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Bi-Orthogonalization of Toomre's Surface-Density Functions for Flat Galaxies

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Abstract

We have attempted bi-orthogonalization of Toomre's (1963) series of the surface-density and the corresponding potential functions for flat galaxies by Schmidt's method, in order to carry out a stability analysis prospectively. The result is given in a compact form such that both density and potential functions are monomials having as a common factor an associated Legendre function of an argument, which has a simple geometrical meaning. Comparison shows that the result of a bi-orthogonalization for flat galaxies obtained by Clutton-Brock (1972) from an entirely different viewpoint agrees with ours essentially, although he did not give an explicit expression but for a recurrence formula.

Key words: Bi-orthogonalization of surface-density functions; Flat galaxies; Potential theory.

1. *Introduction*

Kalnajs (1971) has introduced a useful concept of bi-orthogonal pairs of the surface-density and the potential functions in galactic dynamics. The bi-orthogonal pairs of functions are useful in order to study the global modes of small-amplitude oscillations of flat galaxies. The use of a suitable set of bi-orthogonal functions much reduces the computational difficulty involved and makes it possible to obtain a prospective insight into the stability problem.

For flat galaxies with finite radius, there are a few sets of bi-orthogonal pairs of functions. Hunter (1963) first found such a set: the associated Legendre functions in the oblate spheroidal coordinates. Unfortunately, the oblate spheroidal coordinates have a singularity at the finite radius. This fact does not favor the application of these functions to the normal mode analysis. Yabushita (1969) found an orthogonal pair of functions in the cylindrical coordinates, Bessel functions. There remains, however, a problem of the effects caused by the truncation at the finite radius. Moreover, these functions for disk models with finite radius are rather inconvenient for the purpose of applying the normal mode analysis to the observed models of flat galaxies, because we cannot definitely specify the outer edge of flat galaxies. It is desirable, therefore, to establish a bi-orthogonal set for disk models with infinite radius.

Toomre (1963) showed an elegant formula giving a relation between the surface density for a flat galaxy and the corresponding potential and/or the rotation velocity by Fourier–Bessel transforms (simply called hereinafter Bessel transforms) through a *weighting function* $g(k)$ [his notation is $S(k)$, which may be called *distribution function in k-space* (see appendix 1 for the explicit relation)]. Clutton-Brock (1972) has constructed a complete set of bi-orthogonal functions by the Bessel transforms of Laguerre polynomials which are orthogonal functions by themselves. He presented the method to generate such a pair of functions in terms of recurrence formulas, but not by an explicit expression of the functions. Kalnajs (1976) has shown similarly that a complete set of bi-orthonormal surface-density–potential pairs can be generated in the logarithmic spiral coordinates by means of the Fourier-Abel transforms of orthonormal set of functions. He could, however, find an explicit expression of functions only in the case of disk models with finite radius through the Abel transforms of Jacobi polynomials.

On the other hand, Toomre (1963) considered a series of surface-density–potential function pairs starting from Kuzmin's (1956) model. This series (called hereinafter Toomre's series) is not bi-orthogonalized in its original form. In the present paper we show its bi-orthogonalization by Schmidt's method (see Kaczmarz and Steinhaus, 1951, p. 63). The bi-orthogonal pair can be, as a result, expressed compactly by using spherical harmonics in certain coordinates that are defined by a projection of the infinite plane on the surface of a sphere. It is the purpose of the present paper to show that the series thus obtained is nothing but the series of bi-orthogonal pairs of functions found by Clutton-Brock (1972).

In sections 2 and 3, we deal with the bi-orthogonalization of Toomre's series. In section 4, we discuss a comparison of our result with Clutton-Brock's (1972) and find differences only in the numbering and the numerical factors. In appendix 1, we discuss a restriction on k-space, *which enables us the development in terms of the orthogonal functions we are concerned*; in other words, we restrict the general k-space so that it has only countable bases. This fact makes Clutton-Brock's (1972) result equivalent to ours.

2. *Schmidt's Method*

Let (r, z, θ) be the cylindrical coordinates, a a parameter having the dimension of length, G the gravitational constant, and M a representative mass, not necessarily the total mass. We change slightly the notation and by $\phi_0{}^0$ and $\mu_0{}^0$ we denote the potential function and the corresponding surface-density function for Toomre's (1963) first model,

$$\phi_0{}^0 = -GM(r^2+a^2)^{-1/2} \quad \text{and} \quad \mu_0{}^0 = (2\pi)^{-1}Ma(r^2+a^2)^{-3/2} . \tag{2.1}$$

We modify his operator const$\times a^{2n-1}(\partial/\partial a^2)^{n-1}(1/a)$ to $(a\partial/\partial a)^n$ in order to obtain another model of the series; this does not change the series as a whole. We may call the series thus obtained also *Toomre's series*.

