

# Notes

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## 1 Self-gravitating thin disk with bulb and dark halo

Consider a self gravitating disk with surface density  $\Sigma$  and gravitational potential  $\Phi$ . Let  $\mathbf{v}$  be its velocity field and  $P$  its pression field. Then it is described by the system

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)(\mathbf{v}) = -\frac{1}{\Sigma} \nabla P - \nabla \Phi, \quad (2)$$

$$\Delta \Phi = 4\pi G \Sigma \delta(z). \quad (3)$$

Here,  $\Phi = \Phi_{\text{bulb}} + \Phi_{\text{disk}} + \Phi_{\text{DH}} = \Phi_{\text{bulb}} + \Phi_{\text{sg}}$ . Furthermore,  $\Phi_{\text{bulb}}$  is such that it doesn't yield any contribution to the surface density, i.e.  $\Delta \Phi_{\text{bulb}} = 0$ , nor the pressure field, hence  $\Sigma = \Sigma_{\text{disk}} + \Sigma_{\text{DH}}$  and  $P = P_{\text{disk}} + P_{\text{DH}}$ . Letting  $q = \Sigma_{\text{disk}}/\Sigma$ , we have

$$\Sigma = \Sigma_{\text{disk}} + \Sigma_{\text{DH}} = \Sigma_{\text{disk}} \left( 1 + \frac{\Sigma_{\text{DH}}}{\Sigma_{\text{disk}}} \right) = \frac{1}{q} \Sigma_{\text{disk}}. \quad (4)$$

Suppose that we have a polytrope gas such that  $P = \kappa \Sigma^\Gamma$  and let

$$\psi = \int \frac{dP(\Sigma)}{\Sigma} \Leftrightarrow \nabla \psi = \frac{\nabla P}{\Sigma}. \quad (5)$$

Then

$$\psi = \frac{\kappa \Gamma}{\Gamma - 1} \Sigma^{\Gamma-1}. \quad (6)$$

Letting  $\Psi = \Phi + \psi$ , we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)(\mathbf{v}) = -\nabla \Psi. \quad (7)$$

If  $\Sigma_{\text{disk}} \propto \Sigma_{\text{DH}}$  then  $q \in [0, 1]$  is constant and this system of equation can be rewritten as

$$\frac{\partial \Sigma_{\text{disk}}}{\partial t} + \frac{1}{r} \frac{\partial (r \Sigma_{\text{disk}} v_r)}{\partial r} + \frac{1}{r} \frac{\partial (\Sigma_{\text{disk}} v_\theta)}{\partial \theta} = 0, \quad (8)$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{(v_\theta)^2}{r} = -\frac{\partial \Psi}{\partial r}, \quad (9)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad (10)$$

$$\Delta \Phi_{\text{sg}} = 4\pi G \frac{1}{q} \Sigma_{\text{disk}} \delta(z). \quad (11)$$

Let  $M$  be the total mass of the galactic disk+bulb. Let  $\Phi_{\text{bulb}} = -GM(1-x)/\sqrt{c^2+r^2}$  with  $x \in [0, 1]$ . An equilibrium state is given by the Plummer equilibrium, such that, letting  $\xi = (r^2 - a^2)/(r^2 + a^2)$ , that is,  $r/a = \sqrt{(1+\xi)/(1-\xi)}$ ,

$$\begin{aligned} v_r^0 &= 0, & (v_\theta^0)^2 &= r \frac{\partial \Psi^0}{\partial r}, \\ \psi^0 &= \frac{\kappa \Gamma}{\Gamma - 1} (\Sigma^0)^{\Gamma-1} = \frac{\kappa \Gamma}{(\Gamma - 1)q^{\Gamma-1}} (\Sigma_{\text{disk}}^0)^{\Gamma-1}, \\ \Sigma_{\text{disk}}^0 &= \frac{xM}{2\pi a^2} \frac{1}{(1 + (r/a)^2)^{3/2}} = \frac{xM}{2\pi a^2} \left( \frac{1-\xi}{2} \right)^{3/2}, \\ \Phi_{\text{sg}}^0 &= -\frac{GMx}{aq} \frac{1}{\sqrt{1 + (r/a)^2}} = -\frac{GMx}{aq} \left( \frac{1-\xi}{2} \right)^{1/2}. \end{aligned}$$

