Roadmap for Special Topics on Consumer Theory

一生的消费 ≤ 一生的财富

$px \le m + w \cdot L_s$ (初始 m, 工资 w, 工作时间 L_s)

闲暇

$\Rightarrow px \le m + w(T - l) \Rightarrow u(x, l)$ s.t. $px + wl \le m + wT$

- Price-dependent endowment
  - Labor supply problem
  - Overtime decisions - saving & borrowing
  - Generalization
- Revealed preference - an alternative framework
  - Assumptions - rational & UMP vs. WARP
  - Predictions - (compensated) law of demands
- Aggregation (of choices)
  - Individual demand vs. aggregate demand
  - Income effect - role of wealth distribution
  - Price effect - does the (compensated) law of demand hold?

Labour Supply Problem (1)

- Strictly increasing utility function: $u(x, \ell)$
  where x represents the consumption of a composite good; and ℓ represents the consumption of leisure (non-market activities).
- Endowment: non-labour income Y and time T
- Total budget: $\mathcal{W} = Y + wT$
- The optimization problem

$$
\begin{aligned}
Max_{x,\ell} \quad & u(x, \ell) \\
s.t. \quad & px = Y + wL_s \\
& \ell + L_s = T \\
& x \geq 0 \\
& \ell \geq 0 \\
& \ell \leq T
\end{aligned}
$$

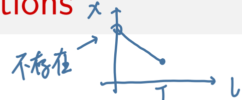
Labour Supply Problem (2)

Assume $lim_{x\to 0} MU_x = +\infty$, the optimization problem becomes

$$\begin{aligned} Max_{x,\ell} \quad & u(x,\ell) \\ s.t. \quad & px + w\ell = Y + wT \\ & \ell \geq 0 \\ & \ell \leq T \end{aligned}$$

First Order Conditions

角点、内点解



Lagrangian function:

非0约束写出

$$L(x, \ell, \lambda, \mu_0, \mu_T) = \underline{u(x,\ell) + \lambda\left[Y + wT - px - w\ell\right] + \mu_0 \ell + \mu_T\left[T - \ell\right]}$$

KT conditions:

$$MU_x = \lambda p; \; MU_\ell = \lambda w - \mu_0 + \mu_T$$

1. $0 < \ell < T \;\Rightarrow\; \frac{MU_x}{p} = \frac{MU_\ell}{w}$
2. $\mu_T > 0 \;\Rightarrow\; \ell = T, L_s = 0$ and $x = \frac{Y}{p}$, $\mu_0 = 0$

$$\frac{MU_x}{p} = \frac{MU_\ell - \mu_T}{w} \;\Rightarrow\; \frac{MU_x}{p} < \frac{MU_\ell}{w}$$

Labour market non-participants

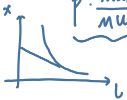
3. $\mu_0 > 0 \;\Rightarrow\; \ell = 0, L_s = T$ and $x = \frac{Y+wT}{p}$, $\mu_T = 0$

$$\frac{MU_x}{p} = \frac{MU_\ell + \mu_0}{w} \;\Rightarrow\; \frac{MU_x}{p} > \frac{MU_\ell}{w}$$

Solutions

Labour market participation decision

- Reservation wage w^R - willingness to work (supply labour)
  - implicit value of non-market activities (leisure)
  - wage rate at which an individual is indifferent between **participating ($\ell < T$)** and **withdrawing from ($\ell = T$)** labour market
  - w^R is the MRS at point $\ell = T$ (assuming $p_x = 1$)
- If $w^R > w$, withdraw, a corner solution $\frac{MU_l}{MU_x} > \frac{w}{P}$ 角解.
- If $w^R \leq w$, participate, an interior solution



$\underbrace{P \cdot \frac{MU_l}{MU_x}}_{w^R}$.

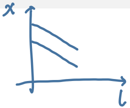
Labour supply conditional on participation ($w^R < w$)

- Marshallian demand for leisure $\ell(w, p, Y) = \ell(w, p, Y + wT)$
- Labour supply function $L_s = T - \ell(w, p, Y) = T - \ell(w, p, Y + wT)$

Comparative Static Analysis

- How does the budget line change?
- Impact on labour market participation
- Impact on labour supply (leisure demand) conditional on participating

Change in Non-labour Income Y



$w^R = p \cdot \frac{MU_L}{MU_x} \downarrow \quad \uparrow$

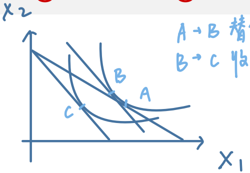
参加工作的意愿↓

If leisure is a normal good, when non-labor income Y increases

- ◇ Impact on labour supply, conditional on participating
  $Y \uparrow \Rightarrow$ demand for leisure $\uparrow \Rightarrow$ supply of labour $\downarrow$
- ◇ Impact on labour market participation
  $Y \uparrow \Rightarrow$ demand for leisure $\uparrow \Rightarrow$ implicit value of leisure $\uparrow \Rightarrow$ reservation wage $\uparrow$
  - Non-participants - still do not participate
  - Participants - may withdraw

  Thus, the overall participating rate $\downarrow$

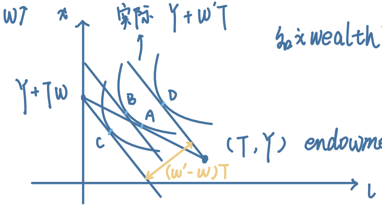
Change in Wage - Impact on Labour Market Participation



A→B 替代。
B→C 收入。

- Can a wage increase make a non-participant participate?
- Can a wage increase make a participant withdraw from labour market?
- Overall impact on participating rate?

假设$p_x=1$，线的夹角为w。



名义wealth变化 (w'-w)T (假设w不变)

(T, Y) endowment point

Change in Wage - Impact on Intensive Labour Supply (1)

和之后的genernal模型有关

Conditional on participating

$$L_s = T - \ell(w, p, \mathcal{W}) = T - \ell(w, p, Y + wT)$$

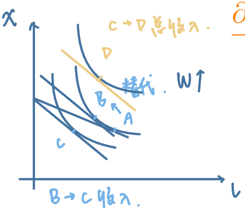
$$\frac{\partial \ell(w, p, Y + wT)}{\partial w} = \frac{\partial \ell(w, p, \mathcal{W})}{\partial w} + \frac{\partial \ell(w, p, \mathcal{W})}{\partial \mathcal{W}} T$$

收入.

wealth项

The standard Slutsky equation

收入效应 and 替代效应.



$$\frac{\partial \ell(w, p, \mathcal{W})}{\partial w} = \frac{\partial \ell^h}{\partial w} - \frac{\partial \ell(w, p, \mathcal{W})}{\partial \mathcal{W}} \ell^*$$

替代

收入.