Outside the plane $z=0$, we have a potential given by

$$\tilde{\phi}_0{}^0 = \frac{-GM}{[r^2+(a+|z|)^2]^{1/2}} . \tag{2.2}$$

This satisfies obviously the Laplace equation,

$$\nabla^2\tilde{\phi}_0{}^0=0 \quad \text{for} \quad z\neq 0 . \tag{2.3}$$

Now, a single application of operator $a\partial/\partial a$ to $\phi_0{}^0$ gives rise to a function

$$a\frac{\partial}{\partial a}\phi_0{}^0=\frac{GMa^2}{(r^2+a^2)^{3/2}} . \tag{2.4}$$

Using Schmidt's method of orthogonalization, we obtain the *next* function

$$\phi_1{}^0=\frac{-GM}{(r^2+a^2)^{1/2}}\left(1-\frac{2a^2}{r^2+a^2}\right), \tag{2.5}$$

which is orthogonal to $\mu_0{}^0$ such that

$$\int_0^{2\pi}\int_0^\infty \phi_1{}^0\mu_0{}^0 r dr d\theta=0 . \tag{2.6}$$

The corresponding surface density is given by

$$\mu_1{}^0=\frac{3Ma}{2\pi(r^2+a^2)^{3/2}}\left(1-\frac{2a^2}{r^2+a^2}\right). \tag{2.7}$$

We also have

$$\int_0^{2\pi}\int_0^\infty \phi_0{}^0\mu_1{}^0 r dr d\theta=0 . \tag{2.6'}$$

This process may be performed successively. As a result we express ϕ and μ with numerical coefficients λ_n and c_{nj} in the following forms:

$$\left.\begin{aligned}\phi_n{}^0&=-\frac{GM}{(r^2+a^2)^{1/2}}\sum_{j=0}^{n}c_{nj}\left(\frac{a^2}{r^2+a^2}\right)^j \\ \text{and}\qquad \mu_n{}^0&=\frac{\lambda_n Ma}{(r^2+a^2)^{3/2}}\sum_{j=0}^{n}c_{nj}\left(\frac{a^2}{r^2+a^2}\right)^j,\end{aligned}\right\} \tag{2.8}$$

where the numerical coefficients are given in table 1 for a few leading cases.

By inspection of this table we presume the general formulas for λ_n and c_{nj} to be

$$\left.\begin{aligned}\lambda_n&=2n+1 , \\ \text{and}\qquad c_{nj}&=(-1)^j\frac{(n+j)!}{(n-j)!(j!)^2} .\end{aligned}\right\} \tag{2.9}$$

Table 1. Values of coefficients in equations (2.8).

n	λ_n	c_{nj}			
		$j=0$	$j=1$	$j=2$	$j=3$
0	1	1			
1	3	1	− 2		
2	5	1	− 6	+ 6	
3	7	1	−12	+30	−20

(The validity of this inference will be understood by the end of this section and will be proven in the next section.) Comparison of equation (2.9) with the coefficients of Legendre functions yields

$$\left.\begin{aligned} \psi_n{}^0 &= -\frac{GM}{(r^2+a^2)^{1/2}} P_n(\xi)\,, \\ &\text{and} \\ \mu_n{}^0 &= \frac{(2n+1)Ma}{2\pi(r^2+a^2)^{3/2}} P_n(\xi)\,, \\ &\text{where} \\ \xi &= (r^2-a^2)/(r^2+a^2)\,. \end{aligned}\right\} \tag{2.10}$$

The bi-orthogonalization can be easily obtained using the orthogonality relation of Legendre functions. Moreover, using the recurrence formula and the differentiation formula for Legendre functions, we obtain *parallel* recurrence formulas as follows:

$$\left.\begin{aligned} \psi_n{}^0 &= \frac{1}{n}\left[(n-1)\psi^0_{n-2} + \left(1+2a\frac{\partial}{\partial a}\right)\psi^0_{n-1}\right], \\ &\text{and} \\ \mu_n{}^0 &= \frac{1}{n}\left[(n-1)\mu^0_{n-2} + \left(1+2a\frac{\partial}{\partial a}\right)\mu^0_{n-1}\right]. \end{aligned}\right\} \tag{2.11}$$

It is noted that these formulas do not include r explicitly, but only differentiation with respect to the parameter a; this is a very important point, since we can obtain a bi-orthogonal system within Toomre's series starting from Toomre's (1963) first model.