Therefore, letting

$$\varepsilon = \frac{U(x=1, q=1)}{|E(x=1, q=1)|} = \frac{3a\kappa\Gamma}{GM} \left( \frac{M}{2\pi a^2} \right)^{\Gamma-1},$$

we obtain

$$\begin{aligned} \Psi^0 &= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} - \frac{GMx}{aq} \left( \frac{1-\xi}{2} \right)^{1/2} + \frac{\kappa\Gamma}{(\Gamma-1)} (\Sigma_{\text{disk}}^0/q)^{\Gamma-1}, \\ &= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} + \frac{GM}{a} \left[ -\left( \frac{x}{q} \right) \left( \frac{1-\xi}{2} \right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-1}}{3(\Gamma-1)} \left( \frac{1-\xi}{2} \right)^{3(\Gamma-1)/2} \right], \\ &= \frac{GMx}{aq} \left[ -\frac{a}{c} \frac{q(1-x)}{x} \frac{1}{\sqrt{1+(r/c)^2}} - \left( \frac{1-\xi}{2} \right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-2}}{3(\Gamma-1)} \left( \frac{1-\xi}{2} \right)^{3(\Gamma-1)/2} \right], \\ (v_\theta^0)^2 &= r \frac{\partial \Psi^0}{\partial r} = r \frac{\partial \Psi^0}{\partial \xi} \frac{\partial \xi}{\partial r} = 4 \left( \frac{r}{a} \right)^2 \left( \frac{1-\xi}{2} \right)^2 \frac{\partial \Psi^0}{\partial \xi} = (1+\xi)(1-\xi) \frac{\partial \Psi^0}{\partial \xi}, \\ &= \frac{GM}{a} \left[ \frac{a(1-x)}{c} \left( \frac{r}{c} \right)^2 \left( \frac{1}{1+(r/c)^2} \right)^{3/2} + \frac{x}{q} \left( \frac{1+\xi}{2} \right) \left( \frac{1-\xi}{2} \right)^{1/2} \left( 1 - \varepsilon \left( \frac{x}{q} \right)^{\Gamma-2} \left( \frac{1-\xi}{2} \right)^{\frac{3\Gamma}{2}-2} \right) \right], \\ &= \frac{GMx}{aq} \left[ \frac{a}{c} \frac{q(1-x)}{x} \left( \frac{r}{c} \right)^2 \left( \frac{1}{1+(r/c)^2} \right)^{3/2} + \left( \frac{1+\xi}{2} \right) \left( \frac{1-\xi}{2} \right)^{1/2} \left( 1 - \varepsilon \left( \frac{x}{q} \right)^{\Gamma-2} \left( \frac{1-\xi}{2} \right)^{\frac{3\Gamma}{2}-2} \right) \right], \end{aligned}$$

with

$$\frac{1-\xi}{2} = \frac{a^2}{r^2+a^2} = \frac{1}{1+(r/a)^2}; \quad \frac{1+\xi}{2} = \frac{r^2}{r^2+a^2} = \frac{(r/a)^2}{1+(r/a)^2}.$$

Note that for  $x = 1$  (no bulb) and  $q = 1$  (purely self-gravitating system), we recover the expression of Toomre

$$v_\theta^0 = \left( \frac{GM}{a} \right)^{1/2} \left( \frac{1+\xi}{2} \right)^{1/2} \left( \frac{1-\xi}{2} \right)^{1/4} \sqrt{1 - \varepsilon \left( \frac{1-\xi}{2} \right)^{\frac{3\Gamma}{2}-2}}.$$

