Department of Mathematics and Computing

Mathematics I

Tutorial Sheet-1 Solutions

Sol.1/(i). $f(x) = \sqrt{3+x^2}$ **at** $x = 1$

$$f(x) = \sqrt{3+x^2}$$

First, calculate the derivatives:

$$f'(x) = \frac{x}{\sqrt{3+x^2}}$$

$$f''(x) = \frac{3-x^2}{(3+x^2)^{3/2}}$$

$$f'''(x) = \frac{3x(x^2-6)}{(3+x^2)^{5/2}}$$

Evaluating at $x = 1$:

$$f(1) = \sqrt{4} = 2$$

$$f'(1) = \frac{1}{2}$$

$$f''(1) = \frac{2}{8} = \frac{1}{4}$$

$$f'''(1) = \frac{3 \cdot 1 \cdot (1-6)}{32} = -\frac{15}{32}$$

The Taylor series expansion is:

$$\sqrt{3+x^2} \approx 2 + \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{5}{384}(x-1)^3$$

(ii). $f(x) = \frac{1}{1-x}$ **at** $x = 2$

$$f(x) = \frac{1}{1-x}$$

First, calculate the derivatives:

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

Evaluating at $x = 2$:

$$f(2) = -1$$

$$f'(2) = 1$$

$$f''(2) = 2$$

$$f'''(2) = 6$$

The Taylor series expansion is:

$$\frac{1}{1-x} \approx -1 + (x-2) + 2(x-2)^2 + \frac{6}{6}(x-2)^3$$

(iii). $f(x) = \frac{1}{1+x}$ at $x = 3$

First, calculate the derivatives:

$$f(x) = \frac{1}{1+x}$$

$$f'(x) = -\frac{1}{(x+1)^2} \Rightarrow f'(3) = -\frac{1}{16}$$

$$f''(x) = \frac{2}{(x+1)^3} \Rightarrow f''(3) = \frac{2}{64} = \frac{1}{32}$$

$$f'''(x) = -\frac{6}{(x+1)^4} \Rightarrow f'''(3) = -\frac{6}{256} = -\frac{3}{128}$$

The Taylor series expansion is:

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \cdots$$

$$\frac{1}{1+x} \approx \frac{1}{4} - \frac{1}{16}(x-3) + \frac{1}{64}(x-3)^2 - \frac{1}{128}(x-3)^3$$

(iv). $f(x) = \frac{1}{x}$ at $x = a, a > 0$

First, calculate the derivatives:

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2} \Rightarrow f'(a) = -\frac{1}{a^2}$$

$$f''(x) = \frac{2}{x^3} \Rightarrow f''(a) = \frac{2}{a^3}$$

$$f'''(x) = -\frac{6}{x^4} \Rightarrow f'''(a) = -\frac{6}{a^4}$$

The Taylor series expansion is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\frac{1}{x} \approx \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3$$

(v). $f(x) = \frac{1}{1+x^2}$ at $x = -2$

First, calculate the derivatives:

$$f(x) = \frac{1}{1+x^2}$$

$$f(-2) = \frac{1}{5}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f'(-2) = -\frac{4}{25}$$

$$f''(x) = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \Rightarrow f''(-2) = \frac{22}{125}$$

$$\begin{aligned} f'''(x) &= \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} \\ &= \frac{-16}{125} + \frac{-32}{125} + \frac{-384}{625} \\ &= \frac{-80 - 160 + 384}{125} = \frac{144}{625} \end{aligned}$$

The Taylor series expansion is:

$$f(x) = f(-2) + f'(-2)(x+2) + \frac{f''(-2)}{2!}(x+2)^2 + \frac{f'''(-2)}{3!}(x+2)^3 + \cdots$$

$$\frac{1}{1+x^2} \approx \frac{1}{5} - \frac{4}{25}(x+2) + \frac{11}{125}(x+2)^2 + \frac{24}{625}(x+2)^3$$

(vi). $f(x) = \sin x^2$ at $x = 1$

$$f(x) = \sin x^2$$

First, calculate the derivatives:

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2\cos x^2 - 4x^2 \sin x^2$$

$$f'''(x) = -8x \cos x^2 - 6x \sin x^2$$

Evaluating at $x = 1$:

$$f(1) = \sin 1$$

$$f'(1) = 2\cos 1$$

$$f''(1) = 2\cos 1 - 4\sin 1$$

$$f'''(1) = -8\cos 1 - 6\sin 1$$

The Taylor series expansion is:

$$\sin x^2 \approx \sin 1 + 2\cos 1(x-1) - (4\sin 1 - 2\cos 1)(x-1)^2 - \frac{1}{6}(-8\cos 1 - 6\sin 1)(x-1)^3$$

(vii). $f(x) = \tan x$ **at** $x = 1$

$$f(x) = \tan x$$

First, calculate the derivatives:

$$f'(x) = \sec^2 x$$

$$f''(x) = 2\sec^2 x \tan x$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

