Notes

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Contents

1 Self-gravitating thin disk with bulb and dark halo

Consider a self gravitating disk with surface density Σ and gravitational potential Φ. Let $\boldsymbol{v}$ be its velocity field and P its pression field. Then it is described by the system

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\frac{1}{\Sigma}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi, \tag{2}$$

$$\Delta\Phi = 4\pi G \Sigma \delta(z). \tag{3}$$

Here, $\Phi = \Phi_{\mathrm{bulb}} + \Phi_{\mathrm{disk}} + \Phi_{\mathrm{DH}} = \Phi_{\mathrm{bulb}} + \Phi_{\mathrm{sg}}$. Furthermore, Φ_{bulb} is such that it doesn't yield any contribution to the surface density, i.e. $\Delta\Phi_{\mathrm{bulb}} = 0$, nor the pressure field, hence $\Sigma = \Sigma_{\mathrm{disk}} + \Sigma_{\mathrm{DH}}$ and $P = P_{\mathrm{disk}} + P_{\mathrm{DH}}$. Letting $q = \Sigma_{\mathrm{disk}}/\Sigma$, we have

$$\Sigma = \Sigma_{\mathrm{disk}} + \Sigma_{\mathrm{DH}} = \Sigma_{\mathrm{disk}}\left(1 + \frac{\Sigma_{\mathrm{DH}}}{\Sigma_{\mathrm{disk}}}\right) = \frac{1}{q}\Sigma_{\mathrm{disk}}. \tag{4}$$

Suppose that we have a polytrope gas such that $P = \kappa\Sigma^{\Gamma}$ and let

$$\psi = \int \frac{\mathrm{d}P(\Sigma)}{\Sigma} \Leftrightarrow \boldsymbol{\nabla}\psi = \frac{\boldsymbol{\nabla} P}{\Sigma}. \tag{5}$$

Then

$$\psi = \frac{\kappa\Gamma}{\Gamma - 1}\Sigma^{\Gamma-1}. \tag{6}$$

Letting $\Psi = \Phi + \psi$, we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\boldsymbol{\nabla}\Psi. \tag{7}$$

If $\Sigma_{\mathrm{disk}} \propto \Sigma_{\mathrm{DH}}$ then $q \in [0, 1]$ is constant and this system of equation can be rewritten as

$$\frac{\partial \Sigma_{\mathrm{disk}}}{\partial t} + \frac{1}{r}\frac{\partial (r\Sigma_{\mathrm{disk}} v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\Sigma_{\mathrm{disk}} v_\theta)}{\partial \theta} = 0, \tag{8}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{(v_\theta)^2}{r} = -\frac{\partial \Psi}{\partial r}, \tag{9}$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \tag{10}$$

$$\Delta\Phi_{\mathrm{sg}} = 4\pi G\frac{1}{q}\Sigma_{\mathrm{disk}}\delta(z). \tag{11}$$

Let M be the total mass of the galactic disk+bulb. Let $\Phi_{\text{bulb}} = -GM(1-x)/\sqrt{c^2+r^2}$ with $x \in [0,1]$. An equilibrium state is given by the Plummer equilibrium, such that, letting $\xi = (r^2-a^2)/(r^2+a^2)$, that is, $r/a = \sqrt{(1+\xi)/(1-\xi)}$,

$$
\begin{aligned}
&v_r^0 = 0, \qquad (v_\theta^0)^2 = r\frac{\partial \Psi^0}{\partial r},\\
&\psi^0 = \frac{\kappa\Gamma}{\Gamma-1}(\Sigma^0)^{\Gamma-1} = \frac{\kappa\Gamma}{(\Gamma-1)q^{\Gamma-1}}(\Sigma^0_{\text{disk}})^{\Gamma-1},\\
&\Sigma^0_{\text{disk}} = \frac{xM}{2\pi a^2}\frac{1}{(1+(r/a)^2)^{3/2}} = \frac{xM}{2\pi a^2}\left(\frac{1-\xi}{2}\right)^{3/2},\\
&\Phi^0_{\text{sg}} = -\frac{GMx}{aq}\frac{1}{\sqrt{1+(r/a)^2}} = -\frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}.
\end{aligned}
$$