The perturbative equation at 1st order are

$$\begin{aligned} \frac{\partial v_r^p}{\partial t} + \frac{v_\theta^0}{r} \frac{\partial v_r^p}{\partial \theta} - 2 \frac{v_\theta^0 v_\theta^p}{r} &= -\frac{\partial \Psi^p}{\partial r}, \\ \frac{\partial v_\theta^p}{\partial t} + v_r^p \frac{\partial v_\theta^0}{\partial r} + \frac{v_\theta^0}{r} \frac{\partial v_\theta^p}{\partial \theta} + \frac{v_r^p v_\theta^0}{r} &= -\frac{1}{r} \frac{\partial \Psi^p}{\partial \theta}, \\ \Delta \Phi_{\text{sg}}^p &= 4\pi G \frac{1}{q} \Sigma_{\text{disk}}^p \delta(z), \\ \psi^p &= \frac{\kappa \Gamma}{q^{\Gamma-1}} (\Sigma_{\text{disk}}^0)^{\Gamma-2} \Sigma_{\text{disk}}^p. \end{aligned}$$

We define  $X^p(r, \theta, t) = \sum_{m \in \mathbb{Z}} X_m^p(r, t) e^{im\theta}$  and look for a temporal dependency in  $e^{-i\omega t}$ . Aoki & Iye say that there is this following correspondance between surface density and gravitational potential through the Poisson equation:

$$(\Sigma_{\text{disk}})_m^p(r, \theta) = \frac{xM}{2\pi a^2} \left( \frac{1-\xi}{2} \right)^{3/2} \sum_{n=|m|}^{\infty} a_n^m \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}, \quad (12)$$

$$(\Phi_{\text{sg}})_m^p(r, \theta) = -\frac{GMx}{aq} \left( \frac{1-\xi}{2} \right)^{1/2} \sum_{n=|m|}^{\infty} \frac{a_n^m}{2n+1} \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}, \quad (13)$$

$$(\psi)_m^p(r, \theta) = \kappa \Gamma \left( \frac{M}{2\pi a^2} \right)^{\Gamma-1} \left( \frac{x}{q} \right)^{\Gamma-1} \left( \frac{1-\xi}{2} \right)^{3/2(\Gamma-1)} \sum_{n=|m|}^{\infty} a_n^m \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}, \quad (14)$$

$$(\Psi)_m^p(r, \theta) = \frac{GMx}{aq} \left( \frac{1-\xi}{2} \right)^{1/2} \sum_{n=|m|}^{\infty} \left[ \frac{\varepsilon}{3} \left( \frac{x}{q} \right)^{\Gamma-2} \left( \frac{1-\xi}{2} \right)^{\frac{3\Gamma}{2}-2} - \frac{1}{2n+1} \right] a_n^m \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}. \quad (15)$$

Let us decompose the velocity components based on their equilibrium expression:

$$(v_r)_m^p = i \frac{m}{|m|} \left( \frac{GMx}{aq} \right)^{1/2} \left( \frac{1+\xi}{2} \right)^{-1/2} \left( \frac{1-\xi}{2} \right)^{1/4} \sum_{n=|m|}^{\infty} b_n^m \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}, \quad (16)$$

$$(v_\theta)_m^p = \left( \frac{GMx}{aq} \right)^{1/2} \left( \frac{1+\xi}{2} \right)^{-1/2} \left( \frac{1-\xi}{2} \right)^{1/4} \sum_{n=|m|}^{\infty} c_n^m \widehat{P_n^{|m|}}(\xi) e^{-i\omega t}. \quad (17)$$

Letting  $X_m^p = X^1 e^{-i\omega t}$ , this yields the set of equations

$$i(-\omega + m\Omega)(\Sigma_{\text{disk}})^1 + \frac{1}{r} \frac{d(r\Sigma_0(v_r)^1)}{dr} + \frac{im\Sigma^0(v_\theta)^1}{r} = 0, \quad (18)$$

$$\frac{d(\Psi)^1}{dr} + i(-\omega + m\Omega)(v_r)^1 - 2\Omega(v_\theta)^1 = 0, \quad (19)$$

$$im \frac{(\Psi)^1}{r} + \frac{\alpha^2}{2\Omega}(v_r)^1 + i(-\omega + m\Omega)(v_\theta)^1 = 0, \quad (20)$$

