Notes

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Contents

1 Determination of the local diffusion coefficients

The local diffusion coefficients are the average velocity changes per unit time. We are interested in computing

$$
\begin{aligned}
\langle \Delta v_{||} \rangle (r, v_{\rm r}, v_{\rm t}) &= \frac{\langle \Delta v_{||} \rangle_{\delta t} (r, v_{\rm r}, v_{\rm t})}{\delta t}, \\
\langle (\Delta v_{||})^2 \rangle (r, v_{\rm r}, v_{\rm t}) &= \frac{\langle (\Delta v_{||})^2 \rangle_{\delta t} (r, v_{\rm r}, v_{\rm t})}{\delta t}, \\
\langle (\Delta v_{\perp})^2 \rangle (r, v_{\rm r}, v_{\rm t}) &= \frac{\langle (\Delta v_{\perp})^2 \rangle_{\delta t} (r, v_{\rm r}, v_{\rm t})}{\delta t},
\end{aligned} \tag{1}
$$

where the subscript are relative to the relative velocity of test star (in the referential where the deflecting field star is still). Consider a test star at position r, mass m and initial velocity $\boldsymbol{v}$ which interacts with a field star with impact parameter b, mass $m_{\rm a}$ and velocity $\boldsymbol{v_{\rm a}}$, Binney et Tremaine (2008, eq. (L.7) page 834) gives , with the convention (here, parallel and perpendicular to relative velocity)

$$
\Delta \boldsymbol{v} = -\Delta v_{||} \boldsymbol{e'_1} + \Delta v_{\perp} (-\boldsymbol{e'_2} \cos\phi + \boldsymbol{e'_3} \sin\phi), \tag{2}
$$

where $\boldsymbol{e'_1} \parallel \boldsymbol{V_0}$ and ϕ is the angle between the plane of the relative orbit and $\boldsymbol{e'_2}$,

$$
\begin{aligned}
\Delta v_{\perp} &= \frac{2 m_{\rm a} V_0}{m + m_{\rm a}} \frac{b/b_{90}}{1 + b^2/b_{90}^2}, \\
\Delta v_{||} &= \frac{2 m_{\rm a} V_0}{m + m_{\rm a}} \frac{1}{1 + b^2/b_{90}^2},
\end{aligned} \tag{3}
$$

where $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_{\rm a}}$ and b_{90} is the 90° deflection radius, given by eq (L.8)

$$
b_{90} = \frac{G(m + m_a)}{V_0^2}. \tag{4}
$$

Furthermore, after averaging over the equiprobable angles ϕ (test star can be on either "side" of the field star), we obtain

$$
\begin{aligned}
\langle \Delta v_i \rangle_\phi &= -\Delta v_{||} \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle, \\
\langle \Delta v_i \Delta v_j \rangle_\phi &= (\Delta v_{||})^2 \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_1} \rangle \\
&\quad + \tfrac{1}{2} (\Delta v_{\perp})^2 \left[\langle \boldsymbol{e_i}, \boldsymbol{e'_2} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_2} \rangle + \langle \boldsymbol{e_i}, \boldsymbol{e'_3} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_3} \rangle \right]
\end{aligned} \tag{5}
$$

where $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ is an fixed, arbitrary coordonnate system. Here, note that when considering a test star with energy and angular momentum (per unit mass) (E, L), using the choise $v_{\rm r} \geq 0$ or the choice $v_{\rm r} \leq 0$ has an impact on the local change of velocity through V_0.

We sum the effects of all the encounter up. Number density of field stars (at position r) within velocity space volume $\mathrm{d}^3\boldsymbol{v_{\rm a}}$ is $f(r, \boldsymbol{v_{\rm a}}) \mathrm{d}^3\boldsymbol{v_{\rm a}}$ (remember that $f(r, \boldsymbol{v_{\rm a}}) = f(r, v_{\rm ar}, v_{\rm at})$). The number of encounters in a time δt with impact parameters between b and $b + \mathrm{d}b$ is just this density times the volume of an annulus with inner radius b, outer radius $b + \mathrm{d}b$, and length $V_0 \delta t$, that is (eq. L9) $2\pi b \mathrm{d}b V_0 \delta t f(r, \boldsymbol{v_{\rm a}}) \mathrm{d}^3\boldsymbol{v_{\rm a}}$.

We sum up over the velocities and the impact parameters. For the latter, we consider impact parameters between 0 and a cut-off $b_{\rm max}$, traditionally given approximately by the radius of the subject star orbit.

Recall that $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_{\rm a}}$. Since we assume that Λ is large, we do not make any significant additional error by replacing the factor V_0 in Λ by some typical stellar speed $v_{\rm typ}$, that is,

$$
\Lambda = \frac{b_{\rm max} v_{\rm typ}^2}{G(m + m_{\rm a})}. \tag{6}
$$

This yields (Binney & Tremaine, eq. L14)

$$
\begin{aligned}
\langle \Delta v_i \rangle &= -4\pi \frac{m_{\rm a}}{m + m_{\rm a}} \int \mathrm{d}^3\boldsymbol{v_{\rm a}} V_0^2 b_{90}^2 f(r, \boldsymbol{v_{\rm a}}) \ln\Lambda \langle \boldsymbol{e_i}, \boldsymbol{e'_1} \rangle, \\
\langle \Delta v_i \Delta v_j \rangle &= 4\pi \left(\frac{m_{\rm a}}{m + m_{\rm a}} \right)^2 \int \mathrm{d}^3\boldsymbol{v_{\rm a}} V_0^3 b_{90}^2 f(r, \boldsymbol{v_{\rm a}}) \ln\Lambda \left[\langle \boldsymbol{e_i}, \boldsymbol{e'_2} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_2} \rangle + \langle \boldsymbol{e_i}, \boldsymbol{e'_3} \rangle \langle \boldsymbol{e_j}, \boldsymbol{e'_3} \rangle \right]
\end{aligned} \tag{7}
$$

where we defined the Coulomb parameter $\Lambda = b_{\max}/b_{90}$. Remark that the scalar products depend on $\boldsymbol{v_a}$. Take $\Lambda = \lambda N$ (Binney et Tremaine, page 581) with $N \sim 10^5$ and $\lambda = 0.059$ (Hamilton et al. (2018), eq. (B37)) for a globular cluster.

Using (Binney & Tremaine, eq. L17 and L18), we obtain

$$\begin{aligned} \langle \Delta v_i \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_{\mathrm{a}} (m + m_{\mathrm{a}}) \ln \Lambda \frac{\partial h}{\partial v_i}(r, \boldsymbol{v}), \\ \langle \Delta v_i \Delta v_j \rangle (r, \boldsymbol{v}) &= 4\pi G^2 {m_{\mathrm{a}}}^2 \ln \Lambda \frac{\partial^2 g}{\partial v_i \partial v_j}(r, \boldsymbol{v}) \end{aligned} \tag{8}$$

where the Rosenbluth potentials are defined as (Binney & Tremaine, eq. L19)

$$\begin{aligned} h(r, \boldsymbol{v}) &= \int \mathrm{d}^3 \boldsymbol{v_a} \frac{f(r, \boldsymbol{v_a})}{|\boldsymbol{v} - \boldsymbol{v_a}|}, \\ g(r, \boldsymbol{v}) &= \int \mathrm{d}^3 \boldsymbol{v_a} f(r, \boldsymbol{v_a}) |\boldsymbol{v} - \boldsymbol{v_a}| \end{aligned} \tag{9}$$

1.1 Legendre expansion

Using that the Legendre expansion through its generating function $1/|\boldsymbol{v} - \boldsymbol{v_a}| = 1/\sqrt{v^2 + v_{\mathrm{a}}^2 - 2 v v_{\mathrm{a}} \cos(\theta)}$, we can expand the integration by using $\boldsymbol{v}$ as the (Oz) axis as

$$\begin{aligned} h(r, \boldsymbol{v}) &= \int_0^\infty \mathrm{d}v_{\mathrm{a}} v_{\mathrm{a}}^2 \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi \frac{F(E_{\mathrm{a}}, L_{\mathrm{a}})}{\sqrt{v^2 + v_{\mathrm{a}}^2 - 2 v v_{\mathrm{a}} \cos(\theta)}}, \\ &= \sum_{l=0}^{\infty} \int_0^\infty \mathrm{d}v_{\mathrm{a}} v_{\mathrm{a}}^2 \frac{v_<^l}{v_>^{l+1}} \int_0^\pi \mathrm{d}\theta \sin\theta P_l(\cos\theta) \int_0^{2\pi} \mathrm{d}\phi F(E_{\mathrm{a}}, L_{\mathrm{a}}), \end{aligned} \tag{10}$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{v_a}$, $v_< = \min\{v, v_{\mathrm{a}}\}$, $v_> = \max\{v, v_{\mathrm{a}}\}$, P_l is the l-th Legendre polynomial and

$$\begin{aligned} E_{\mathrm{a}}(r, v_{\mathrm{a}}, \theta, \phi) &= \psi(r) + \frac{v_{\mathrm{a}}^2}{2}, \\ L_{\mathrm{a}}(r, v_{\mathrm{a}}, \theta, \phi) &= r v_{\mathrm{at}}. \end{aligned} \tag{11}$$

Letting (Ox) be the projection of $\hat{\boldsymbol{r}}$, the cartesian coordinates of v_{a} and $\hat{\boldsymbol{r}}$ are respectively $(v_{\mathrm{a}} \sin\theta \cos\phi, v_{\mathrm{a}} \sin\theta \sin\phi, v_{\mathrm{a}} \cos\theta)$ and $(\sin\theta_{\mathrm{r}}, 0, \cos\theta_{\mathrm{r}}) = (v_{\mathrm{t}}/v, 0, v_{\mathrm{r}}/v)$, and we have

$$\begin{aligned} v_{\mathrm{ar}}(r, v_{\mathrm{a}}, \theta, \phi) &= \boldsymbol{v} \cdot \hat{\boldsymbol{r}} = \frac{v_{\mathrm{t}} v_{\mathrm{a}}}{v} \sin\theta \cos\phi + \frac{v_{\mathrm{r}} v_{\mathrm{a}}}{v} \cos\theta, \\ v_{\mathrm{at}}^2(r, v_{\mathrm{a}}, \theta, \phi) &= v_{\mathrm{a}}^2 \left[1 - \left(\frac{v_{\mathrm{t}}}{v} \sin\theta \cos\phi + \frac{v_{\mathrm{r}}}{v} \cos\theta \right)^2 \right], \end{aligned} \tag{12}$$

and therefore

$$L_{\mathrm{a}}(r, v_{\mathrm{a}}, \theta, \phi) = r v_{\mathrm{a}} \sqrt{1 - \left(\frac{v_{\mathrm{r}}}{v} \cos\theta + \frac{v_{\mathrm{t}}}{v} \sin\theta \cos\phi \right)^2}. \tag{13}$$

