2 a $y''-8y'+15y=9t\cdot e^{2t}$ $\quad y(0)=5,\ y'(0)=10$

$\Rightarrow L\{y''\}-8L\{y'\}+15L\{y\}=9L\{te^{2t}\}$

$\Rightarrow s^2Y-5s-10-8(sY-5)+15Y=\frac{9}{(s-2)^2}$

$\Rightarrow Y(s^2-8s+15)-5s+30=\frac{9}{(s-2)^2}$

$\Rightarrow Y(s^2-8s+15)=\frac{9}{(s-2)^2}+5s-30$

$\Rightarrow Y=\frac{9}{(s-2)^2(s^2-8s+15)}+\frac{5s-30}{s^2-8s+15}$

$\Rightarrow Y=\frac{9}{(s-2)^2(s-3)(s-5)}+\frac{5s-30}{(s-3)(s-5)}$

$\frac{9+(5s-30)(s-2)^2}{(s-2)^2(s-3)(s-5)}=\frac{A}{s-3}+\frac{B}{s-5}+\frac{C}{s-2}+\frac{D}{(s-2)^2}$

$9+(5s-30)(s-2)^2=A(s-5)(s-2)^2+B(s-3)(s-2)^2+C(s-3)(s-5)(s-2)+D(s-3)(s-5)$

Let $s=3$

$(15-30)1^2+9=A(3-5)(3-2)^2$

$\Rightarrow -15+9=-2A$

$\Rightarrow -6=-2A$

$\Rightarrow A=3$

Let $s=5$

$(25-30)(3)^2+9=B(5-3)(5-2)^2$

$\Rightarrow -5\times9+9=2\times9\times B$

$\Rightarrow -36=18B$

$\Rightarrow B=-2$

$0=A+B+C \Rightarrow$ coefficient of s^3 on both sides

$\Rightarrow 0=3-2+C$

$\Rightarrow C=-1$

Let $s=2$

$(10-30)\times0+9=D(2-3)(2-5)$

$\Rightarrow 9=D(-1)(-3)$

$\Rightarrow 9=3D$

$\Rightarrow D=3$ $\quad\therefore A=3, B=-2, C=-1, D=3$

$\Rightarrow Y=\frac{3}{s-3}-\frac{2}{s-5}-\frac{1}{s-2}+\frac{3}{(s-2)^2}$

$\Rightarrow L^{-1}\{Y\}=L^{-1}\{\frac{3}{s-3}\}-L^{-1}\{\frac{2}{s-5}\}-L^{-1}\{\frac{1}{s-2}\}+L^{-1}\{\frac{3}{(s-2)^2}\}$

$\Rightarrow y=3e^{3t}-2e^{5t}-e^{2t}+3te^{2t}$

(b) $y''' - 3y'' + 3y' - y = t^2 e^t \qquad y(0)=1,\ y'(0)=0,\ y''(0)=-2$

$\Rightarrow (s^3Y - s^2 + 2) - 3(s^2Y - s) + 3(sY - 1) - Y = \frac{2}{(s-1)^3}$

$\Rightarrow Y(s^3 - 3s^2 + 3s - 1) - s^2 + 3s + 2 - 3 = \frac{2}{(s-1)^3}$

$\Rightarrow Y((s-1)^3) = \frac{2}{(s-1)^3} + s^2 - 3s + 1$

$\Rightarrow Y = \frac{2}{(s-1)^6} + \frac{s^2-3s+1}{(s-1)^3}$

$\Rightarrow L^{-1}\{Y\} = L^{-1}\left\{\frac{2}{(s-1)^6}\right\} + L^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}\right\}$

$L^{-1}\left\{\frac{2}{(s-1)^6}\right\} = 2e^t \frac{t^5}{120} = \frac{e^t t^5}{60}$

$L^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}\right\} = L^{-1}\left\{\frac{s^2-2s+1-s}{(s-1)^3}\right\} = L^{-1}\left\{\frac{(s-1)^2-(s-1)-1}{(s-1)^3}\right\}$

