Lecture #22: Type Inference and Unification

Typing In the Language ML

- Examples from the language ML:

```
fun map f [] = []
 |  map f (a :: y) = (f a) :: (map f y)
fun reduce f init [] = init
 |  reduce f init (a :: y) = reduce f (f init a) y
fun count [] = 0
 |  count (_ :: y) = 1 + count y
fun addt [] = 0
    addt ((a,_,c) :: y) = (a+c) :: addt y
```

- Despite lack of explicit types here, this language is statically typed!
- Compiler will reject the calls `map 3 [1, 2]` and `reduce (op +) [] [3, 4, 5]`.
- Does this by *deducing* types from their uses.

Type Inference

- In simple case:

```
fun add [] = 0
 |  add (a ::  L) = a + add L
```

compiler deduces that `add` has type `int list` $\rightarrow$ `int`.

- Uses facts that (a) 0 is an `int`, (b) `[]` and `a::L` are lists (`::` is cons), (c) + yields `int`.
- More interesting case:

```
fun count [] = 0
 |  count (_ ::  y) = 1 + count y
```

(_ means "don't care" or "wildcard"). In this case, compiler deduces that `count` has type α `list` $\rightarrow$ `int`.

- Here, α is a *type parameter* (we say that `count` is *polymorphic*).

Aside: Runtime Implementation of Polymorphism

- The last example works for any value of α:

```
fun count [] = 0
 |  count (_ ::  y) = 1 + count y
```

- As is also the case here, where the type of `x` is known to be `bool`, but the types of `z` and `y` are unknown.

```
fun iffy x y z = if x then z else y;
```

- When we get to implementation, we'll see that no special run-time testing is required to bring this about.
- In typical implementations, all types have the same representation at the machine-code level—they are words containing pointers (or possibly integers), for which assignment and parameter passing involve the same instructions regardless of contents.
- Hence, a single translation works for all types.

Doing Type Inference

- Given a definition such as

```
fun add [] = 0
 |  add (a ::  L) = a + add L
```

- First give each named entity here an unbound type parameter as its type: `add:`α`,  a:`β`,  L:`γ`.`
- Now use the type rules of the language to give types to everything and to *relate* the types:
  - 0: `int`, `[]:` δ `list`.
  - Since `add` is function and applies to `int`, must be that $\alpha = \iota \rightarrow \ \kappa$, and $\iota = \delta$ `list`
  - etc.
- Gives us a large set of *type equations*, which can be solved to give types.
- Solving involves *pattern matching*, known formally as *unification*.

Type Expressions

- For this lecture, a type expression can be
  - A *primitive type* (`int`, `bool`);
  - A *type variable* (today we'll use ML notation: 'a, 'b, 'c$_1$, etc.);
  - The *type constructor* T `list`, where T is a type expression (what we'll write as `list of` $[T]$ for the project);
  - A *function type* $D \rightarrow C$, where D and C are type expressions.
- Will formulate our problems as systems of *type equations* between pairs of type expressions.
- Need to find the substitution (the *unifier*) that solves the system (simultaneously makes all the equations true).

Solving Simple Type Equations

- Simple example: solve

  $$\texttt{'a list} = \texttt{int list}$$

- Easy: $\texttt{'a} = \texttt{int}$.
- How about this:

  $$\texttt{'a list} = \texttt{'b list list};\ \ \texttt{'b list} = \texttt{int list}$$

- Also easy: $\texttt{'a} = \texttt{int list};\ \ \texttt{'b} = \texttt{int}$.
- On the other hand:

  $$\texttt{'a list} = \texttt{'b} \rightarrow\ \texttt{'b}$$

  is unsolvable: lists are not functions.
- Also, if we require *finite* solutions, then

  $$\texttt{'a} = \texttt{'b list}; \texttt{'b} = \texttt{'a list}$$

  is unsolvable. However, our algorithm will allow infinite solutions.

Most General Solutions

- Rather trickier:

  `'a list= 'b list list`

- Clearly, there are lots of solutions to this: *e.g*,

  $\texttt{'a} = \texttt{int list}; \quad \texttt{'b} = \texttt{int}$

  $\texttt{'a} = (\texttt{int} \rightarrow \texttt{int})\,\texttt{list}; \quad \texttt{'b} = \texttt{int} \rightarrow \texttt{int}$

  etc.

- But prefer a *most general* solution that will be compatible with *any* possible solution.
- Any substitution for `'a` must be some kind of list, and `'b` must be the type of element in `'a`, but otherwise, no constraints
- Leads to solution

  $\texttt{'a} = \texttt{'b list}$

  where `'b` remains a free type variable.

- In *general*, our solutions look like a bunch of equations $\texttt{'a}_i = T_i$, where the T_i are type expressions and none of the $\texttt{'a}_i$ appear in any of the T's.

Finding Most-General Solution by Unification

- To *unify* two type expressions is to find substitutions for all type variables that make the expressions identical.
- The set of substitutions is called a *unifier*.
- Represent substitutions by giving each type variable, $'\tau$, a *binding* to some type expression.
- The algorithm that follows treats type expressions as objects (so two type expressions may have identical content and still be different objects). All type variables with the same name are represented by the same object.
- Initially, each type expression object is *unbound*.

Unification Algorithm, Simple Version (Noncircular)

- For any type expression, T, and unifier u, define

$$u[T] = \begin{cases} u[T'], & \text{if } T \text{ is bound to type expression } T' \\ T, & \text{otherwise} \end{cases}$$

- Now proceed recursively:

```
unify (TA,TB,u):
   """Returns an extension of unifier u that unifies TA and TB or None."""
   TA = u[TA]; TB = u[TB]
   if TA.isFreeTypevar(u):    #  If TA is type variable not bound in u
      return u.bind(TA, TB)
   if TB.isFreeTypeVar(u):
      return u.bind(TB, TA)
   if TA is C(TA_1,TA_2,...,TA_n) and TB is C(TB_1,...,TB_n):
      for i in range(n):
         u = unify(TA_i, TB_i, u)
         if u is None:  return None
      return u
   return None
```

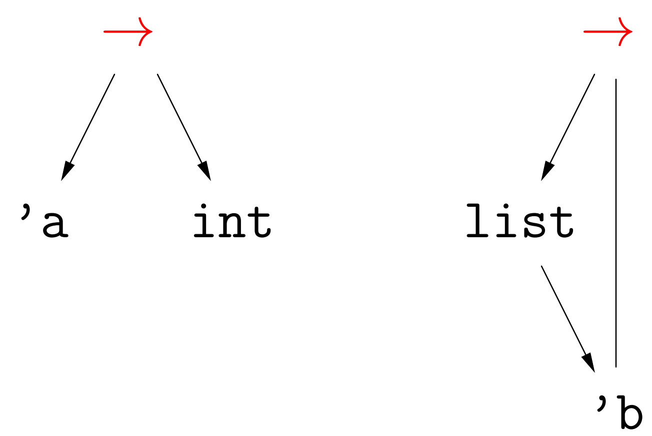
Example of Unification I

- Try to solve $A = B$, where

  A = 'a $\rightarrow$ int; B = 'b list$\rightarrow$ 'b

  by computing unify(A, B).

Dashed arrows are bindings
Red items are current TA and TB



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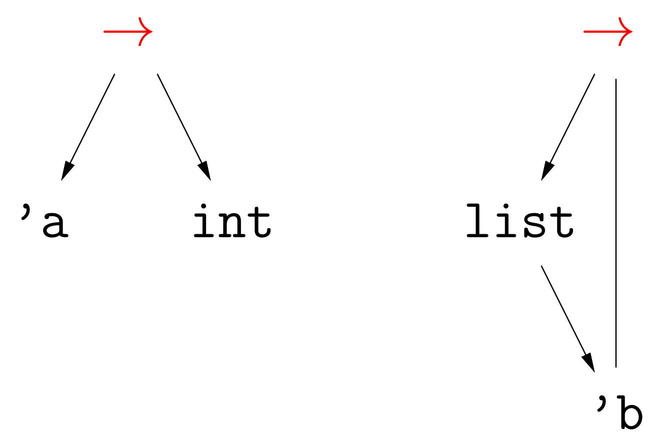

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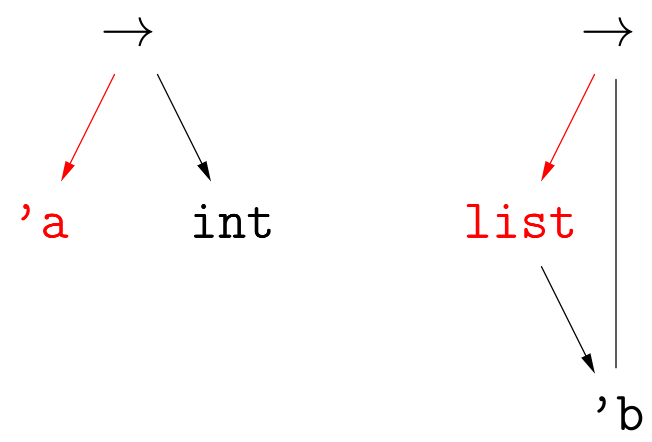

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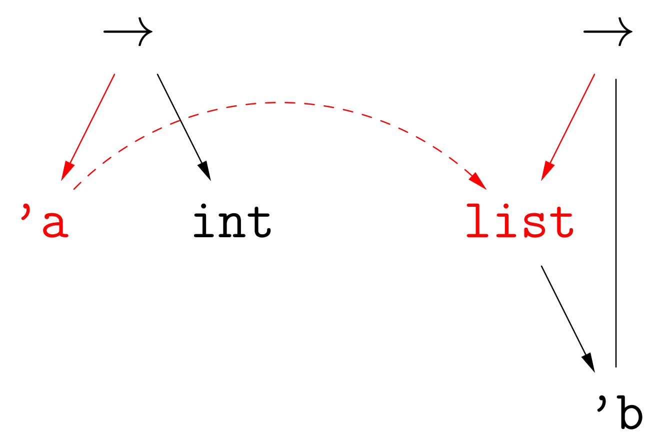

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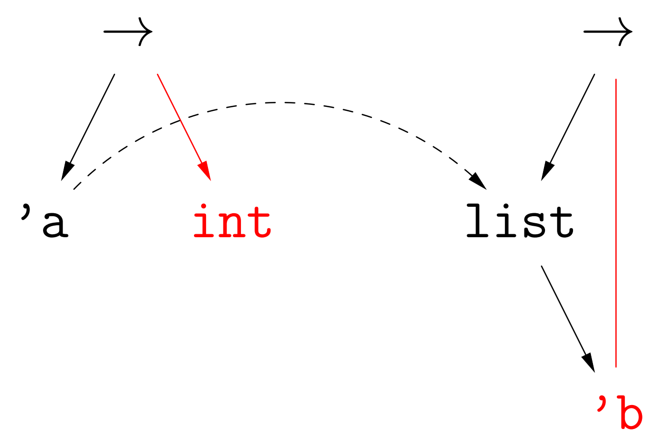

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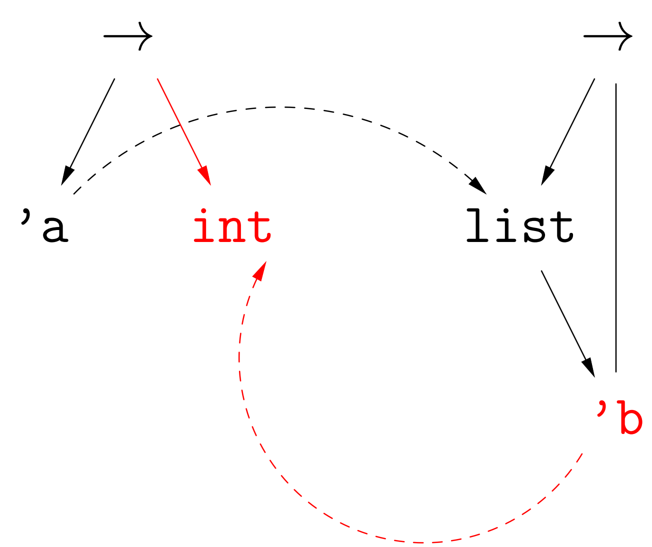

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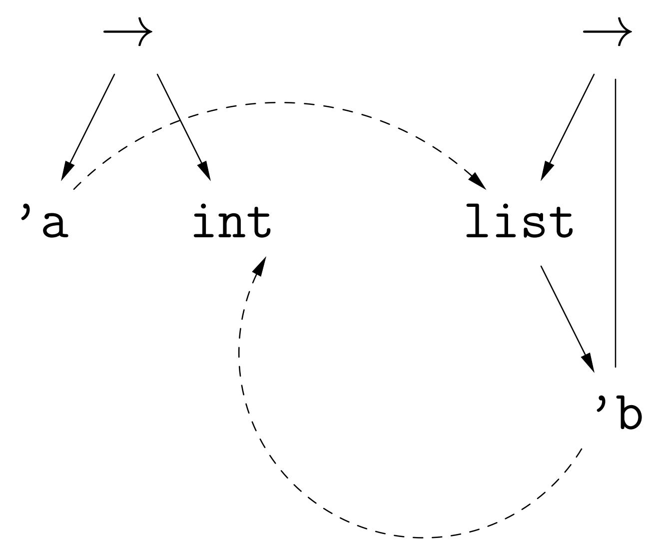


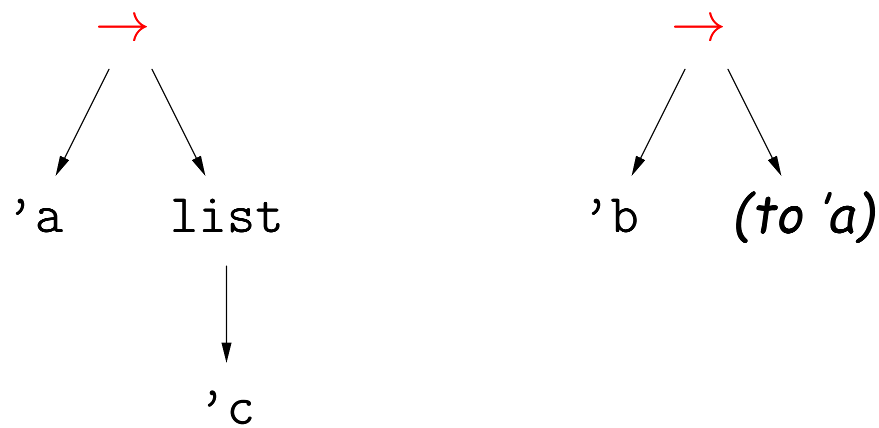
So 'a $=$ int list and 'b $=$ int.