Therefore, letting

$$
\varepsilon = \frac{U(x=1,q=1)}{|E(x=1,q=1)|} = \frac{3a\kappa\Gamma}{GM}\left(\frac{M}{2\pi a^2}\right)^{\Gamma-1},
$$

we obtain

$$
\begin{aligned}
\Psi^0 &= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} - \frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\kappa\Gamma}{(\Gamma-1)}(\Sigma^0_{\text{disk}}/q)^{\Gamma-1},\\
&= -\frac{GM(1-x)}{\sqrt{c^2+r^2}} + \frac{GM}{a}\left[-(x/q)\left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-1}}{3(\Gamma-1)}\left(\frac{1-\xi}{2}\right)^{3(\Gamma-1)/2}\right],\\
&= \frac{GMx}{aq}\left[-\frac{a}{c}\frac{q(1-x)}{x}\frac{1}{\sqrt{1+(r/c)^2}} - \left(\frac{1-\xi}{2}\right)^{1/2} + \frac{\varepsilon(x/q)^{\Gamma-2}}{3(\Gamma-1)}\left(\frac{1-\xi}{2}\right)^{3(\Gamma-1)/2}\right],\\
(v_\theta^0)^2 &= r\frac{\partial\Psi^0}{\partial r} = r\frac{\partial\Psi^0}{\partial\xi}\frac{\partial\xi}{\partial r} = 4\left(\frac{r}{a}\right)^2\left(\frac{1-\xi}{2}\right)^2\frac{\partial\Psi^0}{\partial\xi} = (1+\xi)(1-\xi)\frac{\partial\Psi^0}{\partial\xi},\\
&= \frac{GM}{a}\left[\frac{a(1-x)}{c}\left(\frac{r}{c}\right)^2\left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \frac{x}{q}\left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\left(1-\varepsilon\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right)\right],\\
&= \frac{GMx}{aq}\left[\frac{a}{c}\frac{q(1-x)}{x}\left(\frac{r}{c}\right)^2\left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\left(1-\varepsilon\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}\right)\right],
\end{aligned}
$$

with

$$
\frac{1-\xi}{2} = \frac{a^2}{r^2+a^2} = \frac{1}{1+(r/a)^2}; \quad \frac{1+\xi}{2} = \frac{r^2}{r^2+a^2} = \frac{(r/a)^2}{1+(r/a)^2}.
$$

Note that for $x=1$ (no bulb) and $q=1$ (purely self-gravitating system), we recover the expression of Toomre

$$
v_\theta^0 = \left(\frac{GM}{a}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sqrt{1-\varepsilon\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2}}.
$$

The perturbative equation at 1st order are

$$
\begin{aligned}
&\frac{\partial v_r^p}{\partial t} + \frac{v_\theta^0}{r}\frac{\partial v_r^p}{\partial\theta} - 2\frac{v_\theta^0 v_\theta^p}{r} = -\frac{\partial\Psi^p}{\partial r},\\
&\frac{\partial v_\theta^p}{\partial t} + v_r^p\frac{\partial v_\theta^0}{\partial r} + \frac{v_\theta^0}{r}\frac{\partial v_\theta^p}{\partial\theta} + \frac{v_r^p v_\theta^0}{r} = -\frac{1}{r}\frac{\partial\Psi^p}{\partial\theta},\\
&\Delta\Phi^p_{\text{sg}} = 4\pi G\frac{1}{q}\Sigma^p_{\text{disk}}\delta(z),\\
&\psi^p = \frac{\kappa\Gamma}{q^{\Gamma-1}}(\Sigma^0_{\text{disk}})^{\Gamma-2}\Sigma^p_{\text{disk}}.
\end{aligned}
$$

