Tutorial Sheet 2 (Solutions)

1. Evaluate, when possible, the following integrals:

(i)

$$\int_0^\infty \frac{x}{x^2+4}\,dx$$

Solution: First,we apply the limit as $a \to \infty$:

$$\lim_{a\to\infty}\int_0^a \frac{x}{x^2+4}\,dx = \lim_{a\to\infty}\int_4^{a^2+4}\frac{1}{2u}\,du$$

Where we have made the substitution:

Let $x^2+4=u$

2x dx = du

$x\,dx = \frac{du}{2}$

When

$$x=0 \to u=4, \quad x=a \to u=a^2+4$$

Now we evaluate the integral:

$$= \lim_{a\to\infty}\int_4^{a^2+4}\frac{1}{2}u^{-1}du$$

$$= \lim_{a\to\infty}\frac{1}{2}\left[\log(u)\right]_4^{a^2+4}$$

$$= \lim_{a\to\infty}\frac{1}{2}\left(log(a^2+4)-\log(4)\right)$$

$$= \frac{1}{2}\left(\log(\infty^2+4)-\log(4)\right)$$

As $a \to \infty$,$\log(a^2+4) \to \infty$, so the integral diverges.

$$\to \infty \quad \text{(Divergent)}$$

(ii)

$$\int_1^\infty \frac{dx}{x(1+x)}$$

Solution: First,we apply the limit as $a \to \infty$:

$$= \lim_{a\to\infty}\int_1^a \frac{dx}{x(1+x)}$$

$$= \lim_{a\to\infty}\int_1^a \frac{1+x-x}{x(1+x)}\,dx$$

Now we use partial fraction decomposition:

$$= \lim_{a\to\infty}\int_1^a \left(\frac{1}{x}-\frac{1}{1+x}\right)dx$$

$$= \lim_{a\to\infty} [\log(x) - \log(1+x)]_1^a$$

$$= \lim_{a\to\infty} [\log a - \log 1 - \log(1+a) + \log 2]$$

$$= \lim_{a\to\infty} \left[\log(\frac{a}{1+a}) - \log(\frac{1}{2})\right]$$

Now by L'Hospital's rule:

$$\lim_{a\to\infty} \left(\frac{a}{1+a}\right) = 1$$

$$= \log(1) - \log(\frac{1}{2}) = -\log(\frac{1}{2})$$

(iii)

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^4}\, dx$$

Solution: First we make the substitution: $x^2 = t$, and thus $2xdx = dt$

$$= \int_{-\infty}^{+\infty} \frac{dt}{2(1+t^2)} = \lim_{a\to\infty} \int_{-a}^{+a} \frac{dt}{2(1+t^2)} = 0$$

or

Since the integrand is an odd function, and the limits of integration are symmetric about zero, the integral of an odd function over a symmetric interval is zero. Therefore,

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^4}\, dx = 0$$

(iv)

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$$

Solution: To evaluate this integral, we use partial fraction decomposition:

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{\frac{1}{b^2-a^2}}{x^2+a^2} - \frac{\frac{1}{b^2-a^2}}{x^2+b^2}.$$

Now, integrate each term:

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2} \left(\int_0^{\infty} \frac{dx}{x^2+a^2} - \int_0^{\infty} \frac{dx}{x^2+b^2}\right)$$

Use the standard integral result:

$$\int_0^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$

$$\int_0^{\infty} \frac{dx}{x^2+b^2} = \frac{\pi}{2b}$$

Substitute these results:

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2b}\right).$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(b+a)}$$

.

(v)

$$\int_0^\infty \frac{x}{(x^2+a^2)(x^2+b^2)} dx \; a,b>0$$

Solution: Let $x^2 = u$, so that $2xdx = du$, giving

$$\frac{1}{2}\int_0^\infty \frac{du}{(u+a^2)(u+b^2)} = \frac{1}{2}\int_0^\infty \left(\frac{1}{u+a^2}-\frac{1}{u+b^2}\right)\frac{du}{b^2-a^2}$$

We now compute:

$$\frac{1}{2(b^2-a^2)}\lim_{c\to\infty}\left[\log(u+a^2)-\log(u+b^2)\right]_0^c$$

$$= \frac{1}{2(b^2-a^2)}\lim_{c\to\infty}\left[\log(a^2+c)-\log(b^2+c)-\log(a^2)+\log(b^2)\right]$$

As $c \to \infty$, this simplifies to:

$$= \frac{1}{2(b^2-a^2)}\left(0+\log\left(\frac{b^2}{a^2}\right)\right)$$

$$= \frac{1}{2(b^2-a^2)}\log\left(\frac{b^2}{a^2}\right)$$

$$= \frac{1}{b^2-a^2}\log\left(\frac{b}{a}\right) \; or \; \frac{1}{a^2-b^2}\log\left(\frac{a}{b}\right)$$

(vi)

$$\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}, \textit{ where n is an integer greater than 1.}$$

Solution: To evaluate this integral,we use the substitution $x = \sinh(t)$.