Example of Unification II

- Try to solve $A = B$, where

  A = `'a` $\rightarrow$ `'c list`; B = `'b` $\rightarrow$ `'a`

  by computing `unify`(A, B).



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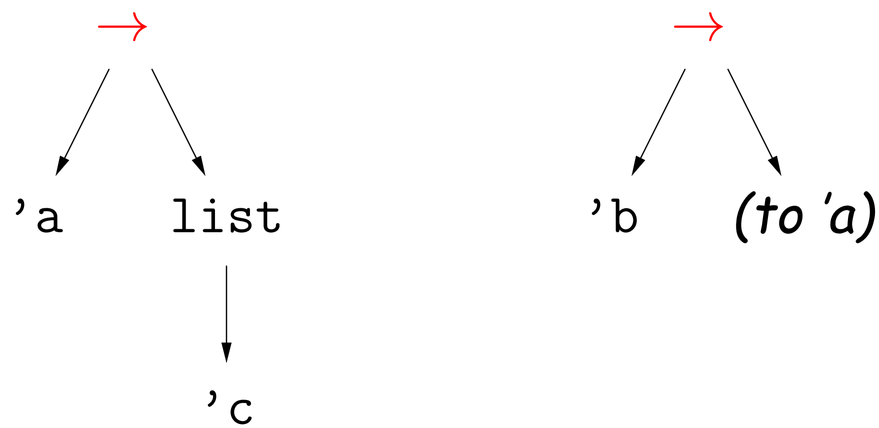

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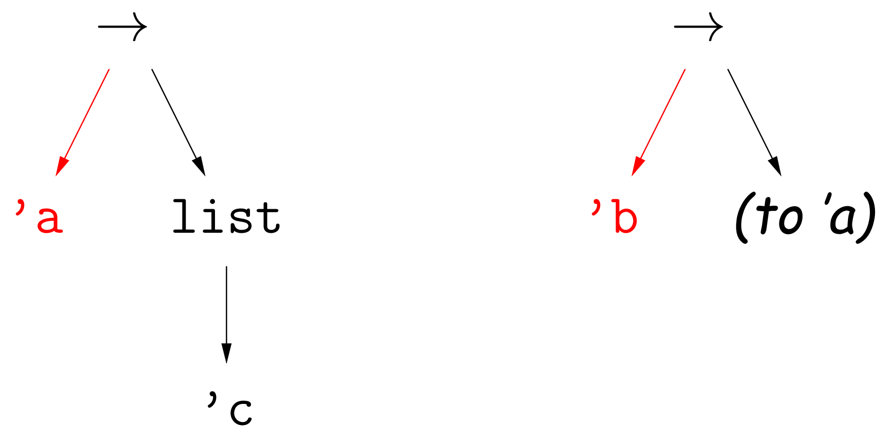

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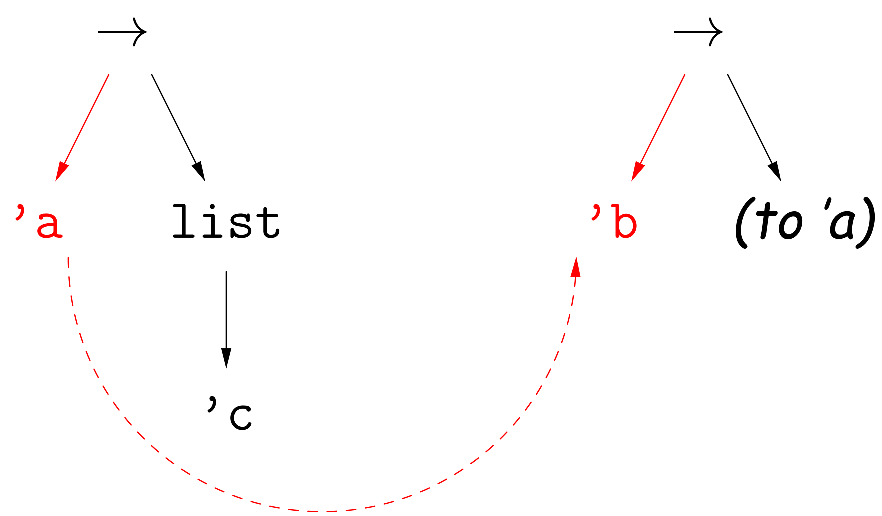

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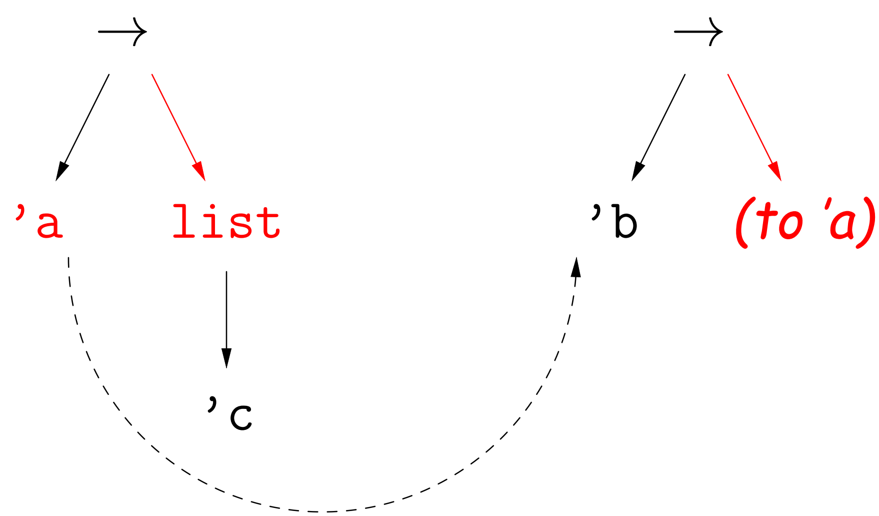

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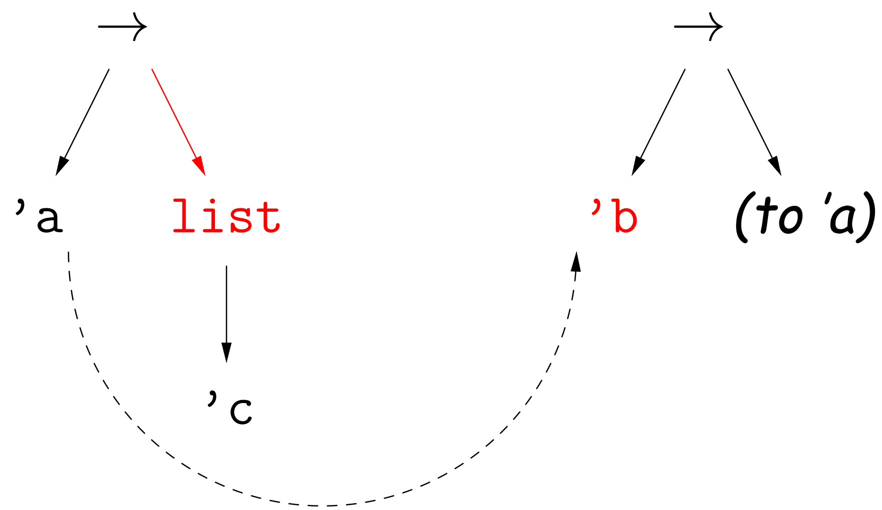

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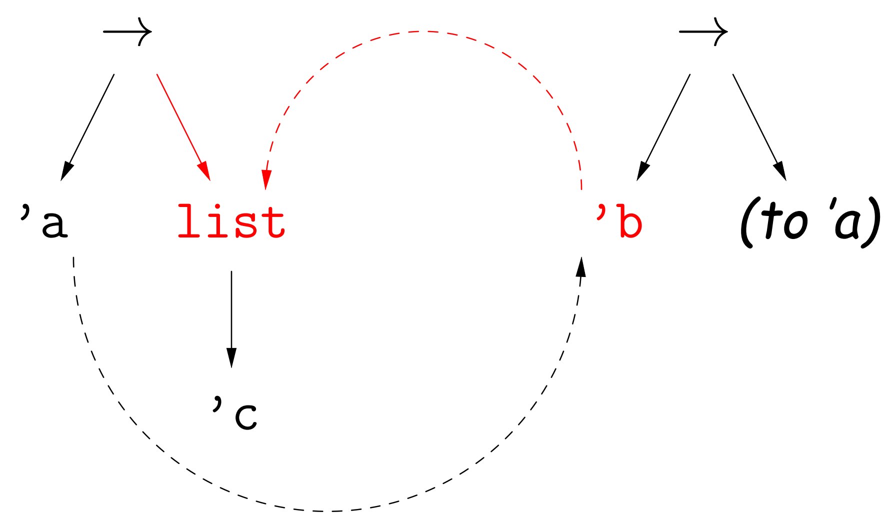

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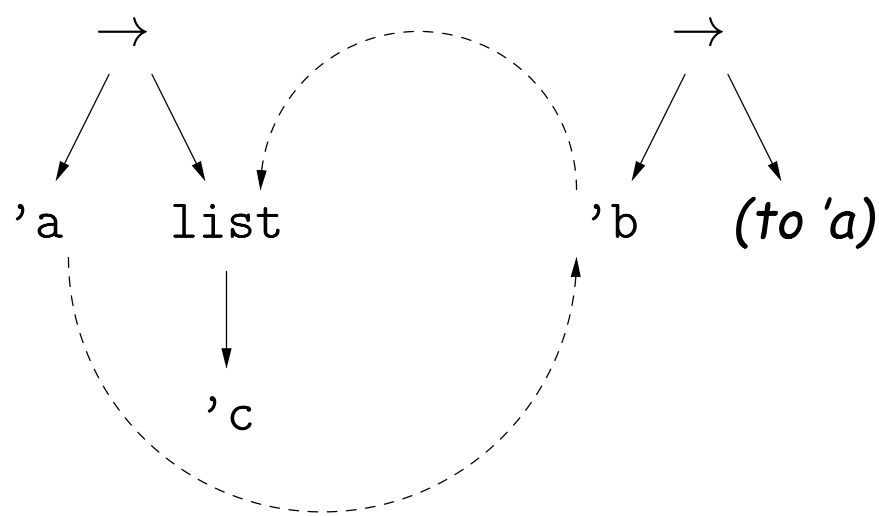


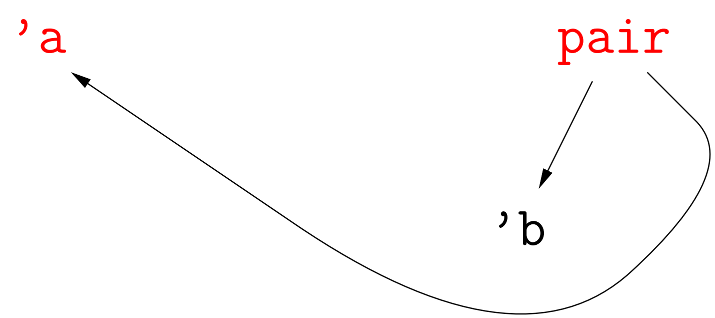
So 'a = 'b = 'c list and 'c is free.

Example of Unification III: Simple Recursive Type

- Introduce a new type constructor: `('h,'t) pair`, which is intended to model typed Lisp cons-cells (or nil). The `car` of such a pair has type `'h`, and the `cdr` has type `'t`.
- Try to solve $A = B$, where

  A = `'a`; B = (`'b`, `'a`) `pair`

  by computing `unify`(A, B).
- This one is very easy:



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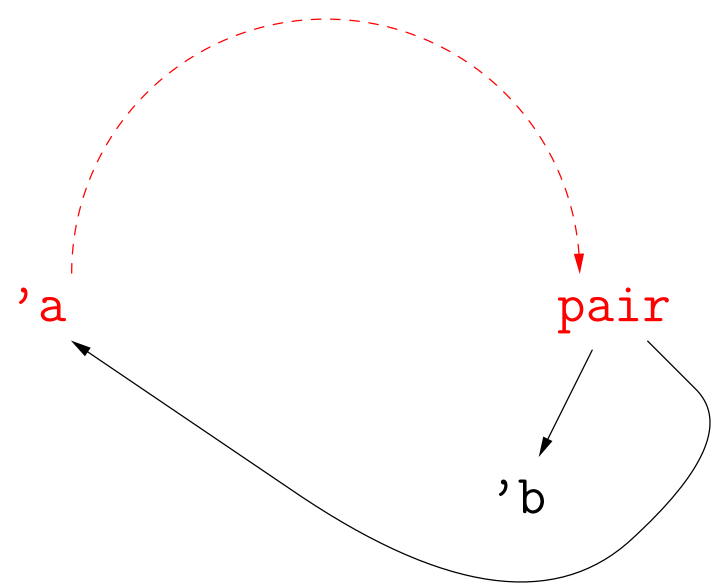

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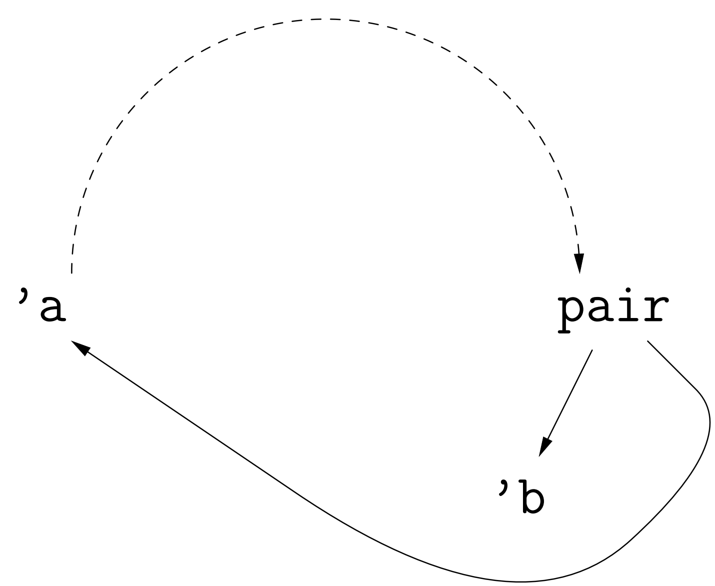

So `'a` = (`'b`, `'a`) `pair`; `'b` is free.

