Notes

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Contents

1 Self-gravitating thin disk with constant stellar mass

1.1 Case $\Gamma = 4/3$

Consider a self gravitating disk with surface density Σ and gravitational potential Φ. Let $\boldsymbol{v}$ be its velocity field and P its pressure. Then it is described by the system

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\frac{1}{\Sigma} \boldsymbol{\nabla} P - \boldsymbol{\nabla} \Phi, \tag{2}$$

$$\Delta \Phi_{\mathrm{disk}} = 4\pi G \Sigma \delta(z). \tag{3}$$

Here, $\Phi = \Phi^0_{\mathrm{bulge}} + \Phi_{\mathrm{disk}} + \Phi^0_{\mathrm{DH}}$, and we assume that the gas is polytropic so that $P = \alpha \Sigma^{\Gamma} = \alpha \Sigma^{4/3}$ and let

$$\psi = \int \frac{\mathrm{d}P(\Sigma)}{\Sigma} \Leftrightarrow \boldsymbol{\nabla} \psi = \frac{\boldsymbol{\nabla} P}{\Sigma}. \tag{4}$$

Then

$$\psi = 4\alpha \Sigma^{1/3}. \tag{5}$$

Letting $\Psi = \Phi + \psi$, we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{v}) = -\boldsymbol{\nabla} \Psi. \tag{6}$$

Let $x = M_{\mathrm{bulge}}/M$ and $q = M_{\mathrm{disk}}/(M_{\mathrm{DH}} + M_{\mathrm{disk}}) \in [0, 1]$, where M is the total stellar mass of the galactic disk+bulge. Then

$$\begin{aligned} M_{\mathrm{bulge}} &= xM, \\ M_{\mathrm{disk}} &= (1-x)M, \\ M_{\mathrm{DH}} &= \left(\frac{1}{q} - 1\right)(1-x)M. \end{aligned}$$

The system of equation can be rewritten as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial (r\Sigma v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\Sigma v_\theta)}{\partial \theta} = 0, \tag{7}$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{(v_\theta)^2}{r} = -\frac{\partial \Psi}{\partial r}, \tag{8}$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \tag{9}$$

$$\Delta \Phi_{\rm disk} = 4\pi G \Sigma \delta(z). \tag{10}$$

Let $\Phi^0_{\rm bulge} = -GM_{\rm bulge}/\sqrt{a_{\rm b}^2 + r^2} = -GMx/\sqrt{a_{\rm b}^2 + r^2}$ with $x \in [0,1]$. An equilibrium state is given by the Plummer equilibrium, such that, letting $\xi = (r^2 - a_{\rm d}^2)/(r^2 + a_{\rm d}^2)$, that is, $r/a_{\rm d} = \sqrt{(1+\xi)/(1-\xi)}$,

$$v_r^0 = 0, \qquad (v_\theta^0)^2 = r\frac{\partial \Psi^0}{\partial r}, \tag{11}$$

$$\psi^0 = 4\alpha (\Sigma^0)^{1/3}, \tag{12}$$

$$\Sigma^0 = \frac{M_{\rm disk}}{2\pi a_{\rm d}^2}\frac{1}{(1+(r/a_{\rm d})^2)^{3/2}} = \frac{(1-x)M}{2\pi a_{\rm d}^2}\left(\frac{1-\xi}{2}\right)^{3/2}, \tag{13}$$

$$\Phi^0_{\rm DH} = -\frac{GM_{\rm DH}}{a_{\rm h}}\frac{1}{\sqrt{1+(r/a_{\rm h})^2}} = -\frac{GM(1-x)}{a_{\rm d}}\left(\frac{1}{q} - 1\right)\frac{a_{\rm d}}{a_{\rm h}\sqrt{1+(r/a_{\rm h})^2}}, \tag{14}$$

$$\Phi^0_{\rm disk} = -\frac{GM_{\rm disk}}{a_{\rm d}}\frac{1}{\sqrt{1+(r/a_{\rm d})^2}} = -\frac{GM(1-x)}{a_{\rm d}}\left(\frac{1-\xi}{2}\right)^{1/2}, \tag{15}$$

