Calculation →

$$f(x) = f(a + (x - a))$$

$$= f(a) + (x-a) f^{1}(a) + \frac{(x-a)^2}{2!} f^{2}(a) + \cdots$$

$$+ \frac{(x-a)^n}{n!} f^{n}(a) + R_n(x)$$

$f(x) \rightarrow$ express $f(x)$ in terms of polynomial

condition → $f(x)$ must has continuous derivatives up to $(n+1)^{th}$ order

$f(x)$ is polynomial about a point $x = a$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \underline{R_n(x)}$$

$\underline{x = a}$ $\quad a_0 = f(a)$

$f'(x) = a_1 + 2(x-a).a_2 + \dots$

$a_1 = f'(a)$

$\vdots \; a_n = \dfrac{f^n(a)}{n!}$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) \dots \frac{(x-a)^n}{n!}f^n(a) + R_n(x)$$

$$f(x+h) =$$

$$f(x) = f(a + (x-a))$$

$$= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f^2(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a)$$

$$+ R_n(x)$$

$R_n(x) \rightarrow$ Error term (Lagrange's form of Remainder)

$$= \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}\left(a + \theta(x-a)\right) \quad 0 < \theta < 1$$

$$R_4 = \frac{h^5}{5!} f^5(a+\theta h)$$

$$f(a+h)$$

$$= f(a) + h f'(a) + \frac{h^2}{2!} f^2(a) + \cdots + \frac{h^n}{n!} f^n(a) + R_n(x)$$

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta h)$$ (Lagrange's form of Remainder)

Maximum Error

$$f(x) = f(a + (x-a))$$

$$= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f^2(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + R_n(x)$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(a + \theta(x-a))$$

$$|R_n(x)| = \left| \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(a + \theta(x-a)) \right| \le \left| \frac{(x-a)^{n+1}}{(n+1)!} \right| \left| f^{n+1}(a + \theta(x-a)) \right| \le \left| \max \frac{(x-a)^{n+1}}{(n+1)!} \right| \left| \max f^{n+1}(a + \theta(x-a)) \right|$$

Obtain the 4th degree Taylor's polynomial approximation of $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \le x \le 0.5$

$$f(x) = f(0 + (x - 0)) = f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(0)$$

$$+ \frac{x^4}{4!} f^4(0) + R_4(x)$$

$f(x) = e^{2x}$

$f^n(x) = 2^n e^{2x}$

$f^n(0) = 2^n$

$$f(x) = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + R_4(x)$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(a+\theta(x-a))$$

$0 < \theta < 1$

(a, x)

$$R_4(x) = \frac{x^5}{5!} f^5(\theta x)$$

$$= \frac{x^5}{5!} 2^5 \cdot e^{2\cdot\theta x}$$

$x \in (0, 0.5)$

$0 < \theta x < 0.5$

$e^{2\cdot\theta x} = e^{2\cdot 0.5} = e$

maximum valueof $R_4(x)$

$$|R_4(x)| = \left|\frac{x^5}{5!}\cdot 32\right| \left|e^{2\cdot\theta x}\right|$$

$$\leq \left|\max \frac{x^5}{5!}\cdot 32\right| \left|\max e^{2\cdot\theta x}\right| = \frac{e}{120}$$

$\left(\frac{1}{2}\right)^5 \cdot \frac{1}{120} \cdot 32$

Maximum error $= \frac{e}{120}$

For the Taylor's polynomial approximation of degree $\leq n$ about a point $x = 0$ for the function e^x. Determine the value of n such that error satisfies $|R_n(x)| \leq \underline{0.005}$ when $\underline{-1 \leq x \leq 1}$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(\theta x)$$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{\theta x}.$$

$$\text{Maximum error} = \left|\frac{\max x^{n+1}}{(n+1)!}\right| \left|\max e^{\theta x}\right|$$

$$\left| \max \frac{x^{n+1}}{(n+1)!} \right| \left| \max e^{a x} \right| \leq 0.005$$

$$\frac{1}{(n+1)!} \cdot e \leq \frac{1}{200}$$

$$(n+1)! \geq 200e$$

$n+1 = 6$

$n = 5$

up to 5^{th} degree polynomial

$$R_5(x) = \frac{x^6}{6!} f^6(a_x)$$

The function $f(x) = \sin x$ is approximated by Taylor's polynomial of degree 3 about a point $x = 0$

Find c, such that error satisfies $|R_3(x)| \leq 0.001$ for all x in the interval $[0, c]$

$$\sin x = x - \frac{x^3}{3!} + R_3(x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_5(x)$$

$$R_3(x) = \frac{x^4}{4!} f^4(\theta x)$$

$$f(x) = \sin x$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$R_4(x)$ θx

$$\max \frac{x^4}{4!} \left| \sin(\theta x + 4\pi/2) \right|_{\max} \leq 0.001$$

$$\frac{c^4}{4!} \leq 0.001$$

$$c^4 \leq 0.024$$

$$c = 0.3936 \qquad [0, 0.3936]$$