

## DOUBLE INTEGRAL

Let  $D$  be a closed\* and bounded\*\* domain in the  $XY$ -plane bounded by a simple closed curve  $c$ . Let

$f(x, y)$  be a given continuous function in  $D$ . Divide  $D$  into  $n$  parts and form the sum

$$\sum_{i=1}^n f(P_i) \Delta S_i$$

where  $f(P_i)$  is the value of  $f$  at an arbitrary point  $P_i$  of the subdomain whose area is  $\Delta S_i$ . Double integral of  $f(x, y)$  over the domain  $D$  is the limit of the above sum as  $n \rightarrow \infty$  and is denoted by  $I_D$  as

$$\begin{aligned} I_D &= \iint_D f(P) dS = \iint_D f(x, y) dx dy \\ &= \lim_{\Delta S_i \rightarrow 0} \sum_{i=1}^n f(P_i) \Delta S_i \end{aligned} \tag{1}$$

Here  $D$  is known as the domain (region) of integration.

The following properties of double integrals follow from the definition (1).

## Properties of Double Integral

1.  $\iint_D a f(x, y) dS = a \iint_D f(x, y) dS, a = \text{constant.}$
2.  $\iint_D [f(x, y) + g(x, y)] dS = \iint_D f(x, y) dS + \iint_D g(x, y) dS$
3.  $\iint_D f(x, y) dS = \iint_{D_1} f(x, y) dS + \iint_{D_2} f(x, y) dS$

where  $D$  is the union of disjoint domains  $D_1$  and  $D_2$ .

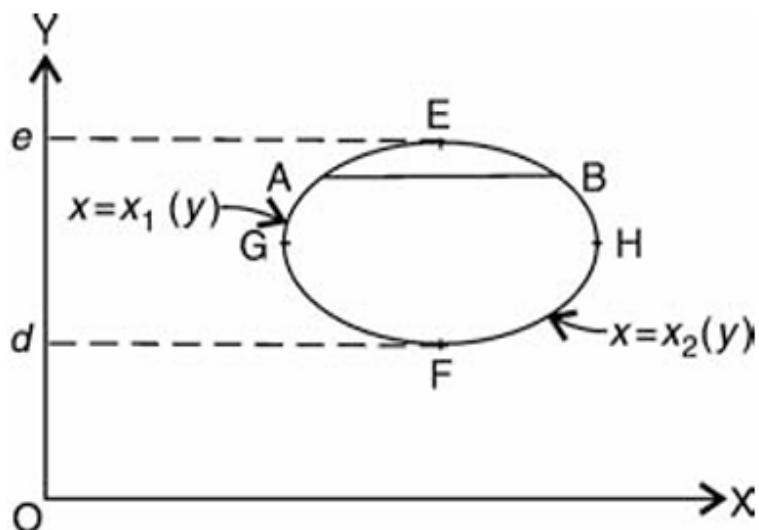
## Evaluation of a Double Integral

A double integral can be evaluated by successive single integrations i.e., as a two-fold iterated (repeated) integral as follows (if  $D$  is regular in  $y$ -direction):

$$I_D = \int_{x=a}^b \left\{ \int_{y=y_1(x)}^{y_2(x)} f(x, y) dy \right\} dx \quad (2)$$

where integration is performed *first* with respect to  $y$  (within the braces). With the substitution of the limits  $y_1(x)$  and  $y_2(x)$ , the integrand becomes a function of  $x$  alone, which is *then* integrated with respect to  $x$  from  $a$  to  $b$ .

In a similar way, for a domain  $D$  (regular in x-direction) which is bounded above by  $EHF$   $x = x_2(y)$  and bounded below by  $EGF$ :  $x = x_1(y)$  and the abscissa  $y = d$  and  $y = e$  (Fig. 7.2).



**Fig. 7.2**

The double integral is evaluated as

$$I_D = \int_{y=d}^e \left\{ \int_{x=x_1(y)}^{x_2(y)} f(x, y) dx \right\} dy \quad (3)$$

In this case the integration is *first* performed with respect to  $x$  and *then* later with respect to  $y$ .