$$\Phi^0_{\rm bulge} + \Phi^0_{\rm DH} = -\frac{GM}{a_{\rm d}}\left[\frac{a_{\rm d}}{a_{\rm b}}\frac{x}{\sqrt{1+(r/a_{\rm b})^2}} + \frac{a_{\rm d}}{a_{\rm h}}\left(\frac{1}{q} - 1\right)\frac{1-x}{\sqrt{1+(r/a_{\rm h})^2}}\right], \tag{16}$$

$$\tag{17}$$

Therefore, letting

$$\varepsilon_0 = \frac{U(x=0, q=1)}{|E(x=0,q=1)|} = \frac{4a_{\rm d}\alpha}{GM}\left(\frac{M}{2\pi a_{\rm d}^2}\right)^{1/3},$$

and

$$\varepsilon = \frac{U_{\rm disk}}{|E_{\rm disk}|} = \frac{4a_{\rm d}\alpha}{GM_{\rm disk}}\left(\frac{M_{\rm disk}}{2\pi a_{\rm d}^2}\right)^{1/3} = \frac{\varepsilon_0}{(1-x)^{2/3}},$$

we obtain

$$\Psi^0 = -\frac{GM}{a_\mathrm{d}}\left[\frac{a_\mathrm{d}}{a_\mathrm{b}}\frac{x}{\sqrt{1+(r/a_\mathrm{b})^2}} + \frac{a_\mathrm{d}}{a_\mathrm{h}}\left(\frac{1}{q}-1\right)\frac{1-x}{\sqrt{1+(r/a_\mathrm{h})^2}} + (1-x)\left(\frac{1-\xi}{2}\right)^{1/2}\right] + 4\alpha(\Sigma^0)^{1/3}, \tag{18}$$

$$= -\frac{GM}{a_\mathrm{d}}\left[\frac{a_\mathrm{d}}{a_\mathrm{b}}\frac{x}{\sqrt{1+(r/a_\mathrm{b})^2}} + \frac{a_\mathrm{d}}{a_\mathrm{h}}\left(\frac{1}{q}-1\right)\frac{1-x}{\sqrt{1+(r/a_\mathrm{h})^2}} + \left(1-x-\varepsilon_0(1-x)^{1/3}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\right], \tag{19}$$

$$= -\frac{GM(1-x)}{a_\mathrm{d}}\left[\frac{a_\mathrm{d}}{a_\mathrm{b}}\frac{x}{1-x}\frac{1}{\sqrt{1+(r/a_\mathrm{b})^2}} + \frac{a_\mathrm{d}}{a_\mathrm{h}}\left(\frac{1}{q}-1\right)\frac{1}{\sqrt{1+(r/a_\mathrm{h})^2}} + \left(1-\frac{\varepsilon_0}{(1-x)^{2/3}}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\right], \tag{20}$$

$$(v_\theta^0)^2 = r\frac{\partial \Psi^0}{\partial r} = r\frac{\partial \Psi^0}{\partial \xi}\frac{\partial \xi}{\partial r} = 4\left(\frac{r}{a_\mathrm{d}}\right)^2\left(\frac{1-\xi}{2}\right)^2\frac{\partial \Psi^0}{\partial \xi} = (1+\xi)(1-\xi)\frac{\partial \Psi^0}{\partial \xi}, \tag{21}$$

$$= \frac{GM(1-x)}{a_\mathrm{d}}\left[\frac{a_\mathrm{d}x}{a_\mathrm{b}(1-x)}\left(\frac{r}{a_\mathrm{b}}\right)^2\left(\frac{1}{1+(r/a_\mathrm{b})^2}\right)^{3/2} + \frac{a_\mathrm{d}}{a_\mathrm{h}}\left(\frac{1}{q}-1\right)\left(\frac{r}{a_\mathrm{h}}\right)^2\left(\frac{1}{1+(r/a_\mathrm{h})^2}\right)^{3/2}\right.$$
$$\left. + \left(1-\frac{\varepsilon_0}{(1-x)^{2/3}}\right)\left(\frac{1+\xi}{2}\right)\left(\frac{1-\xi}{2}\right)^{1/2}\right], \tag{22}$$

with

$$\frac{1-\xi}{2} = \frac{a_\mathrm{d}^2}{r^2+a_\mathrm{d}^2} = \frac{1}{1+(r/a_\mathrm{d})^2}; \quad \frac{1+\xi}{2} = \frac{r^2}{r^2+a_\mathrm{d}^2} = \frac{(r/a_\mathrm{d})^2}{1+(r/a_\mathrm{d})^2}.$$