We define $X^p(r,\theta,t) = \sum_{m\in\mathbb{Z}} X^p_m(r,t)e^{im\theta}$ and look for a temporal dependency in $e^{-i\omega t}$. Aoki & Iye say that there is this following correspondance between surface density and gravitational potential through the Poisson equation:

$$(\Sigma_{\text{disk}})^p_m(r,\theta) = \frac{xM}{2\pi a^2}\left(\frac{1-\xi}{2}\right)^{3/2}\sum_{n=|m|}^{\infty} a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{12}$$

$$(\Phi_{\text{sg}})^p_m(r,\theta) = -\frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty} \frac{a^m_n}{2n+1} \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{13}$$

$$(\psi)^p_m(r,\theta) = \kappa\Gamma\left(\frac{M}{2\pi a^2}\right)^{\Gamma-1}\left(\frac{x}{q}\right)^{\Gamma-1}\left(\frac{1-\xi}{2}\right)^{3/2(\Gamma-1)}\sum_{n=|m|}^{\infty} a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{14}$$

$$(\Psi)^p_m(r,\theta) = \frac{GMx}{aq}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{\varepsilon}{3}\left(\frac{x}{q}\right)^{\Gamma-2}\left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} - \frac{1}{2n+1}\right] a^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}. \tag{15}$$

Let us decompose the velocity components based on their equilibrium expression:

$$(v_r)^p_m = i\frac{m}{|m|}\left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} b^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{16}$$

$$(v_\theta)^p_m = \left(\frac{GMx}{aq}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} c^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}. \tag{17}$$

Letting $X^p_m = X^1 e^{-i\omega t}$, this yields the set of equations

$$i(-\omega + m\Omega)(\Sigma_{\text{disk}})^1 + \frac{1}{r}\frac{\mathrm{d}(r\Sigma_0(v_r)^1)}{\mathrm{d}r} + \frac{im\Sigma^0(v_\theta)^1}{r} = 0, \tag{18}$$

$$\frac{\mathrm{d}(\Psi)^1}{\mathrm{d}r} + i(-\omega + m\Omega)(v_r)^1 - 2\Omega(v_\theta)^1 = 0, \tag{19}$$

$$im\frac{(\Psi)^1}{r} + \frac{\alpha^2}{2\Omega}(v_r)^1 + i(-\omega + m\Omega)(v_\theta)^1 = 0, \tag{20}$$

where $\Omega = v^0_\theta/r$ is the angular velocity and $\alpha^2 = 4\Omega^2[1 + r/(2\Omega)\cdot(\mathrm{d}\Omega/\mathrm{d}r)]$ is the epicyclic frequency. Using the relation

$$\int_{-1}^{1}\mathrm{d}\xi \widehat{P^{|m|}_n}(\xi)\widehat{P^{|m|}_l}(\xi) = \delta_{nl},$$

and defining $\Omega_{\text{ref}} = (GMx/(a^3q))^{1/2}$, $\Sigma_{\text{ref}} = xM/(2\pi a^2)$ such that $\widehat{\omega} = \omega/\Omega_{\text{ref}}$, $\widehat{\Omega} = \Omega/\Omega_{\text{ref}}$, $\widehat{\alpha} = \alpha/\Omega_{\text{ref}}$, $\widehat{\Sigma} = \Sigma/\Sigma_{\text{ref}}$ and $\lambda = \frac{|m|}{m}\widehat{\omega}$, we obtain the matrix equations

$$\sum_{n=|m|}^{\infty} A_{ln}a^m_n + \sum_{n=|m|}^{\infty} B_{ln}b^m_n + \sum_{n=|m|}^{\infty} C_{ln}c^m_n = \lambda a^m_l,$$

$$\sum_{n=|m|}^{\infty} D_{ln}a^m_n + \sum_{n=|m|}^{\infty} A_{ln}b^m_n + \sum_{n=|m|}^{\infty} F_{ln}c^m_n = \lambda b^m_l,$$

$$\sum_{n=|m|}^{\infty} G_{ln}a^m_n + \sum_{n=|m|}^{\infty} H_{ln}b^m_n + \sum_{n=|m|}^{\infty} A_{ln}c^m_n = \lambda b^m_l,$$

