly increasing and concave $\Rightarrow$

$$\begin{aligned} \frac{\partial q^2}{\partial q^1} &< 0 \\ \frac{\partial^2 q^2}{\partial {q^1}^2} &< 0 \end{aligned}$$

Interpretation:
The opportunity cost of product 1 (in terms of q^2) increases with q^1; in other words, one needs to sacrifice more and more units of 2 to get one additional unit of 1.

Efficiency - Revenue Maximization

- In equilibrium
$$\frac{p^1}{p^2} = \frac{c^1}{c^2} = \frac{MP_L^2}{MP_L^1} = \frac{MP_K^2}{MP_K^1}$$
- This is the tangency condition between iso-revenue line and the PPF

Comparative Statics 1 - Change in Product Prices

- $p^1 \uparrow \Rightarrow$ IV1 shifts to the left
- $p^1 \uparrow \Rightarrow$ IC1 shifts to the right
- $w \uparrow$ and $r \downarrow \Rightarrow \frac{w}{r} \uparrow$
- $\frac{w}{p^1}$?

Comparative Statics 1 - Change in Product Prices

- $p^1 \uparrow \Rightarrow$ IV1 shifts to the left
- $p^1 \uparrow \Rightarrow$ IC1 shifts to the right
- $w \uparrow$ and $r \downarrow \Rightarrow \frac{w}{r} \uparrow$
- $\frac{w}{p^1}$?

  $\uparrow$ *as well - magnification effect*

Stolper-Samuelson Theorem (1)

An increase in the relative price of a good will increase the real return to the factor used intensively in that good, and reduce the real return to the other factor.

$$p^i = c^i(r, w)$$

Take total derivative w.r.t. r and w

$$dp^i = k^i dr + l^i dw$$

Use $\widehat{x}$ for percentage change in variable x and θ_{fi} for the share of the cost of factor f by firm i,

$$\widehat{p}^i = \theta_{ki}\widehat{r} + \theta_{li}\widehat{w}$$

Solve for $\widehat{w}$ and $\widehat{r}$ as functions of $\widehat{p}^1$ and $\widehat{p}^2$

Stolper-Samuelson Theorem (2)

Suppose $\widehat{p}^1 > \widehat{p}^2$

$$\begin{aligned}
\widehat{w} &= \frac{(\theta_{k2}-\theta_{k1})\widehat{p}^1 + \theta_{k1}(\widehat{p}^1 - \widehat{p}^2)}{\theta_{k2}-\theta_{k1}} > \widehat{p}^1 \\
\widehat{r} &= \frac{(\theta_{l2}-\theta_{l1})\widehat{p}^2 + \theta_{l2}(\widehat{p}^1 - \widehat{p}^2)}{\theta_{l2}-\theta_{l1}} < \widehat{p}^2
\end{aligned}$$

- Wage increases by more than the price of the labour intensive good
- $\widehat{w} > \widehat{p}^1 > \widehat{p}^2$: real wage increases $\Rightarrow$ workers are better off
- $\widehat{p}^1 > \widehat{p}^2 > \widehat{r}$: interest rate decreases $\Rightarrow$ capital owners are worse off

Comparative Statics 2 - Change in Factor Endowment

- Change in the size of the Edgeworth box
- The new allocation point? Assuming
    - No factor intensity reversal
    - Both endowment points are in the cone of diversification
    - What is going to happen to factor prices?
- Given the factor prices, how would the factor allocation adjust?

Rybczynski Theorem (1)

An increase in a factor endowment will increase output of the industry using it intensively and decrease the output of the other industry

$$\begin{aligned} k^{1*}q^1 + k^{2*}q^2 &= \overline{K} \\ l^{1*}q^1 + l^{2*}q^2 &= \overline{L} \end{aligned}$$

Thus

$$\begin{aligned} k^{1*}dq^1 + k^{2*}dq^2 &= d\overline{K} \\ l^{1*}dq^1 + l^{2*}dq^2 &= d\overline{L} \end{aligned}$$

Rybczynski Theorem (2)

