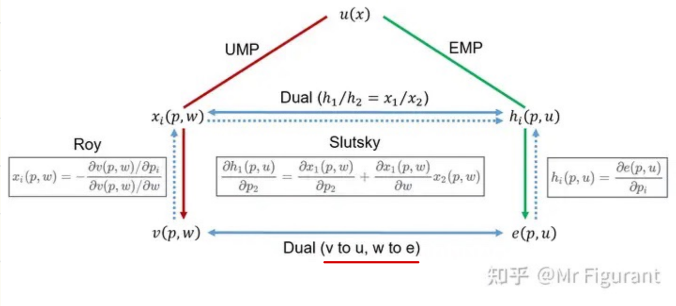
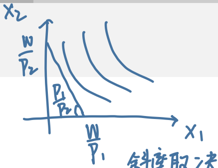
$$\begin{tikzcd}
\text{UMP} \arrow[dd, "\text{分类讨论}"'] & & & \text{Exp} \arrow[dd] & \\
& v(p,w) \arrow[dl, shift left, "\text{Roy}\ -\frac{\partial v/\partial p_i}{\partial v/\partial w}"] \arrow[rrr, leftrightarrow, bend left, "\text{对偶}", "\text{互相替换}"'] & & & e(p,u) \arrow[dl, shift left, "\text{Shephard}\ \frac{\partial e}{\partial p_i}=h_i"] \\
x(p,w) \arrow[ur, shift left, "\text{代入}u"] & & & h(p,u) \arrow[ur, shift left, "\text{代入}p^Tx"] &
\end{tikzcd}$$

包络的结果，把v换成λ就好理解了



Consumer's Problem



$MU_1 \Delta X_1 + MU_2 \cdot \Delta X_2 = 0.$

$\left|\frac{\Delta X_2}{\Delta X_1}\right| = \frac{MU_1}{MU_2} = \frac{P_1}{P_2}.$

斜率取决于商品的类别

$u \to x^* \to V.$

来回推

- Utility function: $U(x_1, x_2)$
  - *indifference curves*
- Budget constraint: $p_1x_1 + p_2x_2 = w$
  - *budget line*
- Utility maximization: $max_{x_1,x_2}\ U(x_1, x_2)$ subject to $p_1x_1 + p_2x_2 = w$
  - *tangency condition*

  get demand functions: $x_1^*(p_1, p_2, w)$, $x_2^*(p_1, p_2, w)$
- Comparative static analysis
  - Price change $\frac{\partial x_i^*}{\partial p_j}, i = 1, 2, j = 1, 2$
    - *price consumption curve/demand curve*
  - Income elasticity of demand $\frac{\partial x_i^*}{\partial w}, i = 1, 2$
    - *income consumption curve/Engel curve*
- Measuring the welfare impact of a price change: *EV* or *CV*
  - *measuring distance between indifference curves*

Moving Forward

- Why utility function (indifference curve) is the way we have assumed? (What assumptions do economists impose on human behaviour?)
- More rigorous investigation of properties of important functions
- Extensions of the standard framework
- Applications in research articles

Consumer Theory Basics Roadmap

- Notations
- Understanding preference
- From preference to utility function
- UMP - Utility Maximization Problem
- EMP - Expenditure Minimization Problem
- Duality - connection between UMP & EMP
- Welfare analysis using the UMP & EMP framework

Set Stage

- Individual consumer: $n = 1, ..., N$
- Commodity: $\ell = 1, ..., L$
- Consumption bundle:
  - $\mathbf{x} = (x_1, ..., x_L)$, $x_\ell \geq 0$ for $\forall \ell = 1, ..., L$
  - consumption bundles $\mathbf{x}^1, \mathbf{x}^2, ...$ 人 or 组合.
  - or for consumer n, $\mathbf{x}^n = (x_1^n, ..., x_L^n)$
- Consumption/choice set: $\mathbf{X} \subseteq R_+^L$
  - closed
  - convex
  - $\mathbf{0} \in \mathbf{X}$

Set Stage

- Price: $\mathbf{p} = (p_1, ..., p_L)$, with $p_\ell > 0, \forall \ell = 1, ..., L$ or $\mathbf{p} >> 0$
- Wealth to spend:
  - $w > 0$
  - For consumer n, w^n
- Budget/feasible set:
  - $B_{p,w} = \left\{ \mathbf{x} \in R^L_+ : \underline{\mathbf{px} \leq w} \right\}$;
  - For consumer n: $\underline{B^n_{p,w} = \left\{ \mathbf{x}^n \in R^L_+ : \mathbf{px}^n \leq w^n \right\}}$

Preference as a Binary Relation

- Orderings of alternatives in the choice set **X** can be thought of as binary relations
- A **preference relation** $\succsim$, is a special binary relation meaning ”at least as good as”; the opposite equivalence $\precsim$ means ”no better than”.
- **Rational preference relations** (making choice in a consistent way) has two properties

  没有选择困难.

  - Complete: for any pair $(\mathbf{x}^1, \mathbf{x}^2) \in \mathbf{X} \times \mathbf{X}$, either $\mathbf{x}^1 \succsim \mathbf{x}^2$ or $\mathbf{x}^2 \succsim \mathbf{x}^1$.
  - Transitive: for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbf{X}$ such that $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$, we have $\mathbf{x}^1 \succsim \mathbf{x}^3$.

连续 $\{x^n\}_{n=1}^{\infty}$ $\{y^n\}$ $\forall n, x^n \succsim y^n$, 且 $\lim_{n\to\infty} x^n = x$ $\lim_{n\to\infty} y^n = y$

$\Rightarrow x \succsim y$ 偏好不能突变.

Preference Relation Examples

- An agent wants to choose one cell phone from three alternatives: iPhone 5 (A), Samsung Galaxy S3 (S) and Huawei Ascend P1(H). Consider the following preference relations, are they rational (complete and transitive)?
  - The agent prefers larger screen size (ss)
    $ss_S = 4.8in$, $ss_H = 4in$ and $ss_A = 3.5in$

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  - The agent would prefer x to y if x has both larger screen and longer talk time (tt) than y
    $tt_S = 22hr$, $tt_H = 5hr$ and $tt_A = 8hr$

Huawei ? Apple ?

Preference Relation Examples

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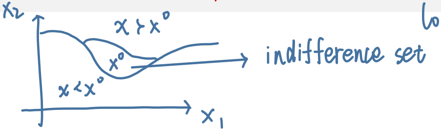
    - The agent would prefer x to y if x has both larger screen and longer talk time (tt) than y
      $tt_S = 22hr$, $tt_H = 5hr$ and $tt_A = 8hr$
    - The agent prefers S to A because S has larger screen, prefers A to H because A has longer talk time and prefers H to S because H is a national brand!

Indifference Curve/Set



local non-satiation 有可能变更好

- An indifference curve going through $\mathbf{x}$ can be represented by set $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succsim \mathbf{x} \text{ and } \mathbf{x} \succsim \mathbf{y}\}$.
- Strict preference: $\mathbf{x}^1 \succ \mathbf{x}^2$ if $\mathbf{x}^1 \succsim \mathbf{x}^2$ but not $\mathbf{x}^2 \succsim \mathbf{x}^1$.
- Given $\mathbf{x}^0 \in \mathbf{X}$, any point in $\mathbf{X}$ can be put in one of three exclusive sets according to its preference relation to $\mathbf{x}^0$: the "worse than" set, the "preferred to" set or the "indifference" set.
- A diagram with one possible layout of three sets that is consistent with rational preference...how is it different from the diagram you saw in the intermediate level class?

More Assumptions on Preference Relations

局部非饱合. 很小的调整.