Note that for $x = 0$ (only disk) and $q = 1$ (purely self-gravitating system), we recover the expression of Toomre

$$v_\theta^0 = \left(\frac{GM}{a_\mathrm{d}}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sqrt{1-\varepsilon_0}. \tag{23}$$

The perturbative equation at 1st order are

$$\frac{\partial v_r^p}{\partial t} + \frac{v_\theta^0}{r}\frac{\partial v_r^p}{\partial \theta} - 2\frac{v_\theta^0 v_\theta^p}{r} = -\frac{\partial \Psi^p}{\partial r}, \tag{24}$$

$$\frac{\partial v_\theta^p}{\partial t} + v_r^p\frac{\partial v_\theta^0}{\partial r} + \frac{v_\theta^0}{r}\frac{\partial v_\theta^p}{\partial \theta} + \frac{v_r^p v_\theta^0}{r} = -\frac{1}{r}\frac{\partial \Psi^p}{\partial \theta}, \tag{25}$$

$$\Delta\Phi_\mathrm{disk}^p = 4\pi G\Sigma^p\delta(z), \tag{26}$$

$$\psi^p = \kappa\Gamma(\Sigma^0)^{-2/3}\Sigma^p. \tag{27}$$

We define $X^p(r,\theta,t) = \sum_{m\in\mathbb{Z}} X_m^p(r,t)e^{im\theta}$ and look for a temporal dependency in $e^{-i\omega t}$. Aoki & Iye say that there is this following correspondance between surface density and gravitational potential through the Poisson equation:

$$(\Sigma)_m^p(r,t) = \frac{M(1-x)}{2\pi a_\mathrm{d}^2}\left(\frac{1-\xi}{2}\right)^{3/2}\sum_{n=|m|}^{\infty} a_n^m\widehat{P_n^{|m|}}(\xi)e^{-i\omega t}, \tag{28}$$

$$(\Phi_\mathrm{disk})_m^p(r,t) = -\frac{GM(1-x)}{a_\mathrm{d}}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\frac{a_n^m}{2n+1}\widehat{P_n^{|m|}}(\xi)e^{-i\omega t}, \tag{29}$$

$$(\psi)_m^p(r,t) = \frac{4\alpha}{3}\left(\frac{M}{2\pi a_\mathrm{d}^2}\right)^{1/3}(1-x)^{1/3}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty} a_n^m\widehat{P_n^{|m|}}(\xi)e^{-i\omega t}, \tag{30}$$

$$(\Psi)_m^p(r,t) = \frac{GM(1-x)}{a_\mathrm{d}}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{\varepsilon_0}{3(1-x)^{2/3}} - \frac{1}{2n+1}\right]a_n^m\widehat{P_n^{|m|}}(\xi)e^{-i\omega t}. \tag{31}$$

Let us decompose the velocity components based on their equilibrium expression:

$$(v_r)^p_m = i\frac{m}{|m|}\left(\frac{GM(1-x)}{a_\mathrm{d}}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} b^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}, \tag{32}$$

$$(v_\theta)^p_m = \left(\frac{GM(1-x)}{a_\mathrm{d}}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} c^m_n \widehat{P^{|m|}_n}(\xi)e^{-i\omega t}. \tag{33}$$