Under this substitution:

$$x = \sinh(t),$$

$$dx = \cosh(t)\,dt,$$

$$\sqrt{1+x^2} = \cosh(t),$$

$$x+\sqrt{1+x^2} = \sinh(t)+\cosh(t) = e^t.$$

The integral becomes:

$$I = \int_0^\infty \frac{1}{(e^t)^n}\cdot\cosh(t)\,dt$$

$$I = \int_0^\infty \frac{\cosh(t)}{e^{nt}}\,dt$$

Using the fact that $\cosh(t) = \frac{e^t+e^{-t}}{2}$, we can rewrite the integral as:

$$I = \int_0^\infty \frac{1}{2} \left(\frac{e^t + e^{-t}}{e^{nt}} \right) dt$$
$$I = \frac{1}{2} \int_0^\infty \left(e^{(1-n)t} + e^{-(n+1)t} \right) dt$$

Separate the integral into two parts:

$$I = \frac{1}{2} \left[\int_0^\infty e^{(1-n)t} \, dt + \int_0^\infty e^{-(n+1)t} \, dt \right]$$

Evaluate each integral: i. For the first term:

$$\int_0^\infty e^{(1-n)t} \, dt = \frac{1}{n-1}$$

ii. For the second term:

$$\int_0^\infty e^{-(n+1)t} \, dt = \frac{1}{n+1}$$

Combining these results:

$$I = \frac{1}{2} \left(\frac{1}{n-1} + \frac{1}{n+1} \right)$$

Thus, the final result for the integral is:

$$I = \frac{1}{2} \left(\frac{1}{n-1} + \frac{1}{n+1} \right)$$
$$I = \frac{n}{n^2-1}$$

2. Examine the convergence of following integrals:

(i)

$$\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$$

Solution: First,we apply the limit as $a \to \infty$:

$$\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} = \lim_{a\to\infty} \int_1^a \frac{dx}{x\sqrt{1+x^2}}$$

Clearly,

$$0 < \frac{1}{x\sqrt{1+x^2}} \forall x \geq 1$$

Also,

$$1 + x^2 > x^2$$

$$\Rightarrow \sqrt{1+x^2} > \sqrt{x^2} = |x| = x[\Rightarrow x\sqrt{1+x^2} > x^2$$

$$\Rightarrow \frac{1}{x\sqrt{1+x^2}} < \frac{1}{x^2}$$

So,

$$\int_1^a \frac{dx}{x\sqrt{1+x^2}} < \int_1^a \frac{dx}{x^2} = \left[\frac{-1}{x}\right]_1^a = \frac{-1}{a} + 1$$
$$\Rightarrow \lim_{a\to\infty} \int_1^a \frac{dx}{x\sqrt{1+x^2}} \leq 1$$

$$\textit{Hence}, 0 \leq \int_1^\infty \frac{dx}{x\sqrt{1+x^2}} \leq 1$$

i.e. the integral converges to a finite limit value.

(ii)

$$\int_1^\infty \frac{\log x}{x^2+1}\, dx$$

Solution: (a) Using a comparison test, we first note that for x ≥ 1, we have:

$$\log x \geq 0 \quad \text{and} \quad \frac{1}{x^2+1} \leq \frac{1}{x^2}$$

Thus, we consider:

$$0 \leq \int_1^\infty \frac{\log x}{x^2+1}\, dx \leq \int_1^\infty \frac{\log x}{x^2}\, dx$$

(b) Now, we calculate $\int_1^\infty \frac{\log x}{x^2} \quad dx$.

First, we apply the limit $\lim a \to \infty$

$$\int_1^\infty \frac{\log x}{x^2}\, dx = \lim_{a\to\infty} \int_1^a \frac{logx}{x^2} dx$$

Now we make the substitution: $z = \log x$, so that, $dz = \frac{dx}{x}$

$$Hence, x = e^z \Rightarrow \frac{1}{x} = e^{-z}$$

This is a standard integral, and it evaluates to:

$$\lim_{a\to\infty} \int_0^{\log a} ze^{-z} dz = 1$$

Thus, we have:

$$\int_1^\infty \frac{\log x}{x^2}\, dx = 1$$

(c) Therefore, by comparison, we have:

$$0 \leq \int_1^\infty \frac{\log x}{x^2+1}\, dx \leq 1$$

This implies that the integral converges to a finite value, and hence the integral converges.

(iii)

$$\int_a^\infty \frac{\sin^2 x}{x^2} dx$$

Solution: Case (1): $If a < 0$, Then,

$$\int_a^\infty \frac{\sin^2 x}{x^2}\,dx = \int_a^0 \frac{\sin^2 x}{x^2}\,dx + \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

First, we show the convergence of the integral:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

We begin by evaluating the integral:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1-\cos 2x}{x^2}\,dx$$
$$= \frac{1}{2}\int_0^\infty \frac{\cos(0x)-\cos(2x)}{x}\,dx$$
$$= \frac{1}{2}\int_0^\infty \frac{1}{x}\left(\int_0^2 \sin(tx)\,dt\right)\,dx$$
$$= \frac{1}{2}\int_0^2 \left(\int_0^\infty \frac{\sin(tx)}{x}\,dx\right)dt$$
$$= \frac{1}{2}\int_0^2 t\left(\int_0^\infty \frac{\sin(tx)}{tx}\,dx\right)dt$$
$$= \frac{1}{2}\int_0^2 \left(\int_0^\infty \frac{\sin(\theta)}{\theta}\,d\theta\right)dt$$

where we make the substitution: $tx = \theta, \Rightarrow$ tdx = dθ

Since $\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$, we obtain:

$$\frac{1}{2}\int_0^2 \frac{\pi}{2}\,dt = \frac{\pi}{4}\cdot 2 = \frac{\pi}{2}$$

Therefore,

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

Thus, $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ converges

Since the limit $\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1$, 0 is not a point of infinite discontinuity for the integrand. Therefore:

$$\int_a^0 \frac{\sin^2 x}{x^2}\,dx \text{ is convergent.}$$

So,

$$\int_a^\infty \frac{\sin^2 x}{x^2}\,dx \text{ is convergent.}$$

Clearly, for other values of $a \neq 0$, this integral also converges.