1.2 Anisotropic case

Since this result is valid for any arbitrary coordinate system, we can fix it to the one where $\boldsymbol{e_1} = \hat{v}$ and $\boldsymbol{e_2}$ is the projection of $\hat{\boldsymbol{r}}$ onto the equatorial plane orthogonal to $\boldsymbol{e_1}$. Then we'll have the relations

$$\begin{aligned} \langle \Delta v_{||} \rangle (r, \boldsymbol{v}) &= \langle \Delta v_1 \rangle (r, \boldsymbol{v}), \\ \langle (\Delta v_{||})^2 \rangle (r, \boldsymbol{v}) &= \langle (\Delta v_1)^2 \rangle (r, \boldsymbol{v}) \\ \langle (\Delta v_\perp)^2 \rangle (r, \boldsymbol{v}) &= \langle (\Delta v_2)^2 \rangle (r, \boldsymbol{v}) + \langle (\Delta v_3)^2 \rangle (r, \boldsymbol{v}) \end{aligned} \tag{14}$$

where the subscripts are relative of the velocity of the test star.

and a tedious by straightforward computation see appendix) yields

$$
\boxed{
\begin{aligned}
\langle \Delta v_{||} \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_\mathrm{a} (m + m_\mathrm{a}) \ln \Lambda \left(\frac{v_\mathrm{r}}{v} \frac{\partial h}{\partial v_\mathrm{r}} + \frac{v_\mathrm{t}}{v} \frac{\partial h}{\partial v_\mathrm{t}} \right), \\
\langle (\Delta v_{||})^2 \rangle (r, \boldsymbol{v}) &= 4\pi G^2 {m_\mathrm{a}}^2 \ln \Lambda \left(\frac{v_\mathrm{r}^2}{v^2} \frac{\partial^2 g}{\partial v_\mathrm{r}^2} + \frac{2 v_\mathrm{r} v_\mathrm{t}}{v^2} \frac{\partial^2 g}{\partial v_\mathrm{t} \partial v_\mathrm{r}} + \left(\frac{v_\mathrm{t}}{v} \right)^2 \frac{\partial^2 g}{\partial v_\mathrm{t}^2} \right) \\
\langle (\Delta v_{\perp})^2 \rangle (r, \boldsymbol{v}) &= 4\pi G^2 m_\mathrm{a}^2 \ln \Lambda \left(\left(\frac{v_\mathrm{t}}{v} \right)^2 \frac{\partial^2 g}{\partial v_\mathrm{r}^2} - \frac{2 v_\mathrm{r} v_\mathrm{t}}{v^2} \frac{\partial^2 g}{\partial v_\mathrm{t} \partial v_\mathrm{r}} + \left(\frac{v_\mathrm{r}}{v} \right)^2 \frac{\partial^2 g}{\partial v_\mathrm{t}^2} + \frac{1}{v_\mathrm{t}} \frac{\partial g}{\partial v_\mathrm{t}} \right)
\end{aligned}
}
\tag{15}
$$

where $h(r, \boldsymbol{v_\mathrm{a}}) = h(r, v_\mathrm{r}, v_\mathrm{t})$ and $g(r, \boldsymbol{v}) = g(r, v_\mathrm{r}, v_\mathrm{t})$.

Applying the change of variable $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_\mathrm{a}}$ and using spherical coordinates with axis $(Oz) = \hat{\boldsymbol{r}}$ the unit radius vector (parallel or antiparallel to the radial component of $\boldsymbol{v}$ by definition) yields

$$
\begin{aligned}
h(r, v_\mathrm{r}, v_\mathrm{t}) &= \int \mathrm{d}^3 \boldsymbol{V_0} \frac{f(r, \boldsymbol{v} - \boldsymbol{V_0})}{V_0} &&= \int_0^\infty \mathrm{d}V_0 V_0 \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f(r, \boldsymbol{v} - \boldsymbol{V_0}), \\
g(r, v_\mathrm{r}, v_\mathrm{t}) &= \int \mathrm{d}^3 \boldsymbol{V_0} f(r, \boldsymbol{v} - \boldsymbol{V_0}) V_0 &&= \int_0^\infty \mathrm{d}V_0 V_0^3 \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi f(r, \boldsymbol{v} - \boldsymbol{V_0})
\end{aligned}
\tag{16}
$$

where

$$
f(r, \boldsymbol{v} - \boldsymbol{V_0}) = f(r, v_\mathrm{ar}, v_\mathrm{at}) = F(E(r, v_\mathrm{ar}, v_\mathrm{at}), L(r, v_\mathrm{ar}, v_\mathrm{at})) \tag{17}
$$

with E, L given by eq (56).

For a given convention $+$ or $-$ of the choice of v_r, and given (E, L) the parameters of the test star, obtain the vectors $\boldsymbol{v_+} = (|v_\mathrm{r}|, \boldsymbol{v}_\mathrm{t})$ and $\boldsymbol{v_-} = (-|v_\mathrm{r}|, \boldsymbol{v}_\mathrm{t})$, which are symmetric with respect to the tangent plane where $\boldsymbol{v}_\mathrm{t}$ lives. In terms of spherical coordinates, we have that $\boldsymbol{v_+} = (v, \theta_0, 0)$ and $\boldsymbol{v_-} = (v, \pi - \theta_0, 0)$. Remember that the integration over the velocities $\boldsymbol{V_0} = \boldsymbol{v} - \boldsymbol{v_\mathrm{a}}$ of the field stars cover the whole $\boldsymbol{V_0}$-space. Given a velocity $\boldsymbol{V_0}$ corresponds bijectively a field star velocity $\boldsymbol{v_\mathrm{a}}$. The overall integration will in fact not depend on the convention we used. The $E(r, v_\mathrm{ar}, v_\mathrm{at})$ component depends on the sign of v_r since

$$
E_\mathrm{a}(r, V_0, \theta, \phi) = \psi(r) + \frac{1}{2} \left[v^2 + V_0^2 - 2V_0 (v_\mathrm{r} \cos\theta + v_\mathrm{t} \sin\theta \cos\phi) \right] \tag{18}
$$

but $L_\mathrm{a}(r, v_\mathrm{ar}, v_\mathrm{at})$ does not. When doing the integration, we will evaluate the integrand at both arguments (V_0, θ, ϕ) and $(V_0, \pi - \theta, \phi)$, and their summed contribution doesn't depend on the convention choice. In the following, we decide to use $v_\mathrm{r} \geq 0$.

For an actual computation, we also need to compute the various velocity-partial derivatives of those integrals, meaning that we need to compute the velocity-partial derivatives of $f(r, \boldsymbol{v_\mathrm{a}}) = F(E_\mathrm{a}, L_\mathrm{a})$ (exchange derivation and integral). Those are (function are evaluated at $(E_\mathrm{a}, L_\mathrm{a})$)

$$
\boxed{
\begin{aligned}
\frac{\partial}{\partial v_\mathrm{r}} \left[f(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_\mathrm{r} + V_0 \cos\theta) \frac{\partial F}{\partial E}, \\
\frac{\partial}{\partial v_\mathrm{t}} \left[f(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_\mathrm{t} + V_0 \sin\theta \cos\phi) \left(\frac{\partial F}{\partial E} - \frac{r}{L_\mathrm{a}} \frac{\partial F}{\partial L} \right), \\
\frac{\partial^2}{\partial v_\mathrm{r}^2} \left[f(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= -\frac{\partial F}{\partial E} + (-v_\mathrm{r} + V_0 \cos\theta)^2 \frac{\partial^2 F}{\partial E^2}, \\
\frac{\partial^2}{\partial v_\mathrm{t} \partial v_\mathrm{r}} \left[f(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= (-v_\mathrm{r} + V_0 \cos\theta)(-v_\mathrm{t} + V_0 \sin\theta \cos\phi) \left(\frac{\partial^2 F}{\partial E^2} - \frac{r}{L_\mathrm{a}} \frac{\partial^2 F}{\partial L \partial E} \right), \\
\frac{\partial^2}{\partial v_\mathrm{t}^2} \left[f(r, \boldsymbol{v} - \boldsymbol{V_0}) \right] &= -\frac{\partial F}{\partial E} + \frac{r}{L_\mathrm{a}} \frac{\partial F}{\partial L} + (-v_\mathrm{t} + V_0 \sin\theta \cos\phi)^2 \\
&\times \left(\frac{\partial^2 F}{\partial E^2} - \frac{2r}{L_\mathrm{a}} \frac{\partial^2 F}{\partial L \partial E} - \frac{r^2}{L_\mathrm{a}^3} \frac{\partial F}{\partial L} + \frac{r^2}{L_\mathrm{a}^2} \frac{\partial^2 F}{\partial L^2} \right).
\end{aligned}
}
\tag{19}
$$

The DF and its derivative vanish when $E_{\rm a} > 0$. Obviously, $v_{\rm a}(r, V_0, \theta, \phi)$ is minored by the polynomial in V_0 given by $v^2 + V_0^2 - 2V_0(v_{\rm r} + v_{\rm t})$. We have $E_{\rm a} > 0$ when $v_{\rm a} > -\psi(r)$, which happens outside of the roots of $v^2 + V_0^2 - 2V_0(v_{\rm r} + v_{\rm t}) + 2\psi(r)$. Those roots are

$$V_{0\pm} = (v_{\rm t} + v_{\rm r}) \pm \sqrt{2(v_{\rm r}v_{\rm t} - \psi(r))}. \tag{20}$$

For $E > 0$, the inferior root is always negative whereas the superior root is always positive. Let's call it $V_{\rm max}$. In the end, the Rosenbluth potentials can be computed over compact domains

$$\boxed{\begin{array}{rcl} h(r, v_{\rm r}, v_{\rm t}) & = & \displaystyle\int_0^{V_{\rm max}} {\rm d}V_0 V_0 \int_0^{\pi} {\rm d}\theta \sin\theta \int_0^{2\pi} {\rm d}\phi f(r, \boldsymbol{v} - \boldsymbol{V_0}), \\ g(r, v_{\rm r}, v_{\rm t}) & = & \displaystyle\int_0^{V_{\rm max}} {\rm d}V_0 V_0^3 \int_0^{\pi} {\rm d}\theta \sin\theta \int_0^{2\pi} {\rm d}\phi f(r, \boldsymbol{v} - \boldsymbol{V_0}) \end{array}} \tag{21}$$

and so do its partial derivatives.

