Regularized Precision Matrix Estimation via ADMM

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Abstract

`ADMMsigma` is an R package that estimates a penalized precision matrix via the alternating direction method of multipliers (ADMM) algorithm. It currently supports a general elastic-net penalty that allows for both ridge and lasso-type penalties as special cases. This report will provide a brief overview of the algorithm and detail how it can be utilized to estimate precision matrices of jointly normal distributions. In addition, examples and simulation results will be provided.

Contents

1 Introduction

Suppose we want to solve the following optimization problem:

$$\begin{aligned} &\text{minimize } f(x) + g(z) \\ &\text{subject to } Ax + Bz = c \end{aligned}$$

where $x \in \mathbb{R}^n, z \in \mathbb{R}^m, A \in \mathbb{R}^{p \times m}, B \in \mathbb{R}^{p \times m}, c \in \mathbb{R}^p$ – though we will later consider cases where x and z are matrices. Further, we will assume f and g are convex. The *augmented lagrangian* is constructed as follows:

$$L_\rho(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|_2^2$$

where $y \in \mathbb{R}^p$ is the lagrange multiplier. The optimal value is

$$p^* = inf \{f(x) + g(z) | Ax + Bz = c\}$$

Clearly, the minimization under the augmented lagrangian is equivalent to that of the usual lagrangian since any feasible point (x, z) satisfies the constraint $\rho \|Ax + Bz - c\|_2^2 / 2 = 0$.

The alternating direct method of multipliers (ADMM) algorithm consists of the following repeated iterations:

$$x^{k+1} := \arg\min_x L_\rho(x, z^k, y^k) \tag{1}$$

$$z^{k+1} := \arg\min_z L_\rho(z^{k+1}, z, y^k) \tag{2}$$

$$y^{k+1} := y^k + \rho(Ax^{k+1} + Bz^{k+1} - c) \tag{3}$$

A more complete introduction to the algorithm – specifically how it arose out of *dual ascent* and *method of multipliers* – can be found in Boyd et al. (2011).

2 Regularized Precision Matrix Estimation

We now consider the case where $X_1, ..., X_n$ are iid $N_p(\mu, \Sigma)$ and we are tasked with estimating the precision matrix, denoted $\Omega \equiv \Sigma^{-1}$. The maximum likelihood estimator for Ω is

$$\hat{\Omega} = \arg\min_{\Omega \in S_+^p} \{Tr(S\Omega) - \log\det(\Omega)\}$$

where $S = \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^T/n$. It is straight forward to show that when the solution exists, $\hat{\Omega} = S^{-1}$. We can construct a *penalized* likelihood estimator by adding a penalty term, $P(\Omega)$, to the likelihood:

$$\hat{\Omega}_\lambda = \arg\min_{\Omega \in S_+^p} \{Tr(S\Omega) - \log\det(\Omega) + P(\Omega)\}$$

Throughout the rest of this document we will take $P(\Omega)$ to be $P(\Omega) = \lambda\left[\frac{1-\alpha}{2}\|\Omega|_F^2 + \alpha\|\Omega\|_1\right]$ so that the full penalized likelihood is as follows:

$$\hat{\Omega}_\lambda = \arg\min_{\Omega \in S_+^p} \left\{Tr(S\Omega) - \log\det(\Omega) + \lambda\left[\frac{1-\alpha}{2}\|\Omega|_F^2 + \alpha\|\Omega\|_1\right]\right\}$$

where $0 \leq \alpha \leq 1$, $\lambda > 0$, $\|\cdot\|_F^2$ is the Frobenius norm and we define $\|A\|_1 = \sum_{i,j}|A_{ij}|$. This *elastic-net* penalty was explored by Hui Zou and Trevor Hastie (Zou and Hastie 2005) and is identical to the penalty used in the popular penalized regression package `glmnet`. Clearly, when $\alpha = 0$ the elastic-net reduces to a ridge-type penalty and when $\alpha = 1$ this reduces to a lasso-type penalty.

By letting f be equal to the non-penalized likelihood and g equal to $P(\Omega)$, our goal is to minimize the full augmented lagrangian where the constraint is that $\Omega - Z$ is equal to zero:

$$L_\rho(\Omega, Z, \Lambda) = f(\Omega) + g(Z) + Tr[\Lambda(\Omega - Z)] + \frac{\rho}{2}\|\Omega - Z\|_F^2$$

The ADMM algorithm for regularized precision matrix estimation is

$$\Omega^{k+1} = \arg\min_{\Omega} \left\{ Tr\left(\Omega\right) - \log\det\left(\Omega\right) + Tr\left[\Lambda^k\left(\Omega - Z^k\right)\right] + \frac{\rho}{2}\left\|\Omega - Z^k\right\|_F^2 \right\} \tag{4}$$

$$Z^{k+1} = \arg\min_{Z} \left\{ \lambda\left[\frac{1-\alpha}{2}\left\|Z\right\|_F^2 + \alpha\left\|Z\right\|_1\right] + Tr\left[\Lambda^k\left(\Omega^{k+1} - Z\right)\right] + \frac{\rho}{2}\left\|\Omega^{k+1} - Z\right\|_F^2 \right\} \tag{5}$$

$$\Lambda^{k+1} = \Lambda^k + \rho\left(\Omega^{k+1} - Z^{k+1}\right) \tag{6}$$

2.1 Condensed-Form ADMM

An alternate form of the ADMM algorithm can constructed by scaling the dual variable. Let us define $R^k = \Omega - Z^k$ and $U^k = \Lambda^k/\rho$. Then

$$\begin{aligned} Tr\left[\Lambda^k\left(\Omega - Z^k\right)\right] + \frac{\rho}{2}\left\|\Omega - Z^k\right\|_F^2 &= Tr\left[\Lambda^k R^k\right] + \frac{\rho}{2}\left\|R^k\right\|_F^2 \\ &= \frac{\rho}{2}\left\|R^k + \Lambda^k/\rho\right\|_F^2 - \frac{\rho}{2}\left\|\Lambda^k/\rho\right\|_F^2 \\ &= \frac{\rho}{2}\left\|R^k + U^k\right\|_F^2 - \frac{\rho}{2}\left\|U^k\right\|_F^2 \end{aligned}$$

