Notes

Kerwann

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1 Plummer model

Consider a Plummer model (Dejonghe, H.1987, MNRAS 224, 13) with potential with units $r_{\rm s}$ the Plummer scale radius (which sets the size of the cluster core), M the total mass of the cluster and $\bar{\tau}$ some unit time. Let $\psi_{\rm s}$ be defined by

$$\psi_{\rm s} = \frac{GM}{r_{\rm s}}, \tag{1}$$

for the central potential

$$\psi(r) = \frac{\psi_{\rm s}}{\sqrt{1+r^2}}. \tag{2}$$

Let use fix $G = 1\, r_{\rm s}^3.M^{-1}.\bar{\tau}^{-2}$ in the new units so that $\psi_{\rm s} = 1\, r_{\rm s}^2 \cdot \bar{\tau}^{-2}$. This fixes the time unit $\bar{\tau}$, as we have the relation. Therefore, in those units the potential (per unit mass) is given by

$$\psi(r) = \frac{1}{\sqrt{1+r^2}}. \tag{3}$$

Define, given a radius r, the angular momentum $L(r, v_{\rm r}, v_{\rm t})$ and binding energy per unit mass $E(r, v_{\rm r}, v_{\rm t})$, functions of the radial velocity $v_{\rm r}$ and the tangential velocity $v_{\rm t} \geq 0$ (defined as $\boldsymbol{v} = v_{\rm r}\hat{\boldsymbol{r}} + \boldsymbol{v_{\rm t}}$), as

$$\begin{array}{rcl} E(r, v_{\rm r}, v_{\rm t}) & = & \psi(r) - \frac{1}{2}v_{\rm r}^2 - \frac{1}{2}v_{\rm t}^2, \\ L(r, v_{\rm r}, v_{\rm t}) & = & r \cdot v_{\rm t}, \end{array} \tag{4}$$

whose Jacobian is

$$\mathrm{Jac}_{(r,v_{\rm r},v_{\rm t})\to(r,E,L)} = \begin{pmatrix} \dfrac{\partial E}{\partial v_{\rm r}} & \dfrac{\partial E}{\partial v_{\rm t}} \\ \dfrac{\partial L}{\partial v_{\rm r}} & \dfrac{\partial L}{\partial v_{\rm t}} \end{pmatrix} = \begin{pmatrix} -v_{\rm r} & -v_{\rm t} \\ 0 & r \end{pmatrix} \Rightarrow |\mathrm{Jac}| = r|v_{\rm r}|. \tag{5}$$

To obtain a bijective transformation, we must chose wether to chose $v_{\rm r} \geq 0$ or $v_{\rm r} \leq 0$. A priori, this choice might have an impact on the result, but we will should that the local and orbit-averaged diffusion coefficients are not that. The coordinate system is spherical, its origin being at the center of the globular cluster. Finally, we consider the corresponding anisotropic distribution functions of the field stars $F_q(r, E, L) = F_q(E, L)$ in (E, L)-space. Since (E, L) and $(v_{\rm r}, v_{\rm t})$ are linked, we can make use of the following equalities (for the moment, $v_{\rm r}$ is defined modulo the sign)

$$\begin{array}{rcl} F_q(E, L) & = & f_q(r, v_{\rm r}(r, E, L), v_{\rm t}(r, E, L)), \\ f_q(r, v_{\rm r}, v_{\rm t}) & = & F_q(E(r, v_{\rm r}, v_{\rm t}), L(r, v_{\rm r}, v_{\rm t})), \end{array} \tag{6}$$

where f_q is the distribution fonction (DF)in the $(v_{\rm r}, v_{\rm t})$ space and where q is an anisotropy parameter:

- $q \in]0, 2]$: radially anisotropic
- $q = 0$: isotropic

- $q \in]-\infty, 0[$: tangentially anisotropic.