$= e^t L^{-1}\left\{\frac{s^2-s-1}{s^3}\right\}$

$= e^t\left(1 - t - \frac{t^2}{2}\right)$

$\Rightarrow y = e^t\left(\frac{t^5}{60} + 1 - t - \frac{t^2}{2}\right)$

(c) $y'' - 6y' + 9y = 0, \quad y(0)=2,\ y'(0)=9$

$\Rightarrow (s^2Y - 2s - 9) - 6(sY - 2) + 9Y = 0$

$\Rightarrow Y(s^2 - 6s + 9) - 2s + 3 = 0$

$\Rightarrow Y = \frac{2s-3}{s^2-6s+9} = \frac{2s-3}{s^2 - 2.s.3 + 3^2 - 3^2 + 9}$

$\Rightarrow Y = \frac{2s-3}{(s-3)^2}$

$\frac{2s-3}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$

$\Rightarrow 2s - 3 = As - 3A + B$

$\Rightarrow 2s = As \qquad -3A + B = -3$

$\Rightarrow A = 2 \qquad \Rightarrow B = 3$

$\Rightarrow Y = \frac{2}{s-3} + \frac{3}{(s-3)^2}$

$\Rightarrow L^{-1}\{Y\} = L^{-1}\left\{\frac{2}{s-3}\right\} + L^{-1}\left\{\frac{3}{(s-3)^2}\right\}$

$\Rightarrow y = 2e^{3t} + 3te^{3t}$

2 (a) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$F(s)=\frac{s}{s^2+a^2}$

$\Rightarrow f(t)=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at$

$G(s)=\frac{s}{s^2+b^2}$

$\Rightarrow g(t)=L^{-1}\left\{\frac{s}{s^2+b^2}\right\}=\cos bt$

Now according to convolution theorem.

$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$=L^{-1}\{\cos at * \cos bt\}$

$=\int_0^t \cos au \cdot \cos b(t-u)\,du$

$=\frac{1}{2}\int_0^t 2\cos au \cdot \cos b(t-u)\,du$

$=\frac{1}{2}\int_0^t \{\cos(au+bt-bu)+\cos(au-bt+bu)\}\,du$

$=\frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b}+\sin\frac{au-bt+bu}{a+b}\right]_0^t$

$=\frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin(-bt)}{a+b}\right]$

$=\frac{1}{2}\left[\frac{\sin at}{a-b}-\frac{\sin bt}{a-b}+\frac{\sin at}{a+b}+\frac{\sin bt}{a+b}\right]$

$=\frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right]$

$=\frac{1}{2}\left[\frac{a\sin at-a\cancel{\sin bt}+b\cancel{\sin at}-b\sin bt+a\sin at+\cancel{a\sin bt}-b\cancel{\sin at}-b\sin at}{a^2-b^2}\right]$

$=\frac{1}{2}\left[\frac{2a\sin \cancel{at}-2b\sin bt}{a^2-b^2}\right]$

$=\frac{a\sin\cancel{at}-b\sin bt}{a^2-b^2}$

(b) $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

$F(s) = \frac{1}{s+1}$ $\qquad$ $G(s) = \frac{1}{s^2+1}$

$f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ $\qquad$ $g(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Now, applying convolution Theorem

$$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$$

$$= L^{-1}\left\{e^{-t}\sin(t)\right\}$$

$$= \int_0^t \left\{e^{-(t-u)} \sin u\right\} du$$

$$= e^{-t}\int_0^t e^{u} \sin u \, du$$

$$= e^{-t}\left[\frac{e^{u}}{2}(\sin u - \cos u)\right]_0^t$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) - \frac{e^0}{2}(\sin 0 - \cos 0)\right]$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}\right]$$

$$= \frac{1}{2}(\sin t - \cos t + e^{-t})$$

(c) $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$

$F(s) = \frac{1}{s^2+4}$ $\qquad$ $G(s) = \frac{1}{(s+1)^2}$

$f(t) = \frac{1}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$ $\qquad$ $g(t) = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