The condensed-form can now be written as follows:

$$\Omega^{k+1} = \arg\min_{\Omega} \left\{ Tr\left(\Omega\right) - \log\det\left(\Omega\right) + \frac{\rho}{2}\left\|\Omega - Z^k + U^k\right\|_F^2 \right\} \tag{7}$$

$$Z^{k+1} = \arg\min_{Z} \left\{ \lambda\left[\frac{1-\alpha}{2}\left\|Z\right\|_F^2 + \alpha\left\|Z\right\|_1\right] + \frac{\rho}{2}\left\|\Omega^{k+1} - Z + U^k\right\|_F^2 \right\} \tag{8}$$

$$U^{k+1} = U^k + \Omega^{k+1} - Z^{k+1} \tag{9}$$

More generally (in vector form),

$$x^{k+1} := \arg\min_{x} \left\{ f(x) + \frac{\rho}{2}\left\|Ax + Bz^k - c + u^k\right\|_2^2 \right\} \tag{10}$$

$$z^{k+1} := \arg\min_{z} \left\{ g(z) + \frac{\rho}{2}\left\|Ax^{k+1} + Bz - c + u^k\right\|_2^2 \right\} \tag{11}$$

$$u^{k+1} := u^k + Ax^{k+1} + Bz^{k+1} - c \tag{12}$$

Note that there are limitations to using this method. For instance, because the dual variable is scaled by ρ (the step size), this form limits one to using a constant step size (without making further adjustments to U^k) – a limitation that could prolong the convergence rate. Because of this, we will only consider the non-condensed form for the remainder of this report.

2.2 Algorithm

$$\Omega^{k+1} = \arg\min_{\Omega} \left\{ Tr\left(\Omega\right) - \log\det\left(\Omega\right) + Tr\left[\Lambda^k\left(\Omega - Z^k\right)\right] + \frac{\rho}{2}\left\|\Omega - Z^k\right\|_F^2 \right\}$$
$$Z^{k+1} = \arg\min_{Z} \left\{ \lambda\left[\frac{1-\alpha}{2}\left\|Z\right\|_F^2 + \alpha\left\|Z\right\|_1\right] + Tr\left[\Lambda^k\left(\Omega^{k+1} - Z\right)\right] + \frac{\rho}{2}\left\|\Omega^{k+1} - Z\right\|_F^2 \right\}$$
$$\Lambda^{k+1} = \Lambda^k + \rho\left(\Omega^{k+1} - Z^{k+1}\right)$$

1. Decompose $S + \Lambda^k - \rho Z^k = VQV^T$.

$$\Omega^{k+1} = \frac{1}{2\rho} V\left[-Q + \left(Q^2 + 4\rho I_p\right)^{1/2}\right] V^T$$

2. Elementwise soft-thresholding for all $i = 1, ..., p$ and $j = 1, ..., p$.

$$\begin{aligned} Z_{ij}^{k+1} &= \frac{1}{\lambda(1-\alpha)+\rho} sign\left(\rho\Omega_{ij}^{k+1} + \Lambda_{ij}^k\right)\left(\left|\rho\Omega_{ij}^{k+1} + \Lambda_{ij}^k\right| - \lambda\alpha\right)_+ \\ &= \frac{1}{\lambda(1-\alpha)+\rho} Soft\left(\left(\rho\Omega_{ij}^{k+1} + \Lambda_{ij}^k\right), \lambda\alpha\right) \end{aligned}$$

3. Update Λ.

$$\Lambda^{k+1} = \Lambda^k + \rho\left(\Omega^{k+1} - Z^{k+1}\right)$$

2.2.1 Proof of (1):

$$\Omega^{k+1} = \arg\min_{\Omega} \left\{ Tr\left(\Omega\right) - \log\det\left(\Omega\right) + Tr\left[\Lambda^k\left(\Omega - Z^k\right)\right] + \frac{\rho}{2}\left\|\Omega - Z^k\right\|_F^2 \right\}$$

Code snippet:

Note this is not the actual code. The real code is written in c++.

```
# ridge penalized precision matrix
# function
RIDGEsigma = function(S, lam) {

    # dimensions
    p = dim(S)[1]

    # gather eigen values of S (spectral
    # decomposition)
    e.out = eigen(S, symmetric = TRUE)
```

```
    # augment eigen values for omega hat
    new.evs = (-e.out$val + sqrt(e.out$val^2 +
        4 * lam))/(2 * lam)

    # compute omega hat for lambda (zero
    # gradient equation)
    omega = tcrossprod(e.out$vec * rep(new.evs,
        each = p), e.out$vec)

    # compute gradient
    grad = S - qr.solve(omega) + lam * omega

    return(list(omega = omega, gradient = grad))
}
```

2.2.2 Proof of (2)

$$Z^{k+1} = \arg\min_{Z} \left\{ \lambda \left[\frac{1-\alpha}{2} \|Z\|_F^2 + \alpha \|Z\|_1 \right] + Tr\left[\Lambda^k \left(\Omega^{k+1} - Z\right)\right] + \frac{\rho}{2} \left\|\Omega^{k+1} - Z\right\|_F^2 \right\}$$

Code snippet:

Note this is not the actual code. The real code is written in c++.

```
# ADMMsigma function
ADMMsigma = function(X = NULL, S = NULL,
    lam, alpha = 1, rho = 2, mu = 10, tau1 = 2,
    tau2 = 2, tol1 = 1e-04, tol2 = 1e-04,
    maxit = 1000) {

    # compute sample covariance matrix, if
    # necessary
    if (is.null(S)) {

        # covariance matrix
        n = dim(X)[1]
        S = (n - 1)/n * cov(X)

    }

    # allocate memory
    p = dim(S)[1]
    criterion = TRUE
    iter = lik = s = r = eps1 = eps2 = 0
    new.Z = Y = Omega = matrix(0, nrow = p,
        ncol = p)

    # loop until convergence
```

```
    while (criterion && (iter <= maxit)) {

        # ridge equation (1) gather eigen values
        # (spectral decomposition)
        Z = new.Z
        Omega = sigma_ridge(S + Y - rho *
            Z, lam = rho)$omega

        # penalty equation (2) soft-thresholding
        new.Z = soft(Y + rho * Omega, lam *
            alpha)/(lam * (1 - alpha) + rho)

        # update U (3)
        Y = Y + rho * (Omega - new.Z)

        # calculate new rho
        s = sqrt(sum((rho * (new.Z - Z))^2))
        r = sqrt(sum((Omega - new.Z)^2))
        rho = rho * (tau1 * (r > mu * s) +
            (s > mu * r)/tau2 + (s/mu <=
            r & r <= mu * s))
        iter = iter + 1

        # stopping criterion
        eps1 = p * tol1 + tol2 * max(sqrt(sum(Omega^2)),
            sqrt(sum(new.Z^2)))
        eps2 = p * tol1 + tol2 * sqrt(sum(Y^2))
        criterion = (r >= eps1 || s >= eps2)

    }
    return(list(Iterations = iter, Omega = Omega))
}
```

3 R Package

3.1 Installation

```
# The easiest way to install is from CRAN
install.packages("ADMMsigma")

# You can also install the development
# version from GitHub:
# install.packages('devtools')
devtools::install_github("MGallow/ADMMsigma")
```

If there are any issues/bugs, please let me know: github. You can also contact me via my website. Pull requests are welcome!