Note that the DF has spherical symmetry in position. Its expression for $E \geq 0, L \geq 0$ is (for $q \neq 0$):

$$F_q(E,L) = \frac{3\Gamma(6-q)}{2(2\pi)^{5/2}\Gamma(q/2)} E^{7/2-q} \mathbb{H}(0, \frac{q}{2}, \frac{9}{2} - q, 1; \frac{L^2}{2E}) \tag{7}$$

where

$$\mathbb{H}(a,b,c,d;x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(c-a)\Gamma(a+d)} x^a \, {}_2F_1(a+b, 1+a-c, a+d; x) & \text{if} \quad x \leq 1, \\ \frac{\Gamma(a+b)}{\Gamma(d-b)\Gamma(b+c)} x^{-b} \, {}_2F_1(a+b, 1+b-d, b+c; \frac{1}{x}) & \text{if} \quad x \geq 1, \end{cases} \tag{8}$$

which reduces in the isotropic case ($q = 0$) to

$$F_0(E) = \frac{3}{7\pi^3}(2E)^{7/2}, \tag{9}$$

and in the extreme radially anisotropic ($q = 2$) to

$$F_2(E) = \begin{cases} \frac{6}{(2\pi)^3}(2E - L)^{3/2} & \text{if} \quad 2E \leq L^2, \\ 0 & \text{if} \quad 2E \geq L^2. \end{cases} \tag{10}$$

When $E \leq 0$ or $L \leq 0$ then $F_q(E, L) = 0$.

2 Determination of the local diffusion coefficients

The local diffusion coefficients are the average velocity changes per unit time. We are interested in computing

$$\begin{aligned} \langle \Delta v_{||} \rangle (r, v_\mathrm{r}, v_\mathrm{t}) &= \frac{\langle \Delta v_{||} \rangle_{\delta t}(r, v_\mathrm{r}, v_\mathrm{t})}{\delta t}, \\ \langle (\Delta v_{||})^2 \rangle (r, v_\mathrm{r}, v_\mathrm{t}) &= \frac{\langle (\Delta v_{||})^2 \rangle_{\delta t}(r, v_\mathrm{r}, v_\mathrm{t})}{\delta t}, \\ \langle (\Delta v_\perp)^2 \rangle (r, v_\mathrm{r}, v_\mathrm{t}) &= \frac{\langle (\Delta v_\perp)^2 \rangle_{\delta t}(r, v_\mathrm{r}, v_\mathrm{t})}{\delta t}, \end{aligned} \tag{11}$$

where the subscript are relative to the relative velocity of test star (in the referential where the deflecting field star is still). Consider a test star at position r, mass m and initial velocity $\boldsymbol{v}$ which interacts with a field star with impact parameter b, mass m_a and velocity $\boldsymbol{v_\mathrm{a}}$, Binney et Tremaine (2008, eq. (L.7) page 834) gives , with the convention (here, parallel and perpendicular to relative velocity)

$$\Delta \boldsymbol{v} = -\Delta v_{||} \boldsymbol{e_1'} + \Delta v_\perp (-\boldsymbol{e_2'} \cos\phi + \boldsymbol{e_3'} \sin\phi), \tag{12}$$

where $\boldsymbol{e_1'} \parallel \boldsymbol{V_0}$ and ϕ is the angle between the plane of the relative orbit and $\boldsymbol{e_2'}$,

$$\begin{aligned} \Delta v_\perp &= \frac{2m_a V_0}{m + m_a} \frac{b/b_{90}}{1 + b^2/b_{90}^2}, \\ \Delta v_{||} &= \frac{2m_a V_0}{m + m_a} \frac{1}{1 + b^2/b_{90}^2}, \end{aligned} \tag{13}$$

where $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_\mathrm{a}}$ and b_{90} is the 90° deflection radius, given by eq (L.8)

$$b_{90} = \frac{G(m + m_a)}{V_0^2}. \tag{14}$$

Furthermore, after averaging over the equiprobable angles ϕ (test star can be on either "side" of the field star), we obtain

$$
\begin{aligned}
\langle \Delta v_i \rangle_\phi &= -\Delta v_{||} \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle, \\
\langle \Delta v_i \Delta v_j \rangle_\phi &= (\Delta v_{||})^2 \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_1} \rangle \\
&\quad + \tfrac{1}{2} (\Delta v_\perp)^2 \left[\langle \boldsymbol{e_i}, \boldsymbol{e'_2} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_2} \rangle + \langle \boldsymbol{e_i}, \boldsymbol{e'_3} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_3} \rangle \right]
\end{aligned}
\tag{15}
$$

where (e_1, e_2, e_3) is an fixed, arbitrary coordonnate system. Here, note that when considering a test star with energy and angular momentum (per unit mass) (E, L), using the choise $v_r \geq 0$ or the choice $v_r \leq 0$ has an impact on the local change of velocity through V_0.