替代效应 < 0.

Change in Wage - Impact on Intensive Labour Supply (2)

Put them together

反过来

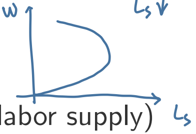
$$
\begin{aligned}
\frac{\partial \ell(w,p,Y+wT)}{\partial w} &= \frac{\partial \ell^h}{\partial w} - \frac{\partial \ell(w,p,\mathcal{W})}{\partial \mathcal{W}}\ell^* + \frac{\partial \ell(w,p,\mathcal{W})}{\partial \mathcal{W}} T \\
&= \frac{\partial \ell^h}{\partial w} + \frac{\partial \ell(w,p,\mathcal{W})}{\partial \mathcal{W}}(T-\ell^*) \\
\frac{\partial L_s}{\partial w} = -\frac{\partial \ell}{\partial w} &= \boxed{-\frac{\partial \ell^h}{\partial w}} - \boxed{\frac{\partial \ell(w,p,\mathcal{W})}{\partial \mathcal{W}}(T-\ell^*)}
\end{aligned}
$$

L_s 小 $\frac{\partial L_s}{\partial w} > 0$

L_s 大 $\frac{\partial L_s}{\partial w} < 0.$

工资越高反而 $L_s \downarrow$

> 0 数量 > 0



Substitution effect $\frac{\partial \ell^h}{\partial w} < 0$.

Income effect for leisure as a normal good $\frac{\partial \ell(w,p,\mathcal{W})}{\partial \mathcal{W}} > 0$.

The overall impact of an increase in w on leisure demand (labor supply) can be positive (negative) for large enough $T - \ell^*$.

- Upward sloping leisure demand curve.
- Backward bending labor supply curve.

Relevant Policy Issues

- Optimal income tax and the elasticity of labour supply
  $\uparrow$ income tax rate $\Rightarrow \downarrow$ labour supply and tax base $\Rightarrow$ ambiguous impact on tax revenue
- The debate between the conservatives and the liberals
  - Liberals: the high-income earners are **insensitive** to tax rates thus it is OK to raise their income tax and use the revenue to subsidize the poor
  - Conservatives: the high-income earners should not be taxed heavily as they work harder when the tax rate is low and the wage is high
- *If the conservatives are correct, what is the likely labour supply outcome of the following tax and transfer scheme?*
  - *Increase income tax rate*
  - *Return the tax revenue to households as lump-sum transfer*

Missing/Imperfect Market for Endowment

先天禀赋.

Example: Impacts of constraints on off-farm job opportunity

A simplified version of "Household composition, labor markets, and labor demand: testing for separation in agricultural household models", by Dwayne Benjamin, Econometrica 1992

An agriculture household's problem

雇佣. 能去外面工作.

$$
\begin{aligned}
max_{c,\ell,L^F,L^O,L^H} \quad & u(c,\ell) \\
s.t. \quad & c = F(L) - wL^H + wL^O + y \\
& L = L^F + L^H \\
& \ell + L^F + L^O = L^T
\end{aligned}
$$

能在农场.

- c - consumption; ℓ - leisure; $u(c,\ell)$ - hh utility function
- w - prevailing wage; y - other income; $F(L)$ - hh farm production function
- L^F - hh labour on farm; L^O - hh off-farm labour supply; L^H - hired labour on farm; L^T - total hh labour endowment

Lagrangian Function and F.O.C.s

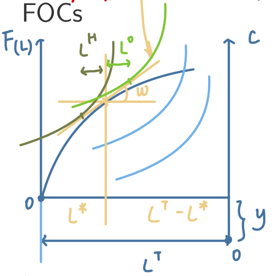
$p_c = 1$

$$
\begin{aligned}
L(c,\ell,L^F,L^O,L^H,\lambda_1,\lambda_2) \;=\;& u(c,\ell) \\
&+ \lambda_1\left[F(L^H+L^F) - wL^H + wL^O + y - c\right] \\
&+ \lambda_2\left[L^T - \ell - L^F - L^O\right]
\end{aligned}
$$

λ_1: marginal effect 多买goods

λ_2: leisure的 marginal effect.

FOCs



原点不同.

$$
\begin{array}{rcl}
c & : & MU_c = \lambda_1 \\
\ell & : & MU_\ell = \lambda_2 \\
\boxed{L^O} & : & \lambda_1 w = \lambda_2 \\
\boxed{L^H} & : & \cancel{\lambda_1} F'(L^*) = \cancel{\lambda_1} w \Rightarrow \text{得到}L^* \\
\boxed{L^F} & : & \lambda_1 F'(L^*) = \lambda_2 \\
 & & \ldots
\end{array}
$$

$$
\begin{aligned}
c &= F(L^*) - wL^H + wL^O + y \\
&= F(L^*) - w(L^* - L^F) + wL^O + y \\
&= F(L^*) - wL^* + w\underbrace{(L^F + L^O)}_{L^T - \ell} + y
\end{aligned}
$$

$c + w\ell = F(L^*) + y + w(L^T - L^*)$

过定点 $(L^T - L^*, F(L^*) + y)$

L^* is determined by $F'(L^*) = w$, that is, the optimal input in production (farming) L^* is independent of the choice of consumption.

Consumption Budget with $F(L^*)$

With L^*, the consumption budget becomes

$$\begin{aligned} c &= F(L^*) - wL^H + wL^O + y \\ &= F(L^*) - w(L^* - L^F) + wL^O + y \\ &= F(L^*) - wL^* + w(L^F + L^O) + y \\ &= F(L^*) - wL^* + w(L^T - l) + y \\ c + w\ell &= F(L^*) - wL^* + wL^T + y \end{aligned}$$

Thus,

- The slope of the budget line is w.
- $c = F(L^*) + y$, $\ell = L^T - L^*$ is one *feasible* bundle on the BL.
- Use tangency condition to find the *optimal* bundle (c^*, ℓ^*).
- Find L^{F^*} and L^{O^*}(or L^{H^*})

Constraint on Off-farm Job Opportunity L^O

Add one more constraint $\underline{L^O \le \bar{L}^O}$ 工作机会.