A (possibly incomplete) list of functions contained in the package can be found below:

- `ADMMsigma()` computes the estimated precision matrix (ridge, lasso, and elastic-net type regularization optional)
- `RIDGEsigma()` computes the estimated ridge penalized precision matrix via closed-form solution
- `plot.ADMMsigma()` produces a heat map for cross validation errors
- `plot.RIDGEsigma()` produces a heat map for cross validation errors

3.2 Usage

```
library(ADMMsigma)

# generate data from a sparse matrix
# first compute covariance matrix
S = matrix(0.7, nrow = 5, ncol = 5)
for (i in 1:5) {
    for (j in 1:5) {
        S[i, j] = S[i, j]^abs(i - j)
    }
}

# print oracle precision matrix
# (shrinkage might be useful)
(Omega = qr.solve(S) %>% round(3))
```

```
##        [,1]   [,2]   [,3]   [,4]   [,5]
## [1,]  1.961 -1.373  0.000  0.000  0.000
## [2,] -1.373  2.922 -1.373  0.000  0.000
## [3,]  0.000 -1.373  2.922 -1.373  0.000
## [4,]  0.000  0.000 -1.373  2.922 -1.373
## [5,]  0.000  0.000  0.000 -1.373  1.961
```

```
# generate 1000 x 5 matrix with rows
# drawn from iid N_p(0, S)
Z = matrix(rnorm(1000 * 5), nrow = 1000,
    ncol = 5)
out = eigen(S, symmetric = TRUE)
S.sqrt = out$vectors %*% diag(out$values^0.5) %*%
    t(out$vectors)
X = Z %*% S.sqrt


# print sample precision matrix (perhaps
# a bad estimate)
(qr.solve(cov(X)) %>% round(5))
```

```
##          [,1]     [,2]     [,3]     [,4]     [,5]
```

```
## [1,]  1.99991 -1.41471  0.09295 -0.08643  0.06678
## [2,] -1.41471  2.98077 -1.39085 -0.00180 -0.02666
## [3,]  0.09295 -1.39085  2.93466 -1.57182  0.13492
## [4,] -0.08643 -0.00180 -1.57182  3.28802 -1.50476
## [5,]  0.06678 -0.02666  0.13492 -1.50476  1.95149
```

```
# elastic-net type penalty (set tolerance
# to 1e-8)
ADMMsigma(X, tol1 = 1e-08, tol2 = 1e-08)
```

```
##
## Iterations:
## [1] 38
##
## Tuning parameters:
##       log10(lam)  alpha
## [1,]          -3    0.6
##
## Omega:
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,]  1.98552 -1.39091  0.07533 -0.07544  0.06043
## [2,] -1.39091  2.94059 -1.35892 -0.02041 -0.01718
## [3,]  0.07533 -1.35892  2.89522 -1.53331  0.11552
## [4,] -0.07544 -0.02041 -1.53331  3.23632 -1.47668
## [5,]  0.06043 -0.01718  0.11552 -1.47668  1.93602
```

```
# lasso penalty (default tolerance)
ADMMsigma(X, alpha = 1)
```

```
##
## Iterations:
## [1] 16
##
## Tuning parameters:
##       log10(lam)  alpha
## [1,]          -3      1
##
## Omega:
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,]  1.98897 -1.39520  0.07501 -0.07195  0.05755
## [2,] -1.39520  2.94945 -1.36518 -0.02047 -0.01395
## [3,]  0.07501 -1.36518  2.90753 -1.54178  0.11625
## [4,] -0.07195 -0.02047 -1.54178  3.24849 -1.48219
## [5,]  0.05755 -0.01395  0.11625 -1.48219  1.93985
```

```
# elastic-net penalty (alpha = 0.5)
ADMMsigma(X, alpha = 0.5)
```

```
##
## Iterations:
## [1] 16
##
## Tuning parameters:
##       log10(lam)  alpha
## [1,]          -3    0.5
##
```

```
## Omega:
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,]  1.98401 -1.38850  0.07368 -0.07468  0.06041
## [2,] -1.38850  2.93561 -1.35372 -0.02388 -0.01640
## [3,]  0.07368 -1.35372  2.88711 -1.52619  0.11302
## [4,] -0.07468 -0.02388 -1.52619  3.22819 -1.47294
## [5,]  0.06041 -0.01640  0.11302 -1.47294  1.93396
```

