(1) Form a Partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— (i)

Here, the number of arbitrary constant is more than the number of independent variable.

Now, Differentiating eqn (i) partially w.r.t x we get,

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{pz}{c^2} = 0 \text{ —— (2)}$$

Differentiating eqn (i) partially w.r.t y we get,

$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{y}{b^2} + \frac{qz}{c^2} = 0 \text{ —— (3)}$$

Differentiating eqn (2) partially w.r.t x we get,

$$\frac{1}{a^2} + \frac{1}{c^2}\left[p \cdot \frac{\partial z}{\partial x} + z \cdot \frac{\partial p}{\partial x}\right] = 0$$

$$\Rightarrow \frac{1}{a^2} + \frac{p^2}{c^2} + \frac{zr}{c^2} = 0 \text{ —— (4)}$$

Differentiating eqn (3) partially w.r.t y we get,

$$\frac{1}{b^2} + \frac{1}{c^2}\left[q \cdot \frac{\partial z}{\partial y} + z \cdot \frac{\partial q}{\partial y}\right] = 0$$

$$\Rightarrow \frac{1}{b^2} + \frac{q^2}{c^2} + \frac{zt}{c^2} = 0 \text{ —— (5)}$$

Differentiating eqn (2) partially w.r.t y we get,

$$\frac{1}{c^2}\left[p \cdot \frac{\partial z}{\partial y} + z \cdot \frac{\partial p}{\partial y}\right] = 0$$

$$\Rightarrow \frac{pq}{c^2} + \frac{zs}{c^2} = 0 \text{ —— (6)}$$

From eqn (6) we can get,

$$\boxed{pq + zs = 0 \text{ (Ans)}}$$

(2) Form a partial differential equation by eliminating the arbitrary functions f and g from $z = y \cdot f(x) + x \cdot g(y)$

Given, $z = y \cdot f(x) + x \cdot g(y)$ —— (i)

Now, Differentiating eqn (i) partially w.r.t x we get,

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) \Rightarrow p = yf'(x) + g(y) \text{ —— (2)}$$

Differentiating eq^n (1) partially w.r.t y we get,

$\frac{\partial z}{\partial y} = f(x) + xg'(y) \Rightarrow q = f(x) + xg'(y)$ —— (3)

Multiply x and y in both sides of eq^n (2) and eq^n (3) respectively we can get,

$$xp = xyf'(x) + xg(y) \quad \text{—— (4)}$$

$$xq = yf(x) + xyg'(y) \quad \text{—— (5)}$$

Now, adding eq^n (4) and (5) we can get,

$$xp + yq = xy[f'(x) + g'(y)] + xg(y) + yf(x)$$

$$\Rightarrow xp + yq = xy[f'(x) + g'(y)] + z \quad \text{—— (6)}$$

Differentiating eq^n (2) partially w.r.t y we can get,

$$\frac{\partial p}{\partial y} = f'(x) + g'(y)$$

$$\Rightarrow s = f'(x) + g'(y)$$

Now, substituting this value in eq^n (6) we can get,

$$\boxed{xp + yq = xys + z \text{ (Ans)}}$$

(3) If $f(x+ay) + \phi(x-ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2\left(\frac{\partial^2 z}{\partial x^2}\right)$

Given, $z = f(x+ay) + \phi(x-ay)$

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$

Now $\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2[f''(x+ay) + \phi''(x-ay)]$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2\left(\frac{\partial^2 z}{\partial x^2}\right) \text{ (proved)}$$

(4) Form a partial differential equation by eliminating the arbitary functions from $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$.

Given $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$.

$$u = \frac{x-a}{z-c}, \quad v = \frac{y-b}{z-c}.$$

$$P = -\frac{\partial\left(\frac{x-a}{z-c}\right)}{\partial z} \cdot \frac{\partial\left(\frac{y-b}{z-c}\right)}{\partial y}.$$

$$\Rightarrow P = -\left[\frac{(z-c).0-(x-a)}{(z-c)^2} \cdot \frac{(z-c)-(y-b).0}{(z-c)^2}\right]$$

$$\Rightarrow P = \frac{(x-a)(z-c)}{(z-c)^4}$$

$$\Rightarrow P = \frac{x-a}{(z-c)^3}$$

$$Q = -\frac{\partial\left(\frac{x-a}{z-c}\right)}{\partial x} \cdot \frac{\partial\left(\frac{y-b}{z-c}\right)}{\partial z}$$