where  $\Omega = v_\theta^0/r$  is the angular velocity and  $\alpha^2 = 4\Omega^2[1 + r/(2\Omega) \cdot (d\Omega/dr)]$  is the epicyclic frequency. Using the relation

$$\int_{-1}^1 d\xi \widehat{P_n^{|m|}}(\xi) \widehat{P_l^{|m|}}(\xi) = \delta_{nl},$$

and defining  $\Omega_{\text{ref}} = (GMx/(a^3q))^{1/2}$ ,  $\Sigma_{\text{ref}} = xM/(2\pi a^2)$  such that  $\widehat{\omega} = \omega/\Omega_{\text{ref}}$ ,  $\widehat{\Omega} = \Omega/\Omega_{\text{ref}}$ ,  $\widehat{\alpha} = \alpha/\Omega_{\text{ref}}$ ,  $\widehat{\Sigma} = \Sigma/\Sigma_{\text{ref}}$  and  $\lambda = \frac{|m|}{m} \widehat{\omega}$ , we obtain the matrix equations

$$\begin{aligned} \sum_{n=|m|}^{\infty} A_{ln} a_n^m + \sum_{n=|m|}^{\infty} B_{ln} b_n^m + \sum_{n=|m|}^{\infty} C_{ln} c_n^m &= \lambda a_l^m, \\ \sum_{n=|m|}^{\infty} D_{ln} a_n^m + \sum_{n=|m|}^{\infty} A_{ln} b_n^m + \sum_{n=|m|}^{\infty} F_{ln} c_n^m &= \lambda b_l^m, \\ \sum_{n=|m|}^{\infty} G_{ln} a_n^m + \sum_{n=|m|}^{\infty} H_{ln} b_n^m + \sum_{n=|m|}^{\infty} A_{ln} c_n^m &= \lambda c_l^m, \end{aligned}$$

where we defined

$$\begin{aligned}
A_{ln} &= |m| \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{[m]}}(\xi), \\
B_{ln} &= 4 \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \left(\frac{1-\xi}{2}\right)^{1/2} \frac{d}{d\xi} \left[ \left(\frac{1-\xi}{2}\right)^{5/4} \widehat{P_n^{[m]}}(\xi) \right], \\
C_{ln} &= |m| \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} \widehat{P_n^{[m]}}(\xi), \\
D_{ln} &= 4 \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \left(\frac{1-\xi}{2}\right)^{5/4} \left(\frac{1+\xi}{2}\right) \frac{d}{d\xi} \left[ \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right) \left(\frac{1-\xi}{2}\right)^{1/2} \widehat{P_n^{[m]}}(\xi) \right], \\
F_{ln} &= 2 \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{[m]}}(\xi), \\
G_{ln} &= -|m| \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right) \widehat{P_n^{[m]}}(\xi), \\
H_{ln} &= \int_{-1}^1 d\xi \widehat{P_l^{[m]}}(\xi) \frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} \widehat{P_n^{[m]}}(\xi).
\end{aligned}$$

with

$$\begin{aligned}
\widehat{\Omega}(\xi) &= \sqrt{\frac{1-\xi}{1+\xi}} \sqrt{\frac{a}{c} \frac{q(1-x)}{x} \left(\frac{r}{c}\right)^2 \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left(\frac{1+\xi}{2}\right) \left(\frac{1-\xi}{2}\right)^{1/2} \left[1 - \varepsilon \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right]}, \\
\frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} &= 2\widehat{\Omega}(\xi) \left[1 + \frac{r}{2\widehat{\Omega}} \frac{d\widehat{\Omega}}{dr}\right] = 2\widehat{\Omega}(\xi) \left[1 + \frac{(1+\xi)(1-\xi)}{2\widehat{\Omega}} \frac{d\widehat{\Omega}}{d\xi}\right],
\end{aligned}$$

where

$$\frac{d(r/a)}{d\xi} = \frac{d}{d\xi} \left[ \sqrt{\frac{1+\xi}{1-\xi}} \right] = (1-\xi)^{-3/2} (1+\xi)^{-1/2}$$

