Firm's Problem

p1:基本概念 p2:成本最小化

A firm

- purchases inputs from input markets - **cost**
- produces and sells its product on the output market - **revenue**
- with the goal of maximizing its profit - **revenue-cost**

Optimization

- Objective - maximizing profits
- Constraints: technology & conditions on input and output markets
- General rule: equalize marginal cost and marginal revenue

$$MC(Q) = MR(Q)$$

Profit Maximizing Rule

$$
\begin{aligned}
\text{总的}\; MC &= MR \\
\frac{\partial}{\partial Q} \underline{TC}(Q) &= \frac{\partial}{\partial Q} \underline{TR}(Q) \\
\frac{\partial}{\partial Q}\left(\sum_{i=1}^{n} w_i(Q)x_i^*(Q)\right) &= \frac{\partial}{\partial Q}\left(p(Q)Q\right)
\end{aligned}
$$

- x_i - quantity of input i
- w_i - the price of input i
- Q - output level
- $p(Q)$ - the inverse demand function for the output

MC in Profit Maximization

$$\frac{\partial}{\partial Q}\left(\sum_{i=1}^{n} w_i(Q)x_i^*(Q)\right) = \frac{\partial}{\partial Q}\left(p(Q)Q\right)$$

LHS involves input markets

- $TC(Q) = \sum_{i=1}^{n} w_i(Q)x_i^*(Q)$
- Optimal choice of inputs under technology constraint and conditions on input markets
  - Technology: production function
  - Input markets: input prices or input supply conditions
- Special case of perfect competition $w_i(Q) = w_i$

MR in Profit Maximization

$$\frac{\partial}{\partial Q}\left(\sum_{i=1}^{n} w_i(Q)x_i^*(Q)\right) = \frac{\partial}{\partial Q}\left(p(Q)Q\right)$$

RHS involves output market

- $TR(Q) = p(Q)Q$
- Optimal choice of output given conditions on the output market
- Firm's individual demand curve, two determinants
  - Market demand - aggregation over consumers
  - Market structure - relationship between firms

Roadmap for Producer Theory

- Production function
- Cost minimization problem - cost function
- Duality - connection between production and cost functions
- Profit maximization under perfect competition

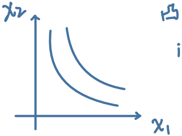
General Technology

- Technological feasibility: production possibility set
- Production plan
  - a vector of inputs and outputs $\mathbf{y} = (y_1, .., y_m) \in \mathbf{Y}$
  - inputs (-) and outputs (+)
- Single output production function: $f : \Re^n_+ \rightarrow \Re_+$

$$y = f(\mathbf{x}) = f(x_1, ..., x_n)$$

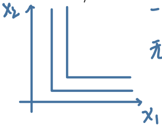
where $\mathbf{x} \geq 0$ and $y \geq 0$

Properties of Production Function



凸向原点，右上方效用增加.
iso-quant 等产量.

- Continuous
- Strictly increasing
- $f(0) = 0$
- Quasi-concave: complementarity between inputs 等产量线.
  - Strongest complementarity - Leontief production function/fixed proportion technology



一定在角点生产.
无法互相替代

$$f(x_1, x_2) = min(\alpha x_1, \beta x_2)$$

  - when $\alpha x_1 = \beta x_2$, one can NOT produce more by increasing x_1 or x_2 ALONE.*

  - Weakest complementarity - linear production function/perfectly substitutable inputs 补偿性. 是补偿而非替代.

$$f(x_1, x_2) = \alpha x_1 + \beta x_2$$

降低 xx，增加 xx.

  - *no matter what the combination of inputs, one can always substitute $\frac{\beta}{\alpha}$ units of x_1 for 1 unit of x_2.* 补偿性强，不容易被取代.

Production Function Graphs

- Isoquant: similar to indifference curves
  - iso - equal
  - quant - quantity
  - take total derivative of the production function at a given output level to find the slope
- Quasi-concave $\Rightarrow$ convex upper contour set
  $\Rightarrow$ isoquants convex to the origin
  $\Rightarrow$ diminishing slope when increasing along the x-axis
- Marginal rate of technical substitution ($MRTS$)

产量 $\Leftrightarrow$ utility

$$MRTS_{ij} = \frac{MP_i}{MP_j}$$

Special Production Functions - Separable

Separable production functions

- Classify inputs into a small number of groups 要素分组.

$$\underline{g_1 = (x_1, x_2, ..., x_{n_1})}, \underline{g_2 = (x_{n_1+1}, ..., x_{n_2})}, ..., \underline{g_m = (x_{n_{m-1}+1}, ..., x_n)}$$

  Then allow within-group substitutability to be different from between-group substitutability.