$$\Rightarrow Q = -\left[\frac{(z-c)-(x-a).0}{(z-c)^2} \cdot \frac{(z-c)0-(y-b)}{(z-c)^2}\right]$$

$$\Rightarrow Q = \frac{(z-c)(y-b)}{(z-c)^4}$$

$$\Rightarrow Q = \frac{y-b}{(z-c)^3}$$

$$R = \frac{\partial\left(\frac{x-a}{z-c}\right)}{\partial x} \cdot \frac{\partial\left(\frac{y-b}{z-c}\right)}{\partial y}$$

$$\Rightarrow R = \frac{(z-c)-(x-a).0}{(z-c)^2} \cdot \frac{(z-c)-(y-b).0}{(z-c)^2}$$

$$\Rightarrow R = \frac{(z-c)^2}{(z-c)^4}$$

$$\Rightarrow R = \frac{1}{(z-c)^2}$$

Therefore, the required PDE is

$$\frac{(x-a)}{(z-c)^3}p + \frac{(y-b)}{(z-c)^3}q = \frac{1}{(z-c)^2}$$

$$\Rightarrow \boxed{(x-a)p + (y-b)q = (z-c)} \text{ (Ans)}.$$

(5) Solve the PDE $y^2p - xyq = x(z-2y)$.

Given, $y^2p - xyq = x(z-2y)$ ——— (1)

Here, $P = y^2$, $Q = -xy$, $R = x(z-2y)$

Now, the lagrange's auxiliary eq^n is $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ ——— (2)

By taking the first and second ratio from eq^n (2) we get,

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow -\int x\,dx = \int y\,dy$$

$$\Rightarrow -\frac{x^2}{2} + c_1 = \frac{y^2}{2} + c_2 \Rightarrow \boxed{x^2 + y^2 = c} \quad \text{——— (3)}$$

By taking the second and last ratio from eq^n (2) we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{2y-z} \Rightarrow \frac{dz}{dy} = \frac{2y-z}{y} \Rightarrow \frac{dz}{dy} = 2 - \frac{z}{y}$$

$$\Rightarrow \frac{dz}{dy} + \frac{z}{y} = 2$$

Integrating factor

$$= e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

Now, $yz = \int 2.y\,dy$

$$\Rightarrow yz = 2 \cdot \frac{y^2}{2} + k$$

$$\Rightarrow yz - y^2 = k \quad \text{——— (4)}$$

Therefore, the required solution is $\phi(x^2+y^2, yz-y^2) = 0$ (Ans)

(6) Solve the PDE $py + qx = xyz^2(x^2-y^2)$.

Given, $py + qx = xyz^2(x^2-y^2)$ ——— (1)

Here, $P = y$, $Q = x$, $R = xyz^2(x^2-y^2)$

Now, the lagrange's auxiliary eq^n is

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)} \quad \text{——— (2)}$$

By taking first and second ratio from eq^n (2) we get, $\frac{dx}{y} = \frac{dy}{x} \Rightarrow \int x\,dx = \int y\,dy \Rightarrow \frac{x^2}{2} + c_1 = \frac{y^2}{2} + c_2$

$$\Rightarrow x^2 - y^2 = c \quad \text{——— (3)}$$

By taking second and last ratio from eq^n (2) we get,

$$\frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$$

$$\Rightarrow dy = \frac{dz}{yz^2(x^2-y^2)}.$$

$\Rightarrow \int y\,dy = \int \frac{dz}{z^2 \cdot c}$

$\Rightarrow \frac{y^2}{2} + c_3 = -\frac{1}{zc} + c_4$

$\Rightarrow \frac{y^2}{2} + \frac{1}{z(x^2-y^2)} = k$ —— (4)

Therefore, the required solution is $\phi\left(x^2-y^2, \frac{y^2}{2} + \frac{1}{z(x^2-y^2)}\right) = 0$. (Ans).

(7) Solve the PDE $(y-z)p+(z-x)q=(x-y)$.

Given, $(y-z)p+(z-x)q=(x-y)$ —— (1)

Here, $P = y-z$, $Q = z-x$, $R = x-y$.

Now, the lagrange's auxiliary eq^n is

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \quad \text{—— (2)}$$

Choosing 1, 1, 1 as multipliers of each fraction for eq^n (2) we get,

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$\Rightarrow dx + dy + dz = 0$

$\Rightarrow \int dx + \int dy + \int dz = 0$.