1.3 Isotropic case

We may want to check that the integrals yield the correct result. To that end, it can be of interest to consider the simple case $q = 0$, where $F(E, L) = F(E)$, i.e. $f(r, \boldsymbol{v}) = f(r, v) = F(E)$. Then according Binney & Tremaine, eq. (L26),

$$\begin{array}{rcl} \langle \Delta v_{||} \rangle (r, v) & = & -\dfrac{16\pi^2 G^2 m_{\rm a}(m + m_{\rm a}) \ln\Lambda}{v^2} K_1(r, v), \\ \langle (\Delta v_{||})^2 \rangle (r, v) & = & \dfrac{32\pi^2 G^2 m_{\rm a}^2 \ln\Lambda}{3} \left(K_0(r, v) + \dfrac{1}{v^3} K_3(r, v) \right) \\ \langle (\Delta v_{\perp})^2 \rangle (r, v) & = & \dfrac{32\pi^2 G^2 m_{\rm a}^2 \ln\Lambda}{3} \left(2K_0(r, v) + \dfrac{3}{v} K_1(r, v) - \dfrac{1}{v^3} K_3(r, v) \right) \end{array} \tag{22}$$

where

$$\begin{array}{rcl} K_0(r, v) & = & \displaystyle\int_0^E {\rm d}E_a F(E_a), \\ K_1(r, v) & = & \displaystyle\int_E^{\psi(r)} {\rm d}E_a v_a F(E_a) \\ K_3(r, v) & = & \displaystyle\int_E^{\psi(r)} {\rm d}E_a v_a^3 F(E_a) \end{array} \tag{23}$$

and the correspondance $E = \psi(r) - v^2/2$

In the appendix, we recompute the formulae of the isotropic case from the arbitrary anisotropic case, with $v^2 = v_{\rm r}^2 + v_{\rm t}^2$, $h(r, v_{\rm r}, v_{\rm t}) = h(r, v)$ and $g(r, v_{\rm r}, v_{\rm t}) = g(r, v)$.

2 Local orbital parameter changes

Now, switch to (E, L) space and using eq. (C15) to (C19) of Bar-Or & Alexander (2016), which doesn't rely on an isotropy assumption, we obtain (evaluate at $(r, \boldsymbol{v}(r, E, L))$) at first order in $\Delta v/v$

$$\begin{array}{rcl} \langle \Delta E \rangle (r, E, L) & = & \dfrac{1}{2} \langle (\Delta v_{||})^2 \rangle + \dfrac{1}{2} \langle (\Delta v_{\perp})^2 \rangle + v \langle \Delta v_{||} \rangle, \\ \langle (\Delta E)^2 \rangle (r, E, L) & = & v^2 \langle (\Delta v_{||})^2 \rangle \\ \langle \Delta L \rangle (r, E, L) & = & \dfrac{L}{v} \langle \Delta v_{||} \rangle + \dfrac{r^2}{4L} \langle (\Delta v_{\perp})^2 \rangle, \\ \langle (\Delta L)^2 \rangle (r, E, L) & = & \dfrac{L^2}{v^2} \langle \Delta v_{||} \rangle + \dfrac{1}{2} \left(r^2 - \dfrac{L^2}{v^2} \right) \langle (\Delta v_{\perp})^2 \rangle \\ \langle \Delta E \Delta L \rangle (r, E, L) & = & L \langle (\Delta v_{||})^2 \rangle \end{array} \tag{24}$$

Due to our analysis, those quantities are well defined and we can use the bijective transformation $(r, E, L) \leftrightarrow (r, v_\mathrm{r}, v_\mathrm{t})$

3 Orbit of a test star in a globular cluster

We can now compute the local diffusion coefficients $\langle \Delta E \rangle$, $\langle (\Delta E)^2 \rangle$, $\langle \Delta L \rangle$, $\langle (\Delta L)^2 \rangle$ and $\langle \Delta E \Delta L \rangle$. Since we are interested in the secular evolution of the system, we can average over the dynamical time and smear out the star along its orbit. This leads us to consider the orbit-average diffusion coefficients

$$\begin{aligned} D_X(E, L) &= \langle \Delta X \rangle_{\circlearrowleft} &= \frac{2}{T} \int_{r_\mathrm{min}}^{r_\mathrm{max}} \langle \Delta X \rangle(r, E, L) \frac{\mathrm{d}r}{v_\mathrm{r}(r, E, L)}, \\ D_{XY}(E, L) &= \langle \Delta X \Delta Y \rangle_{\circlearrowleft} &= \frac{2}{T} \int_{r_\mathrm{min}}^{r_\mathrm{max}} \langle \Delta X \Delta Y \rangle(r, E, L) \frac{\mathrm{d}r}{v_\mathrm{r}(r, E, L)}. \end{aligned} \tag{25}$$

where $v_\mathrm{r}(r)$ is the radial velocity of the orbiting star at r. It is of interest to define the effective potential

$$\psi_\mathrm{eff}(r, L) = \psi(r) + \frac{L^2}{2r^2}. \tag{26}$$

3.1 Study of an orbit

See Kurth (1955), Astronomische Nachrichten, volume 282, Issue 6, p.241.

Consider a test star described by its position vector $\boldsymbol{r}$, its energy $E(t)$ and its angular momentum vector $\boldsymbol{L}(t)$, per unit mass. Then by Newton's law, those two quantities are conserved along an orbit, allowing us to drop the t parameter.

Consider a bound orbit with $E \leq 0$ and $L \geq 0$. Then its ascending radial velocity is given by

$$v_\mathrm{r}(r) = \sqrt{2(E - \psi_\mathrm{eff}(r; L))}, \tag{27}$$

its bounds r_min and r_max are given by the solution of the equation $v_\mathrm{r}(r) = 0$, which has two solutions, and its orbital period T is defined by

$$\frac{T}{2} = \int_{r_\mathrm{min}}^{r_\mathrm{max}} \frac{\mathrm{d}r}{v_\mathrm{r}(r)}. \tag{28}$$

The "type" of an orbit is determined by E and L. Graphically, r_max and r_min are given by the intersection points of $\psi_\mathrm{eff}(r, L)$ and E. There are a few different cases:

- $E \geq 0$: unbounded orbit,
- $E \in]E_\mathrm{c}(L), 0[$: bound "rosette-like" orbit,
- $E = E_\mathrm{c}(L)$: circular orbit,
- $E < E_\mathrm{c}(L)$: impossible,

where $E_\mathrm{c}(L) = \min_{r>0} \psi_\mathrm{eff}(r, L)$.

All the integral are finite, since they are integrable at the endpoints

$$\begin{aligned} \frac{1}{v_\mathrm{r}(r_\mathrm{max} - \epsilon)} &\sim |2\psi'_\mathrm{eff}(r_\mathrm{max})|^{-1/2} \frac{1}{\sqrt{\epsilon}}, \\ \frac{1}{v_\mathrm{r}(r_\mathrm{min} + \epsilon)} &\sim |2\psi'_\mathrm{eff}(r_\mathrm{min})|^{-1/2} \frac{1}{\sqrt{\epsilon}}. \end{aligned} \tag{29}$$

with strictly positive prefactor for bounded, non-circular orbits.

3.2 Circular orbit

The allowed region in (E, L) space is composed of the $E < 0, L > 0$ such that there exists $r > 0$ verifying the inequality $\psi_{\text{eff}}(r; L) \leq E$. For $L > 0$, this function is decreasing until a global minimum before increasing towards 0. Raising the value of L increases this minimum value, meaning that there exists a value $L_{\text{c}}(E)$ such that $\psi_{\text{eff}}(r; L_{\text{c}}(E)) = E$. Then, the forbidden angular momenta (for a given E) are the $L > L_{\text{c}}(E)$. Due to the discussion in the previous section, this couple $(E, L_{\text{c}}(E))$ determines a circular orbit.

3.2.1 Plummer model

(TODO)

It is given by $E_{\text{c}}(L) = \min_{r>0} \psi_{\text{eff}}(r; L) = \psi_{\text{eff}}(r_*^L; L)$. To approximate this r_*^L, we may look for it using Newton's method applied to ψ'_{eff}, since $\psi'_{\text{eff}}(r_*^L; L) = 0$. Start at $r_0^L = L^{2/3}$, where the evaluation yields

$$\psi'_{\text{eff}}(L^{2/3}; L) = -1 + \frac{L^{2/3}}{(1 + L^{4/3})^{3/2}} \in [-1, \sqrt{4/27} - 1] \simeq [-1, -0.615] \tag{30}$$

in order not to be too far away from $\psi'_{\text{eff}}(r_*^L; L) = 0$, and apply the recursion

$$r_{n+1}^L = r_n^L - \frac{\psi'_{\text{eff}}(r_n^L; L)}{\psi''_{\text{eff}}(r_n^L; L)}, \tag{31}$$

where

$$\begin{aligned} \psi'_{\text{eff}}(r; L) &= \frac{r}{(1 + r^2)^{3/2}} - \frac{L^2}{r^3}, \\ \psi''_{\text{eff}}(r_n; L) &= \frac{(1 + r^2)^{3/2} - 3r^2\sqrt{1 + r^2}}{(1 + r^2)^3} + 3\frac{L^2}{r^4}. \end{aligned} \tag{32}$$

Then $r_n^L \to r_*^L$. We can show that (r_n^L) is increasing since $\psi'_{\text{eff}}(r_n^L; L) < 0$ and $\psi''_{\text{eff}}(r_n^L; L) > 0$ (and convexity of ψ'_{eff} where it matters). Therefore a good stopping condition is to get the lowest N such that $\psi'_{\text{eff}}(r_N^L + \epsilon) > 0$ for some precision $\epsilon > 0$. Then, taking $\tilde{r}_*^L = (r_N^L + r_N^L + \epsilon)/2 = r_N^L + \epsilon/2$ we will have $E_{\text{c}}(L) \simeq \psi_{\text{eff}}(\tilde{r}_*^L; L)$, with precision

$$\delta E_{\text{c}}(L) \simeq |\psi_{\text{eff}}(r_*^L; L) - \psi_{\text{eff}}(\tilde{r}_*^L; L)| \simeq \frac{1}{2} |\underbrace{\psi_{\text{eff}}^{(2)}(r_*^L)}_{>0}| \cdot |r_*^L - \tilde{r}_*^L|^2 \simeq |\psi_{\text{eff}}^{(2)}(r_*^L)| \frac{\epsilon^2}{8} \tag{33}$$