Use $\widehat{x}$ for percentage change in variable x and use λ_{fi} for the share of factor f employed in sector i

$$\begin{aligned} \lambda_{k1}\widehat{q}^1 + \lambda_{k2}\widehat{q}^2 &= \widehat{\overline{K}} \\ \lambda_{l1}\widehat{q}^1 + \lambda_{l2}\widehat{q}^2 &= \widehat{\overline{L}} \end{aligned}$$

Solve for q^1 and q^2 as functions of $\widehat{\overline{L}}$ and $\widehat{\overline{K}}$

$$\begin{aligned} \widehat{q}^1 &= \frac{\lambda_{k2}\widehat{\overline{L}} - \lambda_{l2}\widehat{\overline{K}}}{\lambda_{k2} - \lambda_{l2}} \\ \widehat{q}^2 &= \frac{\lambda_{l1}\widehat{\overline{K}} - \lambda_{k1}\widehat{\overline{L}}}{\lambda_{l1} - \lambda_{k1}} \end{aligned}$$

Rybczynski Theorem (3)

Note that $\frac{k^{2*}}{l^{2*}} > \frac{\overline{K}}{\overline{L}} > \frac{k^{1*}}{l^{1*}}$.
Therefore $\lambda_{k1} < \lambda_{l1}$ and $\lambda_{k2} > \lambda_{l2}$.
With $\widehat{\overline{L}} > 0$ and $\widehat{\overline{K}} = 0$, we have

$$\begin{aligned} \widehat{q}^1 &= \frac{\lambda_{k2}}{\lambda_{k2} - \lambda_{l2}} \widehat{\overline{L}} > \widehat{\overline{L}} > 0 \\ \widehat{q}^2 &= -\frac{\lambda_{k1}}{\lambda_{l1} - \lambda_{k1}} \widehat{\overline{L}} < 0 \end{aligned}$$

PPF and Rybczynski Line

- Change in PPF associated with an increase in labour endowment
- Rybczynski line for labour
    - No change in factor prices
    - No change in unit factor demand
    - Capital endowment is constant

$$\begin{aligned} & k^{1*}q^1 + k^{2*}q^2 = \overline{K} \\ \Rightarrow \quad & k^{1*}\Delta q^1 + k^{2*}\Delta q^2 = 0 \\ \Rightarrow \quad & \frac{\Delta q^2}{\Delta q^1} = -\frac{k^{1*}}{k^{2*}} \end{aligned}$$

- By the same logic, Rybczynski line for capital $\frac{\Delta q^2}{\Delta q^1} = -\frac{l^{1*}}{l^{2*}}$
- Furthermore, $\frac{l^{1*}}{l^{2*}} > \frac{p^1}{p^2} > \frac{k^{1*}}{k^{2*}}$

GE with Production - Multiple Inputs and Outputs

GE with Production - Assumptions in Firms

- There are fixed number of commodities, n
- There are fixed number of firms, J
- Firms are owned by consumers
  - represented by one representative consumer
- $\mathbf{y}^j$, an n-dimension vector, is a production plan of firm j (negative numbers represent inputs and positive numbers represent outputs)
- $\mathbf{Y}^j$ is the set of production plans of firm j

General Properties of Production Sets (1)

- Non-empty: the firm has something to do
- Closed: all boundary points included; the limit of a sequence of technologically feasible plans is also feasible.
- No free lunch: impossible to produce something with nothing
  - $\mathbf{Y} \cap \Re^n{}_+ \subset \{\mathbf{0}\}$
  - when $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{y} \geq \mathbf{0}$,then $\mathbf{y} = \mathbf{0}$
- Possibility of inaction: complete shutdown is possible
  - $\mathbf{0} \subset \{\mathbf{Y}\}$
  - What if there is sunk investment? Fixed quantity or committed minimum?