- Local non-satiation: for $\forall \mathbf{x}^0 \in R_+^L$ and $\forall \epsilon > 0$, $\exists \mathbf{x}$ such that $d(\mathbf{x}^0, \mathbf{x}) \boxed{\leq \epsilon}$ and $\underline{\mathbf{x} \succ \mathbf{x}^0}$

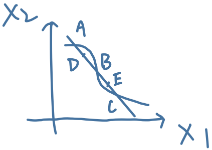
单调.

- Monotone: if $x_{\underline{\underline{\ell}}}^1 \boxed{>} x_{\underline{\underline{\ell}}}^2$ for all $l = 1, ..., L$ then $\underline{\mathbf{x}^1 \boxed{\succ} \mathbf{x}^2}$
- Strongly monotone: if $x_\ell^1 \geq x_\ell^2$ for all $l = 1, ..., L$ and $x_j^1 > x_j^2$ for $\underline{\text{some}}$ $j \in \{1, ..., L\}$ then $\mathbf{x}^1 \succ \mathbf{x}^2$

某些是严格更大的

- Strictly monotonic: if $x_\ell^1 \boxed{\geq} x_\ell^2$ for all $l = 1, ..., L$ then $\mathbf{x}^1 \boxed{\succsim} \mathbf{x}^2$; while if $x_\ell^1 > x_\ell^2$ for all $l = 1, ..., L$ then $\mathbf{x}^1 \succ \mathbf{x}^2$

More Assumptions on Preference Relations



优于

- ◇ Convexity: for $\forall \mathbf{x} \in \mathbf{X}$, the upper level set is convex

严格大于的.

  - Strict convexity: $\mathbf{x}^2 \succsim \mathbf{x}^1$ and $\mathbf{x}^3 \succsim \mathbf{x}^1$, then $t\mathbf{x}^2 + (1-t)\mathbf{x}^3 \succ \mathbf{x}^1$ for $\forall t \in (0,1)$

差不多的就行. 上标!

  - $\Rightarrow \mathbf{x}^2 \sim \mathbf{x}^1$ and $\mathbf{x}^3 \sim \mathbf{x}^1$ then $t\mathbf{x}^2 + (1-t)\mathbf{x}^3 \succ \mathbf{x}^1$
  - Indifference curves convex to the origin - diminishing marginal rate of substitution

完备
传递
连续
局
凸

单调.

Representation Theorem

找到一个function，省得比较bundle

Binary preference relation ⇒ utility function

- ◇ Function $U : \mathbf{X} \to R$ represents preference relation $\succsim$ if for $\forall(\mathbf{x}^1, \mathbf{x}^2) \in \mathbf{X} \times \mathbf{X}$, $U(\mathbf{x}^1) \geq U(\mathbf{x}^2) \Leftrightarrow \mathbf{x}^1 \succsim \mathbf{x}^2$
- ◇ Rational & continuous preference ⇒ continuous utility function
  - ○ Complete: ✓
  - ○ Transitive: ✓ Utility function maps consumption bundles to real numbers: $f : \mathbf{X} \to R$; to represent a preference, it must be the case that $\mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow f(\mathbf{x}^1) \geq f(\mathbf{x}^2)$. Can you find a function that can represent a preference relation that is not transitive?
  - ○ Continuous: ✓ ?

Continuous Preference

- (Definition 1) Preference relation $\succsim$ in $\mathbf{X}$ is continuous if the relation can be preserved under the limit operation.
- For every converging sequence of pairs of consumption bundles

$$\{(\mathbf{x}^n; \mathbf{y}^n)\}_{n=1}^{+\infty} \text{ where } lim_{n\to\infty}\mathbf{x}^n = \mathbf{x}, lim_{n\to\infty}\mathbf{y}^n = \mathbf{y}$$

continuity $\Leftrightarrow$ if $\mathbf{x}^n \succsim \mathbf{y}^n$ for $\forall n$, then $\mathbf{x} \succsim \mathbf{y}$.

- *Lexicographic preference* is not continuous by this definition. We will show it cannot be represented by any real-valued function.

Unrepresentable Lexicographic Preference

$\{(1+\frac{1}{n}, 1)\}_{n=1}^{\infty}$ 无穷!

$(1+\frac{1}{n}, 1) \succsim (1,2)$.

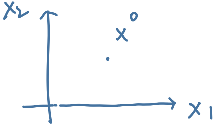
↓ 极限

$(1,1) \precsim (1,2)$

preference 反转

先把 x_1 类似于奖牌榜比金牌.

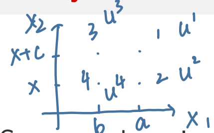
- ◇ $(x_1, x_2) \succsim (x_1', x_2') \Leftrightarrow x_1 > x_1'$ or if $x_1 = x_1', x_2 \geq x_2'$
- ◇ Not continuous - "sudden change" of preference ordering:
  - ◦ Consider sequence $\{a^k\}_{k=1}^{\infty}$ where $a_k = 1 + \frac{1}{k}$ and $\{(a_k, 1)\}_{k=1}^{\infty}$
  - ◦ For $\forall k$, we have $(a_k, 1) \succsim (1, 2)$, but at the limit $(1, 1) \preceq (1, 2)$
- ◇ If $(a_1, a_2) \sim (b_1, b_2)$, i.e, $(a_1, a_2) \succsim (b_1, b_2)$ and $(b_1, b_2) \succsim (a_1, a_2)$, then $a_1 = b_1$ and $a_2 = b_2$ 只有完全相等才有一样的utility ⇒ 不存在 utility function
  $\Rightarrow$ There is NO indifference curve!
- ◇ Want to show that these is no function f that can map $(x_1, x_2) \in R \times R$ to R in a way such that $f(a_1, a_2) \geq f(b_1, b_2) \Leftrightarrow a_1 > b_1$ or $a_1 = b_1, a_2 \geq b_2$



Some Math

- In between any two real numbers, there exist rational numbers.
- Countable means injection (one-to-one mapping) to the set of natural numbers
- The set of rational numbers is countable
- The set of irrational numbers is not countable
- Thus the set of real numbers is not countable
- There can be no injection from an uncountable set to a countable set.

No Utility Function for Lexicographic Preference



$(a, x+c) \succsim (a, x)$

$u(a, x+c) \succsim u(a, x)$

- Suppose there is a utility function $U : R^2_+ \to R$ that represents this Lexicographic preference.
- Then for $\forall a, b, x, c \in R_+$, with $a > b$, we can find rational numbers r_a and r_b such that real ⇒ rational
  $U(a, x + c) > r_a > U(a, x) > U(b, x + c) > r_b > U(b, x)$.
- That is, for any pair of different real numbers, we can find a pair of different rational numbers - an injection (one-to-one mapping) from the set of real numbers to the set or rational numbers.
- This can not be true. Thus the utility function does not exit.

Alternative Definitions of Preference Continuity

$x^1 \succsim y$

$x^2 \succsim y$

$\vdots$

$x \succsim y$

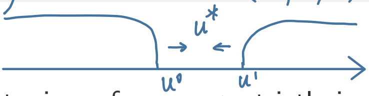
- (Definition 2) For $\forall \mathbf{x} \in \mathbf{X}$, the "at least as good as" set (upper contour set) and the "no better than" set (lower contour set) are both closed

strictly 的是 open set.