where we defined

$$
\begin{aligned}
A_{ln} &= |m| \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{|m|}}(\xi), \\
B_{ln} &= 4 \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1-\xi}{2}\right)^{5/4} \widehat{P_n^{|m|}}(\xi) \right], \\
C_{ln} &= |m| \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} \widehat{P_n^{|m|}}(\xi), \\
D_{ln} &= 4 \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{5/4} \left(\frac{1+\xi}{2}\right) \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} \right) \left(\frac{1-\xi}{2}\right)^{1/2} \widehat{P_n^{|m|}}(\xi) \right], \\
F_{ln} &= 2 \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{|m|}}(\xi), \\
G_{ln} &= -|m| \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} \right) \widehat{P_n^{|m|}}(\xi), \\
H_{ln} &= \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} \widehat{P_n^{|m|}}(\xi).
\end{aligned}
$$

with

$$
\begin{aligned}
\widehat{\Omega}(\xi) &= \sqrt{\frac{1-\xi}{1+\xi}} \sqrt{\frac{a}{c} \frac{q(1-x)}{x} \left(\frac{r}{c}\right)^2 \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left(\frac{1+\xi}{2}\right) \left(\frac{1-\xi}{2}\right)^{1/2} \left[1 - \varepsilon \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} \right]}, \\
\frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} &= 2\widehat{\Omega}(\xi) \left[1 + \frac{r}{2\widehat{\Omega}} \frac{\mathrm{d}\widehat{\Omega}}{\mathrm{d}r} \right] = 2\widehat{\Omega}(\xi) \left[1 + \frac{(1+\xi)(1-\xi)}{2\widehat{\Omega}} \frac{\mathrm{d}\widehat{\Omega}}{\mathrm{d}\xi} \right],
\end{aligned}
$$

where

$$
\frac{\mathrm{d}(r/a)}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\sqrt{\frac{1+\xi}{1-\xi}} \right] = (1-\xi)^{-3/2} (1+\xi)^{-1/2}
$$

Since $(r/c)^2 = (a/c)^2 \cdot (r/a)^2 = (a/c)^2 \cdot (1+\xi)/(1-\xi)$, we have that

$$
\widehat{\Omega}(\xi) = \left(\frac{1-\xi}{2}\right)^{3/4} \sqrt{ \left(\frac{a}{c}\right)^3 \frac{q(1-x)}{x} \left(\frac{1-\xi}{2}\right)^{-3/2} \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left[1 - \varepsilon \left(\frac{x}{q}\right)^{\Gamma-2} \left(\frac{1-\xi}{2}\right)^{\frac{3\Gamma}{2}-2} \right] }.
$$

2 Case $\Gamma = 4/3$

The expressions become

$$
\begin{aligned}
\widehat{\Omega}(\xi) &= \left(\frac{1-\xi}{2}\right)^{3/4} \sqrt{ \left(\frac{a}{c}\right)^3 \frac{q(1-x)}{x} \left(\frac{1-\xi}{2}\right)^{-3/2} \left(\frac{1}{1+(r/c)^2}\right)^{3/2} + \left[1 - \varepsilon \left(\frac{q}{x}\right)^{\frac{2}{3}} \right] }, \\
\frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} &= 2\widehat{\Omega}(\xi) \left[1 + \frac{(1+\xi)(1-\xi)}{2\widehat{\Omega}} \frac{\mathrm{d}\widehat{\Omega}}{\mathrm{d}\xi} \right],
\end{aligned}
$$

with the matrix elements

$$
\begin{aligned}
A_{ln} &= |m| \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{|m|}}(\xi), \\
B_{ln} &= 4 \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1-\xi}{2}\right)^{5/4} \widehat{P_n^{|m|}}(\xi) \right], \\
C_{ln} &= |m| \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} \widehat{P_n^{|m|}}(\xi), \\
D_{ln} &= 4 \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{q}{x}\right)^{\frac{2}{3}} \right) \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{5/4} \left(\frac{1+\xi}{2}\right) \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1-\xi}{2}\right)^{1/2} \widehat{P_n^{|m|}}(\xi) \right], \\
F_{ln} &= 2 \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \widehat{\Omega}(\xi) \widehat{P_n^{|m|}}(\xi), \\
G_{ln} &= -|m| \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{q}{x}\right)^{\frac{2}{3}} \right) \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \widehat{P_n^{|m|}}(\xi), \\
H_{ln} &= \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi) \frac{\widehat{\alpha}^2(\xi)}{2\widehat{\Omega}(\xi)} \widehat{P_n^{|m|}}(\xi).
\end{aligned}
$$