We sum the effects of all the encounter up. Number density of field stars (at position r) within velocity space volume $\mathrm{d}^3\boldsymbol{v_a}$ is $f_q(r, \boldsymbol{v_\mathrm{a}})\mathrm{d}^3\boldsymbol{v_\mathrm{a}}$ (remember that $f_q(r, \boldsymbol{v_\mathrm{a}}) = f_q(r, v_\mathrm{ar}, v_\mathrm{at})$). The number of encounters in a time δt with impact parameters between b and $b + \mathrm{d}b$ is just this density times the volume of an annulus with inner radius b, outer radius $b + \mathrm{d}b$, and length $V_0 \delta t$, that is (eq. L9) $2\pi b \mathrm{d}b V_0 \delta t f_a(r, \boldsymbol{v_a})\mathrm{d}^3\boldsymbol{v_a}$.

We sum up over the velocities and the impact parameters. For the latter, we consider impact parameters between 0 and a cut-off b_max, traditionally given approximately by the radius of the subject star orbit.

Recall that $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_\mathrm{a}}$. Since we assume that Λ is large, we do not make any significant additional error by replacing the factor V_0 in Λ by some typical stellar speed v_typ, that is,

$$
\Lambda = \frac{b_\mathrm{max} v_\mathrm{typ}^2}{G(m + m_a)}. \tag{16}
$$

This yields (Binney & Tremaine, eq. L14)

$$
\begin{aligned}
\langle \Delta v_i \rangle &= -4\pi \frac{m_\mathrm{a}}{m + m_\mathrm{a}} \int \mathrm{d}^3\boldsymbol{v_\mathrm{a}} V_0^2 b_{90}^2 f_q(r, \boldsymbol{v_\mathrm{a}}) \ln \Lambda \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle, \\
\langle \Delta v_i \Delta v_j \rangle &= 4\pi \left(\frac{m_\mathrm{a}}{m + m_\mathrm{a}} \right)^2 \int \mathrm{d}^3\boldsymbol{v_\mathrm{a}} V_0^3 b_{90}^2 f_a(r, \boldsymbol{v_\mathrm{a}}) \ln \Lambda \left[\langle \boldsymbol{e_i}, \boldsymbol{e'_2} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_2} \rangle + \langle \boldsymbol{e_i}, \boldsymbol{e'_3} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_3} \rangle \right]
\end{aligned}
\tag{17}
$$

where we defined the Coulomb parameter $\Lambda = b_\mathrm{max}/b_{90}$. Remark that the scalar products depend on $\boldsymbol{v_a}$. Take $\Lambda = \lambda N$ (Binney et Tremaine, page 581) with $N \sim 10^5$ and $\lambda = 0.059$ (Hamilton et al. (2018), eq. (B37)) for a globular cluster.

Using (Binney & Tremaine, eq. L17 and L18), we obtain

$$
\begin{aligned}
\langle \Delta v_i \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_\mathrm{a} (m + m_\mathrm{a}) \ln \Lambda \frac{\partial h}{\partial v_i}(r, \boldsymbol{v}), \\
\langle \Delta v_i \Delta v_j \rangle (r, \boldsymbol{v}) &= 4\pi G^2 {m_\mathrm{a}}^2 \ln \Lambda \frac{\partial^2 g}{\partial v_i \partial v_j}(r, \boldsymbol{v})
\end{aligned}
\tag{18}
$$

where the Rosenbluth potentials are defined as (Binney & Tremaine, eq. L19)

$$
\begin{aligned}
h(r, \boldsymbol{v}) &= \int \mathrm{d}^3\boldsymbol{v_\mathrm{a}} \frac{f_q(r, \boldsymbol{v_\mathrm{a}})}{|\boldsymbol{v} - \boldsymbol{v_\mathrm{a}}|}, \\
g(r, \boldsymbol{v}) &= \int \mathrm{d}^3\boldsymbol{v_\mathrm{a}} f_q(r, \boldsymbol{v_\mathrm{a}}) |\boldsymbol{v} - \boldsymbol{v_\mathrm{a}}|
\end{aligned}
\tag{19}
$$

