Notes

Kerwann

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1 Plummer model

Consider a Plummer model (Dejonghe, H.1987, MNRAS 224, 13) with potential with units r_s the Plummer scale radius (which sets the size of the cluster core), M the total mass of the cluster and $\bar{\tau}$ some unit time. Let ψ_s be defined by

$$\psi_\mathrm{s} = \frac{GM}{r_\mathrm{s}}, \tag{1}$$

for the central potential

$$\psi(r) = \frac{\psi_\mathrm{s}}{\sqrt{1+r^2}}. \tag{2}$$

Let use fix $G = 1\, r_\mathrm{s}^3.M^{-1}.\bar{\tau}^{-2}$ in the new units so that $\psi_\mathrm{s} = 1\, r_\mathrm{s}^2 \cdot \bar{\tau}^{-2}$. This fixes the time unit $\bar{\tau}$, as we have the relation. Therefore, in those units the potential (per unit mass) is given by

$$\psi(r) = \frac{1}{\sqrt{1+r^2}}. \tag{3}$$

Define, given a radius r, the angular momentum $L(r, v_\mathrm{r}, v_\mathrm{t})$ and binding energy per unit mass $E(r, v_\mathrm{r}, v_\mathrm{t})$, functions of the radial velocity v_r and the tangential velocity $v_\mathrm{t} \geq 0$ (defined as $\boldsymbol{v} = v_\mathrm{r}\hat{\boldsymbol{r}} + \boldsymbol{v_\mathrm{t}}$), as

$$\begin{array}{rcl} E(r, v_\mathrm{r}, v_\mathrm{t}) & = & \psi(r) - \frac{1}{2}v_\mathrm{r}^2 - \frac{1}{2}v_\mathrm{t}^2, \\ L(r, v_\mathrm{r}, v_\mathrm{t}) & = & r \cdot v_\mathrm{t}, \end{array} \tag{4}$$

whose Jacobian is

$$\mathrm{Jac}_{(r,v_\mathrm{r},v_\mathrm{t}) \to (r,E,L)} = \begin{pmatrix} \frac{\partial E}{\partial v_\mathrm{r}} & \frac{\partial E}{\partial v_\mathrm{t}} \\ \frac{\partial L}{\partial v_\mathrm{r}} & \frac{\partial L}{\partial v_\mathrm{t}} \end{pmatrix} = \begin{pmatrix} -v_\mathrm{r} & -v_\mathrm{t} \\ 0 & r \end{pmatrix} \Rightarrow |\mathrm{Jac}| = r|v_\mathrm{r}|. \tag{5}$$

To obtain a bijective transformation, we must chose wether to chose $v_\mathrm{r} \geq 0$ or $v_\mathrm{r} \leq 0$. A priori, this choice might have an impact on the result, but we will should that the local and orbit-averaged diffusion coefficients are not that. The coordinate system is spherical, its origin being at the center of the globular cluster. Finally, we consider the corresponding anisotropic distribution functions of the field stars $F_q(r, E, L) = F_q(E, L)$ in (E, L)-space. Since (E, L) and $(v_\mathrm{r}, v_\mathrm{t})$ are linked, we can make use of the following equalities (for the moment, v_r is defined modulo the sign)

$$\begin{array}{rcl} F_q(E, L) & = & f_q(r, v_\mathrm{r}(r, E, L), v_\mathrm{t}(r, E, L)), \\ f_q(r, v_\mathrm{r}, v_\mathrm{t}) & = & F_q(E(r, v_\mathrm{r}, v_\mathrm{t}), L(r, v_\mathrm{r}, v_\mathrm{t})), \end{array} \tag{6}$$

where f_q is the distribution fonction (DF)in the $(v_\mathrm{r}, v_\mathrm{t})$ space and where q is an anisotropy parameter:

- $q \in]0, 2]$: radially anisotropic
- $q = 0$: isotropic
- $q \in] - \infty, 0[$: tangentially anisotropic.

Note that the DF has spherical symmetry in position. Its expression for $E \geq 0, L \geq 0$ is (for $q \neq 0$):

$$F_q(E, L) = \frac{3\Gamma(6-q)}{2(2\pi)^{5/2}\Gamma(q/2)} E^{7/2-q} \mathbb{H}(0, \frac{q}{2}, \frac{9}{2} - q, 1; \frac{L^2}{2E}) \tag{7}$$

where

$$\mathbb{H}(a,b,c,d;x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(c-a)\Gamma(a+d)} x^a \, {}_2F_1(a+b, 1+a-c, a+d; x) & \text{if} \quad x \leq 1, \\ \frac{\Gamma(a+b)}{\Gamma(d-b)\Gamma(b+c)} x^{-b} \, {}_2F_1(a+b, 1+b-d, b+c; \frac{1}{x}) & \text{if} \quad x \geq 1, \end{cases} \tag{8}$$

which reduces in the isotropic case $(q = 0)$ to

$$F_0(E) = \frac{3}{7\pi^3}(2E)^{7/2}, \tag{9}$$

and in the extreme radially anisotropic $(q = 2)$ to

$$F_2(E) = \begin{cases} \frac{6}{(2\pi)^3}(2E - L)^{3/2} & \text{if} \quad 2E \leq L^2, \\ 0 & \text{if} \quad 2E \geq L^2. \end{cases} \tag{10}$$

When $E \leq 0$ or $L \leq 0$ then $F_q(E, L) = 0$.

2 Determination of the local diffusion coefficients

The local diffusion coefficients are the average velocity changes per unit time. We are interested in computing (bad notation: those are relative to the test star velocity, as opposed to the relative velocity!!!)