(iv)

$$\int_0^\infty \frac{x^{3/2}}{3x^2+5}\,dx$$

Solution: Given:

$$\int_0^\infty \frac{x^{3/2}}{3x^2+5}\,dx \quad \left(\frac{x^{3/2}}{3x^2+5} \geq 0 \quad \forall x > 0\right)$$

$$= \int_0^\infty \frac{x^2}{\sqrt{x}\cdot(3x^2+5)}\,dx$$

$$= \frac{1}{3}\int_0^\infty \left(\frac{3x^2+5-5}{3x^2+5}\right)\frac{dx}{\sqrt{x}}$$

$$= \frac{1}{3}\int_0^\infty \frac{dx}{\sqrt{x}} - \frac{5}{3}\int_0^\infty \frac{dx}{\sqrt{x}(3x^2+5)}$$

$$= I_1 + I_2, \quad \text{where} \quad I_1 = \frac{1}{3}\int_0^\infty \frac{dx}{\sqrt{x}} \quad \text{and} \quad I_2 = -\frac{5}{3}\int_0^\infty \frac{dx}{\sqrt{x}(3x^2+5)}$$

Now,

$$I_1 = \frac{1}{3}\int_0^\infty \frac{dx}{x^{1/2}} = \frac{1}{3}\lim_{\epsilon\to\infty}\int_0^\epsilon x^{-1/2}dx$$

$$= \frac{1}{3}\lim_{\epsilon\to\infty}\left[2x^{1/2}\right]_0^\epsilon = \frac{2}{3}\lim_{\epsilon\to\infty}\epsilon^{1/2} = \infty$$

Clearly, I_1 diverges.

Hence, this integral also diverges.

(v)

$$\int_1^\infty \frac{1}{x^{1/3}(x+1)^{1/2}}\,dx$$

Solution:

$$\int_1^\infty \frac{1}{x^{1/3}(x+1)^{1/2}}\,dx = \int_1^\infty \frac{1}{x^{5/6}}\sqrt{\frac{x}{x+1}}\,dx, Let f(x) = \frac{1}{x^{5/6}}\sqrt{\frac{x}{x+1}}; g(x) = \frac{1}{x^{5/6}}$$

$\implies \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\sqrt{\frac{x}{1+x}} = 1$ (non-zero and finite); By comparison test $\int_1^\infty \frac{1}{x^{5/6}}\,dx$ and $\int_1^\infty f(x)\,dx$ converges or diverges together.

$$\int_1^\infty \frac{1}{x^{5/6}}\,dx = \frac{x^{-5/6+1}}{-5/6+1}\Big|_1^\infty = \infty$$

so it diverges and so $\int_1 \infty f(x)\,dx$ also diverges.

(vi)

$$\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$$

Solution:

$$\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx = \int_1^\infty \frac{x}{(1+x)x^{3/2}}\,dx. Take f(x) = \frac{x}{(1+x)x^{3/2}} and g(x) = \frac{1}{x^{3/2}}$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1(\neq 0)$ and $\int_1^\infty \frac{1}{x^{3/2}}\,dx = 2$ which converges. So, $\int_1^\infty f(x)\,dx$ also converges.

(vii)

$$\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$$

Solution:

$$\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx = \int_2^\infty \frac{x}{x\sqrt{x^2-1}}\,dx. Let f(x) = \frac{x}{x\sqrt{x^2-1}} and g(x) = \frac{1}{x};\ \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}} = 1(\neq 0)$$

$\int_2^\infty \frac{1}{x}\,dx = \int_2^\infty g(x)\,dx = \log x\Big|_2^\infty = \infty$ which diverges. So by comparison test $\int_2^\infty f(x)\,dx$ also diverges.

(viii)

$$\int_1^\infty \frac{x^{m-1}}{x+1}\,dx$$

Solution:

$$\int_1^\infty \frac{x^{m-1}}{x+1}\,dx = I(say)$$

Case 1: $m=1; I = \int_1^\infty \frac{dx}{1+x} = \log(1+x)\Big|_1^\infty = \infty$; which diverges if m=1
Case 2: $m>1; I = \int_1^\infty \frac{x(x)^{m-2}}{x+1}\,dx$, let $f(x) = \frac{x(x)^{m-2}}{x+1}; g(x) = x^{m-2}$, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1(\neq 0)$; $\int_1^\infty g(x)\,dx = \int_1^\infty x^{m-2}\,dx = \infty(m>1)$ which diverges. so by comparison test $\int_1^\infty f(x)\,dx$ diverges for $m>1$
Case 3: $m<1; I = \int_1^\infty \frac{x^{m-1}}{x+1}\,dx$, let $f(x) = \frac{x(x)^{m-2}}{x+1}; g(X) = x^{m-2}$, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1(\neq 0)$, $\int_1^\infty g(x)\,dx = \int_1^\infty x^{m-2}\,dx = finite(m<1)$ so, $\int_1^\infty \frac{x^{m-1}}{x+1}\,dx$ converges when $m<1$.