2.1 Anisotropic case

Since this result is valid for any arbitrary coordinate system, we can fix it to the one where $\boldsymbol{e_1} = \hat{v}$ and $\boldsymbol{e_2}$ is the projection of $\hat{\boldsymbol{r}}$ onto the equatorial plane orthogonal to $\boldsymbol{e_1}$. Then we'll have the relations

$$
\begin{aligned}
\langle \Delta v_{||} \rangle (r, \boldsymbol{v}) &= \langle \Delta v_1 \rangle (r, \boldsymbol{v}), \\
\langle (\Delta v_{||})^2 \rangle (r, \boldsymbol{v}) &= \langle (\Delta v_1)^2 \rangle (r, \boldsymbol{v}) \\
\langle (\Delta v_\perp)^2 \rangle (r, \boldsymbol{v}) &= \langle (\Delta v_2)^2 \rangle (r, \boldsymbol{v}) + \langle (\Delta v_3)^2 \rangle (r, \boldsymbol{v})
\end{aligned}
\tag{20}
$$

where the subscripts are relative of the velocity of the test star.

and a tedious by straightforward computation see appendix) yields

$$\begin{aligned}
\langle \Delta v_{||} \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_{\mathrm{a}} (m + m_{\mathrm{a}}) \ln \Lambda (\frac{v_{\mathrm{r}}}{v} \frac{\partial h}{\partial v_{\mathrm{r}}} + \frac{v_{\mathrm{t}}}{v} \frac{\partial h}{\partial v_{\mathrm{t}}}), \\
\langle (\Delta v_{||})^2 \rangle (r, \boldsymbol{v}) &= 4\pi G^2 {m_{\mathrm{a}}}^2 \ln \Lambda \left(\frac{v_{\mathrm{r}}^2}{v^2} \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} + \frac{2 v_{\mathrm{r}} v_{\mathrm{t}}}{v^2} \frac{\partial^2 g}{\partial v_{\mathrm{t}} \partial v_{\mathrm{r}}} + \left(\frac{v_{\mathrm{t}}}{v} \right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} \right) \\
\langle (\Delta v_{\perp})^2 \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_{\mathrm{a}}^2 \ln \Lambda \left(\left(\frac{v_{\mathrm{t}}}{v} \right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} - \frac{2 v_{\mathrm{r}} v_{\mathrm{t}}}{v^2} \frac{\partial^2 g}{\partial v_{\mathrm{t}}\, partial v_{\mathrm{r}}} + \left(\frac{v_{\mathrm{r}}}{v} \right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} + \frac{1}{v_{\mathrm{t}}} \frac{\partial g}{\partial v_{\mathrm{t}}} \right)
\end{aligned} \tag{21}$$

where $h(r, \boldsymbol{v_a}) = h(r, v_{\mathrm{r}}, v_{\mathrm{t}})$ and $g(r, \boldsymbol{v}) = g(r, v_{\mathrm{r}}, v_{\mathrm{t}})$.

Applying the change of variable $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_a}$ and using spherical coordinates with axis $(Oz) = \hat{\boldsymbol{r}}$ the unit radius vector (parallel or antiparallel to the radial component of $\boldsymbol{v}$ by definition) yields

$$\begin{aligned}
h(r, v_{\mathrm{r}}, v_{\mathrm{t}}) &= \int \mathrm{d}^3 \boldsymbol{V_0} \frac{f_q(r, \boldsymbol{v} - \boldsymbol{V_0})}{V_0} &= \int_0^\infty \mathrm{d}V_0 V_0 \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_q(r, \boldsymbol{v} - \boldsymbol{V_0}), \\
g(r, v_{\mathrm{r}}, v_{\mathrm{t}}) &= \int \mathrm{d}^3 \boldsymbol{V_0} f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) V_0 &= \int_0^\infty \mathrm{d}V_0 V_0^3 \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f_q(r, \boldsymbol{v} - \boldsymbol{V_0})
\end{aligned} \tag{22}$$

where

$$f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) = f_q(r, v_{\mathrm{ar}}, v_{\mathrm{at}}) = F_q(E(r, v_{\mathrm{ar}}, v_{\mathrm{at}}), L(r, v_{\mathrm{ar}}, v_{\mathrm{at}})) \tag{23}$$

with E, L given by eq (4).