Evaluating at $x = 1$:

$$f(1) = \tan 1$$

$$f'(1) = \sec^2 1$$

$$f''(1) = 2\sec^2 1 \tan 1$$

$$f'''(1) = 4\sec^2 1 \tan^2 1 + 2\sec^4 1$$

The Taylor series expansion is:

$$\tan x \approx \tan 1 + \sec^2 1(x-1) + \frac{2\sec^2 1 \tan 1}{2}(x-1)^2 + \frac{4\sec^2 1 \tan^2 1 + 2\sec^4 1}{6}(x-1)^3$$

(viii). $f(x) = e^{-2x}$ **at** $x = \frac{1}{2}$

$$f(x) = e^{-2x}$$

First, calculate the derivatives:

$$f'(x) = -2e^{-2x}$$

$$f''(x) = 4e^{-2x}$$

$$f'''(x) = -8e^{-2x}$$

Evaluating at $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}\right) = e^{-1}$$

$$f'\left(\frac{1}{2}\right) = -2e^{-1}$$

$$f''\left(\frac{1}{2}\right) = 4e^{-1}$$

$$f'''\left(\frac{1}{2}\right) = -8e^{-1}$$

The Taylor series expansion is:

$$e^{-2x} \approx e^{-1} - 2e^{-1}(x - \frac{1}{2}) + 2e^{-1}(x - \frac{1}{2})^2 - \frac{8e^{-1}}{6}(x - \frac{1}{2})^3$$

(ix). $f(x) = \cosh x$ at $x = 1$

$$f(x) = \cosh x$$

First, calculate the derivatives:

$$f'(x) = \sinh x$$

$$f''(x) = \cosh x$$

$$f'''(x) = \sinh x$$

Evaluating at $x = 1$:

$$f(1) = \cosh 1$$

$$f'(1) = \sinh 1$$

$$f''(1) = \cosh 1$$

$$f'''(1) = \sinh 1$$

The Taylor series expansion is:

$$\cosh x \approx \cosh 1 + \sinh 1(x-1) + \frac{\cosh 1}{2}(x-1)^2 + \frac{\sinh 1}{6}(x-1)^3$$

Sol. 2/(i). $f(x) = \frac{1}{1-2x}$

$$f(x) = \frac{1}{1-2x}$$

The Maclaurin series is:

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

(ii). $f(x) = \frac{1}{1+x^3}$

$$f(x) = \frac{1}{1+x^3}$$

The Maclaurin series is:

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots$$

(iii). $f(x) = \sin \pi x$

$$f(x) = \sin \pi x$$

The Maclaurin series is:

$$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots$$

(iv). $f(x) = \cos x^2$

$$f(x) = \cos x^2$$

The Maclaurin series is:

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

(v). $f(x) = e^{x^2}$

$$f(x) = e^{x^2}$$

The Maclaurin series is:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

(vi). $f(x) = \ln(1 + x)$

$$f(x) = \ln(1 + x)$$

The Maclaurin series is:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Sol. 3/(i)

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

Now take $a = 0$, $f(x) = e^x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

Take $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$$

with

$$R_n(1) = \frac{e^c}{(n + 1)!}, \text{ for some } c \in (0, 1).$$

We know, $e < 3 \Rightarrow \frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$ because $1 < e^c < 3$ for $0 < c < 1$.

We find that,

$$\frac{1}{9!} > 10^{-6}, \quad \text{where} \quad \frac{3}{10!} < 10^{-6}.$$

Therefore, $n \leq 9$, with error less than 10^{-6}.

$$e = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{9!} \approx 2.7182.$$

(ii)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

After $x^3/3!$ is no greater than,

$$\left|\frac{x^5}{5!}\right| < 3 \cdot 10^{-4}.$$

or,

$$|x| < \left(\frac{3 \cdot 10^{-4} \cdot 120}{1}\right)^{1/5} \approx 0.514.$$

By Remainder Estimation Theorem, then,

$$|R_4| \leq \frac{|x|^5}{5!} = \frac{|x|^5}{120}.$$

(iii)

Error of alternating series,

$$S_n = U_1 - U_2 + U_3 - \cdots + (-1)^{n-1} U_n$$

is

$$|S - S_n| < U_{n+1}.$$

Error $< \frac{|x|^5}{5!} \Rightarrow |x|^5 < 5! \cdot (6 \cdot 10^{-4}) \Rightarrow |x|^5 < 0.072 \Rightarrow |x| < 0.59084$.

(iv)

If $\sin x = x$ and $|x| < 10^{-3}$. Error $< \frac{|x|^3}{3!} = \frac{(10^{-3})^3}{3!} \approx 1.67 \cdot 10^{-10}$.

By Alternating Series Estimation Theorem, $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. $x < \sin x \Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0$.

(v)

The Taylor series expansion of $\sqrt{1+x}$ is:

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

Error of the alternating series:

$$|\text{error}| < \left|-\frac{x^2}{8}\right| < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}$$

(vi)

For the function $f(x) = e^x$, with $n = 2$ and $a = 0$:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

$$|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{e^2 (0.1)^3}{6} < 1.87 \times 10^{-4}$$

where c lies between 0 and x.

