1 Plummer model

Consider a Plummer model (Dejonghe, H.1987, MNRAS 224, 13) with potential with units r_s the Plummer scale radius (which sets the size of the cluster core), M the total mass of the cluster and $\bar{\tau}$ some unit time. Let ψ_s be defined by

$$\psi_s = \frac{GM}{r_s},$$

for the central potential

$$\psi(r) = \frac{\psi_s}{\sqrt{1+r^2}}.$$

Let use fix $G = 1\, r_s^3.M^{-1}.\bar{\tau}^{-2}$ in the new units so that $\psi_s = 1\, r_s^2 \cdot \bar{\tau}^{-2}$. This fixes the time unit $\bar{\tau}$, as we have the relation.

$$G = \tilde{G}\mathrm{m}^3.\mathrm{kg}^{-1}.\mathrm{s}^{-2} = \tilde{G}\frac{\mathrm{m}^3}{r_s^3}\frac{\mathrm{kg}^{-1}}{M^{-1}}\frac{s^{-2}}{\bar{\tau}^{-2}} r_s^3.M_\odot^{-1}.\tau^{-2} = \bar{G} r_s^3.M^{-1}.\bar{\tau}^{-2},$$

where $\tilde{G} = 6.67430 \times 10^{-11}$. Consequently we can deduce from $1 = \tilde{G}\frac{\mathrm{m}^3}{r_s^3}\frac{M}{\mathrm{kg}}\frac{\bar{\tau}^2}{s^2}$ that

$$\frac{\bar{\tau}}{s} = \sqrt{\frac{1}{\tilde{G}}\frac{r_s^3}{\mathrm{m}^3}\frac{\mathrm{kg}}{M}}.$$

Therefore, in those units the potential (per unit mass) is given by

$$\psi(r) = \frac{1}{\sqrt{1+r^2}}.$$

We use those units from now on. For example, M3 has total mass $M = 4.5 \cdot 10^5 M_\odot$ and radius $r_s = 90\,\mathrm{ly}$, hence $\bar{\tau} = 1.0167 \cdot 10^{14}\,\mathrm{s} = 3.2218\,\mathrm{Myr}$.

Define, given a radius r, the angular momentum $L(r, v_r, v_t)$ and binding energy per unit mass $E(r, v_r, v_t)$, functions of the radial velocity v_r and the tangential velocity $v_t \geq 0$ (defined as $\boldsymbol{v} = \boldsymbol{v_r} + \boldsymbol{v_t} = v_r\hat{\boldsymbol{r}} + \boldsymbol{v_t}$), as

$$E(r, v_r, v_t) = \psi(r) - \frac{1}{2}v_r^2 - \frac{1}{2}v_t^2$$

$$L(r, v_r, v_t) = r v_t$$

which is a transformation with Jacobian

$$\mathrm{Jac}_{(r,v_r,v_t)\to(r,E,L)} = \begin{pmatrix} \frac{\partial E}{\partial v_r} & \frac{\partial E}{\partial v_t} \\ \frac{\partial L}{v_{ar}} & \frac{\partial L}{\partial v_t} \end{pmatrix} = \begin{pmatrix} -v_r & -v_t \\ 0 & r \end{pmatrix} \Rightarrow |\mathrm{Jac}| = r|v_r|$$

To obtain a bijective transformation, we must chose wether to chose $v_r \geq 0$ or $v_r \leq 0$. A priori, this choice might have an impact on the result, but we will should that the local and orbit-averaged diffusion coefficients are not that. The coordinate system is spherical, its origin being at the center of the globular cluster. Finally, we consider the corresponding anisotropic distribution functions of the field stars $F_q(r, E, L) = F_q(E, L)$ in (E, L)-space. Since (E, L) and (v_r, v_t) are linked, we can make use of the following equalities (for the moment, v_r is determined modulo the sign)

$$F_q(r, E, L) = f_a(v_r(r, E, L), v_t(r, E, L))$$

and its converse

$$f_a(r, v_r, v_t) = F_q(E(r, v_r, v_t), L(r, v_r, v_t))$$

where q is an anisotropy parameter:

- $q \in]0, 2]$: radially anisotropic
- $q = 0$: isotropic

- $q \in]-\infty, 0[$: tangentially anisotropic.

Note that the DF has spherical symmetry in position. Its expression is (for $q \neq 0$):

$$F_q(E, L) = \frac{3\Gamma(6-q)}{2(2\pi)^{5/2}\Gamma(q/2)} E^{7/2-q} \mathbb{H}(0, \frac{q}{2}, \frac{9}{2} - q, 1; \frac{L^2}{2E})$$

$$\mathbb{H}(a, b, c, d; x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(c-a)\Gamma(a+d)} x^a \cdot {}_2F_1(a+b, 1+a-c, a+d; x) & x \leq 1 \\ \frac{\Gamma(a+b)}{\Gamma(d-b)\Gamma(b+c)} x^{-b} \cdot {}_2F_1(a+b, 1+b-d, b+c; \frac{1}{x}) & x \geq 1 \end{cases}$$

which reduces in the isotropic case ($q = 0$) to:

$$F(E) = \frac{3}{7\pi^3}(2E)^{7/2}$$

1.1 Determination of the local diffusion coefficients

The local diffusion coefficients are the average velocity changes per unit time. We are interested in computing

$$\langle \Delta v_{||} \rangle (r, v_r, v_t) = \frac{\langle \Delta v_{||} \rangle_{\delta t}}{\delta t}$$

$$\langle (\Delta v_{||})^2 \rangle (r, v_r, v_t) = \frac{\langle (\Delta v_{||})^2 \rangle_{\delta t}}{\delta t}$$

$$\langle (\Delta v_{\perp})^2 \rangle (r, v_r, v_t) = \frac{\langle (\Delta v_{\perp})^2 \rangle_{\delta t}}{\delta t}$$

Consider a test star at position r, mass m and initial velocity $\boldsymbol{v}$ which interacts with a field star with impact parameter b, mass m_a and velocity $\boldsymbol{v}$, Binney et Tremaine (2008, eq. (L.7) page 834) gives

$$\Delta v_{\perp} = \frac{2m_a V_0}{m + m_a} \frac{b/b_{90}}{1 + b^2/b_{90}^2}$$

$$\Delta v_{||} = \frac{2m_a V_0}{m + m_a} \frac{1}{1 + b^2/b_{90}^2}$$

where $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_a}$ and b_{90} is the 90° deflection radius, given by eq (L.8)

$$b_{90} = \frac{G(m + m_a)}{V_0^2}$$

Remember that in our units, $G = 1$ and m, m_a are given in fraction of the total mass M of the cluster.

Here, note that when considering a test star with energy and angular momentum (per unit mass) (E, L), using the choise $v_r \geq 0$ or the choice $v_r \leq 0$ has an impact on the local change of velocity through V_0.