For a given convention $+$ or $-$ of the choice of v_{r}, and given (E, L) the parameters of the test star, obtain the vectors $\boldsymbol{v_+} = (|v_{\mathrm{r}}|, \boldsymbol{v_t})$ and $\boldsymbol{v_-} = (-|v_{\mathrm{r}}|, \boldsymbol{v_t})$, which are symmetric with respect to the tangent plane where $\boldsymbol{v_t}$ lives. In terms of spherical coordinates, we have that $\boldsymbol{v_+} = (v, \theta_0, 0)$ and $\boldsymbol{v_-} = (v, \pi - \theta_0, 0)$. Remember that the integration over the velocities $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_a}$ of the field stars cover the whole $\boldsymbol{V_0}$-space. Given a velocity $\boldsymbol{V_0}$ corresponds bijectively a field star velocity $\boldsymbol{v_a}$. The overall integration will in fact not depend on the convention we used. The $E(r, v_{\mathrm{ar}}, v_{\mathrm{at}})$ component depends on the sign of v_{r} since

$$E_{\mathrm{a}}(r, V_0, \theta, \phi) = \psi(r) - \frac{1}{2} \left[v^2 + V_0^2 - 2V_0 (v_{\mathrm{r}} \cos\theta + v_{\mathrm{t}} \sin\theta \cos\phi) \right] \tag{24}$$

but $L_{\mathrm{a}}(r, v_{\mathrm{ar}}, v_{\mathrm{at}})$ does not. When doing the integration, we will evaluate the integrand at both arguments (V_0, θ, ϕ) and $(V_0, \pi - \theta, \phi)$, and their summed contribution doesn't depend on the convention choice. In the following, we decide to use $v_{\mathrm{r}} \geq 0$.

For an actual computation, we also need to compute the various velocity-partial derivatives of those integrals, meaning that we need to compute the velocity-partial derivatives of $f_q(r, \boldsymbol{v_a}) = F_q(E_{\mathrm{a}}, L_{\mathrm{a}})$ (exchange derivation and integral). The calculation is done in the appendix, and the results are (function are evaluated at $(E_{\mathrm{a}}, L_{\mathrm{a}})$)

$$\begin{aligned}
\frac{\partial}{\partial v_{\mathrm{r}}} \left[f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_{\mathrm{r}} + V_0 \cos\theta) \frac{\partial F}{\partial E}, \\
\frac{\partial}{\partial v_{\mathrm{t}}} \left[f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_{\mathrm{t}} + V_0 \sin\theta \cos\phi) \left(\frac{\partial F}{\partial E} - \frac{r}{L_{\mathrm{a}}} \frac{\partial F}{\partial L} \right), \\
\frac{\partial^2}{\partial v_{\mathrm{r}}^2} \left[f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= -\frac{\partial F}{\partial E} + (-v_{\mathrm{r}} + V_0 \cos\theta)^2 \frac{\partial^2 F}{\partial E^2}, \\
\frac{\partial^2}{\partial v_{\mathrm{t}} \partial v_{\mathrm{r}}} \left[f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_{\mathrm{r}} + V_0 \cos\theta)(-v_{\mathrm{t}} + V_0 \sin\theta \cos\phi) \left(\frac{\partial^2 F}{\partial E^2} - \frac{r}{L_{\mathrm{a}}} \frac{\partial^2 F}{\partial L \partial E} \right), \\
\frac{\partial^2}{\partial v_{\mathrm{t}}^2} \left[f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= -\frac{\partial F}{\partial E} + \frac{r}{L_{\mathrm{a}}} \frac{\partial F}{\partial L} + (-v_{\mathrm{t}} + V_0 \sin\theta \cos\phi)^2 \\
&\quad \times \left(\frac{\partial^2 F}{\partial E^2} - \frac{2r}{L_{\mathrm{a}}} \frac{\partial^2 F}{\partial L \partial E} - \frac{r^2}{L_{\mathrm{a}}^3} \frac{\partial F}{\partial L} + \frac{r^2}{L_{\mathrm{a}}^2} \frac{\partial^2 F}{\partial L^2} \right).
\end{aligned} \tag{25}$$