- (Definition 3) For $\forall \mathbf{x} \in \mathbf{X}$, the "preferred to" set (strict upper contour set) and the "worse than" set (strict lower contour set) are both open
  - *The complement of a closed set is an open set.*
- Lexicographic preference: the upper and lower contour sets are neither closed nor open

More on Utility Function

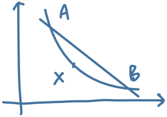
$(x_1, \cdots, x_L) \succ (u^0, \dots, u^0)$ $(u^1, \cdots, u^1) \succ (x_1, \dots, x_2)$



- ◇ + strictly monotonic preference ⇒ strictly increasing utility function
  - ◦ Reduce dimension: comparing vectors ⇒ comparing real numbers

$$\underline{u(\mathbf{x})\mathbf{e} \sim \mathbf{x} \text{ where } e = (1, ..., 1)^T}$$

  - ◦ If such a $u(.)$ exists, it indeed represents $\succsim$: $\underline{\mathbf{x} \succsim \mathbf{x}' \Leftrightarrow u(\mathbf{x}) \geq u(\mathbf{x}')}$
    - – Does such number $u(\mathbf{x})$ exist?
      Yes, by continuity, strict monotonicity and completeness.
    - – Is it unique?
      Yes, by transitivity and strict monotonicity.
    - – Is $u(.)$ continuous? (continuity + some more math)
- ◇ + (strictly) convex preference ⇒ (strictly) quasi-concave utility function

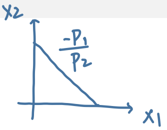


Summary - Preference Relation and Utility Function

- For every rational (complete and transitive), continuous, strictly monotonic and (strictly) convex preference, we can find a continuous, strictly increasing and (strictly) quasi-concave utility function to represent it.
- Preference is an ordering binary relation
  $\Rightarrow$ utility is ordinal
  $\Rightarrow$ A positive monotonic transform of a utility function can represent the same preference

Utility Maximization Problem

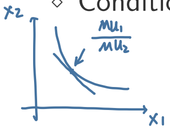
K.T. 考虑 binding 还是 slack



横轴的价格.

转置

- $max_{\mathbf{x}}\ u(\mathbf{x})$ subject to $\mathbf{p}^T\mathbf{x} \leq w$ and $\mathbf{x} \geq \mathbf{0}$
- Does a solution exist?
  Yes, because $u(.)$ is continuous and $B_{\mathbf{p},w}$ is compact.
- Conditions? Suppose $u(.)$ is twice differentiable



斜率表示切换商品的意愿.

$$L(\mathbf{x},\lambda,\boldsymbol{\mu}) = u(\mathbf{x}) + \lambda(w - \mathbf{p}^T\mathbf{x}) + \sum_{\ell=1}^{L} \mu_\ell x_\ell$$

>0 的条件

(1) + (3)

$MU_l - \lambda P_l \leq 0$

在 (1) 中乘 x_l.

$x_l (MU_l - \lambda P_l) = 0$.

① $x_l > 0$ $MU_l = \lambda P_l$ $\Rightarrow$ $\lambda = \frac{MU_l}{P_l} = \frac{MU_j}{P_j}$

花钱在 l 上带来的 u 和花在 j 上一样.

② $MU_l < \lambda P_l$ $x_l = 0$.

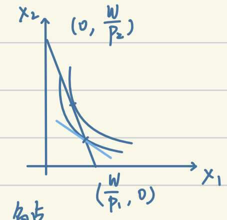
$\Downarrow$ $\lambda = \frac{MU_j}{P_j} > \frac{MU_l}{P_l}$ $\Rightarrow$ 把钱花在 j 上更好

FOCs

求x的偏导

$$
\begin{aligned}
u_\ell(\mathbf{x}^*) - \lambda^* p_\ell + \mu_\ell^* &= 0 \text{ for all } \ell & (1)\\
\lambda^* \geq 0 \text{ and } \mathbf{p}^T\mathbf{x}^* - w \leq 0 \text{ and } \lambda^*(w - \mathbf{p}^T\mathbf{x}^*) &= 0 & (2)\\
\mu_\ell^* \geq 0 \text{ and } x_\ell^* \geq 0 \text{ and } \mu_\ell^* x_\ell^* &= 0 \text{ for all } \ell & (3)
\end{aligned}
$$

Corner & Inner 解.



indifference curve 的斜率 $<$ budget line : 点在 opt 下面

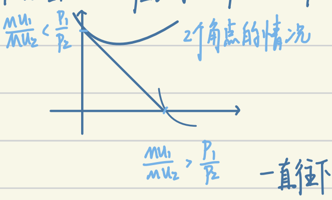
… $>$ … … 上

斜率越往上越大，如果在角点 indifference curve 的斜率比 budget line 小，说明只能取角点

角点

解马歇尔 demanding，要分类讨论，计算 $\frac{MU_1}{MU_2}$. 在 $(0, \frac{W}{P_2})$ 和 $(\frac{W}{P_1}, 0)$ 的值.

效用函数带未知数.



一直往下挪还是没法这么小

quasi-linear

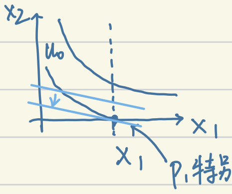
$u(x_1, x_2) = \ln x_1 + x_2$

$MU_1 = \frac{1}{x_1}$ $MU_2 = 1$

$P_1 = \gamma \frac{1}{x_1}$ $P_2 = \gamma$

$\Rightarrow \frac{1}{x_1} = P_1$

$MRS = \frac{MU_1}{MU_2} = \frac{1}{x_1}$



P_1 特别小的时候会取到.

First Order Conditions

- (1) + (3) $\Rightarrow$

  $x_\ell \geq 0,\ u_\ell(x^*) - \lambda^* p_\ell = -\mu_\ell^* \leq 0$ and $x_\ell^* \left(u_\ell(x^*) - \lambda^* p_\ell\right) = 0$
  - $x_\ell^* > 0$ and $\frac{MU_\ell^*}{p_\ell} = \lambda^*$: when $\mathbf{x}^* >> 0$, $\nabla u(\mathbf{x}) = \lambda^* \mathbf{p}$
  - $\frac{MU_\ell^*}{p_\ell} < \lambda^*$ and $x_\ell^* = 0$: when $x_1^* = 0$ and $x_2^* > 0$, $\frac{MU_1(0,w/p_2)}{MU_2(0,w/p_2)} < \frac{p_1}{p_2}$
- (2) $\Rightarrow$
  - $\lambda^* > 0$ and $\mathbf{p}^T \mathbf{x}^* = w$
    We are here if the preference is strongly monotone
  - $\mathbf{p}^T \mathbf{x}^* < w$ and $\lambda^* = 0$

FOCs Sufficient?

Yes, when

- $u(.)$ is twice differentiable and quasi-concave
- all constraints are linear
- $u_\ell(\mathbf{x}^*) \neq 0$ for some $\ell = 1, ..., L$

Marshallian Demand and Indirect Utility Functions

$x^*(p,w)$: Homo 0 in (p,w)

根据$\mathcal{L}$.

$$\begin{cases} \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 = w \end{cases} \Rightarrow x_1、x_2\text{是}p_1、p_2、w\text{的函数}.$$

- Marshallian demand funciton: solution to the utility maximization problem

$$\underline{\mathbf{x}^*(\mathbf{p}, w) = (x_1^*(\mathbf{p}, w), \ldots, \mathbf{x}_L^*(\mathbf{p}, w))}$$

- Indirect utility function: substituting the $\mathbf{x}^*$ into the objective function

解代回去，u变成p和w的函数

$$v(\mathbf{p}, w) = u\left(\mathbf{x}^*(\mathbf{p}, w)\right)$$

Indirect Utility Function Properties (1)

$$
\begin{aligned}
v(\mathbf{p}, w) &= L\left(\mathbf{x}^*(\mathbf{p}, w), \lambda^*(\mathbf{p}, w), \boldsymbol{\mu}^*(\mathbf{p}, w)\right) \\
&= \underline{u\left(\mathbf{x}^*(\mathbf{p}, w)\right)} + \lambda^*(\mathbf{p}, w)\left(w - \mathbf{p}^T\mathbf{x}^*(\mathbf{p}, w)\right) \\
&\quad + \sum_{\ell=1}^{L} \mu_\ell^*(\mathbf{p}, w) x_\ell^*(\mathbf{p}, w)
\end{aligned}
$$

indirect ⇒ x^*.