- Weak: within group subst. indpt. of inputs in other groups

$$\frac{\partial (MRTS_{ij})}{\partial x_k} = \frac{\partial (MP_i/MP_j)}{\partial x_k} = 0 \text{ for } \forall i, j \in g_s \text{ and } k \notin g_s$$

不重要. 和另一组中的要素无关.

- Strong: subst. between any two inputs from g_s and g_t indpt. of inputs in groups other than g_s and g_t

$$\frac{\partial (MRTS_{ij})}{\partial x_k} = \frac{\partial (MP_i/MP_j)}{\partial x_k} = 0 \text{ for } \underline{\forall\ i \in g_s,\ j \in g_t \text{ and } k \notin g_s \cup g_t}$$

  - *when $g_s = g_t$, it is the same as the case of weak separation*

Special Production Functions - CES

CES - constant elasticity of substitution

Elasticity of substitution

替代率的弹性.

纵轴的比值

$$\sigma_{ij}(\mathbf{x}^0) = \frac{dln(x_j/x_i)}{dlnMRTS_{ij}} = \frac{dln(x_j/x_i)}{dln(MP_i/MP_j)} = \frac{\frac{d(x_j/x_i)}{x_j/x_i}}{\frac{d(MP_i/MP_j)}{(MP_i/MP_j)}}$$

百分比变化的影响.

横轴的比例.

- % change in *MRTS* vs. % change in factor ratio
- σ measures how easily input factors can be substituted for one another (holding other inputs and output constant).
- Relate to the curvature of the isoquants - how fast does *MRTS* diminish along an isoquant
  Strong(weak) substitutability: when increasing x_1 and reducing x_2 along one isoquant, $MRTS_{ij}$ - the ability of x_1 to substitute for x_2 - drops a little(a lot)
- $\sigma \in [0, +\infty)$

CES Production Function

$$\frac{\partial y}{\partial x_1} = \frac{1}{\rho} \cdot \alpha_1 \rho x_1^{\rho-1} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}-1}$$

- General form of CES

$$MRTS = \frac{\alpha_1}{\alpha_2}\left(\frac{x_1}{x_2}\right)^{\rho-1}$$

$$y = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}, \ \alpha_1 + \alpha_2 = 1$$

$$\sigma = \frac{1}{1-\rho}$$

$$\rho < 1 \quad \text{and} \quad \rho \neq 0$$

- Special cases
  - Leontief production function $\sigma \to 0$ and $\rho \to -\infty$
  - Linear production function $\sigma \to +\infty$ and $\rho \to 1$ 越大越好代替.
  - Cobb-Douglas production function $\sigma \to 1$ and $\rho \to 0$

Returns to Scale

HOD α.

$$f(tx) = t^{\alpha} f(x)$$

- Returns to scale: long run concept
- Global returns to scale: for all $t > 1$ and all $\mathbf{x}$
  - Increasing: $f(t\mathbf{x}) > tf(\mathbf{x}) \Leftarrow$ homogeneous of degree α & $\alpha > 1$
  - Constant: $f(t\mathbf{x}) = tf(\mathbf{x}) \Leftrightarrow$ homogeneous of degree 1
  - Decreasing: $f(t\mathbf{x}) < tf(\mathbf{x}) \Leftarrow$ homogeneous of degree α & $\alpha < 1$
- Quasi-concave + H.O.D $\alpha \leq 1 \Rightarrow$ Concave 产量的增加规模配不上要素的增加规模
  - When $\alpha = 1$, it is called linear homogeneous function
  - Quasi-concave production function that is H.O.D. 1 is concave
  - $h(x) = g(f(x))$ is concave if g and f are both concave functions; $g(z) = z^{\alpha}, \alpha < 1$ is concave.