Now, applying Convolution Theorem

$$L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{2}\sin 2t \cdot te^{-t}\right\}$$

$$= \frac{1}{2}\int_0^t \left\{\sin 2u \, e^{-(t-u)}(t-u)\right\} du$$

$$= \frac{1}{2} e^{-t} \int_0^t \sin 2u \; e^{u} (t-u)\, du$$

$$= \frac{1}{2} e^{-t} \left[t \int_0^t \sin 2u \; e^{u}\, du - \int_0^t u e^{u} \sin 2u \, du \right]$$

$$= \frac{1}{2} e^{-t} \left\{ t \left[\frac{e^{u}}{5} (\sin 2u - 2\cos 2u) \right]_0^t - \frac{e^{u}}{25} \left[(5u-2)\sin(2u) - (10u-1)\cos(2u) \right]_0^t \right\}$$

$$= \frac{1}{2} e^{-t} \left\{ t \left[\frac{e^{t}}{5} \sin(2t) - 2\cos(2t) + \frac{2}{5} \right] - \frac{e^{t}}{25} \left[(5t-2)\sin(2u) - (10t-1)\cos(2t) \right] + \frac{1}{25} \right\}$$

$$= \frac{1}{2} e^{-t} \left[\frac{t}{5} e^{t} \sin(2t) - \frac{2t}{5} e^{t} \cos(2t) + \frac{2t}{5} - \frac{e^{t}}{25} \left(5t \sin(2t) - 2\sin(2t) - 10t\cos(2t) + \cos(2t) \right) - \frac{1}{25} \right]$$

$$\cancel{= \frac{1}{50} e^{-t}\; 10 t e^{t} \sin(2t) - 20 t e^{t}}$$

$$= \frac{1}{10} e^{-t} \left[t e^{t} \sin(2t) - 2t e^{t} \cos(2t) + 2t - \frac{e^{t}}{5} \left(5t \cdot \sin(2t) - 2\sin(2t) - 10t\cos(2t) + \cos(2t) \right) - \frac{1}{5} \right]$$

$$= \frac{1}{10} e^{-t} \left[\frac{\cancel{5te^{t}\sin(2t)} - \cancel{10te^{t}\cos(2t)} + 10t - \cancel{e^{t}\,5t\sin(2t)} + 2e^{t}\sin(2t) + \cancel{e^{t}\,10t\cos(2t)} - e^{t}\cos(2t) - 1}{5} \right]$$

$$= \frac{1}{10} e^{-t} \left[\frac{10t + 2e^{t}\sin(2t) - e^{t}\cos(2t) - 1}{5} \right]$$

$$= \frac{1}{50} \left(10te^{-t} + 2\sin(2t) - \cos(2t) - e^{-t} \right)$$

(d) $L^{-1}\left\{ \frac{s}{(s^2+a^2)^2} \right\}$

$F(s) = \frac{s}{(s^2+a^2)}$ $\qquad$ $G(s) = \frac{1}{s^2+a^2}$

$F(t) = L^{-1}\left\{ \frac{s}{(s^2+a^2)} \right\} = \cos at$ $\qquad$ $g(t) = \frac{1}{a} L^{-1}\left\{ \frac{a}{s^2+a^2} \right\} = \frac{1}{a} \sin at$

Now according to convolution theorem

$$L^{-1}\left\{ \frac{s}{(s^2+a^2)^2} \right\}$$

$$= L^{-1}\left\{ \cos(at) \cdot \frac{1}{a} \sin(at) \right\}$$

$$=\frac{1}{a}\int_0^t \{\cos au \sin a(t-u)\} du$$

$$=\frac{1}{2a}\int_0^t 2\{\cos au \sin(at-au)\} du$$

$$=\frac{1}{2a}\int_0^t \{\sin(au+at-au)-\sin(au-at+au)\} du$$

$$=\frac{1}{2a}\int_0^t \sin(at)-\sin(2au-at) du$$

$$=\frac{1}{2a}\int_0^t \sin(at)du-\int_0^t \sin(2au-at) du$$

$$=\frac{1}{2a}\left\{\sin(at)[u]_0^t-\left[\frac{-\cos(2au-at)}{2a}\right]_0^t\right\}$$

$$=\frac{1}{2a}\, t\sin(at)+\cancel{\frac{\cos(2at)}{2a}}-\cancel{\frac{\cos(2at)}{2a}}$$