$\Rightarrow x + y + z + c_1 + c_2 + c_3 = 0$

$\Rightarrow x + y + z = c$ —— (3)

Choosing x, y, z as multipliers of each fraction for eq^n (2) we get,

$$\frac{x\,dx}{x(y-z)} = \frac{y\,dy}{y(z-x)} = \frac{z\,dz}{z(x-y)} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{xy}+\cancel{xz}-\cancel{yz}}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c_4 + c_5 + c_6 = 0$

$\Rightarrow x^2 + y^2 + z^2 = k$ —— (4)

Therefore, the required solution is

$\phi(x+y+z, x^2+y^2+z^2) = 0$ (Ans)

(8) Solve the PDE $(y+z)p + (z+x)q = (x+y)$

Given $(y+z)p + (z+x)q = (x+y)$ —— (1)

Here, $P = y+z$, $Q = z+x$, $R = x+y$

Now, the lagrange's auxiliary eq^n is $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ —— (2)

hoosing 1, -1, 0 as multipliers of each fraction of eq^n we get,

$$\frac{dx}{y+z} = \frac{-dy}{-(z+x)} = \frac{0}{0} = \frac{dx-dy}{y+z-z-x} = \frac{dx-dy}{y-x}$$

hoosing 0, 1, -1 as multipliers of each fraction of eq^n ② we get,

$$\frac{0}{0} = \frac{dy}{z+x} = \frac{-dz}{-(x+y)} = \frac{dy-dz}{z+x-x-y} = \frac{dy-dz}{z-y} \quad \text{——} ④$$

Choosing 1, 1, 1 as multipliers of each fraction of eq^n ② we get,

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx+dy+dz}{2x+2y+2z} \quad \text{——} ⑤$$

Now $$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2x+2y+2z} \quad \text{——} ⑥$$

By taking the first and second ratio from eq^n ⑥ we get,

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} \Rightarrow \int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\Rightarrow \ln|x-y| + \ln c_1 = \ln|y-z| + \ln c_2$$

$$\Rightarrow \ln\left|\frac{x-y}{y-z}\right| = \ln\left|\frac{c_2}{c_1}\right|$$

$$\Rightarrow \frac{x-y}{y-z} = c \quad \text{——} ⑦$$

By taking the first and third ratio from eq^n ⑥ we get,

$$\frac{dx-dy}{y-x} = \frac{dx+dy+dz}{2x+2y+2z}$$

$$\Rightarrow \frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\Rightarrow -\int \frac{d(x-y)}{(x-y)} = \int \frac{d(x+y+z)}{2(x+y+z)}$$

$$\Rightarrow -\ln|x-y| + \ln c_3 = \frac{1}{2}\ln|x+y+z| + \ln c_4$$

$$\Rightarrow \frac{1}{2}\ln|x+y+z| + \ln|x-y| = \ln c_3 - \ln c_4$$

$$\Rightarrow \ln|(x+y+z)^{1/2}(x-y)| = \ln\left|\frac{c_3}{c_4}\right|$$

$$\Rightarrow \sqrt{x+y+z}\,(x-y) = k \quad \text{——} ⑧$$

Therefore, the required solution is

$$\phi\left(\frac{x-y}{y-z}, \sqrt{x+y+z}\,(x-y)\right) = 0 \quad \text{(Ans).}$$

(9) Solve the partial differential equation
$px(x+y) - qy(x+y) + (x-y)(2x+2y+z) = 0$.

Given, $px(x+y) - qy(x+y) + (x-y)(2x+2y+z) = 0$

$\Rightarrow px(x+y) - qy(x+y) = -(x-y)(2x+2y+z)$ —— (1)

Here, $P = x(x+y)$, $Q = -y(x+y)$, $R = -(x-y)(2x+2y+z)$

Now, the Lagrange's auxiliary eqn is

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} \quad \text{—— (2)}$$

By taking first and second ratio from eqn (2) we get,

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{-y} \Rightarrow \int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\Rightarrow \ln x + \ln c_1 = -\ln y + \ln c_2$$

$$\Rightarrow \ln x + \ln y = \ln c_2 - \ln c_1$$

$$\Rightarrow \ln|xy| = \ln\left|\frac{c_2}{c_1}\right|$$

$$\Rightarrow xy = c \quad \text{—— (3)}$$

Choosing 1, 1, 0 as multipliers of each fraction of eqn (2) we get,

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{0}{0} = \frac{dx+dy}{x(x+y)-y(x+y)} \quad \text{—— (4)}$$