As for the non-axisymmetric modes one may presume a surface-density function and the corresponding potential on the surface by an extension of equations (2.10), but it is difficult to prove generally the relation between them. In order to save space we introduce instead a potential which is valid also outside the surface ($z=0$) as is expressed by equation (3.1). From this function we can show all the necessary relations which are given in the next section. A direct derivation of equation (3.1) itself, however, includes an inference similar to but more complicated than that done when we obtained equations (2.9), and is given in appendix 2.

Surprisingly enough, each function of the bi-orthogonal series is given by a simple form using associated Legendre functions; this is just an extension of equations (2.10), as is presumed.

3. *Bi-Orthogonal Functions Associated with Toomre's Series*

Consider a potential function given by

$$\tilde{\psi}_n{}^m = -\frac{GM}{\tilde{R}}\left(\frac{a}{a+|z|}\right)^n \sum_{j=0}^{n-m} \frac{(n+m)!}{j!(n+m-j)!}\left(\frac{|z|}{a}\right)^j P^m_{n-j}(\tilde{\xi}) \exp(im\theta)\,. \tag{3.1}$$

Here we used the notations,

$$\tilde{R}=[r^2+(a+|z|)^2]^{1/2}\ , \tag{3.2}$$

$$\tilde{\xi}=[r^2-(a+|z|)^2]/\tilde{R}^2\ , \tag{3.3}$$

and P_n^m is an associated Legendre function given by

$$P_n^m(x)=\frac{(1-x^2)^{m/2}}{n!2^n}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n\ , \tag{3.4}$$

where m and n are non-negative integers such that $m\leqq n$. It is easily shown that functions (3.1) (for n, $n-1$, and $n-2$) satisfy a recurrence formula,

$$\tilde{\Phi}_n{}^m=\frac{1}{n-m}\left[(n+m-1)\tilde{\Phi}^m_{n-2}+\left(1+2a\frac{\partial}{\partial a}\right)\tilde{\Phi}^m_{n-1}\right], \tag{3.5}$$

which is an extension of equations (2.11). Conversely, we can prove that expression (3.1) for n is obtained using the recurrence formula (3.5) assuming expressions (3.1) for $n-1$ and for $n-2$. In this case, we assume

$$\tilde{\Phi}_n{}^m=0\ ,\quad n<m\ , \tag{3.6}$$

and

$$\tilde{\Phi}_m{}^m=-[(2m)!/m!]GMa^mr^m\exp(im\theta)/\tilde{R}^{2m+1}\ ,\quad m\geqq 0\ . \tag{3.6$'$}$$

Thus by a mathematical induction, we know the consistency of the general expression (3.1), for any $n\geqq m$, with equations (3.5), (3.6), and (3.6′).

Now, since we have

$$\nabla^2\tilde{\Phi}_m{}^m=0\quad\text{for}\quad z\neq 0\ , \tag{3.7}$$

we can easily show that

$$\nabla^2\tilde{\Phi}_n{}^m=0\quad\text{for}\quad n\geqq m\ ,\quad z\neq 0\ , \tag{3.8}$$

where ∇^2 is the Laplacian.

In fact, the recurrence formulas are independent of the coordinates but concerned with only the parameter a. Therefore equation (3.8) is satisfied, provided that

$$\nabla^2\tilde{\Phi}^m_{n-1}=\nabla^2\tilde{\Phi}^m_{n-2}=0\quad\text{for}\quad z\neq 0\ . \tag{3.8$'$}$$

On the other hand, equation (3.8) shows that $\tilde{\Phi}_n{}^m$ represents a potential produced by matter being confined within the plane $z=0$. The surface density of this matter can be obtained by

$$\begin{aligned}\mu^m_n&=\frac{1}{2\pi G}\left(\frac{\partial\tilde{\Phi}^m_n}{\partial z}\right)_{z=+0}\\&=\frac{(2n+1)Ma}{2\pi R^3}P^m_n(\xi)\exp(im\theta)\ ,\end{aligned} \tag{3.9}$$

where

$$R=(\tilde{R})_{z=0}=(r^2+a^2)^{1/2}=a\left(\frac{1-\xi}{2}\right)^{-1/2}\ , \tag{3.10}$$