General Properties of Production Sets (2)

- Free disposal: extra amounts of inputs (outputs) can be disposed of or eliminated at no cost
  - If $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{y}' \leq \mathbf{y}$,then $\mathbf{y}' \subset \mathbf{Y}$
  - $\mathbf{Y} - \Re^{n}{}_{+} \subset \{\mathbf{Y}\}$
- Irreversibility: impossible to reverse a technologically possible production vector
  - $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{y} \neq \mathbf{0}$, then $-\mathbf{y} \notin \mathbf{Y}$

General Properties of Production Sets (3)

Returns to scale

- Nonincreasing returns to scale: any feasible input-output vector can be scaled down
  - For any $\mathbf{y} \in \mathbf{Y}$, $\alpha\mathbf{y} \in \mathbf{Y}$ for all scalars $\alpha \in [0, 1]$
- Nondecreasing returns to scale: any feasible input-output vector can be scaled up
  - For any $\mathbf{y} \in \mathbf{Y}$, $\alpha\mathbf{y} \in \mathbf{Y}$ for all scalars $\alpha \geq 1$
- Constant returns to scale: any feasible input-output vector can be scaled up and down
  - For any $\mathbf{y} \in \mathbf{Y}$, $\alpha\mathbf{y} \in \mathbf{Y}$ for all scalars $\alpha \geq 0$

General Properties of Production Sets (4)

- Additivity: multiple plants that do not interfere with each other; free entry
  - $\mathbf{Y} + \mathbf{Y} \subset \mathbf{Y}$
  - $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$, then $\mathbf{y} + \mathbf{y}' \in \mathbf{Y}$
- Convexity: better to have more balanced input combinations
  - $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$ and $\alpha \in [0, 1]$, then $\alpha\mathbf{y} + (1 - \alpha)\mathbf{y}' \in \mathbf{Y}$
  - Together with Inaction implies nonincreasing returns to scale
- $\mathbf{Y}$ a convex cone: convexity + constant returns to scale
  - $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$ and $\alpha \geq 0, \beta \geq 0$, then $\alpha\mathbf{y} + \beta\mathbf{y}' \in \mathbf{Y}$

GE with Production - Firm Assumption and Behavior

For each firm j

- $\mathbf{0} \in \mathbf{Y}^j \subseteq \Re^n$
- $\mathbf{Y}^j$ is closed and bounded
- $\mathbf{Y}^j$ is strongly convex
  - $\forall \mathbf{y}, \mathbf{y}' \in \mathbf{Y}$ and $\forall \alpha \in (0, 1)$, there exists $\bar{\mathbf{y}} \in \mathbf{Y}$ such that $\bar{\mathbf{y}} \geq \alpha \mathbf{y} + (1 - \alpha)\mathbf{y}'$ and equality does not hold

Profit maximization given $\mathbf{p} \geq \mathbf{0}$

$$\begin{aligned} & max_{\mathbf{y}^j \in \mathbf{Y}^j} \ \mathbf{p}'\mathbf{y}^j \\ \Rightarrow \quad & \pi^j(\mathbf{p}) \equiv max_{\mathbf{y}^j \in \mathbf{Y}^j} \ \mathbf{p}'\mathbf{y}^j \end{aligned}$$

Solution $y^j(\mathbf{p})$ exists; it is unique and continuous when $\mathbf{p} \gg \mathbf{0}$.

Firm Behavior

- Aggregate supply $\mathbf{Y} \equiv \left\{ \mathbf{y} | \mathbf{y} = \sum_{j \in J} \mathbf{y}^j,\ where\ \mathbf{y}^j \in \mathbf{Y}^j \right\}$.
- $\mathbf{Y}$ has all the properties of $\mathbf{Y}^j$
  The aggregate production set is simply the aggregation of individual firms' production sets.
- Aggregate profit maximization

$$
\begin{aligned}
& max_{\overline{\mathbf{y}} \in \mathbf{Y}}\ \mathbf{p}'\overline{\mathbf{y}} \\
\Rightarrow \qquad & \pi(\mathbf{p}) \equiv max_{\overline{\mathbf{y}} \in \mathbf{Y}}\ \mathbf{p}'\overline{\mathbf{y}}
\end{aligned}
$$

  Solution $\overline{y}$ maximizes aggregate profit. That is, for $\mathbf{p} \geq \mathbf{0}$, there exists $\mathbf{y}^j$ such that
  - $\sum_{j \in J} \mathbf{y}^j = \overline{y}$
  - $\mathbf{y}^j$ maximizes the profit of firm j