Quasi-concavity, Linear Homogeneity and Concavity

Production function $y = f(\mathbf{x})$; f is quasi-concave and homogeneous of degree 1. Show that

$$f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{x}') = ty + (1-t)y'$$

Proof:
Linear homogeneity $\Rightarrow$

$$f\left(\frac{t\mathbf{x}}{ty}\right) = f\left(\frac{(1-t)\mathbf{x}'}{(1-t)y'}\right) = 1$$

Quasi-concave $\Rightarrow$

$$f\left(\lambda\frac{t\mathbf{x}}{ty} + (1-\lambda)\frac{(1-t)\mathbf{x}'}{(1-t)y'}\right) \geq 1 \text{ for } \forall\lambda \in [0,1]$$

Let $\lambda = \frac{ty}{ty+(1-t)y'}$

$$\begin{aligned} \Rightarrow \quad & f\left(\lambda\frac{t\mathbf{x}}{ty} + (1-\lambda)\frac{(1-t)\mathbf{x}'}{(1-t)y'}\right) = f\left(\frac{t\mathbf{x}}{ty+(1-t)y'} + \frac{(1-t)\mathbf{x}'}{ty+(1-t)y'}\right) \geq 1 \\ \Rightarrow \quad & f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \geq ty + (1-t)y' \end{aligned}$$

Local Returns to Scale

$$\frac{df(tx)/f(tx)}{dt/t} = \frac{f'(tx)\cdot x}{f(tx)/t} \quad t\to 1 \quad \frac{f'(x)x}{f(x)}$$

x里是很多项

Elasticity of scale at $\mathbf{x}$

规模弹性.

$$\mu \equiv lim_{t\to 1}\frac{dlnf(t\mathbf{x})}{dln(t)} = \frac{\sum_{i=1}^{n} MP_i x_i}{f(\mathbf{x})}$$

$\mu(x) > 1$: increasing.

Define output elasticity of input i

产量的弹性

$$\mu_i(\mathbf{x}) \equiv \frac{\partial f(\mathbf{x})}{\partial x_i}\frac{x_i}{f(\mathbf{x})} = \frac{MP_i x_i}{f(\mathbf{x})}$$

Thus

规模弹性是投入弹性之和.

$$\mu(\mathbf{x}) = \sum_{i=1}^{n} \mu_i(\mathbf{x})$$

Sometimes it can be written as $\mu^*(y)$, so you can say the technology displays *locally increasing/constant/decreasing return to scale* at output level y.

Example 3.2 in JR(3rd).

Short Run: *MP* and *AP* of Variable Input

$$\frac{\partial AP_i}{\partial x_i} = \frac{MP_i x_i - f(x)}{x_i^2} = \frac{MP_i - AP_i}{x_i}$$

Two input K and L, K is fixed at $\overline{K}$ in the short run.

$MP > AP$, $x \uparrow \Rightarrow AP \uparrow$

How does $y = f(L; \overline{K})$ change with L?

- Average product of labour $AP_L = \frac{f(L;\overline{K})}{L}$
- Marginal product of labour $MP_L = \frac{\partial f(L;\overline{K})}{\partial L}$
  - $MP_L \uparrow$ in L when L is small
    - *efficiency gain from the division of labour*
  - $MP_L \downarrow$ in L when L is large
    - *exhaust the benefit from the division of labour and MP_L may become negative due to the constraint on capital input*

MP, AP and TP in the Short Run

- For the first unit of labour input $MP_L = AP_L$
- How does AP_L change with L when $MP_L > AP_L$?
- How does AP_L change with L when $MP_L < AP_L$?
- Output elasticity of input: $\mu_i(\mathbf{x}) = \frac{MP_i x_i}{f(\mathbf{x})} = \frac{MP_i}{AP_i}$
- What happens to $TP(L; \overline{K})$ when $MP_L = 0$?
- When does MP_L achieve maximum?
- $AP(L; \overline{K})$ and $TP(L; \overline{K})$ curves
- When a production function is concave, there is diminishing MP_L.

Cost Minimization

Firm's cost minimization problem (similar to consumers' EMP) - assuming perfectly competitive input markets

要素价格

$$
\begin{aligned}
c(\mathbf{w}, y) \quad &\equiv \quad min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} \\
s.t. \quad &f(\mathbf{x}) \geq y
\end{aligned}
$$

排除角点解

Assume the Inada conditions so that $lim_{x_i \to 0} f_i = +\infty$, the FOC is

$$\frac{MP_i}{w_i} = \frac{MP_j}{w_j}$$

$$\begin{cases} x_1^* = x_1^*(w_1, w_2, y) \\ x_2^* = x_2^*(w_1, w_2, y) \end{cases}$$

要素的价格

- Solve for conditional input demand function, $\mathbf{x}^*(\mathbf{w}, y)$, and cost function, $c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{x}^*$
- Interpretation of the Lagrange multiplier: $MC(y)$
- Minimum cost and isocost lines

Cost & Conditional Input Demand Functions

$\mathcal{L} = -wx + \lambda(f-y)$

$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda MP_1 = 0$

$\lambda = \frac{w_1}{MP_1}$ λ的计算!