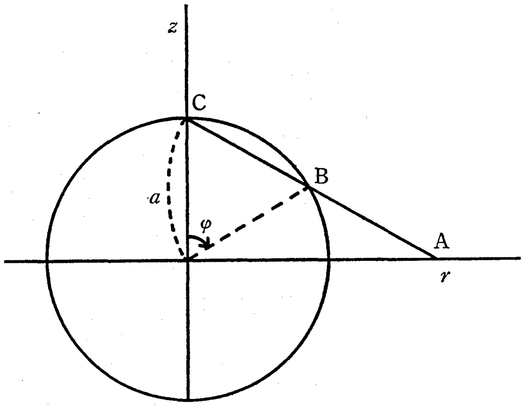

Fig. 1. Geometrical implication of ξ given by equation (3.10′). Make a straight line through A(r, 0, θ) and C(0, a, θ) and find its intersection B on the sphere having radius a. If φ denotes the angle between the positive z-axis and the straight line toward B from the origin O, then we obtain easily $\xi=\cos\varphi$.

and

$$\xi=(\tilde{\xi})_{z=0}=(r^2-a^2)/(r^2+a^2)\ . \tag{3.10′}$$

A geometrical implication of ξ is given in figure 1.

It is easily shown that the bi-orthogonality relation is given by

$$\int_0^{2\pi}\int_0^{\infty}(\mu_n^m)^*\psi_{n'}^{m'}rdrd\theta=-\frac{GM^2(2n+1)}{4a}\delta_{mm'}\int_{-1}^{1}P_n{}^m(\xi)P_{n'}^{m'}(\xi)d\xi$$
$$=\frac{-(n+m)!}{2(n-m)!}\,\frac{GM^2}{a}\delta_{mm'}\delta_{nn'}\ , \tag{3.11}$$

since

$$rdr=(1/4)(R^4/a^2)d\xi=a^2(1-\xi)^{-2}d\xi\ . \tag{3.12}$$

Here we used the notation

$$\psi_n{}^m=(\tilde{\psi}_n{}^m)_{z=0}=-(GM/R)P_n{}^m(\xi)\exp(im\theta)\ . \tag{3.13}$$

It is noted that there exists a parallelism between $\mu_n{}^m$ and $\psi_n{}^m$; in other words, $\mu_n{}^m$ is an eigenfunction of an integral equation,

$$f(r,\theta)=\lambda\frac{a}{2\pi(r^2+a^2)}\int_0^{2\pi}\int_0^{\infty}\frac{f(r',\theta')r'dr'd\theta'}{[r^2+(r')^2-2rr'\cos(\theta-\theta')]^{1/2}}\ , \tag{3.14}$$

with $\lambda=2n+1$ being the corresponding eigenvalue. Thus the bi-orthogonality is attained compactly as mentioned above.

4. *Comparison with Clutton-Brock's (1972) Result*

The discussion in appendix 1 shows the equivalence of our system with Clutton-Brock's (1972) as a whole. However, there may be a linear transformation between the two systems. In this section we deal with such a point, as well as the condition for the convergency of expansions for $m \geqq 0$.

First of all, we have from Clutton-Brock's (1972) expressions (2.22) and (4.2),

$$\Phi_0{}^m = -\frac{(2m-1)!!\hat{r}^m \exp(im\theta)}{(\hat{r}^2+1)^{m+1/2}}, \tag{4.1}$$

where we have used the notation,

$$\hat{r} = r/a, \tag{4.2}$$

instead of his r, and

$$(2m-1)!! = \frac{(2m)!}{2^m m!} = (2m-1)(2m-3)\cdots 3\cdot 1. \tag{4.3}$$

Expression (4.1) can be written in terms of our notation

$$\begin{aligned}\Phi_0{}^m &= -\frac{(2m-1)!!(1-\xi^2)^{m/2}}{2^m(\hat{r}^2+1)^{1/2}} \exp(im\theta) \\ &= -\frac{P_m{}^m(\xi)\exp(im\theta)}{2^m(\hat{r}^2+1)^{1/2}}.\end{aligned} \tag{4.4}$$

Thus we have an equivalence relation,

$$\psi_m{}^m(r,\theta) = \frac{2^m GM}{a}\Phi_0{}^m\left(\frac{r}{a},\theta\right). \tag{4.5}$$