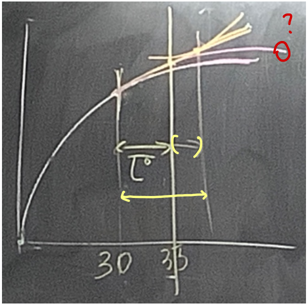
$$
\begin{aligned}
L(c, \ell, L^O, L^F, \lambda_1, \lambda_2, \mu) &= u(c, \ell) + \lambda_1 \left[F(L) + wL^O + y - c \right] \\
&+ \lambda_2 \left[L^T - \ell - L^F - L^O \right] + \underline{\mu \left[\bar{L}^O - L^O \right]}
\end{aligned}
$$

$\bar{L}^O$ is small and the constraint is binding. Let $L^{H*} = 0$.

F.O.C.s become

$$
\begin{aligned}
c &: \quad MU_c = \lambda_1 \\
\ell &: \quad MU_\ell = \lambda_2 \\
L^O &: \quad \lambda_1 w = \lambda_2 + \mu \\
L^F &: \quad \lambda_1 F' = \lambda_2
\end{aligned}
$$

$\frac{\lambda_2}{\lambda_1} = w - \frac{\mu}{\lambda_1}$



From the two equalities

$$F(L^T - \bar{L}^O - \ell^*) + w\bar{L}^O + y - c^* = 0$$

First Order Conditions

Production and consumption decisions are NOT separable.

$$\begin{aligned}
\frac{MU_\ell}{MU_c} = \frac{\lambda_2}{\lambda_1} &= F'(L^T - \overline{L^O} - \ell^*) \\
&= w - \frac{\mu}{\lambda_1} \\
&\equiv w^* \ \ \text{(shadow wage)} \\
&< w
\end{aligned}$$

Refer to w^* in balancing consumption of c and l

$$\frac{MU_\ell}{w*} = MU_c$$

The difference between market wage and shadow wage reflects the tightness of outside employment constraint

$$w - w^* = w - F'(L^*) = \frac{\mu}{\lambda_1}$$

Moral of the Story

- When the market for endowment is perfect and efficient, the household's production decision is separable from its consumption decision, which means the household can make decisions in two steps
  1. Maximize the value of the household endowment
  2. Optimize household consumption with the budget from Step 1
- When some market is missing or imperfect, the household would have to make production and consumption decisions simultaneously and end up with lower welfare level.
- A well functioning market system improves welfare.

Over-time Consumption Decision

- Decision over consumption today vs. consumption tomorrow
- Budget conditions
  - Endowment: income today and income tomorrow
  - Relative price and interest rate
    - Borrowing rate the same as saving rate
    - Borrowing rate higher than saving rate
- Optimization
  - Borrow or save?
  - How much?
- Comparative static analysis
  borrower vs. saver vs. "P-to-P" (paycheck to paycheck)
  - Increase in borrowing rate
  - Increase in saving rate

Problems with Endowment

- Endowment is a vector of goods $\mathbf{a} \geq \mathbf{0}$ the consumer owns.
- *UMP* with endowment

$$\begin{aligned} & \max_{\mathbf{x}} \; u(\mathbf{x}) \\ s.t. \quad & \mathbf{p} \cdot \mathbf{x} \leq y + \mathbf{p} \cdot \mathbf{a} \end{aligned}$$

- *EMP* with endowment

$$\begin{aligned} & \min_{\mathbf{x}} \; \mathbf{p} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{a} \\ s.t. \quad & u(\mathbf{x}) \geq u \end{aligned}$$

  Note the optimal choice is the same as the solution to

$$\begin{aligned} & \min_{\mathbf{x}} \; \mathbf{p} \cdot \mathbf{x} \\ s.t. \quad & u(\mathbf{x}) \geq u \end{aligned}$$

  But the value functions differ.

UMP with Endowment

- Lagrangian

$$\mathcal{L} = u(\mathbf{x}) + \lambda(y + \mathbf{p} \cdot \mathbf{a} - \mathbf{p} \cdot \mathbf{x}) + \sum_{\ell=1}^{L} \mu_\ell x_\ell$$

- KKT conditions

$$\begin{aligned} u_\ell(\mathbf{x}^*) - \lambda^* p_\ell + \mu_\ell^* &= 0 \\ \lambda^* \geq 0, w - \mathbf{p} \cdot \mathbf{x}^* + \mathbf{p} \cdot \mathbf{a} \geq 0, \lambda^*(y - \mathbf{p} \cdot \mathbf{a} - \mathbf{p} \cdot \mathbf{x}^*) &= 0 \\ \mu_\ell^* \geq 0, x_\ell^* \geq 0, \mu_\ell^* x_\ell^* &= 0 \end{aligned}$$

- Solution

$$\mathbf{x}^*(\mathbf{p}, y, \mathbf{a}), v(\mathbf{p}, y, \mathbf{a})$$

EMP with Endowment

- Lagrangian

$$\mathcal{L} = \mathbf{p} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{a} + \gamma(u - u(\mathbf{x})) - \sum_{\ell=1}^{L} \eta_\ell x_\ell$$

- The same KKT conditions as before. Solution

$$\mathbf{h}(\mathbf{p}, u), e(\mathbf{p}, u, \mathbf{a})$$

- Shephard's Lemma:

$$\frac{\partial e}{\partial p_\ell}\Big|_{\mathbf{x}^*} = h_\ell(\mathbf{p}, u) - a_\ell$$

Slutsky Equation with Endowment (1)

- The two demand functions $\mathbf{h}(\cdot)$ and $\mathbf{x}(\cdot)$ satisfy

$$h_\ell(\mathbf{p}, u) \equiv x_\ell(\mathbf{p}, e(\mathbf{p}, u, \mathbf{a}), \mathbf{a})$$

- Slutsky decomposition with endowment

$$\begin{aligned}
\frac{\partial h_i}{\partial p_j} &= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial p_j} \\
&= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}(h_j - a_j) \\
&= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}(x_j - a_j) \\
\frac{\partial x_i}{\partial p_j} &= \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial w}(x_j - a_j)
\end{aligned}$$

Slutsky Equation with Endowment (2)

Focus on the own price effect when $j = i$

- Key difference with endowment

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial w}(x_i - a_i)$$

- Even if x_i is a normal good, when a_i is large enough, $\partial x_i / \partial p_i$ may be positive.
  - when $x_i - a_i > 0$, the consumer is a *net buyer* of i; otherwise, he is a *net seller*.
  - Without endowment ($a_i = 0$), an increase in p_i lowers the consumer's purchasing power and induces welfare loss.
  - With endowment ($a_i > 0$), for a net seller of i ($x_i < a_i$), an increase in p_i raises the consumer's purchasing power through endowment effects.

