1 (c) $y'' - 6y' + 9y = 0$, $y(0) = 2$, $y'(0) = 9$

Applying L.T to D.E we get

$\cancel{s^2 Y - s f(0) - s f'(0)}$

$s^2 Y - s\,y(0) - y'(0) - 6\left[sY - y(0)\right] + 9Y = 0.$

$\Rightarrow s^2 Y - 2s - 9 - 6\left[sY - 2\right] + 9Y = 0$

$\Rightarrow s^2 Y - 2s - 9 - 6sY + 12 + 9Y = 0$

$\Rightarrow Y\left(s^2 - 6s + 9\right) = 2s - 3$

$\Rightarrow Y = \dfrac{2s - 3}{s^2 - 6s + 9}$

$y = L^{-1}(Y) = L^{-1}\left\{\dfrac{2s-3}{s^2-6s+9}\right\}$

$= L^{-1}\left\{\dfrac{2s-3}{s^2 - 2.3.s + 3^2}\right\}$

$= L^{-1}\left\{\dfrac{2s-3}{(s-3)^2}\right\}$

$= L^{-1}\left\{\dfrac{2(s-3)+3}{(s-3)^2}\right\}$

$= 2L^{-1}\left\{\dfrac{1}{(s-3)}\right\} + 3L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$

$= 2e^{3t} + 3e^{3t}.t$

$= e^{3t}(2 + 3t)$

Q)

(a) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$\Rightarrow$ Here $F(s) = \frac{s}{s^2+a^2}$ $\quad G(s) = \frac{s}{(s^2+b^2)}$

$f(t) = \cos at$ & $g(t) = \cos bt$

$L^{-1}\left\{\frac{s^2}{\ldots}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\, du$

$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du$

$= \frac{1}{2}\left[\frac{\sin au+bt-bu}{a-b} + \frac{\sin au-bt+bu}{a+b}\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin at+\cancel{bt}-\cancel{bt}}{a-b} + \frac{\sin at-\cancel{bt}+\cancel{bt}}{a+b} - \frac{\sin 0+bt-0}{a-b} - \frac{\sin 0-bt+0}{a+b}\right]$

$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$

$= \frac{1}{2}\left[\frac{a\sin at + b\sin at + a\sin at - b\sin at - a\sin bt - b\sin bt + a\sin bt - b\sin bt}{a^2-b^2}\right]$

e)

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2 - b^2}\right]$$

$$= \left[\frac{a\sin at - b\sin bt}{a^2 - b^2}\right]$$

e) $$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$

Here $F(s) = \frac{1}{(s+2)^2}$, $G(s) = \frac{1}{(s-2)}$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}\cdot t$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

Now

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \int_0^t e^{-2u}\cdot u\, e^{2(t-u)}\,du$$

$$= \int_0^t \left(e^{-2u}\cdot e^{-2u}\cdot u e^{2t}\right)du$$

$$= e^{2t}\int_0^t e^{-4u}u\,du$$

$$= e^{2t}\left[\frac{ue^{-4u}}{-4} - \int\frac{e^{-4u}}{-4}du\right]_0^t$$

$$= e^{2t}\left[\frac{-ue^{-4u}}{4} - \frac{e^{-4u}}{4\times4}\right]_0^t$$

$$= e^{2t}\left(-\frac{te^{-4t}}{4} - \frac{e^{-4t}}{4\times4} + 0 + \frac{e^0}{4\times4}\right) = \cancel{e^{2t}e^{-4t}}$$

$$= \frac{e^{2t}}{4} \left(-t e^{-4t} \cancel{+} \frac{e^{-4t}}{4} + \frac{1}{4} \right) \quad \text{(Ans)}$$