The expression can be extended for any $n \geqq m$; *namely we have*

$$\psi_n{}^m(r,\theta) = \frac{2^m GM}{a}\Phi_{n-m}^m\left(\frac{r}{a},\theta\right), \tag{4.6}$$

since his recurrence formulas (4.9) are rewritten in the following forms,

$$(n-m)\Phi_l{}^m = (2n-1)\xi\Phi_{l-1}^m - (n+m-1)\Phi_{l-2}^m \quad \text{for} \quad l \geqq 2, \tag{4.7}$$

$$\Phi_1{}^m = (2m+1)\xi\Phi_0{}^m, \tag{4.8}$$

where $l = n-m$. In fact the recurrence formulas have the same forms as the associated Legendre functions, and by a mathematical induction we obtain

$$\Phi_l{}^m = AP_n{}^m(\xi) = AP_{m+l}^m(\xi) \quad \text{for} \quad l \geqq 0, \tag{4.9}$$

where A does not depend on n, and is given from the case $n=m$ as

$$A = -\frac{\exp(im\theta)}{2^m(\hat{r}^2+1)^{1/2}}. \tag{4.10}$$

It is noted that the factor $(1-\xi^2)^{m/2}$ in equation (4.4) is treated as an essential part of associated Legendre functions in our expression, whereas Clutton-Brock's (1972) expression deals with it separately. Thus we have a new mathematical formula

$$\int_0^\infty J_m(k\hat{r})L_l{}^{2m}(2k)\exp(-k)k^m dk = \frac{2^{-m}}{(\hat{r}^2+1)^{1/2}}P_{m+l}^m\left(\frac{\hat{r}^2-1}{\hat{r}^2+1}\right) \quad \text{for} \quad m \geqq 0, \quad l \geqq 0, \tag{4.11}$$

from equation (4.6) together with his equation (2.16) and our expression (3.13), where J_m denotes the m-th order Bessel function, L_l^{2m} the associated Laguerre polynomial (see Erdélyi et al. 1953). Similarly we have, by comparison with the density functions, another formula

$$\int_0^\infty J_m(k\hat{r})L_l^{2m}(2k)\exp(-k)k^{m+1}dk=\frac{(2m+2l+1)2^{-m}}{(\hat{r}^2+1)^{3/2}}P_{m+l}^m\left(\frac{\hat{r}^2-1}{\hat{r}^2+1}\right). \quad (4.12)$$

Relation (4.11) can also be given directly from the generating function

$$\begin{aligned}\hat{r}^m G_m^{[\phi]}(t,\hat{r})&\equiv\hat{r}^m\sum_{l=0}^\infty t^l\phi_l^m(\hat{r})\\&=\frac{(2m-1)!!\hat{r}^m}{[(1+t)^2+\hat{r}^2(1-t)^2]^{m+1/2}}\\&=2^{-m}\sum_{l=0}^\infty P_{m+l}^m(\xi)\frac{t^l}{(\hat{r}^2+1)^{1/2}}.\end{aligned} \quad (4.13)$$

The derivation can be made in a similar method to that for ultraspherical polynomials by Clutton-Brock (1973), since we have

$$(1+t)^2+\hat{r}^2(1-t)^2=(1+\hat{r}^2)[1-2t\xi+t^2]. \quad (4.14)$$

Now, if we take the orthonormal systems given by

$$\hat{L}_l^{2m}(2k)=\left[\frac{l!2^{2m+1}}{(2m+l)!}\right]^{1/2}L_l^{2m}(2k), \quad (4.15)$$

and

$$\hat{P}_{m+l}^m(\xi)=\left[\frac{l!(2m+2l+1)}{2(2m+l)!}\right]^{1/2}P_{m+l}^m(\xi), \quad (4.16)$$

instead of $L_l^{2m}(2k)$ and $P_{m+l}^m(\xi)$, respectively, and let the Fourier coefficient of $g(k)$ be c_l^m, then it is easy to show that the corresponding coefficients of μ and ϕ are given by

$$\left.\begin{aligned}&a_l^m=2(2m+2l+1)^{1/2}c_l^m\\ \text{and}\quad&\\&b_l^m=2(2m+2l+1)^{-1/2}c_l^m,\end{aligned}\right\} \quad (4.17)$$

respectively. The completeness condition for k-space is given by

$$\int_0^\infty|g^{(m)}|^2dk=\sum_{l=0}^\infty|c_l^m|^2<\infty, \quad (4.18)$$

for fixed m, where $g^{(m)}$ is the g's component which has $\exp(im\theta)$ as a factor. Then we have also the completeness condition in r-space given by

$$\int_0^{2\pi}\int_0^\infty(\mu^{(m)})^*\phi^{(m)}rdrd\theta=-\frac{GM^2}{4a}\sum_{l=0}^\infty(a_l^m)^*b_l^m=-\frac{GM^2}{a}\sum_{l=0}^\infty|c_l^m|^2. \quad (4.19)$$