We sum the effects of all the encounter up. Number density of field stars (at position r) within velocity space volume $\mathrm{d}^3\boldsymbol{v_a}$ is $f_a(r, \boldsymbol{v_a})\mathrm{d}^3\boldsymbol{v_a}$ (remember that $f_a(r, \boldsymbol{v_a}) = f_a(r, v_{ar}, v_{at})$). The number of encounters in a time δt with impact parameters between b and $b + \mathrm{d}b$ is just this density times the volume of an annulus with inner radius b, outer radius $b + \mathrm{d}b$, and length $V_0\delta t$, that is (eq. L9)

$$2\pi b \mathrm{d}b V_0 \delta t f_a(r, \boldsymbol{v_a})\mathrm{d}^3\boldsymbol{v_a}$$

We sum up over the velocities and the impact parameters. For the latter, we consider impact parameters between 0 and a cut-off $b_{\max}$, traditionally given approximately by the radius of the subject star orbit. This yields

$$\langle \Delta v_{\|} \rangle (r, \boldsymbol{v}) = 2\pi \int \mathrm{d}^3\boldsymbol{v_a} f_a(r, \boldsymbol{v_a}) \frac{m_a V_0^2 b_{90}^2}{m + m_a} \ln(1 + \Lambda^2)$$

$$\langle(\Delta v_\parallel)^2\rangle(r,\boldsymbol{v}) = 2\pi \int \mathrm{d}^3\boldsymbol{v_a} f_a(r,\boldsymbol{v_a}) 2V_0 \left(\frac{m_a V_0 b_{90}}{m+m_a}\right)^2 \left(1-\frac{1}{1+\Lambda^2}\right)$$

$$\langle(\Delta v_\perp)^2\rangle(r,\boldsymbol{v}) = 2\pi \int \mathrm{d}^3\boldsymbol{v_a} f_a(r,\boldsymbol{v_a}) 2V_0 \left(\frac{m_a V_0 b_{90}}{m+m_a}\right)^2 \left(\ln(1+\Lambda^2)+\frac{1}{1+\Lambda^2}-1\right)$$

where we defined the Coulomb parameter $\Lambda = b_{\max}/b_{90}$. Take $\Lambda = \lambda N$ (Binney et Tremaine, page 581) with $N \sim 10^5$ for a globular cluster.

Recall that $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_a}$. Since we assume that Λ is large, we do not make any significant additional error by replacing the factor V_0 in Λ by some typical stellar speed v_{typ}, that is,

$$\Lambda = \frac{b_{\max} v_{\mathrm{typ}}^2}{G(m+m_a)}.$$

Doing all the replacement gives and applying the change of variable $\boldsymbol{v'} = \boldsymbol{v} - \boldsymbol{v_a} = \boldsymbol{V_0}$

$$\langle\Delta v_\parallel\rangle(r,\boldsymbol{v}) = 2\pi \ln(1+\Lambda^2) G^2 m_a(m_a+m) \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'^2}$$

$$\langle(\Delta v_\parallel)^2\rangle(r,\boldsymbol{v}) = 4\pi \left(1-\frac{1}{1+\Lambda^2}\right) G^2 m_a^2 \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'}$$

$$\langle(\Delta v_\perp)^2\rangle(r,\boldsymbol{v}) = 4\pi \left(\ln(1+\Lambda^2)+\frac{1}{1+\Lambda^2}-1\right) G^2 m_a^2 \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'}$$

Since Λ is very large

$$\langle\Delta v_\parallel\rangle(r,\boldsymbol{v}) = 4\pi \ln\Lambda G^2 m_a(m_a+m) \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'^2}$$

$$\langle(\Delta v_\parallel)^2\rangle(r,\boldsymbol{v}) = 4\pi G^2 m_a^2 \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'}$$

$$\langle(\Delta v_\perp)^2\rangle(r,\boldsymbol{v}) = 4\pi(2\ln\Lambda-1) G^2 m_a^2 \int \mathrm{d}^3\boldsymbol{v'} \frac{f_a(r,\boldsymbol{v}-\boldsymbol{v'})}{v'}$$

Using spherical coordinates with axis $(Oz) = \hat{\boldsymbol{r}}$ the unit radius vector (parallel or antiparallel to the radial component of $\boldsymbol{v}$ by definition), we have

$$\langle\Delta v_\parallel\rangle(r,\boldsymbol{v}) = 4\pi \ln\Lambda G^2 m_a(m_a+m) \int_0^\infty \mathrm{d}v' \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r,\boldsymbol{v}-\boldsymbol{v'})$$

$$\langle(\Delta v_\parallel)^2\rangle(r,\boldsymbol{v}) = 4\pi G^2 m_a^2 \int_0^\infty \mathrm{d}v' v' \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r,\boldsymbol{v}-\boldsymbol{v'})$$

$$\langle(\Delta v_\perp)^2\rangle(r,\boldsymbol{v}) = 4\pi(2\ln\Lambda-1) G^2 m_a^2 \int_0^\infty \mathrm{d}v' v' \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r,\boldsymbol{v}-\boldsymbol{v'})$$

where we have

$$f_a(r,\boldsymbol{v}-\boldsymbol{v'}) = f_a(r,v_{ar},v_{at}) = F_q(E(r,v_{ar},v_{at}), L(r,v_{ar},v_{at}))$$

with

$$E(r,v_{ar},v_{at}) = \psi(r) - \frac{1}{2}{v_{ar}}^2 - \frac{1}{2}{v_{at}}^2$$

$$L(r,v_{ar},v_{at}) = r v_{at}$$

For a given convention $+$ or $-$ of the choice of v_r, and given (E,L) the parameters of the test star, obtain the vectors $\boldsymbol{v_+} = (|v_r|, \boldsymbol{v_t})$ and $\boldsymbol{v_-} = (-|v_r|, \boldsymbol{v_t})$, which are symmetric with respect to the tangent plane where $\boldsymbol{v_t}$ lives. In terms of spherical coordinates, we have that $\boldsymbol{v_+} = (v, \theta_0, 0)$ and $\boldsymbol{v_-} = (v, \pi-\theta_0, 0)$. Remember that the integration over the velocities $\boldsymbol{v'} = \boldsymbol{v} - \boldsymbol{v_a} = \boldsymbol{V_0}$of the field stars cover the whole $\boldsymbol{V_0}$-space. Given a velocity $\boldsymbol{V_0}$ corresponds bijectively a field star velocity $\boldsymbol{v_a}$.

We need to compute

$$I_1(r, \boldsymbol{v}) = \int_0^{\infty} \mathrm{d}v' \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r, \boldsymbol{v} - \boldsymbol{v'})$$

$$I_2(r, \boldsymbol{v}) = \int_0^{\infty} \mathrm{d}v' v' \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r, \boldsymbol{v} - \boldsymbol{v'})$$

where we need to compute the radial and tangential components of the vector $\boldsymbol{v_a} = \boldsymbol{v} - \boldsymbol{v'}$. This is where we need to make sure that the choice of convention for v_r doesn't change the overall result. The radial component is quite straightforward since

$$v_{ar} = (\boldsymbol{v} - \boldsymbol{v'})_r = v_r - v'_r = v_r - v' \cos\theta.$$