3(c) $\frac{s^3 + 6s^2 + 14s}{(s+2)^4}$

$$= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$\Rightarrow (s+2)^3 A + (s+2)^2 B + (s+2) C + D = s^3 + 6s^2 + 14s$$

put $s = -2$

$$D = (-2)^3 + 6(-2)^2 + 14(-2) = -8 + 24 - 28 = -12$$

$$\Rightarrow \boxed{D = -12}$$

put $s = 0$

$$\Rightarrow 8A + 4B + 2C \cancel{+(-12)} - 12 = 0 \quad \text{—— (1)}$$

put $s = 1$

$$\Rightarrow 27A + 9B + 3C - 12 = 21$$

$$\Rightarrow 27A + 9B + 3C = 33 \quad \text{—— (2)}$$

put $s = -1$

$$\Rightarrow A + B + C - 12 = -1 + 6 - 14$$

$$\Rightarrow A + B + C = 3 \quad \text{—— (3)}$$

Solving eqn (1), (2) and (3) we get

~~A = 0~~
~~B = 4~~
~~C = -1~~

$A = 1$
$B = 0$
$C = 2$

Now

$$L^{-1}\left\{\frac{s^3+6s^2+14s}{(s+2)^4}\right\} =$$

$$L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\{0\} + L^{-1}\left\{\frac{2}{(s+2)^3}\right\} + L^{-1}\left\{\frac{-12}{(s+2)^4}\right\}$$

$$= e^{-2t} + \frac{\cancel{2}e^{-2t}}{\cancel{2}}\cdot t^2 - \frac{\cancel{12}^{\,2}}{\cancel{6}} e^{-2t}t^3$$

$$\cancel{= e^{-2t}\left(1 + e^{2}\right.}$$

$$= e^{-2t}\left(1 + t^2 - 2t^3\right)$$

(d) $L^{-1}\left\{\dfrac{s+1}{(s^2+1)(s^2+4)}\right\}$

~~Other~~

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Ds+C}{(s^2+4)}$$

$$\Rightarrow As+B(s^2+4) + Ds+C(s^2+1) = s+1$$

$$\Rightarrow s^3A + 4As + Bs^2 + 4B + Ds^3 + Ds + Cs^2 + C = s+1$$

$$\Rightarrow s^3(A+D) + s(4A+D) + s^2(B+C) + 4B+C = s+1$$

$$\Rightarrow A+D = 0 \quad \text{—— (1)}$$

$$4A+D = 1 \quad \text{—— (2)} \Rightarrow D = 1-4A$$

$$B+C = 0 \quad \text{—— (3)}$$

$$4B+C = 1 \quad \text{—— (4)} \Rightarrow C = 1-4B$$

putting values of D & C in eq^n (1) and 3 respectively

we get $A + 1 - 4A = 0 \Rightarrow -3A = -1 \Rightarrow A = 1/3$

similarly $B = 1/3$, $C = D = -1/3$

so $L^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

$$= L^{-1}\left\{\frac{\frac{s}{3}+\frac{1}{3}}{s^2+1}\right\} - L^{-1}\left\{\frac{\frac{s}{3}+\frac{1}{3}}{s^2+4}\right\}$$

$$= \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$- \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$= \frac{1}{3}\left[\cos t + \sin t - \cos 2t - \frac{\sin 2t}{2}\right]$$

6) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Here $F(s) = \frac{s}{s^2+a^2}$ $G(s) = \frac{1}{s^2+a^2}$

$f(t) = \cos at$ $g(t) = \sin at$

So $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ ~~$= L^{-1}\{$~~

$$= \int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\sin(\cancel{au}+at-\cancel{au}) - \sin(au-at+au)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\sin at - \sin(2au-at)\right]du$$

8)

$= \frac{1}{2}\left[\sin at \cdot t + \left[\frac{\cos(2au - at)}{2a}\right]_0^t\right]$

$= \frac{1}{2}\left\{t \sin at + \left[\frac{\cos(2at - at) - \cos(-at)}{2a}\right]\right\}$

$= \frac{1}{2} t \sin at + \left[\frac{\cancel{\cos at} - \cancel{\cos at}}{2a}\right]$

$= \frac{t \sin at}{2}$

8) $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

$= L^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\}$

$= L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$

$= 4e^{-4t} L^{-1}\left\{\frac{s}{s^2}\right\} - 4L^{-1}\left\{\frac{1}{(s+4)^2}\right\}$