- Cost function $c(\mathbf{w}, y)$
  - Expenditure function
  - $c(\mathbf{w}, y) = 0$ as $f(\mathbf{0}) = 0$
  - Continuous
  - Increasing in $\mathbf{w}$
  - Homogeneous of degree 1 in $\mathbf{w}$
  - Concave in $\mathbf{w}$
  - Shepard lemma: $\nabla_{\mathbf{w}} c(\mathbf{w}, y) = \mathbf{x}(\mathbf{w}, y)$
- Conditional input demand function $\mathbf{x}(\mathbf{w}, y)$
  - Hicksian demand function
  - Homogeneous of degree 0 in $\mathbf{w}$
  - Symmetric and negative semi-definite substitution matrix

三个重要的性质

包络 ⇒ 对L的求导.

$MC = \frac{\partial c}{\partial y} = \frac{\partial \mathcal{L}}{\partial y} = \lambda.$

$\frac{\partial c(w,y)}{\partial w_i} = x_i^*(w,y) \geq 0$

好理解!工资对成本的影响就是看人头.

要素价格上涨,成本肯定同步上涨.

看对谁求导.

维持产量,x不能变.

Cost Function for Homothetic Technology

When the production function is homothetic, $c(\mathbf{w}, y)$ is multiplicatively separable in input prices and output. It can be written as

$$c(\mathbf{w}, y) = h(y)c(\mathbf{w}, 1)$$

$f(x) = m(g(x)) \geq y$

$g(x) \geq m^{-1}(y)$

$g\left(\frac{1}{m^{-1}(y)} x\right) = \frac{1}{m^{-1}(y)} g(x) \geq 1$

where $h(y)$ is strictly increasing.

HOD 1

Proof:

$f(x)$ is homothetic $\Rightarrow f(x) = m(g(x))$, homogenous $g(.)$ and monotone $m(.)$.

$m^{-1}(1)\, g\left[\frac{x}{m^{-1}(y)}\right] \geq m^{-1}(1)$

$$c(\mathbf{w}, y) = min_{\mathbf{x}}\ \mathbf{w} \cdot \mathbf{x}\ s.t.\ m\left(\frac{m^{-1}(1)}{m^{-1}(y)} g(\mathbf{x})\right) \geq 1$$

$g\left[\frac{m^{-1}(1)}{m^{-1}(y)} x\right] \geq m^{-1}(1)$

取m

$f\left[\frac{m^{-1}(1)}{m^{-1}(y)} x\right] \geq 1$

Let $\widehat{\mathbf{x}} = \frac{m^{-1}(1)}{m^{-1}(y)} \mathbf{x}$,

$f(\hat{x}) \geq 1$

$$\begin{aligned} c(\mathbf{w}, y) &= \frac{m^{-1}(y)}{m^{-1}(1)} min_{\widehat{\mathbf{x}}}\ \mathbf{w} \cdot \widehat{\mathbf{x}}\ s.t.\ f(\widehat{\mathbf{x}}) \geq 1 \\ &= \frac{m^{-1}(y)}{m^{-1}(1)} c(\mathbf{w}, 1) = h(y)c(\mathbf{w}, 1) \end{aligned}$$

$c(w, 1) = w \cdot \hat{x}$

$= \frac{m^{-1}(1)}{m^{-1}(y)} \cdot c(w, y)$

y的函数

Homethetic and Homogeneous Production Functions

- When $f(x)$ is homothetic, $\mathbf{x}^*(\mathbf{w}, y)$ are multiplicatively separable in input prices and output.
  It can be written as $\mathbf{x}^*(\mathbf{w}, y) = h(y)\mathbf{x}(\mathbf{w}, 1)$, where $h(y)$ is strictly increasing.
- When the production function is homogeneous of degree $\alpha > 0$,

$$
\begin{aligned}
c(\mathbf{w}, y) &= y^{\frac{1}{\alpha}} c(\mathbf{w}, 1) \\
\mathbf{x}(\mathbf{w}, y) &= y^{\frac{1}{\alpha}} \mathbf{x}(\mathbf{w}, 1)
\end{aligned}
$$

Use the fact that

$$
f(\mathbf{x}) = y \Leftrightarrow f\left(\frac{\mathbf{x}}{y^{\frac{1}{\alpha}}}\right) = 1
$$

Short-run Cost Functions

长短期的生产关系.