(ix)

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$$

Solution: Take $a\neq 0$. For $a=0$ use μ test

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \int_0^\infty \frac{x^2}{(a^2+x^2)(a^2+x^2)}\,dx. Take f(x) = \frac{x^2}{(a^2+x^2)(a^2+x^2)} and g(x) = \frac{1}{a^2+x^2}$$

$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^2}{a^2+x^2} = 1(\neq 0)$ and $\int_0^\infty \frac{}{(a^2+x^2)}\,dx = \frac{1}{a}\tan^{-1}(x/a)\Big|_0^\infty = \frac{\pi}{2a}$ which is convergent. So by comparison text $\int_0^\infty f(x)\,dx$ also converges.

3. Evaluate, when possible, the following integrals :

(i)

$$\int_0^\pi \frac{dx}{1+\cos x}$$

$$\left[\because \cos\pi = -1, \text{ so the function } \frac{1}{1+\cos x} \text{ is undefined at } x=\pi\right]$$

$$= \lim_{b\to\pi^-}\int_0^b \frac{dx}{1+\cos x} = \lim_{b\to\pi^-}\int_0^b \frac{(1-\cos x)}{1-\cos^2 x}\,dx = \lim_{b\to\pi^-}\int_0^b \frac{(1-\cos x)}{\sin^2 x}\,dx$$

$$= \lim_{b\to\pi^-}\int_0^b \left(\csc^2 x - \csc x\cdot\cot x\right)\,dx = \lim_{b\to\pi^-}\left[-\cot x + \csc x\right]_0^b$$

Since $\cot 0$ and $\csc 0$ are undefined, hence the given integration is divergent.

(ii)

$$\int_{-1}^{1} \frac{1}{x^3}\, dx$$

[at $x = 0$, the function $\dfrac{1}{x^3}$ is not defined]

$$= \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx$$

$$= \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^3}\, dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x^3}\, dx$$

$$= \lim_{b\to 0^-} \left[\frac{-1}{2x^2}\right]_{-1}^{b} + \lim_{a\to 0^+} [\frac{-1}{2x^2}{}_{a}^{1}$$

$$= -\frac{1}{2}[\lim_{b\to 0^-} \left(\frac{1}{b^2} - 1\right) + \lim_{a\to 0^+} \left(1 - \frac{1}{a^2}\right)$$

=undefined

Hence the given integration is divergent.

(iii)

$$\int_{0}^{\pi} \frac{\sin x}{\cos^2 x}\, dx$$

[at $x = \dfrac{\pi}{2}$, $\dfrac{\sin x}{\cos^2 x}$ is undefined]

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\cos^2 x}\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{\cos^2 x}\, dx$$

$$= \lim_{b\to \frac{\pi}{2}^-} \int_{0}^{b} \sec x \tan x\, dx + \lim_{a\to \frac{\pi}{2}^+} \int_{a}^{\pi} \sec x \tan x\, dx$$

$$= \lim_{b\to \frac{\pi}{2}^-} [\sec x]_0^b + \lim_{a\to \frac{\pi}{2}^+} [\sec x]_a^\pi$$

$$= \lim_{b\to \frac{\pi}{2}^-} [\sec b - 1] + \lim_{a\to \frac{\pi}{2}^+} [-1 - \sec a]$$

=undefined.

$\therefore$ The integration is divergent.

(iv)

$$\int\limits_{-\infty}^{\infty} \frac{dx}{x^3}$$

[At$x = 0$,the function$\dfrac{1}{x^3}$is not defined]

$$= \int\limits_{-\infty}^{-1} \frac{dx}{x^3} + \int\limits_{-1}^{1} \frac{dx}{x^3} + \int\limits_{1}^{\infty} \frac{dx}{x^3}$$

From $3(ii)$, we can say that the given integration is divergent

(v) We know,

$$|\sin x| \le 1, \quad \forall x \in \mathbb{R}.$$

Now,

$$\frac{\sin x}{x^p} \leq \frac{1}{x^p} \quad [\text{since } x \in [0, \frac{\pi}{2}]].$$

Thus,

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}\, dx \leq \int_0^{\frac{\pi}{2}} \frac{dx}{x^p}.$$

Now,$\int_0^{\frac{\pi}{2}} \frac{dx}{x^p}$

$$[\text{At } x = 0, \frac{1}{x^p} \text{ is not defined}]$$

$= \lim_{a\to 0^+} \int_a^{\frac{\pi}{2}} \frac{dx}{x^p}$
$= \lim_{a\to 0^+} \left[\frac{1}{-p+1} \cdot \frac{1}{x^{p-1}}\right]_a^{\frac{\pi}{2}}$
$= \frac{1}{1-p} \cdot \lim_{a\to 0^+} \left[\frac{1}{x^{1-p}}\right]_a^{\frac{\pi}{2}}$
$= \frac{1}{1-p} \cdot \lim_{a\to 0^+} \left[\left(\frac{\pi}{2}\right)^{1-p} - \frac{1}{a^{1-p}}\right]$
$= \frac{1}{1-p} \left[\left(\frac{\pi}{2}\right)^{1-p} - \lim_{a\to 0^+} \frac{1}{a^{1-p}}\right].$

Since $\lim_{a\to 0^+} \frac{1}{a^{1-p}}$ exists finitely if $p \geq 1$ and for $p < 1$, it does not exist.