$$=\frac{t}{2a}\sin(at)$$

(e) $L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$

$F(s)=\frac{1}{(s+2)^2}$

$f(t)=L^{-1}\left\{\frac{1}{(s+2)^2}\right\}=te^{-2t}$

$G(s)=\frac{1}{s-2}$

$g(t)=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$

Now according to convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$

$$=L^{-1}\{te^{-2t}\, e^{2t}\}$$

$$=\int_0^t ue^{-2u}\, e^{2(t-u)} du$$

$$=\int_0^t ue^{-2u}\, e^{2t}\, e^{-2u} du$$

$$=e^{2t}\int_0^t u e^{-4u} du$$

$$=e^{2t}\left[\frac{ue^{-4u}}{-4}\right]_0^t-\int_0^t\frac{e^{-4u}}{-4}$$

$$= e^{2t}\left[\frac{u e^{-4u}}{-4}\right]_0^t - \left[\frac{e^{-4u}}{16}\right]_0^t$$

$$= e^{2t}\left[-\frac{t e^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right]$$

$$= \frac{e^{2t}}{4}\left[-t e^{-4t} - \frac{e^{-4t}}{4} + \frac{1}{4}\right]$$

③ a $L^{-1}\left\{\frac{s+17}{(s-1)(s+3)}\right\}$

By using partial fraction

$$\frac{s+17}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

$s+17 = A(s+3) + B(s-1)$

At $s=-3$ At $s=1$

$14 = -4B$ $18 = 4A$

$B = -\frac{14}{4}$ $A = \frac{18}{4}$

$\Rightarrow B = -\frac{7}{2}$ $\Rightarrow A = \frac{9}{2}$

Now, $L^{-1}\left\{\frac{9/2}{s-1}\right\} + L^{-1}\left\{\frac{-7/2}{s+3}\right\}$

$$= \frac{9}{2}e^{t} - \frac{7}{2}e^{-3t}$$

ⓑ $L^{-1}\left\{\frac{2s+12}{s^2+6s+13}\right\}$

$L^{-1}\left\{\frac{2s+12}{(s+3)^2+4}\right\}$

$L^{-1}\left\{\frac{2(s+3)+6}{(s+3)^2+4}\right\}$

$e^{-3t}\left[L^{-1}\left\{\frac{2s}{s^2+2^2}\right\} + \frac{6}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right]$

$e^{-3t}\left[2\cos(2t) + 3\sin(2t)\right]$

(c) $L^{-1}\left\{\frac{s^3+6s^2+14s}{(s+2)^4}\right\}$

By using partial fraction

$$\frac{s^3+6s^2+14s}{(s+2)^4} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3+6s^2+14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

At $s=-2$
$-8+24-28=D$
$\Rightarrow D=-12$

when $s=-1$
$A+B+C+D=-1+6-14$
$A+B+C-12=-9$
$A+B+C=3$

when $s=0$
$8A+4B+2C+D=0$
$8A+4B+2C=12$

when $s=1$
$27A+9B+3C-12=1+6+14$
$27A+9B+3C=33$
$\therefore A=1,\ b=0,\ C=2,\ D=-12$

Now, $L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{2}{(s+2)^3}\right\} - L^{-1}\left\{\frac{12}{(s+2)^4}\right\}$

$$= e^{-2t}\left(1+\frac{2}{2}t^2-\frac{12}{6}t^3\right)$$

$$= e^{-2t}\left(1+t^2-2t^3\right)$$

(d) $L^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

By using partial fraction

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$(As+B)(s^2+4)+(Cs+D)(s^2+1)=s+1$

$(As+B)(s^2+4)+Cs^3+Cs+Ds^2+D=s+1$

$As^3+4As+Bs^2+4B+Cs^3+Cs+Ds^2+D=s+1$

$\Rightarrow s^3(A+C)+s^2(B+D)+s(4A+C)+4B+D=s+1$

$A+C=0$
$C=-A$
$C=-\frac{1}{3}$

$B+D=0$
$D=-B$
$D=-\frac{1}{3}$

$4A+C=1$
$4A-A=1$
$3A=1$
$A=\frac{1}{3}$

$4B+D=1$
$4B-B=1$
$3B=1$
$B=\frac{1}{3}$

Now, $L^{-1}\left\{\frac{\frac{1}{3}s+\frac{1}{3}}{s^2+1}\right\} + L^{-1}\left\{\frac{-\frac{1}{3}s-\frac{1}{3}}{s^2+4}\right\}$