SR中,有的要素不改变
如$K=\bar{K}$
$f(L,\bar{K})\geq y$ 一定binding
$\Rightarrow L^*(\bar{K},y)$
$\frac{\partial L^*}{\partial y}=\frac{1}{MP_L}$

A simple example with two inputs: capital K and labour L
Suppose capital is fixed at $\overline{K}$ in the short run

- Labour requirement function: $L(y;\overline{K}) = f^{-1}(y;\overline{K})$
- $\frac{\partial L(y;\overline{K})}{\partial y}$: the reciprocal of $MP_L = \frac{\partial y}{\partial L}$ 倒数.
- $\frac{L(y;\overline{K})}{y}$: the reciprocal of $AP_L = \frac{y}{L}$
- Marginal cost: $w \cdot \frac{\partial L(y;\overline{K})}{\partial y}$ $\frac{w}{MP_L}=\lambda$ 约束对成本的影响.
- Variable cost: $w \cdot L(y;\overline{K})$
- Average variable cost: $w \cdot \frac{L(y;\overline{K})}{y}$ 比生产量.
- Total cost: $w \cdot L(y;\overline{K}) + r\overline{K}$
- Average cost: $\frac{w \cdot L(y;\overline{K}) + r \cdot \overline{K}}{y}$

Short-run and Long-run Cost Curves

$\mathcal{L} \Rightarrow \begin{cases} L^*(w,r,y) \\ K^*(w,r,y) \end{cases}$

- Short-run cost curves with $\overline{K_1}$, $\overline{K_2}$,...
- Long run cost curve is the lower envelope of the entire family of short-run curves

Suppose (L^*, K^*) is the long-run optimal factor combination to produce y^* given the factor prices. Then L^* is also the required labour input for producing y^* in a short-run situation if the capital input is fixed at K^*

- $STC(y^*; K^*) = LTC(y^*)$, $SAC(y^*; K^*) = LAC(y^*)$
- $\underline{STC(y; K^*) > LTC(y)}$, $\underline{SAC(y; K^*) > LAC(y)}$ when $y \neq y^*$ K最小化.
- $SMC(y^*) = LMC(y^*)$
- $LMC(y) < SMC(y; K^*)$ to the right of y^*; $LMC(y) > SMC(y; K^*)$ to the left of y^*

Comparison in Isoquants Diagram

Start from an optimal point in the long run, given **w** and y

- ⋄ To increase output level to y', additional cost in the short run?
- ⋄ To increase output level to y', additional cost in the long run?
- ⋄ To reduce output level to y'', cost saving in the short run?
- ⋄ To reduce output level to y'', cost saving in the long run?

SMC and *LMC*

Denote by $\mathbf{x}_v$ the vector of variable inputs, and $\mathbf{x}_f$ the vector of fixed inputs.

1. The long-run problem is

$$c(\mathbf{w}, y) \equiv min_{\mathbf{x}_v, \mathbf{x}_f} \mathbf{w}_v \cdot \mathbf{x}_v + \mathbf{w}_f \cdot \mathbf{x}_f \ s.t. \ f(\mathbf{x}_v, \mathbf{x}_f) \geq y$$

$$FOC : w_s = LMC(y)MP_s, \ \forall x_s \ in \ \mathbf{x}$$

2. Let $\mathbf{x}_f = \bar{\mathbf{x}}_f$ in the short run.
   - The short-run cost minimization problem is

$$sc(\mathbf{w}, y, \bar{\mathbf{x}}_f) \equiv min_{\mathbf{x}_v} \mathbf{w}_v \cdot \mathbf{x}_v + \mathbf{w}_f \cdot \bar{\mathbf{x}}_f \ s.t. \ f(\mathbf{x}_v; \bar{\mathbf{x}}_f) \geq y$$

$$ET : \frac{\partial sc}{\partial x_j} = w_j - SMC(y)MP_j, \ \forall x_j \ in \ \mathbf{x}_f$$

   - The long-run problem is equivalent to

$$c(\mathbf{w}, y) \equiv min_{\mathbf{x}_f} \ sc(\mathbf{w}, y, \mathbf{x}_f)$$

$$FOC : \frac{\partial sc}{\partial x_j} = 0 = w_j - SMC(y)MP_j \ \forall x_j \ in \ \mathbf{x}_f \Rightarrow w_j = SMC(y)MP_j$$

Thus $SMC(y) = LMC(y)$ at $(\mathbf{x}^*_v, \mathbf{x}^*_f)$

Comparative Statics

Three firms all use labour L and capital K in their production. They have different technologies

- Firm A has a Leontief production function
- Firm B has a linear production function
- Firm C has a Cobb-Douglas production function

At the initial point, they incur the same total costs in producing a given amount of output.