Accordingly the convergence condition for μ, for example, is given by

$$\begin{aligned}&\lim_{N\to\infty}\iint(\mu^{(m)}-\sum_{l=0}^N a_l^m\hat{\mu}_{m+l}^m)^*S(\mu^{(m)}-\sum_{l=0}^N a_l^m\hat{\mu}_{m+l}^m)rdrd\theta\\&=\lim_{N\to\infty}\iint(\mu^{(m)}-\sum_{l=0}^N a_l^m\hat{\mu}_{m+l}^m)^*(\phi^{(m)}-\sum_{l=0}^N b_l^m\hat{\phi}_{m+l}^m)rdrd\theta=0,\end{aligned} \quad (4.20)$$

where S is a self-adjoint operator such that we obtain the corresponding potential function by applying it to the surface-density function. Thus it is proven that the degree of convergence in r-space coincides mathematically with that in k-space. Here we have used the notations,

$$\hat{\phi}_n{}^m = -\frac{GM}{a}\left(\frac{1-\xi}{2}\right)^{1/2}\hat{P}_n{}^m(\xi)\exp(im\theta)\,, \qquad \hat{\mu}_n{}^m = \frac{M}{2\pi a^2}\left(\frac{1-\xi}{2}\right)^{3/2}\hat{P}_n{}^m(\xi)\exp(im\theta)\,. \tag{4.21}$$

5. *Conclusion*

We have attempted the bi-orthogonalization of Toomre's series of surface-density functions for flat galaxies and of the corresponding potential functions. The result is given in a surprisingly compact form as in equations (3.9) and (3.13) by using associated Legendre functions with argument $\xi=(r^2-a^2)/(r^2+a^2)$.

On the other hand, Clutton-Brock (1972) has attempted to obtain a suitable bi-orthogonalization for further applications. He seems to have extended Toomre's series. The result we have obtained, however, shows the equivalence of Toomre's series as a whole with Clutton-Brock's (1972) series as is discussed in appendix 1 and a comparison of numerical factors between our expressions and his is given in section 4.

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Appendix 1. *Completeness in k-Space (Wavenumber Space)*

It was first shown by Toomre (1963) that the relation between the surface density μ of an axisymmetric flat galaxy and the self-gravitating potential ϕ is given by simple formulas through Bessel transforms:

$$\mu=\int_0^\infty J_0(kr)g(k)kdk \tag{A.1}$$

and

$$\phi=-2\pi G\int_0^\infty J_0(kr)g(k)dk\,, \tag{A.2}$$

where J_0 is the zeroth-order Bessel's function. Bessel's inverse transform yields at once

$$g(k)=\int_0^\infty J_0(kr)\mu(r)rdr\,. \tag{A.3}$$

For this method, Clutton-Brock (1972) introduced a complete set of bi-orthogonal density–potential pairs, using Laguerre functions. Mathematically speaking, however, this cannot be done without some restriction on $g(k)$, the distribution function in k-space. This reflects on the choice of surface-density functions. According to Watson (1922, p. 456f.) the Bessel transforms can be made, if $g(k)$ is a function such that $\int_0^\infty g(k)k^{1/2}dk$ exists and is absolutely convergent, namely

$$\int_0^\infty |g(k)|k^{1/2}dk<\infty\ . \tag{A.4}$$

$F(R)$ of Watson's function is converted here to $g(k)$, for consistency. $g(k)$ in equation (A.3) is to be read as $(1/2)[g(k+0)+g(k-0)]$. On the other hand, in order that $g(k)$ is expanded by a series of complete orthogonal functions in $L^2(0, \infty)$ with (general) Fourier coefficients, it is necessary and sufficient that $g(k)$ also satisfies

$$g(k)\in L^2(0, \infty)\ , \tag{A.5}$$

namely

$$\int_0^\infty |g(k)|^2dk<\infty\ .$$

This statement comes directly from the completeness theorem of $L^2(0, \infty)$ itself. As for the completeness of $L^2(0, \infty)$, see, for example, Kolmogorov and Formin (1970). The system of Laguerre functions, on the other hand, constitutes a complete orthogonal set on $L^2(0, \infty)$ (see, e.g., Kaczmarz and Steinhaus 1951, p. 142).