- ◇ $\frac{\partial v}{\partial w}\Big|_{\mathbf{x}^*} = \lambda^* \geq 0$
- ◇ $\frac{\partial v}{\partial p_\ell}\Big|_{\mathbf{x}^*} = -\lambda^* x_\ell^* \leq 0$ for all ℓ
- ◇ Roy's identity: $\boxed{x_\ell^*(\mathbf{p}, w) = -\frac{\partial v / \partial p_\ell}{\partial v / \partial w}}$ for all ℓ
  strongly monotone preference ⇒ binding budget at optimum

Indirect Utility Function Properties (2)

$$
\begin{aligned}
v(\mathbf{p}, w) &= L\left(\mathbf{x}^*(\mathbf{p}, w), \lambda^*(\mathbf{p}, w), \boldsymbol{\mu}^*(\mathbf{p}, w)\right) \\
&= \underline{\underline{u\left(\mathbf{x}^*(\mathbf{p}, w)\right)}} + \lambda^*(\mathbf{p}, w)\left(w - \mathbf{p}^T\mathbf{x}^*(\mathbf{p}, w)\right) \\
&\quad + \sum_{\ell=1}^{L} \mu_\ell^*(\mathbf{p}, w) x_\ell^*(\mathbf{p}, w)
\end{aligned}
$$

写出定义!

- ◇ $v(\mathbf{p}, w)$ is continuous in $(\mathbf{p}, w)$
- ◇ $v(\mathbf{p}, w)$ is homogeneous of degree zero in $(\mathbf{p}, w)$
- ◇ $v(\mathbf{p}, w)$ is quasi-convex in $(\mathbf{p}, w)$

$V(tp_1, tp_2, tw) = t^0 V(p_1, p_2, w)$

价格组合.

$V(p^t, w^t) \leq \max\{V(p^1, w^1), V(p^2, w^2)\}$

$p^t = tp^1 + (1-t)p^2$

$w^t = tw^1 + (1-t)w^2.$

可以反证.

包络:

$\frac{\partial v}{\partial p_i} = \frac{\partial L}{\partial p_i} < 0$ $\frac{\partial v}{\partial w} = \frac{\partial L}{\partial w} > 0$

Indirect Utility Function Properties (3)

$v(\mathbf{p}, w)$ is quasi-convex in $(\mathbf{p}, w)$

- If $v(\mathbf{p}^1, w^1) \leq v$ and $v(\mathbf{p}^2, w^2) \leq v$, then $v(\mathbf{p}^t, w^t) \leq v$ for $\forall t \in [0, 1]$, where $\mathbf{p}^t = t\mathbf{p}^1 + (1-t)\mathbf{p}^2$ and $w^t = tw^1 + (1-t)w^2$
- Utility level $v(\mathbf{p}^t, w^t)$ is achievable under either $(\mathbf{p}^1, w^1)$ or $(\mathbf{p}^2, w^2)$, i.e, $\mathbf{x}^{t*} \in B_{\mathbf{p}^1, w^1} \cup B_{\mathbf{p}^2, w^2}$
- If $\mathbf{x}^{t*}$ does not belong to $B_{\mathbf{p}^1, w^1} \cup B_{\mathbf{p}^2, w^2}$, i.e, if $\mathbf{p}^1\mathbf{x}^{t*} > w^1$ and $\mathbf{p}^2\mathbf{x}^{t*} > w^2$, then $(t\mathbf{p}^1 + (1-t)\mathbf{p}^2)\mathbf{x}^{t*} > tw^1 + (1-t)w^2 = w^t$. This is not true. So $\mathbf{p}^1\mathbf{x}^{t*} \leq w^1$ or $\mathbf{p}^2\mathbf{x}^{t*} \leq w^2$.
- So $v(\mathbf{p}^t, w^t) = u(\mathbf{x}^{t*}) \leq v(\mathbf{p}^1, w^1)$ or $v(\mathbf{p}^t, w^t) \leq v(\mathbf{p}^2, w^2)$, thus $v(\mathbf{p}^t, w^t) \leq v$ and $v(\mathbf{p}^t, w^t) \leq max\{v(\mathbf{p}^1, w^1), v(\mathbf{p}^2, w^2)\}$
- It is easier to see in a diagram of budget lines
  - Intersection point
  - Range of slope

Marshallian Demand Function Properties

- Continuous in $(\mathbf{p}, w)$: related to strict convexity of preference
  What is the other benefit of having convex preference, thus quasi-concave utility function?
- Differentiable under certain conditions
- Homogeneity of degree zero in $(\mathbf{p}, w)$
- With strongly monotone preference, we have Walras' Law

$$\mathbf{p}^T \mathbf{x}^*(\mathbf{p}, w) = w$$

钱一定会花光.

More on Walras' Law

- Taking the derivative w.r.t. w

在 l 上支出的变化、 $\Sigma \Rightarrow$ 总支出变化 .

$$\sum_{\ell=1}^{L} \boxed{p_\ell \frac{\partial x_\ell}{\partial w}|_{\mathbf{x}^*}} = 1$$

  - This means the change in expenditure and the change in wealth must be the same
  - Can all commodities be inferior goods?
- Taking the derivative w.r.t. $\underline{\underline{p_j}}$

$$x_j^*(\mathbf{p}, w) + \sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell}{\partial p_j}|_{\mathbf{x}^*} = 0 \text{ for } \forall j = 1, ..., n$$

Elasticities

a对b的弹性: b变1%时, a ?%.

- Income elasticity

$$\eta_\ell = \frac{\partial x_\ell / x_\ell}{\partial w / w} = \frac{\partial x_\ell}{\partial w} \frac{w}{x_\ell}$$

- Price elasticity

$$\epsilon_{\boxed{ij}} = \frac{\boxed{\partial x_i / x_i}}{\partial p_j / p_j} = \frac{\partial x_i}{\partial p_j} \frac{p_j}{x_i}$$

对其他的价格的弹性.

- Expenditure share

$$s_\ell = \frac{p_\ell x_\ell}{w}$$

$$\sum_{i=1}^{L} s_i = 1$$

Walras' Law in Elasticity Terms

- Engel aggregation:

对 Walras 的求导.

$$\eta_l = \frac{\partial x / x}{\partial w / w}$$

$$\eta_l \frac{x}{w} = \frac{\partial x}{\partial w}$$

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell}{\partial w}\Big|_{\mathbf{x}^*} = 1$$

$$\Rightarrow \quad \sum_{\ell=1}^{L} \eta_\ell \boxed{s_\ell} = 1$$

- Cournot aggregation:

$$\epsilon_{ij} = \frac{\partial x_i / x_i}{\partial p_j / p_j} \qquad \epsilon_{ij} \frac{x_i}{p_j}$$

$$x_j^*(\mathbf{p}, w) + \sum_{\ell=1}^{L} p_\ell \frac{\partial x_{\underline{\underline{\ell}}}}{\partial p_{\underline{\underline{j}}}}\Big|_{\mathbf{x}^*} = 0$$

$$\Rightarrow \quad \sum_{\ell=1}^{L} s_\ell \epsilon_{\ell j} = -s_j$$

同乘 $\frac{p_j}{w}$.