On the other hand, the tangential component is a bit more tricky. Let $\boldsymbol{v} = (v, \theta_0, 0)$, where $v_r = v\cos\theta_0$ and $v_t = v\sin\theta_0$, and let $\boldsymbol{v'} = (v', \theta, \phi)$. In cartesian coordinates, setting $(Ox) = \boldsymbol{v_t}$, $(Oz) = \boldsymbol{v_r}$ and (Oy) such that $(Oxyz)$ is a direction orthonormal coordinate system, we have

$$v_x = v_t$$

$$v_y = 0$$

$$v_z = v_r$$

and

$$v'_x = v' \sin\theta \cos\phi$$

$$v'_y = v' \sin\theta \sin\phi$$

$$v'_z = v' \cos\theta$$

Then

$$v_{ax} = v_t - v' \sin\theta\cos\phi$$

$$v_{ay} = -v' \sin\theta \sin\phi$$

$$v_{az} = v_r - v' \cos\theta$$

Therefore

$$v_{at}^2 = v_{ax}^2 + v_{ay}^2 = (v_t - v'\sin\theta\cos\phi)^2 + (v'\sin\theta\sin\phi)^2$$

$$v_{at}^2 = v_t^2 + v'^2 \sin^2\theta\cos^2\phi - 2v_r v' \sin\theta\cos\phi + v'^2\sin^2\theta\sin^2\phi$$

$$v_{at}^2 = v_t^2 + v'^2 \sin^2\theta - 2v_t v' \sin\theta\cos\phi$$

The complete norm of $\boldsymbol{v_a}$ is

$$v_a^2 = v_{ar}^2 + v_{at}^2 = (v_r - v'\cos\theta)^2 + v_t^2 + v'^2\sin^2\theta - 2v_t v' \sin\theta\cos\phi$$

$$v_a^2 = v_r^2 + v'^2\cos^2\theta - 2v_r v'\cos\theta + v_t^2 + v'^2\sin^2\theta - 2v_t v'\sin\theta\cos\phi$$

$$v_a^2 = v^2 + v'^2 - 2v'(v_r\cos\theta + v_t\sin\theta\cos\phi)$$

which gives the binding energy per unit mass

$$E(r, v', \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(v_r\cos\theta + v_t\sin\theta\cos\phi)\right]$$

and the angular momentum per unit mass

$$L(r, v', \theta, \phi) = r(v_t^2 + v'^2\sin^2\theta - 2v_t v'\sin\theta\cos\phi)$$

Now, remember that $\boldsymbol{v_+} = (|v_r|, \boldsymbol{v_t})$ and $\boldsymbol{v_-} = (-|v_r|, \boldsymbol{v_t})$ in the $+$ and $-$ convention. For a given $\boldsymbol{v'} = (v', \theta, \phi)$, the angular momentum doesn't depend on the convention we chose since only v_r is impacted. On the other hand, for the bunding energy per unit mass, we obtain

$$E_+(r, v', \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(|v_r|\cos\theta + v_t \sin\theta\cos\phi)\right]$$

$$E_-(r, v', \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(-|v_r|\cos\theta + v_t \sin\theta\cos\phi)\right]$$

which are different. However, when considering $\boldsymbol{\tilde{v}'} = (v', \pi - \theta, \phi)$, which is another vector used in the integration, we have

$$E_+(r, v', \pi - \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(-|v_r|\cos\theta + v_t \sin\theta\cos\phi)\right]$$

$$E_-(r, v', \pi - \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(|v_r|\cos\theta + v_t \sin\theta\cos\phi)\right]$$

meaning that

$$E_+(r, v', \theta, \phi) = E_-(r, v', \pi - \theta, \phi)$$
$$E_-(r, v', \theta, \phi) = E_+(r, v', \pi - \theta, \phi)$$

Furthermore, the prefactor in the integrand $\sin\theta$ becomes $\sin(\pi - \theta) = \sin\theta$ through this transformation. Therefore, the convention doesn't change the result of the overall integration, and we may chose to set $v_r \geq 0$.

Let $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, and consider

$$g(r, v', \theta, \phi) = v^2 + v'^2 - 2v'(v_r \cos\theta + v_t \sin\theta\cos\phi)$$

Then $g(r, v', \theta, \phi) \geq v^2 + v'^2 - 2v'(v_t + |v_r|)$ for all θ, ϕ in range, meaning that $g(r, v', \theta, \phi) \to +\infty$ as $v' \to \infty$ uniformely in angles. Therefore, there exists an bound v'_{max} such that

$$\forall v' > v'_{\text{max}}, \forall \theta \in [0, \pi], \forall \phi \in [0, 2\pi]; g(r, v', \theta, \phi) > 2\psi(r)$$

that is,

$$\forall v' > v'_{\text{max}}, \forall \theta \in [0, \pi], \forall \phi \in [0, 2\pi]; E(r, v', \theta, \phi) < 0$$

Above this bound, as the binding energy par unit mass is negative, the DF of the field stars, evaluated for (r, v', θ, ϕ), vanishes. This shows that the v'-integral is in fact finite. We can obtain this bound by solving the inequation in v'

$$E(r, v', \theta, \phi) < 0 \Leftrightarrow v'^2 - 2v'(v_t + |v_r|) + v^2 - 2\psi(r) > 0$$

The v' which satisfy this inequation are those which yields $E(r, v', \theta, \phi) < 0$ (for any angle), hence a vanishing DF. Since the polynomial in v' has non-negative leading coefficient (it is monic), the polynomial is either always non-negative (negative discriminant) or there is an closed interval over which it is negative (non-negative discriminant). The polynomial's discriminant is

$$\Delta_v = 4(v_t + |v_r|)^2 - 4(v^2 - 2\psi(r)) = 4(v_t^2 + v_r^2 + 2|v_r|v_t) - 4(v^2 - 2\psi(r)) = 8(|v_r|v_t + \psi(r))$$

Since $|v_r|, v_t \geq 0$, it is non-negative. In that case, the v' over which we can integrate are those which are positive ($v' \geq 0$) and between the roots of the polynomial, given by

$$v'_\pm = \frac{2(v_t + |v_r|) \pm \sqrt{8(|v_r|v_t + \psi(r))}}{2} = (v_t + |v_r|) \pm \sqrt{2(|v_r|v_t + \psi(r))}.$$

One may ask if v'_- is positive or negative. To that, recall that $|v_r| = v\cos\theta_0$ and $v_t = v\sin\theta_0$ for $\theta_0 \in [0, \pi/2]$. Then

$$v'_- = v(\cos\theta + \sin\theta) - \sqrt{2(v^2\cos\theta\sin\theta + \psi(r))}$$

We have that $v_- \leq 0$ iif $v(\cos\theta_0 + \sin\theta_0) \leq \sqrt{2(v^2\cos\theta_0\sin\theta_0 + \psi(r))}$. Both sides are positive for $\theta_0 \in [0, \pi/2]$, therefore $v_- \leq 0$ iif

$$v^2(\cos\theta_0 + \sin\theta_0)^2 \leq 2(v^2\cos\theta_0\sin\theta_0 + \psi(r))$$

iff $v^2(1 + 2\cos\theta_0\sin\theta_0) \leq 2(v^2\cos\theta_0\sin\theta_0 + \psi(r))$ iff $v^2 \leq 2\psi(r)$. Since $E = \psi(r) - \frac{1}{2}v^2$, this is equivalent to $2(\psi(r) - E) \leq 2\psi(r)$, i.e. $E \geq 0$. Therefore, since we study systems with $E \geq 0$, it follows that we always have $v_- \leq 0$, meaning that our integration is over $[0, v'_+]$. This upper bound depends on E, L, r and is given by the formula as long as $E \leq \psi(r)$ and $L \leq r\sqrt{2(\psi(r) - E)}$

$$v_+ = \left(\frac{L}{r} + \sqrt{2(\psi(r) - E) - \frac{L^2}{r^2}}\right) + \sqrt{2\left(\frac{L}{r}\sqrt{2(\psi(r) - E) - \frac{L^2}{r^2}} + \psi(r)\right)}$$

