LECTURE - 3

(Also see Tutorial 1)

SECTION - C 07/08/2025

SECTION - D 14/08/2025

SECTION - B 11/08/2025

SECTION - A 11/08/2025

TOPICS :

0. Continuation from L2 about potential.

1. Beyond Second Law

   Why alternative formulations are required?

2. Example of problem with a different coordinate system.

Potential Energy

Scalar potential $V(\vec{r})$ only defined up to a constant.

$$\vec{F} = -\vec{\nabla} V$$

Let's define, a new potential

$$V_2(\vec{r}) = V(\vec{r}) + V_0$$

Then,

$$\vec{F} = -\vec{\nabla}\left(V_2(\vec{r}) - V_0\right)$$

$$= -\vec{\nabla} V_2(\vec{r})$$

$$\because \quad \vec{\nabla} V_0 = 0$$

→ Same force (and hence, same physics) independent of added constant 'V_0'.

** FOR INTERESTED STUDENTS

→ This is like "GAUGE SYMMETRY".

→ We will encounter this in 'ELECTRODYNAMICS' part. (Unknown upto a gauge field)

**

→ Absolute SCALAR POTENTIAL is ambiguous.

LET'S SEE HOW THIS WORKS IN PRACTICE !

$$W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{r} = -\int_{1}^{2} \vec{\nabla} V \cdot d\vec{r} = V_1 - V_2$$

potential

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$; $\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$

Check this out in the 1 dimensional version :

1d version: $$-\int_{x_1}^{x_2} \frac{d}{dx} V(x)\, dx = -\left[V(x_2) - V(x_1)\right]$$

$$W_{12} = T_2 - T_1 = V_1 - V_2 \Leftarrow \text{Conservative force}$$

$$\Rightarrow \boxed{V_1 + T_1 = V_2 + T_2}$$

Total energy at the start of journey at pt. 1 is equal to the total energy at the end of journey at pt. 2.

$$\boxed{E = T + V = \text{Constant}}$$

$\Rightarrow$ CONSERVATION OF ENERGY

Examples of Newtonian Formulation

① In Coordinate systems other than Cartesian

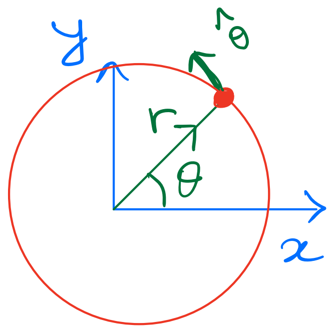
② In moving reference frames ** FOR INTERESTED STUDENTS

Until now, we have been working with Cartesian coordinates. Note that the usefulness of the plane polar coordinates for certain problems is discussed in Tutorial 1.

① Let us find NL $\underline{II}$ in plane polar coordinates (2-dimension).

$\Rightarrow$ Single particle in some potential $V(\vec{r})$

$$\Rightarrow \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$



$x = r\cos\theta$
$y = r\sin\theta$
$\vec{r} = r\hat{r}$

$$= \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Rightarrow \hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

i.e. $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$, $\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

Now,

NL $\underline{II}$ in Cartesian Coordinates:

$$m\ddot{x} = -\frac{\partial V}{\partial x} \quad ; \quad m\ddot{y} = -\frac{\partial V}{\partial y}$$

NL II in polar coordinates:

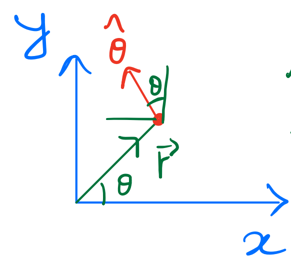
Is it $m\ddot{r} = -\frac{\partial V}{\partial r}$? NO!