Revealed Preference Relationship 显示性偏好

更优的消费点

- Observe choice of consumption bundles $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^n$ under different budget conditions (prices and wealth)

$$(\mathbf{p}^1, w^1), (\mathbf{p}^2, w^2), ..., (\mathbf{p}^n, w^n)$$

  Are these choices consistent with maximizing a (quasi-concave) utility function subject to the budget constraint?

- **Revealed preference**: binary relationship based on **observed** choices (instead of axioms on hypothetical preference)

Revealed Preference Definition

$c(\mathbf{p}, w)$ is the choice function

- $\mathbf{x}$ is chosen when $\mathbf{x}'$ is affordable $\Leftrightarrow$ $\mathbf{x}$ is revealed preferred to $\mathbf{x}'$

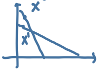
$$\mathbf{x} = c(\mathbf{p}, w) \text{ and } \mathbf{p} \cdot \mathbf{x}' \leq w \ \Leftrightarrow \ \mathbf{x} \succsim^R \mathbf{x}'$$

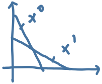
- $\mathbf{x}$ is chosen when $\mathbf{x}'$ is strictly affordable $\Leftrightarrow$ $\mathbf{x}$ is strictly revealed preferred to $\mathbf{x}'$

$$\mathbf{x} = c(\mathbf{p}, w) \text{ and } \mathbf{p} \cdot \mathbf{x}' < w \ \Leftrightarrow \ \mathbf{x} \succ^R \mathbf{x}'$$

Weak Axioms of Revealed Preferences

在p^0下买得起x^1，但还是选了x^0.
在p^1下买不起x^0，买了x^1





$\{p^0, x^0\}$ $\{p^1, x^1\}$

WARP: $\mathbf{p}^0 \cdot \mathbf{x}^1 \leq \mathbf{p}^0 \cdot \mathbf{x}^0 \Rightarrow \mathbf{p}^1 \cdot \mathbf{x}^0 > \mathbf{p}^1 \cdot \mathbf{x}^1$

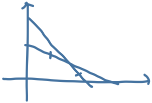
无法判断，但不违反.

- If $\mathbf{x}^0$ is (weakly) revealed preferred to $\mathbf{x}^1$ and they are different consumption bundles, $\mathbf{x}^1$ can not be (weakly) revealed preferred to $\mathbf{x}^0$
- The following example of choices does not satisfy WARP
  - when the budget is $(\mathbf{p}, w)$, the consumer chooses $\mathbf{x}$ and $\mathbf{p} \cdot \mathbf{x}' < w$
  - when the budget is $(\mathbf{p}', w')$, the consumer chooses $\mathbf{x}'$ and $\mathbf{p}' \cdot \mathbf{x} \leq w'$
- What if $\mathbf{x}$ is observed under $(\mathbf{p}, w)$, $\mathbf{x}'$ is observed under $(\mathbf{p}', w')$, while $\mathbf{p}' \cdot \mathbf{x} > w'$ and $\mathbf{p} \cdot \mathbf{x}' > w$?

无传递性

$w=8$

		x^1	x^2	x^3	
$p^1=(2,1,2)$	$x^1=(1,2,2)$	8	9	8	$x^1 \succsim^r x^3$
$p^2=(2,2,1)$	$x^2=(2,1,2)$	8	8	9	$x^2 \succsim^r x^1$
$p^3=(1,2,2)$	$x^3=(2,2,1)$	9	6	8	$x^3 \succsim^r x^2$



违反

form a loop.

Homogeneous of Degree 0 in $(\mathbf{p}, w)$

WARP + Walras' Law $\Rightarrow$ **Choice function H.D.0** in **p** and w

$$\begin{aligned} \mathbf{p}^0, w^0 &\rightarrow \mathbf{x}^0 \\ \mathbf{p}^1 = t\mathbf{p}^0, w^1 = tw^0, t > 0 &\rightarrow \mathbf{x}^1 \end{aligned}$$

If the choice function is not homogeneous of degree 0 in **p** and w, $\mathbf{x}^0$ and $\mathbf{x}^1$ need to be different

$$\underline{\mathbf{p}^1 \cdot \mathbf{x}^1} = w^1 = tw^0 = \underline{t\mathbf{p}^0 \cdot \mathbf{x}^0}$$

$$\begin{aligned} \mathbf{p}^0 \cdot \mathbf{x}^1 = \mathbf{p}^0 \cdot \mathbf{x}^0 &\Leftrightarrow \mathbf{x}^0 \succsim^R \mathbf{x}^1 \\ \mathbf{p}^1 \cdot \mathbf{x}^1 = \mathbf{p}^1 \cdot \mathbf{x}^0 &\Leftrightarrow \mathbf{x}^1 \succsim^R \mathbf{x}^0 \end{aligned}$$