This condition will always be satisfied in a star orbit (see next section). It can be useful to define the effective binding potential

$$\psi_{\text{eff}}(r; L) = \psi(r) - \frac{L^2}{2r^2},$$

so that we can rewrite the upper bound of the integral as

$$v_{\max} = \frac{L}{r} + \sqrt{2(\psi_{\text{eff}}(r; L) - E)} + \sqrt{2\left(\frac{L}{r}\sqrt{2(\psi_{\text{eff}}(r; L) - E)} + \psi(r)\right)}$$

$$K(r, v') = \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a\left(r, v', \theta, \phi\right)$$

where

$$f_a(r, v', \theta, \phi) = F_q(E(r, v', \theta, \phi), L(r, v', \theta, \phi))$$

with

$$E(r, v', \theta, \phi) = \psi(r) - \frac{1}{2}\left[v^2 + v'^2 - 2v'(v_r\cos\theta + v_t\sin\theta\cos\phi)\right]$$

$$L(r, v', \theta, \phi) = r(v_t^2 + v'^2\sin^2\theta - 2v_t v'\sin\theta\cos\phi)$$

If we want to use the Cuba.jl package with the Cuhre method (cuhre() in Julia), we want to reduce our integral over an integration over $[0, 1]^3$. We must compute the two integrals

$$I_1(r, \boldsymbol{v}) = \int_0^{v_{\max}} \mathrm{d}v' \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r, v', \theta, \phi)$$

$$I_2(r, \boldsymbol{v}) = \int_0^{v_{\max}} \mathrm{d}v' v' \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_a(r, v', \theta, \phi)$$

Make the change of variable

$$\tilde{v'} = v'/v_{\max}; \quad \tilde{\theta} = \theta/\pi; \quad \tilde{\phi} = \phi/2\pi.$$

Then

$$I_1(r, \boldsymbol{v}) = 2\pi^2 v_{\max} \int_0^1 \mathrm{d}\tilde{v'} \int_0^1 \mathrm{d}\tilde{\theta}\sin(\pi\tilde{\theta}) \int_0^1 \mathrm{d}\phi f_a(r, v_{\max}v', \pi\tilde{\theta}, 2\pi\tilde{\phi})$$

$$I_2(r, \boldsymbol{v}) = 2\pi^2 v_{\max}^2 \int_0^1 \mathrm{d}\tilde{v'}\tilde{v'} \int_0^{\pi} \mathrm{d}\tilde{\theta}\sin(\pi\tilde{\theta}) \int_0^1 \mathrm{d}\phi f_a(r, v_{\max}v', \pi\tilde{\theta}, 2\pi\tilde{\phi})$$

Note that the Cuba.jl Julia library calls the Cuba C library. The cuhre function can be found:

- https://github.com/JohannesBuchner/cuba/blob/master/src/cuhre

May be interesting to find how to free memory to avoid getting millions (or billions) of allocations. Maybe go to C?

Now, switch to (E, L) space and using eq. (C15) to (C19) of Bar-Or & Alexander (2016), which doesn't rely on an isotropy assumption, we obtain (evaluate at $(r, \boldsymbol{v}(r, E, L))$) at first order in $\Delta v/v$ (with opposite convention for $\Delta v_{||}$, as we defined $\boldsymbol{\Delta v} = \boldsymbol{v'} - \boldsymbol{v} = -\Delta v_{||}\hat{\boldsymbol{v}} + \Delta \boldsymbol{v}_{\perp}$) :

$$\langle \Delta E \rangle(r, E, L) = v\langle \Delta v_{||}\rangle - \frac{1}{2}\langle (\Delta v_{||})^2\rangle - \frac{1}{2}\langle (\Delta v_{\perp})^2\rangle$$

$$\langle (\Delta E)^2 \rangle(r, E, L) = v^2\langle (\Delta v_{||})^2\rangle$$

$$\langle \Delta L \rangle(r, E, L) = -\frac{L}{v}\langle \Delta v_{||}\rangle + \frac{r^2}{4L}\langle (\Delta v_{\perp})^2\rangle$$

$$\langle (\Delta L)^2 \rangle(r, E, L) = \frac{L^2}{v^2}\langle (\Delta v_{||})^2\rangle + \frac{1}{2}\left(r^2 - \frac{L^2}{v^2}\right)\langle (\Delta v_{\perp})^2\rangle$$

$$\langle \Delta E \Delta L \rangle(r, E, L) = -L\langle (\Delta v_{||})^2\rangle$$

Finally, due to our analysis, those quantities are well defined and we can use the bijective transformation $(r, E, L) \leftrightarrow (r, v_r, v_t)$, , through the relations

$$E(r, v_r, v_t) = \psi(r) - \frac{1}{2}{v_r}^2 - \frac{1}{2}{v_t}^2$$

$$L(r, v_r, v_t) = r v_t$$

where $v_r, v_t \geq 0$, yielding $E \in]-\infty, \psi(r)]$ and $L \in [0, r\sqrt{2(\psi(r) - E)}]$, while the converse relations are

$$v_t(r, E, L) = \frac{L}{r}$$

$$v_r(r, E, L) = \sqrt{2(\psi_{\text{eff}}(r; L) - E)}$$

where $E \in]-\infty, \psi(r)]$ and $L \in [0, r\sqrt{2(\psi(r) - E)}]$, yielding $v_r, v_t \geq 0$.

1.2 Orbit of a test star in a globular cluster

We can now compute the local diffusion coefficients $\langle \Delta E \rangle(r, E, L)$, $\langle (\Delta E)^2 \rangle(r, E, L)$, $\langle \Delta L \rangle(r, E, L)$, $\langle (\Delta L)^2 \rangle(r, E, L)$ and $\langle \Delta E \Delta L \rangle(r, E, L)$. Since we are interested in the secular evolution of the system, we can average over the dynamical time and smear out the star along its orbit. This leads us to consider the orbit-average diffusion coefficients

$$\overline{D_X}(E, L) \doteqdot \langle D_X \rangle_{\circlearrowleft}(E, L) = \frac{1}{T}\int_0^T \langle \Delta X \rangle(r(t), E, L)\mathrm{d}t = \frac{2}{T}\int_{r_{min}}^{r_{max}} \langle \Delta X \rangle(r, E, L)\frac{\mathrm{d}r}{v_r(r, E, L)}$$

$$\overline{D_{XY}}(E, L) \doteqdot \langle D_{XY} \rangle_{\circlearrowleft}(E, L) = \frac{1}{T}\int_0^T \langle \Delta X \Delta Y \rangle(r(t), E, L)\mathrm{d}t = \frac{2}{T}\int_{r_{min}}^{r_{max}} \langle \Delta X \Delta Y \rangle(r, E, L)\frac{\mathrm{d}r}{v_r(r, E, L)}$$

where $v_r(r)$ is the radial velocity of the orbiting star at r. However, this suppose that the star follow "nice" trajectories. This is what we will look into in this section. Furthermore, since E, L are chosen so that the trajectory is a circular orbit with radius R, then we apply the "time" formula instead of the radial velocity one and obtain

$$\overline{D_X}(E, L) = \frac{1}{T}\int_0^T \langle \Delta X \rangle(R, E, L)\mathrm{d}t = \langle \Delta X \rangle(R, E, L)$$

$$\overline{D_{XY}}(E, L) = \frac{1}{T}\int_0^T \langle \Delta X \Delta Y \rangle(R, E, L)\mathrm{d}t = \langle \Delta X \Delta Y \rangle(R, E, L)$$

1.2.1 Study of an orbit

-> See Kurth (1955), Astronomische Nachrichten, volume 282, Issue 6, p.241.

