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Time dependented SE :- (V: potential energy)

(i) In 1-D

along x axis : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + (V\psi)$

along y axis : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial y^2}\right) + (V\psi)$

along z axes : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial z^2}\right) + (V\psi)$

(ii) In - 2D

along xy plane : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + V\psi$

along yz plane : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V\psi$

along zx plane : $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2}\right) + V\psi$

(iii) In 3-D

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\underbrace{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}}\right) + V\psi$$

(Or)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla^2 \psi\right) + V\psi$$

For free particle $V = 0$

Time independent SE :- (E - total energy)

(i) In 1-D

along x axis: $\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

along y axis: $\frac{\partial^2\psi}{\partial y^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

along z axis: $\frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

(ii) In 2-D

along xy plane: $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

along yz plane: $\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

along zx plane: $\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

(iii) In 3-D

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$\Rightarrow \nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

For free particle: $V = 0$

Applications of Schrodingen Wave Equation :-

(i) 1- Dimensional potential step

(ii) 1 - Dimensional potential well or particle in a box

(iii) potential barreier

All applications are based on time independent SE in 1- Dimensional.

1. Potential STEP :-

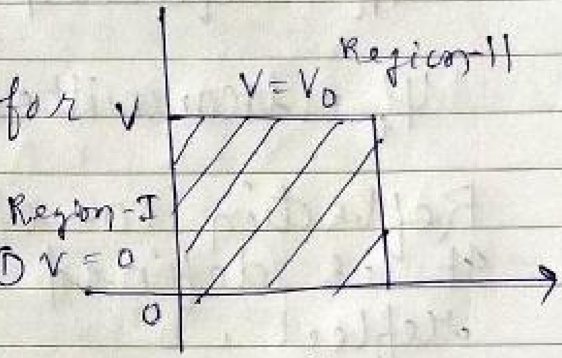
When 2 regions are separated by a potential then it consistudes a potential step problem.

for $E > V$

the potential distribution for the step problem

$V(x) = 0, x < 0$, Region-I

$V(x) = V_0 ; x > 0$, Region-II — (1)



The schrodinger Equations are :

1. $$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE}{\hbar^2} \psi = 0 \text{, Region - I} \quad \text{——— (2)}$$

2. $$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m(E-V_0)}{\hbar^2} \psi = 0 \text{, Region - II} \quad \text{——— (3)}$$

let
$$\left.\begin{aligned} \frac{2mE}{\hbar^2} &= k_1^2 \Rightarrow k_1 = \frac{\sqrt{2mE}}{\hbar} \\ \frac{2m(E-V_0)}{\hbar^2} &= k_2^2 \Rightarrow k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar} \end{aligned}\right\} \text{— (4)}$$

$$\therefore \frac{\partial^2 \psi_1}{\partial x^2} + k_1^2 \psi = 0 \text{ and } \frac{\partial^2 \psi_2}{\partial x^2} + k_2^2 \psi_2 = 0 \text{ — (5)}$$

The sol^n of Eq^n (5) is given by,

$$\left.\begin{aligned} \text{Reg-I} ;\ \psi_1 &= Ae^{ik_1x} + Be^{-ik_1x} \\ \text{Reg-II} ;\ \psi_2 &= Ce^{ik_2x} + \underbrace{\boxed{De^{-ik_2x}}}_{\text{not valid}} \end{aligned}\right\} \quad \text{— (6)}$$

$$\psi_i = \text{incident}\overset{\text{wave}}{_n} = Ae^{ik_1x}$$

$$\psi_r = \text{reflected wave} = Be^{-ik_1x} \qquad (7)$$

$$\psi_t = \text{transmitted} = Ce^{ik_2x}$$

Reflection co-efficient $-(R_C)$:
It is defined as the no. of particles reflect,

$$R_c = \frac{[\sqrt{E} - \sqrt{E-V_0}]^2}{[\sqrt{E} + \sqrt{E-V_0}]^2} \quad \text{— (8)}$$

Transmission coefficient $-(T_C)$
It is defined as the no. of particles transmitted into the second region.

$$T_c = \frac{4\sqrt{E}\sqrt{E-V_0}}{[\sqrt{E} + \sqrt{E-V_0}]^2} \quad \text{— (9)}$$

$$\boxed{R_c + T_c = 1}$$

this shows the $E > V_0$ then according to quantum mechanics few particles are reeflected or few particles are transmitted.