Therefore, the given integration is convergent for $p \geq 1$ and divergent for $p < 1$.

4. Examine the convergence of the following integrals

(i)

$$I = \int_0^1 \frac{dx}{(1+x)\sqrt{x}},$$

Solution:

We want to evaluate the integral

$$I = \int_0^1 \frac{dx}{(1+x)\sqrt{x}},$$

Let's make the substitution $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\, du$.

Now, the limits of integration change accordingly:

- When x = 0, u = 0.
- When x = 1, u = 1.

Substituting into the integral:

$$I = \int_0^1 \frac{2u\, du}{(1+u^2)\sqrt{u^2}} = \int_0^1 \frac{2u\, du}{(1+u^2)u}.$$

Simplifying:

$$I = \int_0^1 \frac{2\,du}{1+u^2}.$$

This is now a standard integral:

$$I = 2\int_0^1 \frac{du}{1+u^2}.$$

The integral $\int \frac{du}{1+u^2}$ is known to be $\tan^{-1}(u)$ so:

$$I = 2\left[\tan^{-1}(u)\right]_0^1.$$

Now, we evaluate the antiderivative at the limits:

$$I = 2\left(\tan^{-1}(1) - \tan^{-1}(0)\right).$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(0) = 0$, we get: $I = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$.

The value of the integral is $I = \frac{\pi}{2}$, Hence the integral is convergent.

(ii)

$$\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$$

Solution:

$$\int_0^1 \frac{\log x}{\sqrt{x}}\,dx = \lim_{a\to 0}\int_1^a \frac{\log x}{\sqrt{x}}\,dx$$

Using integration by parts, we get:

$$= \lim_{a\to 0}\left[\left[2\sqrt{x}\log x\right]_a^1 - \int_a^1 2\sqrt{x}\cdot\frac{1}{x}\,dx\right]$$

$$= \lim_{a\to 0}\left[\left[2\sqrt{x}\log x\right]_a^1 - \int_a^1 \frac{2}{\sqrt{x}}\,dx\right]$$

$$= \lim_{a\to 0}\left[2\sqrt{1}\log 1 - 2\sqrt{a}\log a - [\,4\sqrt{x}]_a^1\,\right]$$

$$= \lim_{a\to 0}\left[0 - 2\sqrt{a}\log a - \left(4 - 4\sqrt{a}\right)\right]$$

$$= \lim_{a\to 0}\left[-4 + 4\sqrt{a} - 2\sqrt{a}\log a\right]$$

As $a \to 0$, the limit simplifies to:

$$= -4$$

Since the limit exists, the integral converges.

(iii)

$$\int_1^2 \frac{\sqrt{x}}{\log x}\,dx,$$

Solution: To examine the convergence of the integral

$$\int_1^2 \frac{\sqrt{x}}{\log x}\,dx,$$

we need to analyze the behavior of the integrand $\frac{\sqrt{x}}{\log x}$ over the interval $[1, 2]$.

The function $\sqrt{x}$ is continuous and bounded on $[1, 2]$.

The function $\log x$ is also continuous and positive on $(1, 2]$

As $x \to 1$, $\log x \to 0$ and $\sqrt{x} \to 1$, leading to $\frac{\sqrt{x}}{\log x} \to \infty$.

Thus, we need to check if the integral diverges as x approaches 1.

We can approximate $\log x$ near $x = 1$

using the Taylor expansion: $\log x \approx x - 1$. Therefore, near $x = 1$,

$$\frac{\sqrt{x}}{\log x} \approx \frac{1}{x-1}.$$

We can analyze the integral $\int_1^{1+\epsilon} \frac{1}{x-1}\,dx$ for small $\epsilon > 0$: $\int_1^{1+\epsilon} \frac{1}{x-1}\,dx = \log|x-1|\Big|_1^{1+\epsilon} = \log\epsilon \to -\infty$ as $\epsilon \to 0$.

Since $\int_1^{1+\epsilon} \frac{1}{x-1}\,dx$ diverges, the integral

$$\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$$

diverges as well.

Thus, the integral diverges.

(iv) Given integration is:

$$\int_a^b \frac{1}{(x-a)\sqrt{b-x}}\,dx$$

This is a second-kind improper integral, and we solve this by using the μ-test.

Let

$$f(x) = \frac{1}{(x-a)\sqrt{b-x}}$$

Test for 'a':

$$\lim_{x\to a}(x-a)\frac{1}{(x-a)\sqrt{b-x}} = \frac{1}{\sqrt{b-a}}$$

which is finite.

Here, $\mu = 1$, so by the μ-test, the integral is divergent.

(v) Consider the integral:

$$\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\,dx = \int_0^{\pi/2} \frac{1}{\frac{\sin x}{x}} \cdot \frac{\sqrt{x}}{x}\,dx$$

Here, $\frac{1}{\frac{\sin x}{x}}$ is bounded and monotone. Also,

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx$$

is convergent by the p-test.

Thus, by Abel's test, the integral

$$\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\,dx$$

is convergent.

(vi) Consider the integral:

$$\int_0^1 \frac{x^{m-1}}{1+x}\,dx$$

The function $\frac{1}{1+x}$ is bounded and monotone in $[0,1]$.