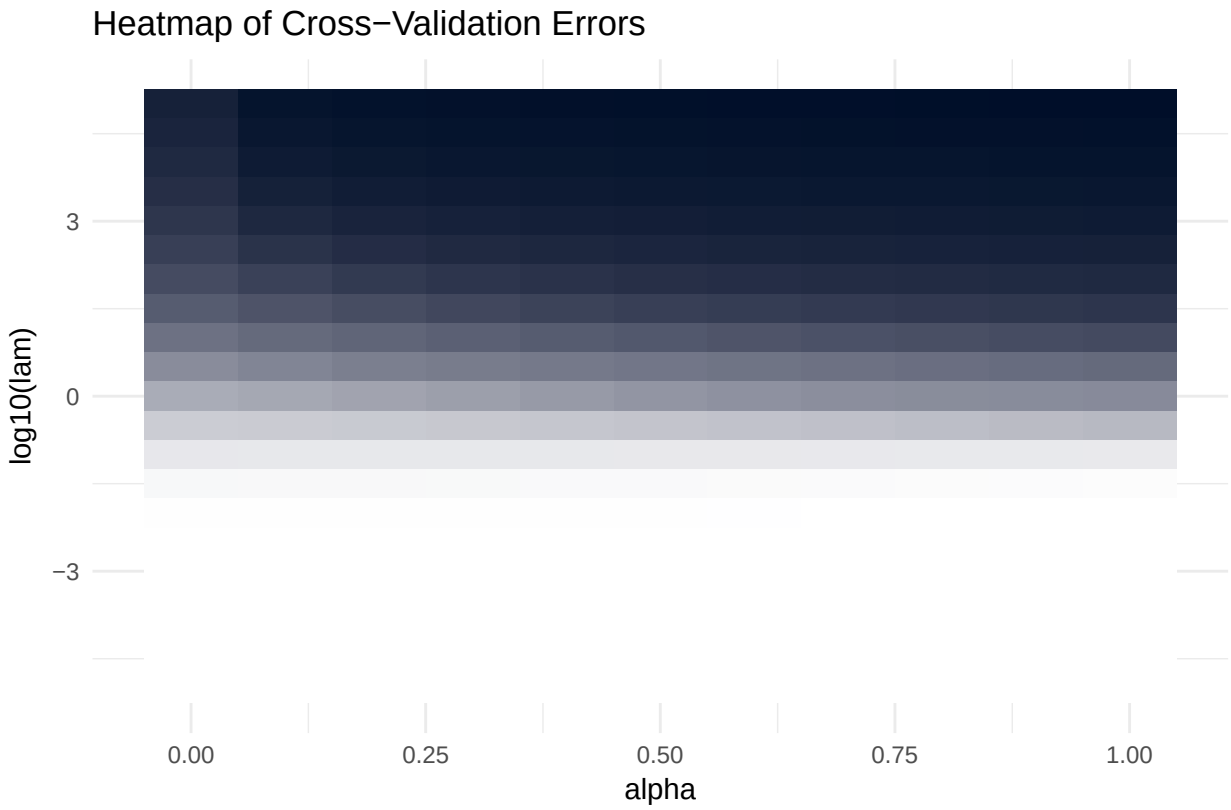
```
# ridge penalty
ADMMsigma(X, alpha = 0)
```

```
##
## Iterations:
## [1] 16
##
## Tuning parameters:
##       log10(lam)  alpha
## [1,]          -3      0
##
## Omega:
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,]  1.97957 -1.38283  0.07351 -0.07837  0.06365
## [2,] -1.38283  2.92392 -1.34491 -0.02497 -0.01979
## [3,]  0.07351 -1.34491  2.87050 -1.51430  0.11142
## [4,] -0.07837 -0.02497 -1.51430  3.21177 -1.46548
## [5,]  0.06365 -0.01979  0.11142 -1.46548  1.92892
```

```
# ridge penalty no ADMM
RIDGEsigma(X, lam = 10^seq(-8, 8, 0.01))
```

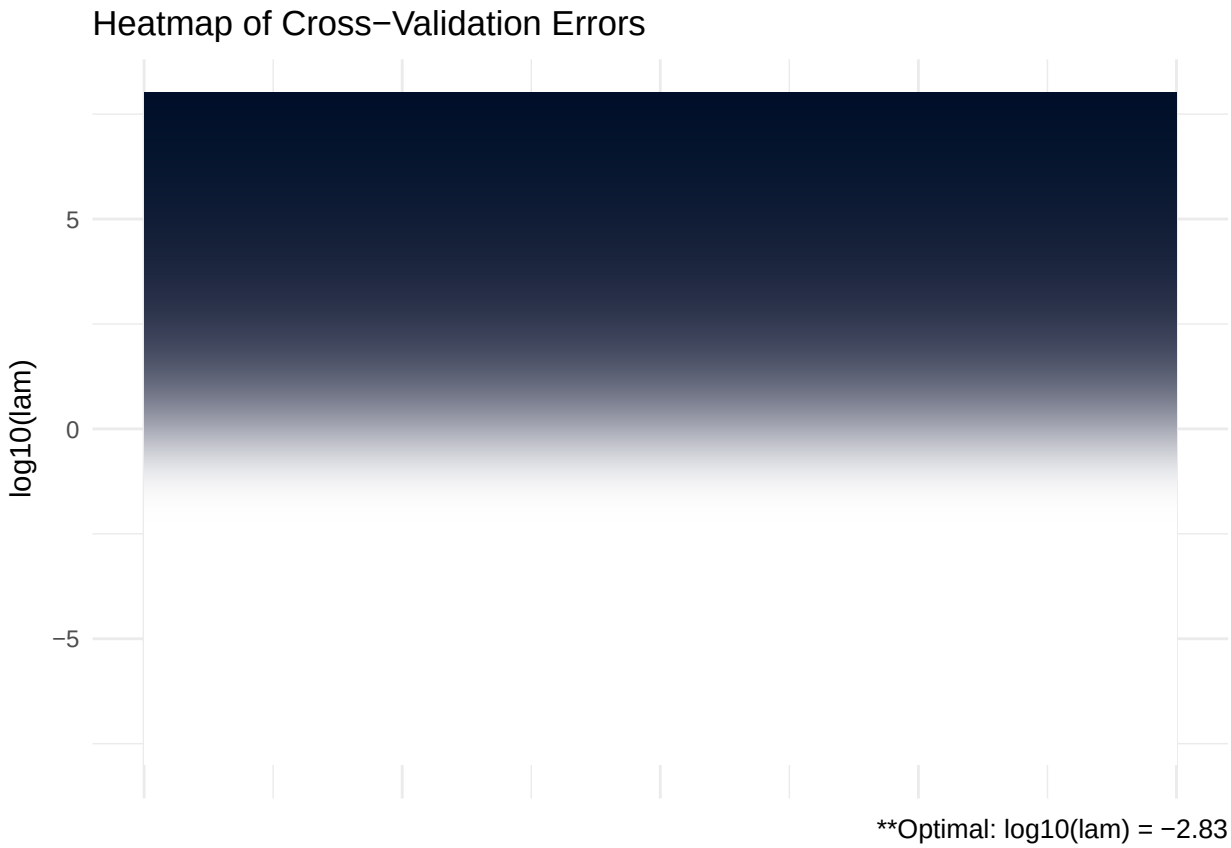
```
##
## Tuning parameter:
##       log10(lam)    lam
## [1,]       -2.82  0.002
##
## Omega:
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,]  1.96954 -1.36852  0.06659 -0.07680  0.06314
## [2,] -1.36852  2.89875 -1.32718 -0.03111 -0.01877
## [3,]  0.06659 -1.32718  2.84508 -1.49281  0.10334
## [4,] -0.07680 -0.03111 -1.49281  3.18037 -1.44902
## [5,]  0.06314 -0.01877  0.10334 -1.44902  1.91851
```

```
# produce CV heat map for ADMMsigma
ADMMsigma(X, tol1 = 1e-08, tol2 = 1e-08) %>%
    plot
```