Consumer Behavior (1)

- There are I consumers
- Each consumer's utility function u^i is continuous, strongly increasing, and strictly quasi-concave on $\Re^n_+$
- The distribution of firm profits $\theta^{ij} \in [0, 1]$ is consumer i's proportion of profits by firm j.

$$\sum_{i \in I} \theta^{ij} = 1$$

- Two sources of income: selling endowment and receiving firms' profits

$$\mathbf{p}'\mathbf{x}^i \leq \mathbf{p}'\mathbf{e}^i + \sum_{j \in J} \theta^{ij} \pi^j(\mathbf{p}) = m^i(\mathbf{p})$$

Consumer Behavior (2)

Utility maximization

$$
\begin{aligned}
& max_{\mathbf{x}^i \in \Re^n_+} \; u^i(\mathbf{x}^i) \; s.t \; \mathbf{p}'\mathbf{x}^i \leq m^i(\mathbf{p}) \\
\Rightarrow \quad & \mathbf{x}^i \left(\mathbf{p}, m^i(\mathbf{p})\right)
\end{aligned}
$$

Given the assumptions on consumer preferences and production sets, thus $m^i(\mathbf{p})$, a unique and continuous solution exits for $\mathbf{p} \gg \mathbf{0}$.

Aggregation and Walras' Law

- The economy can be described as $(u^i, e^i, \theta^{ij}, \mathbf{Y}^j)_{i \in I, j \in J}$
- Excess demand for commodity k

$$z_k(\mathbf{p}) = \sum_{i \in I} \mathbf{x}_k^i \left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{j \in J} \mathbf{y}_k^j(\mathbf{p}) - \sum_{i \in I} e_k^i$$

$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), ..., z_n(\mathbf{p}))$

- Walras' Law holds, i.e, $\mathbf{p}'\mathbf{z}(\mathbf{p}) = 0$

$$\begin{aligned}
\mathbf{p}'\mathbf{z}(\mathbf{p}) &= \mathbf{p}' \left(\sum_{i \in I} \mathbf{x}^i \left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{j \in J} \mathbf{y}^j(\mathbf{p}) - \sum_{i \in I} \mathbf{e}^i \right) \\
&= \sum_{i \in I} \left(\mathbf{p}'\mathbf{e}^i + \sum_{j \in J} \theta^{ij} \pi^j(\mathbf{p}) \right) - \mathbf{p}' \sum_{j \in J} \mathbf{y}^j(\mathbf{p}) - \sum_{i \in I} \mathbf{p}'\mathbf{e}^i \\
&= \sum_{j \in J} \left(\sum_{i \in I} \theta^{ij} \pi^j(\mathbf{p}) - \mathbf{p}'\mathbf{y}^j(\mathbf{p}) \right) = 0
\end{aligned}$$

Existence of Equilibrium

Given the restrictions on preference and production technology

- ◇ Each consumer's utility function is continuous, strongly increasing and strictly quasi-concave
- ◇ Each firm's production set $\mathbf{Y}^j$ is closed, bounded and strongly convex. Inaction is allowed. The aggregate production set shares all these
- ◇ There exists some aggregate production vector $\mathbf{y} \in \sum_{j \in J} \mathbf{Y}^j$ such that $\mathbf{y} + \sum_{i \in I} \mathbf{e}^i \gg 0$

Then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$

Welfare Properties

- First welfare theorem: Walrasian equilibrium $(\mathbf{x}, \mathbf{y}, \mathbf{p}) \Rightarrow (\mathbf{x}, \mathbf{y})$ is Pareto efficient
- Second welfare theorem: For any Pareto efficient allocation $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ there are income transfers $T_1, ..., T_I$ with $\sum_{i \in I} T_i = 0$, and price vector $\overline{\mathbf{p}}$ such that
  - $\widehat{\mathbf{x}^i}$ maximizes consumer i's utility, given budget $\overline{\mathbf{p}}\mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i$
  - $\widehat{\mathbf{y}^j}$ maximizes firm j's profit