However, there is no need to use an approximation method to compute L_{c}, as we can obtain an analytical formula. Indeed, it is given by

$$L_{\text{c}}^2(E) = \eta(r_*^E; E) = \max_{r>0} \eta(r; E). \tag{34}$$

where we defined $\eta(r; E) \doteqdot 2r^2(E - \psi(r))$. Using the change of variable $x = 2r^2$, we obtain $\tilde{\eta}(x, L) = \eta(r, L) = x(E + 1/\sqrt{1 + x/2})$, whose maximum is equal to that of η. It is reached for $x_{\text{c}}(E)$ (defined in the next section), whose computation is done is the appendix and is given by eq(106). Therefore, we have an analytical expression for $L_{\text{c}}^2(E)$, given by

$$L_{\text{c}}^2(E) = \tilde{\eta}(x_{\text{c}}(E), E) = \eta(\sqrt{x_{\text{c}}(E)/2}, E). \tag{35}$$

3.2.2 Isochrone model

In this case, $\psi_{\text{eff}}(r, L) = -2/(1 + \sqrt{1 + r^2}) + L^2/(2r^2)$. Its minimum value is given by

$$E_{\text{c}}(L) = \frac{1}{4}\Big(- L^2 + \sqrt{L^2(L^2 + 8)} - 4\Big). \tag{36}$$

As for L_c^2, we can deduces its expressions by inverting eq.(36). Letting $X = L^2$, we must solve for $X > 0$ the equation

$$4E = -X + \sqrt{X(X+8)} - 4 = h(X), \tag{37}$$

which is an increasing function such that $h(0) = -4$ and $h(+\infty) = 0$. Therefore it has only one strictly positive solution. We can rewrite this equation as the degree-2 equation

$$(4(E+1) + X)^2 = X(X+8), \tag{38}$$

which simplifies a linear equation with the solution $X = -2(E+1)^2/E > 0$. Therefore, the angular momentum of a circular orbit for the isochronous potential is given by

$$L_\mathrm{c}(E) = \sqrt{\frac{2(E+1)^2}{-E}}. \tag{39}$$

3.2.3 Plummer model

We need to compute orbit-averaged quantities of the form

$$I(D) = \int_{r_\mathrm{min}}^{r_\mathrm{max}} \frac{D(r)\mathrm{d}r}{v_\mathrm{r}(r)}, \tag{40}$$

where $v_\mathrm{r}(r) = \sqrt{2(E - \psi(r)) - L^2/r^2}$. Letting $x = 2r^2$ and $Y(x) = 2r^2\psi(r)$ gives

$$I(D) = \frac{1}{4}\int_{x_\mathrm{min}}^{x_\mathrm{max}} \frac{D(r(x))\mathrm{d}x}{\sqrt{Ex - Y(x) - L^2}}. \tag{41}$$

Defining $\ell^2 = L_\mathrm{c}^2(E) - L^2$ and $z^2 = Y(x) - (Ex - L_\mathrm{c}^2)$, we have

$$I(D) = \frac{1}{4}\int_{x_\mathrm{min}}^{x_\mathrm{max}} \frac{D(r(x))\mathrm{d}x}{\sqrt{\ell^2 - z^2(x)}}. \tag{42}$$

The equation $z^2 = Y(x) - (Ex - L_\mathrm{c}^2)$ has two solutions in $[0, x_\mathrm{c}]$ (for $(z < 0)$) and $[x_\mathrm{c}, +\infty[$ (for $z > 0$). This defines a function $x = \varphi(z)$, which is C^1 and strictly increasing with $\varphi(x_\mathrm{c}) = 0$. This change of variable gives

$$I(D) = \frac{1}{4}\int_{-\ell}^{\ell} \frac{D(r(z))\varphi'(z)\mathrm{d}z}{\sqrt{\ell^2 - z^2}}. \tag{43}$$

A last variable of variable $z = \ell\sin\phi$ gives

$$I(D) = \frac{1}{4}\int_{-\pi/2}^{\pi/2} D(r(\phi))\varphi'(\ell\sin\phi)\mathrm{d}\phi, \tag{44}$$

where $r(\phi) = \sqrt{\varphi(\ell\sin\phi)/2}$. Computations given in the appendix gives the following. Define the function $\Upsilon(x) = z^2(x) = Y(x) - (Ex - L_\mathrm{c}^2)$. Then

$$z(x) = \begin{cases} -\sqrt{\Upsilon(x)} & \text{if} \quad x \leq x_\mathrm{c}, \\ +\sqrt{\Upsilon(x)} & \text{if} \quad x \geq x_\mathrm{c}. \end{cases} \tag{45}$$

Since $z^2 = \Upsilon(\varphi(z))$, we have that

$$\varphi'(z) = \frac{2z}{\Upsilon'(\varphi(z))}, \tag{46}$$

which is strictly positive for $z \neq 0$ and evaluates to $\sqrt{2/\Upsilon''(x_\mathrm{c})} > 0$ at $z = 0$ (see appendix). A straighforward computation gives us that $x = \varphi(z)$ is solution to the degree-3 polynomial equation $z^2 - L_\mathrm{c}^2 + Ex)^2(1 +$

$x/2) = x^2$. This equation has at least the two real-positive solutions of $z^2 = \Upsilon(x)$, to which it must have an additional root at some negative x. Cardan's formulae (and more precisely Viète's formulae) gives us analytical expressions for those solutions (in terms of its coefficients), hence analytical expressions in term of z. Using equation (46), we obtain an analytical expression of $\varphi'(z)$ which is regular along z, meaning that the orbit-averaged expression from eq. (44) is nicely integrable.

In particular, we obtain the following expressions

$$\boxed{\begin{aligned} \frac{T}{2} &= \int_{-\pi/2}^{\pi/2} \varphi'(\ell \sin\phi)\mathrm{d}\phi, \\ \langle D \rangle &= \frac{2}{T}\int_{-\pi/2}^{\pi/2} D(r(\phi))\varphi'(\ell\sin\phi)\mathrm{d}\phi, \\ r(\phi) &= \sqrt{\varphi(\ell\sin\phi)/2}. \end{aligned}} \tag{47}$$

3.2.4 Isochrone model

(TODO)

4 Diffusion equation and change of variables

All in all, those coefficients appear in the Fokker-Planck diffusion equation

$$\frac{\partial P}{\partial t}(E,L,t) = -\frac{\partial}{\partial E}\left[D_E P\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[D_{EE}P\right] - \frac{\partial}{\partial L}\left[D_L P\right] + \frac{1}{2}\frac{\partial^2}{\partial L^2}\left[D_{LL}P\right] + \frac{1}{2}\frac{\partial^2}{\partial E \partial L}\left[D_{EL}P\right] \tag{48}$$

If we want to change coordinates $\boldsymbol{x} \to \boldsymbol{x'}(\boldsymbol{x})$, we may use the formulae (C.52) and (C.53) p.25 from Bar-Or & Alexander (2016)

$$\begin{aligned} D'_l &= \frac{\partial x'_l}{\partial x_k} D_k + \frac{1}{2}\frac{\partial^2 x'_l}{\partial x_r \partial x_k} D_{rk}, \\ D'_{lm} &= \frac{\partial x'_l}{\partial x_r}\frac{\partial x'_m}{\partial x_k} D_{rk}. \end{aligned} \tag{49}$$

(error on the sign? should be

$$D'_l = \frac{\partial x'_l}{\partial x_k} D_k - \frac{1}{2}\frac{\partial^2 x'_l}{\partial x_r \partial x_k} D_{rk} \quad ?)$$

Instead of considering the parameters (E, L), we could go to the action parameters (J_r, L) with J_r being the radial action. It is defined as

$$J_\mathrm{r}(E,L) = \frac{1}{\pi}\int_{r_\mathrm{min}(E,L)}^{r_\mathrm{max}(E,L)} v_\mathrm{r}(r,E,L)\mathrm{d}r = \frac{1}{\pi}\int_{r_\mathrm{min}(E,L)}^{r_\mathrm{max}(E,L)} \sqrt{2(\psi_\mathrm{eff}(r,L)-E)}\mathrm{d}r. \tag{50}$$

It is the generating function of the radial period T defined in eq (28) and of the apsidal angle Θ defined by $\Theta(t) = \theta(t+T) - \theta(t)$, which is constant and given by the formula

$$\Theta(E,L) = \int_{r_\mathrm{min}(E,L)}^{r_\mathrm{max}(E,L)} \frac{2L\mathrm{d}r}{r^2 v_\mathrm{r}(r,E,L)} = \int_{r_\mathrm{min}(E,L)}^{r_\mathrm{max}(E,L)} \frac{2L\mathrm{d}r}{r^2\sqrt{2(\psi_\mathrm{eff}(r,L)-E)}}, \tag{51}$$

with $\theta(t) = p(t) + \Theta t/T$ with $p(t)$ is T-periodical. Those quantities are linked by the relations (see appendix)

$$\frac{T}{2\pi} = -\frac{\partial J_\mathrm{r}}{\partial E} \quad \text{and} \quad \frac{\Theta}{2\pi} = -\frac{\partial J_\mathrm{r}}{\partial L} \tag{52}$$

The transformation $(E, L) \to (J_\mathrm{r}, L)$ yields (check that the relation is correct?)

