Department of Mathematics and Computing

Mathematics I

Tutorial Sheet-1

1. Find the first four non-zero terms of the Taylor series generated by the function $f(x)$.
   (i) $f(x) = \sqrt{3+x^2}$ at $x = -1$ (ii) $f(x) = \dfrac{1}{1-x}$ at $x = 2$ (iii) $f(x) = \dfrac{1}{1+x}$ at $x = 3$
   (iv) $f(x) = \dfrac{1}{x}$ at $x = a > 0$ (v) $f(x) = \dfrac{1}{1+x^2}$ at $x = -2$ (vi) $f(x) = \sin(x^2)$ at $x = 1$
   (vii) $f(x) = \tan(x)$ at $x = 1$ (viii) $f(x) = e^{-2x}$ at $x = \frac{1}{2}$ (ix) $f(x) = \cosh(x)$ at $x = 1$

2. Find the Maclaurin series for the following functions.
   (i) $f(x) = \dfrac{1}{1-2x}$ (ii) $f(x) = \dfrac{1}{1+x^3}$ (iii) $f(x) = \sin(\pi x)$ (iv) $f(x) = \sin^2(x)$
   (v) $f(x) = \cos(x^{5/2})$ (vi) $f(x) = \cos(\sqrt{5x})$ (vii) $f(x) = e^{\pi x/2}$ (viii) $f(x) = e^{-x^2}$
   (ix) $f(x) = \log(1+x)$ (x) $f(x) = \dfrac{1}{1+x^2}$ (xi) $f(x) = \sinh(x)$

3. (i) Calculate e with an error of 10^{-6}.
   (ii) For what values of x can we replace $\sin(x)$ by $x - \frac{x^3}{3!}$ with an error of magnitude no greater than $3 \cdot 10^{-4}$?
   (iii) For approximately what values of x can you replace $\sin(x)$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than $5 \cdot 10^{-4}$? Give reasons for your answer.
   (iv) How close is the approximation $\sin(x) = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin(x)$?
   (v) The estimate $\sqrt{1+x} \approx 1 + \frac{x}{2}$ is used when x is small. Estimate the error when $|x| < 0.01$.
   (vi) The approximation $e^x \approx 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.
   (vii) Estimate the error in the approximation $\sinh(x) \approx x + \frac{x^3}{3!}$ when $|x| < 0.5$. (Hint: Use R_4 not R_3.)
   (viii) When $0 \leq h \leq 0.01$, show that e^h may be replaced by $1 + h$ with an error of magnitude no greater than 0.
   (ix) For what values of x can you replace $\ln(1+x)$ by x with an error of magnitude no greater than 1% of the value of x?

4. Find the limits (if they exist):
   (i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{x-y}$ (ii) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ (iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ (iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$

5. Examine the following functions for continuity at $(0,0)$. The expressions below give the values at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:
   (i) $\dfrac{x^3y}{x^6+y^2}$ (ii) $xy\dfrac{x^2-y^2}{x^2+y^2}$ (iii) $||x|-|y|| - |x| - |y|$ (iv) $\dfrac{x^3y}{x^4+y^2}$

6. Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the values at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:
(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$ (ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$ (iii) $\dfrac{xy}{x^2+y^2}$ (iv) $|x|+|y|$ (v) $\dfrac{\sin(x^3+y^4)}{x^2+y^2}$

7. (i) Compute the partial derivatives at $(0,0)$ for

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(ii) Determine the differentiability of the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^3y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

8. Let $u = \frac{y}{x}, v = x^2+y^2, w = w(u,v)$.

(i) Express the partial derivatives w_x and w_y in terms of w_u and w_v (and x and y).
(ii) Express $xw_x + yw_y$ in terms of w_u and w_v and write the coefficients as functions of u and v.
(iii) Find $xw_x + yw_y$ in case $w = v^5$.

9. Consider the curve of points (x,y,z) satisfying $x^5 + yz = 3$ and $xy^2 + yz^2 + zx^2 = 7$. Use the method of total differentials to find $\frac{dx}{dy}$ at $(x,y,z) = (1,1,2)$.

10. Analyze the following functions for local maxima, local minima, and saddle points:
(i) $f(x,y) = (x^2-y^2)\,e^{-(x^2+y^2)/2}$ (ii) $f(x,y) = x^3 - 3xy^2$ (iii) $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$
(iv) $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$ (v) $f(x,y) = x^3 + y^3 - 3xy$

11. Find the absolute maximum and the absolute minimum of

$$f(x,y) = \left(x^2 - 4x\right)\cos y \text{ for } 1 \leq x \leq 3, \frac{-\pi}{4} \leq y \leq \frac{\pi}{4}.$$

12. The temperature at the point (x,y,z) in 3-space is given by $T(x,y,z) = 400xyz$. Find the highest temperature on the unit sphere $x^2+y^2+z^2 = 1$.

13. Maximize $f(x,y,z) = xyz$ subject to the constraints $x+y+z = 40$ and $x+y = z$.

14. Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraints $x+2y+3z = 6$ and $x+3y+4z = 9$.

15. A company manufactures stainless steel right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

16. Find the minimum distance from the cone $z = \sqrt{x^2+y^2}$ to the point $(-6,4,0)$.

17. Find the dimension of the rectangular box of maximum volume that can be inscribed inside the sphere $x^2+y^2+z^2 = 4$.

18. Find the point on the plane $x+2y+3z = 13$ closest to the point $(1,1,1)$.

19. Use Taylor's formula for $f(x,y)$ at the origin to find quadratic and cubic approximations of f near the origin.
(i) $f(x,y) = e^x \cos y$ (ii) $f(x,y) = e^{x^2-y}$ (iii) $f(x,y) = \frac{3}{1-2x-y}$ (iv) $f(x,y) = xe^y$

20. Use Taylor's formula to find a quadratic approximation of $f(x,y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.