Contradict with WARP, so $\mathbf{x}^1 = \mathbf{x}^0$

Compensated Law of Demand

WARP + Walras' Law $\Rightarrow$ Compensated Law of Demand

- price change $\mathbf{p}^2 = \mathbf{p}^1 + \Delta\mathbf{p}$
- Slutsky compensation $\Delta w = \Delta\mathbf{p} \cdot \mathbf{x}^1$ and $w^c = w^1 + \Delta w = \mathbf{p}^2 \cdot \mathbf{x}^1$
- WARP + Walras' Law $\Rightarrow$

$$
\begin{aligned}
\Delta\mathbf{p} \cdot \Delta\mathbf{x} &= (\mathbf{p}^2 - \mathbf{p}^1) \cdot \left(\mathbf{x}^c(\mathbf{p}^2, w^c) - \mathbf{x}^1(\mathbf{p}^1, w)\right) \\
&= \mathbf{p}^2 \cdot (\mathbf{x}^c - \mathbf{x}^1) - \mathbf{p}^1 \cdot (\mathbf{x}^c - \mathbf{x}^1) \\
&= -\mathbf{p}^1 \cdot (\mathbf{x}^c - \mathbf{x}^1) \\
&\leq 0
\end{aligned}
$$

where equality holds only when $\mathbf{x}^1 = \mathbf{x}^c$

- The second last step uses the rule of Slutsky compensation: $\mathbf{p}^2 \cdot \mathbf{x}^c = \mathbf{p}^2 \cdot \mathbf{x}^1$
- The last step is because of WARP: since $\mathbf{p}^2 \cdot \mathbf{x}^c = \mathbf{p}^2 \cdot \mathbf{x}^1$, we must have $\mathbf{p}^1 \cdot \mathbf{x}^c > \mathbf{p}^1 \cdot \mathbf{x}^1$ if $\mathbf{x}^1 \neq \mathbf{x}^c$.

Negative Semi-definite Substitution Matrix (1)

WARP + Walras' Law $\Rightarrow$ negative semi-definite substitution matrix

Assume a differentiable choice function $\mathbf{x}(\mathbf{p}, w)$. The compensated demand associated with a change of price from $\mathbf{p}^1$ to $\mathbf{p}^2$ is

$$\mathbf{x}^c = \mathbf{x}(\mathbf{p}^2, \mathbf{p}^2 \cdot \mathbf{x}^1)$$

With a change in the price of j

$$dx_i^c = (\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j^1) dp_j$$

With changes in the prices of multiple goods

$$dx_i^c = \sum_{j=1}^{L} (\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j^1) dp_j = \mathbf{s}_i \cdot d\mathbf{p}$$

Negative Semi-definite Substitution Matrix (2)

Stack all the compensated demand changes dx_i^c

$$d\mathbf{x}^c = Sd\mathbf{p}, \text{ where } S = \left[\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j^1\right]_{i,j}$$

We have shown that $\Delta\mathbf{p} \cdot \Delta\mathbf{x} \leq 0$ under WARP and Walras's Law.

$$d\mathbf{p} \cdot d\mathbf{x}^c = d\mathbf{p}^T S \, d\mathbf{p} \leq 0$$

S is negative semi-definite.

Symmetry (1)

Some properties of the substitution matrix

$$\mathbf{p}^T S(\mathbf{p}, w) = 0 \text{ and } S(\mathbf{p}, w)\mathbf{p} = 0, \text{ for } \forall \mathbf{p} \text{ and } w.$$

- The jth column of $\mathbf{p}^T S(\mathbf{p}, w)$ is

$$\sum_{i=1,\ldots,L} p_i \left(\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j \right) = \left(\sum_{i=1,\ldots,L} \frac{\partial x_i}{\partial p_j} p_i + x_j \right) + x_j \left(\sum_{i=1,\ldots,L} \frac{\partial x_i}{\partial w} p_i - 1 \right)$$

  0 by Walras' Law: changes in p_j and w for the two terms

- The ith row of $S(\mathbf{p}, w)\,\mathbf{p}$ is

$$\begin{aligned} \sum_{j=1,\ldots,L} \left(\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j \right) p_j &= \sum_{j=1,\ldots,L} \frac{\partial x_i}{\partial p_j} p_j + \frac{\partial x_i}{\partial w} \sum_{j=1,\ldots,L} x_j p_j \\ &= \sum_{j=1,\ldots,L} \frac{\partial x_i}{\partial p_j} p_j + \frac{\partial x_i}{\partial w} w \end{aligned}$$

  0 by Euler's Theorem: consumption choices $\mathbf{x}$ H.D.0 in $(\mathbf{p}, w)$

Symmetry(2)

- When $L = 2$, S is symmetric, i.e, $s_{1,2} = s_{2,1}$

$$\begin{aligned} p_1 s_{1,1} + p_2 s_{2,1} &= 0 \\ s_{1,1} p_1 + s_{1,2} p_2 &= 0 \end{aligned}$$

- When $L > 2$, S is NOT NECESSARILY symmetric

$$\begin{aligned} p_1 s_{1,1} + p_2 s_{2,1} + p_3 s_{3,1} &= 0 \\ s_{1,1} p_1 + s_{1,2} p_2 + s_{1,3} p_3 &= 0 \end{aligned}$$

NOT NECESSARY that $s_{1,2} = s_{2,1}$ and $s_{1,3} = s_{3,1}$

Transitivity(1)

Suppose WARP is satisfied

- When $L = 2$, the revealed preference is also transitive
  - Suppose NOT, then there exist $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ such that

$$\mathbf{a} \succsim^R \mathbf{b}, \quad \mathbf{b} \succsim^R \mathbf{c}, \quad \mathbf{c} \succsim^R \mathbf{a}$$

Without loss of generality, set $p_2^{\mathbf{a}} = p_2^{\mathbf{b}} = p_2^{\mathbf{c}} = 1$.

$$\begin{aligned}
\mathbf{a} \succsim^R \mathbf{b},\, \mathbf{c} \succsim^R \mathbf{a} \quad &\Rightarrow \quad p^{\mathbf{a}} c_1 + c_2 > p^{\mathbf{a}} a_1 + a_2 \geq p^{\mathbf{a}} b_1 + b_2 \\
\mathbf{a} \succsim^R \mathbf{b},\, \mathbf{b} \succsim^R \mathbf{c} \quad &\Rightarrow \quad p^{\mathbf{b}} a_1 + a_2 > p^{\mathbf{b}} b_1 + b_2 \geq p^{\mathbf{b}} c_1 + c_2 \\
\mathbf{b} \succsim^R \mathbf{c},\, \mathbf{c} \succsim^R \mathbf{a} \quad &\Rightarrow \quad p^{\mathbf{c}} b_1 + b_2 > p^{\mathbf{c}} c_1 + c_2 \geq p^{\mathbf{c}} a_1 + a_2
\end{aligned}$$

This is IMPOSSIBLE.