Example of Unification IV: Another Recursive Type

- This time, consider solving $A = B,\ C = D,\ A = C$, where

  A = `'a`; B = (`'b`, `'a`) `pair`; C = `'c`; D = (`'d`, (`'d`, `'c`) `pair`) `pair`.

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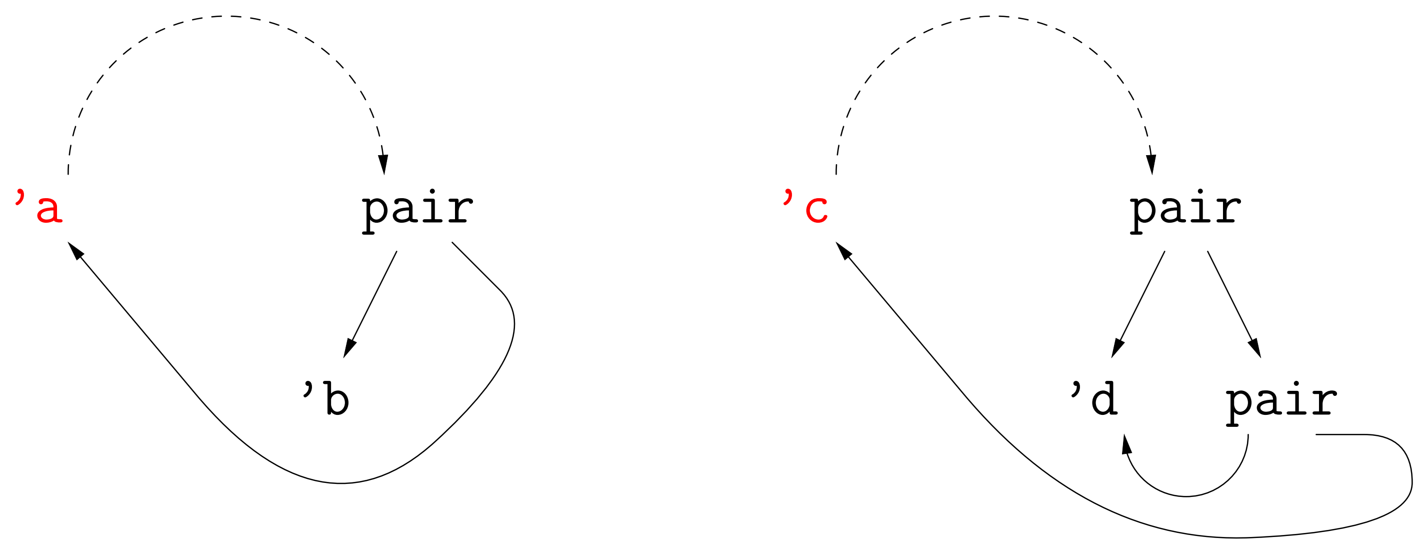

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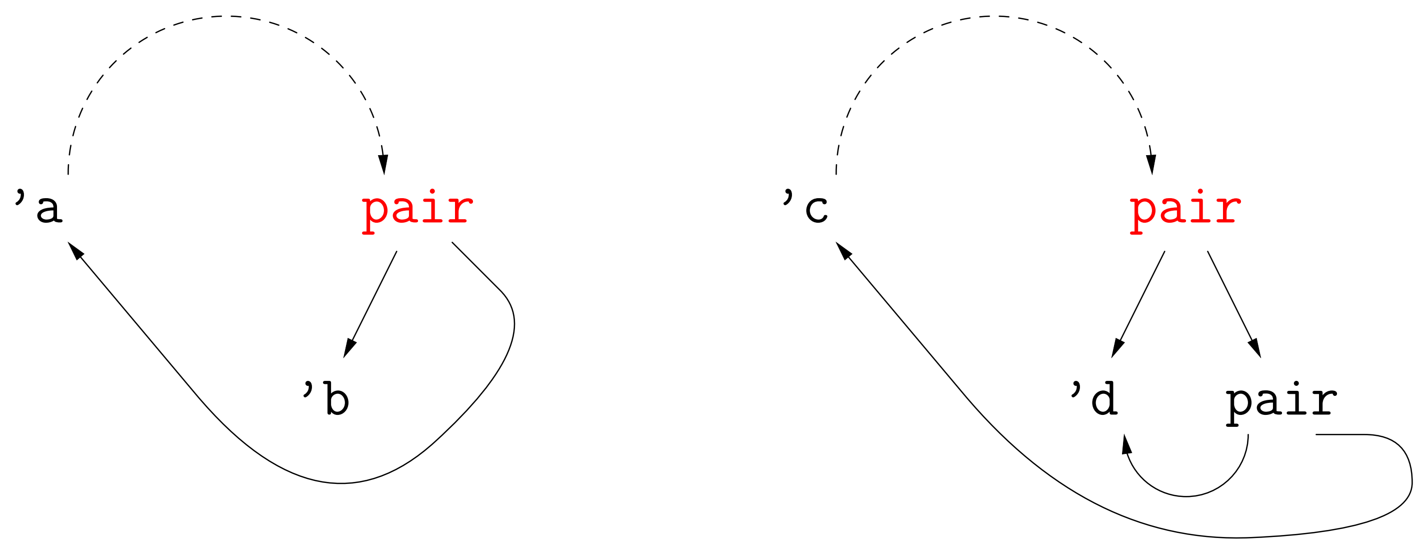

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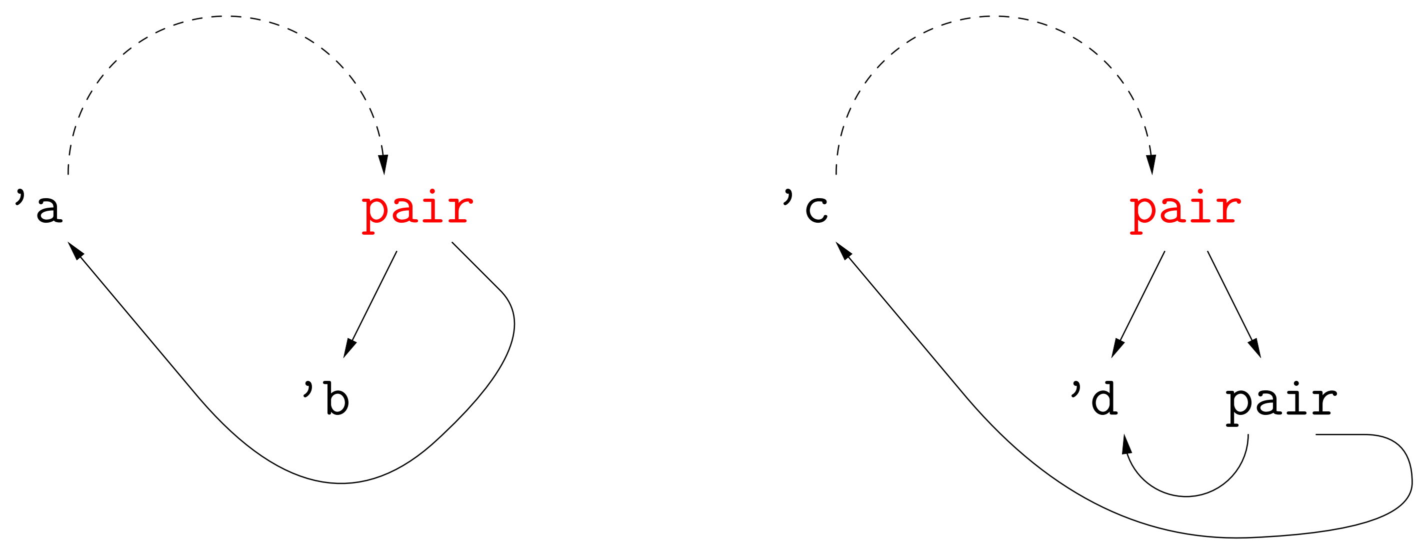

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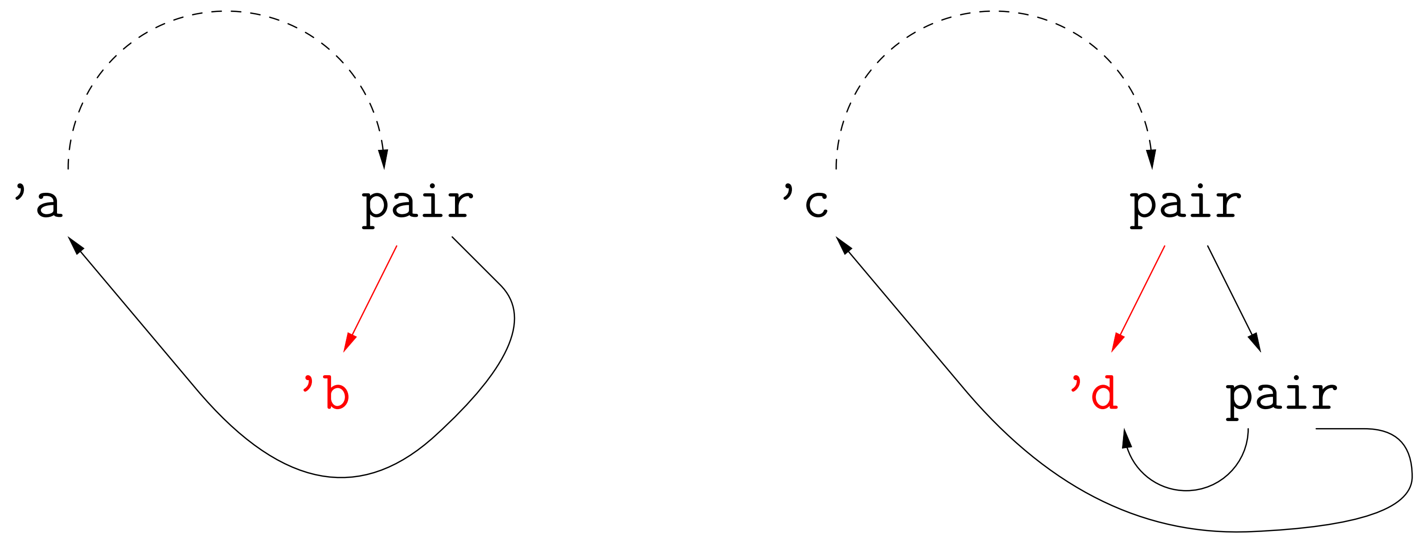

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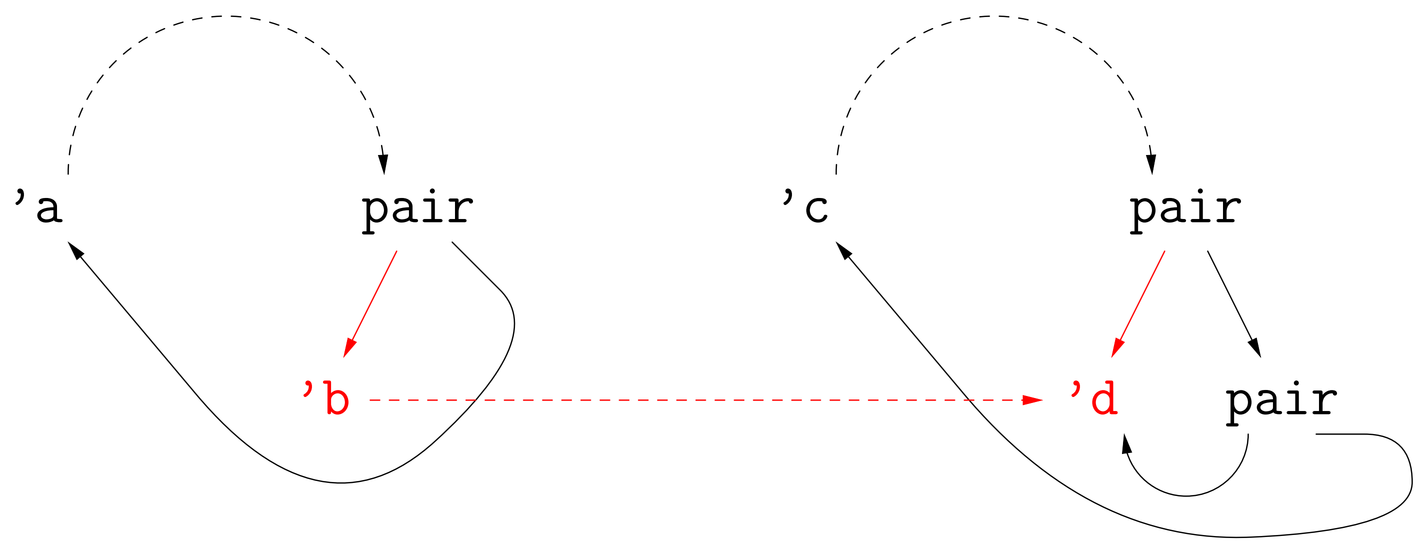

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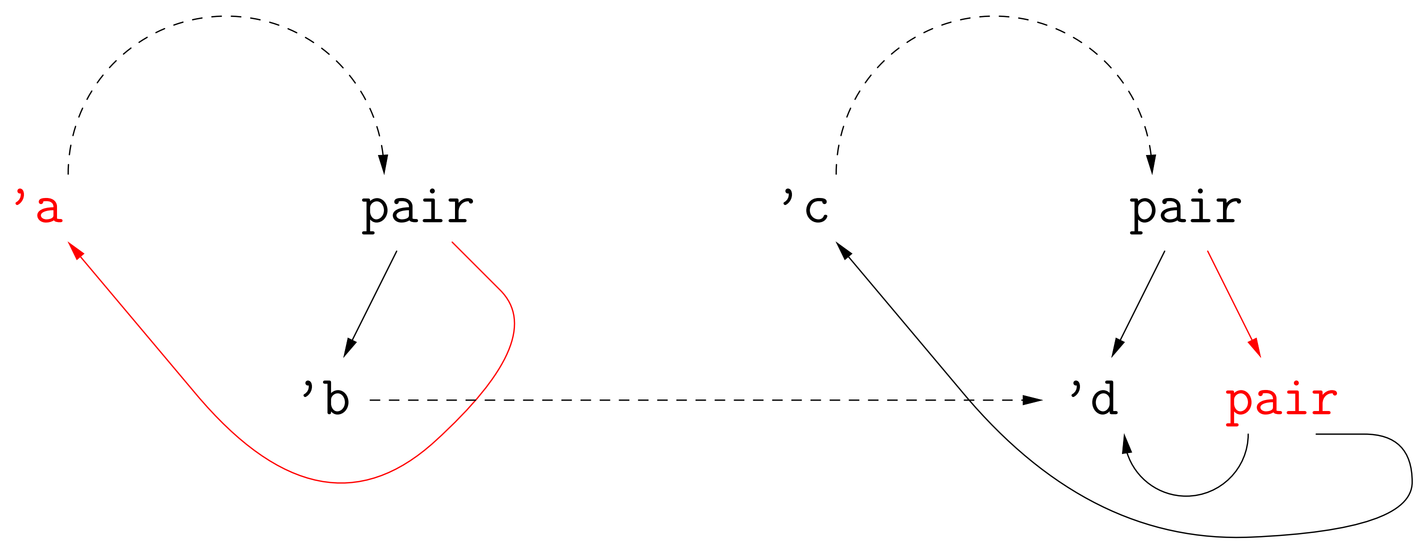

