Notes

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Contents

1 Self-gravitating thin disk with bulb

Consider a self gravitating disk with surface density Σ and gravitational potential Φ. Let $\boldsymbol{v}$ be its velocity field and P its pression field. Then it is described by the system

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\frac{1}{\Sigma}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi, \tag{2}$$

$$\Delta\Phi = 4\pi G \Sigma \delta(z). \tag{3}$$

Here, $\Phi = \Phi_{\mathrm{bulb}} + \Phi_{\mathrm{disk}} + \Phi_{\mathrm{DH}} = \Phi_{\mathrm{bulb}} + \Phi_{\mathrm{sg}}$. Furthermore, Φ_{bulb} is such that it doesn't yield any contribution to the surface density, i.e. $\Delta\Phi_{\mathrm{bulb}} = 0$, nor the pressure field, hence $\Sigma = \Sigma_{\mathrm{disk}} + \Sigma_{\mathrm{DH}}$ and $P = P_{\mathrm{disk}} + P_{\mathrm{DH}}$. Letting $q = \Sigma_{\mathrm{disk}}/\Sigma$, we have

$$\Sigma = \Sigma_{\mathrm{disk}} + \Sigma_{\mathrm{DH}} = \Sigma_{\mathrm{disk}}\left(1 + \frac{\Sigma_{\mathrm{DH}}}{\Sigma_{\mathrm{disk}}}\right) = \frac{1}{q}\Sigma_{\mathrm{disk}}. \tag{4}$$

Suppose that we have a polytrope gas such that $P = \kappa\Sigma^{\Gamma}$ and let

$$\psi = \int \frac{\mathrm{d}P(\Sigma)}{\Sigma} \Leftrightarrow \boldsymbol{\nabla}\psi = \frac{\boldsymbol{\nabla}P}{\Sigma}. \tag{5}$$

Then

$$\psi = \frac{\kappa\Gamma}{\Gamma - 1}\Sigma^{\Gamma-1}. \tag{6}$$

Letting $\Psi = \Phi + \psi$, we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\boldsymbol{\nabla}\Psi. \tag{7}$$

If $\Sigma_{\mathrm{disk}} \propto \Sigma_{\mathrm{DH}}$ then $q \in [0, 1]$ is constant and this system of equation can be rewritten as

$$\frac{\partial \Sigma_{\mathrm{disk}}}{\partial t} + \frac{1}{r}\frac{\partial (r\Sigma_{\mathrm{disk}} v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\Sigma_{\mathrm{disk}} v_\theta)}{\partial \theta} = 0, \tag{8}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{(v_\theta)^2}{r} = -\frac{\partial \Psi}{\partial r}, \tag{9}$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \tag{10}$$

$$\Delta\Phi_{\mathrm{sg}} = 4\pi G\frac{1}{q}\Sigma_{\mathrm{disk}}\delta(z). \tag{11}$$

Let M be the total mass of the galactic disk+bulb. Let $\Phi_{\text{bulb}} = -GM(1-x)/\sqrt{c^2+r^2}$ with $x \in [0,1]$. An equilibrium state is given by the Plummer equilibrium, such that, letting $\xi = (r^2-a^2)/(r^2+a^2)$, that is, $r/a = \sqrt{(1+\xi)/(1-\xi)}$,

$$
\begin{aligned}
& v_r^0 = 0, \qquad (v_\theta^0)^2 = r\frac{\partial \Psi^0}{\partial r}, \\
& \psi^0 = \frac{\kappa\Gamma}{\Gamma-1}(\Sigma^0)^{\Gamma-1} = \frac{\kappa\Gamma}{(\Gamma-1)q^{\Gamma-1}}(\Sigma^0_{\text{disk}})^{\Gamma-1}, \\
& \Sigma^0_{\text{disk}} = \frac{xM}{2\pi a^2}\frac{1}{(1+(r/a)^2)^{3/2}} = \frac{xM}{2\pi a^2}\left(\frac{1-\xi}{2}\right)^{3/2}, \\
& \Phi^0_{\text{sg}} = -\frac{GMx}{aq}\frac{1}{\sqrt{1+(r/a)^2}} = -\frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}.
\end{aligned}
$$