Transitivity(2)

- With $L > 2$, the revealed preference that satisfies WARP is NOT NECESSARILY transitive.
  - Example: cycle with $L = 3$

$$\begin{array}{lcl} \mathbf{p}^1 = (2,1,2) & , & \mathbf{x}^1 = (1,2,2) \\ \mathbf{p}^2 = (2,2,1) & , & \mathbf{x}^2 = (2,1,2) \\ \mathbf{p}^3 = (1,2,2) & , & \mathbf{x}^3 = (2,2,1) \end{array}$$

Strong Axiom of Revealed Preference

WARP + 传递性.

SARP(Houthakker 1950) rules out intransitive revealed preference.

- (JR) SARP is satisfied if, for every sequence of distinct bundles $\{\mathbf{x}^i\}_{i=1}^N$, where

$$\mathbf{x}^1 \succsim^R \mathbf{x}^2, \text{ and } \mathbf{x}^2 \succsim^R \mathbf{x}^3, \ . \ . \ . \ , \text{ and } \mathbf{x}^{k-1} \succsim^R \mathbf{x}^k$$

  it is not the case that

$$\mathbf{x}^k \succsim^R \mathbf{x}^1 \text{ for } \forall\ k = 2, .., N$$

- (MWG) If $\mathbf{x}^1 = \mathbf{x}(\mathbf{p}^1, w^1)$ is *directly or indirectly* revealed preferred to $\mathbf{x}^N = \mathbf{x}(\mathbf{p}^N, w^N)$, then $\mathbf{x}^N$ cannot be (directly) revealed preferred to $\mathbf{x}^1$, i.e, $\mathbf{p}^N \cdot \mathbf{x}^1 > w^N$. With $\mathbf{x}^{n+1} \neq \mathbf{x}^n$,

$$\mathbf{p}^n \cdot \mathbf{x}^{n+1} \leq w^n, \forall n \leq N-1 \Rightarrow \mathbf{p}^N \cdot \mathbf{x}^1 > w^N$$

GARP(Afriat 1967) ... $\Rightarrow \mathbf{p}^N \cdot \mathbf{x}^1 \geq w^N$; ... $\mathbf{x}^N$ not strictly r. p. to $\mathbf{x}^1$.

Application in Welfare Analysis of Tax

没收: x^0, $\{p^0, w^0\}$

T : x^T, $\{p^0, w^0-T\}$ $\quad T = t \cdot x^t$

t : x^t, $\{p^0+t, w^0\}$

$p^0x^0 = (p^0+t)x^t = w^0$

$p^0x^t + T = w^0$

$p^0x^t = w^0 - T = p^0x^T$

Revenue neutral per unit tax (distorting tax)

- Compared to lump-sum tax
- No tax vs. tax & rebate program with balanced budget

退还.

$(p^0+t)x^t = w^0 + c$

$p^0x^t = w^0 = p^0x^0$ 但买的是x^0!

Other Applications

A consumer spends all her income on X and Y. In period 1, she bought 20 units of X at \$5 per unit and 15 units of Y at \$5 per unit. In period 2, she bought 30 units of X at \$5 per unit and 10 units of Y at \$10 per unit.

- Draw the BL and find the consumption bundle for each period
- Which bundle does she prefer?
- Is she better off or worse off in the second period?
- What if in the second period she bought 12 units of X at \$10 per unit and 23 units of Y at \$5 per unit
- What if in the second period she bought 8 units of X at \$10 per unit and 30 units of Y at \$5 per unit

Index Numbers

- Change of interest: price index, quantity index
- Weights: Laspeyres Index (base) and Paasche Index (end)
- Welfare change over time if
  - CPI is lower than nominal income growth rate
  - real GDP growth is negative

Aggregation of Individual Demand

- Adding up individual demand/Stacking up individual demand curves $\Rightarrow$ aggregate demand (curve)
  - N consumers, $1, 2, ..., N$
  - $x_i^n(\mathbf{p}, w^n)$: the demand for i by the nth consumer
  - Aggregate demand for i

商品 i

$$\widehat{D}_i(\mathbf{p}, w^1, ..., w^N) = \sum_{n=1}^{N} x_i^n(\mathbf{p}, w^n)$$

- Aggregate demand function vs. individual demand function

Key Questions about Aggregate Demand Function

Is the aggregate demand derived this way consistent with the behaviour of a utility maximizing "representative consumer"?

- ⋄ When does the aggregate demand depend only on total wealth instead of the distribution of wealth?
- ⋄ What properties does the aggregate demand have? Homogeneous of degree 0 in price and total wealth? Walras' Law? WARP?

Dealing with Wealth (1)

The distribution of wealth does not matter for aggregate demand $\Longleftrightarrow$
There exists $D_i(\mathbf{p}, w)$ such that for $\forall\ (w^1, ..., w^N) > \mathbf{0}$ with $w = \sum_{n=1}^{N} w^n$

不影响!

$$D_i(\mathbf{p}, w) = \widehat{D}_i(\mathbf{p}, w^1, ..., w^N)$$

- The impact of a change in wealth distribution on aggregate demand

$$\sum_{n=1}^{N} \frac{\partial \widehat{D}_i}{\partial w^n}(\mathbf{p}, w^1, ..., w^N) = \sum_{n=1}^{N} \frac{\partial x_i^n(\mathbf{p}, \mathbf{w})}{\partial w^n}$$

- Redistribution of the same w does not affect aggregate demand

$$\text{For } \forall\ d\mathbf{w} \text{ such that } \sum_{n=1}^{N} dw^n = 0,\ \sum_{n=1}^{N} \frac{\partial x_i^n(\mathbf{p}, \mathbf{w})}{\partial w^n} dw^n = 0$$

Dealing with Wealth (2)

Consider the following redistribution plans

$$(1, -1, 0, ...0), (1, 0, -1, ..., 0), ..., (1, 0, 0, ..., -1)$$

If none of the above has any impact on the aggregate demand, we have

$$D_i(\mathbf{p}, w) \text{ exists } \Leftrightarrow \frac{\partial x_i^k(\mathbf{p}, \mathbf{w})}{\partial w^k} = \frac{\partial x_i^j(\mathbf{p}, \mathbf{w})}{\partial w^j} \text{ for } \forall j, k = 1, ..., N$$

- The same wealth effect for **all consumers** at **all wealth levels**.
- The income consumption curves (wealth expansion paths) are **parallel** and **straight** lines.