Let us revisit the position vector $\vec{r} = r\hat{r}$

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\frac{d\theta}{dt}$$

$$= \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}\dot{\theta} = \dot{\theta}\,\hat{\theta}$$

Recall: $\hat{\theta} = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$



$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$

Now,

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = \begin{pmatrix}-\cos\theta \\ -\sin\theta\end{pmatrix}\dot{\theta} = -\dot{\theta}\,\hat{r}$$

$$\dot{\vec{r}} = \frac{d}{dt}\left(\vec{r}\right) = \frac{d}{dt}\left(r\hat{r}\right) = \frac{d(r)}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{\vec{r}}\right) = \frac{d(\dot{r})}{dt}\hat{r} + \dot{r}\frac{d(\hat{r})}{dt} + \frac{d(r)}{dt}\dot{\theta}\,\hat{\theta}$$

$$+ r\,d/dt\,(\dot{\theta})\,\hat{\theta} + r\dot{\theta}\,d/dt\,(\hat{\theta})$$

Substitute all the terms :

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}$$

Also, $\vec{\nabla} = \hat{r}\dfrac{\partial}{\partial r} + \hat{\theta}\,\dfrac{1}{r}\dfrac{\partial}{\partial \theta}$ $\Rightarrow$ Given

[Can be proved by chain rule].

NLII in plane polar coordinates :

$$\vec{F} = m\vec{a} \;\Rightarrow\; -\vec{\nabla} V = m\ddot{\vec{r}} \quad \not\Rightarrow \; m\ddot{r} = -\frac{\partial V}{\partial r}$$

along $\hat{r}$: $\quad -\dfrac{\partial V}{\partial r} = m\ddot{r} - m r\dot{\theta}^2 \longrightarrow$ Centrifugal force

along $\hat{\theta}$: $\quad -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = 2m\dot{r}\dot{\theta} + m r\ddot{\theta}$

TAKE HOME MESSAGE :

$m\ddot{q} = F_q$: no such Newton equation of motion for a generalized coordinate q.

NLII works in Cartesian coordinate.
(when broken into components)

The second law changes if we change the coordinate system.

Recall the statement of second law; it states "acceleration of an object is directly proportional to net force acting on it and inversely proportional to its mass."

Clearly, for the central force problem in plane polar coordinates this is not the case, unless we introduce the concept of some fictitious forces.

In other words, $F = m\dfrac{d^2x}{dt^2}$ in 1D

But now if we replace x by some coordinate θ where x is a function of θ, then, $\Rightarrow \dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$

$$F = m\,d^2x/dt^2 = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{d}{dt}\left(\frac{dx}{d\theta}\dot{\theta}\right)$$

$$= m\left(\frac{dx}{d\theta}\ddot{\theta} + \frac{d^2x}{d\theta^2}\dot{\theta}^2\right)$$

$$\frac{d}{dt}\left(\frac{dx}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{dx}{d\theta}\right)\dot{\theta}$$

$$= \frac{d^2x}{d\theta^2}\dot{\theta}$$

SUMMARY :

- Casting Newton's laws in non-inertial frames and for generalized coordinates becomes extremely complicated.

  In other words Newton's equation of motion is not invariant as we change coordinate systems or work in non-inertial frames.

- Complications will increase as we increase the number of particles in the system.

- But there are alternative formulations of Classical Mechanics where these issues can be overcome.

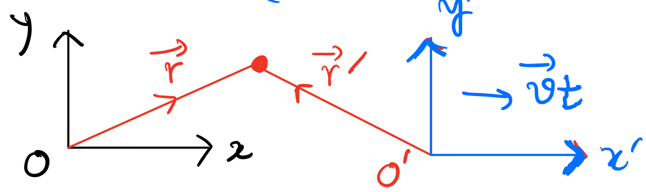
- Lagrangian and Hamiltonian formulations are required to build connection with Quantum Mechanics.

② Moving Reference frames [for Interested Students]

$\Rightarrow$ example of an accelerated frame of reference.