There is a sudden increase of 10% in the capital rental price r. To produce the same level of output, how would their total costs change? Rank the three firms by the change in their total costs.

Application - Recover Market Power

Long tradition...
One recent work by De Loecker & Warzynski (*AER*, 2012)

- Suppose a firm has market power on the output market and is a price taker on the input market of factor i. Let the inverse demand for its output be $p(y)$.
- Market power can be measured by markup $\frac{p}{MC}$. If a firm is a cost minimizer, then its market power can be recovered from production information.

$$\begin{aligned}\text{Cost minimization} \quad &\Rightarrow \quad w_i = MC \cdot MP_i \\ &\Rightarrow \quad \frac{p}{MC} = \frac{pMP_i}{w_i} = \frac{p\Delta Q}{w_i \Delta x_i} = \frac{pQ}{w_i x_i}\frac{x_i \Delta Q}{Q \Delta x_i} = \frac{\mu_i}{\alpha_i}\end{aligned}$$

where μ_i is the output elasticity of factor i and α_i is its share of expenditure in total revenue.

Duality: $f(\mathbf{x}) \Leftrightarrow c(\mathbf{w}, y)$ and $\mathbf{x}(\mathbf{w}, y)$

成本函数如何恢复成生产函数

消掉 w.

- $c(\mathbf{w}, y) \Rightarrow f(\mathbf{x}) = y$?

$$\begin{aligned} f(\mathbf{x}) &\equiv max\,\{y \geq 0 | \mathbf{w}\cdot\mathbf{x} \geq c(\mathbf{w}, y), \forall \mathbf{w} \gg 0\} \\ &= max\left\{y \geq 0 | y \leq c^{-1}(\mathbf{w}, \mathbf{w}\cdot\mathbf{x}), \forall \mathbf{w} \gg 0\right\} \end{aligned}$$

  - Start with input $\mathbf{x}$
  - For any factor price vector $\mathbf{w}$, find all the output levels that can be produced with budget $\mathbf{w}\cdot\mathbf{x}$, denote the set by $\mathbf{Y_w}$
  - Construct the intersection of all these output sets $\mathbf{Y}{=}\cap\mathbf{Y_w}$
  - Find the largest element in $\{\mathbf{Y}\}$ - y
  - y is the value of the production function at $\mathbf{x}$
- $\mathbf{x}(\mathbf{w}, y) \Rightarrow f(\mathbf{x}) = y$?
  - $x_i(\mathbf{w}, y)$ is homogeneous of degree 0 and $\left(\frac{\partial x_i}{\partial w_j}\right)$ is a symmetric negative semi-definite matrix.
  - $\sum_{i=1}^{n} w_i x_i(\mathbf{w}, y)$ has all the properties of a cost function
  - Use this cost function to reconstruct the original technology

Special Cases: Cobb-Douglas and CES

- What technology can generate cost function

$$c(\mathbf{w}, y) = y w_1^{\alpha} w_2^{1-\alpha}$$

  - Conditional factor demand

$$\begin{aligned} x_1 &= \alpha y w_1^{\alpha-1} w_2^{1-\alpha} \\ x_2 &= (1-\alpha) y w_1^{\alpha} w_2^{-\alpha} \end{aligned}$$

  - Try to get rid of **w**

$$\left(\frac{x_1}{\alpha y}\right)^{\alpha} = \left(\frac{x_2}{(1-\alpha) y}\right)^{\alpha-1} \Rightarrow y = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} x_1^{\alpha} x_2^{1-\alpha}$$

- What technology can generate cost function

$$c(w_1, w_2, y) = \left(\left(\frac{w_1}{\alpha_1}\right)^{r} + \left(\frac{w_2}{\alpha_2}\right)^{r}\right)^{\frac{1}{r}} y$$

The CES production function

Isocost Curves in Factor Price Space

Curve of interest in the **w** space:

$$c(\mathbf{w}, y) \equiv \bar{c}$$

Take total derivative to find the slope $\frac{\Delta w_2}{\Delta w_1}$

$$|\frac{\Delta w_1}{\Delta w_2}| = \frac{\partial c/\partial w_2}{\partial c/\partial w_1} = \frac{x_2^*}{x_1^*}$$

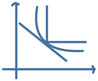
Slope of isoquants at the cost minimizing point in the input space

$$|\frac{\Delta x_2}{\Delta x_1}| = \frac{\partial f/\partial x_1}{\partial f/\partial x_2} = \frac{w_1}{w_2}$$

Curvature of Isoquants vs. Curvature of Isocost Curves

弯曲.