Examples of Common Wealth Effect

- Examples of preference that give "common" income effect?
  - Quasi-linear preference: 0 income effects
  - Identical homothetic preference
    - Homothetic thus the same at all income levels
    - Identical thus the same for all consumers
- Is there a general form?

Gorman Form (1)

- Gorman form: indirect utility function of consumer n

$$v^n(\mathbf{p}, w) = a^n(\mathbf{p}) + b(\mathbf{p})w^n$$

- Separable in prices and wealth
- Marshallian demand for i by consumer n

$$x_i^n(\mathbf{p}, w^n) = -\frac{\partial v^n/\partial p_i}{\partial v^n/\partial w^n} = -\frac{a_i^n(\mathbf{p}) + b_i(\mathbf{p})w^n}{b(\mathbf{p})}$$

Gorman Form (2)

- Income effect

$$\frac{\partial x_i^n}{\partial w^n} = -\frac{b_i(\mathbf{p})}{b(\mathbf{p})}$$

which does not depend on n or w^n. The same linear rate for all consumers at all wealth levels.

- The slope of the income consumption curve (wealth expansion curve)

$$\frac{\partial x_i^n}{\partial x_j^n} = \frac{\partial x_i^n / \partial w^n}{\partial x_j^n / \partial w^n} = \frac{b_i(\mathbf{p})}{b_j(\mathbf{p})}$$

Gorman Form (3)

Aggregation

$$
\begin{aligned}
D_i(\mathbf{p}, w^1, ..., w^N) &= -\sum_{n=1}^{N} \frac{a_i^n(\mathbf{p}) + b_i(\mathbf{p}) w^n}{b(\mathbf{p})} \\
&= -\sum_{n=1}^{N} \frac{a_i^n(\mathbf{p})}{b(\mathbf{p})} - \frac{b_i(\mathbf{p})}{b(\mathbf{p})} \sum_{n=1}^{N} w^n \\
&= -\sum_{n=1}^{N} \frac{a_i^n(\mathbf{p})}{b(\mathbf{p})} - \frac{b_i(\mathbf{p})}{b(\mathbf{p})} w^{total}
\end{aligned}
$$

Gorman Form (4)

- This proves the sufficiency of the Gorman form. The proof for the necessity - Gorman form is the only form that gives straight and parallel wealth expansion paths - is more complicated.
- Include information on income distribution in the AD function
  - Variance or other inequality measures of the wealth distribution
  - Specify wealth distributing rules $w^n = w^n(\mathbf{p}, w)$ so that

$$x_i^n(\mathbf{p}, w^n) = x_i^n(\mathbf{p}, w^n(\mathbf{p}, w)) = x_i^n(\mathbf{p}, w)$$

e.g., $w^n = \alpha^n w$ and $\sum_{n=1}^{N} \alpha^n = 1$.

Properties of Aggregate Demand

- Properties of individual demand shared by aggregate demand: Continuity, H.D.0 in $(\mathbf{p}, w)$, Walras' Law.
- How about compensated Law of demand? or WARP?
  - Compensated Law of Demand:

$$(\mathbf{p}' - \mathbf{p}) \cdot (D(\mathbf{p}', \mathbf{p}' \cdot D(\mathbf{p}, w)) - D(\mathbf{p}, w)) \leq 0$$

  - If WARP is satisfied

$$\mathbf{p} \cdot D(\mathbf{p}', w') \leq w \text{ and } D(\mathbf{p}, w) \neq D(\mathbf{p}', w') \Rightarrow \mathbf{p}' \cdot D(\mathbf{p}, w) > w'$$

WARP (1)

Individual WARP is NOT sufficient for aggregate WARP.

Example: Individuals $i = 1, 2$, $w^i = 0.5w$, *observe choices under* $\mathbf{p}$ *and* $\tilde{\mathbf{p}}$.

$$\mathbf{p} \cdot \mathbf{x}^i = 0.5w \text{ and } \tilde{\mathbf{p}} \cdot \tilde{\mathbf{x}}^i = 0.5w \text{ for } i = 1, 2$$

Suppose the above choices satisfy WARP and

$$\begin{array}{lllll} \mathbf{p} \cdot \tilde{\mathbf{x}}^1 < 0.5w & \& & \tilde{\mathbf{p}} \cdot \mathbf{x}^1 > 0.5w & \textit{thus} & \mathbf{x}^1 \succsim^R \tilde{\mathbf{x}}^1 \\ \mathbf{p} \cdot \tilde{\mathbf{x}}^2 > 0.5w & \& & \tilde{\mathbf{p}} \cdot \mathbf{x}^2 < 0.5w & \textit{thus} & \tilde{\mathbf{x}}^2 \succsim^R \mathbf{x}^2 \end{array}$$

It is possible *that the following is true*

$$\mathbf{p} \cdot (\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2) < w \text{ and } \tilde{\mathbf{p}} \cdot (\mathbf{x}^1 + \mathbf{x}^2) < w$$

WARP is violated in aggregate. Unrestricted wealth effects are crucial.

WARP (2)

Fixing the wealth distribution rule $\{\alpha^n\}_{n=1}^N$, $\alpha^n \geq 0\ \forall n$, $\sum_{n=1}^N \alpha^n = 1$.

- Suppose the price change from $\mathbf{p}$ to $\mathbf{p}'$ is compensated for consumer n by wealth adjustment from $\alpha^n w$ to $\alpha^n w'$, i.e.

$$\alpha^n w' = \mathbf{p}' \cdot \mathbf{x}^n(\mathbf{p}, \alpha^n w)$$

- Individual WARP means

$$(\mathbf{p}' - \mathbf{p}) \cdot \left(\mathbf{x}^n(\mathbf{p}', \alpha^n w') - \mathbf{x}^n(\mathbf{p}, \alpha^n w)\right) \leq 0$$

- Add across n to verify compensated price-wealth change and WARP at the aggregate level

$$w' = \mathbf{p}' \cdot \left(\sum_{n=1}^N \mathbf{x}^n(\mathbf{p}, \alpha^n w)\right) = \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w)$$

$$(\mathbf{p}' - \mathbf{p}) \cdot \left(\mathbf{x}(\mathbf{p}', w') - \mathbf{x}(\mathbf{p}, w)\right) \leq 0$$

WARP (3)