Choosing 1, 1, 1 as multipliers of each fraction of eqn (2) we get,

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} = \frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)} \quad \text{—— (5)}$$

Now, $$\frac{dx+dy}{x(x+y)-y(x+y)} = \frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$\Rightarrow \frac{dx+dy}{(x+y)(x-y)} = \frac{dx+dy+dz}{(x+y)(x-y)-(x-y)(2x+2y+z)}$$

$$\Rightarrow \frac{dx+dy}{(x+y)(x-y)} = \frac{dx+dy+dz}{(x-y)(x+y-2x-2y-z)}$$

$$\Rightarrow \frac{dx+dy}{x+y} = \frac{dx+dy+dz}{-(x+y+z)}$$

$\Rightarrow \int \frac{d(x+y)}{x+y} = -\int \frac{d(x+y+z)}{x+y+z}$

$\Rightarrow \ln|x+y| + \ln c_3 = -\ln|x+y+z| + \ln c_4$

$\Rightarrow \ln|x+y| + \ln|x+y+z| = \ln c_4 - \ln c_3$

$\Rightarrow \ln|(x+y)(x+y+z)| = \ln\left|\frac{c_4}{c_3}\right|$

$\Rightarrow (x+y)(x+y+z) = k$ ——— (6)

Therefore, the required solution is

$\phi(xy, (x+y)(x+y+z)) = 0$

(10) Find the complete integral of $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

Given, $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ ——— (1)

Here, $f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

Now $f_x = 0$, $f_y = 0$, $f_z = 32p^2z^2 + 18q^2z + 8z$,

$f_p = 32pz^2$, $f_q = 18qz^2$

Now, charpit's auxiliary eqn is

$$\frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dz}{-32p^2z^2 - 18q^2z^2} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}$$

By taking first and second ratio of eqn (2) we get,

$$\frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz}$$

$$\Rightarrow \frac{dp}{p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)}$$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{dq}{q}$$

$\Rightarrow \ln p + \ln c_1 = \ln q + \ln c_2$

$\Rightarrow \ln p - \ln q = \ln c_2 - \ln c_1$

$\Rightarrow \ln\left(\frac{p}{q}\right) = \ln\left(\frac{c_2}{c_1}\right) \Rightarrow \frac{p}{q} = a \Rightarrow p = aq$

Now, putting this value in eqn (1) we get,

$16a^2q^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

$\Rightarrow q^2z^2(16a^2 + 9) = 4 - 4z^2$

$$\Rightarrow q^2z^2 = \frac{4 - 4z^2}{16a^2 + 9}$$

$\Rightarrow q^2 = \frac{4}{z^2}\left[\frac{1-z^2}{16a^2+9}\right]$

$\Rightarrow q = \frac{2}{z}\sqrt{\frac{1-z^2}{16a^2+9}}$

$\Rightarrow q = \frac{2}{\sqrt{16a^2+9}} \cdot \frac{\sqrt{1-z^2}}{z}$

Now $p = \frac{2a}{\sqrt{16a^2+9}} \cdot \frac{\sqrt{1-z^2}}{z}$

Now, $dz = p\,dx + q\,dy$

$\Rightarrow dz = \left(\frac{2a}{\sqrt{16a^2+9}} \cdot \frac{\sqrt{1-z^2}}{z}\right)dx + \left(\frac{2}{\sqrt{16a^2+9}} \cdot \frac{\sqrt{1-z^2}}{z}\right)dy$

$\Rightarrow \frac{z}{\sqrt{1-z^2}}\,dz = \frac{2}{\sqrt{16a^2+9}}(a\,dx + dy)$

$\Rightarrow \int \frac{z \cdot dz}{\sqrt{1-z^2}} = \frac{2}{\sqrt{16a^2+9}}\left[a\int dx + \int dy\right]$

let $1-z^2 = v^2$.
$\Rightarrow -2z\,dz = 2v\,dv$
$\Rightarrow -z\,dz = v\,dv$
$\Rightarrow z\,dz = -v\,dv$

$\Rightarrow -\int \frac{v\,dv}{v} = \frac{2}{\sqrt{16a^2+9}}(ax+y) + b$

$\Rightarrow -\sqrt{1-z^2} = \frac{2}{\sqrt{16a^2+9}}(ax+y) + b$

$\Rightarrow \frac{2}{\sqrt{16a^2+9}}(ax+y) + \sqrt{1-z^2} = b$ (Ans).