$= \frac{1}{3}L^{-1}\left\{\frac{s+1}{s^2+1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s+1}{s^2+4}\right\}$

$= \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right]$

$= \frac{1}{3}\left[\cos t + \sin t - \cos 2t - \frac{1}{2}\sin 2t\right]$

(e) $L^{-1}\left\{\frac{1}{s(s+1)^2}\right\}$

By using partial fraction

$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$1 = A(s+1)^2 + Bs(s+1) + Cs$

when $s=-1$ when $s=0$ when $s=1$

$1=-C$ $1=A$ $1=4A+2B+C$

$\Rightarrow C=-1$ $\Rightarrow A=1$ $\Rightarrow 1 = 4+2B-1$

$\Rightarrow 2B = -2$

$\Rightarrow B=-1$

$\therefore A=1, C=-1, B=-1$

Now, $L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= 1 - e^{-t} - te^{-t}$

(f) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$F(s) = \frac{1}{s^2+a^2}$

$f(t) = \frac{1}{a}L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

$G(s) = \frac{1}{s^2+a^2}$

$g(t) = \frac{1}{a}L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

Now, by using convolution

$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$= L^{-1}\left\{\frac{1}{a}\sin at \cdot \frac{1}{a}\sin at\right\}$

$= \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\, du$

$= \frac{1}{2a^2}\int_0^t 2\sin au \cdot \sin a(t-u)\, du$

$= \frac{1}{2a^2} \int_0^t \cos(au - at + au) - \cos(au + at - au)\, du$

$= \frac{1}{2a^2} \int_0^t (\cos(2au - at) - \cos(at))\, du$

$= \frac{1}{2a^2} \left[\frac{\sin(2au - at)}{2a} - \cos(at)\, \cancel{u} \right]_0^t$

$= \frac{1}{2a^2} \left[\frac{\sin(at)}{2a} - t\cos(at) \right]$

$= \frac{1}{2a^3} \sin(at) - \frac{t}{2a^2} \cos(at).$

⑤ $L^{-1}\left\{ \frac{s+2}{s^2(s+3)} \right\}$

By using partical fraction.

$\frac{s+2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$

$s+2 = As(s+3) + B(s+3) + Cs^2$

when $s=-3$: $-1 = 9C \Rightarrow C = \frac{-1}{9}$

when $s=0$: $2 = 3B \Rightarrow B = \frac{2}{3}$

when $s=1$:
$3 = 4A + 4B + C$
$3 = 4A + \frac{8}{3} - \frac{1}{9}$
$A = \frac{1}{4}\left(3 + \frac{1}{9} - \frac{8}{3}\right)$
$A = \frac{1}{4} \times \frac{4}{9}$
$A = \frac{1}{9}$

$\therefore A = \frac{1}{9},\ B = \frac{2}{3},\ C = \frac{-1}{9}$

Now, $L^{-1}\left\{ \frac{\frac{1}{9}}{s} \right\} + L^{-1}\left\{ \frac{\frac{2}{3}}{s^2} \right\} + L^{-1}\left\{ \frac{-\frac{1}{9}}{s+3} \right\}$

$= \frac{1}{9} L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9} L^{-1}\left\{\frac{1}{s+3}\right\}$

$= \frac{1}{9} + \frac{2t}{3} - \frac{1}{9} e^{-3t}$

⑥ $L^{-1}\left\{ \frac{s}{(s^2+a^2)^2} \right\}$

$F(s) = \frac{s}{s^2+a^2}$

$f(t) = L^{-1}\left\{ \frac{s}{s^2+a^2} \right\} = \cos at$

$g(s) = \frac{1}{s^2+a^2}$

$g(t) = \frac{1}{a} L^{-1}\left\{ \frac{a}{s^2+a^2} \right\} = \frac{1}{a} \sin at$

Now, by using convalation

$L^{-1}\left\{ \frac{s}{(s^2+a^2)^2} \right\} = L^{-1}\left\{ \cos at \cdot \frac{1}{a} \sin at \right\}$