(vii)

The Taylor series for $\sinh x$ is:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

For $n = 4$:

$$|R_4(x)| = \left| \frac{(\sinh c)^5 x^5}{5!} \right| < \frac{(\sinh 0.5)^5 \cdot 0.5^5}{120} \approx 0.000294$$

(viii)

For e^h approximated by $1 + h$, where $0 \le h \le 0.01$:

$$|\text{error}| < \left| \frac{e^c h^2}{2} \right| \le \frac{e^{0.01} h^2}{2} \approx \frac{1.01005 h^2}{2} = 0.00505 h^2$$

So:

$$|\text{error}| < 0.006h$$

where c is between 0 and h.

(ix)

The Taylor series expansion of $\ln(1 + x)$ is:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

The remainder term for $n = 1$:

$$|R_1| = \left| \frac{1}{(1 + c)^2} \cdot \frac{x^2}{2!} \right| < \frac{x^2}{2}$$

Given:

$$\frac{(|x|)^2}{2} < 0.01|x|$$

which implies:

$$0 < |x| < 0.02$$

Sol. 4

(i) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{x - y}$

Take $x - y = h$

Then $h \to 0$ as $(x, y) \to (0, 0)$

$$\Rightarrow L = \lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{x - y}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} = 1$$

(ii) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

At $(x,y)=(1,-1),\ x+y=0$ and $x^3+y^3=0$

Use $y=-1+t$ where $t\to 0$

$$\Rightarrow \lim_{t\to 0} \frac{1+(-1+t)^3}{1+(-1+t)}$$

$$= \lim_{t\to 0} \frac{1-1+3(-1)t+3(-1)^2t^2+t^3}{t}$$

$$= \lim_{t\to 0} \frac{-3t+3t^2-t^3}{t}$$

$$= \lim_{t\to 0}(-3+3t-t^2) = -3$$

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| = \left|\frac{(x+y)(x^2+y^2-xy)}{x^2+y^2}\right|$$
$$= \left|(x+y)\left(1-\frac{xy}{x^2+y^2}\right)\right|$$

$$= |x+y|\left|1-\frac{xy}{x^2+y^2}\right|$$

$$< |x+y| \le |x|+|y|$$

Let $\varepsilon>0$ be given. Then, for sufficiently small $\delta>0,$ if $\sqrt{x^2+y^2}<\delta,$ then

$$\left|\frac{x^3+y^3}{x^2+y^2}-0\right| < \varepsilon$$

Choosing $\delta=\sqrt{\dfrac{\varepsilon}{2}},$ we have

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < \varepsilon$$

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$

Take $y=mx^2$

$$= \lim_{x \to 0} \frac{x^4}{x^4 + m^2 x^4}$$

$$= \frac{1}{1+m^2} \text{ (path dependent)}$$

$\Rightarrow$ limit DNE.

Sol. 5

(i)

$$f(x,y) = \frac{x^3 y}{x^6 + y^2} \quad \text{for } (x,y) \neq (0,0)$$

$$f(0,0) = 0$$

If $y = mx^3$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x \to 0} \frac{x^3 \cdot mx^3}{x^6 + (mx^3)^2}$$

$$= \lim_{x \to 0} \frac{mx^6}{x^6 + m^2 x^6}$$

$$= \frac{m}{1+m^2}$$

$\Rightarrow$ limit exists

$\Rightarrow$ not continuous.

(ii) $\quad f(x,y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$

$$|f(x,y) - 0| = |xy| \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \leq |x||y|$$

For every $\varepsilon > 0, \quad \exists \delta = \sqrt{\varepsilon} > 0$ such that

$|f(x,y) - 0| < \varepsilon$ for $|x| < \delta, |y| < \delta$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Also, $f(0,0) = 0$

$\Rightarrow$ function is continuous at $(0,0)$

(iii) $\quad f(x,y) = ||x| - |y|| - |x| - |y|$

$$|||x|-|y||-||x|-|y|| \le ||x|-|y||+||x|+|y||$$

$$= ||x|-|y|+|x|+|y||$$

$$\le |x|+|y|+|x|+|y|$$

$$= 2|x|+2|y|$$

For every $\varepsilon > 0, \quad \exists \delta = \frac{\varepsilon}{4} > 0$ such that

$|f(x,y)-0| < \varepsilon$ for $|x| < \delta, |y| < \delta$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Also, $f(0,0) = 0$

$\Rightarrow f$ is continuous at $(0,0)$

(iv)

$$f(x,y) = \frac{x^3 y}{x^4 + y^2}$$

$$|f(x,y)-0| = \left|\frac{x^3 y}{x^4+y^2}\right|$$

$$= \left|\frac{x^2 y}{x^4+y^2}\right|$$

$$= \frac{|x^2 y|}{x^4+y^2}$$

$$\le \frac{|x^2 y|}{y^2} = \frac{|x^2|}{y} \text{ (for small } x, y\text{)}$$

So, limit exists

Function $f(x,y) = 0$ at $(0,0)$

Sol. 6

(i) $f(x,y) = xy \cdot \dfrac{x^2-y^2}{x^2+y^2}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$
$$= 0$$

