PHI101: Classical Mechanics and Electrodynamics (2025-26 - semester I: Tutorial set 1)

- **Problem 1**

This problem highlights the Usefulness of Polar Coordinates in Classical Mechanics

Consider a particle of mass m that is subject to a central force, such as gravitational attraction by a massive body (e.g., a planet orbiting the Sun). The central force $\mathbf{F}$ depends only on the distance r between the particle and the origin (where the central mass is located), and it has the general form:

$$\mathbf{F} = -\frac{k}{r^2}\hat{r}$$

where k is a constant related to the strength of the force (for gravity, this is $k = GMm$, where G is the gravitational constant, and M and m are the masses of the central body and the particle, respectively).

Cartesian Coordinates Approach

In Cartesian coordinates, the position of the particle is described by (x, y). The gravitational force acts radially, and the particle's motion needs to be broken down into components along the x- and y-axes. The equations of motion require resolving the force into these components, which can quickly become quite complicated.

1. The position vector of the particle is $\mathbf{r} = (x, y)$.
2. The central force must be resolved into components:

$$F_x = -\frac{kx}{(x^2+y^2)^{3/2}}, \quad F_y = -\frac{ky}{(x^2+y^2)^{3/2}}.$$

3. The equations of motion are:

$$m\frac{d^2x}{dt^2} = F_x, \quad m\frac{d^2y}{dt^2} = F_y,$$

which are coupled, nonlinear second-order differential equations.

Solving these equations in Cartesian coordinates is difficult because the force has both an x- and y-component, and they must be solved simultaneously.

Polar Coordinates Approach

In *polar coordinates*, the particle's position is described by the radial distance r and the angle θ (angle with respect to some reference direction). In polar coordinates, the motion under a central force becomes much simpler:

1. The central force only depends on the radial distance r, so the radial component of the acceleration $\ddot{r}$ is the only relevant term to consider for the equation of motion.
2. The radial force is simply:
$$F_r = -\frac{k}{r^2}.$$
3. The equation of motion in polar coordinates can be derived using Newton's second law in the radial direction:
$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{k}{r^2}.$$
4. Additionally, the angular motion (the angular momentum $L = mr^2\dot{\theta}$) remains conserved, so you have another equation:
$$\dot{\theta} = \frac{L}{mr^2}.$$
5. This allows you to reduce the problem to solving for the radial coordinate $r(t)$, which is simpler than working with the coupled x- and y-components in Cartesian coordinates.

Key Points

- In polar coordinates, the central force simplifies the problem significantly because it only acts in the radial direction, so there is no need to resolve it into x- and y-components.
- The angular motion is independent of the radial force and leads to conserved angular momentum, further simplifying the problem.
- The motion in polar coordinates reduces to a second-order differential equation for $r(t)$, whereas in Cartesian coordinates, you would need to solve a system of two coupled second-order differential equations.

Conclusion

The *motion of a particle under a central force* is a perfect example where polar coordinates make the solution much easier. The problem in Cartesian coordinates is complicated by the need to handle two coupled equations, whereas in polar coordinates, the radial and angular components separate, and the problem becomes more tractable. This is a typical example in classical mechanics where polar coordinates are more natural and efficient, especially when dealing with central forces like gravity or electromagnetism. However, if you follow the notes in Lecture 3, you will notice that the equations of motion are not simply like $F_x = m\ddot{x}$, $F_y = m\ddot{y}$, and, $F_z = m\ddot{z}$. In other words, the form of Newton's law changes in a very non-intuitive way as we work with different coordinates systems. But practical situation may sometime demand working in other coordinate systems for a simpler interpretation and easier approach to the actual solution. Hence, we resort to different coordinate systems, but we need to carefully work out the Newtonian formulation.

- **Problem 2: Line integral along different paths. It is like calculating the work done by a force vector along different paths.**

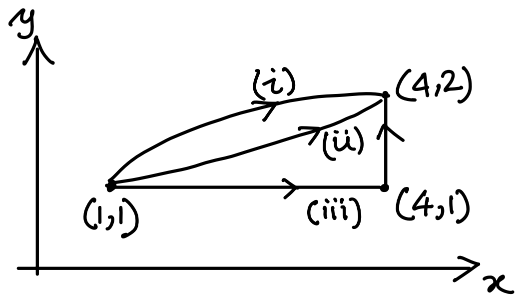
Evaluate the line integral $I = \int_C \vec{F}.d\vec{r}$ where $\vec{F} = (x+y)\hat{x} + (y-x)\hat{y}$, along each of the paths in the xy-plane mentioned below

(i) the parabola $y^2 = x$ from $(1,1)$ to $(4,2)$,

(ii) the curve $x = 2u^2 + u + 1$, $y = 1 + u^2$ from $(1,1)$ to $(4,2)$,

(iii) the line $y = 1$ from $(1,1)$ to $(4,1)$, followed by the line $x = 4$ from $(4,1)$ to $(4,2)$.

$$I = \int_C \vec{A}.\vec{dr} = \int_C \left[(x+y)dx + (y-x)dy\right]$$



Case(i) along $y^2 = x \Rightarrow 2y\,dy = dx$

$$\therefore I = \int_{(1,1)}^{(4,2)} \left[(x+y)dx + (y-x)dy\right]$$

$$= \int \left((y^2+y)2y + (y-y^2)\right)dy = \frac{34}{3}.$$

Case(ii) $x = 2u^2 + u + 1$, $y = 1 + u^2$. $\Rightarrow$ u parameterizes the path

One can eliminate u to obtain $x(y)$ or $y(x)$ as the path and proceed like (i).