(11) Find the complete integral of $p^2+q^2-2px-2qy+2xy=0$.

Given $p^2+q^2-2px-2qy+2xy=0$ ——— (1)

Here $f(x,y,z,p,q) = p^2+q^2-2px-2qy+2xy=0$

Now, $f_x = -2p+2y$, $f_y = -2q+2x$, $f_z = 0$

$f_p = 2p-2x$ $\quad f_q = 2q-2y$

Now, charpit's auxiliary eqn is

$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dz}{-2p^2+2px-2q^2+2qy} = \frac{dx}{-2p+2x} = \frac{dy}{-2q+2y} = 0$ ——— (2)

By taking first and second ratio from eqn (2)

$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dp+dq}{2(-p+y-q+x)}$ ——— (3)

By taking fourth and fifth ratio from eq^n (2) we get,
$$\frac{dx}{-2p+2x} = \frac{dy}{-2q+2y} = \frac{dx+dy}{2(-p+x-q+y)} \quad \text{—— (4)}$$
From eq^n (3) and (4) we get
$$\frac{dp+dq}{2(-p+y-q+x)} = \frac{dx+dy}{2(-p+x-q+y)}$$
$$\Rightarrow \int d(p+q) = \int d(x+y)$$
$$\Rightarrow p+q = x+y+a$$
$$\Rightarrow p-x+q-y = a \quad \text{—— (5)}$$
From eq^n (1) we can write
$$(p-x)^2 + (q-y)^2 = (x-y)^2 \quad \text{—— (6)}$$
From eq^n (5), $q-y = a-(p-x)$
Putting this value in eq^n (6) we can get,
$$(p-x)^2 + [a-(p-x)]^2 = (x-y)^2$$
$$\Rightarrow (p-x)^2 + a^2 + (p-x)^2 - 2a(p-x) = (x-y)^2$$
$$\Rightarrow 2(p-x)^2 - 2a(p-x) + \{a^2-(x-y)^2\} = 0$$
$$\Rightarrow p-x = \frac{2a \pm \sqrt{4a^2 - 8\{a^2-(x-y)^2\}}}{4}$$
$$\Rightarrow p = x + \frac{1}{2}\left[a \pm \sqrt{2(x-y)^2 - a^2}\right]$$
From eq^n (5) we get, $q = a+y-p+x$
$$\Rightarrow q = a+y-(p-x)$$
$$\Rightarrow q = y+a-\frac{2a \pm \sqrt{4a^2-8\{a^2-(x-y)^2\}}}{4}$$
$$\Rightarrow q = y + \frac{2a \pm \sqrt{4a^2-8\{a^2-(x-y)^2\}}}{4}$$
$$\Rightarrow q = y + \frac{1}{2}\left[a \pm \sqrt{2(x-y)^2 - a^2}\right]$$
Putting these value of p and q in $dz = p\,dx + q\,dy$ we get,
$$dz = x\,dx + y\,dy + \frac{1}{2}\left[a \pm \sqrt{2(x-y)^2-a^2}\right]dx + \frac{1}{2}\left[a \pm \sqrt{2(x-y)^2-a^2}\right]dy$$
$$\Rightarrow dz = x\,dx + y\,dy + \frac{a}{2}(dx+dy) \pm \frac{1}{2}\sqrt{(x-y)^2 - \frac{a^2}{2}}\,(dx-dy).$$

$$\Rightarrow \int dz = \int x\,dx + \int y\,dy + \frac{a}{2}\int d(x+y) \pm \frac{1}{\sqrt{2}}\int \sqrt{(x-y)^2 - \frac{a^2}{2}}\,(dx-dy).$$

$$\Rightarrow z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2}(x+y) \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2-\frac{a^2}{2}} - \frac{a^2}{4}\log\left[(x-y)+\sqrt{(x-y)^2-\frac{a^2}{2}}\right]\right)$$

$$\Rightarrow z = \frac{x^2+y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2-\frac{a^2}{2}} - \frac{a^2}{4}\log\left[(x-y)+\sqrt{(x-y)^2-\frac{a^2}{2}}\right]\right)$$

(Ans)