Therefore, letting

$$
\varepsilon = \frac{\text{sg internal energy}}{|\text{total sg energy}|} = \frac{3a\kappa\Gamma}{GM}\left(\frac{M}{2\pi a^2}\right)^{\Gamma-1},
$$

we obtain

$$
\begin{aligned}
\Psi^0 &= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} - \frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\kappa\Gamma}{(\Gamma-1)}(\Sigma^0_{\text{disk}}/q)^{\Gamma-1}, \\
&= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} + \frac{GM}{a}\left[-(x/q)\left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-1}}{3(\Gamma-1)}\left(\frac{1-\xi}{2}\right)^{3(\Gamma-1)/2}\right], \\
&= \frac{GMx}{aq}\left[-\frac{a}{c}\frac{q(1-x)}{x}\frac{1}{\sqrt{1+(r/c)^2}} - \left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-2}}{3(\Gamma-1)}\left(\frac{1-\xi}{2}\right)^{3(\Gamma-1)/2}\right], \\
(v_\theta^0)^2 &= r\frac{\partial\Psi^0}{\partial r} = r\frac{\partial\Psi^0}{\partial\xi}\frac{\partial\xi}{\partial r} = 4\left(\frac{r}{a}\right)^2\left(\frac{1-\xi}{2}\right)^2\frac{\partial\Psi^0}{\partial\xi} = (1+\xi)(1-\xi)\frac{\partial\Psi^0}{\partial\xi}, \\
&= \frac{GM}{a}\left[\frac{a(1-x)}{c}\left(\frac{r}{c}\right)^2\left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \frac{x}{q}\left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\left(1-\varepsilon\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right)\right], \\
&= \frac{GMx}{aq}\left[\frac{a}{c}\frac{q(1-x)}{x}\left(\frac{r}{c}\right)^2\left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\left(1-\varepsilon\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right)\right],
\end{aligned}
$$

with

$$
\frac{1-\xi}{2} = \frac{a^2}{r^2+a^2} = \frac{1}{1+(r/a)^2}; \quad \frac{1+\xi}{2} = \frac{r^2}{r^2+a^2} = \frac{(r/a)^2}{1+(r/a)^2}.
$$

Note that for $x=1$ (no bulb) and $q=1$ (purely self-gravitating system), we recover the expression of Toomre

$$
v_\theta^0 = \left(\frac{GM}{a}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sqrt{1-\varepsilon\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}}.
$$

The perturbative equation at 1st order are

$$
\begin{aligned}
& \frac{\partial v_r^p}{\partial t} + \frac{v_\theta^0}{r}\frac{\partial v_r^p}{\partial\theta} - 2\frac{v_\theta^0 v_\theta^p}{r} = -\frac{\partial\Psi^p}{\partial r}, \\
& \frac{\partial v_\theta^p}{\partial t} + v_r^p\frac{\partial v_\theta^0}{\partial r} + \frac{v_\theta^0}{r}\frac{\partial v_\theta^p}{\partial\theta} + \frac{v_r^p v_\theta^0}{r} = -\frac{1}{r}\frac{\partial\Psi^p}{\partial\theta}, \\
& \Delta\Phi^p_{\text{sg}} = 4\pi G\frac{1}{q}\Sigma^p_{\text{disk}}\delta(z), \\
& \psi^p = \frac{\kappa\Gamma}{q^{\Gamma-1}}(\Sigma^0_{\text{disk}})^{\Gamma-2}\Sigma^p_{\text{disk}}.
\end{aligned}
$$

We define $X^p(r,\theta,t) = \sum_{m\in\mathbb{Z}} X^p_m(r,t)e^{im\theta}$ and look for a temporal dependency in $e^{-i\omega t}$. Aoki & Iye say that there is this following correspondance between surface density and gravitational potential through the Poisson equation:

$$(\Sigma_{\text{disk}})^p_m(r,\theta) = \frac{xM}{2\pi a^2}\left(\frac{1-\xi}{2}\right)^{3/2}\sum_{n=|m|}^{\infty} a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{12}$$

$$(\Phi_{\text{sg}})^p_m(r,\theta) = -\frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty} \frac{a^m_n}{2n+1}\widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{13}$$

$$(\psi)^p_m(r,\theta) = \kappa\Gamma\left(\frac{M}{2\pi a^2}\right)^{\Gamma-1}\left(\frac{x}{q}\right)^{\Gamma-1}\left(\frac{1-\xi}{2}\right)^{3/2(\Gamma-1)}\sum_{n=|m|}^{\infty} a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{14}$$

$$(\Psi)^p_m(r,\theta) = \frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{\varepsilon}{3}\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} - \frac{1}{2n+1}\right] a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}. \tag{15}$$

Let us decompose the velocity components based on their equilibrium expression:

$$(v_r)^p_m = i\frac{m}{|m|}\left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} b^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{16}$$

$$(v_\theta)^p_m = \left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} c^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}. \tag{17}$$