Alternatively, $dx = (4u+1)du \quad dy = 2u\,du$

$$\therefore I = \int_{(1,1)}^{(4,2)} \left[(x+y)dx + (y-x)dy\right] = \int_{u=0}^{u=1} \left[(3u^2+u+2)(4u+1) - (u^2+u)2u\right]du$$

$$= \frac{32}{3}.$$

Case-(iii) $I = \int_{(1,1)}^{(4,2)} \left[(x+y)dx + (y-x)dy\right]$

$$= \underbrace{\int_{(1,1)}^{(4,1)} \left[(x+y)dx + (y-x)dy\right]}_{dy=0,\ y=1} + \underbrace{\int_{(4,1)}^{(4,2)} \left[(x+y)dx + (y-x)dy\right]}_{x=4,\ dx=0}$$

$$= \int_1^4 (x+1)\,dx + \int_1^2 (y-4)\,dy = \frac{21}{2} - \frac{5}{2} = 8$$

If the above integral signified work done along some path what would you infer from this result?

Would you be able to prove that the work done would depend on the path without considering any specific path? What relation would you use to prove it?

TRY AS HOMEWORK PROBLEM

- **Problem 3: This problem will help you revise the calculation of velocity and acceleration, given a position vector.**

  A particle moves so that its position vector is given by

  $$\vec{r}(t) = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$$

  where ω is a constant.

  1. Show that the velocity, $\vec{v}$ of the particle is perpendicular to $\vec{r}(t)$.
  2. Show that the acceleration is directed towards the origin.
  3. Show that $\vec{r} \times \vec{v}$ is a constant.

- **Problem 4: Through this problem you will work with Conservative and Non-Conservative Force Fields**

1. Let us consider a vector $\vec{A}$ given by

$$\vec{A} = \left(2xy + z^3\right)\hat{\imath} + \left(x^2 + 2y\right)\hat{\jmath} + \left(3xz^2 - 2\right)\hat{k}$$

Show that $\vec{\nabla} \times \vec{A} = \vec{0}$.

2. Find a scalar function ϕ such that $\vec{A} = \vec{\nabla}\phi$.
3. Show that the force field, $\vec{F} = \left(y^2z^3 - 6xz^2\right)\hat{\imath} + \left(2xyz^3\right)\hat{\jmath} + \left(3xy^2z^2 - 6x^2z\right)\hat{k}$ is a conservative force field.
4. Find the work done by $\vec{F}$ in moving a particle from position $P_1(-2, 1, 3)$ to position $P_2(1, -2, 1)$.

HW

Solution :

1. $$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2+2y & 3xz^2-2 \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left(3xz^2-2\right) - \frac{\partial}{\partial z}\left(x^2+2y\right)\right] - \hat{y}\left[\frac{\partial}{\partial x}\left(3xz^2-2\right) - \frac{\partial}{\partial z}\left(2xy+z^3\right)\right]$$

$$+ \hat{z}\left[\frac{\partial}{\partial x}\left(x^2+2y\right) - \frac{\partial}{\partial y}\left(2xy+z^3\right)\right]$$

$$= \hat{x}\,0 - \hat{y}\left(3z^2 - 3z^2\right) + \hat{z}\left(2x - 2x\right)$$

$$= \vec{0}$$

(2) Finding scalar function ϕ, such that $\vec{A} = \vec{\nabla}\phi$

Now,

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Comparing with ;

$$\vec{A} = (2xy + z^3)\hat{i} + (x^2+2y)\hat{j} + (3xz^2-2)\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2y$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - 2$$

Integrating :

$$\phi = x^2y + xz^3 + F_1(y,z) \longrightarrow \text{EQ } 1$$

$$\phi = x^2y + y^2 + F_2(x,z) \longrightarrow \text{EQ } 2$$

$$\phi = xz^3 - 2z + F_3(x,y) \longrightarrow \text{EQ } 3$$

You may now already guess the form of $F_1(y,z)$ by connecting the missing pieces. Below is a more systematic way of obtaining it.

Now, consider EQ 1,

$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial}{\partial y} F_1(y, z)$$

$$\Rightarrow \quad x^2 + 2y = x^2 + \frac{\partial F_1}{\partial y}$$

$$\Rightarrow \quad F_1(y,z) = y^2 + F_4(z)$$

$$\Rightarrow \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_4}{\partial z} = -2$$

$$\Rightarrow \quad F_4 = -2z + \text{Const.}$$

Similarly, from EQ 1

$$\frac{\partial \phi}{\partial z} = 3xz^2 + \frac{\partial F_1}{\partial z}$$

$$\Rightarrow 3xz^2 - 2 = 3xz^2 + \frac{\partial F_1}{\partial z}$$

$$\Rightarrow \frac{\partial F_1}{\partial z} = -2$$

$$\therefore \quad \boxed{F_1(y, z) = y^2 - 2z} \quad (\text{Ignore constant})$$

Comparing EQ 1 & EQ 2

$$\phi = x^2y + xz^3 + y^2 - 2z$$

From EQ 1

Also, $\phi = x^2y + y^2 + F_2(x, z)$

Thus, $\boxed{F_2(x, z) = xz^3 - 2z}$

Comparing : EQ 1 & EQ 3

$$x^2y + \cancel{x}z^3 + F_1(y,z) = \cancel{x}z^3 - 2z + F_3(x,y)$$

$$\Rightarrow F_1(y,z) - F_3(x,y) = -2z - x^2y$$

$\rightarrow$ EQ 4

$$\Rightarrow y^2 - 2z - F_3(x,y) = -2z - x^2y$$

$$\Rightarrow \boxed{F_3(x,y) = y^2 + x^2y}$$

Thus,

$$\phi = x^2y + xz^3 + y^2 - 2z + \text{Const.}$$