生产函数 and 成本函数.



用坐标轴表示.

开口小,变化多.

leo 完全反映了变化.

leo ⇔ linear 来回换

- small curvature of isoquants (linear production function)
  $\Rightarrow$ strong substitutability
  $\Rightarrow$ change in factor price $\rightarrow$ big adjustment in factor usage
  $\Rightarrow$ big curvature of isocost curves
- big curvature of isoquants (Leontief production function)
  $\Rightarrow$ weak substitutability
  $\Rightarrow$ change in factor price $\rightarrow$ small adjustment in factor usage
  $\Rightarrow$ small curvature of isocost curves

Cost Function and Returns to Scale

Elasticity of scale at cost minimizing point $\mathbf{x}^*$

$$
\begin{aligned}
\mu(\mathbf{x}^*) &= \sum_{i=1}^{n} \frac{MP_i x_i}{f(\mathbf{x})} \\
&= \sum_{i=1}^{n} \frac{w_i x_i}{MC(y) f(\mathbf{x})} \\
&= \frac{c(\mathbf{w}, y)}{MC(\mathbf{w}, y) f(\mathbf{x})} \\
&= \frac{AC(\mathbf{w}, y)}{MC(\mathbf{w}, y)}
\end{aligned}
$$

$MP_i = \frac{w_i}{MC}$

- Increasing local returns to scale $\Leftrightarrow \underline{AC > MC}$ 边际的更低，生产就完事了.
- Constant local returns to scale $\Leftrightarrow AC = MC$
- Decreasing local returns to scale $\Leftrightarrow AC < MC$

Profit Maximization Problem

Assume the output market is perfectly competitive, use $c(\mathbf{w}, y)$

$$Max_{y\geq 0} \quad py - c(\mathbf{w}, y)$$

$$FOC: \quad p = MC(y^*)$$

$$SOC: \quad \frac{d^2 c(\mathbf{w}, y)}{dy^2}|_{y=y^*} \geq 0$$

最小成本 代入$c(w,y)$

$\Rightarrow y^*(p,w) \Rightarrow c^*(p,w)$

$\Rightarrow \frac{\partial c^*}{\partial w^*} = x^*$.

An alternative method, 拉格朗日的方法.

$$Max_{\mathbf{x}\geq 0} \quad pf(\mathbf{x}) - \mathbf{w}\cdot\mathbf{x}$$

$$FOC: \quad pMP_i - w_i = 0 \ for \ \forall i = 1, ..., n$$

$$SOC: \quad \left(\frac{\partial MP_i}{\partial x_j}\right) \text{negative semi-definite}$$

- FOCs are the same - cost minimization implies $MC = \frac{w_i}{MP_i}$
- SOCs?

  - cost function convex in y and production function concave in $\mathbf{x}$

SOCs

SOCs are the same - concave production function (in x) implies convex cost function (in y)

- Given $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{x}'$ are the cost-minimizing input vectors for producing y and y'. Thus

$$tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}, y') = t\mathbf{w}\cdot\mathbf{x} + (1-t)\mathbf{w}\cdot\mathbf{x}' = \mathbf{w}\cdot\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)$$

- $f(.)$ is concave $\Rightarrow ty + (1-t)y' \leq f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)$
- $c(\mathbf{w}, y)$ is non-decreasing in y. Thus

$$\begin{aligned} c\left(\mathbf{w}, ty + (1-t)y'\right) &\leq c\left(\mathbf{w}, f\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)\right) \\ &\leq \mathbf{w}\cdot\left(t\mathbf{x} + (1-t)\mathbf{x}'\right) \\ &= tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}, y') \end{aligned}$$

Important Functions from Profit Maximization

Profit maximization (if a solution exists) $\Rightarrow$

- Output supply function: $y^*(p, \mathbf{w})$
- Input demand function: $\mathbf{x}^*(p, \mathbf{w})$
- Profit function: $\pi(p, \mathbf{w}) = py^*(p, \mathbf{w}) - \mathbf{w} \cdot \mathbf{x}^*(p, \mathbf{w})$

The maximum may not exist (SOCs not satisfied).
- e.g., increasing returns to scale technology