Letting $X^p_m = X^1 e^{-i\omega t}$, this yields the set of equations

$$i(-\omega + m\Omega)(\Sigma_\mathrm{disk})^1 + \frac{1}{r}\frac{\mathrm{d}(r\Sigma_0(v_r)^1)}{\mathrm{d}r} + \frac{im\Sigma^0(v_\theta)^1}{r} = 0, \tag{34}$$

$$\frac{\mathrm{d}(\Psi)^1}{\mathrm{d}r} + i(-\omega + m\Omega)(v_r)^1 - 2\Omega(v_\theta)^1 = 0, \tag{35}$$

$$im\frac{(\Psi)^1}{r} + \frac{\kappa^2}{2\Omega}(v_r)^1 + i(-\omega + m\Omega)(v_\theta)^1 = 0, \tag{36}$$

where $\Omega = v^0_\theta/r$ is the angular velocity and $\kappa^2 = 4\Omega^2[1 + r/(2\Omega)\cdot(\mathrm{d}\Omega/\mathrm{d}r)]$ is the epicyclic frequency. Using the relation

$$\int_{-1}^{1}\mathrm{d}\xi \widehat{P^{|m|}_n}(\xi)\widehat{P^{|m|}_l}(\xi) = \delta_{nl},$$

and defining $\Omega_\mathrm{ref} = \sqrt{GM/a^3_\mathrm{d}}$, $\Sigma_\mathrm{ref} = M/(2\pi a^2_\mathrm{d})$ such that $\widehat{\omega} = \omega/\Omega_\mathrm{ref}$, $\widehat{\Omega} = \Omega/\Omega_\mathrm{ref}$, $\widehat{\kappa} = \kappa/\Omega_\mathrm{ref}$, $\widehat{\Sigma} = \Sigma/\Sigma_\mathrm{ref}$ and $\lambda = \frac{|m|}{m}\widehat{\omega}$, we obtain the matrix equations

$$\sum_{n=|m|}^{\infty} A_{ln}a^m_n + \sum_{n=|m|}^{\infty} B_{ln}b^m_n + \sum_{n=|m|}^{\infty} C_{ln}c^m_n = \lambda a^m_l, \tag{37}$$

$$\sum_{n=|m|}^{\infty} D_{ln}a^m_n + \sum_{n=|m|}^{\infty} A_{ln}b^m_n + \sum_{n=|m|}^{\infty} F_{ln}c^m_n = \lambda b^m_l, \tag{38}$$

$$\sum_{n=|m|}^{\infty} G_{ln}a^m_n + \sum_{n=|m|}^{\infty} H_{ln}b^m_n + \sum_{n=|m|}^{\infty} A_{ln}c^m_n = \lambda c^m_l, \tag{39}$$

where we defined

$$A_{ln} = |m|\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\widehat{\Omega}(\xi)\widehat{P^{|m|}_n}(\xi), \tag{40}$$

$$B_{ln} = 4\sqrt{1-x}\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\left(\frac{1-\xi}{2}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{1-\xi}{2}\right)^{5/4}\widehat{P^{|m|}_n}(\xi)\right], \tag{41}$$

$$C_{ln} = |m|\sqrt{1-x}\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\left(\frac{1-\xi}{2}\right)^{3/4}\left(\frac{1+\xi}{2}\right)^{-1}\widehat{P^{|m|}_n}(\xi), \tag{42}$$

$$D_{ln} = 4\sqrt{1-x}\left(\frac{1}{2n+1} - \frac{\varepsilon_0}{3}\frac{1}{(1-x)^{2/3}}\right)\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\left(\frac{1-\xi}{2}\right)^{5/4}\left(\frac{1+\xi}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{1-\xi}{2}\right)^{1/2}\widehat{P^{|m|}_n}(\xi)\right], \tag{43}$$

$$F_{ln} = 2\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\widehat{\Omega}(\xi)\widehat{P^{|m|}_n}(\xi), \tag{44}$$

$$G_{ln} = -|m|\sqrt{1-x}\left(\frac{1}{2n+1} - \frac{\varepsilon_0}{3}\frac{1}{(1-x)^{2/3}}\right)\int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\left(\frac{1-\xi}{2}\right)^{3/4}\widehat{P^{|m|}_n}(\xi), \tag{45}$$

$$H_{ln} = \int_{-1}^{1}\mathrm{d}\xi\widehat{P^{|m|}_l}(\xi)\frac{\widehat{\kappa}^2(\xi)}{2\widehat{\Omega}(\xi)}\widehat{P^{|m|}_n}(\xi). \tag{46}$$