(ii) $f(x,y) = \dfrac{\sin^2(x+y)}{|x|+|y|}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin^2(h) - 0}{|h|}$$

$$= \lim_{h\to 0} \frac{\sin^2(h)}{|h|} = 0$$

(if $h \to 0^+$ then $\displaystyle\lim_{h\to 0^+} \frac{\sin^2(h)}{|h|} = \lim_{h\to 0} \frac{\sin^2(h)}{h} = 0$)

if $h \to 0^-$ then $\displaystyle\lim_{h\to 0} \frac{\sin^2(h)}{|h|} = \lim_{h\to 0} \frac{\sin^2(h)}{-h} = 0$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

(iii) $f(x,y) = \dfrac{xy}{x^2+y^2}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = 0$$

(iv) $f(x,y) = |x| + 7y$

$$f_x(0,0) = \lim_{h\to 0} \frac{|h| - 0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \text{ for } h > 0 \text{ and } -1 \text{ for } h < 0$$

$$f_y(0,0) = 7$$

(v) $f(x,y) = \dfrac{\sin(x^3+y^4)}{x^2+y^2}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^3)}{h^2} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^3} \left(h^3 - \frac{(h^3)^3}{3!} - \cdots \right) = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\sin(k^4)}{k^2} \cdot \frac{1}{k}$$

$$= \lim_{k\to 0} \frac{1}{k^3} \left(k^4 - \frac{(k^4)^3}{3!} - \cdots \right) = 0$$

Sol. 7:

(i) $w_x = w_u \cdot \left(-\frac{y}{x^2}\right) + w_v \cdot 2x, \quad w_y = w_u \cdot \frac{1}{x} + w_v \cdot 2y.$

(ii) $xw_x + yw_y = \left(-\frac{y}{x}\right) w_u + 2x^2 w_v + w_u \left(\frac{y}{x}\right) + 2y^2 w_v = 2v w_v.$

(iii)$xw_x + yw_y = 5v^5.$

Sol. 8:

To find $\frac{dx}{dy}$ at the point $(x, y, z) = (1, 1, 2)$ for the given equations $x^5 + yz = 3$ and $xy^2 + yz^2 + zx^2 = 7$ using the method of total differentials, follow these steps:

Total Differentials:

For the first equation $F(x, y, z) = x^5 + yz - 3 = 0.$
$dF = 5x^4\,dx + z\,dy + y\,dz = 0$

For the second equation $G(x, y, z) = xy^2 + yz^2 + zx^2 - 7 = 0.$

$dG = (y^2 + 2xz)\,dx + (2xy + z^2)\,dy + (2yz + x^2)\,dz = 0.$

Evaluating at $(x, y, z) = (1, 1, 2)$:

For $F(x, y, z)$

$dF = 5(1)^4\,dx + 2\,dy + 1\,dz = 0 \implies 5\,dx + 2\,dy + dz = 0$ For $G(x, y, z)$:

$dG = 5\,dx + 6\,dy + 5\,dz = 0.$

System of Equations: $\begin{cases} 5\,dx + 2\,dy + dz = 0 \\ 5\,dx + 6\,dy + 5\,dz = 0 \end{cases}$

Solving for $\frac{dx}{dy}$:

From the first equation, solve for dz:

$dz = -5\,dx - 2\,dy$

Substitute dz into the second equation:

$5\,dx + 6\,dy + 5(-5\,dx - 2\,dy) = 0$

$5\,dx + 6\,dy - 25\,dx - 10\,dy = 0$

$-20\,dx - 4\,dy = 0$

$-20\,dx = 4\,dy$

$dx = -\frac{1}{5}\,dy$

Therefore, $\frac{dx}{dy} = -\frac{1}{5}$.

Sol. 9:

(i) $f(x,y) = \left(x^2 - y^2\right) e^{-\left(x^2+y^2\right)/2}$

$f_x = 2xe^{-\left(x^2+y^2\right)/2} + x^2 e^{-\left(x^2+y^2\right)/2} \cdot (-2x/2) - y^2 e^{-\left(x^2+y^2\right)/2} \cdot (-2x/2)$

$= \left(2x - x^3 + xy^2\right) e^{-\left(x^2+y^2\right)/2}$

$f_x = 0 \Rightarrow 2x - x^3 + xy^2 = 0$

$x\left(2 - x^2 + y^2\right) = 0$

So, we get, $x = 0, 2 - x^2 + y^2 = 0, x^2 - y^2 = 2$

$f_y = (x^2 - y^2)e^{-\left(x^2+y^2\right)/2}(-y) - 2y \cdot e^{-\left(x^2+y^2\right)/2}$

$f_y = 0, -yx^2 - 2y + y^3 = 0 \Rightarrow x^2 - y^2 = 2$ and $y = 0$

Now, $f_{xx} = 2e^{-\left(x^2+y^2\right)/2} + 2xe^{-\left(x^2+y^2\right)/2} \cdot (-x) - 3x^2 e^{-\left(x^2+y^2\right)/2} - x^3 e^{-\left(x^2+y^2\right)/2} \cdot (-x) + y^2 e^{-\left(x^2+y^2\right)/2} + xy^2 e^{-\left(x^2+y^2\right)/2}(-x)$