Expenditure Minimization Problem

$$\begin{cases} \lambda : \dfrac{\Delta u}{\Delta W} \\ \gamma : \dfrac{\Delta E}{\Delta u} \leftarrow \text{Expenditure} \end{cases}$$

- $min_{\mathbf{x}}\ \mathbf{p}^T\mathbf{x}$ subject to $u(\mathbf{x}) \geq u$ and $\mathbf{x} \geq \mathbf{0}$
- Lagrange function

最小化

$$L(\mathbf{x},\gamma,\boldsymbol{\eta}) = \mathbf{p}^T\mathbf{x} + \underline{\gamma\left(u - u(\mathbf{x})\right)} - \underline{\sum_{\ell=1}^{L} \eta_\ell x_\ell}$$

FOCs

① $u(x^*) > u \Rightarrow \gamma^* = 0$

$u(x^*) = P_l$.

?

没有免费的午餐.

微分.

MU_l

$$\begin{aligned} p_\ell - \gamma^* u_\ell(\mathbf{x}^*) - \eta_\ell &= 0 \text{ for all } \ell && (4) \\ \underline{\gamma^* \geq 0, u(\mathbf{x}^*) \geq u \text{ and } \gamma^*\left(u - u(\mathbf{x}^*)\right)} &= 0 && (5) \\ \underline{\eta_\ell^* \geq 0 \text{ and } x_\ell^* \geq 0 \text{ and } \eta_\ell^* x_\ell^*} &= 0 \text{ for all } \ell && (6) \end{aligned}$$

② $\gamma^* > 0 \Rightarrow u(x^*) = u$

$u_l^* > 0 \quad x_l^* = 0$

总有一个为0.

$x_l^* > 0 \quad u_l^* = 0 \quad P_l = \gamma^* \cdot MU_l$

$P_l > \gamma MU_l$.

最小化 ⇒ 取负号最大化.

$\max \quad -p^T x \quad s.t. \quad u(x) \geq u$

$\mathcal{L} = -p^T x + \gamma(u(x) - u)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0 \\ \gamma \frac{\partial \mathcal{L}}{\partial \gamma} = 0 \end{cases}$$ ↓ 一定是binding的

$$\Rightarrow \begin{cases} \frac{mu_1}{mu_2} = \frac{p_1}{p_2} \\ u(x) = u \end{cases}$$ ⇒ h_1、h_2可以用p_1、p_2、u来表示 ⇒ 希克斯需求.

First Order Conditions

- Start from (5), if $u(\mathbf{x}^*) > u$ and $\gamma^* = 0$, then $\eta_\ell^* = p_\ell > 0$ and $x_\ell^* = 0$ for all ℓ
- Thus $\gamma^* > 0 \;\Rightarrow\; u(\mathbf{x}^*) = u$
- If $x_\ell^* > 0$ then $\eta_\ell^* = 0$ and (4) $\Rightarrow p_\ell = \gamma^* u_\ell(\mathbf{x}^*)$
    - When $\mathbf{x}^* >> 0$, $\gamma^* \nabla u(\mathbf{x}) = \mathbf{p}$
- If $\eta_\ell^* > 0$ then $x_\ell^* = 0$ and (4) $\Rightarrow p_\ell > \gamma^* u_\ell(\mathbf{x}^*)$
    - $x_1^* = 0, x_2^* > 0$ and $\frac{MU_1(0,x_2^*)}{MU_2(0,x_2^*)} < \frac{p_1}{p_2}$

Hicksian Demand and Expenditure Functions

- Hicksian demand funciton: solution to the expenditure minimization problem

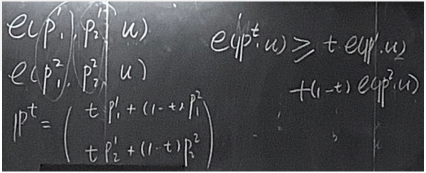
$$\mathbf{h}(\mathbf{p}, u) = (x_1^*(\mathbf{p}, u), ..., x_L^*(\mathbf{p}, u))$$

- Expenditure function: substituting the x^* into the objective function

代入.

$$e(\mathbf{p}, u) = \mathbf{p}^T \mathbf{h}(\mathbf{p}, u)$$

Expenditure Function Properties



$e(p_1^1, p_2^1, u)$

$e(p_1^2, p_2^2, u)$

$\mathbb{P}^t = \begin{pmatrix} t p_1^1 + (1-t) p_1^2 \\ t p_2^1 + (1-t) p_2^2 \end{pmatrix}$

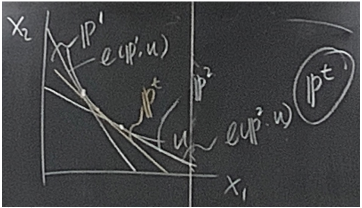
$e(p^t, u) \geq t\, e(p^1, u) + (1-t)\, e(p^2, u)$

$\frac{\partial e}{\partial u} > 0$ $\quad \mathcal{L} = px_1 + px_2 + \gamma [u - u(x_1, x_2)]$

$\hookrightarrow \frac{\partial \mathcal{L}}{\partial u} = \gamma > 0$ (包络)

只要u增加，花费肯定增加.

$$
\begin{aligned}
e(\mathbf{p}, u) &= \boxed{L}(\mathbf{h}^*(\mathbf{p}, u), \gamma^*(\mathbf{p}, u), \boldsymbol{\mu}^*(\mathbf{p}, u)) \\
&= \mathbf{p}^T \mathbf{h}^*(\mathbf{p}, u) + \gamma^*(\mathbf{p}, u)\left(u - u\left(\mathbf{h}^*\left(\mathbf{p}, u\right)\right)\right) - \sum_{\ell=1}^{L} \eta_\ell^*(\mathbf{p}, u) h_\ell^*(\mathbf{p}, u)
\end{aligned}
$$



- ◇ Continuous in $(\mathbf{p}, u)$
- ◇ $\frac{\partial e}{\partial u}|_{\mathbf{x}^*} = \gamma^* > 0$ 用包络定理
- ◇ Shephard's Lemma: $\frac{\partial e}{\partial p_\ell}|_{\mathbf{x}^*} = h_\ell^*(\mathbf{p}, u) \geq 0$
- 是e的性质 ◇ Homogeneous of degree $\boxed{1}$ in $\mathbf{p}$ = 不是在 (p, u) !
- ◇ Concave in $\boxed{\mathbf{p}}$

$e(tp, u) = te(p, u)$. 价格↑, expenditure↑

Expenditure Function Properties

$e(p^1, u) = p_1^1 a + p_2^1 b$
$e(p^2, u) = p_1^2 a + p_2^2 b$

- Suppose $\mathbf{h}^1$ and $\mathbf{h}^2$ are solutions to the expenditure minimization problems with $\mathbf{p}^1$ and $\mathbf{p}^2$ respectively. For $\forall t \in [0,1]$, define $\underline{\mathbf{p}^t = t\mathbf{p}^1 + (1-t)\mathbf{p}^2}$. Being concave means

$$e(\mathbf{p}^t, \underset{\triangle}{u}) \geq \underline{t}e(\mathbf{p}^1, \underset{\triangle}{u}) + \underline{(1-t)}e(\mathbf{p}^2, \underset{\triangle}{u})$$

- Note that $u(\boxed{\mathbf{h}^t}) = u$, and that

相同的u!

$$\begin{aligned} e(\mathbf{p}^t, u) &\equiv \mathbf{p}^t\mathbf{h}^t \\ &= t\mathbf{p}^1\mathbf{h}^t + (1-t)\mathbf{p}^2\mathbf{h}^t \\ &\geq t\underline{e(\mathbf{p}^1, u)} + (1-t)\underline{e(\mathbf{p}^2, u)} \end{aligned}$$

用p^t, u表示的需求.