Also,

$$\int_0^1 x^{m-1}\,dx = \lim_{b\to 0}\int_b^1 x^{m-1}\,dx$$

$$= \lim_{b\to 0}\left[\frac{x^m}{m}\right]_b^1$$

$$= \lim_{b\to 0}\frac{1}{m}\left[1-b^m\right]$$

$= \{ \frac{1}{m}$ if $m>0$, ∞ if $m<0$, 0 if $m=0$ $\}$

Therefore, the integral $\int_0^1 x^{m-1}\,dx$ converges if $m>0$.

So, the integral

$$\int_0^1 \frac{x^{m-1}}{1+x}\,dx$$

converges if $m>0$.

(vii)

$$\int_0^{\pi} \frac{dx}{\sqrt{\sin x}} = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} + \int_{\pi/2}^{\pi} \frac{dx}{\sqrt{\sin x}}$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right) + I$$

Now,

$$I = \int_{\pi/2}^{\pi} \frac{dx}{\sqrt{\sin x}}$$

Take $x = z + \frac{\pi}{2} \Rightarrow dx = dz$

$$\therefore \sin x = \sin\left(\frac{\pi}{2} + z\right) = \cos z$$

$$\therefore I = \int_0^{\pi/2} \frac{dz}{\sqrt{\cos z}}$$

$$= \int_0^{\pi/2} \frac{1}{\cos^{1/2} z} dz$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \left(\sin^0 z\right) \left(\cos^{-\left(\frac{1}{2}\right)} z\right) dz$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \left(\sin^{2\cdot\frac{1}{2}-1} z\right) \left(\cos^{2\cdot\frac{1}{4}-1} z\right) dz$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

Therefore the given integral converges.

(viii)

$$\int_0^1 x^{n-1} \log x dx = \lim_{b \to 0} \int_b^1 x^{n-1} \log x dx$$

$$= \lim_{b \to 0} \left[\frac{x^n}{n} \log x - \int_b^1 \frac{x^n}{n} \frac{1}{x} dx\right]$$

$$= \lim_{b \to 0} \left[\frac{x^n}{n} \log x - \frac{x^n}{n^2}\right]_b^1$$

$$= \lim_{b \to 0} \left[0 - \frac{1}{n^2} - \frac{b^n}{n} \log b + \frac{b^n}{n^2}\right]$$

$$= -\frac{1}{n^2}$$

(ix)

$$\int_1^\infty \frac{dx}{x \log x}$$

Put $\log x = t \Rightarrow \frac{dx}{x} = dt$

Now, as $x \to 1 \Rightarrow t \to 0$ and $x \to \infty \Rightarrow t \to \infty$

$$\therefore \int_1^\infty \frac{dx}{x \log x} = \int_0^\infty \frac{dt}{t} = \int_0^1 \frac{dt}{t} + \int_1^\infty \frac{dt}{t}$$

$$\int_0^1 \frac{dt}{t} = \lim_{b\to 0}\int_b^1 \frac{dt}{t} = \lim_{b\to 0}[\log t]_b^1 = \lim_{b\to 0}[\log 1 - \log b] = \infty$$

$$\therefore \int_1^\infty \frac{dx}{x\log x} \text{ diverges.}$$

(x)

$$\int_0^{\pi/2} \frac{\log x}{1+x^2}dx$$

Put $x = \tan\theta, \quad dx = \sec^2\theta d\theta$

$$I = \int_0^{\pi/2} \frac{\log(\tan\theta)\sec^2\theta}{\sec^2\theta}d\theta = \int_0^{\pi/2}\log(\tan\theta)d\theta$$

Again,

$$I = \int_0^{\pi/2}\log\tan\left(\frac{\pi}{2}-\theta\right)d\theta = \int_0^{\pi/2}\log\cot\theta d\theta$$

$$\therefore 2I = \int_0^{\pi/2}\log\tan\theta d\theta + \int_0^{\pi/2}\log\cot\theta d\theta$$

$$= \int_0^{\pi/2}\log 1\cdot d\theta = 0$$

$$\therefore I = 0$$

So, the given integral converges.

5. Discuss the convergence of

$$\int_0^1 log(\Gamma x)dx.$$

Solution: Let

$$I = \int_0^1 log(\Gamma x)dx. \tag{1}$$

Let $x = 1-t$ then $dx = -dt$. Putting in (1), we get

$$I = \int_1^0 log(\Gamma 1 - t)(-dt)$$

$$I = \int_0^1 log(\Gamma 1 - t)(dt) \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} 2I &= \int_0^1 log(\Gamma x)dx + \int_0^1 log(\Gamma 1 - x)dx \\ &= \int_0^1 log(\Gamma x\, \Gamma(1-x))dx \end{aligned}$$

From Gamma function we know that,

$$\Gamma x\, \Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$\begin{aligned} 2I &= \int_0^1 log(\frac{\pi}{\sin \pi x})dx \\ &= \int_0^1 log(\pi)dx - \int_0^1 log(\sin \pi x)dx. \end{aligned}$$