Letting $X^p_m = X^1 e^{-i\omega t}$, this yields the set of equations

$$i(-\omega + m\Omega)(\Sigma_{\text{disk}})^1 + \frac{1}{r}\frac{\mathrm{d}(r\Sigma_0(v_r)^1)}{\mathrm{d}r} + \frac{im\Sigma^0(v_\theta)^1}{r} = 0, \tag{18}$$

$$\frac{\mathrm{d}(\Psi)^1}{\mathrm{d}r} + i(-\omega + m\Omega)(v_r)^1 - 2\Omega(v_\theta)^1 = 0, \tag{19}$$

$$im\frac{(\Psi)^1}{r} + \frac{\alpha^2}{2\Omega}(v_r)^1 + i(-\omega + m\Omega)(v_\theta)^1 = 0, \tag{20}$$

where $\Omega = v^0_\theta/r$ is the angular velocity and $\alpha^2 = 4\Omega^2[1 + r/(2\Omega)\cdot(\mathrm{d}\Omega/\mathrm{d}r)]$ is the epicyclic frequency. Using the relation

$$\int_{-1}^{1}\mathrm{d}\xi\,\widehat{P^{|m|}_n}(\xi)\widehat{P^{|m|}_l}(\xi) = \delta_{nl},$$

and defining $\Omega_{\text{ref}} = (GMx/(a^3q))^{1/2}$, $\Sigma_{\text{ref}} = xM/(2\pi a^2)$ such that $\widehat{\omega} = \omega/\Omega_{\text{ref}}$, $\widehat{\Omega} = \Omega/\Omega_{\text{ref}}$, $\widehat{\alpha} = \alpha/\Omega_{\text{ref}}$, $\widehat{\Sigma} = \Sigma/\Sigma_{\text{ref}}$ and $\lambda = \frac{|m|}{m}\widehat{\omega}$, we obtain the matrix equations

$$\sum_{n=|m|}^{\infty} A_{ln}a^m_n + \sum_{n=|m|}^{\infty} B_{ln}b^m_n + \sum_{n=|m|}^{\infty} C_{ln}c^m_n = \lambda a^m_l$$

$$\frac{\mathrm{d}(\Psi)^1}{\mathrm{d}r} + i(-\omega + m\Omega)(v_r)^1 - 2\Omega(v_\theta)^1 = 0$$

$$\frac{\mathrm{d}\left(\frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{\varepsilon}{3}\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}-\frac{1}{2n+1}\right]a_n^m\widehat{P_n^{|m|}}(\xi)\right)}{\mathrm{d}r}$$
$$+\,i(m\Omega)i\frac{m}{|m|}\left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty}b_n^m\widehat{P_n^{|m|}}(\xi)$$
$$-\,2\Omega\left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty}c_n^m\widehat{P_n^{|m|}}(\xi)$$
$$=i(\omega)i\frac{m}{|m|}\left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty}b_n^m\widehat{P_n^{|m|}}(\xi)$$

$$\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{-1/4}\frac{\mathrm{d}\left(\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{1}{2n+1}-\frac{\varepsilon}{3}\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right]a_n^m\widehat{P_n^{|m|}}(\xi)\right)}{\mathrm{d}(r/a)}$$
$$+\,(m\widehat{\Omega})\frac{m}{|m|}\sum_{n=|m|}^{\infty}b_n^m\widehat{P_n^{|m|}}(\xi)$$
$$+\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{-1/4}2\widehat{\Omega}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty}c_n^m\widehat{P_n^{|m|}}(\xi)$$
$$=\lambda\sum_{n=|m|}^{\infty}b_n^m\widehat{P_n^{|m|}}(\xi)$$

$$\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{-1/4}\frac{\mathrm{d}\left(\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{1}{2n+1}-\frac{\varepsilon}{3}\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right]a_n^m\widehat{P_n^{|m|}}(\xi)\right)}{\mathrm{d}(r/a)}$$
$$+\sum_{n=|m|}^{\infty}A_{ln}b_n^m+\sum_{n=|m|}^{\infty}F_{ln}c_n^m=\lambda b_l^m$$

where we defined

$$A_{ln}=|m|\int_{-1}^{1}\mathrm{d}\xi\widehat{P_l^{|m|}}(\xi)\widehat{\Omega}(\xi)\widehat{P_n^{|m|}}(\xi),$$
$$B_{ln}=4\int_{-1}^{1}\mathrm{d}\xi\widehat{P_l^{|m|}}(\xi)\left(\frac{1-\xi}{2}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{1-\xi}{2}\right)^{5/4}\widehat{P_n^{|m|}}(\xi)\right],$$
$$C_{ln}=|m|\int_{-1}^{1}\mathrm{d}\xi\widehat{P_l^{|m|}}(\xi)\left(\frac{1-\xi}{2}\right)^{3/4}\left(\frac{1+\xi}{2}\right)^{-1}\widehat{P_n^{|m|}}(\xi),$$
$$F_{ln}=2\int_{-1}^{1}\mathrm{d}\xi\widehat{P_l^{|m|}}(\xi)\widehat{\Omega}(\xi)\widehat{P_n^{|m|}}(\xi),$$