Consider a test star described by its position vector $\boldsymbol{r}$, its binding energye (opposite of its energy) $E(t)$ and its angular momentum vector $\boldsymbol{L}(t)$, per unit mass. Then

$$E(t) = \psi(r(t)) - \frac{1}{2}\dot{\boldsymbol{r}}^2(t) = \psi(r(t)) - \frac{1}{2}\dot{r}^2(t) - \frac{1}{2}r^2(t)\dot{\theta}^2(t)$$

$$\boldsymbol{L}(t) = \boldsymbol{r}(t) \times \boldsymbol{v}(t)$$

Differentiating the binding energy and using Newton's law shows that it is conserved. Let $E(t) = E$. On the other hand, differentiating $\boldsymbol{L}(t)$ and using the fact that the potential is central shows that this quantity is also conserved. Therefore the star's orbit is kept within a fixed plane determined by its initial conditions. Let $L(t) = L$ be its conserved norm. We also have

$$L = r(t)v_t(t) = r(t)^2\dot{\theta}(t)$$

In our case, consider an orbit with binding energy $E \geq 0$ and angular momentum $L \geq 0$. We can rewriting the energy conservation equation (on one orbit) as

$$E = \psi(r) - \frac{1}{2}\dot{r}^2 - \frac{L^2}{2r^2} \Leftrightarrow \dot{r}^2 = 2(\psi(r) - E) - \frac{L^2}{r^2} = 2(\psi_{\text{eff}}(r; L) - E)$$

Define

$$v_r(r) \doteqdot \sqrt{2(\psi_{\text{eff}}(r; L) - E)} \geq 0$$

Consider starting the motion from a radius with position initial radial velocity. Then as long as the radial velocity is positive, we have

$$\int_0^t \frac{\dot{r}(t)\mathrm{d}t}{v_r(r(t))} = t \Leftrightarrow \int_{r_{\min}}^{r(t)} \frac{\mathrm{d}r}{v_r(r)} = t.$$

This motion goes on until $\dot{r}(\tau) = 0$ for some time τ defined by

$$\tau = \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)}$$

where $r_{\max}$ is the radius reached at τ. Then the $\ddot{r}$ (negative) decreases r until a radius $r_{\min}$ which has vanishing radial velocity (but positive $\ddot{r}$), and the process repeats itself (see next section for a proof of those accelerations). Note that the motion from $r_{\max}$ to $r_{\min}$ is symmetrical to that from $r_{\min}$to $r_{\max}$, as it follows the relation

$$\int_\tau^t \frac{\dot{r}(t)\mathrm{d}t}{-v_r(r(t))} = t - \tau \Leftrightarrow \int_{r_{\max}}^{r(t)} \frac{\mathrm{d}r}{v_r(r)} = \tau - t$$

hence

$$\int_{r_{\min}}^{r(t)} \frac{\mathrm{d}r}{v_r(r)} - \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)} = \tau - t \Leftrightarrow \int_{r_{\min}}^{r(t)} \frac{\mathrm{d}r}{v_r(r)} = 2\tau - t.$$

Letting the orbit start at $r_{\min}$ and setting

$$F(r) = \int_{r_{\min}}^{r} \frac{dx}{v_r(x)},$$

we obtain an expression for the radius of the orbit

$$r(t) = \begin{cases} F^{-1}(t) & t \in [0, \tau] \\ F^{-1}(2\tau - t) & t \in [\tau, 2\tau] \end{cases},$$

with the symmetry $r(t) = r(2\tau - t)$ and the 2τ-periodicity of the radius. Furthermore, notice that $1/v_r(r)$ is integrable at $r_{\min}$ and $r_{\max}$. To show this, consider what happens near $r_{\max}$. Let $\epsilon > 0$.

$$v_r^2(r_{\max} - \epsilon) = 2(\psi(r_{\max} - \epsilon) - E) - \frac{L^2}{(r_{\max} - \epsilon)^2} = 2(\psi(r_{\max}) - \psi'(r_{\max})\epsilon + o(\epsilon) - E) - \frac{L^2}{r_{\max}^2}\frac{1}{(1 - \frac{\epsilon}{r_{\max}})^2}$$

$$v_r^2(r_{\max} - \epsilon) = 2(\psi(r_{\max}) - \psi'(r_{\max})\epsilon + o(\epsilon) - E) - \frac{L^2}{r_{\max}^2}(1 + \frac{2\epsilon}{r_{\max}} + o(\epsilon))$$

$$v_r^2(r_{\max} - \epsilon) = \underbrace{v_r^2(r_{\max})}_{=0} - 2\psi'(r_{\max})\epsilon - \frac{2L^2\epsilon}{r_{\max}^3} + o(\epsilon)$$

$$v_r^2(r_{\max} - \epsilon) = \frac{2r_{\max}}{(1 + r_{\max}^2)^{3/2}}\epsilon - \frac{2L^2\epsilon}{r_{\max}^3} + o(\epsilon)$$

Thus

$$\frac{1}{v_r(r_{\max} - \epsilon)} = \left(\underbrace{\frac{2r_{\max}}{(1 + r_{\max}^2)^{3/2}} - \frac{2L^2}{r_{\max}^3}}_{=-2\psi'_{\text{eff}}(r_{\max}) \geq 0}\right)^{-1/2} \frac{1}{\sqrt{\epsilon}}(1 + o(1))$$

A similar calculation yields

$$\frac{1}{v_r(r_{\min} + \epsilon)} = \left(\underbrace{-\frac{2r_{\min}}{(1 + r_{\min}^2)^{3/2}} + \frac{2L^2}{r_{\min}^3}}_{=2\psi'_{\text{eff}}(r_{\min}) \geq 0}\right)^{-1/2} \frac{1}{\sqrt{\epsilon}}(1 + o(1))$$

which is integrable since $1/\sqrt{\epsilon}$ is integrable at 0^+ if the $()^{-1/2}$ term in strictly positive. Once we have shown this, we can conclude that the orbit is "rosette-like", with periodical radius of periode $T = 2\tau$ with

$$\tau = \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)}.$$

Maybe isolate the borders and use change of variable $\psi_{\text{eff}}(r; L) - E = \sin^2(\theta)$. Then

$$\int_{r_{\min}} \frac{\mathrm{d}r}{v_r(r)} = \int_0 \frac{2\cos\theta\sin\theta\mathrm{d}\theta}{\psi'_{\text{eff}}(r; L)\sqrt{2\sin^2\theta}} = \int_0 \frac{\sqrt{2}\cos\theta\mathrm{d}\theta}{\psi'_{\text{eff}}(r; L)}$$