**Optimal: log10(lam) = −3, alpha = 0.8

```
# produce CV heat map for RIDGEsigma
RIDGEsigma(X, lam = 10^seq(-8, 8, 0.01)) %>%
    plot
```



3.3 Benchmark

3.3.1 Computer Specs:

- MacBook Pro (Late 2016)
- Processor: 2.9 GHz Intel Core i5
- Memory: 8GB 2133 MHz
- Graphics: Intel Iris Graphics 550

```
# generate data from tri-diagonal
# (sparse) matrix compute covariance
# matrix (can confirm inverse is
# tri-diagonal)
S = matrix(0, nrow = 100, ncol = 100)

for (i in 1:100) {
    for (j in 1:100) {
        S[i, j] = 0.7^(abs(i - j))
    }
}

# generate 1000 x 100 matrix with rows
# drawn from iid N_p(0, S)
```

```
Z = matrix(rnorm(1000 * 100), nrow = 1000,
    ncol = 100)
out = eigen(S, symmetric = TRUE)
S.sqrt = out$vectors %*% diag(out$values^0.5) %*%
    t(out$vectors)
X = Z %*% S.sqrt


# glasso (for comparison)
microbenchmark(glasso(s = S, rho = 0.1))
```

```
## Unit: milliseconds
##                       expr      min       lq     mean   median       uq
##  glasso(s = S, rho = 0.1) 49.49746 51.2968 55.21674 53.17891 56.54074
##       max neval
##  94.10694   100
```

```
# benchmark ADMMsigma - default tolerance
microbenchmark(ADMMsigma(S = S, lam = 0.1,
    alpha = 1, tol1 = 1e-04, tol2 = 1e-04))
```

```
## Unit: milliseconds
##                                                                  expr
##  ADMMsigma(S = S, lam = 0.1, alpha = 1, tol1 = 1e-04, tol2 = 1e-04)
##       min       lq     mean   median       uq      max neval
##  40.48786 41.80211 45.23219 42.58309 44.18543 212.5204   100
```

```
# benchmark ADMMsigma - tolerance 1e-8
microbenchmark(ADMMsigma(S = S, lam = 0.1,
    alpha = 1, tol1 = 1e-08, tol2 = 1e-08))
```

```
## Unit: milliseconds
##                                                                  expr
##  ADMMsigma(S = S, lam = 0.1, alpha = 1, tol1 = 1e-08, tol2 = 1e-08)
##       min       lq     mean   median       uq      max neval
##  185.7857 190.2324 198.1983 193.4001 198.3315 256.4238   100
```

```
# benchmark ADMMsigma CV - default
# parameter grid
microbenchmark(ADMMsigma(X), times = 5)
```

```
## Unit: seconds
##          expr      min       lq    mean   median       uq      max neval
##  ADMMsigma(X) 16.30647 16.30802 16.3881 16.40673 16.45475 16.46454     5
```

```
# benchmark ADMMsigma parallel CV
microbenchmark(ADMMsigma(X, cores = 3), times = 5)
```

```
## Unit: seconds
##                     expr      min       lq     mean   median       uq
##  ADMMsigma(X, cores = 3) 12.95143 13.07855 13.29565 13.29788 13.54759
##       max neval
##  13.60278     5
```

```
# benchmark ADMMsigma CV - likelihood
# convergence criteria
microbenchmark(ADMMsigma(X, crit = "loglik"),
```

```
    times = 5)
```

```
## Unit: seconds
##                           expr      min       lq     mean   median
##  ADMMsigma(X, crit = "loglik") 52.37604 52.58107 52.74499 52.68818
##        uq      max neval
##  52.77534 53.30433     5
```