The DF and its derivative vanish when $E_{\rm a} < 0$. Obviously, $v_{\rm a}(r, V_0, \theta, \phi)$ is minored by the polynomial in V_0 given by $v^2 + V_0^2 - 2V_0(v_{\rm r} + v_{\rm t})$. We have $E_{\rm a} < 0$ when $v_{\rm a} > \psi(r)$, which happens outside of the roots of $v^2 + V_0^2 - 2V_0(v_{\rm r} + v_{\rm t}) - 2\psi(r)$. Those roots are

$$V_{0\pm} = (v_{\rm t} + v_{\rm r}) \pm \sqrt{2(v_{\rm r}v_{\rm t} + \psi(r))}. \tag{26}$$

For $E > 0$, the inferior root is always negative whereas the superior root is always positive. Let's call it $V_{\rm max}$. In the end, the Rosenbluth potentials can be computed over compact domains

$$\begin{aligned} h(r, v_{\rm r}, v_{\rm t}) &= \int_0^{V_{\rm max}} {\rm d}V_0 V_0 \int_0^{\pi} {\rm d}\theta \sin\theta \int_0^{2\pi} {\rm d}\phi f_q(r, \boldsymbol{v} - \boldsymbol{V_0}), \\ g(r, v_{\rm r}, v_{\rm t}) &= \int_0^{V_{\rm max}} {\rm d}V_0 V_0^3 \int_0^{\pi} {\rm d}\theta \sin\theta \int_0^{2\pi} {\rm d}\phi f_q(r, \boldsymbol{v} - \boldsymbol{V_0}) \end{aligned} \tag{27}$$

and so do its partial derivatives.

2.2 Isotropic case

We may want to check that the integrals yield the correct result. To that end, it can be of interest to consider the simple case $q = 0$, where $f(E, L) = f(E)$, i.e. $f(r, \boldsymbol{v}) = f(r, v) = F_q(E)$. Then according Binney & Tremaine, eq. (L26),

$$\begin{aligned} \langle \Delta v_{||} \rangle(r, v) &= -\frac{16\pi^2 G^2 m_{\rm a}(m + m_{\rm a}) \ln\Lambda}{v^2} K_1(r, v), \\ \langle (\Delta v_{||})^2 \rangle(r, v) &= \frac{32\pi^2 G^2 m_{\rm a}^2 \ln\Lambda}{3} \left(K_0(r, v) + \frac{1}{v^3} K_3(r, v) \right) \\ \langle (\Delta v_{\perp})^2 \rangle(r, v) &= \frac{32\pi^2 G^2 m_{\rm a}^2 \ln\Lambda}{3} \left(2K_0(r, v) + \frac{3}{v} K_1(r, v) - \frac{1}{v^3} K_3(r, v) \right) \end{aligned} \tag{28}$$

where

$$\begin{aligned} K_0(r, v) &= \frac{1}{21\pi^3}(2E)^{9/2}, \\ K_1(r, v) &= \int_E^{\psi(r)} {\rm d}E_a v_a F_0(E_a) \\ K_3(r, v) &= \int_E^{\psi(r)} {\rm d}E_a v_a^3 F_0(E_a) \end{aligned} \tag{29}$$

and the correspondance $E = \psi(r) - v^2/2$

In the appendix, we recompute the formulae of the isotropic case from the arbitrary anisotropic case, with $v^2 = v_{\rm r}^2 + v_{\rm t}^2$, $h(r, v_{\rm r}, v_{\rm t}) = h(r, v)$ and $g(r, v_{\rm r}, v_{\rm t}) = g(r, v)$.

3 Local orbital parameter changes

Now, switch to (E, L) space and using eq. (C15) to (C19) of Bar-Or & Alexander (2016), which doesn't rely on an isotropy assumption, we obtain (evaluate at $(r, \boldsymbol{v}(r, E, L))$) at first order in $\Delta v/v$

$$\begin{aligned} \langle \Delta E \rangle(r, E, L) &= -\frac{1}{2}\langle (\Delta v_{||})^2 \rangle - \frac{1}{2}\langle (\Delta v_{\perp})^2 \rangle - v\langle \Delta v_{||} \rangle, \\ \langle (\Delta E)^2 \rangle(r, E, L) &= v^2 \langle (\Delta v_{||})^2 \rangle \\ \langle \Delta L \rangle(r, E, L) &= \frac{L}{v}\langle \Delta v_{||} \rangle + \frac{r^2}{4L}\langle (\Delta v_{\perp})^2 \rangle, \\ \langle (\Delta L)^2 \rangle(r, E, L) &= \frac{L^2}{v^2}\langle \Delta v_{||} \rangle + \frac{1}{2}\left(r^2 - \frac{L^2}{v^2} \right) \langle (\Delta v_{\perp})^2 \rangle \\ \langle \Delta E \Delta L \rangle(r, E, L) &= -L\langle (\Delta v_{||})^2 \rangle \end{aligned} \tag{30}$$