with $\psi_{\text{eff}}(r; L) - E = \sin^2(\theta)$. This change of variable is posible because $\psi_{\text{eff}}(r; L) \geq E$ and $E > 0$, hence $\psi_{\text{eff}}(r; L) - E \geq 0$ and $\psi_{\text{eff}}(r; L) - E \leq \psi_{\text{eff}}(r; L) \leq 1$. This transformation is bijective on $r \in [r_{\min}, r_*^L]$ and $r \in [r_*^L, r_{\max}]$ where $\psi'_{\text{eff}}(r_*^L; L) = 0$. This way we have

$$\frac{1}{\sqrt{1 + r^2}} = \frac{L^2}{2r^2} + E + \sin^2\theta = \frac{L^2 + 2r^2(E + \sin^2\theta)}{2r^2}$$

$$\frac{1}{1 + r^2} = \frac{(L^2 + 2r^2(E + \sin^2\theta))^2}{4r^4}$$

Let $X = r^2$. Then

$$4X^2 = (1 + X)(L^2 + 2X(E + \sin^2\theta))^2,$$

degree-3 equation hence with the roots X with an analytical expression. Two solution between $r_{\min}$ and $r_{\max}$, each separated by r_*^L. Therefore, letting $r_{\min} < r_1 < r_*^L$ and $r_*^L < r_2 < r_{\max}$, decompose the integral into

$$\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)} = \int_{r_{\min}}^{r_1} \frac{\mathrm{d}r}{v_r(r)} + \int_{r_1}^{r_*^L} \frac{\mathrm{d}r}{v_r(r)} + \int_{r_*^L}^{r_2} \frac{\mathrm{d}r}{v_r(r)} + \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)}$$

$$\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_r(r)} = \int_0^{\theta_1} \frac{\sqrt{2}\cos\theta \mathrm{d}\theta}{\psi'_{\text{eff}}(r(\theta);L)} + \int_{r_1}^{r_*^L} \frac{\mathrm{d}r}{v_r(r)} + \int_{r_*^L}^{r_2} \frac{\mathrm{d}r}{v_r(r)} + \int_0^{\theta_2} \frac{\sqrt{2}\cos\theta \mathrm{d}\theta}{|\psi'_{\text{eff}}(r(\theta);L)|}$$

where $\psi_{\text{eff}}(r_1;L) - E = \sin^2(\theta_1)$ and $\psi_{\text{eff}}(r_2;L) - E = \sin^2(\theta_2)$.

As for the angle, its derivative $\dot{\theta}$ is T-periodical since $L = r^2\dot{\theta}$ with $r \geq 0$ T-periodical. Therefore it can be decomposed as

$$\theta(t) = \omega t + p(t)$$

where $p(t)$ is T-periodical and ω is a real constant. Indeed, let $\omega = \frac{1}{T}\int_0^T \dot{\theta}(t)\mathrm{d}t$ and $p(t) = \theta(t) - \omega t$. Then

$$p(t+T) = \theta(t+T) - \omega(t+T) = \int_0^t \dot{\theta}(t)\mathrm{d}t + \int_t^{t+T} \dot{\theta}(t)\mathrm{d}t - \omega t - \omega T$$

$$p(t+T) = \left(\int_0^t \dot{\theta}(t)\mathrm{d}t - \omega t\right) + \left(\int_0^T \dot{\theta}(t)\mathrm{d}t - \int_0^T \dot{\theta}(t)\mathrm{d}t\right) = (\theta(t) - \omega t) = p(t)$$

showing that $p(t)$ is T-periodical.

Now, to find what $r_{\max}$ and $r_{\min}$ are, we need to solve $v_r(r) = 0$, i.e. $\psi(r) - E - \frac{L^2}{2r^2} = 0$, i.e.

$$\frac{1}{\sqrt{1+r^2}} = \frac{L^2}{2r^2} + E = \frac{2Er^2 + L^2}{2r^2} \Leftrightarrow \frac{1}{1+r^2} = \frac{(2Er^2+L^2)^2}{4r^4} \Leftrightarrow 4r^4 = (2Er^2+L^2)^2(1+r^2)$$

Let $X = r^2$. Then

$$4X^2 = (2EX + L^2)^2(1+X) \Leftrightarrow 4E^2X^3 + 4(E^2 - 1 + EL^2)X^2 + (4EL^2 + L^4)X + L^4 = 0$$

This is a degree-3 polynomial in X. It has 3 real roots iff its discriminant Δ is strictly positive (two real roots, one of which is double, if $\Delta = 0$). Let

$$\alpha = 4E^2; \quad \beta = 4(E^2 - 1 + EL^2); \quad \gamma = 4EL^2 + L^4; \quad \delta = L^4.$$

Then the polynomial has the form $\alpha X^3 + \beta X^2 + \gamma X + \delta$. Suppose $E > 0$. Setting

$$X = Y - \frac{\beta}{3\alpha}; \quad p = \frac{3\alpha\gamma - \beta^2}{3\alpha^2}; \quad q = \frac{2\beta^3 - 9\alpha\beta\gamma + 27\alpha^2\delta}{27\alpha^3},$$

we have that $\alpha X^3 + \beta X^2 + \gamma X + \delta = 0$ iff $Y^3 + pY + q = 0$ where the roots of the two polynomials are linked by the formula $X_i = Y_i - \frac{\beta}{3\alpha}$. As for the discriminant of the Y polynomial, it is

$$\Delta = -(4p^3 + 27q^2).$$

Since only its sign matter on looking for the behavior of the solutions, we may only compute Δ.

If $\Delta < 0$, i.e. $4p^3 + 27q^2 > 0$, then the polynomial has only one real root given by

$$Y = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}$$

If $\Delta \geq 0$, i.e. $4p^3 + 27q^2 \geq 0$, then there are three real roots given by

$$Y_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right], \quad k \in \{0,1,2\}.$$

Note that when $\Delta = 0$, this reduces to the two roots

$$Y_0 = 2\sqrt{-\frac{p}{3}}, \quad Y_{1,2} = -\sqrt{-\frac{p}{3}}.$$

Recall $\dot{r}^2/2 = \psi_{\text{eff}}(r;L) - E$ (≥ 0 on the orbit). $\psi_{\text{eff}}(r;L)$ has derivative is

$$\psi'_{\text{eff}}(r;L) = \frac{L^2}{r^3} - \frac{r}{(1+r^2)^{3/2}}$$

with $\psi'_{\text{eff}}(r;L) \leq 0$ iif $r^4/L^2 \geq (1+r^2)^{3/2}$. Left term grows more quickly ($\sim r^4$) that the second term ($\sim r^3$) but stars at 0 whereas second term starts at $1/2 > 0$. Therefore the two curves cross at a unique point, and this inequality is satisfied after this point. This shows that $\psi_{\text{eff}}(r;L)$ is increasing until some point, then decreases. In this maximum is stricly below E, then there are no solution. If the maximum is exactly E, then there is only one solution and the orbit is circular. If the maximum is strictly above E, then they are two solutions which are $r_{\min}$and $r_{\max}$. The latter is because $\lim_{r\to\infty}\psi_{\text{eff}}(r;L) = 0 < E$.

If we are in the case of no solution, then $v_r(r) = 2(\psi_{\text{eff}}(r;L) - E) < 0$, which is impossible on an orbit.