Put $\pi x = u \Longrightarrow \pi dx = du$, then

$$2I = log(\pi) - \frac{1}{\pi}\int_0^{\pi} log(\sin u)du. \tag{3}$$

Let

$$\begin{aligned} I_1 &= \int_0^{\pi} log(\sin x)dx. \\ &= \int_0^{\frac{\pi}{2}} log(\sin x)dx + \int_{\frac{\pi}{2}}^{\pi} log(\sin x)dx \\ &= \int_0^{\frac{\pi}{2}} log(\sin x)dx + \int_{\frac{\pi}{2}}^{0} log(\sin(\pi - x))(-dx) \\ &= 2\int_0^{\frac{\pi}{2}} log(\sin x)dx. \end{aligned}$$

From question (5), we have

$$\int_0^{\frac{\pi}{2}} log(\sin x)dx = -\frac{\pi}{2}log(2)$$

so, $I_1 = -\pi log(2)$

Putting in (3), we get

$$2I = log(\pi) + log(2).$$

$\Longrightarrow I = \frac{1}{2}log(2\pi)$. So, integral converges to $\frac{1}{2}log(2\pi)$.

6. Show that $\int_0^{\frac{\pi}{2}} log(\sin x)dx$ converges and hence evaluate it.

Solution: We know that $\sin x < x, \quad x \in (0, \pi/2]$, logarithmic function is always increasing in $x \in (0, \pi/2]$, so $log(\sin x) < log(x)$.

We will check convergence of $\int_0^{\frac{\pi}{2}} log(x)dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} log(x)dx &= \lim_{t\to 0}\int_t^{\frac{\pi}{2}} log(x)dx. \\
&= \lim_{t\to 0}\Big[x\, logx - x\Big]_t^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}\, log(\frac{\pi}{2}) - \frac{\pi}{2}.
\end{aligned}$$

Thus the integral is convergent.

Let

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} log(\sin x)dx \qquad (4)\\
&= \int_0^{\frac{\pi}{2}} log\Big(\sin(\pi/2 - x)\Big)dx,
\end{aligned}$$

Thus

$$I = \int_0^{\frac{\pi}{2}} log(\cos x)dx. \qquad (5)$$

Adding (4) and (5), we get

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} log(\frac{2\sin x\cos x}{2})dx \\
&= \int_0^{\frac{\pi}{2}} log(sin2x)dx - \int_0^{\frac{\pi}{2}} log(2)dx.
\end{aligned}$$

Put $2x = t \Longrightarrow 2dx = dt$, then

$$2I = \frac{1}{2}\int_0^{\pi} log(\sin t)dt - log(2)\int_0^{\frac{\pi}{2}} dx.$$

We have the property, if $f(2a - x) = f(x)$ then

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx.$$

Thus,

$$\int_0^{\pi} log(\sin t)dt = 2\int_0^{\frac{\pi}{2}} log(\sin t)dt.$$

$\Longrightarrow \quad 2I = \frac{1}{2}[2I] - log(2)\frac{\pi}{2}.$

Hence $\quad I = -\frac{\pi}{2}log(2).$

7. Using substitution $x = e^{-n}$, show that $\int_0^1 x^{m-1}(\log x)^n dx$, converges for $m > 0$,$n > -1$

   Solution: The given integral is,

$$I = \int_0^1 x^{m-1}(\log x)^n \, dx$$

   where $m > 0$ and $n > -1$.

   Using the substitution $x = e^{-t}$, which simplifies the integral

   Let $x = e^{-t}$

$$dx = -e^{-t} \, dt,$$

   As x goes from 0 to 1, t goes from ∞ to 0.

   Thus, the integral becomes:

$$I = \int_{\infty}^{0} (e^{-t})^{m-1} \left(\log(e^{-t})\right)^n (-e^{-t}) \, dt.$$

   We now simplify the terms inside the integral:

$$x^{m-1} = (e^{-t})^{m-1} = e^{-(m-1)t},$$

$$\log(x) = \log(e^{-t}) = -t,$$

$$dx = -e^{-t} \, dt.$$

   Thus, the integral becomes:

$$I = \int_{\infty}^{0} e^{-(m-1)t}(-t)^n(-e^{-t}) \, dt,$$

   which simplifies to:

$$I = (-1)^n \int_0^{\infty} e^{-mt}t^n \, dt.$$

   Now put $mt = x$

   $mdt = dx$

   $dt = \frac{dx}{m}$

$$I = (-1)^n \frac{1}{m^{n+1}} \int_0^{\infty} e^{-x}x^n \, dx$$

$$I = (-1)^n \frac{1}{m^{n+1}} \int_0^\infty e^{-x} x^{n+1-1}\, dx$$
$$I = (-1)^n \frac{\Gamma(n+1)}{m^{n+1}}$$

$$I = (-1)^n \frac{n!}{m^{n+1}}$$

Also as we know The resulting integral is a standard form of the Gamma function,

$$\int_0^\infty e^{-mt} t^n\, dt = \frac{\Gamma(n+1)}{m^{n+1}}.$$

This integral converges for $n > -1$ because $\Gamma(n+1)$ is finite when $n > -1$. The original integral

$$\int_0^1 x^{m-1} (\log x)^n\, dx = (-1)^n \frac{\Gamma(n+1)}{m^{n+1}}.$$

converges when $m > 0$ and $n > -1$.