5 Plummer model

Consider a Plummer model (Dejonghe, H.1987, MNRAS 224, 13) with potential with units $r_{\rm s}$ the Plummer scale radius (which sets the size of the cluster core), M the total mass of the cluster and $\bar{\tau}$ some unit time. Let $\psi_{\rm s}$ be defined by

$$\psi_{\rm s} = \frac{GM}{r_{\rm s}}, \tag{53}$$

for the central potential

$$\psi(r) = -\frac{\psi_{\rm s}}{\sqrt{1+r^2}}. \tag{54}$$

Let use fix $G = 1\, r_{\rm s}^3.M^{-1}.\bar{\tau}^{-2}$ in the new units so that $\psi_{\rm s} = 1\, r_{\rm s}^2 \cdot \bar{\tau}^{-2}$. This fixes the time unit $\bar{\tau}$, as we have the relation. Therefore, in those units the potential (per unit mass) is given by

$$\psi(r) = -\frac{1}{\sqrt{1+r^2}}. \tag{55}$$

Define, given a radius r, the angular momentum $L(r, v_{\rm r}, v_{\rm t})$ and energy per unit mass $E(r, v_{\rm r}, v_{\rm t})$, functions of the radial velocity $v_{\rm r}$ and the tangential velocity $v_{\rm t} \geq 0$ (defined as $\boldsymbol{v} = v_{\rm r}\hat{\boldsymbol{r}} + \boldsymbol{v}_{\rm t}$), as

$$\begin{array}{rcl} E(r, v_{\rm r}, v_{\rm t}) & = & \psi(r) + \frac{1}{2}v_{\rm r}^2 + \frac{1}{2}v_{\rm t}^2, \\ L(r, v_{\rm r}, v_{\rm t}) & = & r \cdot v_{\rm t}, \end{array} \tag{56}$$

whose Jacobian is

$$\mathrm{Jac}_{(r,v_{\rm r},v_{\rm t})\to(r,E,L)} = \left(\begin{array}{cc} \frac{\partial E}{\partial v_{\rm r}} & \frac{\partial E}{\partial v_{\rm t}} \\ \frac{\partial L}{\partial v_{\rm r}} & \frac{\partial L}{\partial v_{\rm t}} \end{array} \right) = \left(\begin{array}{cc} v_{\rm r} & v_{\rm t} \\ 0 & r \end{array} \right) \Rightarrow |\mathrm{Jac}| = r|v_{\rm r}|. \tag{57}$$

To obtain a bijective transformation, we must chose whether to chose $v_{\rm r} \geq 0$ or $v_{\rm r} \leq 0$. A priori, this choice might have an impact on the result, but we will should that the local and orbit-averaged diffusion coefficients are not that. The coordinate system is spherical, its origin being at the center of the globular cluster. Finally, we consider the corresponding anisotropic distribution functions of the field stars $F_q(r, E, L) = F_q(E, L)$ in (E, L)-space. Since (E, L) and $(v_{\rm r}, v_{\rm t})$ are linked, we can make use of the following equalities (for the moment, $v_{\rm r}$ is defined modulo the sign)

$$\begin{array}{rcl} F_q(E, L) & = & f_q(r, v_{\rm r}(r, E, L), v_{\rm t}(r, E, L)), \\ f_q(r, v_{\rm r}, v_{\rm t}) & = & F_q(E(r, v_{\rm r}, v_{\rm t}), L(r, v_{\rm r}, v_{\rm t})), \end{array} \tag{58}$$

where f_q is the distribution fonction (DF)in the $(v_{\rm r}, v_{\rm t})$ space and where q is an anisotropy parameter:

- $q \in]0, 2]$: radially anisotropic
- $q = 0$: isotropic
- $q \in] - \infty, 0[$: tangentially anisotropic.

Note that the DF has spherical symmetry in position. Its expression for $-E \geq 0, L \geq 0$ is (for $q \neq 0$):

$$F_q(E, L) = \frac{3\Gamma(6-q)}{2(2\pi)^{5/2}\Gamma(q/2)}(-E)^{7/2-q}\mathbb{H}(0, \frac{q}{2}, \frac{9}{2} - q, 1; -\frac{L^2}{2E}) \tag{59}$$

where

$$\mathbb{H}(a, b, c, d; x) = \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(c-a)\Gamma(a+d)} x^a\, {}_2F_1(a+b, 1+a-c, a+d; x) & \text{if} \quad x \leq 1, \\ \dfrac{\Gamma(a+b)}{\Gamma(d-b)\Gamma(b+c)} x^{-b}\, {}_2F_1(a+b, 1+b-d, b+c; \dfrac{1}{x}) & \text{if} \quad x \geq 1, \end{cases} \tag{60}$$

which reduces in the isotropic case ($q = 0$) to

$$F_0(E) = \frac{3}{7\pi^3}(-2E)^{7/2}, \tag{61}$$

and in the extreme radially anisotropic ($q = 2$) to

$$F_2(E) = \begin{cases} \dfrac{6}{(2\pi)^3}(-2E - L)^{3/2} & \text{if} \quad -2E \leq L^2, \\ 0 & \text{if} \quad -2E \geq L^2. \end{cases} \tag{62}$$

When $-E \leq 0$ or $L \leq 0$ then $F_q(E, L) = 0$.

6 Isochrone model

The isochronous model is associated with the gravitationnal potential

$$\psi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}} = \frac{2E_{\mathrm{o}}}{1 + \sqrt{1 + x^2}} \tag{63}$$

where $x = r/b$ and $E_{\mathrm{o}} = -GM/(2b)$.

We consider the orbital plane of the motion. Within it, we can describe the motion by its hamiltonian H and its actions $\boldsymbol{J} = (J_{\mathrm{r}}, L)$

$$H(\boldsymbol{J}) = -\frac{(GM)^2}{2\left[J_{\mathrm{r}} + \frac{1}{2}\left(L + \sqrt{L^2 + 4GMb}\right)\right]^2} \tag{64}$$

From this, we can obtain a closed expression for J_{r}

$$J_{\mathrm{r}}(E, L) = \frac{GM}{\sqrt{-2E}} - \frac{1}{2}\left(L + \sqrt{L^2 + 4GMb}\right) \tag{65}$$

from the generic radial action

$$J_{\mathrm{r}}(E, L) = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} v_{\mathrm{r}}(r)\mathrm{d}r = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} \sqrt{2(E - \psi_{\mathrm{eff}}(r, L))}\mathrm{d}r. \tag{66}$$

From the generic definition $\mathrm{d}E = \Omega_1 \mathrm{d}J_{\mathrm{r}} + \Omega_2 \mathrm{d}L$, which yields equivalently $\mathrm{d}J_{\mathrm{r}} = (1/\Omega_1)\mathrm{d}E - (\Omega_2/\Omega_1)\mathrm{d}L$, we obtain

$$\frac{2\pi}{\Omega_1} = \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v_{\mathrm{r}}(r)} = T$$
$$\frac{\Omega_2}{\Omega_1} = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} \frac{L\mathrm{d}r}{r^2 v_{\mathrm{r}}(r)} = \frac{\Theta}{2\pi}$$

where T is the radial period and Θ is the apsidal angle. In particular, for the isochronous potential,

$$\Omega_1(E, L) = \omega(E)\Omega_o$$
$$\Omega_2(E, L) = \eta(L)\omega(E)\Omega_o,$$

where $\omega(E) = (E/E_{\mathrm{o}})^{3/2}$, $\Omega_o = \sqrt{GM/b^3}$ and $\eta(L) = \Omega_2/\Omega_1 = (1/2)(1 + L/\sqrt{L^2 + 4GMb})$.

The angles associated to the actions are (cf. main.pdf Fouvry eq (A3) : mistake on denominator: r instead of r^2?)

$$\theta_1 = \int_{r_{\min}}^{r(\theta_1)} \frac{\Omega_1}{v_{\mathrm{r}}(r)} \mathrm{d}r$$
$$\theta_2 - \varphi = \int_{r_{\min}}^{r(\theta_1)} \frac{\Omega_2 - L/r^2}{v_{\mathrm{r}}(r)} \mathrm{d}r,$$

where φ is the angle from the ascending node to the current location of the particle along the orbital motion.

Appendices

A Local diffusion coefficients

Consider a fixed, arbitrary coordinate system when we let $\boldsymbol{e_1}$ be the z-axis and let $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ be arbitrary basis vector in the orthogonal plane. Let $\boldsymbol{r} = (r, \theta, \phi)$ such that

$$\begin{aligned} r_1 &= r\cos\theta, \\ r_2 &= r\sin\theta\cos\phi, \\ r_3 &= r\sin\theta\sin\phi. \end{aligned} \tag{67}$$

and letting $\boldsymbol{v} = (v_1, v_2, v_3)$ be the velocity cartesian coordinates. Then

$$\begin{aligned} v_{\mathrm{r}} &= \boldsymbol{v}\cdot\hat{\boldsymbol{r}} &&= v_1\cos\theta + v_2\sin\theta\cos\phi + v_3\sin\theta\sin\phi, \\ v_{\mathrm{t}}^2 &= v^2 - v_{\mathrm{r}}^2 &&= v^2 - (v_1\cos\theta + v_2\sin\theta\cos\phi + v_3\sin\theta\sin\phi). \end{aligned} \tag{68}$$

Let $h(\boldsymbol{r}, \boldsymbol{v}) = h(r, v_{\mathrm{r}}, v_{\mathrm{t}})$. Note that $\partial r/\partial v_i = 0$. Thus

$$\frac{\partial h}{\partial v_1} = \frac{\partial v_{\mathrm{r}}}{\partial v_1}\frac{\partial h}{\partial v_{\mathrm{r}}} + \frac{\partial v_{\mathrm{t}}}{\partial v_1}\frac{\partial h}{\partial v_{\mathrm{t}}} \tag{69}$$

We have, (at fixed $\boldsymbol{r}$, i.e. fixed r, θ, ϕ),

$$\begin{aligned} \frac{\partial v_{\mathrm{r}}}{\partial v_1} &= \cos\theta &;\quad \frac{\partial v_{\mathrm{t}}}{\partial v_1} &= \frac{v_1 - v_{\mathrm{r}}\cos\theta}{v_{\mathrm{t}}}, \\ \frac{\partial v_{\mathrm{r}}}{\partial v_2} &= \sin\theta\cos\phi &;\quad \frac{\partial v_{\mathrm{t}}}{\partial v_2} &= \frac{v_2 - v_{\mathrm{r}}\sin\theta\cos\phi}{v_{\mathrm{t}}}, \\ \frac{\partial v_{\mathrm{r}}}{\partial v_3} &= \sin\theta\sin\phi &;\quad \frac{\partial v_{\mathrm{t}}}{\partial v_3} &= \frac{v_3 - v_{\mathrm{r}}\sin\theta\sin\phi}{v_{\mathrm{t}}}. \end{aligned} \tag{70}$$

In the special coordinate system where $\boldsymbol{e_1} = \hat{\boldsymbol{v}}$ and $\boldsymbol{e_2}$ is the projection of $\hat{\boldsymbol{r}}$ on the equatorial plane ($\phi = 0$), we have that $\Delta v_1 = \Delta v_{||}$ and $\Delta v_2^2 + \Delta v_3^2 = \Delta v_{\perp}^2$, meaning that

$$\begin{aligned} \langle \Delta v_{||} \rangle &= \langle \Delta v_1 \rangle, \\ \langle (\Delta v_{||})^2 \rangle &= \langle (\Delta v_1)^2 \rangle, \\ \langle (\Delta v_{\perp})^2 \rangle &= \langle (\Delta v_2)^2 \rangle + \langle (\Delta v_3)^2 \rangle. \end{aligned} \tag{71}$$