- Start with a price-wealth change that is compensated at the aggregated level, i.e,

$$w' = \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w)$$

- Consumers might be over or under compensated. It is possible that

$$\alpha^n w' = \alpha^n \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w) \neq \mathbf{p}' \cdot \mathbf{x}^n(\mathbf{p}, \alpha^n w)$$

  Despite WARP at the individual level, we may have

$$(\mathbf{p}' - \mathbf{p}) \cdot \left(\mathbf{x}^n(\mathbf{p}', \alpha^n w') - \mathbf{x}^n(\mathbf{p}, \alpha^n w)\right) \geq 0$$

- Adding up across n does not necessarily deliver

$$(\mathbf{p}' - \mathbf{p}) \cdot \left(\mathbf{x}(\mathbf{p}', w') - \mathbf{x}(\mathbf{p}, w)\right) \leq 0$$

Uncompensated Law of Demand and Aggregate WARP

- Uncompensated Law of Demand (ULD)

$$(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}(\mathbf{p}', w) - \mathbf{x}(\mathbf{p}, w)) \leq 0$$

  with equality only when $\mathbf{x}(\mathbf{p}', w) = \mathbf{x}(\mathbf{p}, w)$
- Individual ULD $\Rightarrow$ Aggregate ULD $\Rightarrow$ Aggregate WARP.

Individual ULD $\Rightarrow$ Aggregate ULD

$$
\begin{aligned}
(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}^n(\mathbf{p}', w^n) - \mathbf{x}^n(\mathbf{p}, w^n)) &\leq 0 \\
\sum_{n=1}^{N} (\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}^n(\mathbf{p}', w^n) - \mathbf{x}^n(\mathbf{p}, w^n)) &\leq 0 \\
(\mathbf{p}' - \mathbf{p}) \cdot (\sum_{n=1}^{N} \mathbf{x}^n(\mathbf{p}', w^n) - \sum_{n=1}^{N} \mathbf{x}^n(\mathbf{p}, w^n)) &\leq 0 \\
(\mathbf{p}' - \mathbf{p}) \cdot (D(\mathbf{p}', w) - D(\mathbf{p}, w)) &\leq 0
\end{aligned}
$$

with equality only when $D(\mathbf{p}', w) = D(\mathbf{p}, w)$

ULD $\Rightarrow$ CLD (1)

Choice function $\mathbf{x}(\mathbf{p}, w)$ is homogeneous of degree 0, satisfies Walras Law. The choice functin also satisfies ULD, that is, for $\forall \mathbf{p}^1$, w^1 and $\mathbf{p}$

$$(\mathbf{p} - \mathbf{p}^1) \cdot \left(\mathbf{x}(\mathbf{p}, w^1) - \mathbf{x}(\mathbf{p}^1, w^1)\right) \leq 0$$

with equality only when $\mathbf{x}(\mathbf{p}, w^1) = \mathbf{x}(\mathbf{p}^1, w^1)$.

We want to show it also satisfies WARP (therefore CLD), i.e,

$$\text{for any } \mathbf{x}^2 = \mathbf{x}(\mathbf{p}^2, w^2) \neq \mathbf{x}(\mathbf{p}^1, w^1)$$

$$\text{if } \mathbf{p}^1 \cdot \mathbf{x}^2 \leq w^1, \text{ then } \mathbf{p}^2 \cdot \mathbf{x}^1 > w^2$$

ULD $\Rightarrow$ CLD (2)

With H.D.0 choice function,

$$\mathbf{x}^2 = \mathbf{x}(\mathbf{p}^2, w^2) = \mathbf{x}(\frac{w^1}{w^2}\mathbf{p}^2, w^1)$$

With ULD,

$$\begin{aligned}
(\frac{w^1}{w^2}\mathbf{p}^2 - \mathbf{p}^1) \cdot \left(\mathbf{x}(\frac{w^1}{w^2}\mathbf{p}^2, w^1) - \mathbf{x}(\mathbf{p}^1, w^1)\right) &\leq 0 \\
(\frac{w^1}{w^2}\mathbf{p}^2 - \mathbf{p}^1) \cdot (\mathbf{x}^2 - \mathbf{x}^1) &\leq 0 \\
(w^1 - \frac{w^1}{w^2}\mathbf{p}^2 \cdot \mathbf{x}^1) + (w^1 - \mathbf{p}^1 \cdot \mathbf{x}^2) &\leq 0 \\
w^1 - \frac{w^1}{w^2}\mathbf{p}^2 \cdot \mathbf{x}^1 &\leq 0
\end{aligned}$$

With $\mathbf{x}^1 \neq \mathbf{x}^2$

$$w^1 - \frac{w^1}{w^2}\mathbf{p}^2 \cdot \mathbf{x}^1 < 0 \text{ and } \mathbf{p}^2 \cdot \mathbf{x}^1 > w^2$$

Homothetic Preference and ULD (1)

Demand functions based on a homothetic preference satisfies ULD.

- Homethetic preference $\Rightarrow$ unitary income elasticity for all ℓ

$$\frac{\partial x_i}{\partial w}\frac{w}{x_i} = 1 \;\Rightarrow\; \frac{\partial x_i}{\partial w}x_j = \frac{x_i x_j}{w}$$

- Denote by S^M the Marshallian price effect matrix composed of $\frac{\partial x_i}{\partial p_j}$

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial w}x_j = \frac{\partial h_i}{\partial p_j} - \frac{x_i x_j}{w}$$

- Denote by S^H the Hicksian substitution matrix of $\frac{\partial h_i}{\partial p_j}$. Denote by $M = \mathbf{x}\mathbf{x}^T$ the Kronecker product of $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ and $\mathbf{x}^T = (x_1, x_2, ..., x_n)$; $x_i x_j$ is the ij-th entry.

$$S^M = S^H - \frac{1}{w}M$$

Homothetic Preference and ULD (2)

- Examine the definiteness of S^M

$$d\mathbf{p}^T S^M d\mathbf{p} = d\mathbf{p}^T S^H d\mathbf{p} - \frac{1}{w} d\mathbf{p}^T M d\mathbf{p}$$

- S^H is negative semi-definite.
- What about M?

$$d\mathbf{p}^T M d\mathbf{p} = d\mathbf{p}^T \mathbf{x}\mathbf{x}^T d\mathbf{p} = \left(d\mathbf{p}^T \mathbf{x}\right)^2 \geq 0$$

Therefore S^M is negative semi-definite and ULD is satisfied.