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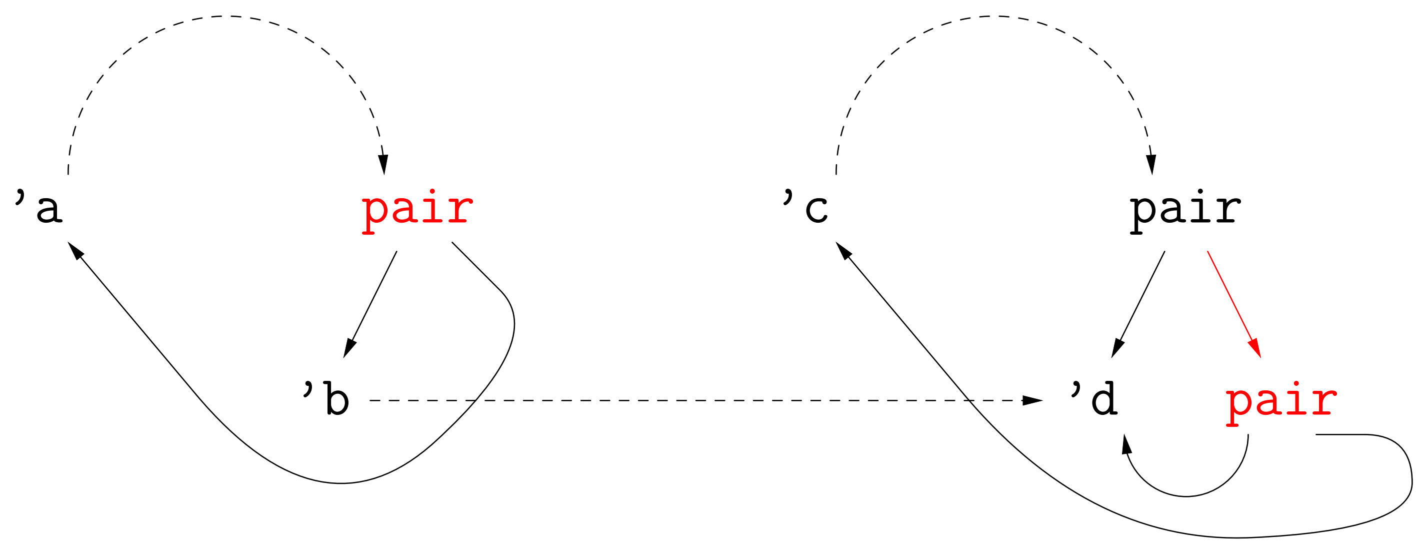

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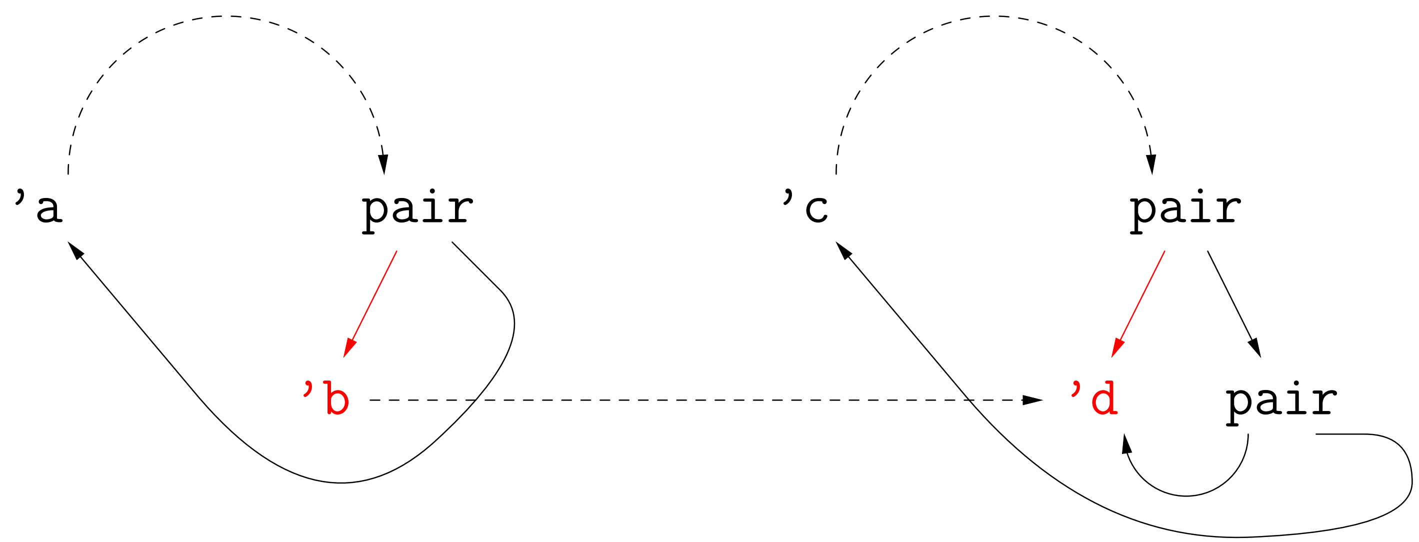

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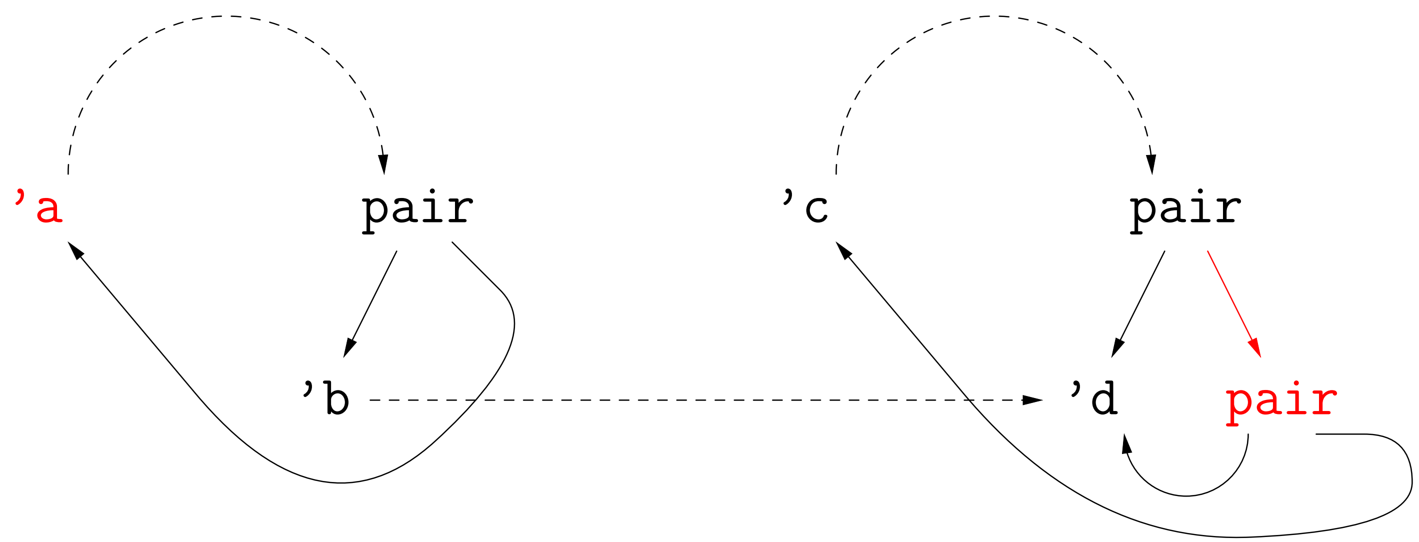

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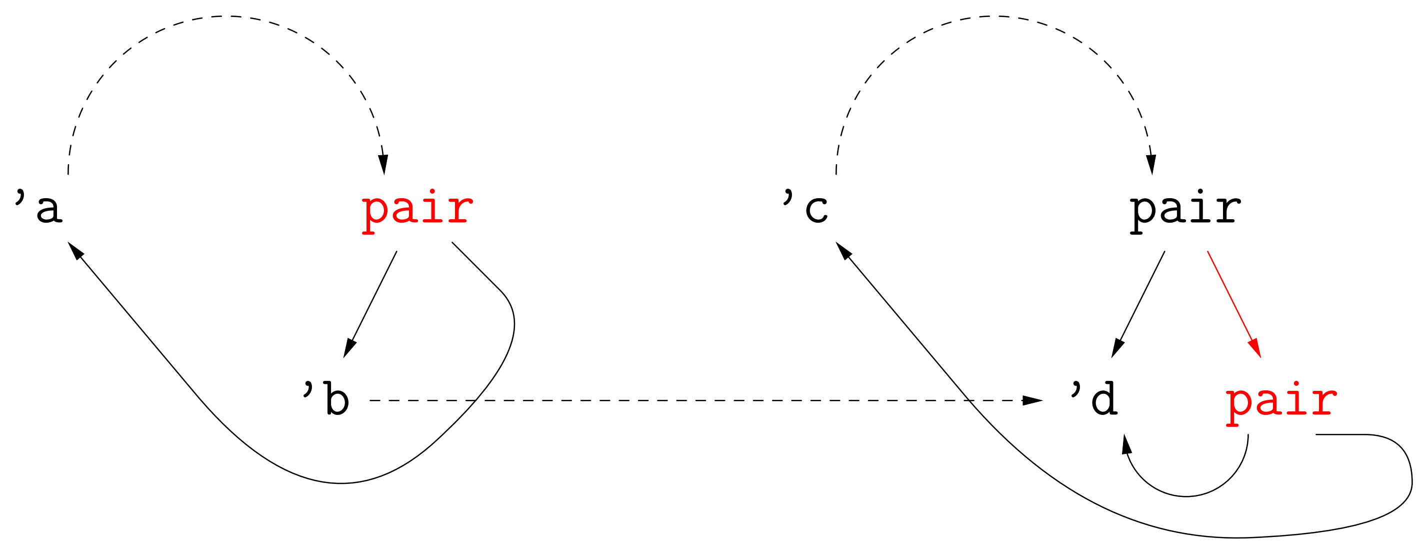

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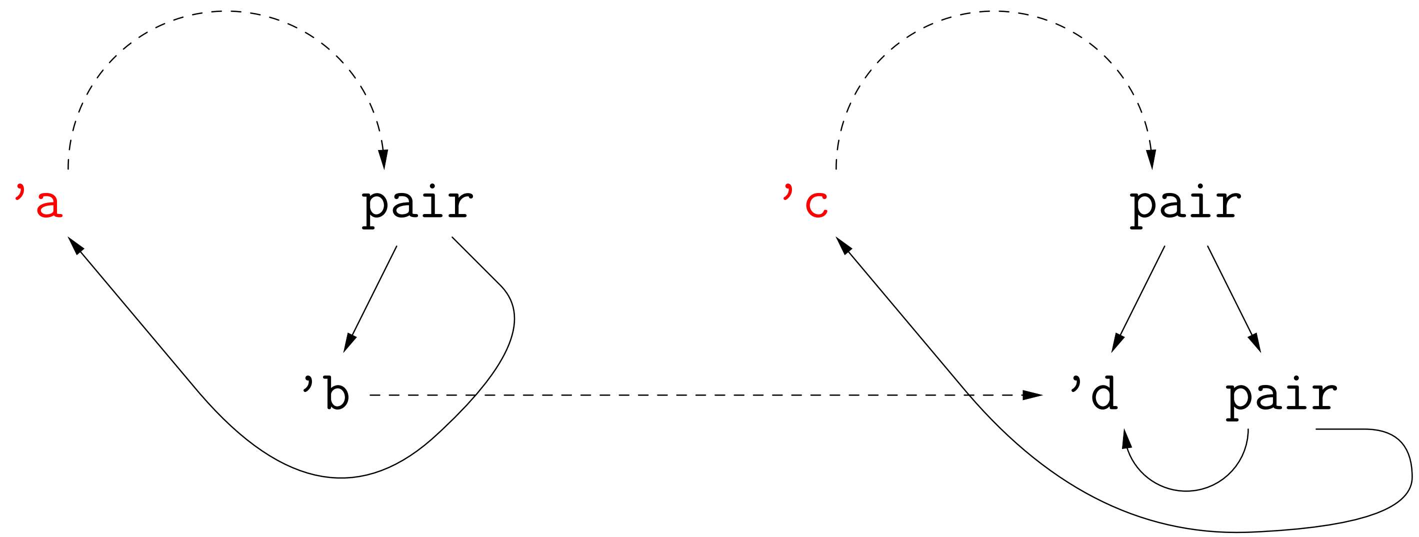