(12) Solve the PDE $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

Given $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

$$\Rightarrow p^2q^2 + x^2y^2 = x^4q^2 + x^2y^2q^2$$

$$\Rightarrow p^2q^2 = x^4q^2 + x^2y^2(q^2-1) \quad \text{--- (1)}$$

Dividing q^2 both sides of the eqn (1) we get,

$$p^2 = x^4 + x^2y^2\left(1-\frac{1}{q^2}\right)$$

$$\Rightarrow p^2 - x^4 = x^2y^2\left(1-\frac{1}{q^2}\right) \quad \text{--- (2)}$$

Dividing x^2 both sides of the eqn (2) we get,

$$\frac{p^2}{x^2} - x^2 = y^2\left(1-\frac{1}{q^2}\right)$$

$$\Rightarrow \frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2} = a^2$$

Now, $\frac{p^2}{x^2} - x^2 = a^2$

$$\Rightarrow \frac{p^2}{x^2} = a^2 + x^2$$

$$\Rightarrow p^2 = (a^2+x^2)x^2$$

$$\Rightarrow p = x\sqrt{a^2+x^2}$$

$y^2 - \frac{y^2}{q^2} = a^2$

$$\Rightarrow \frac{y^2}{q^2} = y^2 - a^2$$

$$\Rightarrow q^2 = \frac{y^2}{y^2-a^2}$$

$$\Rightarrow q = \frac{y}{\sqrt{y^2-a^2}}$$

Now, $dz = p\,dx + q\,dy$

$\Rightarrow dz = x\sqrt{a^2+x^2}\,dx + \frac{y}{\sqrt{y^2-a^2}}\,dy$ $\qquad$ let $y^2-a^2 = v^2$ $\Rightarrow 2y\,dy = 2v\,dv$ $\Rightarrow y\,dy = v\,dv$

$\Rightarrow \int dz = \int \sqrt{a^2+x^2}\,x\,dx + \int \frac{y\,dy}{\sqrt{y^2-a^2}}$

$\Rightarrow \int dz = \int u \cdot u\,du + \int \frac{v\,dv}{v}$ $\qquad$ let $u^2 = a^2+x^2$

$\Rightarrow \int dz = \int u^2\,du + \int dv$ $\qquad$ $\Rightarrow 2u\,du = 2x\,dx$ $\Rightarrow u\,du = x\,dx$

$\Rightarrow z = \frac{u^3}{3} + v + b$

$\Rightarrow z = \frac{1}{3}(a^2+x^2)^{3/2} + \sqrt{y^2-a^2} + b$ (Ans)

(13) Find the complementary function of $(D_x + 2D_y - 3)(D_x + D_y - 1)z = 0$.

Given $(D_x + 2D_y - 3)(D_x + D_y - 1)z = 0$

Here $a = 1,\ b = 2,\ c = -3,\quad a = 1,\ b = 1,\ c = -1$.

So $z = e^{3x}\phi(y+2x) + e^{x}\phi(y-x)$ (Ans).

(14) Find the complementary function of $(D_x^3 - D_x^2 D_y - 8D_x D_y^2 + 12D_y^3)z = 0$.

Given, $(D_x^3 - D_x^2 D_y - 8D_x D_y^2 + 12D_y^3)z = 0$

Take $D_x = m,\ D_y = 1$

So auxiliary eq^n is $m^3 - m^2 - 8m + 12 = 0$

$\Rightarrow m = -3, 2, 2$

$m_1 = -3,\ m_2 = 2,\ m_3 = 2$

So, $z = \phi_1(y - 3x) + \phi_2(y+2x) + x\phi_3(y+2x)$ (Ans)

(15) Find the complementary function of $(D_x^4 - D_y^4)z = 0$

Given $(D_x^4 - D_y^4)z = 0$

Take $D_x = m,\ D_y = 1$

So auxiliary eq^n is $(m^4 - 1) = 0$

$\Rightarrow m = 1, -1, i, -i$

So $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_3(y-ix)$
$+ i[\phi_4(y+ix) - \phi_4(y-ix)]$ (Ans).