$= \frac{1}{2a}\int_0^t 2\cos(au)\sin a(t-u)\,du$

$= \frac{1}{2a}\int_0^t \{\sin(au+at-au) - \sin(au-at+au)\}\,du$

$= \frac{1}{2a}\int_0^t \{\sin(at) - \sin(2au-at)\}\,du$

$= \frac{1}{2a}\left[\sin(at).u + \frac{\cos(2au-at)}{2a}\right]_0^t$

$= \frac{1}{2a}\left(t\sin(at) + \cancel{\frac{\cos at}{2a}} - \cancel{\frac{\cos at}{2a}}\right)$

$= \frac{t}{2a}\sin(at)$

(7) $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

$= L^{-1}\left\{\frac{s+1}{((s+1)^2+1)^2}\right\}$

$= e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$\left[\because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at\right]$

$= \frac{e^{-t}t}{2}\sin t$

(8) $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

$= L^{-1}\left\{\frac{4(s+3)}{(s+4)^2}\right\}$

$= L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$

$= e^{-4t}L^{-1}\left\{\frac{4s-4}{(s)^2}\right\}$

$= e^{-4t}\,L^{-1}\left\{\frac{4s}{s^2}\right\} - L^{-1}\left\{\frac{4}{s^2}\right\}$

$= e^{-4t}(4-4t)$

$= 4e^{-4t}(1-t)$

(9) $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$

$L^{-1}\left\{\frac{1}{((s+1)^2+4)^2}\right\}$

$e^{-t}\, L^{-1}\left\{\frac{1}{(s^2+2^2)^2}\right\}$

$F(s) = \frac{1}{s^2+2^2}$ $\qquad$ $G(s) = \frac{1}{s^2+2^2}$

$f(t) = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$ $\qquad$ $g(t) = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$

Now, by using convolution

$e^{-t}\, L^{-1}\left\{\frac{1}{(s^2+2^2)^2}\right\}$

$= e^{-t}\, L^{-1}\left\{\frac{\sin 2t}{2} \cdot \frac{\sin 2t}{2}\right\}$

$= \frac{e^{-t}}{4} \int_0^t \sin 2u \cdot \sin 2(t-u)\, du$

$= \frac{e^{-t}}{8} \int_0^t 2\sin(2u) \cdot \sin(2t-2u)\, du$

$= \frac{e^{-t}}{8} \int_0^t \cos(2u-2t+2u) - \cos(2u+2t-2u)\, du$

$= \frac{e^{-t}}{8} \int_0^t \cos(4u-2t) - \cos(2t)\, du$

$= \frac{e^{-t}}{8} \left[\frac{\sin(4u-2t)}{4} - \cos(2t).\,u\right]_0^t$

$= \frac{e^{-t}}{8} \left[\frac{\sin(4t-2t)}{4} - t\cos(2t) + \frac{\sin(2t)}{4}\right]$

$= \frac{e^{-t}}{32} \left[\sin(2t) - 4t\cos(2t) + \sin(2t)\right]$

$= \frac{e^{-t}}{32} \left[2\sin(2t) - 4t\cos(2t)\right]$

$= \frac{e^{-t}}{16} \left[\sin(2t) - 2t\cos(2t)\right]$

10 $L^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\}$

$L^{-1}\left\{\frac{(s+a)-a}{(s+a)^2+b^2}\right\}$

$= e^{-at} L^{-1}\left\{\frac{s-a}{s^2+b^2}\right\}$

$= e^{-at}\left[L^{-1}\left\{\frac{s}{s^2+b^2}\right\} - \frac{a}{b} L^{-1}\left\{\frac{b}{s^2+b^2}\right\}\right]$