能的最小.

- Comparison of budget lines in a diagram illustrates this better.

没乘价格

Hicksian Demand Function Properties

e concave in p. ⇒ 2阶导半负定

$\frac{\partial e}{\partial p_l} = h_l$

max: $x^*(p, w)$

min: $h^*(p, u)$

demand function.

$\frac{\partial h_l}{\partial p_j}$

数量

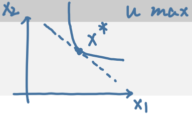
- ◇ Homogeneous of degree 0 in **p**
- ◇ No excess utility: $u(\mathbf{h}(\mathbf{p}, \underline{u})) = \underline{u}$, binding constraint and $\gamma^* > 0$
- ◇ Cross price effect: $\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j}$
- ◇ The matrix of own and cross-partial derivatives w.r.t **p** is symmetric and negative semi-definite - Hicksian substitution matrix
- ◇ Since $e(\mathbf{p}, u)$ is concave in **p**, the own-price elasticities are non-positive.

主对角的元素 ≤0 是负半定的条件

$$\frac{\partial h_\ell(\mathbf{p}, u)}{\partial p_\ell} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_\ell^2} \leq 0$$

$\frac{\partial e^2}{\partial p_i^2}$

替代效应.



Dual Problems

对偶问题.

e min: 给定u, 找最小的e.

When $\underline{\mathbf{x}^* >> 0}$

$MU_L = \lambda P_L$

$P_L = \gamma MU_L$

- In expenditure minimization: $\gamma^* \nabla u(\mathbf{x}^*) = \mathbf{p}$
- In utility maximization: $\nabla u(\mathbf{x}^*) = \lambda^* \mathbf{p}$
- Both give $MRS_{ij} = \frac{p_i}{p_j}$
- In terms of demand functions

utility target

数量 $X(p,W) \equiv h(p, \overset{\downarrow}{V(p,w)})$

花销 or $W \equiv e(p, V(p,w))$

$$\begin{aligned} \mathbf{h}(\mathbf{p}, v(\mathbf{p}, w)) &\equiv \mathbf{x}(\mathbf{p}, w) \\ \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u)) &\equiv \mathbf{h}(\mathbf{p}, u) \end{aligned}$$

- In terms of value functions

得到一致的信息.

$$\left\{ \begin{aligned} u &\equiv v(\mathbf{p}, e(\mathbf{p}, \textcircled{u})) \\ w &\equiv e(\mathbf{p}, v(\mathbf{p}, w)) \end{aligned} \right.$$

- In terms of Lagrange multipliers: $\gamma^* = \frac{1}{\lambda^*}$
- They are dual problems in the sense that they contain the same information

已知 $v(p,w)$，求 $e(p,u)$

$$\begin{cases} v(p,w) \\ e(p,u) \end{cases}$$

①将 w 代为 e

② $v[p,e(p,u)]=u \Rightarrow e(p,u)=?$

e.g. $v(p,w)=\frac{1}{p_1^{\alpha}p_2^{1-\alpha}}\alpha^{\alpha}(1-\alpha)^{1-\alpha}w$，求 $e(p,u)$

$$\frac{1}{p_1^{\alpha}p_2^{1-\alpha}}\alpha^{\alpha}(1-\alpha)^{1-\alpha}e(p,u)=u$$

$$e(p,u)=u\cdot\left[\frac{1}{p_1^{\alpha}p_2^{1-\alpha}}\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right]^{-1}$$

各种代换

or 已知 $e(p,u)$，求 $v(p,w)$

① 将 u 转化为 v

② 另 $e=w$

Duality - Recovering Utility Function

$$\begin{tikzcd}
& U(x) \arrow[dl, dash, "UMP"'] \arrow[dr, dash, "EMP"] & \\
x(p,w) \arrow[rr, leftrightarrow] \arrow[d, dash] & & h(p,w) \arrow[d, dash] \\
V(p,w) \arrow[rr, leftrightarrow] & & e(p,w)
\end{tikzcd}$$

x和h不能直接转化.
但是可以用 u、v进行转化.
e、w.

v是间接效用

- Any function of $(\mathbf{p}, u)$ with the properties of an expenditure function - taking zero with minimum u, continuous in u, strictly increasing and unbounded above in u, increasing in $\mathbf{p}$, homogeneous of degree 1 in $\mathbf{p}$ and concave in $\mathbf{p}$ - is an expenditure function
- Thus a solution to an expenditure minimization problem with some $u(\mathbf{x})$
- $u(\mathbf{x})$ is actually implied by $e(\mathbf{p}, u)$

Recovering Utility Function

$e(p,u) \to u(x).$

$p^T \cdot x \Rightarrow$ money

$\geq$

$e(p,u).$ $e(p^*,u) = p^* \cdot x$

$u(\mathbf{x}) = u$, how to define these $\mathbf{x}$?

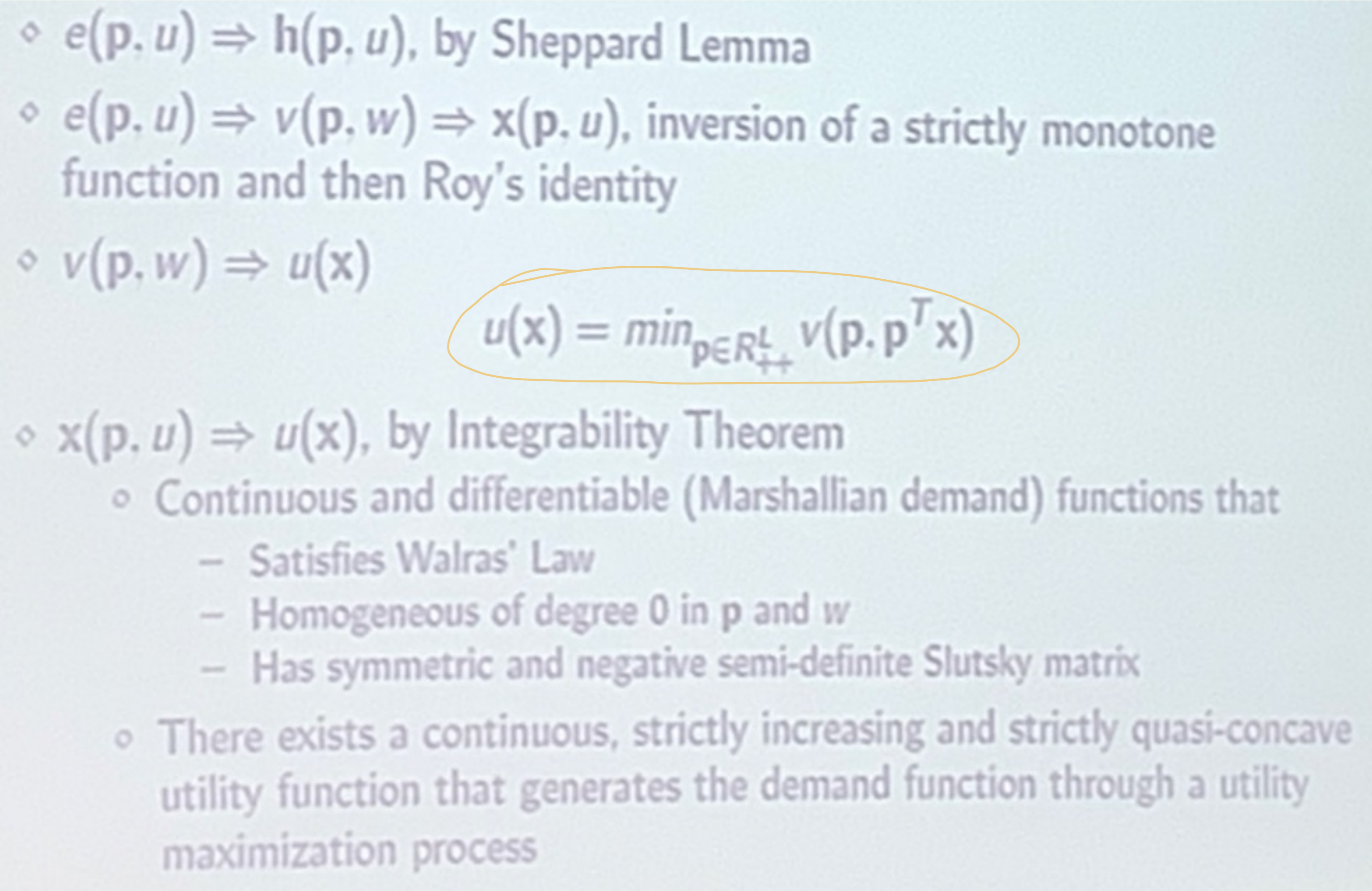
- Notice that for $\forall \mathbf{p} >> 0$
  - $\underline{\mathbf{px}\text{: one way to achieve } u(\mathbf{x})}$
  - $e(\mathbf{p}, u(\mathbf{x}))$: $\underline{\text{the cheapest way to achieve } u(\mathbf{x})}$
- Define $u(\mathbf{x}) = max \left\{u \geq 0 | \underline{e(\mathbf{p}, u) \leq \mathbf{px} \; for \; \forall \mathbf{p} >> 0}\right\}$, it is
  - increasing, quasi-concave, etc.
  - $e(\mathbf{p}, u)$ is the solution to the expenditure minimization problem with $u(\mathbf{x})$