## APPLICATION OF DOUBLE INTEGRAL

### Area of a Plane Region

The area  $A$  of a plane regular region (domain)  $D$  is given by a two-fold iterated integral

$$\begin{aligned} A &= \iint_D ds = \iint_D dx dy \\ &= \int_{x=a}^b \int_{y=y_1(x)}^{y_2(x)} dy dx = \int_{y=d}^e \int_{x=x_1(y)}^{x_2(y)} dx dy \end{aligned}$$

### Mass Contained in a Plane Region

Let  $f(x, y) > 0$  be the surface density (mass/unit area) of a given plane region  $D$ . Then the amount (quantity) of mass  $M$  contained in the plane region

$D$  is given by

$$M = \iint_D f(x, y) dx dy$$

## **Centre of Gravity (Centroid) of a Plane Region $D$**

The coordinates  $(x_c, y_c)$  of the centre of gravity (centroid) of a plane region  $D$  with surface density  $f(x, y)$  and containing mass  $M$  are

$$x_c = \frac{\iint_D x \cdot f(x, y) dx dy}{M}, \quad y_c = \frac{\iint_D y \cdot f(x, y) dx dy}{M}$$

## **Moment of Inertia of a Plane Region**

Moments of inertia of a plane region  $D$  (with surface density  $f(x, y)$ ) relative to  $x$ -axis,  $y$ -axis and origin O are respectively given by

$$I_{xx} = \iint_D y^2 f(x, y) dx dy$$

$$I_{yy} = \iint_D x^2 f(x, y) dx dy$$

$$I_o = I_{xx} + I_{yy} - \iint_D (x^2 + y^2) f(x, y) dx dy$$

$I_o$  is also known as polar moment of inertia.

## **Volume under a Surface of a Solid as a Double Integral**

Let  $z = f(x, y) > 0$  be the equation of a surface. Let the curve  $c$  be the boundary of the plane domain  $D$  in the  $XY$ -plane. Further let  $V$  be the volume of the solid  $Q$ , under the surface  $z = f(x, y)$  (i.e., bounded above by the surface) and above the  $XY$ -plane (i.e., bounded below by  $z = 0$ ) and a cylindrical surface whose generator are parallel to the  $z$ -axis, while the directrix  $Q$  is  $c$  (see Fig. 7.7).

Then the double integral of  $f(x, y)$  taken over  $D$  gives the volume  $V$  under the surface  $z = f(x, y)$

$$V = \iint_D f(x, y) dx dy.$$

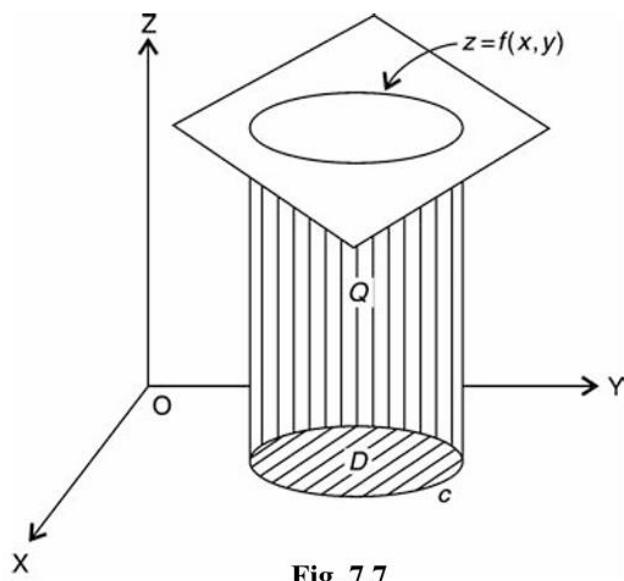


Fig. 7.7

## 7.3 CHANGE OF ORDER OF INTEGRATION: DOUBLE INTEGRAL

As already seen, for the double integral with variable limits

$$I_D = \iint_D f(x, y) ds \quad (1)$$

The limits of integration can be fixed from a rough sketch of the domain of integration. Then (1) can be evaluated as a two-fold iterated integral using either

$$I_D = \int_{x=a}^b \int_{y=y_1(x)}^{y_2(x)} f(x, y) dy dx \quad (2)$$

or

$$I_D = \int_{y=d}^e \int_{x=x_1(y)}^{x_2(y)} f(x, y) dx dy \quad (3)$$

In each specific problem, depending upon the type of the domain  $D$  and/or the nature of integrand, choose either of the form (2) or (3) whichever is easier to evaluate. Thus in several problems, the evaluation of double integral becomes easier with the change of order of integration, which of course, changes the limits of integration also.