with

$$\widehat{\Omega}(\xi) = \sqrt{1-x}\left(\frac{1-\xi}{2}\right)^{3/4}\left[\left(\frac{a_\mathrm{d}}{a_\mathrm{b}}\right)^3\frac{x}{1-x}\left(\frac{1-\xi}{2}\right)^{-3/2}\left(\frac{1}{1+(r/a_\mathrm{b})^2}\right)^{3/2}\right.$$

$$\left. + \left(\frac{a_\mathrm{d}}{a_\mathrm{h}}\right)^3\left(\frac{1}{q}-1\right)\left(\frac{1-\xi}{2}\right)^{-3/2}\left(\frac{1}{1+(r/a_\mathrm{h})^2}\right)^{3/2} + \left(1-\frac{\varepsilon_0}{(1-x)^{2/3}}\right)\right]^{1/2}, \tag{47}$$

$$\frac{\widehat{\kappa}^2(\xi)}{2\widehat{\Omega}(\xi)} = 2\widehat{\Omega}(\xi)\left[1+\frac{(1+\xi)(1-\xi)}{2\widehat{\Omega}}\frac{\mathrm{d}\widehat{\Omega}}{\mathrm{d}\xi}\right], \tag{48}$$

Setting $a_\mathrm{d} = a_\mathrm{h} = a_\mathrm{b}$, the angular frequency reads as

$$\frac{\widehat{\Omega}(\xi)}{\sqrt{1-x}} = \left(\frac{1-\xi}{2}\right)^{3/4}\sqrt{\frac{x}{1-x}+\left(\frac{1}{q}-\frac{\varepsilon_0}{(1-x)^{2/3}}\right)} = \left(\frac{1-\xi}{2}\right)^{3/4}\sqrt{\frac{x}{1-x}+\left(\frac{1}{q}-\varepsilon\right)}, \tag{49}$$

where ε is the temperature of the disk. Echanging $M_\mathrm{bulge} \leftrightarrow M_\mathrm{DH}$ with $M_\mathrm{disk} \propto 1-x$ fixed leaves the angular frequency unchanged since

$$\frac{x}{1-x}+\frac{1}{q}-\varepsilon = \frac{M_\mathrm{bulge}}{M_\mathrm{disk}}+\frac{M_\mathrm{disk}+M_\mathrm{DH}}{M_\mathrm{disk}}-\varepsilon = \frac{M_\mathrm{disk}+M_\mathrm{DH}+M_\mathrm{bulge}}{M_\mathrm{disk}}-\varepsilon.$$

NB: ε_0 is fixed when M is kept constant, while ε is fixed when M_disk is kept constant.

1.2 Computation of matrix elements

Integrals A_{ln} and F_{ln} are proportional. With the addition of H_{ln}, those 3 integrals must be computed numerically because of the non-trivial shift in their expression induced by the bulge potential. As for B_{ln}, C_{ln}, D_{ln} and G_{ln}, their can be can expressed in terms of the two following integrals

$$\widehat{I}(l,n) = \int_{-1}^{1}\mathrm{d}\xi\left(\frac{1-\xi}{2}\right)^{3/4}\widehat{P_l^{|m|}}(\xi)\widehat{P_n^{|m|}}(\xi), \tag{50}$$

$$\widehat{J}(l,n) = \int_{-1}^{1}\mathrm{d}\xi\left(\frac{1-\xi}{2}\right)^{3/4}\left(\frac{1+\xi}{2}\right)^{-1}\widehat{P_l^{|m|}}(\xi)\widehat{P_n^{|m|}}(\xi), \tag{51}$$

as

$$B_{ln} = \frac{\sqrt{1-x}}{2}\left[\sqrt{\frac{(2l+1)(l+m+1)(l-m+1)}{2l+3}}\widehat{J}(l+1,n)+\widehat{J}(l,n)\right. \tag{52}$$

$$\left. -\sqrt{\frac{(2l+1)(l+m)(l-m)}{2l-1}}\widehat{J}(l-1,n)\right], \tag{53}$$

$$C_{ln} = m\sqrt{1-x}\widehat{J}(l,n), \tag{54}$$

$$D_{ln} = \frac{\sqrt{1-x}}{2}\left(\frac{1}{2n+1}-\frac{\varepsilon_0}{3}\frac{1}{(1-x)^{2/3}}\right)\left[-\sqrt{\frac{(2n+1)(n+m+1)(n-m+1)}{2n+3}}\widehat{I}(l,n+1)\right. \tag{55}$$