Original coordinates : $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ Cartesian

NLII : $-\vec{\nabla} V = m \ddot{\vec{r}}$



New coordinates : MOVING REFERENCE FRAME

$$\vec{r}' = \vec{r} - \vec{v}\, t$$

NOW, $\vec{v}$ = constant , then, $\ddot{\vec{r}}' = \ddot{\vec{r}}$

NLII : $-\vec{\nabla} V = m \ddot{\vec{r}} = m \ddot{\vec{r}}'$ [SAME EXPRESSION!]

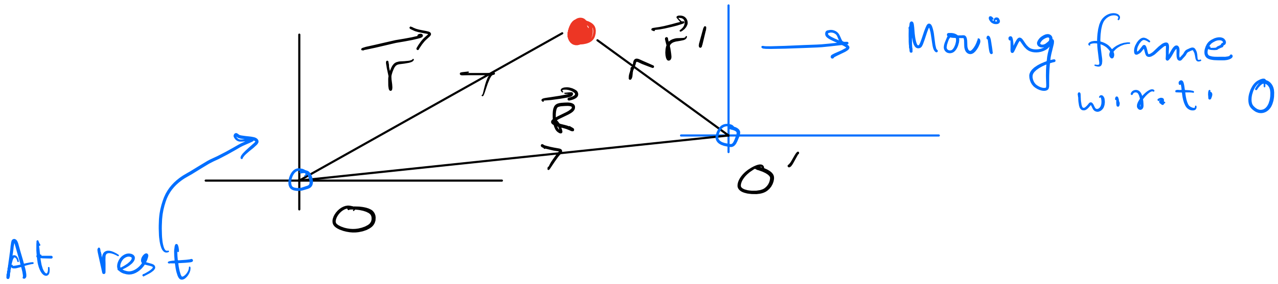
If $\vec{v}$ is NOT constant, [Non-Inertial]

NLII : ? ?

Applications

$\Rightarrow$ Weather events on Earth.

a. Ex. Linearly accelerating reference frames:



- At $t=0$, the origin of O & O' coincide. Then O' moves w.r.t. O. Let $\vec{R}(t)$ denote the position vector of the origin of O' as seen by observer in frame O.

- The velocity of O' is $\frac{d\vec{R}(t)}{dt} = \vec{v}_{o'}(t)$ and acceleration of O' is $\frac{d\vec{v}_{o'}(t)}{dt} = \vec{a}_{o'}(t)$ with respect to O frame.

- Now position vector $\vec{r}(t)$ of an object in O is related to the position vector $\vec{r}'(t)$ of the object in O':

$$\vec{r}(t) = \vec{r}\,'(t) + \vec{R}(t)$$

- Differentiating the above :

$$\dot{\vec{r}}(t) = \dot{\vec{r}}\,'(t) + \dot{\vec{R}}(t)$$

↳ Since O' is moving w.r.t. O !

Similarly,

$$\Rightarrow \quad \ddot{\vec{r}}(t) = \ddot{\vec{r}}\,'(t) + \ddot{\vec{R}}(t)$$

Recall! In an inertial frame, O.

$$m\ddot{\vec{r}} = \vec{F}_{inertial}$$

In the non-inertial frame NLII of object needs to be fixed :

$$m\ddot{\vec{r}}(t) = m\ddot{\vec{r}}\,'(t) + m\ddot{\vec{R}}(t)$$

$$\Rightarrow \quad m\ddot{\vec{r}}\,'(t) = m\ddot{\vec{r}}(t) - m\ddot{\vec{R}}(t)$$

$$\Rightarrow \quad m\vec{a}\,' = m\vec{a} - m\vec{A}$$

$$\Rightarrow \quad m\vec{a}\,' = \vec{F}_{inertial} - m\vec{A}$$

Define : $-m\vec{A} = F_{fictitious}$

$$\Rightarrow \quad \boxed{m\vec{a}\,' = \vec{F}_{inertial} + \vec{F}_{fictitious}}$$

MODIFIED NEWTON'S SECOND LAW IN NON-INERTIAL FRAME

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