$= e^{-at}\left[\cos bt - \frac{a}{b}\sin bt\right]$

11 $L^{-1}\left\{\frac{a^3}{s^4-a^4}\right\}$

$a^3 L^{-1}\left\{\frac{1}{(s^2-a^2)(s^2+a^2)}\right\}$

By using partial fraction

$\frac{1}{(s^2-a^2)(s^2+a^2)} = \frac{As+B}{s^2-a^2} + \frac{Cs+D}{s^2+a^2}$

$(As+B)(s^2+a^2) + (Cs+D)(s^2-a^2) = 1$

$\Rightarrow (As+B)(s^2+a^2) + Cs^3 - Ca^2s + Ds^2 - Da^2 = 1$

$\Rightarrow As^3 + Aa^2s + Bs^2 + Ba^2 + Cs^3 - Ca^2s + Ds^2 - Da^2 = 1$

$\Rightarrow s^3(A+C) + s^2(B+D) + s(Aa^2 - Ca^2) + Ba^2 - Da^2 = 1$

$A+C=0$ $\quad B+D=0$ $\quad Aa^2-Ca^2=0$ $\quad Ba^2-Da^2=1$

$\quad\quad\quad\quad B=-D$ $\quad A-C=0$ $\quad a^2(-D-D)=1$

$A=0$ $\quad\quad B=\frac{1}{2a^2}$ $\quad C=0$ $\quad a^2(-2D)=1$

$D=\frac{-1}{2a^2}$

Now,

$a^3\left\{L^{-1}\left\{\frac{\frac{1}{2a^2}}{s^2-a^2}\right\} + L^{-1}\left\{\frac{\frac{-1}{2a^2}}{s^2+a^2}\right\}\right\}$

$\frac{\cancel{a^3}}{2\cancel{a^3}} L^{-1}\left\{\frac{a^3}{s^2-a^2}\right\} - \frac{1}{2a^3} L^{-1}\left\{\frac{a^3}{s^2+a^2}\right\}$

$\frac{1}{2}(\sin hat - \sin at)$ (Hence proved)

$$L^{-1}\left\{\frac{64}{81s^4 - 256}\right\}$$

$$= L^{-1}\left\{\frac{4^3}{(3s)^4 - 4^4}\right\}$$

$$= \frac{1}{2}(\sinh 4t - \sin 4t)$$

(12) $$L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$$

$$= \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right\}$$

$$= \frac{3}{2}\left\{1 - \frac{4t^2}{2} + \frac{4t^4}{24}\right\}$$

$$= \frac{3}{2}\left(1 - 2t^2 + \frac{t^4}{6}\right)$$

$$= \frac{3}{2} - 3t^2 + \frac{t^4}{4}$$

(13) $$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$$

$$L\{e^{-at} - e^{-bt}\} = \left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$= \int_s^\infty \frac{1}{u+a} - \frac{1}{u+b}\, du$$

$$= \left[\ln(u+a) - \ln(u+b)\right]_s^\infty$$

$$= -\ln(s+a) + \ln(s+b)$$

$$= \ln\left(\frac{s+b}{s+a}\right) \text{ (Hence Proved)}$$

$$\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt \qquad L\{e^{-3t} - e^{-6t}\} = \frac{1}{s+3} - \frac{1}{s+6}$$

$$= \int_0^\infty \frac{1}{u+3} - \frac{1}{u+6}\, du$$

$$= \left[\ln(u+3) - \ln(u+6)\right]_0^\infty$$

$$= \left[\ln\left(\frac{u+3}{u+6}\right)\right]_0^\infty$$

$$= \ln\frac{6}{3}$$

$$= \ln 2$$

(14) $L\left\{\frac{\cos at - \cos bt}{t}\right\}$ $L(\cos at - \cos bt) = \frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}$

$\int_s^\infty \frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\, du$

$\frac{1}{2}\int_s^\infty \frac{2u}{u^2+a^2} - \frac{2u}{u^2+b^2}\, du$

$\frac{1}{2}\left[\ln|u^2+a^2| - \ln|u^2+b^2|\right]_s^\infty$

$= \frac{1}{2}\left[-\ln|s^2+a^2| + \ln|s^2+b^2|\right]$

$= \frac{1}{2}\ln\left|\frac{s^2+b^2}{s^2+a^2}\right|$ (Hence proved)

$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$ $-L\{\cos 6t - \cos 4t\} = -\frac{s}{s^2+36} + \frac{s}{s^2+16}$