If we are in the case with two solutions, then the solution are not the maximum of $\psi_{\text{eff}}(r;L)$, meaning that the derivative evaluated at the solutions are non-zero. This completes the proof that $1/v_r(r)$ is integrable at $r_{\max}$ and $r_{\min}$ as the integral of $1/\sqrt{r - r_{\text{m}}}$. Furthermore, since $\psi_{\text{eff}}(r;L)$ must have two positive distincts solutions $r_{\min}, r_{\max}$, then the polynomial should also have two distincts positive solutions $X_{\max} = r^2_{\max}$ and $X_{\min} = r^2_{\min}$ (and the third one being negative).

One should check whether a given couple (E,L) allows for bound orbits. That that end, we should find if there exists at least one r that that $\psi_{\text{eff}}(r;L) \geq E$, i.e. if there are solution to the polynomial. Equivalently, this reduces to computing the discriminant Δ and testing if it is positive.

1.2.2 Computing the binding energy of a circular orbit given L

We have now access to the NR, orbit-averaged diffusion coefficients in (E,L)-space for the allowed bound orbits: $\bar{D}_E$, $\bar{D}_{EE}$, $\bar{D}_L$, $\bar{D}_{LL}$ and $\bar{D}_{EL}$, functions of (E,L). The allowed region in (E,L) space is composed of the $E, L \geq 0$ such that there exists $r > 0$ verifying the inequality $\psi_{\text{eff}}(r;L) \geq E$, where we defined the effective potential

$$\psi_{\text{eff}}(r;L) = \psi(r) - \frac{L^2}{2r^2} = \frac{1}{\sqrt{1+r^2}} - \frac{L^2}{2r^2}.$$

As shown before, for $L > 0$, this function has limits $\lim_0 \psi_{\text{eff}} = -\infty$ and $\lim_{+\infty}\psi_{\text{eff}} = 0$, is increasing until a global maximum before decreasing towards 0. Raising the value of L lowers this maximum value, meaning that there exists a value $L_{\text{c}}(E)$ such that $\psi_{\text{eff}}(r;L_{\text{c}}(E)) = E$. Then, the forbidden angular momenta (for a given E) are the $L > L_{\text{c}}(E)$. Due to the discussion in the previous section, this couple $(E, L_{\text{c}}(E))$ determines a circular orbit.

There are a priori no analytical formula composed only of basic operations and radical for $L_{\text{c}}(E)$. Indeed, we noted that an orbit with (E,L) was circular if the discriminant $\Delta = 18\alpha\beta\gamma\delta - 4\beta^3\delta + \beta^2\gamma^2 - 4\alpha\gamma^3 - 27\alpha^2\delta^2$, where $\alpha = 4E^2; \beta = 4(E^2 - 1 + EL^2); \gamma = 4EL^2 + L^4; \delta = L^4$, was zero. This is a degree-6 polynomial equation in the variable L^2, which has no such formula for its solutions (Abel, 1826). However, we may approximate it. For simplicity's sake, look for $E_c(L)$ at a given L. It is given by $E_c(L) = \max_{r>0}\psi_{\text{eff}}(r;L) = \psi_{\text{eff}}(r^L_*;L)$. To approximate this r^L_*, we may look for it using Newton's method applied to ψ'_{eff}, since $\psi'_{\text{eff}}(r^L_*;L) = 0$. Start at $r^L_0 = L^{2/3}$, where the evaluation yields

$$\psi'_{\text{eff}}(L^{2/3};L) = 1 - \frac{L^{2/3}}{(1+L^{4/3})^{3/2}} \in [1 - \sqrt{4/27}, 1] \simeq [0.615, 1]$$

in order not to be too far away from $\psi'_{\rm eff}(r^L_*; L) = 0$, and apply the recursion

$$r^L_{n+1} = r^L_n - \frac{\psi'_{\rm eff}(r_n; L)}{\psi''_{\rm eff}(r_n; L)},$$

where

$$\psi'_{\rm eff}(r; L) = -\frac{r}{(1+r^2)^{3/2}} + \frac{L^2}{r^3}; \quad \psi''_{\rm eff}(r_n; L) = -\frac{(1+r^2)^{3/2} - 3r^2\sqrt{1+r^2}}{(1+r^2)^3} - 3\frac{L^2}{r^4}.$$

Then $r^L_n \to r^L_*$. We can show that (r^L_n) is increasing since $\psi'_{\rm eff}(r^L_n; L) > 0$ and $\psi''_{\rm eff}(r^L_n; L) < 0$ (and convexity of $\psi'_{\rm eff}$ where it matters). Therefore a good stopping condition is to get the lowest N such that $\psi'_{\rm eff}(r^L_N + \epsilon) < 0$ for some precision $\epsilon > 0$. Then, taking $\tilde{r}^L_* = (r^L_N + r^L_N + \epsilon)/2 = r^L_N + \epsilon/2$ we will have $E_{\rm c}(L) \simeq \psi_{\rm eff}(\tilde{r}^L_*; L)$, with precision

$$\delta E_{\rm c}(L) \simeq |\psi_{\rm eff}(r^L_*; L) - \psi_{\rm eff}(\tilde{r}^L_*; L)| \simeq \frac{1}{2}|\underbrace{\psi^{(2)}_{\rm eff}(r^L_*)}_{<0}| \cdot |r^L_* - \tilde{r}^L_*|^2 \simeq |\psi^{(2)}_{\rm eff}(r^L_*)|\frac{\epsilon^2}{8}$$

1.2.3 Study of the radiux acceleration $\ddot{r}$ for non-circular orbits

A condition to the periodicity of $r(t)$ is that $\ddot{r}(t)$ should be strictly negative when reaching $r_{\max}$ and strictly positive when reaching $r_{\min}$, so that for small $\epsilon > 0$ (time just after $t_{\max}$ and $t_{\min}$)

$$r(t_{\max} + \epsilon) = r(t_{\max}) + \dot{r}(t_{\max})\epsilon + \ddot{r}(t_{\max})\frac{\epsilon^2}{2} + o(\epsilon^2) = r_{\max} + \ddot{r}_{\max}\frac{\epsilon^2}{2} + o(\epsilon^2) < r_{\max},$$

$$r(t_{\min} + \epsilon) = r(t_{\min}) + \dot{r}(t_{\min})\epsilon + \ddot{r}(t_{\min})\frac{\epsilon^2}{2} + o(\epsilon^2) = r_{\min} + \ddot{r}_{\min}\frac{\epsilon^2}{2} + o(\epsilon^2) > r_{\min}.$$

Newton's law asserts that $a_r = \ddot{r} - r\dot{\theta}^2 = \psi'(r)$, where $L = r^2\dot{\theta}$, hence $\ddot{r} = \psi'(r) + \frac{L^2}{r^3} = \psi'_{\rm eff}(r; L)$. In the case of a non-circular orbits, $r_{\max}$ and $r_{\min}$ don't correspond to maxima of $\psi_{\rm eff}(r; L)$, therefore $\psi'_{\rm eff}(r; L)$ doesn't vanish at those points. Furthermore, we showed that $\psi_{\rm eff}(r; L)$ was strictly increasing until its maximum, then was strictly decreasing. This proves that $\ddot{r}_{\min} > 0$ and that $\ddot{r}_{\max} < 0$.