$f_{xx}(0,0) = 2 > 0$

$f_{yy} = -yx^2 e^{-\left(x^2+y^2\right)/2}(-y) - x^2 e^{-\left(x^2+y^2\right)/2} - 2e^{-\left(x^2+y^2\right)/2} - 2ye^{-\left(x^2+y^2\right)/2}(-y) + 3y^2 e^{-\left(x^2+y^2\right)/2} + y^3 e^{-\left(x^2+y^2\right)/2}(-y)$

$f_{yy}(0,0) = -2$

$f_{xy} = 2xe^{-\left(x^2+y^2\right)/2}(-y) - x^3 e^{-\left(x^2+y^2\right)/2}(-y) + 2xye^{-\left(x^2+y^2\right)/2} + xy^2 e^{-\left(x^2+y^2\right)/2}(-y)$

$f_{xy}(0,0) = 0$

$f_{xx}f_{yy} - f_{xy}^2 = 2(-2) - 0 = -4 < 0, \quad D < 0$

At $(0,0)$, saddle point.

(ii) $f(x,y) = x^3 - 3xy^2$

$f_x = 3x^2 - 3y^2, f_{xx} = 6x, f_y = -6xy, f_{yy} = -6x, f_{xy} = -6y$

Putting values of x and y, we get, $f_x = 0, f_y = 0, 3x^2 - 3y^2 = 0, x^2 = y^2, f_y = 0, -6xy = 0, xy = 0 \Rightarrow x = 0, y = 0$

$f_{xx}(0,0) = 0, f_{yy}(0,0) = 0, f_{xy}(0,0) = 0$

At $(0,0), D = 0$. So, no information. Further investigation is needed.

(iii) $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$

$f_x = 2y - 2x + 3, f_{xx} = -2 < 0$

$f_y = 2x - 4y, f_{yy} = -4, \quad f_{xy} = 2$

$f_x = 0 \Rightarrow 2y - 2x + 3 = 0$

$f_y = 0 \Rightarrow 2x - 4y = 0$ So, $x = 3, y = 3/2$

Also, $D = f_{xx}.f_{yy} - (f_{xy})^2 = (-2).(-4) - 2^2 = 4 > 0$

$f_{xx} < 0, D > 0$. So, local maxima.

(iv) $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

$f_x = 4x + 3y - 5, f_{xx} = 4 > 0$

$f_y = 3x + 8y + 2, f_{yy} = 8$

$f_{xy} = 3$ So, $D = 32 - 3^2 > 0$

$f_{xx} > 0, D > 0 \rightarrow$ minima

The point is $x = 2, y = -1$. The function is having local minima.

(v) $f(x, y) = x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y, f_{xx} = 6x$

$f_y = 3y^2 - 3x, f_{yy} = 6y, f_{xy} = -3$

$3x^2 - 3y = 0, y = x^2$

$3y^2 - 3x = 0 \Rightarrow 3 \cdot x^4 - 3x = 0$

So, $x = 0, y = 0$ or $x = 1, y = 1$

$f_{xx} = 6x$ at $(1, 1), f_{xx} = 6 > 0$ $D = 36 - (-3)^2 = 27 > 0$

local minima at $(1, 1)$

$D = 0.0 - (-3)^2 = -9 < 0$ i.e., D < 0

At $(0, 0)$, the function has a saddle point.

Sol. 10:

$f(x, y) = \left(x^2 - 4x\right) \cos y$

$f_x = (2x - 4) \cos y = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$

$f_y = \left(x^2 - 4x\right) \sin y = 0 \Rightarrow \sin y = 0 \Rightarrow y = 0$

$f(2, 0) = (4 - 8) \cos 0 = -4$

$(2, 0)$-minimum. Also, computing the valuess, we get $(1, -\pi/4), (1, \pi/4), (3, \pi/4), (3, -\pi/4)$ maximum.

Sol. 11:

Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$ $\phi(x, y, z) = 400xyz^2$
$F(x, y, z) = \left(x^2 + y^2 + z^2 - 1\right) + \lambda \left(400xyz^2\right)$

$\frac{\delta F}{\delta x} = 0, \frac{\delta F}{\delta y} = 0, \frac{\delta F}{\delta z} = 0$ gives $x = 1/2, y = 1/2, z^2 = 1/2$

Highest temp $= 400(1/2)(1/2)(1/2) = 400/8 = 50$

Sol. 12:

To maximize the function $f(x, y, z) = xyz$ subject to the constraints $x + y + z = 40$ and $x + y = z$ using Lagrange multipliers, we follow these steps:

First, we set up the Lagrangian function:

$\mathcal{L}(x, y, z, \lambda_1, \lambda_2) = xyz + \lambda_1(x + y + z - 40) + \lambda_2(x + y - z)$

Now we take the partial derivatives of $\mathcal{L}$ with respect to x, y, z, λ_1, and λ_2 and set them equal to zero.