Duality

- ⋄ Expenditure function ⇒ Hicksian demand
- ⋄ Expenditure function ⇒ indirect utility function ⇒ Marshallian demand
- ⋄ Indirect utility function ⇒ utility function

$$u(\mathbf{x}) = min_{\mathbf{p}\in R^L_{++}} v(\mathbf{p}, \mathbf{px})$$

- ⋄ Marshallian demand function ⇒ utility function
  Integrability Theorem
  - ◦ Continuous and differentiable (Marshallian demand) functions that
    - Satisfies Walras' Law
    - Homogeneous of degree 0 in **p** and w
    - Has symmetric and negative semi-definite Slutsky matrix
  - ◦ There exists a continuous, strictly increasing and strictly quasi-concave utility function that generates the demand function through a utility maximization process



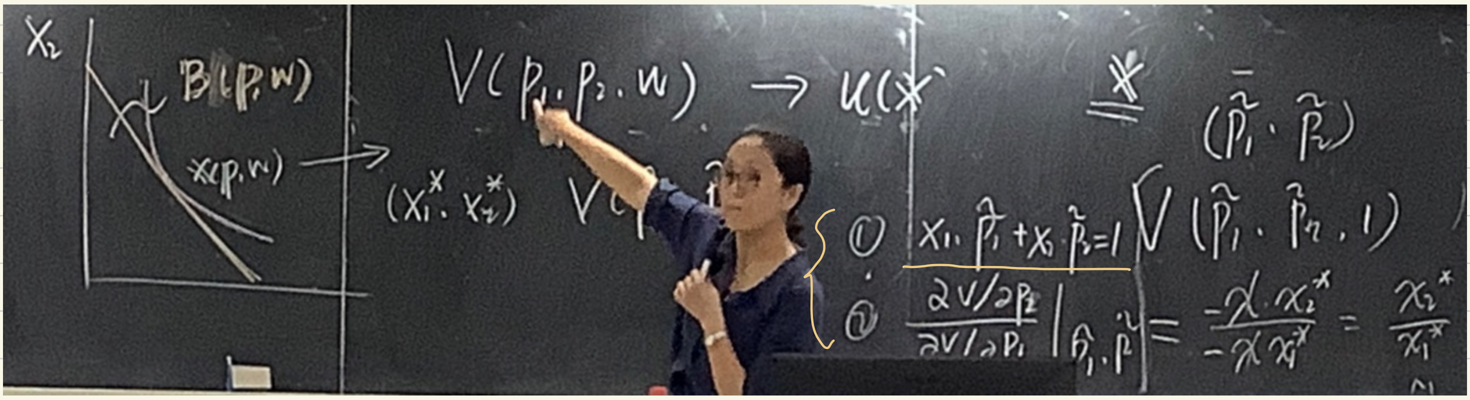
- $e(\mathbf{p}, u) \Rightarrow \mathbf{h}(\mathbf{p}, u)$, by Sheppard Lemma
- $e(\mathbf{p}, u) \Rightarrow v(\mathbf{p}, w) \Rightarrow \mathbf{x}(\mathbf{p}, u)$, inversion of a strictly monotone function and then Roy's identity
- $v(\mathbf{p}, w) \Rightarrow u(\mathbf{x})$

$$u(\mathbf{x}) = min_{\mathbf{p} \in R^L_{++}} v(\mathbf{p}, \mathbf{p}^T \mathbf{x})$$

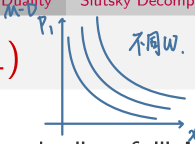
- $\mathbf{x}(\mathbf{p}, u) \Rightarrow u(\mathbf{x})$, by Integrability Theorem
  - Continuous and differentiable (Marshallian demand) functions that
    - Satisfies Walras' Law
    - Homogeneous of degree 0 in $\mathbf{p}$ and w
    - Has symmetric and negative semi-definite Slutsky matrix
  - There exists a continuous, strictly increasing and strictly quasi-concave utility function that generates the demand function through a utility maximization process

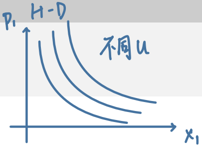
$V(p_1, p_2, w) \to u(x)$

consumption bundle 和 indirect… 联系.



Slutsky Decomposition (1)





p_i变化对j的影响 { 替代 / 收入

Linking the observable $\mathbf{x}(\mathbf{p}, w)$ to the "useful" $\mathbf{h}(\mathbf{p}, u)$.

代入支出.

收入水平 w下的 x(p,w)和效用水平 u = v(p,w)下的 h(p,w)相等.

For any l and i

$$\frac{\partial x}{\partial w} = \begin{cases} >0 : \text{normal} \\ =0 : \text{neutral} \\ <0 : \text{inferior} \end{cases}$$

研究和工资的关系.

$\frac{\partial x}{\partial p} > 0 \Rightarrow$ 吉芬.

$$\begin{aligned} h_\ell(\mathbf{p}, u) &\equiv x_\ell(\mathbf{p}, e(\mathbf{p}, u)) \\ \frac{\partial h_\ell}{\partial p_i} &= \frac{\partial x_\ell}{\partial p_i} + \frac{\partial x_\ell}{\partial w}\frac{\partial e}{\partial p_i} \\ \frac{\partial h_\ell}{\partial p_i} &= \frac{\partial x_\ell}{\partial p_i} + \frac{\partial x_\ell}{\partial \underline{w}} x_i \\ \frac{\partial x_\ell}{\partial p_i} &= \frac{\partial h_\ell}{\partial p_i} - \boxed{\frac{\partial x_\ell}{\partial w} x_i} \end{aligned}$$

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} \cdot x_j$$

Decomposition of Total Effect ($\frac{\partial x_\ell}{\partial p_i}$) into Substitution Effect ($\frac{\partial h_\ell}{\partial p_i}$) and Income Effect ($-\frac{\partial x_\ell}{\partial w} x_i$).

固定u不变.