Integrals A_{ln} and F_{ln} are proportional. With the addition of H_{ln}, those 3 integrals must be computed numerically because of the non-trivial shift in their expression induced by the bulb potential. As for B_{ln}, C_{ln}, D_{ln} and G_{ln}, their can be can expressed in terms of the two following integrals

$$
\widehat{I}(l,n) = \int_{-1}^{1} \mathrm{d}\xi \left(\frac{1-\xi}{2}\right)^{3/4} \widehat{P_l^{|m|}}(\xi) \widehat{P_n^{|m|}}(\xi), \tag{21}
$$

$$
\widehat{J}(l,n) = \int_{-1}^{1} \mathrm{d}\xi \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} \widehat{P_l^{|m|}}(\xi) \widehat{P_n^{|m|}}(\xi), \tag{22}
$$

as

$$
\begin{aligned}
B_{ln} &= \frac{1}{2} \left[\sqrt{\frac{(2l+1)(l+m+1)(l-m+1)}{2l+3}} \widehat{J}(l+1,n) + \widehat{J}(l,n) \right. \\
&\qquad \left. + \sqrt{\frac{(2l+1)(l+m)(l-m)}{2l-1}} \widehat{J}(l-1,n) \right], \\
C_{ln} &= m \widehat{J}(l,n), \\
D_{ln} &= \frac{1}{2} \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{q}{x}\right)^{\frac{2}{3}} \right) \left[- \sqrt{\frac{(2n+1)(n+m+1)(n-m+1)}{2n+3}} \widehat{I}(l,n+1) \right. \\
&\qquad + \widehat{I}(l,n) \\
&\qquad \left. + \sqrt{\frac{(2n+1)(n+m)(n-m)}{2n-1}} \widehat{I}(l,n-1) \right], \\
G_{ln} &= -|m| \left(\frac{1}{2n+1} - \frac{\varepsilon}{3} \left(\frac{q}{x}\right)^{\frac{2}{3}} \right) \widehat{I}(l,n).
\end{aligned}
$$

where $\widehat{I}(l,n)$ and $\widehat{J}(l,n)$ can be computed by recursion and using the symmetry $l \leftrightarrow n$. Defining

$$
\widehat{I}'(l,n) = \int_{-1}^{1} \mathrm{d}\xi \, \xi \left(\frac{1-\xi}{2}\right)^{3/4} \widehat{P_l^{|m|}}(\xi) \widehat{P_n^{|m|}}(\xi), \tag{23}
$$

$$
\widehat{J}'(l,n) = \int_{-1}^{1} \mathrm{d}\xi \, \xi \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} \widehat{P_l^{|m|}}(\xi) \widehat{P_n^{|m|}}(\xi), \tag{24}
$$

Starting from(Aoki79, A16)

$$\widehat{I}(l,n) = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\widehat{I}'(l,n-1) + \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}}\widehat{I}(l,n-2),$$
$$\widehat{I}'(l,n-1) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\widehat{I}(l+1,n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\widehat{I}(l-1,n-1),$$

hence to compute until $l, n = m + N$, we need to compute $\widehat{I}(l', m)$ until $l' = m + 2N$. By convention (for the recursion), we have set $\widehat{I}(l', n') = 0$ for $l' < m$ or $n' < m$. We initialize with

$$\widehat{I}(l,m) = \frac{l-3/4-m-1}{l+3/4+m+1}\sqrt{\frac{(l+m)(2l+1)}{(l-m)(2l-1)}}\widehat{I}(l-1,m),$$
$$\widehat{I}(m,m) = 2^m \prod_{k=0}^{m} \frac{2k+1}{3/4+m+1+k}.$$

We proceed as follows:
- Compute the line $n = m$: $\widehat{I}(m,m)$, $\widehat{I}(m+1,m)$, ... , $\widehat{I}(m+2N,m)$
- Complete the line $l = m$ by symmetry
- Compute the line $n = m+1$: $\widehat{I}(m+1,m+1)$, $\widehat{I}(m+2,m+1)$, ... , $\widehat{I}(m+2N-1,m+1)$
- Complete the line $l = m+1$ by symmetry
- Compute the line $n = m+2$: $\widehat{I}(m+2,m+2)$, $\widehat{I}(m+3,m+2)$, ... , $\widehat{I}(m+2N-2,m+2)$
- ...
- Compute the line $n = m+N-1$: $\widehat{I}(m+N-1,m+N-1)$, $\widehat{I}(m+N+1,m+N-1)$.
- Complete the line $l = m+N-1$ by symmetry
- Compute the line $n = m+N$: $\widehat{I}(m+N,m+N)$.