Since  $(r/c)^2 = (a/c)^2 \cdot (r/a)^2 = (a/c)^2 \cdot (1+\xi)/(1-\xi)$ , we have that

$$\widehat{\Omega}(\xi) = \left(\frac{1-\xi}{2}\right)^{3/4} \sqrt{\left(\frac{a}{c}\right)^3 \frac{q(1-x)}{x} \left(\frac{1-\xi}{2}\right)^{-3/2} \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left[1 - \varepsilon \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right]}.$$

## 2 Case $\Gamma = 4/3$

The expressions become

$$\begin{aligned}
\widehat{\Omega}(\xi) &= \left(\frac{1-\xi}{2}\right)^{3/4} \sqrt{\left(\frac{a}{c}\right)^3 \frac{q(1-x)}{x} \left(\frac{1-\xi}{2}\right)^{-3/2} \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left[1 - \varepsilon \left(\frac{q}{x}\right)^{\frac{2}{3}}\right]}, \\
\frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} &= 2\widehat{\Omega}(\xi) \left[1 + \frac{(1+\xi)(1-\xi)}{2\widehat{\Omega}} \frac{d\widehat{\Omega}}{d\xi}\right],
\end{aligned}$$

with the matrix elements

$$\begin{aligned}
A_{ln} &= |m| \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \widehat{\Omega(\xi)} \widehat{P_n^{[m]}(\xi)}, \\
B_{ln} &= 4 \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \left( \frac{1-\xi}{2} \right)^{1/2} \frac{d}{d\xi} \left[ \left( \frac{1-\xi}{2} \right)^{5/4} \widehat{P_n^{[m]}(\xi)} \right], \\
C_{ln} &= |m| \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \left( \frac{1-\xi}{2} \right)^{3/4} \left( \frac{1+\xi}{2} \right)^{-1} \widehat{P_n^{[m]}(\xi)}, \\
D_{ln} &= 4 \left( \frac{1}{2n+1} - \frac{\varepsilon}{3} \left( \frac{q}{x} \right)^{\frac{2}{3}} \right) \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \left( \frac{1-\xi}{2} \right)^{5/4} \left( \frac{1+\xi}{2} \right) \frac{d}{d\xi} \left[ \left( \frac{1-\xi}{2} \right)^{1/2} \widehat{P_n^{[m]}(\xi)} \right], \\
F_{ln} &= 2 \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \widehat{\Omega(\xi)} \widehat{P_n^{[m]}(\xi)}, \\
G_{ln} &= -|m| \left( \frac{1}{2n+1} - \frac{\varepsilon}{3} \left( \frac{q}{x} \right)^{\frac{2}{3}} \right) \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \left( \frac{1-\xi}{2} \right)^{3/4} \widehat{P_n^{[m]}(\xi)}, \\
H_{ln} &= \int_{-1}^1 d\xi \widehat{P_l^{[m]}(\xi)} \frac{\widehat{\alpha^2(\xi)}}{2\widehat{\Omega(\xi)}} \widehat{P_n^{[m]}(\xi)}.
\end{aligned}$$

Integrals  $A_{ln}$  and  $F_{ln}$  are proportional. With the addition of  $H_{ln}$ , those 3 integrals must be computed numerically because of the non-trivial shift in their expression induced by the bulb potential. As for  $B_{ln}$ ,  $C_{ln}$ ,  $D_{ln}$  and  $G_{ln}$ , their can be expressed in terms of the two following integrals

$$\widehat{I}(l, n) = \int_{-1}^1 d\xi \left( \frac{1-\xi}{2} \right)^{3/4} \widehat{P_l^{[m]}(\xi)} \widehat{P_n^{[m]}(\xi)}, \quad (21)$$

$$\widehat{J}(l, n) = \int_{-1}^1 d\xi \left( \frac{1-\xi}{2} \right)^{3/4} \left( \frac{1+\xi}{2} \right)^{-1} \widehat{P_l^{[m]}(\xi)} \widehat{P_n^{[m]}(\xi)}, \quad (22)$$

