$$f(x) = f(a + (x-a))$$

$$= f(a) + (x-a) f'(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + R_n(x)$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(a + \theta(x-a))$$

Lagrange's form of Remainder $0 < \theta < 1$

$$= \frac{(x-a)^{n+1}}{n!} (1-\theta)^n f^{n+1}(a + \theta(x-a)) \quad 0 < \theta < 1$$

Cauchy form of Remainder

Find the Taylor's Series of $f(x) = \sin x$

about a point $x = \pi/4$

$$f^n(x) = \sin(x + n\pi/2)$$

$$f^n(\pi/4) = \sin(\pi/4 + n\pi/2)$$

$$f(x) = f(\pi/4 + (x - \pi/4))$$

$$= f(\pi/4) + (x - \pi/4) f'(\pi/4) + \frac{(x - \pi/4)^2}{2!} f^2(\pi/4) + \cdots \infty$$

$$\boxed{\text{if } \lim_{n \to \infty} \frac{(x - \pi/4)^{n+1}}{(n+1)!} f^{n+1}\left(\pi/4 + \theta(x - \pi/4)\right) \to 0}$$

$$\lim_{n \to \infty} \frac{(x - \pi/4)^{n+1}}{(n+1)!} f^{n+1}\left(\pi/4 + \theta(x - \pi/4)\right)$$

$$\lim_{n \to \infty} \frac{(x - \pi/4)^{n+1}}{(n+1)!} \sin\left(\pi/4 + \theta(x - \pi/4) + (n+1)\pi/2\right)$$

For fixed x

$$\lim_{n \to \infty} \frac{(x-a)^{n+1}}{(n+1)!} \to 0$$

$$\left| f^{n+1}\left(\pi/4 + \theta(x - \pi/4)\right)\right| < K$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$$

about a point $x = a$

$$\lim_{n \to \infty} \frac{(x-a)^{n+1}}{(n+1)!} \to 0$$

$$\left| f^{n+1}(a + \theta(x-a)) \right| < K$$

→ Sequence

log,

If $f(x) = \ln(1+x)$, $x > 0$, using Maclaurin's Theorem (Taylor's theorem about a point $x = 0$), show that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

$$f(x) = f(0 + (x-0))$$

$$= f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + R_2(x)$$

$$R_2(x) = \frac{x^3}{3!} f^3(\theta x)$$

Hence show that $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$.

$$1 + \theta x > 1$$

$$\frac{1}{(1+\theta x)} < 1$$

$$f(x) = \ln(1+x)$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(\theta x)$$

$$f(x) = \ln(1+x), \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f^2(x) = -\frac{1}{(1+x)^2} \qquad f^2(0) = -1$$

$$f^3(x) = 2\,\frac{1}{(1+x)^3}$$

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdot \frac{1}{(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdot \frac{1}{(1+\theta x)^3}$$

as $x > 0$, $0 < \theta < 1$

$1 + \theta x > 1$

$$\frac{1}{1+\theta x} < 1$$

$$\frac{1}{(1+\theta x)^3} < 1$$

$$\frac{x^3}{3(1+\theta x)^3} < \frac{x^3}{3}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f^2(a) + \cdots + \frac{h^n}{n!} f^n(a)$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta h)$$

$$0 < \theta < 1$$

Taylor's Theorem (Generalised Mean Value Theorem)

If $f(x)$ and it's first n derivatives be continuous in $[a, a+h]$ and (ii) $f^{n+1}(x)$ exist for every value of x in $(a, a+h)$, then there exists at one θ, $0 < \theta < 1$ such that

$$f(a+h) = f(a) + h f'(a) + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta h)$$

$\varphi'(a+\theta h)=0$

$K = f^{n+1}(a+\theta h)$

proof : Consider the function $\varphi(x)$

$$\varphi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f^2(x)$$

$\varphi(x) \to [a, a+h]$
$\varphi(x) \to (a, a+h)$
$\boxed{\varphi(a) = \varphi(a+h)}$

$$+ \cdots + \frac{(a+h-x)^n}{n!}f^n(x) + \frac{(a+h-x)^{n+1}}{(n+1)!}K$$

K is a constant such that

$$\underbrace{f(a+h)}_{\varphi(a+h)} = \underbrace{f(a) + hf'(a) + \frac{h^2}{2!}f^2(a) \cdots + \frac{h^n}{n!}f^n(a)}_{\varphi(a)} + \frac{h^{n+1}}{(n+1)!}K$$

$$\varphi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x)$$

$$\cdots + \frac{(a+h-x)^n}{n!} f^n(x) + \frac{(a+h-x)^{n+1}}{(n+1)!} K$$

$$\varphi'(x) = \cancel{f'(x)} - \cancel{f'(x)} + (a+h-x) f''(x) + \frac{2(a+h-x)}{2}(-1) f''(x)$$

$$+ \frac{(a+h-x)^2}{2!} f'''(x) + \cdots + \frac{(a+h-x)^n}{n!} f^{n+1}(x) - \frac{(a+h-x)^n}{n!} K$$

$$= \frac{(a+h-x)^n}{n!}\left(f^{n+1}(x) - K\right). \quad \boxed{K = f^{n+1}(a+\theta h)}$$

① Find first 4 non-zero terms of Taylor series for the following functions

(1) $f(x) = \dfrac{1}{1+x^2}$, $x = 1$

(2) $f(x) = \cos 2x$ $\quad x = \pi/6$

(3) $f(x) = (1+x+x^2)^{-2}$ $\quad x = 1$,

$f(x) = \sin^2 x$

$f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$

⋮