In that system, we have that $v_1 = v$, $v_2 = v_3 = 0$ and $\phi = 0$. We also have that $\cos\theta = v_{\mathrm{r}}/v$ because θ is the angle between $\hat{\boldsymbol{r}}$ and $(Oz) = \boldsymbol{e_1} = \hat{\boldsymbol{v}}$, and v_{r} is the orthogonal projection of $\boldsymbol{v}$ on $\hat{\boldsymbol{r}}$. For the same reason, $\sin\theta = v_{\mathrm{t}}/v$. Therefore, in that special coordinate system:

$$\frac{\partial v_{\mathrm{r}}}{\partial v_1} = \frac{v_{\mathrm{r}}}{v}; \quad \frac{\partial v_{\mathrm{t}}}{\partial v_1} = \frac{v - v_{\mathrm{r}}^2/v}{v_{\mathrm{t}}} = \frac{v^2 - v_{\mathrm{r}}^2}{v_{\mathrm{t}} v} = \frac{v_{\mathrm{t}}^2}{v_{\mathrm{t}} v} = \frac{v_{\mathrm{t}}}{v} \tag{72}$$

and we have

$$\frac{\partial h}{\partial v_1} = \frac{v_{\mathrm{r}}}{v}\frac{\partial h}{\partial v_{\mathrm{r}}} + \frac{v_{\mathrm{t}}}{v}\frac{\partial h}{\partial v_{\mathrm{t}}} \tag{73}$$

As for second order derivatives, we must compute $\partial^2 g/\partial v_1^2$, $\partial^2 g/\partial v_2^2$ and $\partial^2 g/\partial v_3^2$. Let us compute those in an arbitrary coordinate system.

$$\begin{aligned}\frac{\partial^2 g}{\partial v_1^2} &= \frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right)\frac{\partial g}{\partial v_\mathrm{r}} + \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2} + \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}, \\ &+ \frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial g}{\partial v_\mathrm{t}} + \left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2} + \left(\frac{\partial v_\mathrm{t}}{\partial v_1}\frac{\partial v_\mathrm{r}}{\partial v_1}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}.\end{aligned} \tag{74}$$

Its coefficients are

$$\begin{aligned}\frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right) &= 0, \\ \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2} &= \cos^2\theta\frac{\partial^2 g}{\partial v_\mathrm{r}^2}, \\ \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}} &= \cos\theta\left(\frac{v_1 - v_\mathrm{r}\cos\theta}{v_\mathrm{t}}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}, \\ \frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial g}{\partial v_\mathrm{t}} &= \cos\theta\left(\frac{\sin^2\theta v_\mathrm{t} - (v_1 - v_\mathrm{r}\cos\theta)\frac{v_1 - v_\mathrm{r}\cos\theta}{v_\mathrm{t}}}{v_\mathrm{t}^2}\right)\frac{\partial g}{\partial v_\mathrm{t}}, \\ \left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2} &= \left(\frac{v_1 - v_\mathrm{r}\cos\theta}{v_\mathrm{t}}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2},\end{aligned} \tag{75}$$

which in the special coordinate system yields

$$\begin{aligned}\frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right) &= 0, \\ \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2} &= \left(\frac{v_\mathrm{r}}{v}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2}, \\ \left(\frac{\partial v_\mathrm{r}}{\partial v_1}\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}} &= \left(\frac{v_\mathrm{r} v_\mathrm{t}}{v^2}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}, \\ \frac{\partial}{\partial v_1}\left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)\frac{\partial g}{\partial v_\mathrm{t}} &= 0, \\ \left(\frac{\partial v_\mathrm{t}}{\partial v_1}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2} &= \left(\frac{v_\mathrm{t}}{v}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2},\end{aligned} \tag{76}$$

hence

$$\frac{\partial^2 g}{\partial v_1^2} = \left(\frac{v_\mathrm{r}}{v}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2} + \frac{2v_\mathrm{r}v_\mathrm{t}}{v^2}\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}} + \left(\frac{v_\mathrm{t}}{v}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2}. \tag{77}$$

For the v_2 partial derivative, we have:

$$\begin{aligned}\frac{\partial^2 g}{\partial v_2^2} &= \frac{\partial}{\partial v_2}\left(\frac{\partial v_\mathrm{r}}{\partial v_2}\right)\frac{\partial g}{\partial v_\mathrm{r}} + \left(\frac{\partial v_\mathrm{r}}{\partial v_2}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{r}^2} + \left(\frac{\partial v_\mathrm{r}}{\partial v_2}\frac{\partial v_\mathrm{t}}{\partial v_2}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}, \\ &+ \frac{\partial}{\partial v_2}\left(\frac{\partial v_\mathrm{t}}{\partial v_2}\right)\frac{\partial g}{\partial v_\mathrm{t}} + \left(\frac{\partial v_\mathrm{t}}{\partial v_2}\right)^2\frac{\partial^2 g}{\partial v_\mathrm{t}^2} + \left(\frac{\partial v_\mathrm{t}}{\partial v_2}\frac{\partial v_\mathrm{r}}{\partial v_2}\right)\frac{\partial^2 g}{\partial v_\mathrm{t}\partial v_\mathrm{r}}.\end{aligned} \tag{78}$$

Its coefficients are

$$
\begin{aligned}
\frac{\partial}{\partial v_2}\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\right) &= 0, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} &= \sin^2\theta\cos^2\phi \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2}, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}} &= \sin\theta\cos\phi\left(\frac{v_2 - v_{\mathrm{r}}\sin\theta\cos\phi}{v_{\mathrm{t}}}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}}, \\
\frac{\partial}{\partial v_2}\left(\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)\frac{\partial g}{\partial v_{\mathrm{t}}} &= \left(\frac{(1-\sin^2\theta\cos^2\phi)v_{\mathrm{t}} - (v_2 - v_{\mathrm{r}}\sin\theta\cos\phi)\frac{v_2 - v_{\mathrm{r}}\sin\theta\cos\phi}{v_{\mathrm{t}}}}{v_{\mathrm{t}}^2}\right)\frac{\partial g}{\partial v_{\mathrm{t}}}, \\
\left(\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} &= \left(\frac{v_2 - v_{\mathrm{r}}\sin\theta\cos\phi}{v_{\mathrm{t}}}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{t}}^2},
\end{aligned}
\tag{79}
$$

which in the special coordinate system yields

$$
\begin{aligned}
\frac{\partial}{\partial v_2}\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\right) &= 0, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} &= \left(\frac{v_{\mathrm{t}}}{v}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2}, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_2}\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}} &= \left(-\frac{v_{\mathrm{r}}v_{\mathrm{t}}}{v^2}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}}, \\
\frac{\partial}{\partial v_2}\left(\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)\frac{\partial g}{\partial v_{\mathrm{t}}} &= 0, \\
\left(\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} &= \left(\frac{v_{\mathrm{r}}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{t}}^2},
\end{aligned}
\tag{80}
$$

hence

$$
\frac{\partial^2 g}{\partial v_2^2} = \left(\frac{v_{\mathrm{t}}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} - \frac{2v_{\mathrm{r}}v_{\mathrm{t}}}{v^2}\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}} + \left(\frac{v_{\mathrm{r}}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{t}}^2}. \tag{81}
$$

For the v_3 partial derivative, we have :

$$
\begin{aligned}
\frac{\partial^2 g}{\partial v_3^2} &= \frac{\partial}{\partial v_3}\left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\right)\frac{\partial g}{\partial v_{\mathrm{r}}} + \left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} + \left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\frac{\partial v_{\mathrm{t}}}{\partial v_3}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}}, \\
&+ \frac{\partial}{\partial v_3}\left(\frac{\partial v_{\mathrm{t}}}{\partial v_3}\right)\frac{\partial g}{\partial v_{\mathrm{t}}} + \left(\frac{\partial v_{\mathrm{t}}}{\partial v_3}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} + \left(\frac{\partial v_{\mathrm{t}}}{\partial v_3}\frac{\partial v_{\mathrm{r}}}{\partial v_3}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}}.
\end{aligned}
\tag{82}
$$

Its coefficients are

$$
\begin{aligned}
\frac{\partial}{\partial v_3}\left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\right) &= 0, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2} &= \sin^2\theta\sin^2\phi \frac{\partial^2 g}{\partial v_{\mathrm{r}}^2}, \\
\left(\frac{\partial v_{\mathrm{r}}}{\partial v_3}\frac{\partial v_{\mathrm{t}}}{\partial v_3}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}} &= \sin\theta\sin\phi\left(\frac{v_3 - v_{\mathrm{r}}\sin\theta\sin\phi}{v_{\mathrm{t}}}\right)\frac{\partial^2 g}{\partial v_{\mathrm{t}}\partial v_{\mathrm{r}}}, \\
\frac{\partial}{\partial v_2}\left(\frac{\partial v_{\mathrm{t}}}{\partial v_2}\right)\frac{\partial g}{\partial v_{\mathrm{t}}} &= \left(\frac{(1-\sin^2\theta\sin^2\phi)v_{\mathrm{t}} - (v_3 - v_{\mathrm{r}}\sin\theta\sin\phi)\frac{v_3 - v_{\mathrm{r}}\sin\theta\sin\phi}{v_{\mathrm{t}}}}{v_{\mathrm{t}}^2}\right)\frac{\partial g}{\partial v_{\mathrm{t}}}, \\
\left(\frac{\partial v_{\mathrm{t}}}{\partial v_3}\right)^2 \frac{\partial^2 g}{\partial v_{\mathrm{t}}^2} &= \left(\frac{v_3 - v_{\mathrm{r}}\sin\theta\sin\phi}{v_{\mathrm{t}}}\right)^2\frac{\partial^2 g}{\partial v_{\mathrm{t}}^2},
\end{aligned}
\tag{83}
$$

which in the special coordinate system yields

$$
\begin{aligned}
\frac{\partial}{\partial v_3}\left(\frac{\partial v_{\rm r}}{\partial v_3}\right) &= 0,\\
\left(\frac{\partial v_{\rm r}}{\partial v_3}\right)^2\frac{\partial^2 g}{\partial v_{\rm r}^2} &= 0,\\
\left(\frac{\partial v_{\rm r}}{\partial v_3}\frac{\partial v_{\rm t}}{\partial v_3}\right)\frac{\partial^2 g}{\partial v_{\rm t}\partial v_{\rm r}} &= 0,\\
\frac{\partial}{\partial v_3}\left(\frac{\partial v_{\rm t}}{\partial v_3}\right)\frac{\partial g}{\partial v_{\rm t}} &= \left(\frac{1}{v_{\rm t}}\right)\frac{\partial g}{\partial v_{\rm t}},\\
\left(\frac{\partial v_{\rm t}}{\partial v_3}\right)^2\frac{\partial^2 g}{\partial v_{\rm t}^2} &= 0,
\end{aligned}
\tag{84}
$$

hence

$$
\frac{\partial^2 g}{\partial v_3^2} = \frac{1}{v_{\rm t}}\frac{\partial g}{\partial v_{\rm t}}. \tag{85}
$$