As for $\widehat{J}$, let

$$\widehat{I}_\alpha(l,m) = \frac{(-1)^{l-m}(2m-1)!!2^{m+1}\Gamma(\alpha+1)\Gamma(\alpha+m+1)(l+m)!}{\Gamma(\alpha+1-l+m)\Gamma(\alpha+m+l+2)(l-m)!} \tag{25}$$

with $\widehat{I}_{3/4}(l,m) = \widehat{I}(l,m)$. We can also compute it by recursion using the formulae

$$\widehat{J}(l,n) = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\widehat{J}'(l,n-1) + \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}}\widehat{J}(l,n-2),$$
$$\widehat{J}'(l,n-1) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\widehat{J}(l+1,n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\widehat{J}(l-1,n-1),$$

hence to compute until $l, n = m + N$, we need to compute $\widehat{I}(l', m)$ until $l' = m + 2N$. By convention (for the recursion), we have set $\widehat{I}(l', n') = 0$ for $l' < m$ or $n' < m$. We initialize with

$$\widehat{J}(l,m) = \sqrt{\frac{(l-m)(l-m-1)(2l+1)}{(l+m)(l+m-1)(2l-3)}}\widehat{J}(l-2,m) + 4\sqrt{\frac{(2m+1)(2l+1)(2l-1)}{2m(l+m)(l+m-1)}}\widehat{I}_{7/3}(l-1,m-1),$$
$$\widehat{J}(m,m) = \frac{2^m}{m}\prod_{k=1}^{m}\frac{2k+1}{3/4+m+k},$$
$$\widehat{J}(m+1,m) = -\frac{7}{4}\frac{2^m\sqrt{2m+3}}{m}\prod_{k=0}^{m}\frac{2k+1}{3/4+m+1+k} = -\frac{7\sqrt{2m+3}}{4(3/4+2m+1)}\widehat{J}(m,m).$$

We can compute the $\widehat{I}_{7/4}$ part by recursion. Indeed,

$$\widehat{I}_\alpha(l,m) = \frac{l-\alpha-m-1}{l+\alpha+m+1}\sqrt{\frac{(l+m)(2l+1)}{(l-m)(2l-1)}}\widehat{I}_\alpha(l-1,m), \tag{26}$$

$$\widehat{I}_\alpha(m,m) = 2^m \prod_{k=0}^{m} \frac{2k+1}{\alpha+m+1+k}. \tag{27}$$

Hence, we need to compute beforehand the values $\widehat{I}_\alpha(l', m-1)$ for $l = m-1, ..., m+2N-1$, and then apply the same process as for $\widehat{I}$.

As for the numerical integral, we use a simple midpoint rule with K points. Those integrals have the form

$$I_{ln} = \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi)\phi(\xi)\widehat{P_n^{|m|}}(\xi) \approx \frac{2}{K}\sum_{k=1}^{K} \widehat{P_l^{|m|}}(\xi_k)\phi(\xi_k)\widehat{P_n^{|m|}}(\xi_k),$$

where $\xi_k = -1 + (2/K)(k-0.5)$. As we with to compute those elements for $m \leq l, n \leq m+N$, we have to compute the $\widehat{P_n^{|m|}}(\xi_k)$ for $n = m, ..., m+N$ and $k = 1, ..., K$. To that end, We compute before hand a table of the values $\{\widehat{P_n^{|m|}}(\xi_k)\}_{(n,k)}$ and of the values $\{\phi(\xi_k)\}_k$. The Legendre associated functions can be efficiently computed by using the Julia library "", in which we use the function "" which compute $P_n^{|m|}(\cos(\theta))$ for all $n = 0, ..., n_{\max}$ at a given m.