The conditions (A.4) and (A.5) are not identical. We can choose easily an example, which satisfies condition (A.4) but not condition (A.5):

$$g(k)=ck^{-1/2}\exp(-ak)\ , \tag{A.6}$$

where c is a constant. From equations (A.1) and (A.2) we have in this case

$$\left.\begin{aligned} \mu&=\frac{c\Gamma(3/2)}{(r^2+a^2)^{3/4}}\,{}_2F_1\left(\frac{3}{4},\ -\frac{1}{4};\ 1;\ \frac{r^2}{r^2+a^2}\right) \\ &\text{and} \\ \psi&=\frac{-2\pi Gc\Gamma(1/2)}{(r^2+a^2)^{1/4}}\,{}_2F_1\left(\frac{1}{4}\ ,\ \frac{1}{4};\ 1;\ \frac{r^2}{r^2+a^2}\right)\ , \end{aligned}\right\} \tag{A.7}$$

according to Watson (1922, p. 385), where ${}_2F_1$ is a hypergeometric function. In this case μ and ψ cannot be expanded by Toomre's series in the sense of convergence in the mean [i.e., in the sense of equation (4.20)].

Apart from a mathematical complexity, we can first show the equivalence of Toomre's series with Clutton-Brock's (1972) system, as a whole, in spite of Clutton-Brock's (1972) statement that the former is *ill-conditioned.*

In fact, each Laguerre function, say $L_n(2k)$, is a polynomial in k of the n-th order and the integral is given by

$$\begin{aligned} &\int_0^\infty J_0(kr)\exp(-ak)k^{n_1}dk \\ &=\frac{\Gamma(n_1+1)}{a^{n_1+1}(r^2/a^2+1)^{n_1+1/2}}\,{}_2F_1\left(\frac{-n_1}{2}\ ,\ \frac{-n_1+1}{2};\ 1;\ -\frac{r^2}{a^2}\right)\ , \end{aligned} \tag{A.8}$$

where $0\leqq n_1\leqq n$. This form reveals that the number of terms is finite and the highest power in r^2/a^2 is $n_1/2$ (n_1 being even) or $(n_1-1)/2$ (n_1 being odd). Therefore, the potential corresponding to $g(k)=L_n(2k)\exp(-ak)$ is, anyhow, composed of the terms

$$(r^2/a^2+1)^{-n_2-1/2}\ ,$$

where n_2's are non-negative integers not greater than n. This proves our statement, at least for $m=0$.

Appendix 2. *Derivation of Equation (3.1) from Recurrence Formula (3.5)*

We have omitted to derive equation (3.1) from recurrence formula (3.5) directly in the text, but have given only a suggestion for the proof by a mathematical induction. Here, we present how we have arrived at the explicit expression. The method is not deductive but includes an inductive inference.

We can obtain the following expressions of $\tilde{\phi}_n{}^m$ for $n=m+1,\ m+2,\ m+3, \cdots$ [here we have omitted a common factor $-GM\exp(im\theta)$ for brevity], by applying recurrence formula (3.5) to $\tilde{\phi}_m{}^m$, given by equation (3.6′), successively,

$$\tilde{\phi}^m_{m+1}=\frac{(2m+1)!a^m r^m}{\tilde{R}^{2m+1}}\left[+\frac{1}{m!}-\frac{(2m+2)a(a+|z|)}{(m+1)!\tilde{R}^2}\right], \tag{A.9}$$

$$\tilde{\phi}^m_{m+2}=\frac{(2m+2)!a^m r^m}{\tilde{R}^{2m+1}}\left[+\frac{1}{0!0!2!m!}-\frac{(2m+2)a(a+|z|)}{0!1!1!(m+1)!\tilde{R}^2}+\frac{(2m+4)(2m+3)a^2(a+|z|)^2}{0!2!0!(m+2)!\tilde{R}^4}-\frac{a^2}{1!0!0!(m+1)!\tilde{R}^2}\right], \tag{A.10}$$

$$\begin{aligned}\tilde{\phi}^m_{m+3}=\frac{(2m+3)!a^m r^m}{\tilde{R}^{2m+1}}&\times\left[+\frac{1}{0!0!3!m!}-\frac{(2m+2)a(a+|z|)}{0!1!2!(m+1)!\tilde{R}^2}+\frac{(2m+4)(2m+3)a^2(a+|z|)^2}{0!2!1!(m+2)!\tilde{R}^4}-\frac{a^2}{1!0!1!(m+1)!\tilde{R}^2}\right.\\&\left.-\frac{(2m+6)(2m+5)(2m+4)a^3(a+|z|)^3}{0!3!0!(m+3)!\tilde{R}^6}+\frac{(2m+4)a^3(a+|z|)}{1!1!0!(m+2)!\tilde{R}^4}\right],\end{aligned} \tag{A.11}$$

and so on. From these regularly arranged expressions we presume the following general expression, as we have done for equation (2.9),

$$\tilde{\phi}_n{}^m=\frac{(n+m)!a^m r^m}{\tilde{R}^{2m+1}}\sum_{\alpha=0}^{n-m}\sum_{i=0}^{[\alpha/2]}\frac{(-1)^{\alpha-i}(2m+2\alpha-2i)!a^\alpha(a+|z|)^{\alpha-2i}}{i!(\alpha-2i)!(n-m-\alpha)!(m+\alpha-i)!(2m+\alpha)!\tilde{R}^{2\alpha-2i}}\ . \tag{A.12}$$