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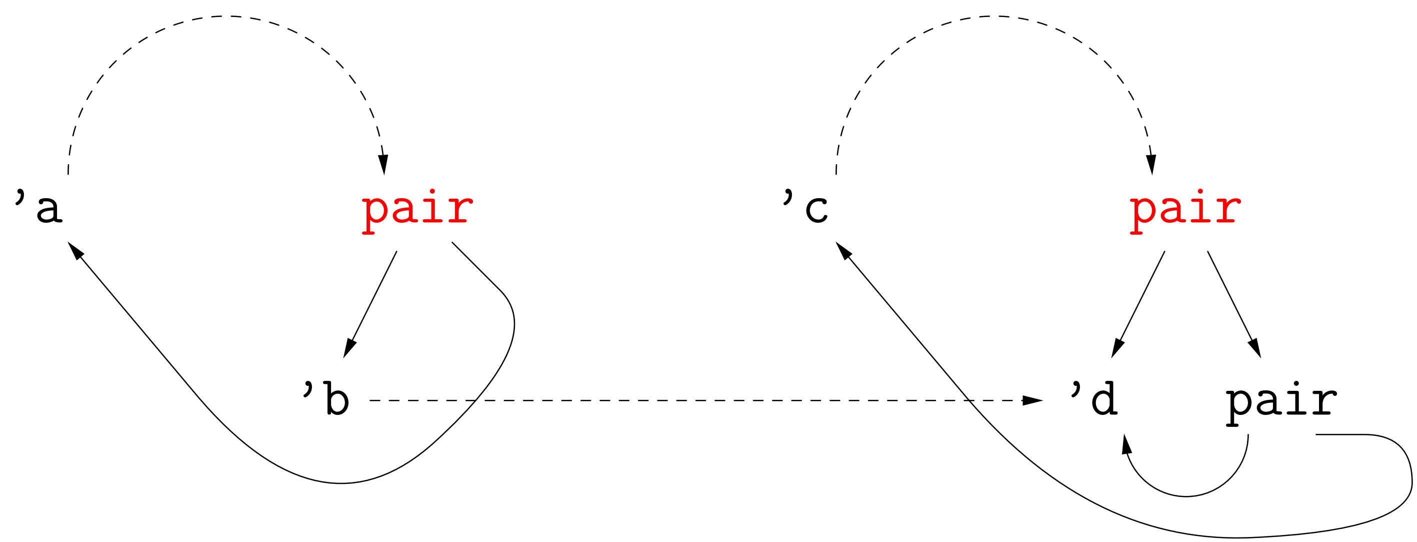


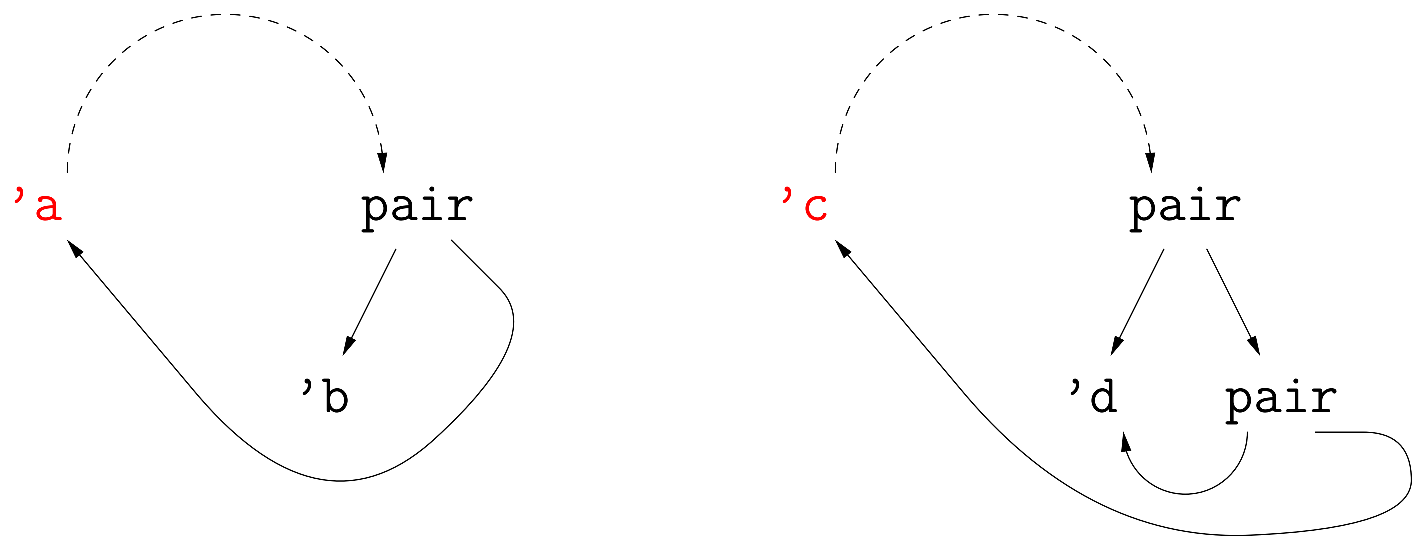
Now we're in trouble: infinite recursion.

Unification Algorithm for Circular Types

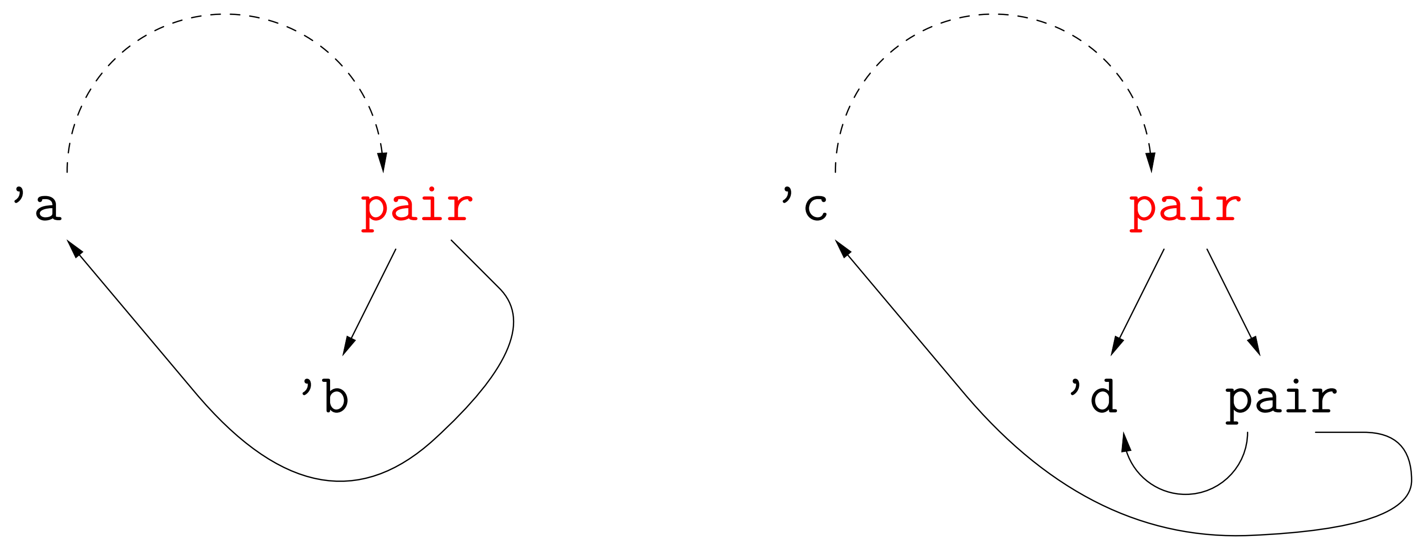
- The major change is that *any* type node, not just type variables, can be bound.
- Something is bound or unification ends at each step, so that process must terminate.

```
unify (TA,TB,u):
   """Returns an extension of unifier u that unifies TA and TB or None."""
   TA = u[TA]; TB = u[TB]
   if TA is TB:          #  True if TA and TB are the same object
      return u
   if TA.isFreeTypevar(u):
      return u.bind(TA, TB)
   if TB.isFreeTypevar(u):
      return u.bind(TB, TA)
   u = u.bind(TA, TB)  #  Prevents infinite recursion: TA marked as matched
   if TA is C(TA_1,TA_2,...,TA_n) and TB is C(TB_1,...,TB_n):
      for i in range(n):
         u = unify(TA_i, TB_i, u)
         if u is None:  return None
      return u
   return None
```