$$-\widehat{I}(l,n) \tag{56}$$

$$\left. +\sqrt{\frac{(2n+1)(n+m)(n-m)}{2n-1}}\widehat{I}(l,n-1)\right], \tag{57}$$

$$G_{ln} = -|m|\sqrt{1-x}\left(\frac{1}{2n+1}-\frac{\varepsilon_0}{3}\frac{1}{(1-x)^{2/3}}\right)\widehat{I}(l,n). \tag{58}$$

where $\widehat{I}(l,n)$ and $\widehat{J}(l,n)$ can be computed by recursion and using the symmetry $l \leftrightarrow n$. Defining

$$\widehat{I}'(l,n) = \int_{-1}^{1}\mathrm{d}\xi\,\xi\left(\frac{1-\xi}{2}\right)^{3/4}\widehat{P_l^{|m|}}(\xi)\widehat{P_n^{|m|}}(\xi), \tag{59}$$

$$\widehat{J}'(l,n) = \int_{-1}^{1}\mathrm{d}\xi\,\xi\left(\frac{1-\xi}{2}\right)^{3/4}\left(\frac{1+\xi}{2}\right)^{-1}\widehat{P_l^{|m|}}(\xi)\widehat{P_n^{|m|}}(\xi), \tag{60}$$

Starting from(Aoki79, A16)

$$\widehat{I}(l,n) = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\widehat{I}'(l,n-1) - \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}}\widehat{I}(l,n-2), \quad (61)$$

$$\widehat{I}'(l,n-1) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\widehat{I}(l+1,n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\widehat{I}(l-1,n-1), \quad (62)$$

hence to compute until $l, n = m + N$, we need to compute $\widehat{I}(l', m)$ until $l' = m + 2N$. By convention (for the recursion), we have set $\widehat{I}(l', n') = 0$ for $l' < m$ or $n' < m$. We initialize with

$$\widehat{I}(l,m) = \frac{l - 3/4 - m - 1}{l + 3/4 + m + 1}\sqrt{\frac{(l+m)(2l+1)}{(l-m)(2l-1)}}\widehat{I}(l-1,m), \quad (63)$$

$$\widehat{I}(m,m) = 2^m \prod_{k=0}^{m} \frac{2k+1}{3/4 + m + 1 + k}. \quad (64)$$

We proceed as follows:
- Compute the line $n = m$: $\widehat{I}(m,m)$, $\widehat{I}(m+1,m)$, ... , $\widehat{I}(m+2N,m)$
- Complete the line $l = m$ by symmetry
- Compute the line $n = m+1$: $\widehat{I}(m+1,m+1)$, $\widehat{I}(m+2,m+1)$, ... , $\widehat{I}(m+2N-1,m+1)$
- Complete the line $l = m+1$ by symmetry
- Compute the line $n = m+2$: $\widehat{I}(m+2,m+2)$, $\widehat{I}(m+3,m+2)$, ... , $\widehat{I}(m+2N-2,m+2)$
- ...
- Compute the line $n = m+N-1$: $\widehat{I}(m+N-1,m+N-1)$, $\widehat{I}(m+N+1,m+N-1)$.
- Complete the line $l = m+N-1$ by symmetry
- Compute the line $n = m+N$: $\widehat{I}(m+N,m+N)$.

As for $\widehat{J}$, let

$$\widehat{I}_\alpha(l,m) = \frac{(-1)^{l-m}(2m-1)!!2^{m+1}\Gamma(\alpha+1)\Gamma(\alpha+m+1)(l+m)!}{\Gamma(\alpha+1-l+m)\Gamma(\alpha+m+l+2)(l-m)!} \quad (65)$$

with $\widehat{I}_{3/4}(l,m) = \widehat{I}(l,m)$. We can also compute it by recursion using the formulae

$$\widehat{J}(l,n) = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\widehat{J}'(l,n-1) - \sqrt{\frac{(n+m-1)(n-m-1)(2n+1)}{(n+m)(n-m)(2n-3)}}\widehat{J}(l,n-2), \quad (66)$$