Profit Function Properties

- Increasing in p
- Decreasing in $\mathbf{w}$
- Homogeneous of degree 1 in $(p, \mathbf{w})$
- Hotelling's lemma (Envelope Theorem)

$$\begin{aligned} \frac{\partial \pi(p, \mathbf{w})}{\partial p} &= y^*(p, \mathbf{w}) \\ -\frac{\partial \pi(p, \mathbf{w})}{\partial w_i} &= x_i^*(p, \mathbf{w}) \textit{ for } \forall i = 1, \dots, n \end{aligned}$$

凸函数！

- Convex in $(p, \mathbf{w})$

Convex in Prices

$$
\begin{aligned}
\pi(p^t, \mathbf{w}^t) &= p^t y^t - \mathbf{w}^t \cdot \mathbf{x}^t \\
&= \left(tp + (1-t)p'\right) y^t - \left(t\mathbf{w} + (1-t)\mathbf{w}' \cdot\right) \mathbf{x}^t \\
&= t(py^t - \mathbf{w} \cdot \mathbf{x}^t) + (1-t)\left(p'y^t - \mathbf{w}' \cdot \mathbf{x}^t\right) \\
&\le t(py - \mathbf{w} \cdot \mathbf{x}) + (1-t)\left(p'y' - \mathbf{w}' \cdot \mathbf{x}'\right) \\
&= \pi(p, \mathbf{w}) + (1-t)\pi(p', \mathbf{w}')
\end{aligned}
$$

可以各自最大化利润.

Because of the maximization process,

- when there is a decrease in $\mathbf{w}$(or an increase in p), the profit is going to increase at least as fast as a linear function does
- when there is an increase in $\mathbf{w}$ (or a decrease in p), the profit is going to decrease at most as fast as a linear function does
- The Hessian matrix is positive semi-definite

Output Supply and Input Demand Functions

f(x)不能是规模回报递增. 永远不是最优的.
否则π不存在. 会不断增加 x^*

产量函数 y(p,w) 和 unconditional input demand x(p,w) 都 HOD 0 in (p,w)

- Both homogeneity of degree 0 in $(p, \mathbf{w})$
- Output supply: non-negative own price effects on profit

$$\frac{\partial y^*(p, \mathbf{w})}{\partial p} = \frac{\partial^2 \pi(p, \mathbf{w})}{\partial p^2} \geq 0$$

- Input demand: non-positive own price effects on profit

$$\frac{\partial x_i^*(p, \mathbf{w})}{\partial w_i} = -\frac{\partial^2 \pi(p, \mathbf{w})}{\partial w_i^2} \leq 0 \; for \; \forall i = 1, ..., n$$

- Hessian matrix of the profit function is symmetric and positive semi-definite

Short-run Profit Maximization

- Fixed costs: do not vary with output level
  - Sunk fixed costs: predetermined and cannot be changed *no matter* $y = 0$ *or* $y > 0$
  - Non-sunk fixed costs: not incurred if $y = 0$
- Total Costs and average costs
  - $STC = FC + TVC = (SC + \underbrace{NSFC) + TVC} = SC + NSC$
  - $SAC = AFC + AVC = (ASC + \underbrace{ANSFC) + AVC} = ASC + ANSC$
- $\pi = TR - STC = TR - (SC + NSC) = \underbrace{TR - NSC} - SC = PS - SC$
- Conditional on operating: $p = MC(y)$
- When to shut down?

Short-run Profit Maximization, continue

1. $P_1 = MC(q_1^*) < AVC(q_1^*) \Rightarrow TR(q_1^*) < TVC(q_1^*)$
   Shut down because of negative surplus
2. $AVC(q_2^*) < P_2 = MC(q_2^*) < ANSC(q_2^*) \Rightarrow$
   $TR(q_2^*) < NSC(q_2^*) = NSFC + TVC(q_2^*)$
   Shut down because of negative surplus
3. $P_3 = MC(q_3^*) = ANSC(q_3^*)$, thus the minimum point of $ANSC$
   $\Rightarrow TR(q_3^*) = NSC(q_3^*) = NSFC + TVC(q_3^*)$
   Shut down or stay (with negative profit and zero surplus)
4. $ANSC(q_4^*) < P_4 = MC(q_4^*) \leq SAC(q_4^*) \Rightarrow$
   $NSC(q_4^*) = NSFC + TVC(q_4^*) < TR(q_4^*) < STC(q_4^*)$
   Stay in business (with negative profit but positive surplus)
5. $P_5 = MC(q_5^*) > SAC(q_5^*) \Rightarrow TR(q_5^*) > STC(q_5^*)$
   Stay in business (with positive profit)