as

$$\begin{aligned}
B_{ln} &= \frac{1}{2} \left[ \sqrt{\frac{(2l+1)(l+m+1)(l-m+1)}{2l+3}} \widehat{J}(l+1, n) + \widehat{J}(l, n) \right. \\
&\quad \left. - \sqrt{\frac{(2l+1)(l+m)(l-m)}{2l-1}} \widehat{J}(l-1, n) \right], \\
C_{ln} &= m \widehat{J}(l, n), \\
D_{ln} &= \frac{1}{2} \left( \frac{1}{2n+1} - \frac{\varepsilon}{3} \left( \frac{q}{x} \right)^{\frac{2}{3}} \right) \left[ - \sqrt{\frac{(2n+1)(n+m+1)(n-m+1)}{2n+3}} \widehat{I}(l, n+1) \right. \\
&\quad \left. - \widehat{I}(l, n) \right. \\
&\quad \left. + \sqrt{\frac{(2n+1)(n+m)(n-m)}{2n-1}} \widehat{I}(l, n-1) \right], \\
G_{ln} &= -|m| \left( \frac{1}{2n+1} - \frac{\varepsilon}{3} \left( \frac{q}{x} \right)^{\frac{2}{3}} \right) \widehat{I}(l, n).
\end{aligned}$$

where  $\widehat{I}(l, n)$  and  $\widehat{J}(l, n)$  can be computed by recursion and using the symmetry  $l \leftrightarrow n$ . Defining

$$\widehat{I}'(l, n) = \int_{-1}^1 d\xi \xi \left( \frac{1-\xi}{2} \right)^{3/4} \widehat{P_l^{[m]}(\xi)} \widehat{P_n^{[m]}(\xi)}, \quad (23)$$

$$\widehat{J}'(l, n) = \int_{-1}^1 d\xi \xi \left( \frac{1-\xi}{2} \right)^{3/4} \left( \frac{1+\xi}{2} \right)^{-1} \widehat{P_l^{[m]}(\xi)} \widehat{P_n^{[m]}(\xi)}, \quad (24)$$

Starting from(Aoki79, A16)

$$\begin{aligned}\hat{I}(l, n) &= \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}} \hat{I}(l, n-1) - \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}} \hat{I}(l, n-2), \\ \hat{I}'(l, n-1) &= \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}} \hat{I}(l+1, n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} \hat{I}(l-1, n-1),\end{aligned}$$

hence to compute until  $l, n = m + N$ , we need to compute  $\hat{I}(l', m)$  until  $l' = m + 2N$ . By convention (for the recursion), we have set  $\hat{I}(l', n') = 0$  for  $l' < m$  or  $n' < m$ . We initialize with

$$\begin{aligned}\hat{I}(l, m) &= \frac{l - 3/4 - m - 1}{l + 3/4 + m + 1} \sqrt{\frac{(l+m)(2l+1)}{(l-m)(2l-1)}} \hat{I}(l-1, m), \\ \hat{I}(m, m) &= 2^m \prod_{k=0}^m \frac{2k+1}{3/4 + m + 1 + k}.\end{aligned}$$

We proceed as follows:

- Compute the line  $n = m$ :  $\hat{I}(m, m), \hat{I}(m+1, m), \dots, \hat{I}(m+2N, m)$
- Complete the line  $l = m$  by symmetry
- Compute the line  $n = m+1$ :  $\hat{I}(m+1, m+1), \hat{I}(m+2, m+1), \dots, \hat{I}(m+2N-1, m+1)$
- Complete the line  $l = m+1$  by symmetry
- Compute the line  $n = m+2$ :  $\hat{I}(m+2, m+2), \hat{I}(m+3, m+2), \dots, \hat{I}(m+2N-2, m+2)$
- ...
- Compute the line  $n = m+N-1$ :  $\hat{I}(m+N-1, m+N-1), \hat{I}(m+N+1, m+N-1)$ .
- Complete the line  $l = m+N-1$  by symmetry
- Compute the line  $n = m+N$ :  $\hat{I}(m+N, m+N)$ .