3.4 Simulations

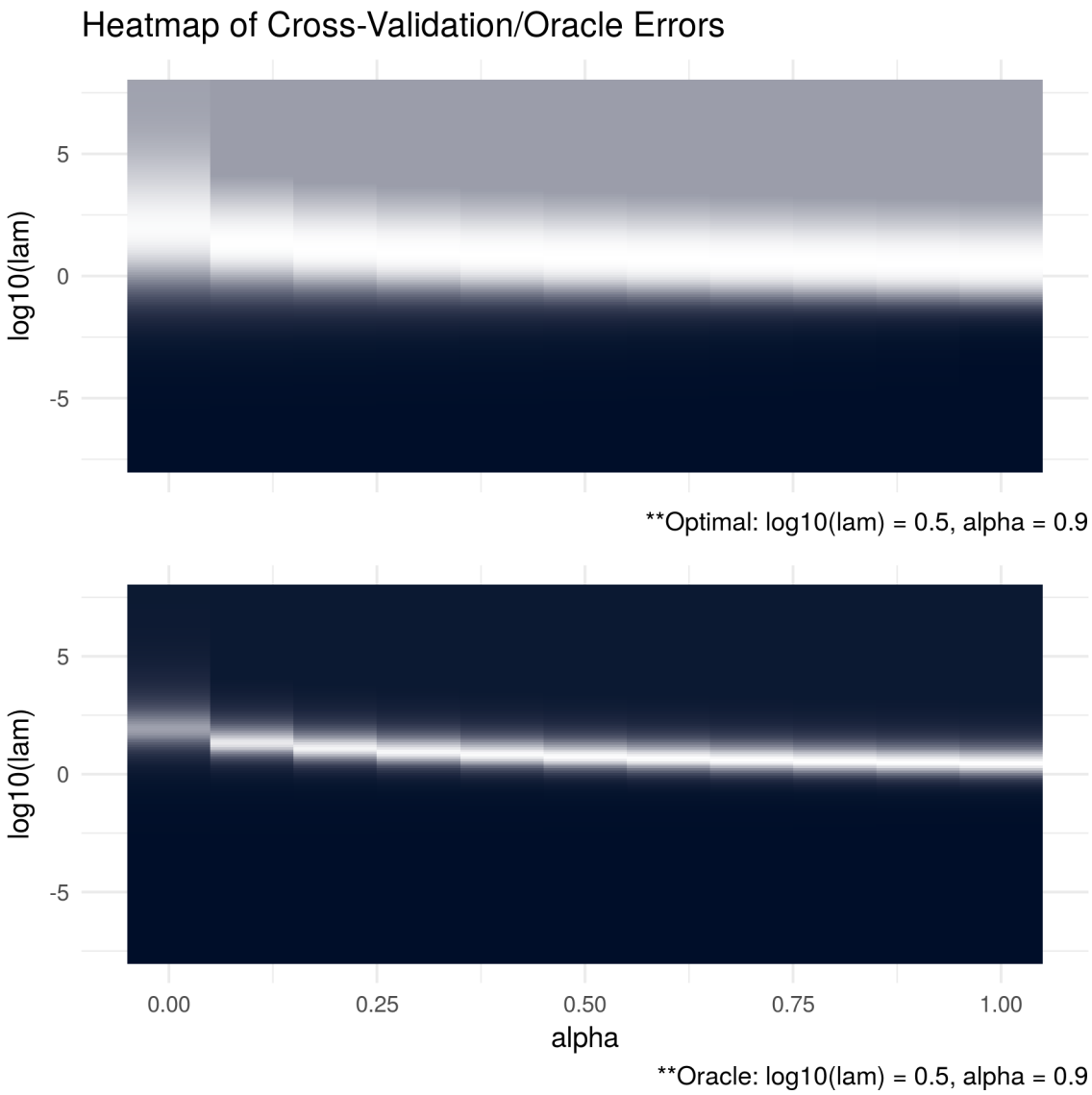

Heatmap of Cross-Validation/Oracle Errors
log10(lam)
5
0
-5
**Optimal: log10(lam) = 0.5, alpha = 0.9
log10(lam)
5
0
-5
0.00
0.25
0.50
0.75
1.00
alpha
**Oracle: log10(lam) = 0.5, alpha = 0.9

Figure 1:

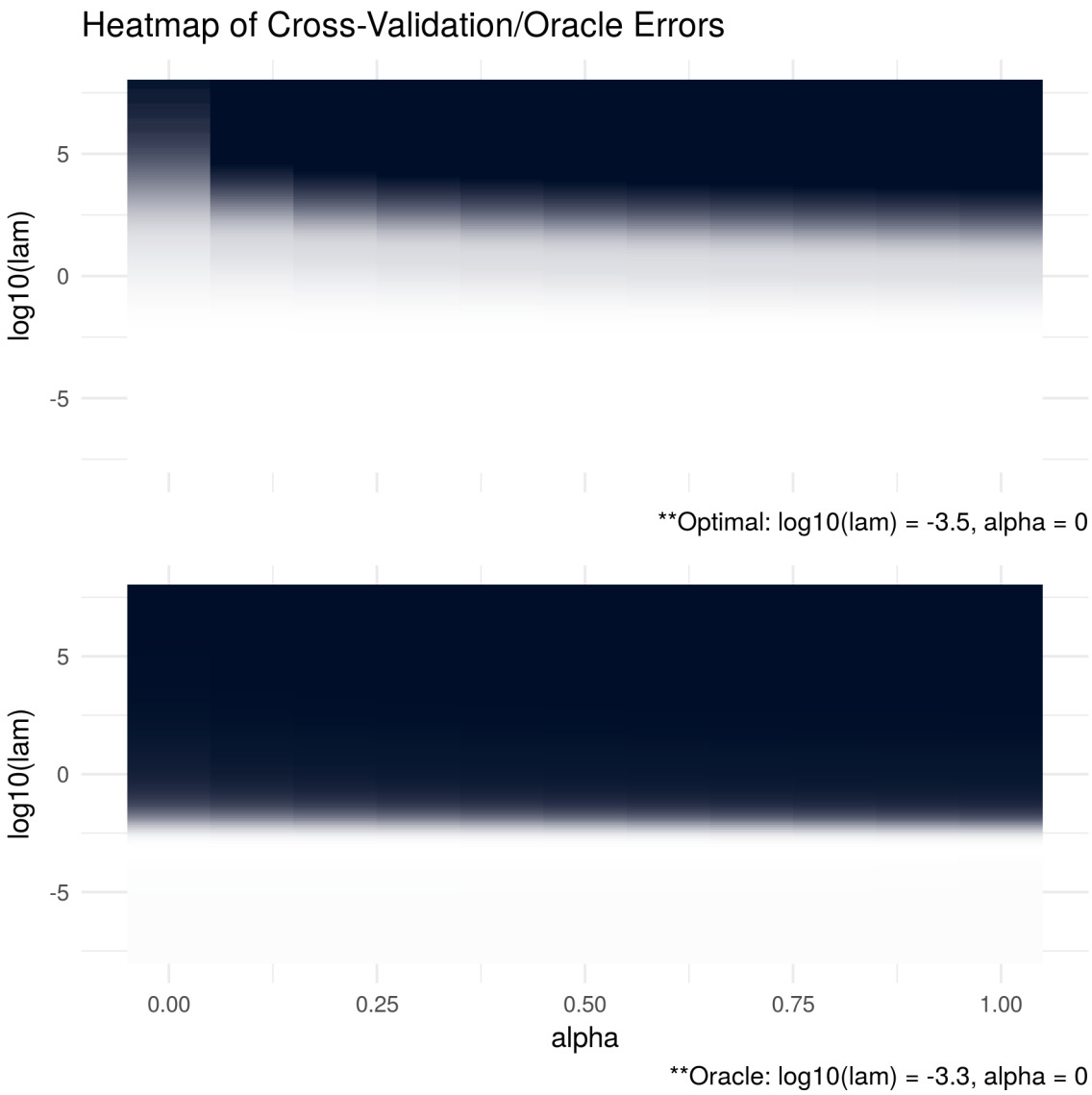

Heatmap of Cross-Validation/Oracle Errors
log10(lam)
5
0
-5
**Optimal: log10(lam) = -3.5, alpha = 0
log10(lam)
5
0
-5
0.00
0.25
0.50
0.75
1.00
alpha
**Oracle: log10(lam) = -3.3, alpha = 0

Figure 2:

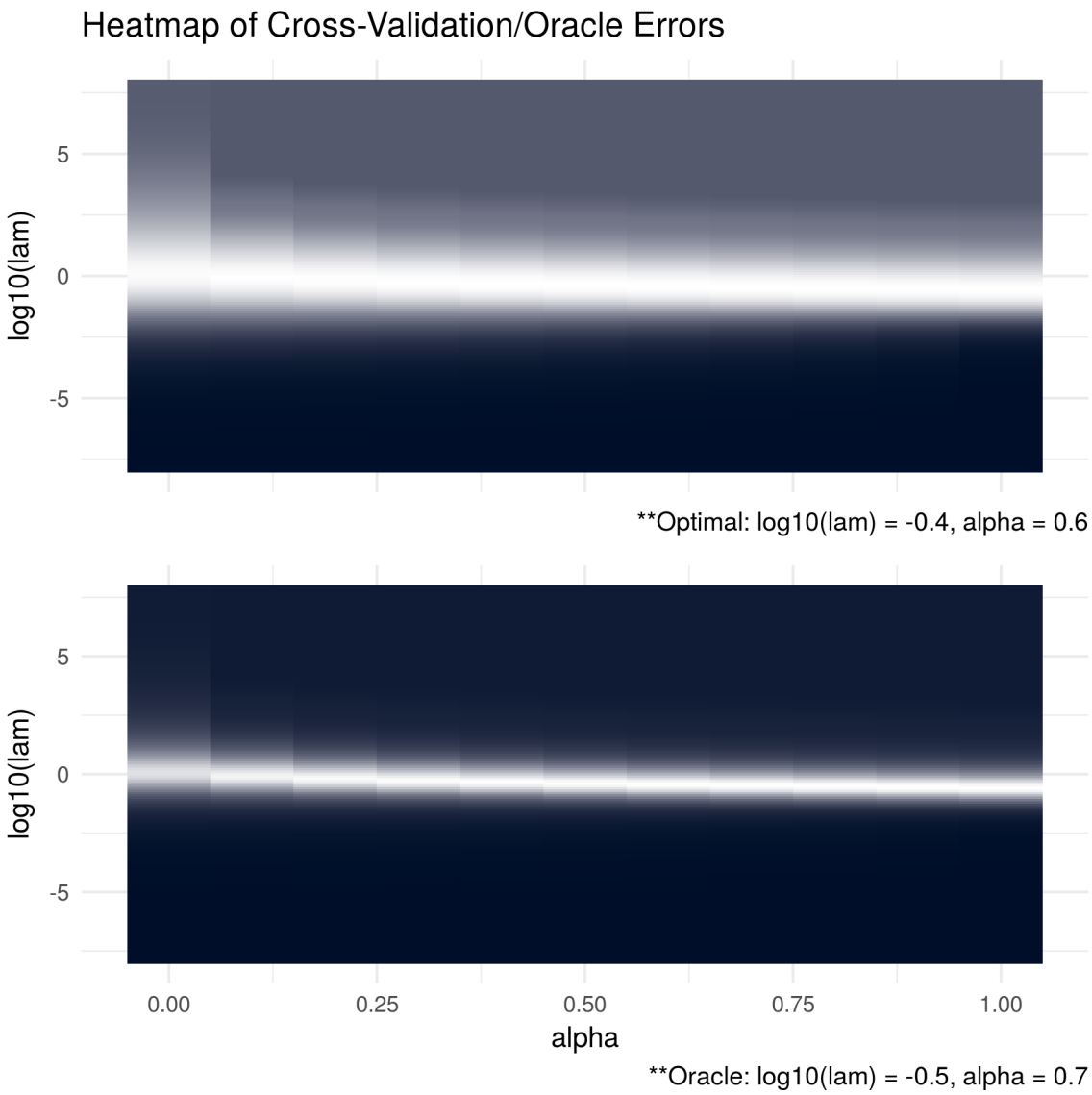

Heatmap of Cross-Validation/Oracle Errors
log10(lam)
5
0
-5
**Optimal: log10(lam) = -0.4, alpha = 0.6
log10(lam)
5
0
-5
0.00
0.25
0.50
0.75
1.00
alpha
**Oracle: log10(lam) = -0.5, alpha = 0.7

Figure 3:

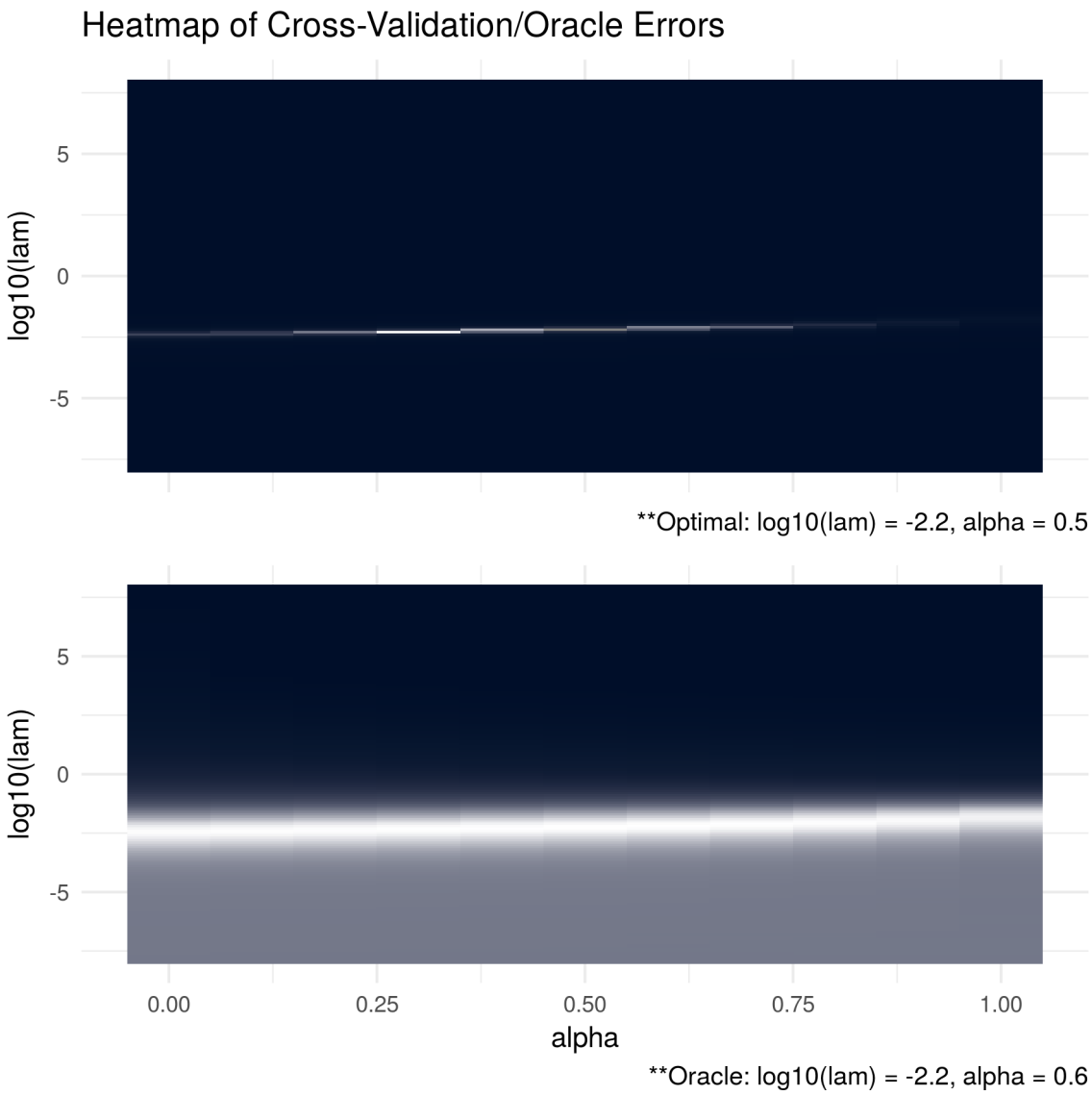

Heatmap of Cross-Validation/Oracle Errors
log10(lam)
5
0
-5
**Optimal: log10(lam) = -2.2, alpha = 0.5
log10(lam)
5
0
-5
0.00
0.25
0.50
0.75
1.00
alpha
**Oracle: log10(lam) = -2.2, alpha = 0.6

Figure 4:

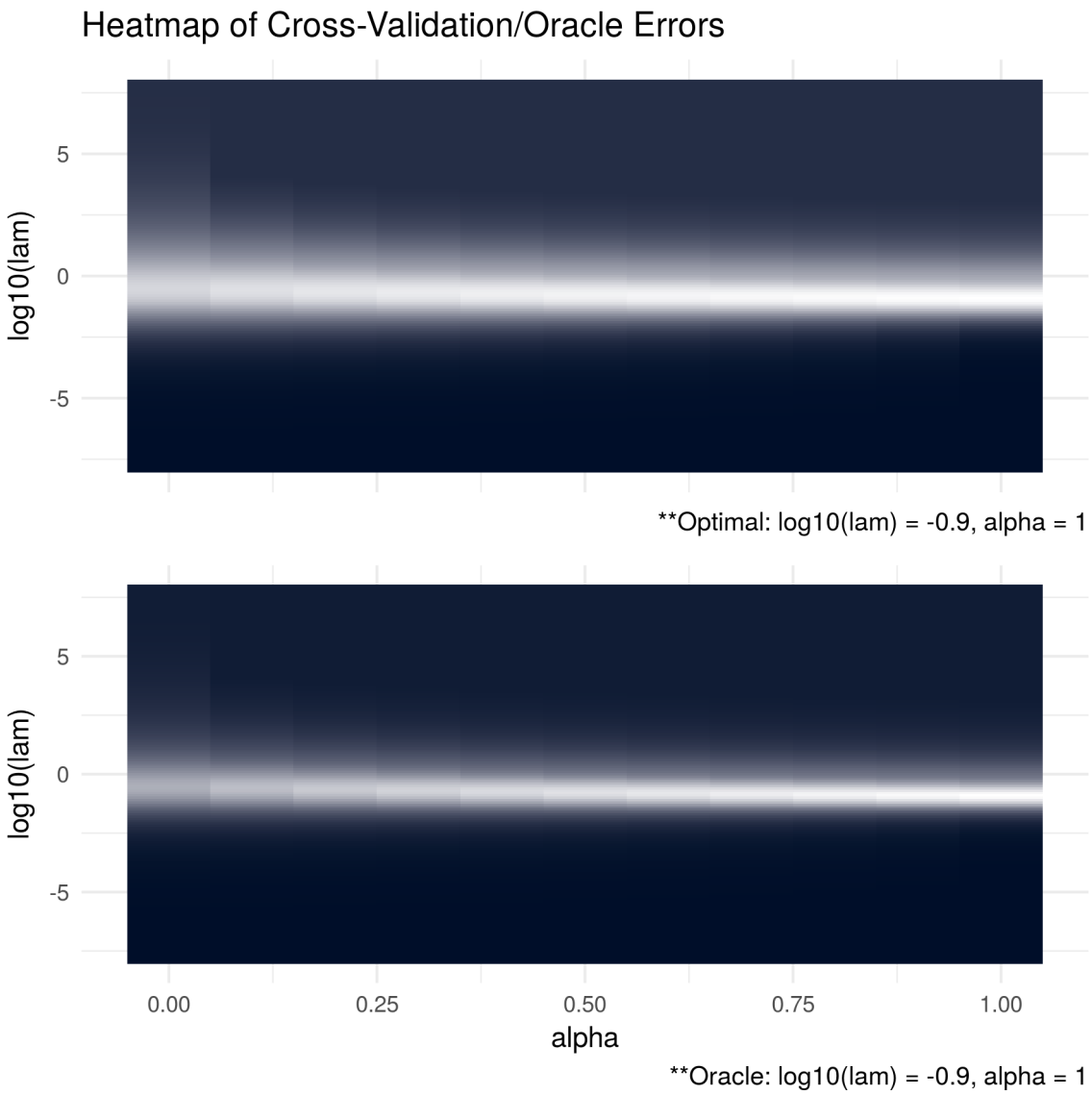

Heatmap of Cross-Validation/Oracle Errors
log10(lam)
5
0
-5
**Optimal: log10(lam) = -0.9, alpha = 1
log10(lam)
5
0
-5
0.00
0.25
0.50
0.75
1.00
alpha
**Oracle: log10(lam) = -0.9, alpha = 1

Figure 5:

References

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Zou, Hui, and Trevor Hastie. 2005. "Regularization and Variable Selection via the Elastic Net." *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 67 (2). Wiley Online Library: 301–20.