## 7.4 GENERAL CHANGE OF VARIABLES IN DOUBLE INTEGRAL

In several cases, the evaluation of double integrals becomes easy when there is a change of variables.

Let  $D$  be domain in  $xy$ -plane and let  $x, y$  be the rectangular cartesian coordinates of any point  $P$  in  $D$ . Let  $u, v$  be new variables in domain  $D^*$  such that  $x, y$  and  $u, v$  are connected through the continuous functions (transformations).

$$x = g(u, v), y = h(u, v) \quad (1)$$

Then  $u, v$  are said to be curvilinear coordinates of point  $P^*$  in  $D^*$  which uniquely corresponds to  $P$  in  $D$ . Solving (1) for  $u$  and  $v$ , we get

$$u = g^*(x, y), v = h^*(x, y) \quad (2)$$

Then a given double integral in the given (old) variables  $x, y$  can be transformed to a double integral

in terms of new variables  $u, v$  as follows:

$$\iint_D f(x, y) dx dy = \iint_{D^*} F(u, v) |J| du dv \quad (3)$$

Here  $f(x, y) = f(x(u, v), y(u, v)) = F(u, v)$  and  $J$  is the Jacobian (functional determinant) defined as

$$J = J \left( \frac{x, y}{u, v} \right) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(3) is known as formula for transformation of coordinates in double integral.

## Change of Variables: Cartesian to Polar Coordinates

### Double Integrals in Polar Coordinates

For a double integral in cartesian coordinates  $x, y$ , the change of variables to polar coordinates  $r, \theta$  can be done through the transformation

$$x = r \cos \theta, y = r \sin \theta$$

(i.e.,  $u = r, v = \theta$  in general change of variables.)

The Jacobian in this case is

$$\begin{aligned} J &= J\left(\frac{x, y}{r, \theta}\right) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ J &= r(\cos^2 \theta + \sin^2 \theta) = r \end{aligned}$$

Thus the double integral in cartesian coordinates  $x, y$  gets transformed to double integral in polar coordinates as follows:

$$\begin{aligned} \iint_D f(x, y) dx dy &= \iint_{D^*} f(r \cos \theta, r \sin \theta) |r| dr d\theta \\ &= \int_{\theta=\alpha}^{\beta} \int_{r=\phi_1(\theta)}^{\phi_2(\theta)} F(r, \theta) r dr d\theta \end{aligned}$$

where  $F(r, \theta) = f(r \cos \theta, r \sin \theta) = f(x, y)$  and  $D^*$  is the corresponding domain in polar coordinates.  
Area in polar coordinates

$$A = \iint_D ds = \iint_D dx dy = \iint_{D^*} r dr d\theta$$

## 7.5 TRIPLE INTEGRALS

Triple integral is a generalization of a double integral. Let  $V$  be a given three-dimensional domain in space, bounded by a closed surface  $S$ . Let  $f(x, y, z)$  be a continuous function in  $V$  of the rectangular coordinates  $x, y, z$ .

Divide  $V$  into subdomains  $\Delta v_i$ . Let  $f(P_i)$  be the value of  $f$  at an arbitrary point  $P_i$  of  $\Delta v_i$ . Then a triple integral of  $f$  over the domain  $V$ , denoted by  $\iiint_V f(P)dV$ , is defined as

$$\begin{aligned}\lim_{\Delta v_i \rightarrow 0} \sum f(P_i) \Delta v_i &= \iiint_V f(P)dV \\ &= \iiint_V f(x, y, z)dx dy dz \quad (1)\end{aligned}$$

However, the triple integral is seldom evaluated directly from its definition (1) as a limit of a sum.

## Evaluation of a Triple Integral

### Regular three-dimensional domain

$V$  is said to be a regular three-dimensional domain if (i) every straight line parallel to  $z$ -axis and drawn through an interior (i.e., not lying on the boundary  $S$ ) point of  $V$  cuts the surface  $S$  at two points (ii) entire  $V$  is projected on the  $xy$ -plane into a regular two-dimensional domain  $D$ .