购买力变化

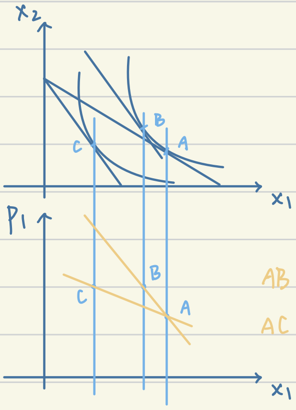
Slutsky Decomposition (2)

With $l = i$ we have the following in elasticity terms

$$\frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i} = \frac{\partial h_i}{\partial p_i}\frac{p_i}{x_i} - \frac{\partial x_i}{\partial w}\frac{w}{x_i}\frac{p_i x_i}{w} = \frac{\partial h_i}{\partial p_i}\frac{p_i}{x_i} - \frac{\partial x_i}{\partial w}\frac{w}{x_i}s_i$$

Examples of Giffen behaviour

> *"Giffen behavior and subsistence consumption", by Robert T. Jensen and Nolan H. Miller, American Economic Review 2008, 98:4, 1553 - 1577*



AB : H-D

AC : M-D

38:07

Slutsky Decomposition (3)

- Slutsky matrix, Hicksian substitution matrix, and the Hessian Matrix of the expenditure function w.r.t. prices are the same thing
- Hicksian compensation and Slutsky compensation are different ways of compensating a consumer after a price change to **maintain her purchasing power**
  - Hicksian compensation - achieve the same utility (utility framework)
  - Slutsky compensation - can afford the same consumption bundle (revealed preference)

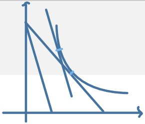
Marshallian and Hicksian Demand Schedules

价格高,数量少

- Hicksian demand (schedule/curve) is also called compensated demand
- The (Compensated) Law of Demand -
  (Hicksian) Marshallian demand curve is downward sloping
- Slope comparison
  - Normal goods: Hicksian demand is steeper
  - Inferior goods: Marshallian demand is steeper
  - Income neutral goods: two curves coinside
- Demand curve shifters

$$h_\ell(\mathbf{p}, u) = x_\ell(\mathbf{p}, e(\mathbf{p}, u)) \Rightarrow \frac{\partial h_\ell}{\partial u} = \frac{\partial x_\ell}{\partial w}\frac{\partial e}{\partial u}$$

Welfare Evaluation



CV:补偿变动 价格变动，需要多付出以达到同样的u

EV:等价变动.

- Welfare change associated with price change from p_ℓ^1 to p_ℓ^2 (for $\forall j \neq l$, $p_j^1 = p_j^2 = p_j$)
  - $CV(\mathbf{p}^1, \mathbf{p}^2, w) = w - e(\mathbf{p}^2, v(\mathbf{p}^1, w))$
  - $EV(\mathbf{p}^1, \mathbf{p}^2, w) = e(\mathbf{p}^1, v(\mathbf{p}^2, w)) - w$ 财富的变化.
- Notice that $e(\mathbf{p}^2, v(\mathbf{p}^2, w)) = e(\mathbf{p}^1, v(\mathbf{p}^1, w)) = \underline{\underline{w}}$
- Substitute in
  - $CV(\mathbf{p}^1, \mathbf{p}^2, w) = e(\mathbf{p}^2, v(\mathbf{p}^2, w)) - e(\mathbf{p}^2, v(\mathbf{p}^1, w))$
  - $EV(\mathbf{p}^1, \mathbf{p}^2, w) = e(\mathbf{p}^1, v(\mathbf{p}^2, w))$-$e(\mathbf{p}^1, v(\mathbf{p}^1, w))$

Thus CV and EV are indeed monetary measures of welfare change.

Hicksian Demand and Welfare Measure

w -

记住p↑ CV,EV<0 即可
顺序自然定了.

C前E后

W

维持.

- Substitute in
  - $CV(\mathbf{p}^1, \mathbf{p}^2, w) = e(\mathbf{p}^1, v(\mathbf{p}^1, w)) - e(\mathbf{p}^2, \underline{v(\mathbf{p}^1}, w))$
  - $EV(\mathbf{p}^1, \mathbf{p}^2, w) = e(\mathbf{p}^1, v(\mathbf{p}^2, w))\text{-}\underline{e(\mathbf{p}^2, v(\mathbf{p}^2, w))}_{W}$
- $h_\ell = \frac{\partial e}{\partial p_\ell}$
  - $CV(\mathbf{p}^1, \mathbf{p}^2, w) = -\int_{p_\ell^1}^{p_\ell^2} h_\ell(p_\ell, p_{-l}, v(\underline{\mathbf{p}^1}, w))dp_\ell$
  - $EV(\mathbf{p}^1, \mathbf{p}^2, w) = -\int_{p_\ell^1}^{p_\ell^2} h_\ell(p_\ell, p_{-l}, v(\underline{\mathbf{p}^2}, w))dp_\ell$

故用不同.

- Change in consumer surplus: $\Delta CS = -\int_{p_\ell^1}^{p_\ell^2} \underline{x}_\ell(p_\ell, p_{-\ell}, \underline{w})dp_\ell$

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Welfare Measure Comparison

Consider the welfare loss associated with **a price increase**

- For normal goods: $|CV| > |\Delta CS| > |EV|$
- For income neutral goods: $|CV| = |\Delta CS| = |EV|$
  Quasi-linear utility
- For inferior goods: $|EV| > |\Delta CS| > |CV|$

Application Example (1)

① Which measure to use if you are asked to compare two alternative price changes to $\mathbf{p}^a$ and $\mathbf{p}^b$ respectively?

② Deadweight Loss associated with a tax t on commodity 1, i.e, $\mathbf{p}^2 = (t, 0, ..., 0) + \mathbf{p}^1$

  ① Expression using EV: $-EV(\mathbf{p}^1, \mathbf{p}^2, w) - T$
  - *Monetary measure of consumers' welfare loss is larger than the government's tax revenue - a lump sum tax of the same amount leaves consumers better off than the consumption tax.*

  ② Expression using CV?
  - *Returning the tax revenue as a lump sum subsidy is not enough to compensate for the welfare loss due to the consumption tax and price increase; equivalently, to fully compensate consumers for the welfare loss, the government would have to fund a deficit.*

Application Example (1)

Expression using CV - the crucial zero deficit budget line

- The optimal consumption after price change and subsidy is exactly affordable under the original budge
- The MRS of this consumption point is the same as the new relative price ratio; thus the point is to the left of the intersection of the initial budget line and the auxiliary budge line for calculating CV
- Thus the zero deficit budget line is below the auxiliary budget line for CV
- Full compensation (a consumption point above the initial budget line) leads to deficit.

Application Example (2)

$x = a$, $p = b$, $\frac{\partial x}{\partial p} = -e$ and $\frac{\partial x}{\partial w} = i$, with $a, b, e, i > 0$
How much compensation is needed to make the consumer feel equally well off after the price increases from p_1 from b to b'?

- CV or EV?
- Slope of Hicksian demand curve at point $x = a$, $p = b$
- Linear approximation of Hicksian demand curve
- Calculate compensation needed

Application Example (2)

$x = a$, $p = b$, $\frac{\partial x}{\partial p} = -e$ and $\frac{\partial x}{\partial w} = i$, with $a, b, e, i > 0$
How much compensation is needed to make the consumer feel equally well off after the price increases from p_1 from b to b'?

- ◇ CV or EV?
- ◇ Slope of Hicksian demand curve at point $x = a$, $p = b$
- ◇ Linear approximation of Hicksian demand curve
- ◇ Calculate compensation needed
  $\frac{b'-b}{2}(2a - (b' - b)(e - ia))$