Partial derivative with respect to x:

$\frac{\partial \mathcal{L}}{\partial x} = yz + \lambda_1(1) + \lambda_2(1) = 0$

$yz + \lambda_1 + \lambda_2 = 0 \quad (1)$

Partial derivative with respect to y:

$\frac{\partial \mathcal{L}}{\partial y} = xz + \lambda_1(1) + \lambda_2(1) = 0$

$xz + \lambda_1 + \lambda_2 = 0 \quad (2)$

Partial derivative with respect to z:

$\frac{\partial \mathcal{L}}{\partial z} = xy + \lambda_1(1) - \lambda_2(1) = 0$

$xy + \lambda_1 - \lambda_2 = 0 \quad (3)$

Partial derivative with respect to λ_1:

$\frac{\partial \mathcal{L}}{\partial \lambda_1} = x + y + z - 40 = 0$

$x + y + z = 40 \quad (4)$

Partial derivative with respect to λ_2:

$\frac{\partial \mathcal{L}}{\partial \lambda_2} = x + y - z = 0$

$x + y = z \quad (5)$

From equation (5), we have:

$z = x + y$

Substitute $z = x + y$ into equation (4):

$x + y + (x + y) = 40$

$2x + 2y = 40$

$x + y = 20$

Now we have: $z = 20$

Next, substituting $z = 20$ into equations (1), (2), and (3):

From equation (1):

$20y + \lambda_1 + \lambda_2 = 0 \quad (1)$

From equation (2):

$20x + \lambda_1 + \lambda_2 = 0 \quad (2)$
From equation (3):

$xy + \lambda_1 - \lambda_2 = 0 \quad (3)$

Since $20y = 20x$, this implies $x = y$.

Using $x + y = 20$:

$x + x = 20$

$2x = 20$

$x = 10$

Therefore, $y = 10$ and $z = 20$.

Finally, substitute these values back into the function:

$f(x, y, z)=10 \cdot 10 \cdot 20=2000$

Thus, the maximum value of $f(x, y, z)$ is 2000 [0.2cm]when $x=10$, $y=10$, and $z=20$.

Sol. 13:

$f(x, y, z)=x^2+y^2+z^2$ given by $x+2y+3z=6$ & $x+3y+4z=9$

$F=x^2+y^2+z^2+\lambda_1(x+2y+3z-6)+\lambda_2(x+3y+4z-9)$

$\frac{\delta F}{\delta x}=2x+\lambda_1+\lambda_2=0$

$\frac{\delta F}{\delta y}=2y+2\lambda_1+3\lambda_2=0$

$\frac{\delta F}{\delta z}=2z+3\lambda_1+4\lambda_2=0$

$x=-\left(\lambda_1+\lambda_2\right)/2, y=-\left(2\lambda_1+3\lambda_2\right)/2, z=-\left(3\lambda_1+4\lambda_2\right)/2$

Solving we get, $\lambda_1=10, \lambda_2=-8$. Putting these values in equations, we get $x=-1, y=2, z=1$ and the maximum value is 6.

Sol. 14:

To determine the sensitivity of the volume V of the tanks to small variations in height h and radius r, we can use the concept of partial derivatives. The volume of a right circular cylindrical tank is given by:

$$V=\pi r^2 h$$

The sensitivity of the volume with respect to small changes in height and radius can be analyzed by finding the partial derivatives of V with respect to h and r. **Partial derivative with respect to height** h:

$$\frac{\partial V}{\partial h}=\pi r^2$$

This partial derivative tells us how the volume changes with a small change in height when the radius is fixed. **Partial derivative with respect to radius** r:

$$\frac{\partial V}{\partial r}=2\pi r h$$

This partial derivative tells us how the volume changes with a small change in radius when the height is fixed.

Now, we can evaluate these partial derivatives at the given dimensions of the tank:

Height $h=25$ ft ,Radius $r=5$ ft

Evaluate $\frac{\partial V}{\partial h}$ at $r=5$:

$$\frac{\partial V}{\partial h}=\pi(5)^2=25\pi \text{ cubic feet per foot}$$

Evaluate $\frac{\partial V}{\partial r}$ at $r=5$ and $h=25$:

$$\frac{\partial V}{\partial r}=2\pi(5)(25)=250\pi \text{ cubic feet per foot}$$

These results show that:

For every 1-foot increase in height, the volume of the tank increases by 25π cubic feet.

For every 1-foot increase in radius, the volume of the tank increases by 250π cubic feet.

The volume is more sensitive to changes in the radius than to changes in the height. This is because the partial derivative with respect to the radius (250π) is significantly larger than the partial derivative with respect to the height (25π). Therefore, small variations in the radius will have a more substantial impact on the volume of the tank compared to small variations in the height.