Q. Let in a step problem, $E = 16eV$, $V_0 = 7eV$ find R_c and T_c.

$$R_c = \frac{(\sqrt{16} - \sqrt{16-7})^2}{\sqrt{16} + \sqrt{16-7})^2} = \frac{(4-3)^2}{(4+3)^2} = \frac{1}{49} = 0.02 \approx 2\%$$

$$T_c = 1 - R_c = 1 - 0.02 = 0.98 \approx 98\%$$

Q. $E = 25eV$, $V_0 = 9eV$

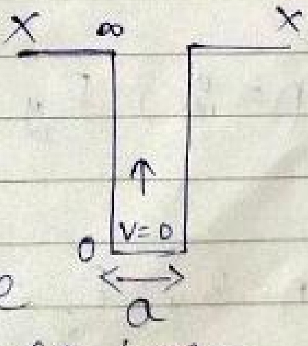
$$R_c = \frac{(\sqrt{25} - \sqrt{25-9})^2}{(\sqrt{25} + \sqrt{25+9})^2} = \frac{(5-4)^2}{(5+4)^2} = \frac{1}{81} = 0.01 \approx 1\%$$

$$T_c = 1 - R_c = 1 - 0.01 = 99\%$$

Q. Potential well - (a particle in a box) :-
When a particle is trapped by infinite barriers then it constitutes a potential well problem

potential distribution is given by

$$v(x) = 0, \quad 0 < x < a \quad —(1)$$



The Schrodinger eqn for the particle in the well is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE}{\hbar^2}\psi = 0 \quad —(2)$$

Let $k = \frac{\sqrt{2mE}}{\hbar}$ —— (3)

$\therefore \frac{\partial^2\psi}{\partial x^2} + k^2\psi = 0$ —— (4)

Sol^n $\psi = Ae^{ikx} + Be^{-ikx}$

or $\psi = C\sin kx + D\cos kx$ —— (5)

For boundary - I , $x = 0 \Rightarrow \psi = 0 \Rightarrow D = 0$

$\psi = C\sin kx$

boundary II , $x = a, \psi = 0 \Rightarrow \sin ka = 0 \Rightarrow \sin n\pi$

$ka = n\pi$

$\Rightarrow k^2a^2 = n^2\pi^2$

$\Rightarrow \frac{2mE}{\hbar^2} a^2 = n^2\pi^2$

$\Rightarrow E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}$ $(\because n = 1, 2, \cdots)$

Where a = width of the well

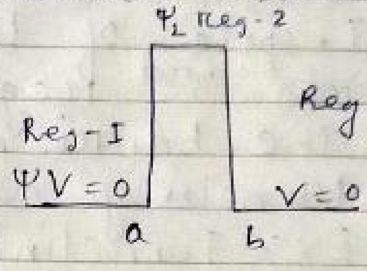
For $n = 1, E_1 = \frac{\hbar^2\pi^2}{2ma^2} \rightarrow$ ground state Energy

$n = 2, E_2 = 4E_1 \rightarrow 1^{st}$ Excited state energy

$n = 3, E_3 = 9E_1 \rightarrow 2^{nd}$ Excited state energy

Potential Barrier :-
when 2 regions are separated by a barrier then it constitues a potential barrier problem.

$V(x) = 0 \ , \ x < 0$
$V(x) = V_0 \ , \ 0 < x < b$
$V(x) = 0 \ , \ x > b$



For $E < V_0$
Here energy is less and potential is high.
The Schrodinger equations are :-

Reg-I = $\frac{\partial^2 \psi_1}{\partial x^2} + \frac{2ME}{\hbar^2} \psi_1 = 0$

Reg-II = $\frac{\partial^2 \psi_2}{\partial x^2} - \frac{2m(V_0 - E)}{\hbar^2} \psi_2 = 0$

Reg-III = $\frac{\partial^2 \psi_3}{\partial x^2} + \frac{2ME}{\hbar^2} \psi_3 = 0$

Sol^n - $\psi_1 = Ae^{ikx} + Be^{-ikx}$
$\psi_2 = Ce^{\alpha x} + De^{-\alpha x}$
$\psi_3 = Fe^{ikx} + \boxed{Ge^{-ikx}}$ → not valid

Transmission probability / Flux :-

$$\boxed{T_c = e^{-2\alpha b}}$$

$\alpha = \frac{2m}{\hbar^2}(V_0 - E)$

b = width of barrier

Conclusion :-

Quantum Tunnelling :-

Inspite of low energy particles transmitte into 3^{rd} Region by making tunnels in the barrier this phenomenon is called tunneling effect.

Ex:- Tunneling diodes, tunneling microscopes, Josephson diode.