As for  $\hat{J}$ , let

$$\hat{I}_\alpha(l, m) = \frac{(-1)^{l-m} (2m-1)!! 2^{m+1} \Gamma(\alpha+1) \Gamma(\alpha+m+1) (l+m)!}{\Gamma(\alpha+1-l+m) \Gamma(\alpha+m+l+2) (l-m)!} \quad (25)$$

with  $\hat{I}_{3/4}(l, m) = \hat{I}(l, m)$ . We can also compute it by recursion using the formulae

$$\begin{aligned}\hat{J}(l, n) &= \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}} \hat{J}'(l, n-1) - \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}} \hat{J}(l, n-2), \\ \hat{J}'(l, n-1) &= \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}} \hat{J}(l+1, n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} \hat{J}(l-1, n-1),\end{aligned}$$

hence to compute until  $l, n = m + N$ , we need to compute  $\hat{I}(l', m)$  until  $l' = m + 2N$ . By convention (for the recursion), we have set  $\hat{I}(l', n') = 0$  for  $l' < m$  or  $n' < m$ . We initialize with

$$\begin{aligned}\hat{J}(l, m) &= \sqrt{\frac{(l-m)(l-m-1)(2l+1)}{(l+m)(l+m-1)(2l-3)}} \hat{J}(l-2, m) + 4 \sqrt{\frac{(2m+1)(2l+1)(2l-1)}{2m(l+m)(l+m-1)}} \hat{I}_{7/3}(l-1, m-1), \\ \hat{J}(m, m) &= \frac{2^m}{m} \prod_{k=1}^m \frac{2k+1}{3/4 + m + k}, \\ \hat{J}(m+1, m) &= -\frac{7}{4} \frac{2^m \sqrt{2m+3}}{m} \prod_{k=0}^m \frac{2k+1}{3/4 + m + 1 + k} = -\frac{7\sqrt{2m+3}}{4(3/4 + 2m + 1)} \hat{J}(m, m).\end{aligned}$$

We can compute the  $\widehat{I}_{7/4}$  part by recursion. Indeed,

$$\widehat{I}_\alpha(l, m) = \frac{l - \alpha - m - 1}{l + \alpha + m + 1} \sqrt{\frac{(l + m)(2l + 1)}{(l - m)(2l - 1)}} \widehat{I}_\alpha(l - 1, m), \quad (26)$$

$$\widehat{I}_\alpha(m, m) = 2^m \prod_{k=0}^m \frac{2k + 1}{\alpha + m + 1 + k}. \quad (27)$$

Hence, we need to compute beforehand the values  $\widehat{I}_\alpha(l', m - 1)$  for  $l = m - 1, \dots, m + 2N - 1$ , and then apply the same process as for  $\widehat{I}$ .

As for the numerical integral, we use a simple midpoint rule with  $K$  points. Those integrals have the form

$$I_{ln} = \int_{-1}^1 d\xi \widehat{P}_l^{[m]}(\xi) \phi(\xi) \widehat{P}_n^{[m]}(\xi) \approx \frac{2}{K} \sum_{k=1}^K \widehat{P}_l^{[m]}(\xi_k) \phi(\xi_k) \widehat{P}_n^{[m]}(\xi_k),$$

where  $\xi_k = -1 + (2/K)(k - 1/2)$ . As we wish to compute those elements for  $m \leq l, n \leq m + N$ , we have to compute the  $\widehat{P}_l^{[m]}(\xi_k)$  for  $n = m, \dots, m + N$  and  $k = 1, \dots, K$ . To that end, We compute beforehand a table of the values  $\{\widehat{P}_l^{[m]}(\xi_k)\}_{(n,k)}$  and of the values  $\{\phi(\xi_k)\}_k$ . The Legendre associated functions can be efficiently computed by using the Julia library "SphericalHarmomics", in which we use the function "computePlmcostheta( $\theta, l_{\max}, m$ )" which compute  $\widehat{P}_l^{[m]}(\cos(\theta))/\sqrt{\pi}$  for all  $n = 0, \dots, l_{\max}$  at a given  $m$ .