Sol. 15:

$z^2 = x^2 + y^2$, the point is $(-6, 4, 0)$

The distance formula is $d = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = \sqrt{(x + 6)^2 + (y - 4)^2 + z^2}$
$d^2 = (x + 6)^2 + (y - 4)^2 + z^2$

$\Rightarrow d^2 = 2x^2 + 2y^2 + 12x - 8y + 52$

$f_x(x, y) = 4x + 12 = 0 \Rightarrow x = -3$

$f_y(x, y) = 4y - 8 = 0 \Rightarrow y = 2$

Critical point $= (-3, 2)$

$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 0$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 16 > 0$

Also, $f_{xx} > 0$ and $D > 0$. Therefore, we have minima at the critical point $(-3, 2)$

$z^2 = 13, z = \sqrt{13}$ So, points on the cone are, $(-3, 2, \sqrt{13})$ and$(-3, 2, -\sqrt{13})$

So, minimum distance $= \sqrt{(-3 + 6)^2 + (2 - 4)^2 + (\sqrt{13})^2} = \sqrt{26} = 5.099.$

Sol. 16:

Let $2x, 2y, 2z$ be the length, breadth & height of the rectangular box.

Its volume $= 8xyz$

Now, the sphere is given as $x^2 + y^2 + z^2 = 4$

$F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - 4)$

$\frac{\partial F}{\partial x} = 8yz + 2x\lambda = 0$

$\frac{\partial F}{\partial y} = 8xz + 2y\lambda = 0$

$\frac{\partial F}{\partial z} = 8xy + 2z\lambda = 0$

Solving we get, $2x^2\lambda = 2y^2\lambda = 2z^2\lambda = -8xyz$

Thus for a maximum volume, $x = y = z$ i.e., the rectangular solid is a cube.

Sol. 17:

The normal vector to the plane is $(1, 2, 3)$. The point we assume would have to be multiple of this vector added to $(1, 1, 1)$

$P = (1, 1, 1) + \alpha(1, 2, 3) = (1 + \alpha, 1 + 2\alpha, 1 + 3\alpha)$

The point has to satisfy the palne's equation. the equation of the plane is $x + 2y + 3z = 13$

$1 + \alpha + 2(1 + 2\alpha) + 3(1 + 3\alpha) = 13$

$\Rightarrow 6 + 14\alpha = 13 \Rightarrow \alpha = 1/2$

The point on the plane is $P = (1 + 1/2, 1 + 1, 1 + 3/2) = (3/2, 2, 5/2)$

Sol. 18:

(i) $e^x \cos y = f(x,y)$

$f(0,0) = e^0 \cos 0 = 1, f_x(x,y) = e^x \cos y, f_x(0,0) = 1, f_y = -e^x \sin y, f_y(0,0) = 0,$

$f_{xx} = e^x \cos y, f_{xx} = 1, f_{yy} = -e^x \cos y, f_{yy}(0,0) = -1$

$f_{xyy} = -e^x \cos y = -1, f_{yyy} = e^x \sin y = 0$

$f_{xxx} = e^x \cos y = 1, f_{xxy} = -e^x \sin y, f_{xxy}(0,0) = 0$

$f_{xy} = -e^x \sin y, f_{xy}(0,0) = 0$

The expression becomes $f(0,0)+xf_x+yf_y+\frac{1}{2}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right)+\frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + \ 3xy^2 f_{xyy} + y^3 f_{yyy}\right)$

$= 1 + x \cdot 1 + y.0 + \frac{1}{2}\left(x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)\right) + \frac{1}{6}\left(x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right)$

Quadratice form $= 1 + x + \frac{1}{2}\left(x^2 - y^2\right)$, cubic form $= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left(x^3 - 3xy^2\right)$

(ii) $f(x,y) = e^{x^2-y}$

$f(0,0) = 1, f_x = e^{x^2-y}(2x) = 0, f_y = e^{x^2-y}(-1), f_y(0,0) = -1$

$f_{xx} = (2x)e^{x^2-y}(2x) + 2e^{x^2-y}, f_{xx}(0,0) = 2$

$f_{xy}(0,0) = 0, f_{yy}(0,0) = 1$

$f_{xxx} = e^{x^2-y} \cdot 4x^2(2x) + 8xe^{x^2-y} + 2e^{x^2-y}(2x) = 0$

$f_{yyy} = e^{x^2-y}(-1) = -1$

$f_{xxy} = -2, f_{xyy}(0,0) = 0$

$f(x,y) = f(0,0)+xf_x+yf_y+\frac{1}{2}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right)+\frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)+\ldots$

$= 1 + x \cdot 0 + y \cdot (-1) + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 1\right) + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot (-2) + 3xy^2 \cdot 0 + y^3 \cdot (-1)\right)$

$= 1 - y + \frac{1}{2}\left(2x^2 + y^2\right) + \frac{1}{6}\left(-6x^2 y - y^3\right)$

Quadratic: $1 - y + \frac{1}{2}\left(2x^2 + y^2\right)$ or, $1 - y + x^2 - y^2/2$

Cubic: $1 - y + x^2 - y^2/2 - x^2 y - y^3/6$

(iii) $f(x,y) = 3/(1 - 2x - y)$

$f(0,0) = 3/(1 - 0 - 0) = 3$

$f_x = \frac{-3}{(1-2x-y)^2} \cdot (-2)$

$f_x(0,0) = 6, \quad f_y = \frac{-3}{(1-2x-y)^2}(-1) = 3 \ f_{xx} = \frac{(-2)\cdot 6 \cdot (-2)}{(1-2x-y)^3} = \frac{24}{(1-2x-y)^3}, \quad f_{xx} = 24$