$$\widehat{J}'(l,n-1) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\widehat{J}(l+1,n-1) + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\widehat{J}(l-1,n-1), \quad (67)$$

hence to compute until $l, n = m + N$, we need to compute $\widehat{I}(l', m)$ until $l' = m + 2N$. By convention (for the recursion), we have set $\widehat{I}(l', n') = 0$ for $l' < m$ or $n' < m$. We initialize with

$$\widehat{J}(l,m) = \sqrt{\frac{(l-m)(l-m-1)(2l+1)}{(l+m)(l+m-1)(2l-3)}}\widehat{J}(l-2,m) + 4\sqrt{\frac{(2m+1)(2l+1)(2l-1)}{2m(l+m)(l+m-1)}}\widehat{I}_{7/3}(l-1,m-1), \quad (68)$$

$$\widehat{J}(m,m) = \frac{2^m}{m}\prod_{k=1}^{m}\frac{2k+1}{3/4+m+k}, \quad (69)$$

$$\widehat{J}(m+1,m) = -\frac{7}{4}\frac{2^m\sqrt{2m+3}}{m}\prod_{k=0}^{m}\frac{2k+1}{3/4+m+1+k} = -\frac{7\sqrt{2m+3}}{4(3/4+2m+1)}\widehat{J}(m,m). \quad (70)$$

We can compute the $\widehat{I}_{7/4}$ part by recursion. Indeed,

$$\widehat{I}_\alpha(l,m) = \frac{l-\alpha-m-1}{l+\alpha+m+1}\sqrt{\frac{(l+m)(2l+1)}{(l-m)(2l-1)}}\widehat{I}_\alpha(l-1,m), \tag{71}$$

$$\widehat{I}_\alpha(m,m) = 2^m \prod_{k=0}^{m} \frac{2k+1}{\alpha+m+1+k}. \tag{72}$$

Hence, we need to compute beforehand the values $\widehat{I}_\alpha(l', m-1)$ for $l = m-1, ..., m+2N-1$, and then apply the same process as for $\widehat{I}$.

As for the numerical integral, we use a simple midpoint rule with K points. Those integrals have the form

$$I_{ln} = \int_{-1}^{1} \mathrm{d}\xi \widehat{P_l^{|m|}}(\xi)\phi(\xi)\widehat{P_n^{|m|}}(\xi) \approx \frac{2}{K}\sum_{k=1}^{K} \widehat{P_l^{|m|}}(\xi_k)\phi(\xi_k)\widehat{P_n^{|m|}}(\xi_k), \tag{73}$$

where $\xi_k = -1 + (2/K)(k-1/2)$. As we with to compute those elements for $m \leq l, n \leq m+N$, we have to compute the $\widehat{P_l^{|m|}}(\xi_k)$ for $n = m, ..., m+N$ and $k = 1, ..., K$. To that end, We compute before hand a table of the values $\{\widehat{P_l^{|m|}}(\xi_k)\}_{(n,k)}$ and of the values $\{\phi(\xi_k)\}_k$. The Legendre associated functions can be efficiently computed by using the Julia library "SphericalHarmomics", in which we use the function "computePlmcostheta($\theta, l_{\max}, m$)" which compute $\widehat{P_l^{|m|}}(\cos(\theta))/\sqrt{\pi}$ for all $n = 0, ..., l_{\max}$ at a given m.

1.3 Eigenmode representation

Among the eigenvalues of the truncated matrix, we select the physical ones (those which converge) within some precision threshold (in distance and in variance) and select the growth rate (i.e. the eigenvalue with the maximum imaginary part). Figure 1 and 2 represent the growth rate w.r.t x the bulge fraction in the disk and q the self-gravity parameter. The relevant region seems to within the triangle with the 3 vertices $(x, q) = (0.0, 1.0), (0.0, 0.5), (0.5, 1.0)$.

Associated with the fastest growing eigenmode is an eigenvector, from which we can recover the over-density through

$$(\Sigma)_m^p(r,\theta,t) = \frac{M(1-x)}{2\pi a_\mathrm{d}^2}\left(\frac{1-\xi}{2}\right)^{3/2}\sum_{n=|m|}^{\infty} a_n^m \widehat{P_n^{|m|}}(\xi)e^{i(m\theta-\omega t)}. \tag{74}$$

The corotation radius r_corot is defined by the equation $m\widehat{\Omega}(r_\mathrm{corot}) = \Re(\widehat{\omega})$.