It is noted that the upper limits for summations may be taken as $+\infty$, since, even if so chosen, the superfluous terms become automatically zero owing to the factorials in the denominator. In this respect, we do not need to take care of the upper limits explicitly, and may change the order of summations, if necessary, without paying any attention to the upper limits. The lower limits are always zero.

On the other hand, we presume that we otabin

$$\phi_n{}^m = P_n{}^m(\xi)/(r^2+a^2)^{1/2}$$
$$= \frac{a^m r^m}{R^{2m+1}} \sum_{k=0}^{n-m} \frac{(-1)^k(n+m+k)!}{(n-m-k)!(m+k)!k!} \left(\frac{a}{R}\right)^{2k} , \tag{A.13}$$

when $z \to 0$.

In order to obtain this relation, it is necessary and sufficient to have an identity (by replacing $\alpha = k+i$),

$$\sum_i \frac{(n+m)!(2m+2k)!}{i!(k-i)!(n-m-k-i)!(2m+k+i)!} = \frac{(n+m+k)!}{(n-m-k)!k!} . \tag{A.14}$$

This is proven, in fact, if we compare the ε^{n-m-k} terms on both sides of the identity,

$$(1+\varepsilon)^k(1+\varepsilon)^{n+m} = (1+\varepsilon)^{n+m+k} ,$$

since equation (A.14) is equivalent to

$$\sum_i \binom{k}{i} \binom{n+m}{n-m-k-i} = \binom{n+m+k}{n-m-k} . \tag{A.14$'$}$$

Now, we rearrange equation (A.12) slightly, using equations (A.14) and (A.13). First of all we have (for $\alpha = k+i$)

$$\tilde{\phi}_n{}^m = \frac{(n+m)!a^n}{(a+|z|)^n} \frac{(a+|z|)^m r^m}{\tilde{R}^{2m+1}} \sum_k (-1)^k \frac{(2m+2k)!}{(m+k)!} \left(\frac{a+|z|}{\tilde{R}}\right)^{2k}$$
$$\times \sum_i \frac{[(a+|z|)/a]^{n-m-k-i}}{i!(k-i)!(n-m-k-i)!(2m+k+i)!} . \tag{A.15}$$

The factor in the second line can be expanded and rearranged as follows:

$$\sum_j \frac{1}{j!} \left(\frac{|z|}{a}\right)^j \sum_i \frac{1}{i!(k-i)!(n-m-k-j-i)!(2m+k+i)!}$$
$$= \sum_j \frac{1}{j!} \left(\frac{|z|}{a}\right)^j \frac{(n+m+k-j)!}{(n-m-k-j)!(2m+2k)!(n+m-j)!k!} . \tag{A.16}$$

Here we have used equation (A.14), where n is replaced by $n-j$. Thus, finally we have the equation,

$$\tilde{\phi}_n{}^m = \left(\frac{a}{a+|z|}\right)^n \sum_j \frac{(n+m)!}{j!(n+m-j)!} \left(\frac{|z|}{a}\right)^j$$
$$\times \frac{(a+|z|)^m r^m}{\tilde{R}^{2m+1}} \sum_k \frac{(-1)^k(n-j+m+k)!}{(n-j-m-k)!(m+k)!k!} \left(\frac{a+|z|}{\tilde{R}}\right)^{2k}$$
$$= \left(\frac{a}{a+|z|}\right)^n \frac{1}{\tilde{R}} \sum_{j=0}^{n-m} \frac{(n+m)!}{j!(n+m-j)!} \left(\frac{|z|}{a}\right)^j P^m_{n-j}(\tilde{\xi}) , \tag{A.17}$$

which is equation (3.1) omitting the factor $-GM \exp(im\theta)$. Here we have used equation (A.13), where a is replaced by $a+|z|$, ξ by, $\tilde{\xi}$, R by $\tilde{R}$, and n by $n-j$.

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