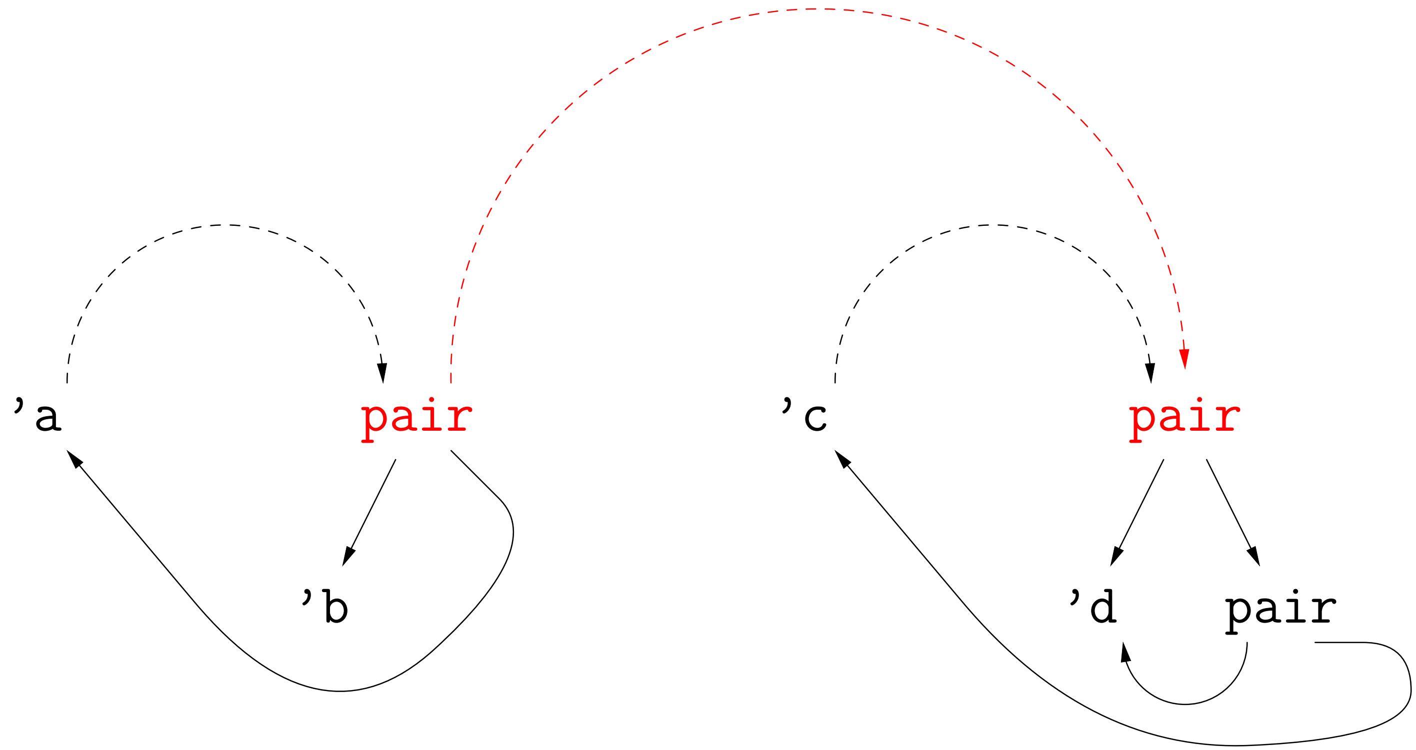
Example of Unification IV, Completed



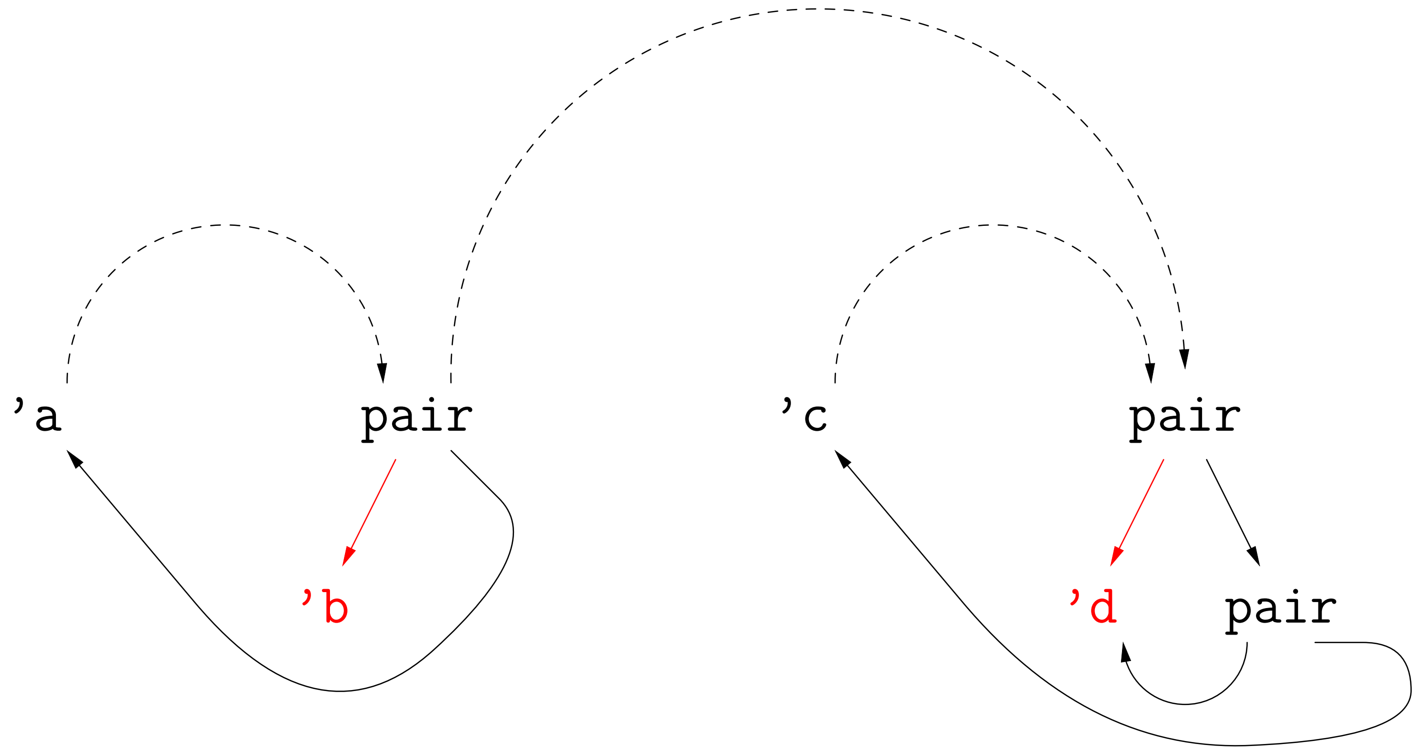
Example of Unification IV, Completed



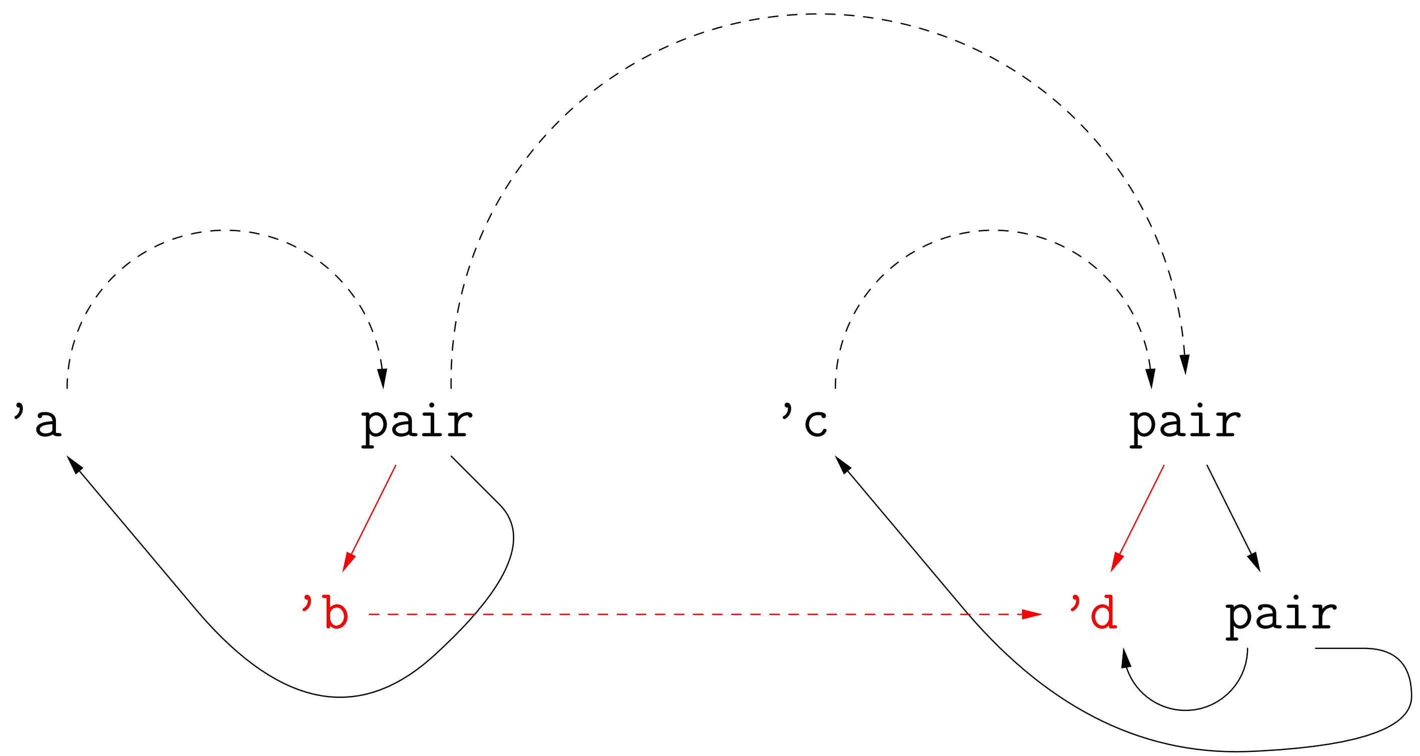
Example of Unification IV, Completed



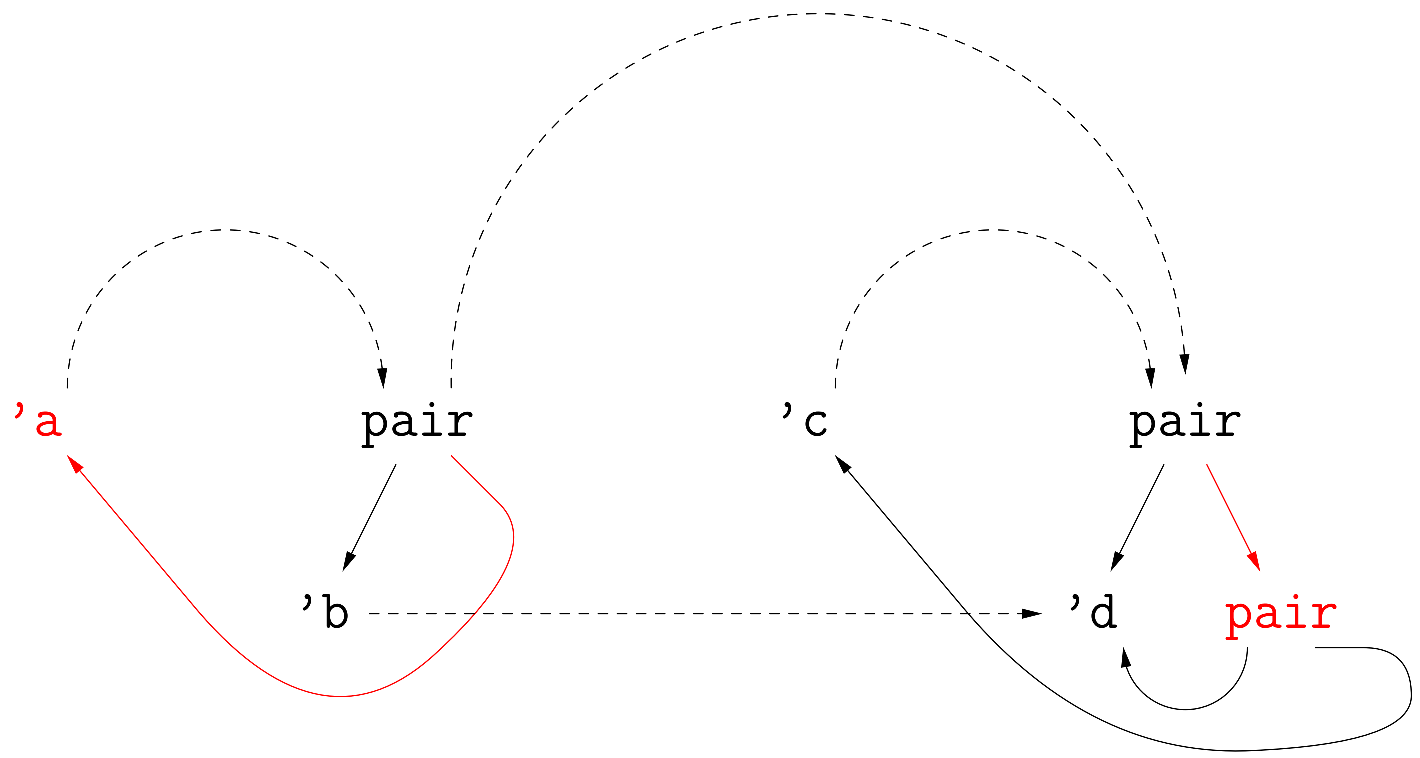
Example of Unification IV, Completed



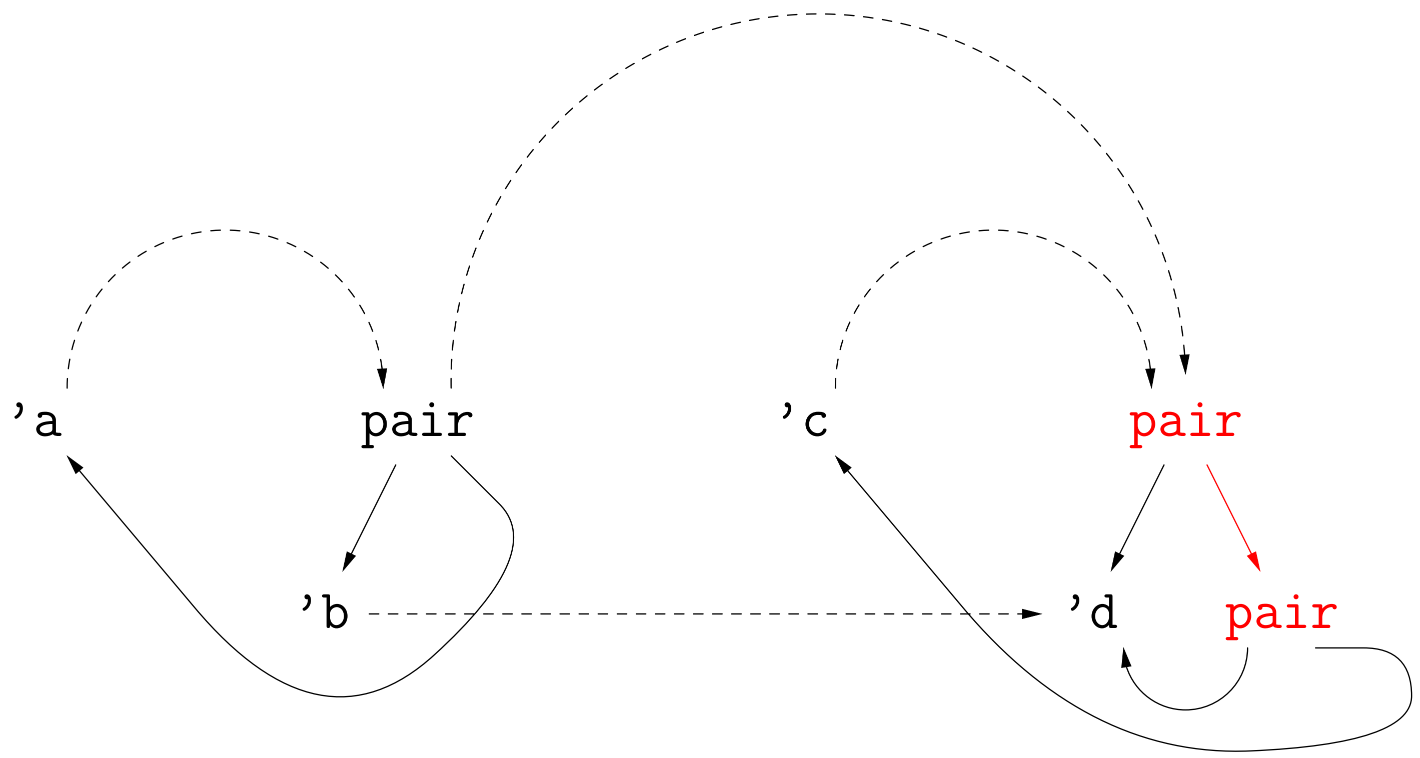
Example of Unification IV, Completed



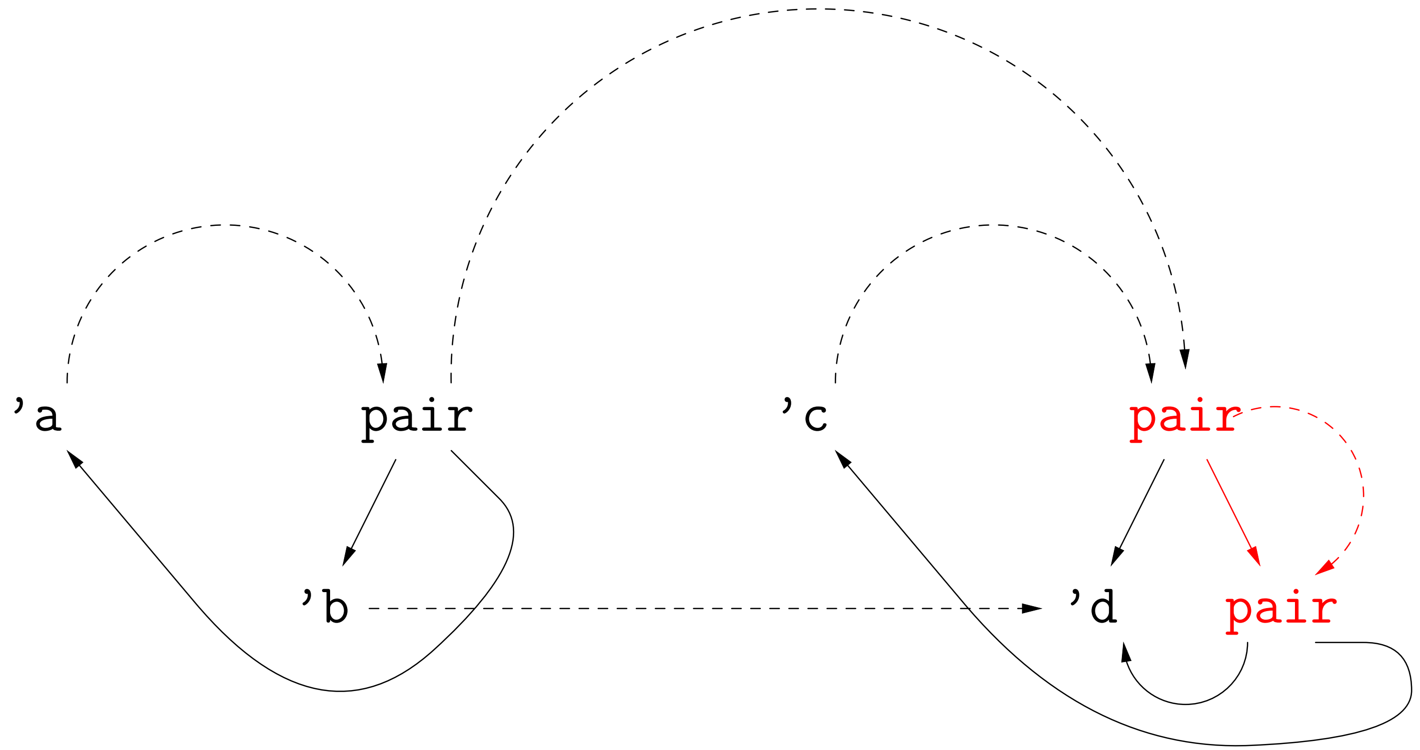
Example of Unification IV, Completed



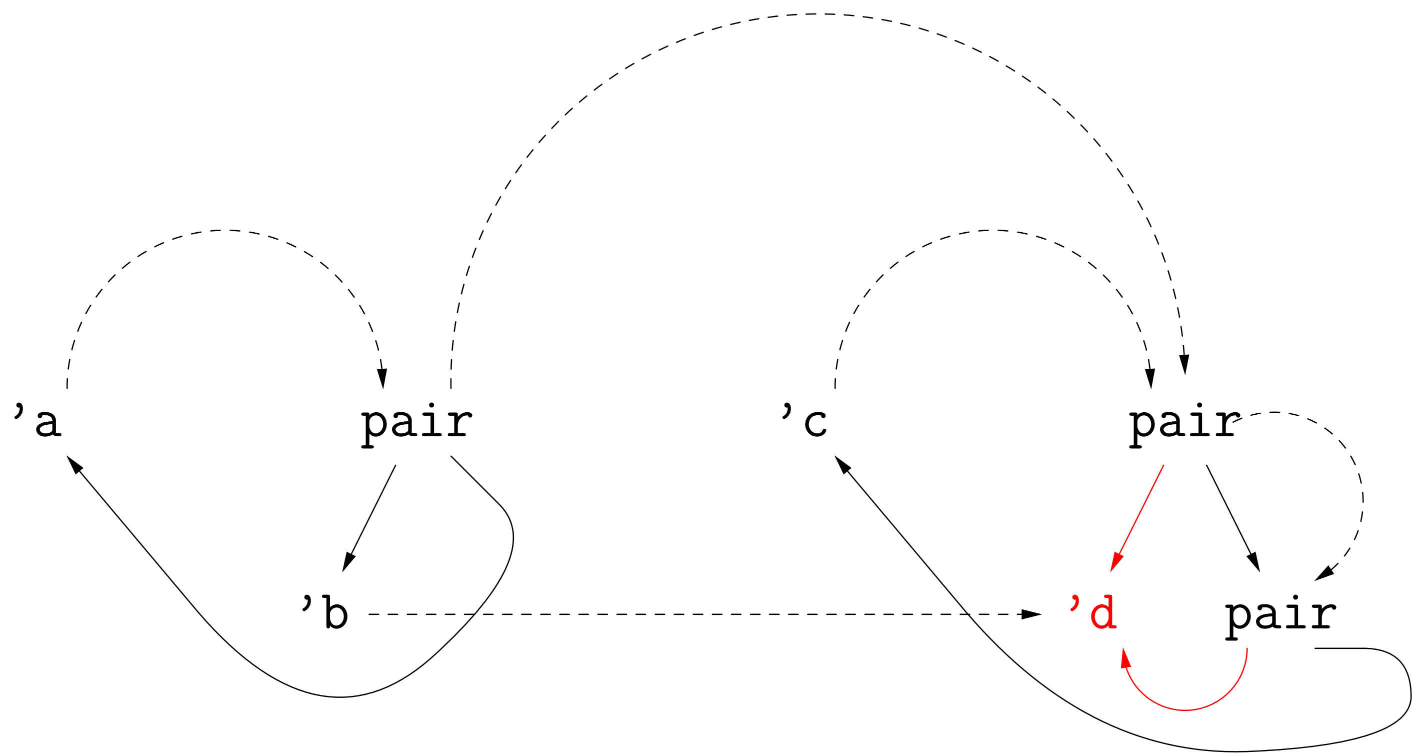
Example of Unification IV, Completed



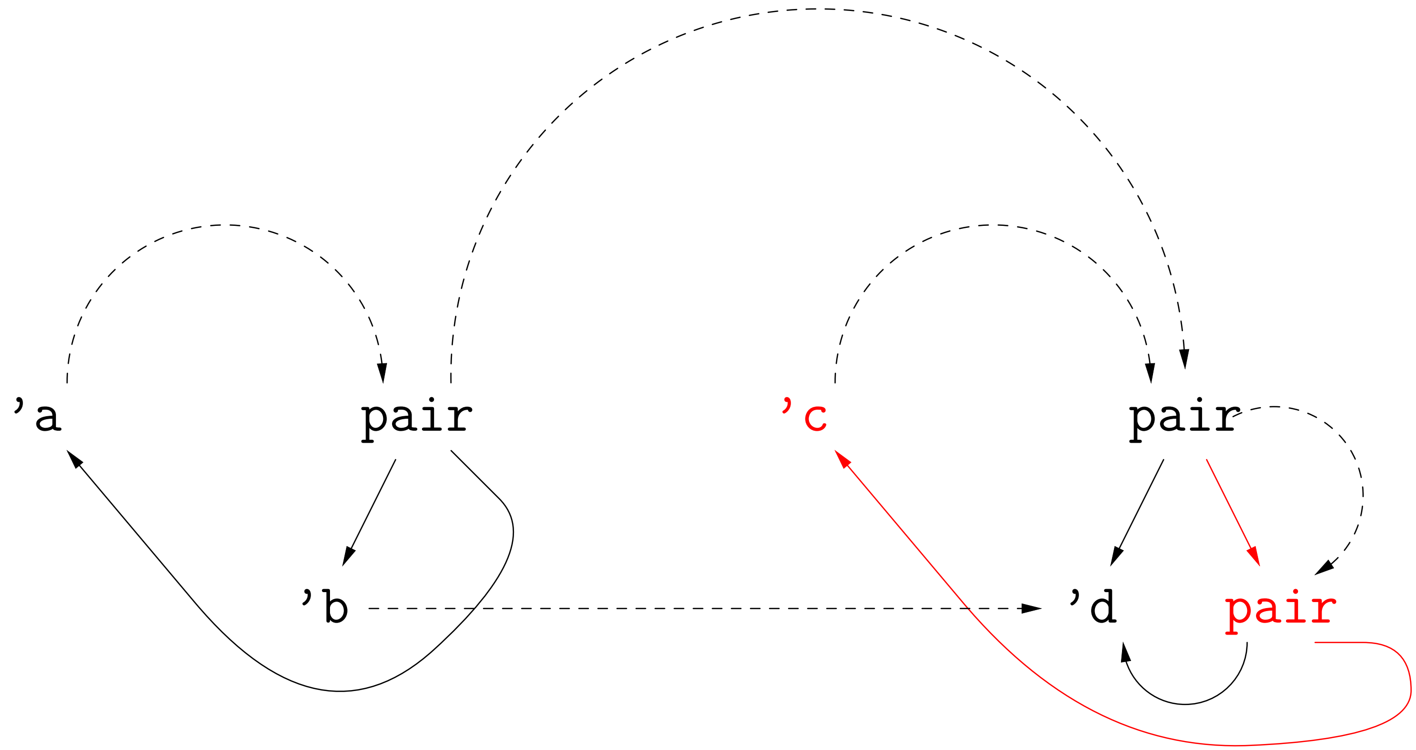
Example of Unification IV, Completed



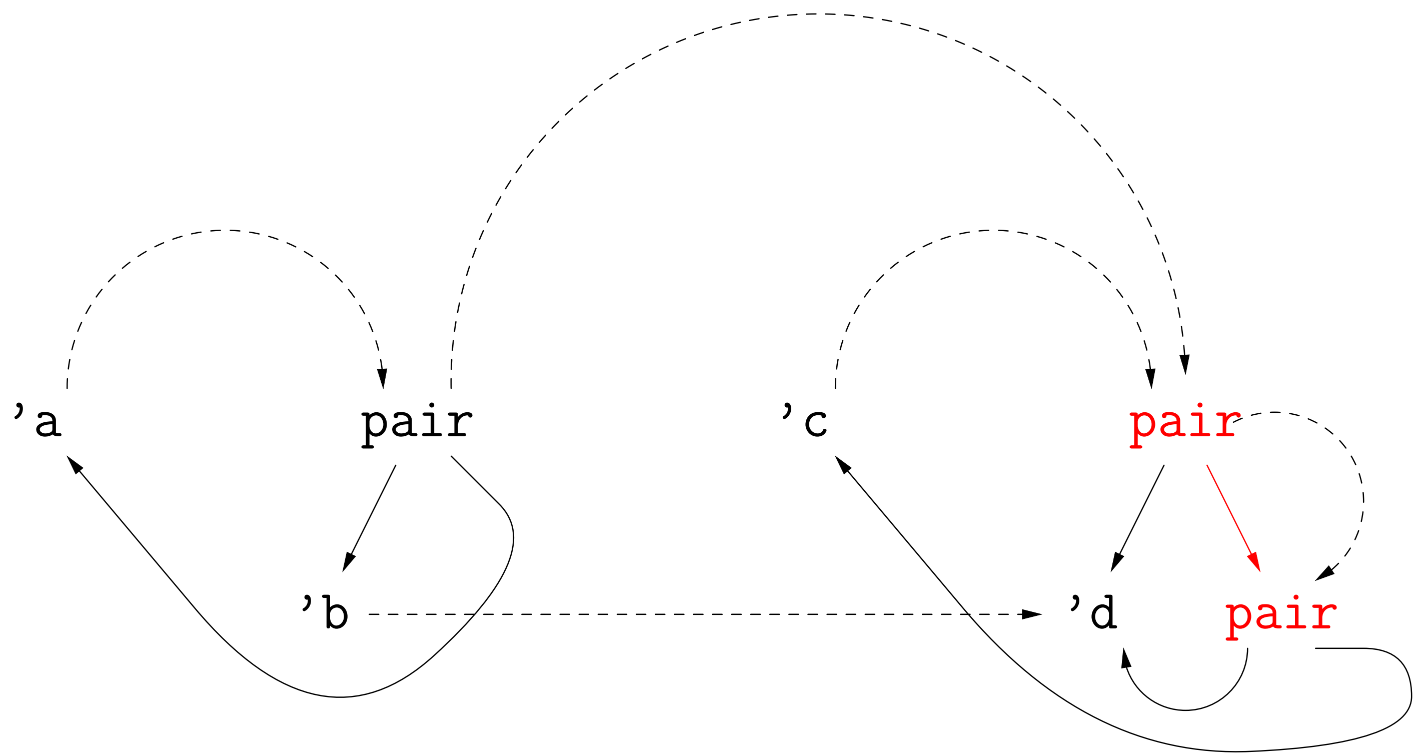
Example of Unification IV, Completed



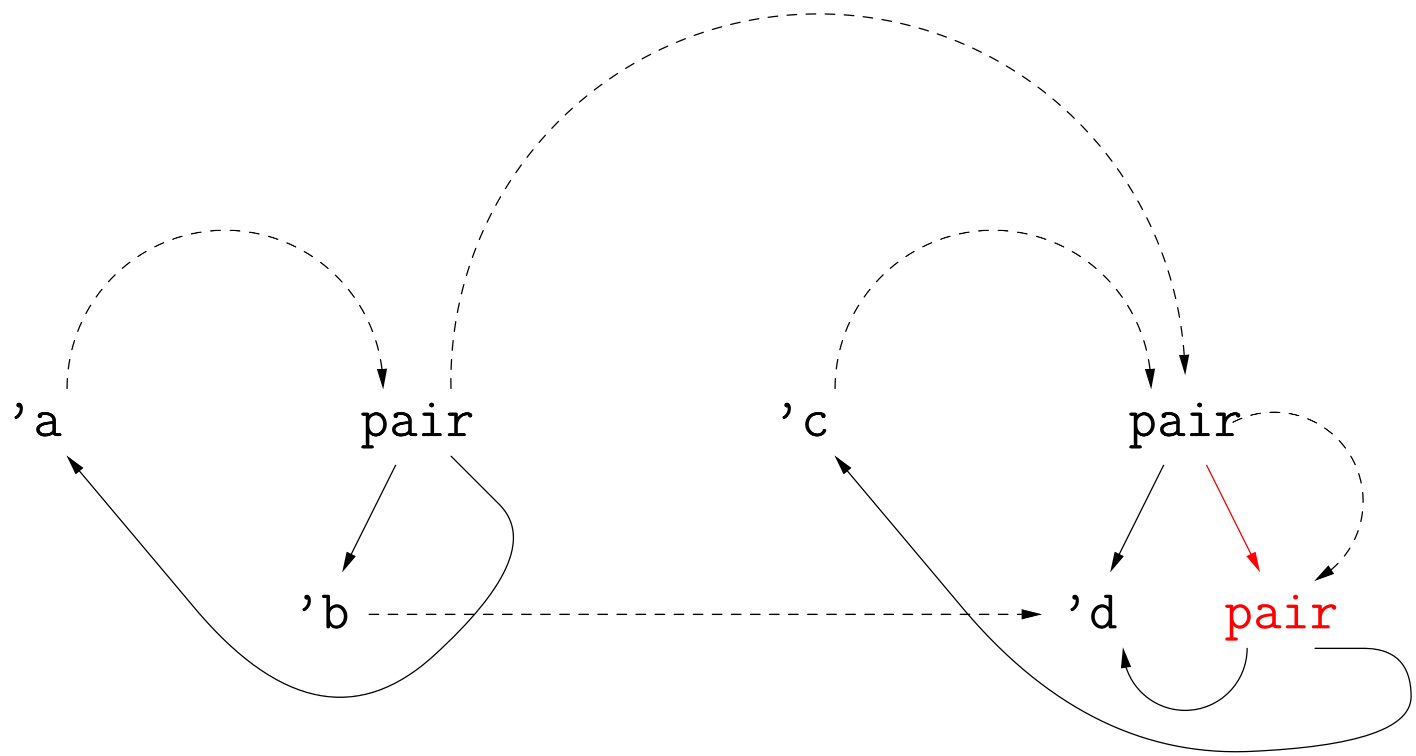
Example of Unification IV, Completed



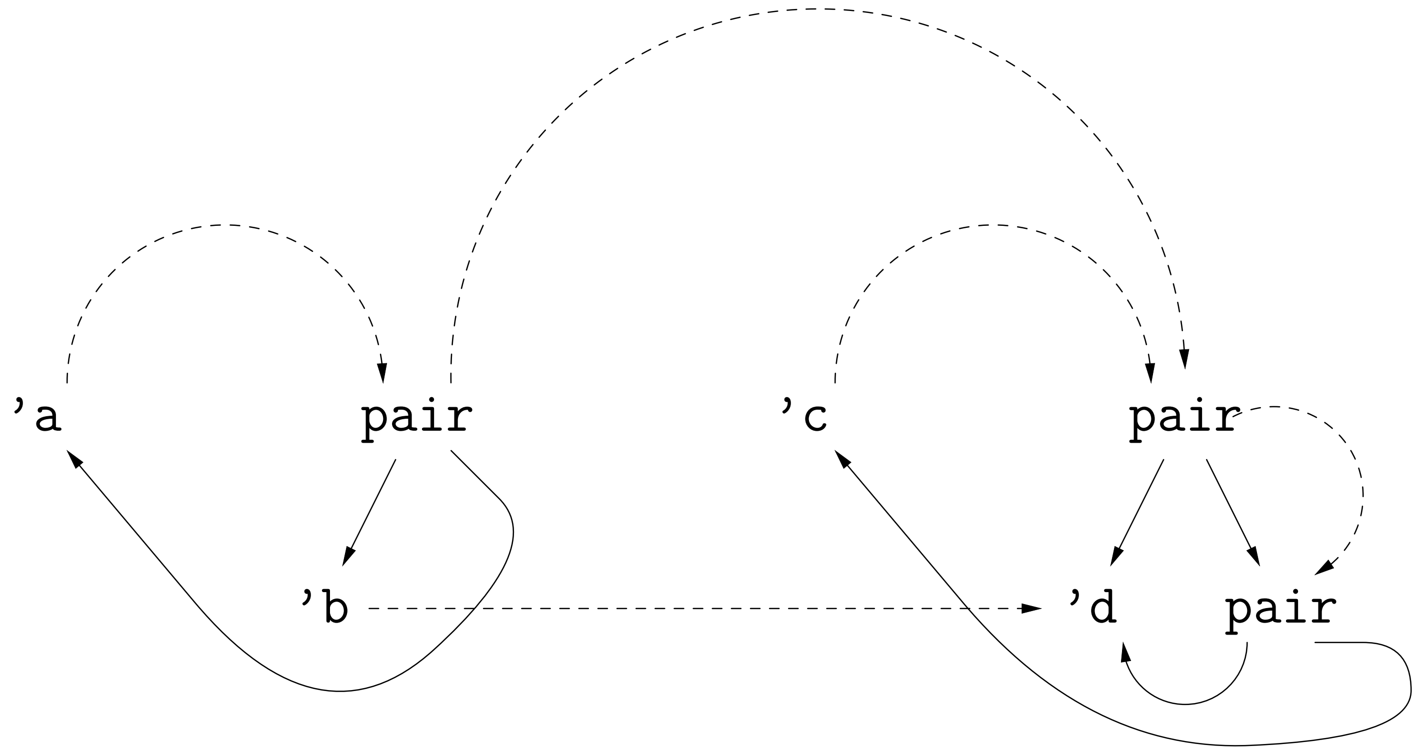
Example of Unification IV, Completed



Example of Unification IV, Completed

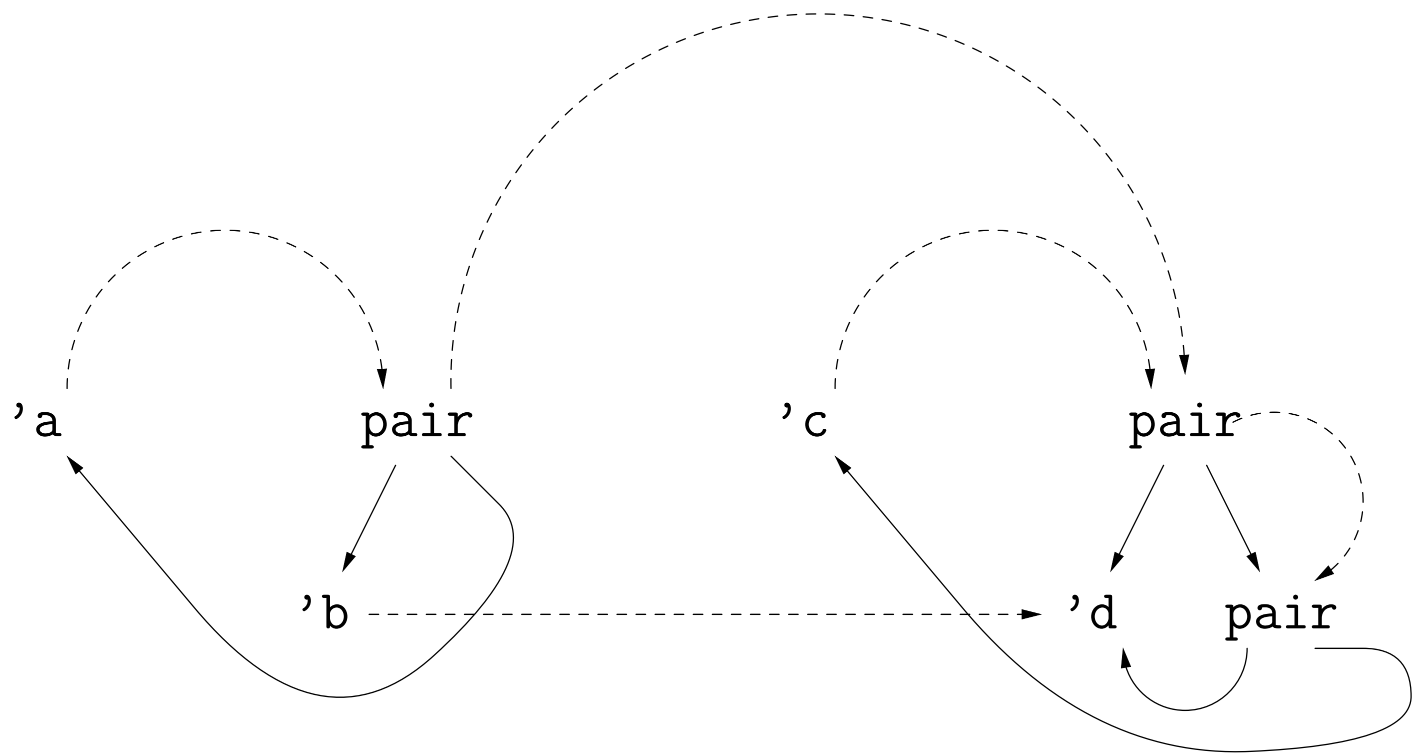


Example of Unification IV, Completed



And now, TA and TB are both pointing at the same object: we're done

Example of Unification IV, Completed



And now, TA and TB are both pointing at the same object: we're done

So `'a` = `'c` = (`'d`, `'a`) `pair`; `'b='d`; `'d` is free.

Example of Unification V

- Try to solve

  `'b list= 'a list; 'a→ 'b = 'c;`
  `'c → bool= (bool→ bool) → bool`

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

`'a:`

`'b:`

`'c:`

Example of Unification V

- Try to solve

  ```
  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool
  ```

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                    Unify 'b  list, 'a  list:

'b:


'c:
```

Example of Unification V

- Try to solve

  ’b list= ’a list; ’a$\rightarrow$ ’b = ’c;
  ’c $\rightarrow$ bool= (bool$\rightarrow$ bool) $\rightarrow$ bool

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                        Unify 'b  list, 'a  list:
                             Unify 'b, 'a
'b:  'a


'c:
```

Example of Unification V

- Try to solve

  ```
  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool
  ```

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                       Unify 'b  list, 'a  list:
                            Unify 'b, 'a
'b:  'a                   Unify 'a→ 'b, 'c


'c:  'a →  'b
```

Example of Unification V

- Try to solve

  ```
  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool
  ```

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                     Unify 'b list, 'a list:
                          Unify 'b, 'a
'b:  'a                 Unify 'a→ 'b, 'c
                        Unify 'c →  bool, (bool →  bool) →  bool

'c:  'a →  'b
```

Example of Unification V

- Try to solve

  `'b list= 'a list; 'a→ 'b = 'c;`
  `'c → bool= (bool→ bool) → bool`

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                     Unify 'b  list, 'a  list:
                          Unify 'b, 'a
'b:  'a                 Unify 'a→ 'b, 'c
                        Unify 'c →  bool, (bool →  bool) →  bool
                          Unify 'c, bool →  bool:
'c:  'a →  'b
```

Example of Unification V

- Try to solve

  $$\text{'b list} = \text{'a list};\ \text{'a} \to \text{'b} = \text{'c};$$
  $$\text{'c} \to \text{bool} = (\text{bool} \to \text{bool}) \to \text{bool}$$

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:                       Unify 'b  list, 'a  list:
                            Unify 'b, 'a
'b:  'a                   Unify 'a→ 'b, 'c
                          Unify 'c →  bool, (bool →  bool) →  bool
                            Unify 'c, bool →  bool:
'c:  'a →  'b                 Unify 'a →  'b, bool →  bool:
```

Example of Unification V

- Try to solve

  ```
  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool
  ```

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:  bool              Unify 'b  list, 'a  list:
                         Unify 'b, 'a
'b:  'a                Unify 'a→ 'b, 'c
                       Unify 'c →  bool, (bool →  bool) →  bool
                         Unify 'c, bool →  bool:
'c:  'a →  'b              Unify 'a →  'b, bool →  bool:
                             Unify 'a, bool
```

Example of Unification V

- Try to solve

  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:  bool              Unify 'b  list, 'a  list:
                         Unify 'b, 'a
'b:  'a                Unify 'a→ 'b, 'c
                       Unify 'c →  bool, (bool →  bool) →  bool
                         Unify 'c, bool →  bool:
'c:  'a →  'b              Unify 'a →  'b, bool →  bool:
                             Unify 'a, bool
                             Unify 'b, bool:
```

Example of Unification V

- Try to solve

  'b list= 'a list; 'a→ 'b = 'c;
  'c → bool= (bool→ bool) → bool

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:  bool               Unify 'b  list, 'a  list:
                          Unify 'b, 'a
'b:  'a                 Unify 'a→ 'b, 'c
                        Unify 'c →  bool, (bool →  bool) →  bool
                          Unify 'c, bool →  bool:
'c:  'a →  'b               Unify 'a →  'b, bool →  bool:
                              Unify 'a, bool
                              Unify 'b, bool:
                                Unify bool, bool
```

Example of Unification V

- Try to solve

```
'b list= 'a list; 'a→ 'b = 'c;
'c → bool= (bool→ bool) → bool
```

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

```
'a:  bool                 Unify 'b  list, 'a  list:
                            Unify 'b, 'a
'b:  'a                   Unify 'a→ 'b, 'c
    bool                  Unify 'c →  bool, (bool →  bool) →  bool
                            Unify 'c, bool →  bool:
'c:  'a →  'b                 Unify 'a →  'b, bool →  bool:
    bool  →  bool                 Unify 'a, bool
                                  Unify 'b, bool:
                                    Unify bool, bool
                            Unify bool, bool
```

Some Type Rules (reprise)

Construct	**Type**	**Conditions**
Integer literal	`int`	
[]	`'a list`	
hd (L)	'a	L: 'a list
tl (L)	'a list	L: 'a list
E_1+E_2	int	E_1: int, E_2: int
E_1::E_2	'a list	E_1: 'a, E_2: 'a list
$E_1 = E_2$	bool	E_1: 'a, E_2: 'a
E_1!=E_2	bool	E_1: 'a, E_2: 'a
if E_1 then E_2 else E_3 fi	'a	E_1: bool, E_2: 'a, E_3: 'a
E_1 E_2	'b	E_1: 'a $\rightarrow$ 'b, E_2: 'a
`def f x1 ...xn = E`		x1: 'a$_1$, ..., xn: 'a$_n$ E:'a$_0$, f: 'a$_1$ $\rightarrow \ldots \rightarrow$ 'a$_n$ $\rightarrow$ 'a$_0$.

Using the Type Rules

- Interpret the notation $E : T$, where E is an expression and T is a type, as

  type(E) = T

- Seed the process by introducing a set of fresh type variables to describe the types of all the variables used in the program you are attempting to process. For example, given

```
def f x = x
```

  we might start by saying that

```
type(f) = 'a0, type(x) = 'a1
