Discretization of Common Filters

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1 Introduction

This document contains difference equation implementations for transfer functions of common filters. Continuous-time transfer functions can be discretized using the relationship $z = e^{sT}$, where T is the sampling period, s is imaginary frequency and z is discretized imaginary frequency.

In order for this relationship to be helpful, we need to solve for s, then replace all instances of s in our continuous-time transfer functions with the result. Solving for s, however, results in a relationship involving a natural logarithm. We make a first-order approximation of the logarithm instead.

$$s = \frac{1}{T}\ln(z) \approx \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{1}$$

To go from a discrete-time transfer function to a difference equation, variables that are multiplied by z^{-n} are replaced with the values of those variables from n time steps in the past.

We use the variable u to represent input and y to represent output from the filters. We give all coefficients in the form of a_0 through a_n and b_0 through b_m where a coefficients are applied to input quantities and b coefficients are applied to output quantities. All coefficients can be scaled by $1/b_0$, so then $b_0 = 1$ and the first b coefficient required to be stored will be b_1. These can be used in a difference equation as shown below.

$$y_k = \frac{a_0 u_k + a_1 u_{k-1} + ... + a_n u_{k-n} - b_1 y_{k-1} - ... - b_m y_{k-m}}{b_0} \tag{2}$$

2 First Order Low-Pass Filter

Continuous-time transfer function:

$$\frac{Y}{U} = \frac{\omega}{s + \omega} \tag{3}$$

3 Second Order Low-Pass Filter

Continuous-time transfer function:

$$\frac{Y}{U} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} \tag{4}$$

Difference equation coefficients:

$$\begin{aligned}
a_0 &= \omega^2 T^2 \\
a_1 &= 2a_0 \\
a_2 &= a_0 \\
b_0 &= 4 + 4\zeta\omega T + \omega^2 T^2 \\
b_1 &= 2\omega^2 T^2 - 8 \\
b_2 &= 4 - 4\zeta\omega T + \omega^2 T^2
\end{aligned}$$

4 First Order High-Pass Filter

Continuous-time transfer function:

$$\frac{Y}{U} = \frac{s}{s + \omega} \tag{5}$$