* Find the complete integral of $p^2+q^2-2px-2qy+2xy=0$

Soln:- Given equation is,

$$f(x,y,z,p,q)=p^2+q^2-2px-2qy+2xy=0 \quad \text{——— (1)}$$

Charpit's auxillary equations are,

$$\frac{dp}{\frac{\partial f}{\partial x}+p\cdot\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\cdot\frac{\partial f}{\partial z}}=\frac{dz}{-p\cdot\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{-2p+2y}=\frac{dq}{-2q+2x}=\frac{dz}{-2p^2+2xp-2q^2+2qy}=\frac{dx}{2x-2p}=\frac{dy}{2y-2q}$$

$$\Rightarrow \frac{dp+dq}{-2p+2y-2q+2x}=\frac{dx+dy}{2x-2p+2y-2q}$$

$$\Rightarrow dp+dq=dx+dy$$

Integrating the above equation,

$$\Rightarrow p+q=x+y+\text{Constant}$$

$$\Rightarrow (p-x)+(q-y)=a \quad \text{——— (2)}$$

Re-writing equation - (1) we get,

$$p^2+q^2-2px-2qy+x^2+y^2=x^2+y^2-2xy$$

$$\Rightarrow (p-x)^2+(q-y)^2=(x-y)^2 \quad \text{——— (3)}$$

from eq^n (2) and (3)

$$q-y=a-(p-x)$$

$$\Rightarrow (p-x)^2+\left[a-(p-x)\right]^2=(x-y)^2$$

$$\Rightarrow (p-x)^2+a^2+(p-x)^2-2a(p-x)=(x-y)^2$$

$$\Rightarrow 2(p-x)^2-2a(p-x)+\{a^2-(x-y)^2\}=0$$

$$\Rightarrow p-x=\frac{2a\pm\sqrt{4a^2-4\cdot 2\{a^2-(x-y)^2\}}}{4}$$

$\Rightarrow p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}]$

$\Rightarrow q = a + y - p + x$

$= a + y + \cancel{x} - \cancel{x} - \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}]$

$= y + \frac{1}{2}[a \mp \sqrt{\{2(x-y)^2 - a^2\}}]$

put $x - y = z$

$dx - dy = dz$

putting the values of p and q in

$dz = p\,dx + q\,dy$, we get,

$\Rightarrow dz = x\,dx + y\,dy + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}](dx - dy)$

Integration,

$\Rightarrow z = \frac{x^2}{2} + \frac{y^2}{2} + \int \frac{1}{2}[a \pm \sqrt{2z^2 - a^2} \cdot dz]$

$= \frac{x^2}{2} + \frac{y^2}{2} + \int \frac{1}{2}[a \pm \sqrt{2}\sqrt{z^2 - \left(\frac{a}{\sqrt{2}}\right)^2}]\,dz$

$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2}\left[\int a \pm \sqrt{2} \cdot \sqrt{z^2 - \left(\frac{a}{\sqrt{2}}\right)^2}\,dz\right]$

$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2}\left[az \pm \sqrt{2}\left\{\frac{z}{2}\sqrt{z^2 - \left(\frac{a}{\sqrt{2}}\right)^2} - \frac{a^2}{2 \cdot 2}\ln\left|z + \sqrt{z^2 - \left(\frac{a}{\sqrt{2}}\right)^2}\right|\right\}\right] + C$

$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2}a(x-y) \pm \frac{1}{\sqrt{2}}\left\{\frac{x-y}{2}\sqrt{(x-y)^2 - \frac{a^2}{2}} - \frac{a^2}{4}\ln\left|(x-y) + \sqrt{(x-y)^2 - \frac{a^2}{2}}\right|\right\} + C$

(Ans.)

(08) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Soln:

Here $P = x^2(y-z)$

$Q = y^2(z-x)$

$R = z^2(x-y)$

Here, the Lagrange's Auxillary equation is,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \text{——} \quad (1)$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers of each fraction of equation (1), we get;

$$\frac{\frac{1}{x}dx}{x(y-z)} = \frac{\frac{1}{y}dy}{y^2(z-x)} = \frac{\frac{1}{z}dz}{z(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{xz} - \cancel{yz}}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integration,

$$\Rightarrow \int\frac{dx}{x} + \int\frac{dy}{y} + \int\frac{dz}{z} = 0$$

$$\Rightarrow \ln x + \ln y + \ln z = \ln c$$

$$\Rightarrow \boxed{xyz = c}$$

Choosing $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers of each fraction of equation (1), we get,

$$\frac{\frac{dx}{x^2}}{y-z} = \frac{\frac{dy}{y^2}}{z-x} = \frac{\frac{dz}{z^2}}{(x-y)} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{\cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}}$$

$$\Rightarrow \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integration $\int\frac{dx}{x^2} + \int\frac{dy}{y^2} + \int\frac{dz}{z^2} = 0$ $\quad \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = K$

$\therefore$ Soln: $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$