1.3 Diffusion equation and change of variables

All in all, those coefficients appear in the Fokker-Planck diffusion equation

$$\frac{\partial P}{\partial t}(E, L, t) = -\frac{\partial}{\partial E}\left[\bar{D}_E P\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\bar{D}_{EE} P\right] - \frac{\partial}{\partial L}\left[\bar{D}_L P\right] + \frac{1}{2}\frac{\partial^2}{\partial L^2}\left[\bar{D}_{LL} P\right] + \frac{1}{2}\frac{\partial^2}{\partial E \partial L}\left[\bar{D}_{EL} P\right]$$

If we want to change coordinates $\boldsymbol{x} \to \boldsymbol{x'}(\boldsymbol{x})$, we may use the formulae (C.52) and (C.53) p.25 from Bar-Or & Alexander (2016)

$$D'_l = \frac{\partial x'_l}{\partial x_k} D_k + \frac{1}{2}\frac{\partial^2 x'_l}{\partial x_r \partial x_k} D_{rk},$$

(error on the sign? should be

$$D'_l = \frac{\partial x'_l}{\partial x_k} D_k - \frac{1}{2}\frac{\partial^2 x'_l}{\partial x_r \partial x_k} D_{rk} \quad ?)$$

$$D'_{lm} = \frac{\partial x'_l}{\partial x_r}\frac{\partial x'_m}{\partial x_k} D_{rk}.$$

We might be interested in the change of variable $(E, L) \to (E, \ell)$ where $\ell(E, L) = L/L_{\rm c}(E)$. We have

(E,L)	(E,ℓ)
$(0,0)$	$(0,0)$
$(E,0)$	$(E,0)$
$(1-\epsilon,\alpha)$	$(1-\epsilon,\alpha/L_\mathrm{c}(1-\epsilon))$
$(E,L_\mathrm{c}(E))$	$(E,1)$
$(0,L>0)$	$(0,1)$
$(0,+\infty)$	$(0,1)$

For $(E,L)=(1-\epsilon,\alpha)$ with $\epsilon,\alpha>0$ small, letting $\alpha\to 0$ more quickly than $\epsilon\to 0$ corresponds to taking an arbitry limit in the authorized (E,L)-space towards $(E,L)=(1,0)$. Then $L_\mathrm{c}(1-\epsilon)\sim |L_\mathrm{c}(1)|\epsilon$ and $(E,\ell)\to(1,0)$, meaning that we have

(E,L)	(E,ℓ)
$(0,0)$	$(0,0)$
$(E,0)$	$(E,0)$
$(1,0)$	$(1,0)$
$(E,L_\mathrm{c}(E))$	$(E,1)$
$(0,L)$	$(0,\ell)$
$(0,+\infty)$	$(0,1)$

We have thus transformed the (E,L)-space to a square (E,ℓ)-space, were:

- the side $\ell=0$ is a degenerated bound orbit (straight line through the center with $r_\mathrm{max}=\sqrt{1/E^2-1}$. Because of the definition, its period is $4\int_0^{r_\mathrm{max}}\mathrm{d}r/\sqrt{2(\psi_\mathrm{eff}(r)-E}$ ($r_\mathrm{max}\to 0\to r_\mathrm{max}\to 0\to r_\mathrm{max}$ instead of $r_\mathrm{max}\to r_\mathrm{min}\to r_\mathrm{max}$).
- the side $\ell=1$ is the limit of circular orbits.
- the side $E=0$ is the limit of unbounded orbits.
- the side $E=1$ is the limit of forbidden parameters.

Then we obtain the transformed Fokker-Planck equation

$$\frac{\partial P}{\partial t}(E,\ell,t)=-\frac{\partial}{\partial E}\left[\bar{D}_E P\right]+\frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\bar{D}_{EE}P\right]-\frac{\partial}{\partial \ell}\left[\bar{D}_\ell P\right]+\frac{1}{2}\frac{\partial^2}{\partial \ell^2}\left[\bar{D}_{\ell\ell}P\right]+\frac{1}{2}\frac{\partial^2}{\partial E\partial \ell}\left[\bar{D}_{E\ell}P\right]$$

1.4 Useful Julia packages

- HCuba, with its function cuhre((x,f)->f[1] = integrand, 3, 1), to integrable multidimensional integrals (here 3D) over the unit hypercube. May be useful to compute the local diffusion coefficients since all integration bounds are finite. The cuhre() function is deterministic, fast and globally adaptive.

2 Constant anisotropy parameter

Consider an anisotropy parameter $\beta(r)=1-\frac{\sigma_\theta^2(r)+\sigma_\phi^2(r)}{2\sigma_r^2(r)}$. The different behavior are:

- $\beta>0$: radially based.
- $\beta=0$: isotropic.
- $\beta<0$: tangentially biased.

Consider a simple toy model in which the anisotropy parameter takes some fixed non-zero value β for all radii. This can be generated by the DF of the form (Binney & Tremaine, eq. 4.62)

$$f(E,L) = L^{-2\beta} f_1(E)$$

where $f_1(E)$ is any arbitration non-negative function. Consider the Jaffe, Hernquist and NFW models. Eq. (2.66) gives

$$M(<r) = \begin{cases} \frac{r/a}{1+r/a} & \text{Jaffe} \\ \frac{(r/a)^2}{2(1+r/a)^2} & \text{Hernquist} \\ \ln(1+\frac{r}{a}) - \frac{r/a}{1+r/a} & \text{NFW} \end{cases}$$

which yields (eq. (2.67))

$$\Phi(r) = -4\pi G\rho_0 a^2 \times \begin{cases} \ln(1+a/r) & \text{Jaffe} \\ \frac{1}{2(1+r/a)} & \text{Hernquist} \\ \frac{\ln(1+r/a)}{r/a} & \text{NFW} \end{cases}$$

For Hernquist, eq. (4.50) and (4.51) give, for $\tilde{E} = -Ea/(GM)$

$$f_H(E) = \frac{1}{\sqrt{2}(2\pi)^3(GMa)^{3/2}} \frac{\sqrt{\tilde{E}}}{(1-\tilde{E})^2}$$
$$\times \left[(1-2\tilde{E})(8\tilde{E}^2 - 8\tilde{E} - 3) + \frac{3\sin^{-1}\sqrt{\tilde{E}}}{\sqrt{\tilde{E}(1-\tilde{E})}}\right]$$

For Jaffe, eq. (4.52) and (4.53) give

$$F_J(E) = \frac{1}{(2\pi)^3(GMa)^{3/2}} \left[F_-(\sqrt{2\tilde{E}}) - \sqrt{2}F_-(\sqrt{\tilde{E}})\right.$$
$$\left. -\sqrt{2}F_+(\sqrt{\tilde{E}}) + F_+(\sqrt{2\tilde{E}})\right]$$

with $F_\pm(z)$ the Dawson's integral

$$F_-(z) = \frac{1}{2}\sqrt{\pi}e^{z^2}\operatorname{erf}(z)$$
$$F_+(z) = -\frac{1}{2}\sqrt{\pi}ie^{-z^2}\operatorname{erf}(iz)$$

The main remark is that

$$f(\tilde{E},L) = f_1(\tilde{E})g(L)$$
$$\tilde{E} = \psi(r) - \frac{1}{2}v^2$$
$$L = rv_t$$