**Examples:** Parallelopiped, ellipsoid, tetrahedron.

Let the equations of the surfaces bounding a regular domain  $V$  below and above be  $z = z_1(x, y)$  and  $z = z_2(x, y)$  respectively (see Fig. 7.29).

Let the projection  $D$  of  $V$  onto  $xy$ -plane be bounded by the curves  $y = y_1(x)$  and  $y = y_2(x)$  and  $x = a, x = b$ .

Then the three fold integral  $I_V$  of a continuous function  $f(x, y, z)$  over a regular domain  $V$  is

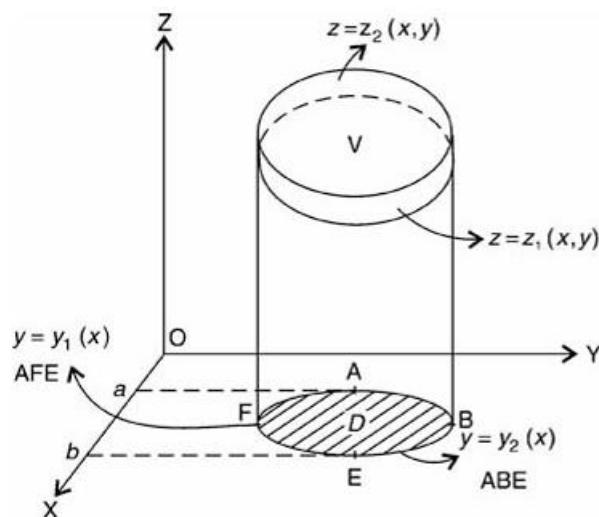


Fig. 7.29

defined as

$$I_V = \int_{x=a}^{x=b} \left[ \int_{y=y_1(x)}^{y=y_2(x)} \left\{ \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z) dz \right\} dy \right] dx$$

Here the limits of integration are chosen to cover the domain  $V$  by varying  $z$  from lower surface  $z = z_1(x, y)$  to the upper surface  $z = z_2(x, y)$  and covering the projection  $D$  by varying  $y$  from  $y_1(x)$  to  $y_2(x)$  and  $x$  from  $a$  to  $b$ .

In the three-fold iterated integral, integration is done first with respect to  $z$  (i.e., within the braces) with the substitution of limits for  $z$ , the next integration is carried with respect to  $y$  (i.e., within the square brackets). This results in an integrand which is a function of  $x$  alone which is then integrated w.r.t.  $x$  between  $a$  and  $b$ .

**Note:** When  $V$  is projected on to  $xz$ -plane or  $yz$ -plane instead of  $xy$ -plane, then the order of integration and the limits are to be rewritten appropriately.

## Applications of Triple Integrals

### Volume

Volume of a solid contained in the domain  $V$  is given by the triple integral (1) with  $f(x, y, z) = 1$  i.e., volume =  $\iiint_V dx dy dz$ .

## **Mass**

If  $\gamma(x, y, z) > 0$  is the volume density (mass/unit volume) of distribution of mass over  $V$  then the triple

integral (1) gives the entire mass contained in  $V$

$$\text{Mass} = \iiint_V \gamma(x, y, z) dx dy dz$$

## **Moment of inertia of a solid**

The moment of inertia of a solid relative to the  $z$ -axis is

$$I_{zz} = \iiint_V (x^2 + y^2) \gamma(x, y, z) dx dy dz$$

where  $\gamma(x, y, z)$  is the density of the substance. Similarly, moment of inertia of a solid relative to  $x$ -axis and  $y$ -axis are respectively

$$I_{xx} = \iiint_V (y^2 + z^2) \gamma \cdot dx dy dz$$

$$I_{yy} = \iiint_V (x^2 + z^2) \gamma dx dy dz$$

## **Centre of gravity of a solid: $(x_c, y_c, z_c)$**

$$x_c = \frac{\iiint_V x \gamma dV}{\iiint_V \gamma dV}, y_c = \frac{\iiint_V y \gamma dV}{\iiint_V \gamma dV}$$

$$z_c = \frac{\iiint_V z \gamma dV}{\iiint_V \gamma dV}.$$