$F(s) = \int_0^\infty e^{-st}\frac{\cos 6t - \cos 4t}{t}\, dt$

$F'(s) = \int_0^\infty -e^{-st}(\cos(6t) - \cos(4t))\, dt$

$s_0 \int_0^\infty \frac{u}{u^2+16} - \frac{u}{u^2+36}\, du$

$= \frac{1}{2}\int_0^\infty \frac{2u}{u^2+16} - \frac{2u}{u^2+36}\, du$

$= \frac{1}{2}\left[\ln|u^2+16| - \ln|u^2+36|\right]_0^\infty$

$= \frac{1}{2} \times \ln\left|\frac{u^2+16}{u^2+36}\right|$

$= \frac{1}{2}\ln\left(\frac{16}{36}\right)$

$= \frac{1}{2}\ln\left(\frac{4}{6}\right)^2$

$= \ln\left(\frac{2}{3}\right)$

$= \ln\left(\frac{2}{3}\right)$ (Hence proved)

(15) $\int_0^\infty t \cdot e^{-3t} \sin t \, dt$

$\int_0^\infty e^{-3t} (t \sin t) dt$ = laplace trans formation of $(t^2 \cdot \sin 2t)$ at $s = +3$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\{t \sin t\} = -\frac{d}{dt}(s^2+1)^{-1} = \frac{2s}{(s^2+1)^2}$

at $s = +3 = \frac{3 \times 2}{(3^2+1)^2} = \frac{6}{(10)^2} = \frac{6}{100} = \frac{3}{50}$ (Hence Proved)

(16) $\int_0^\infty t^2 e^{-4t} \sin 2t \, dt$

$L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{t \sin 2t\} = -2(s^2+4)^{-2} 2s = \frac{-4s}{(s^2+4)^2}$

$L\{t^2 \sin 2t\} = \frac{(s^2+4)^2(4) - (4s)2(s^2+4)2s}{(s^2+4)^4}$

$= s^2+4 \left[\frac{(s^2+4)(4) - (-4s)(4s)}{(s^2+4)^4}\right]$

$= \frac{-4s^2 - 16 + 16s^2}{(s^2+4)^3}$

$= \frac{12s^2-16}{(s^2+4)^3}$

$\therefore$ At $s = 4 = \frac{12(4)^2-16}{(4^2+4)^3} = \frac{12 \times 16 - 16}{(16+4)^3} = \frac{176}{8000} = \frac{11}{500}$

17 $L\{e^{2t}(3 \sin 4t - 4 \cos 4t)\}$

$L\{3 \sin 4t - 4 \cos 4t\}$ at $s = s-2$

$= \frac{3 \cdot 4}{(s-2)^2+16} - \frac{4 \cdot (s-2)}{(s-2)^2+16}$

$= \frac{12 - 4s + 8}{s^2 - 4s + 20}$

$= \frac{20-4s}{s^2-4s+20}$

18 $L\{\cos 3t \cdot \cos 2t \cdot \cos t\}$

$= L\{(\cos 3t \cdot \cos 2t) \cdot \cos t\}$

$= \frac{1}{2}\{L(2\cos 3t \cdot \cos 2t) \cdot \cos t\}$

$= \frac{1}{2}\{L\{(\cos 5t + \cos t)\cos t\}\}$

$= \frac{1}{2}\left[L\{\cos 5t \cos t\} + L\{\cos t \cos t\}\right]$

$= \frac{1}{4}\left[L\{2\cos 5t \cdot \cos t\} + L\{2\cos t \cos t\}\right]$

$= \frac{1}{4}\left[L\{\cos 6t + \cos 4t\} + L\{\cos 2t + \cos 0\}\right]$

$= \frac{1}{4}\left[L\{\cos 6t\} + L\{\cos 4t\} + L\{\cos 2t\} + L\{1\}\right]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right]$

19 $L\{\cos^3 at\}$

$L\left\{\frac{\cos 3at + 3\cos at}{4}\right\}$

$= \frac{1}{4}\left[L\{\cos(3at)\} + 3L\{\cos(at)\}\right]$

$= \frac{1}{4}\left[\frac{s}{s^2+9a^2} + 3\frac{s}{s^2+a^2}\right]$

$\{\cos hat\} = \frac{e^{ax} + e^{-ax}}{2}$

20 $L\{\cos hat \cdot \cos bt\}$

$L\left\{\left(\frac{e^{ax} + e^{-ax}}{2}\right)\cos bt\right\}$

$= \frac{1}{2}\left[L\{e^{ax}\cos bt\} + L\{e^{-ax}\cos bt\}\right]$

$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$