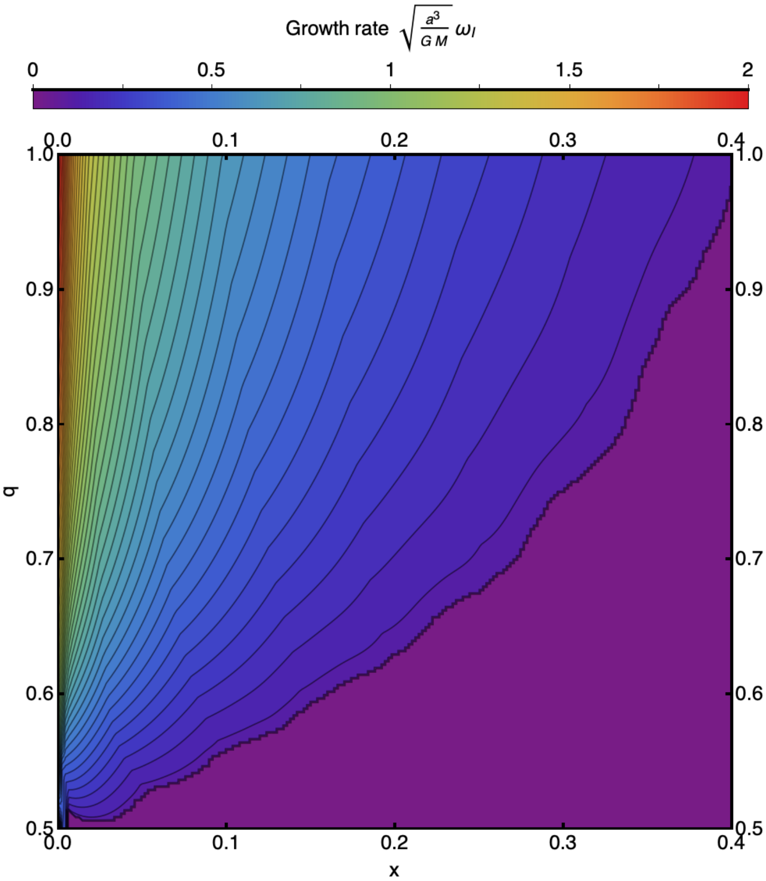

Figure 1: Growth rate (contour plot) at $\epsilon = 0.1$ with $N = 170$.

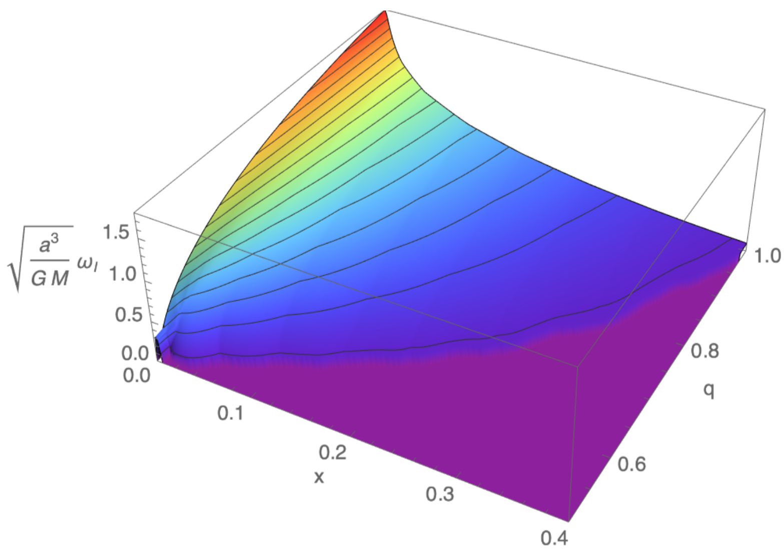

Figure 2: Growth rate (3D plot) at $\epsilon = 0.1$ with $N = 170$.