Due to our analysis, those quantities are well defined and we can use the bijective transformation $(r, E, L) \leftrightarrow (r, v_{\rm r}, v_{\rm t})$

4 Orbit of a test star in a globular cluster

We can now compute the local diffusion coefficients $\langle \Delta E \rangle$, $\langle (\Delta E)^2 \rangle$, $\langle \Delta L \rangle$, $\langle (\Delta L)^2 \rangle$ and $\langle \Delta E \Delta L \rangle$. Since we are interested in the secular evolution of the system, we can average over the dynamical time and smear out the star along its orbit. This leads us to consider the orbit-average diffusion coefficients

$$\begin{aligned} D_X(E,L) &= \langle \Delta X \rangle_{\circlearrowleft} &= \frac{2}{T}\int_{r_{\min}}^{r_{\max}} \langle \Delta X \rangle(r,E,L) \frac{\mathrm{d}r}{v_{\mathrm{r}}(r,E,L)}, \\ D_{XY}(E,L) &= \langle \Delta X \Delta Y \rangle_{\circlearrowleft} &= \frac{2}{T}\int_{r_{\min}}^{r_{\max}} \langle \Delta X \Delta Y \rangle(r,E,L) \frac{\mathrm{d}r}{v_{\mathrm{r}}(r,E,L)}. \end{aligned} \tag{31}$$

where $v_{\mathrm{r}}(r)$ is the radial velocity of the orbiting star at r. It is of interest to define the effective potential

$$\psi_{\mathrm{eff}}(r,L) = \psi(r) - \frac{L^2}{2r^2}. \tag{32}$$

4.1 Study of an orbit

See Kurth (1955), Astronomische Nachrichten, volume 282, Issue 6, p.241.

Consider a test star described by its position vector $\boldsymbol{r}$, its binding energy (opposite of its energy) $E(t)$ and its angular momentum vector $\boldsymbol{L}(t)$, per unit mass. Then by Newton's law, those two quantities are conserved along an orbit, allowing us to drop the t parameter.

Consider a bound orbit with $E \geq 0$ and $L \geq 0$. Then its ascending radial velocity is given by

$$v_{\mathrm{r}}(r) = \sqrt{2(\psi_{\mathrm{eff}}(r;L) - E)}, \tag{33}$$

its bounds $r_{\min}$ and $r_{\max}$ are given by the solution of the equation $v_{\mathrm{r}}(r) = 0$, which has two solutions, and its orbital period T is defined by

$$\frac{T}{2} = \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_{\mathrm{r}}(r)} \tag{34}$$

The "type" of an orbit is determined by E and L. Graphically, $r_{\max}$ and $r_{\min}$ are given by the intersection points of $\psi_{\mathrm{eff}}(r,L)$ and E. There are a few different cases:

- $E \leq 0$: unbounded orbit,
- $E \in]0, E_{\mathrm{c}}(L)[$: bound "rosette-like" orbit,
- $E = E_{\mathrm{c}}(L)$: circular orbit,
- $E > E_{\mathrm{c}}(L)$: impossible,

where $E_{\mathrm{c}}(L) = \max_{r>0} \psi_{\mathrm{eff}}(r,L)$.

All the integral are finite, since they are integrable at the endpoints

$$\begin{aligned} \frac{1}{v_{\mathrm{r}}(r_{\max}-\epsilon)} &\sim |2\psi'_{\mathrm{eff}}(r_{\max})|^{-1/2}\frac{1}{\sqrt{\epsilon}}, \\ \frac{1}{v_{\mathrm{r}}(r_{\min}+\epsilon)} &\sim |2\psi'_{\mathrm{eff}}(r_{\min})|^{-1/2}\frac{1}{\sqrt{\epsilon}}. \end{aligned} \tag{35}$$

with strictly positive prefactor for bounded, non-circular orbits.