B From anisotropy to isotropy

From the anisotropic Rosenbluth formulae, we should recover the isotropic formulae. Let $h(v_r, v_t) = h(v)$ where $v^2 = v_{\rm r}^2 + v_{\rm t}^2$. Then the $\langle \Delta v_{||}\rangle$ partial derivatives yield

$$
\frac{v_{\rm r}}{v}\frac{\partial h}{\partial v_{\rm r}} + \frac{v_{\rm t}}{v}\frac{\partial h}{\partial v_{\rm t}} = \frac{v_{\rm r}}{v}\frac{\partial v}{\partial v_{\rm r}}h' + \frac{v_{\rm t}}{v}\frac{\partial v}{\partial v_{\rm t}}h' = \frac{v_{\rm r}^2}{v^2}h' + \frac{v_{\rm t}^2}{v^2}h' = h' \tag{86}
$$

The $\langle(\Delta v_{||})^2\rangle$ and $\langle(\Delta v_{\perp})^2\rangle$ partial derivatives yield

$$
\begin{aligned}
\frac{\partial^2 g}{\partial v_{\rm r}^2} &= \frac{\partial}{\partial v_{\rm r}}(\frac{\partial v}{\partial v_{\rm r}}g'(v)) = \frac{\partial}{\partial v_{\rm r}}(\frac{v_{\rm r}}{v}g'(v)) = \frac{v - v_{\rm r}^2/v}{v^2}g'(v) + \frac{v_{\rm r}}{v}\frac{\partial v}{\partial v_{\rm r}}g''(v)\\
&= \frac{v_{\rm t}^2}{v^3}g'(v) + \frac{v_{\rm r}^2}{v^2}g''(v)\\
\frac{\partial^2 g}{\partial v_{\rm t}\partial v_{\rm r}} &= \frac{\partial}{\partial v_{\rm t}}(\frac{\partial v}{\partial v_{\rm r}}g') = \frac{\partial}{\partial v_{\rm t}}(\frac{v_{\rm r}}{v}g') = -\frac{v_{\rm r}v_{\rm t}/v}{v^2}g' + \frac{v_{\rm r}}{v}\frac{\partial v}{\partial v_{\rm t}}g''\\
&= -\frac{v_{\rm r}v_{\rm t}}{v^3}g' + \frac{v_{\rm r}v_{\rm t}}{v^2}g''\\
\frac{\partial^2 g}{\partial v_{\rm t}^2} &= \frac{\partial}{\partial v_{\rm t}}(\frac{\partial v}{\partial v_{\rm t}}g') = \frac{\partial}{\partial v_{\rm t}}(\frac{v_{\rm t}}{v}g') = \frac{v - v_{\rm t}^2/v}{v^2}g' + \frac{v_{\rm t}}{v}\frac{\partial v}{\partial v_{\rm t}}g'\\
&= \frac{v_{\rm r}^2}{v^3}g' + \frac{v_{\rm t}^2}{v^2}g''\\
\frac{\partial g}{\partial v_{\rm t}} &= \frac{\partial v}{\partial v_{\rm t}}g' = \frac{v_{\rm t}}{v}g'.
\end{aligned}
\tag{87}
$$

Therefore, after doing the calculations, hence

$$
\left(\frac{v_{\rm r}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\rm r}^2} + \frac{2v_{\rm r}v_{\rm t}}{v^2}\frac{\partial^2 g}{\partial v_{\rm t}\partial v_{\rm r}} + \left(\frac{v_{\rm t}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\rm t}^2} = g'' \tag{88}
$$

and

$$
\left(\frac{v_{\rm t}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\rm r}^2} - \frac{2v_{\rm r}v_{\rm t}}{v^2}\frac{\partial^2 g}{\partial v_{\rm t}\partial v_{\rm r}} + \left(\frac{v_{\rm r}}{v}\right)^2\frac{\partial^2 g}{\partial v_{\rm t}^2} + \frac{1}{v_{\rm t}}\frac{\partial g}{\partial v_{\rm t}} = \frac{2}{v}g' \tag{89}
$$

C Test star velocity components

We have to do an integration over velocity space to compute the Rosenbluth potential and its derivatives. To that end, we use spherical coordinate by fixing $(Oz) = \hat{\boldsymbol{r}}$ and letting $(Ox) \propto \boldsymbol{v}_{\rm t}$. Then $v_{\rm x} = v_{\rm t}$, $v_{\rm y} = 0$ and $v_{\rm z} = v_{\rm r}$.

Since

$$
\begin{aligned}
V_{0\mathrm{x}} &= V_0 \sin\theta\cos\phi, \\
V_{0\mathrm{y}} &= V_0 \sin\theta\sin\phi, \\
V_{0\mathrm{z}} &= V_0 \cos\theta,
\end{aligned}
\tag{90}
$$

then

$$
\begin{aligned}
v_{\mathrm{ax}} &= v_{\mathrm{t}} - V_0 \sin\theta\cos\phi, \\
v_{\mathrm{ay}} &= -V_0 \sin\theta\sin\phi, \\
v_{\mathrm{az}} &= v_{\mathrm{r}} - V_0 \cos\theta,
\end{aligned}
\tag{91}
$$

and therefore

$$
\begin{aligned}
v_{\mathrm{ar}} &= v_{\mathrm{r}} - V_0 \cos\theta, \\
v_{\mathrm{at}}^2 &= (v_{\mathrm{t}} - V_0 \sin\theta\cos\phi)^2 + (V_0 \sin\theta\sin\phi)^2.
\end{aligned}
\tag{92}
$$

The associated energy and angular momentum are

$$
\begin{aligned}
E_{\mathrm{a}} &= \psi(r) - \frac{1}{2}\left[v^2 + V_0^2 - 2V_0\left(v_{\mathrm{r}}\cos\theta + v_{\mathrm{t}}\sin\theta\cos\phi\right)\right], \\
L_{\mathrm{a}} &= r\left(v_{\mathrm{t}}^2 + V_0^2\sin^2\theta - 2v_{\mathrm{t}}V_0\sin\theta\cos\phi\right)^{1/2}.
\end{aligned}
\tag{93}
$$

D Velocity-derivatives of the DF $F(E, L)$

First, we compute the velocity dependency of the energy and the angular momentum

$$
\begin{aligned}
\frac{\partial E_{\mathrm{a}}}{\partial v_{\mathrm{r}}} &= -v_{\mathrm{r}} + V_0\cos\theta \quad ; \quad & \frac{\partial E_{\mathrm{a}}}{\partial v_{\mathrm{t}}} &= -v_{\mathrm{t}} + V_0\sin\theta\cos\phi, \\
\frac{\partial L_{\mathrm{a}}}{\partial v_{\mathrm{r}}} &= 0 \quad ; \quad & \frac{\partial L_{\mathrm{a}}}{\partial v_{\mathrm{t}}} &= \frac{r}{L_{\mathrm{a}}}(v_{\mathrm{t}} - V_0\sin\theta\cos\phi).
\end{aligned}
\tag{94}
$$

Then we just apply the chain rule as many times as we need to to obtain the desire formulae.

E Partial derivatives of F_q (Plummer)

We will need the derivatives of $\mathbb{H}$, which are

$$
\mathbb{H}'(a,b,c,d;x) = \begin{cases}
\dfrac{\Gamma(a+b)x^{a-1}}{\Gamma(c-a)\Gamma(a+d)}\Big[a\,{}_2F_1(a+b, 1+a-c, a+d; x) \\
\quad + \dfrac{(a+b)(1+a-c)}{a+d}x\,{}_2F_1(a+b+1, a-c+2, a+d+1; x)\Big] & \text{if} \quad x \le 1, \\
\\
\dfrac{\Gamma(a+b)x^{-b-1}}{\Gamma(d-b)\Gamma(b+c)}\Big[(-b)\,{}_2F_1(a+b, 1+b-d, b+c; \frac{1}{x}) \\
\quad - \dfrac{(a+b)(1+b-d)}{b+c}x^{-1}\,{}_2F_1(a+b+1, b-d+2, b+c+1; \frac{1}{x})\Big] & \text{if} \quad x \ge 1,
\end{cases}
\tag{95}
$$

and

$$
\mathbb{H}''(a,b,c,d;x) = \begin{cases} \frac{\Gamma(a+b)x^{a-2}}{\Gamma(c-a)\Gamma(a+d)}\Big[a(a-1)\,{}_2F_1(a+b,1+a-c,a+d;x) \\ +\frac{2a(a+b)(1+a-c)}{a+d}x\,{}_2F_1(a+b+1,a-c+2,a+d+1;x) \\ +\frac{(a+b)(1+a-c)}{a+d}\frac{(a+b+1)(a-c+2)}{a+d+1}x^2 \\ \times\,{}_2F_1(a+b+2,a-c+3,a+d+2;x)\Big] & \text{if} \quad x \le 1, \\ \\ \frac{\Gamma(a+b)x^{-b-2}}{\Gamma(d-b)\Gamma(b+c)}\Big[b(b+1)\,{}_2F_1(a+b,1+b-d,b+c;\frac{1}{x}) \\ +\frac{(2b+2)(a+b)(1+b-d)}{b+c}x^{-1}\,{}_2F_1(a+b+1,b-d+2,b+c+1;\frac{1}{x}) \\ +\frac{(a+b)(1+b-d)}{b+c}\frac{(a+b+1)(b-d+2)}{b+c+1}x^{-2} \\ \times\,{}_2F_1(a+b+2,b-d+3,b+c+2;\frac{1}{x})\Big] & \text{if} \quad x \ge 1, \end{cases} \tag{96}
$$

Then, for $q \notin \{0,2\}$, we have for first-order

$$
\begin{aligned}
\frac{\partial F_q}{\partial E}(E,L) &= \frac{3\Gamma(6-q)E^{3/2-q}}{2(2\pi)^{5/2}\Gamma(q/2)}\left[E\left(\frac{7}{2}-q\right)\mathbb{H}(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E}) - \frac{L^2}{2}\mathbb{H}'(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right], \\
\frac{\partial F_q}{\partial L}(E,L) &= \frac{3\Gamma(6-q)}{2(2\pi)^{5/2}\Gamma(q/2)}E^{5/2-q}L\mathbb{H}'(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E}).
\end{aligned} \tag{97}
$$