$= 4e^{-4t} - 4e^{-4t} \cdot t$

$= 4e^{-4t}(1-t)$

$u)] du$

11) $L^{-1}\left\{\frac{a^3}{s^4-a^4}\right\}$

$= a^3 L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2-a^2)}\right\}$

~~here if $F(s) = \frac{1}{s^2+a^2}$, $G(s) = \frac{1}{s^2-a^2}$~~

~~$f(t) = \sin at$, $g(t) = \sinh at$~~

$$\frac{1}{(s^2+a^2)(s^2-a^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2-a^2}$$

$\Rightarrow (s^2-a^2)(As+B) + (s^2+a^2)(Cs+D) = 1$

$\Rightarrow s^3A + Bs^2 - a^2As + Cs^3 + Ds^2 + a^2Cs + a^2D - a^2B = 1$

$\Rightarrow s^3(A+C) + s^2(B+D) + s(a^2C - a^2A) + a^2D - a^2B = 1$

here

$A+C=0$ —— 1

$B+D=0$ —— 2

$C-A=0$ —— 3

$D-B=1/a^2$ —— 4

from eq^n ① and ③

$$A + C = 0$$
$$-A + C = 0$$
$$2C = 0$$
$$\boxed{C = 0}$$
$$\Rightarrow \boxed{A = 0}$$

from eq^n 2 and 4

$$B + D = 0$$
$$-B + D = 1/a^2$$
$$2D = 1/a^2$$
$$\boxed{D = 1/2a^2}$$
$$\boxed{B = -1/2a^2}$$

So

$$a^3 L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2-a^2)}\right\}$$

$$= a^3\left[L^{-1}\left\{\frac{-1/2a^2}{s^2+a^2}\right\} + L^{-1}\left\{\frac{1/2a^2}{s^2-a^2}\right\}\right]$$

$$= \frac{1}{2}\left[-\sin at + \sinh at\right] \text{ (Proved)}$$

Hence $L^{-1}\left\{\frac{64}{81s^4-256}\right\}$

$$= L^{-1}\left\{\frac{\frac{64}{81}}{s^4-\left(\frac{4}{3}\right)^4}\right\} = \frac{1}{3} L^{-1}\left\{\frac{\left(\frac{4}{3}\right)^3}{s^4-\left(\frac{4}{3}\right)^4}\right\}$$

$$= \frac{1}{6}\left(\sinh\frac{4t}{3} - \sin\frac{4t}{3}\right)$$

13)

$$L\left\{\frac{\cos at - \cos bt}{t}\right\}$$

$$L\{\cos at - \cos bt\}$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right) du$$

$$= \frac{1}{2}\left[\ln(u^2+a^2) - \ln(u^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[(\cancel{\ln\infty - \ln\infty})^{0} - \ln(s^2+a^2) + \ln(s^2+b^2)\right]$$

$$= \frac{1}{2}\left[\ln(s^2+b^2) - \ln(s^2+a^2)\right]$$

$$= \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Now $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$ $\cancel{= f}$

~~here a = 6~~

~~b = 4~~

~~here~~

16)

$$= \int_0^\infty e^{-0t} \frac{\cos 6t - \cos 4t}{t} \, dt$$

$$= L\left\{ \frac{\cos 6t - \cos 4t}{t} \right\} \text{ when } s = 0$$

here $a = 6, b = 4$

So it will be

$$\frac{1}{2} \ln \left(\frac{\cancel{16}\,4}{\cancel{36}\,9} \right)$$

$$= \frac{1}{2} \ln \left(\frac{2}{3}\right)^2$$

$$= \ln 2/3 \text{ (Proved)}$$

16) Given L.H.S $= \int_0^\infty t^2 . e^{-4t} \sin 2t \, dt$

$$= L\{ t^2 \sin 2t \} \text{ at } s = 4$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{t \sin 2t\} = \frac{(s^2+4) \cdot 0 - 2 \cdot 2s}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$$

$$L\{t^2 \sin 2t\} = -1 \left[\frac{(s^2+4)^2 \cdot 4 - [2(s^2+4) \cdot 2s] \, 4s}{(s^2+4)^4} \right]$$

at $s = 4$ it will be $= -1 \left[\frac{(16+4)^2 \cdot 4 - [2(16+4) \cdot 2 \cdot 4] \, 4 \cdot 4}{(16+4)^4} \right] = \frac{11}{500}$ (Proved)