$f_{xy} = \frac{6\cdot(-2)}{(1-2x-y)^3}(-1) = 12$

$f_{yy} = 6, \quad f_{xxx} = \frac{24\cdot(-3)\cdot(-2)}{(1-2x-y)^4} = 144$

$f_{yyy} = \frac{6\cdot(-3)\cdot(-1)}{(1-2x-y)^4} = \frac{18}{(1-2x-y)^4} = 18$

$f_{xxy} = 72, \quad f_{xyy} = 36$

The expression is:

$f(0,0) + xf_x + yf_y + \ldots$

$= 3 + x \cdot 6 + y \cdot 3 + \frac{1}{2}(x^2 \cdot 24 + 2xy \cdot 12 + y^2 \cdot 6) + \frac{1}{6}(x^3 \cdot 144 + 3x^2 y \cdot 72 + 3xy^2 \cdot 36 + y^3 \cdot 18)$

$= 3 + 6x + 3y + 12x^2 + 12xy + 3y^2 + 24x^3 + 36x^2 y + 18xy^2 + 3y^3$

Quadratics: $3 + 6x + 3y + 12x^2 + 12xy + 3y^2$

Cubic: $3(1+2x+y+4x^2+4xy+y^2+8x^3+12x^2y+6xy^2+y^3)$

(iv)

Given the function $f(x,y)=xe^y$, we want to find the quadratic and cubic approximations near the origin using Taylor's formula.

The Taylor series expansion of a function $f(x,y)$ around the point (x_0,y_0) is given by:

$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+\frac{1}{2!}(f_{xx}(x_0,y_0)(x-x_0)^2+2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)+f_{yy}(x_0,y_0)(y-y_0)^2)+\frac{1}{3!}(f_{xxx}(x_0,y_0)(x-x_0)^3+3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0)+3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2+f_{yyy}(x_0,y_0)(y-y_0)^3)+\cdots$

For $f(x,y)=xe^y$, evaluate the function and its partial derivatives at $(0,0)$:

$f(x,y)=xe^y$

$f_x=e^y, f_y=xe^y, f_{xx}=0, f_{xy}=e^y, f_{yy}=xe^y, f_{xxx}=0, f_{xxy}=0, f_{xyy}=e^y, f_{yyy}=xe^y$

Evaluate these at $(0,0)$:

$f(0,0)=0, f_x(0,0)=1, f_y(0,0)=0, f_{xx}(0,0)=0, f_{xy}(0,0)=1, f_{yy}(0,0)=0, f_{xxx}(0,0)=0, f_{xxy}(0,0)=0, f_{xyy}(0,0)=1, f_{yyy}(0,0)=0$

The quadratic approximation of $f(x,y)$ near $(0,0)$ is: $f(x,y)\approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)$

Substitute the values:

$f(x,y)\approx 0+1\cdot x+0\cdot y+\frac{1}{2}(0\cdot x^2+2\cdot 1\cdot xy+0\cdot y^2)$
$f(x,y)\approx x+xy$

The cubic approximation of $f(x,y)$ near $(0,0)$ is:
$f(x,y)\approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)+\frac{1}{6}(f_{xxx}(0,0)x^3+3f_{xxy}(0,0)x^2y+3f_{xyy}(0,0)xy^2+f_{yyy}(0,0)y^3)$

Substitute the values:

$f(x,y)\approx 0+1\cdot x+0\cdot y+\frac{1}{2}(0\cdot x^2+2\cdot 1\cdot xy+0\cdot y^2)+\frac{1}{6}(0\cdot x^3+3\cdot 0\cdot x^2y+3\cdot 1\cdot xy^2+0\cdot y^3)$

$f(x,y)\approx x+xy+\frac{1}{6}\cdot 3xy^2$

$f(x,y)\approx x+xy+\frac{1}{2}xy^2$

Sol. 19:

$f(x,y)=\cos x\cos y$

$f(x,y)=f(0,0)=\cos 0\cos 0=1$

$f_x=-\sin x\cos y,\quad f_x=0$

$f_y=-\sin y\cos x,\quad f_y=0$

$f_{xx}=-\cos x\cos y,\quad f_{xx}=-1$

$f_{yy}=-\cos y\cos x,\quad f_{yy}=-1$

$f_{xy}=\sin x\sin y,\quad f_{xy}=0$

The expression is $1+x.0+y\cdot 0+\frac{1}{2}\left(x^2\cdot(-1)+2xy\cdot 0+y^2\cdot(-1)=1-\frac{1}{2}\left(x^2+y^2\right)=1-x^2/2-y^2/2\right.$ Also, $E(x,y)=\frac{1}{6}\left(x^3f_{xx}\ldots\ldots\right)$

The third derivative never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x|\leq 0.1$ and $|y|\leq 0.1$

Hence, $E(x,y) < \frac{1}{6}\left((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3\right) \le 0.00134$

The error will never exceed 0.00134 if $|x| \le 0.1$ and $|y| \le 0.1$