For second order, we have

$$
\begin{aligned}
\frac{\partial^2 F_q}{\partial E^2}(E,L) &= \frac{3\Gamma(6-q)E^{3/2-q}}{2(2\pi)^{5/2}\Gamma(q/2)}\left[\left(\frac{7}{2}-q\right)\left(\frac{5}{2}-q\right)\mathbb{H}(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right. \\
&- \left.(5-2q)\frac{L^2}{2E}\mathbb{H}'(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E}) + \frac{L^4}{4E^2}\mathbb{H}''(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right], \\
\frac{\partial^2 F_q}{\partial E \partial L}(E,L) &= \frac{3\Gamma(6-q)LE^{1/2-q}}{2(2\pi)^{5/2}\Gamma(q/2)}\left[\left(\frac{5}{2}-q\right)E\mathbb{H}'(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right. \\
&- \left.\frac{L^2}{2}\mathbb{H}''(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right], \\
\frac{\partial^2 F_q}{\partial L^2}(E,L) &= \frac{3\Gamma(6-q)E^{3/2-q}}{2(2\pi)^{5/2}\Gamma(q/2)}\left[E\mathbb{H}'(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E}) + L^2\mathbb{H}''(0,\frac{q}{2},\frac{9}{2}-q,1;\frac{L^2}{2E})\right].
\end{aligned} \tag{98}
$$

For the isotropic case $q = 0$, the derivatives are

$$
\begin{aligned}
\frac{\partial F_0}{\partial E}(E) &= \frac{3}{\pi^3}(2E)^{5/2}, \\
\frac{\partial^2 F_0}{\partial E^2}(E) &= \frac{15}{\pi^3}(2E)^{3/2}.
\end{aligned} \tag{99}
$$

while in the highly-radially anisotropic case $q=2$, for $L^2 \leq 2E$ (null if otherwise)

$$\begin{aligned}
\frac{\partial F_2}{\partial E}(E,L) &= \frac{18}{(2\pi)^3}(2E-L^2)^{1/2}, \\
\frac{\partial F_0}{\partial L}(E,L) &= \frac{-18L}{(2\pi)^3}(2E-L^2)^{1/2}, \\
\frac{\partial^2 F_2}{\partial E^2}(E,L) &= \frac{18}{(2\pi)^3}(2E-L^2)^{-1/2}, \\
\frac{\partial^2 F_0}{\partial E \partial L}(E,L) &= \frac{-18L}{(2\pi)^3}(2E-L^2)^{-1/2}, \\
\frac{\partial^2 F_0}{\partial L^2}(E,L) &= -\frac{18}{(2\pi)^3}(2E-L^2)^{1/2} + \frac{18L^2}{(2\pi)^3}(2E-L^2)^{-1/2}.
\end{aligned} \tag{100}$$

F Generating function $J_{\mathrm{r}}(E,L)$

$J_{\mathrm{r}}(E,L)$ is a function of the form

$$J_{\mathrm{r}}(E,L) = \frac{1}{\pi}\Big(G(r_{\max}(E,L),E,L) - G(r_{\min}(E,L),E,L)\Big), \tag{101}$$

where $v_{\mathrm{r}}(r_{\min},E,L) = v_{\mathrm{r}}(r_{\max},E,L) = 0$ and

$$G(x,E,L) = \int_0^x v_{\mathrm{r}}(x,E,L)\mathrm{d}x. \tag{102}$$

Differentiation by E using chain rule and $\partial G/\partial r(x,E,L) = v_{\mathrm{r}}(x,E,L)$, yields

$$\begin{aligned}
\frac{\partial J_{\mathrm{r}}}{\partial E}(E,L) &= \frac{1}{\pi}\left(\frac{\partial G}{\partial E}(r_{\min},E,L) - \frac{\partial G}{\partial E}(r_{\min},E,L)\right) \\
&= \frac{1}{\pi}\int_{r_{\min}(E,L)}^{r_{\max}(E,L)} \frac{\partial v_{\mathrm{r}}}{\partial E}(r,E,L)\mathrm{d}r \\
&= -\frac{1}{\pi}\int_{r_{\min}(E,L)}^{r_{\max}(E,L)} \frac{\mathrm{d}r}{v_{\mathrm{r}}(r,E,L)} = -\frac{T}{2\pi}.
\end{aligned}$$

By the same method,

$$\begin{aligned}
\frac{\partial J_{\mathrm{r}}}{\partial L}(E,L) &= \frac{1}{\pi}\int_{r_{\min}(E,L)}^{r_{\max}(E,L)} \frac{\partial v_{\mathrm{r}}}{\partial L}(r,E,L)\mathrm{d}r \\
&= \frac{1}{\pi}\int_{r_{\min}(E,L)}^{r_{\max}(E,L)} \frac{-L\mathrm{d}r}{r^2 v_{\mathrm{r}}(r,E,L)} = -\frac{\Theta}{2\pi}
\end{aligned}$$

G Plummer model's orbit-averaging

Consider the equation $z^2(x) = Y(x) - (Ex - L_{\mathrm{c}}^2) = \Upsilon(x)$. From its definition, we have that

$$Y(x) = 2r^2\psi(r) = x\psi(\sqrt{x/2}) = -\frac{x}{\sqrt{1+x/2}}. \tag{103}$$

Notice that by definition (see eq. (34)), $\Upsilon(x)$ reaches its extremum at x_{c} with $\Upsilon(x_{\mathrm{c}}) = 0$.

Its derivatives are

$$\Upsilon'(x) = -\frac{4+x}{\sqrt{2}(2+x)^{3/2}} - E,$$
$$\Upsilon''(x) = \frac{x+8}{2\sqrt{2}(2+x)^{5/2}} > 0,$$

where $\Upsilon'(x_\mathrm{c}) = 0$. Therefore $\Upsilon'(x)$ is negative in $[0, x_\mathrm{c}]$ and positive in $[x_\mathrm{c}, +\infty[$, meaning that $\Upsilon(x)$ decreasing down to 0 in $[0, x_\mathrm{c}]$, then increases to infinity in $[x_\mathrm{c}, +\infty[$.

This shows that $z^2 = \Upsilon(x)$ has two solutions in x. We define $x = \varphi(z)$ as done in eq, that is, by taking the solution $x < x_\mathrm{c}$ for $z < 0$ and taking the solution $x > x_\mathrm{c}$ for $z > 0$.

By the bijection theorem $\varphi(z)$ is a continuous inverse bijection. Since $z(x)$ is C^1 with non-vanishing derivative for $x \neq x_\mathrm{c}$, so is its inverse $\varphi(z)$. To prove that it is differentiable at $z = 0$, one only has to show that a finite derivative exists at $z = 0$ (mean value theorem). Let $\epsilon > 0$. Then

$$z(x_\mathrm{c}+\epsilon) = +\sqrt{\Upsilon(x_\mathrm{c}+\epsilon)} = +\sqrt{0 + \epsilon\Upsilon'(x_\mathrm{c}) + \frac{\epsilon^2}{2}\Upsilon''(x_\mathrm{c}) + o(\epsilon^2)} = +\epsilon\sqrt{\frac{\Upsilon''(x_\mathrm{c})}{2}} + o(\epsilon),$$
$$z(x_\mathrm{c}-\epsilon) = -\sqrt{\Upsilon(x_\mathrm{c}-\epsilon)} = -\sqrt{0 - \epsilon\Upsilon'(x_\mathrm{c}) + \frac{\epsilon^2}{2}\Upsilon''(x_\mathrm{c}) + o(\epsilon^2)} = -\epsilon\sqrt{\frac{\Upsilon''(x_\mathrm{c})}{2}} + o(\epsilon),$$

hence the left and right derivatives coincide and $z'(x_\mathrm{c}) = \sqrt{\Upsilon''(x_\mathrm{c})/2} > 0$, which is non-zero. This means that the inversion bijection is differentiable at $z = 0$, with derivative $\varphi'(0) = \sqrt{2/\Upsilon''(x_\mathrm{c})} > 0$.

Furthermore,

$$z^2 = Y(x) - (Ex - L_\mathrm{c}^2),$$

$$z^2 = -\frac{x}{\sqrt{1+x/2}} - Ex + L_\mathrm{c}^2,$$

$$z^2 - L_\mathrm{c}^2 = \Big(-\frac{1}{\sqrt{1+x/2}} - E\Big)x,$$

$$\frac{z^2 - L_\mathrm{c}^2 + Ex}{x} = -\frac{1}{\sqrt{1+x/2}},$$

$$(z^2 - L_\mathrm{c}^2 + Ex)^2(1+x/2) = x^2.$$

This is a degree-3 polynomial equation, with at least the two positive-real solutions of $z^2 = \Upsilon(x)$. There are no other positive real solutions, but because it is a degree-3 polynomial equation with real coefficients and strictly more than 1 real solution, it more have a third real solution, which must be negative. Viète's formulae give us analytical expressions for those solutions.

Defining

$$a = E^2/2,$$
$$b = (z^2 - L_\mathrm{c}^2)E + E^2 - 1,$$
$$c = 2(z^2 - L_\mathrm{c}^2)E + (z^2 - L_\mathrm{c}^2)^2/2,$$
$$d = (z^2 - L_\mathrm{c}^2)^2.$$

the equation becomes $ax^3 + bx^2 + cx + d = 0$. Setting $x = t - b/(3a)$ and

$$
\begin{aligned}
p &= (3ac - b^2)/(3a^2), \\
q &= (2b^3 - 9abc + 27a^2 d)/(27a^3),
\end{aligned}
$$

the equation to solve is $t^3 + pt + q = 0$, with the three roots given by

$$
t_k = 2\sqrt{-\frac{p}{3}} \cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right]. \tag{104}
$$

One must then order the three roots, discard the negative one, and select the smaller or the bigger root depending on the sign of z.

Using the same method, we may also obtain an analytical expression for x_{c}, since the condition $\Upsilon'(x_{\mathrm{c}}) = 0$ (only one positive solution) is equivalent to the cubic equation

$$
2E^2(2 + x)^3 - (4 + x)^2 = 0, \tag{105}
$$

which has only one real solution $(-1 < E < 0)$. Defining

$$
\begin{aligned}
a &= E^2/2, \\
b &= 12E^2 - 1, \\
c &= 24E^2 - 8, \\
d &= 16(E^2 - 1),
\end{aligned}
$$

and defining p, q the same way, we can express the solution as

$$
x_{\mathrm{c}}(E) = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{106}
$$