```

- Apply the type rules to your program to get a bunch of Conditions.
- Whenever two Conditions ascribe a type to the same expression, equate those types.
- Solve the resulting equations.

Aside: Currying

- Writing

```
def sqr x = x*x;
```

means essentially that `sqr` is defined to have the value λ `x. x*x`.

- To get more than one argument, write

```
def f x y = x + y;
```

and `f` will have the value λ `x.` λ `y. x+y`

- Its type will be `int` $\rightarrow$ `int` $\rightarrow$ `int` (Note: $\rightarrow$ is right associative).
- So, `f 2 3 = (f 2) 3 = (`λ `y. 2 + y) (3) = 5`
- Zounds! It's the CS61A substitution model!
- This trick of turning multi-argument functions into one-argument functions is called *currying* (after Haskell Curry).

Example

```
if p L then init else f init (hd L) fi + 3
```

- Let's initially use `'p`, `'L`, etc. as the fresh type variables giving the types of identifiers.
- Using the rules then generates equations like this:

```
'p = 'a0→ 'a1, 'L = 'a0, type(p L) = 'a1 # call rule
'L = 'a2 list, type(hd L) = 'a2          # hd rule
'f = 'a3→ 'a4, 'init = 'a3, type(f init) = 'a4
                                         # call rule
'a4 = 'a5→ 'a6, 'a2 = 'a5, type(f init (hd L)) = 'a6
                                         # call rule
'a1 = bool, 'init = 'a7, 'a6 = 'a7, type(if...  fi) = 'a7
                                         # if rule
'a7 = int, int = int, type(if...  fi+3) = int   # + rule
```

etc.

Example, contd.

Solve all these equations by sequentially unifying the two sides of each equation, in any order, keeping the bindings as you go.

```
'p = 'a0→ 'a1, 'L = 'a0
'L = 'a2 list
    'a0 = 'a2 list
'f = 'a3→ 'a4, 'init = 'a3
'a4 = 'a5→ 'a6, 'a2 = 'a5
'a1 = bool, 'init = 'a7, 'a6 = 'a7
    'a3 = 'a7
'a7 = int, int = int
```

So (eventually),

```
'p = 'a5 list→ bool, 'L = 'a5 list, 'init = int,
'f = int → 'a5→ int
```

Introducing Fresh Variables

- The type rules for the simple language we've been using generally call for introducing fresh type variables for each application of the rule.
- Example: in the expression

  ```
  if x = [] then [] else x::y fi
  ```

  the two [] are treated as having two different types, say `'a0 list` and `'a1 list`, which is a good thing, because otherwise, this expression cannot be made to type-check [why?].
- You'd probably want to do the same with count:

  ```
  fun count [] = 0
   |  count (_ ::  y) = 1 + count y
  ```

  Analyzing this gives a type of `'a list` → `int`. Suppose we have two calls later in the program: `count (0::x)` and `count ([1]::y)`.
- Obviously, we also want to replace `'a` in each case with a fresh type variable, since otherwise, `count` would be specialized to work only on lists of integers or only on lists of lists.

. . . Or not?

- But we *don't* want to introduce a fresh type variable for each call when inferring the type of a function from its definition:

```
fun switcher x y z = if x=0 then y else switcher(x-1,z, y) fi
```

- Here, we want the type of `switcher` to come out to be `int→ 'y→ 'y→ 'y`, but that can't happen if the recursive call to `switcher` can take argument types that are independent of those of `y` and `z`.
- Same problem with a set of mutually recursive definitions.
- So our language must always state which groups of definitions get resolved together, and when calling a function is supposed to create a fresh set of type variables instead.