8. Express the following integrals in terms of Gamma function:

$$(i) \int_0^\infty e^{-k^2x^2} dx (ii) \int_0^\infty x^{p-1} e^{-kx} dx, k > 0 (iii) \int_0^\infty \frac{x^c}{c^x} dx, c > 1 (iv) \int_0^1 (log(\frac{1}{y}))^{n-1} dy$$

Ans: **(i)** Put $kx = t \Rightarrow kdx = dt$

$$\therefore \int_0^\infty e^{-k^2x^2} dx = \int_0^\infty e^{-t^2} \frac{dt}{k} = \frac{\sqrt{\pi}}{2k} = \frac{\Gamma(\frac{1}{2})}{2k}$$
$$Note : \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}, \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

(ii) Put $kx = t \Rightarrow dx = \frac{dt}{k}, k > 0$

$$\therefore \int_0^\infty x^{p-1} e^{-kx} dx = \int_0^\infty \left(\frac{t}{k}\right)^{p-1} e^{-t} \frac{dt}{k} = \frac{1}{k^p} \int_0^\infty t^{p-1} e^{-t} dt \qquad = \frac{\Gamma(p)}{k^p}$$

(iii)

$$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty e^{-x \log c} x^c dx$$

Put $x \log c = t \Rightarrow \log c dx = dt$,

Thus,

$$\int_0^\infty e^{-x \log c} x^c dx = \int_0^\infty e^{-t} \left(\frac{t}{\log c}\right)^c \frac{dt}{\log c}$$
$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^c dt = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

(iv)

$$u = \int_0^1 \left(\log\left(\frac{1}{y}\right)\right)^{n-1} dy$$

We know that,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Put $e^{-x} = y \Rightarrow -e^{-x}dx = dy \Rightarrow dx = -\frac{dy}{e^{-x}} = -\frac{dy}{y}$

Also, $e^{-x} = y \Rightarrow -x = \log y \Rightarrow x = \log\frac{1}{y}$

$$\therefore \Gamma(n) = \int_1^0 y(\log(\frac{1}{y}))^{n-1}(-\frac{dy}{y}) = \int_0^(\log(\frac{1}{y}))^{n-1}dy$$

$$\therefore \int_0^1 (\log(\frac{1}{y}))^{n-1}dy = \Gamma(n)$$

9. Show that:

(i)

$$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta, \quad I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\, d\theta$$

We know that

$$B(x, y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2}B\left(\frac{1}{2}, \frac{3}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(1 + \frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{2\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2}B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}$$

Thus,

$$I = I_1 \cdot I_2 = \frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \cdot \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \pi$$

Finally,

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} d\theta \cdot \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}} d\theta = \pi$$

(ii)

$$\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta)^{-\frac{1}{2}} \sin^{\frac{1}{2}}\theta \, d\theta + \int_0^{\frac{\pi}{2}} \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right) + \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{4}{4}\right)} + \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

$$= \frac{1}{2}\left[\Gamma\left(\frac{1}{4}\right)\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{4}{4}\right)} + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)\right]$$

$$= \frac{1}{2}\left[\Gamma\left(\frac{1}{4}\right)\left(\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right)\right]$$

10. Show that $\int_0^1 x^m (logx)^n = (-1)^n \frac{n!}{(m+1)^{n+1}}$,where n is positive integer and $m > -1$

Solution: The given integral is,

$$I = \int_0^1 x^m (\log x)^n \, dx$$

Let $x = e^{-t}$, where $t \in [0, \infty)$. Then:

$$dx = -e^{-t} \, dt$$

As x ranges from 0 to 1, t will range from ∞ to 0. Therefore, the limits of integration change accordingly:

$$I = \int_0^1 x^m (\log x)^n \, dx = \int_\infty^0 (e^{-t})^m (\log(e^{-t}))^n (-e^{-t}) \, dt$$

$$x^m = e^{-mt}, \quad \log x = \log(e^{-t}) = -t$$

Thus, the integral becomes:

$$I = \int_\infty^0 e^{-mt} (-t)^n (-e^{-t}) \, dt$$

$$I = (-1)^n \int_0^\infty e^{-(m+1)t} t^n \, dt$$

This integral is now in the form of the Gamma function:

$$\int_0^\infty e^{-at} t^b \, dt = \frac{\Gamma(b+1)}{a^{b+1}}$$

where $a = m+1$ and $b = n$. Therefore, the result of the integral is:

$$I = (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

Conclusion

The integral $\int_0^1 x^m (\log x)^n \, dx$ converges for $m > -1$ and n is a positive integer, and its value is:

$$I = (-1)^n \frac{n!}{(m+1)^{n+1}}$$

11. Show that

(i)

$$\int_0^1 \frac{x}{\sqrt{1-x^5}} \, dx = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$$

Solution:

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta, \; m > 0, n > 0$

Take $x = \sin^{\frac{2}{5}}\theta \quad \Rightarrow dx = \frac{2}{5}\sin^{-\frac{3}{5}}\theta \cos\theta \, d\theta$

Therefore, $x^5 = \sin^2\theta$

$\sqrt{1-x^5} = \sqrt{1-\sin^2\theta} = \cos\theta$

$\int_0^1 \frac{x}{\sqrt{1-x^5}} \, dx = \int_0^{\pi/2} \frac{\sin^{\frac{2}{5}}\theta}{\cos\theta} \cdot \frac{2}{5}\sin^{-\frac{3}{5}}\theta \cos\theta \, d\theta$

$= \frac{2}{5}\int_0^{\pi/2} \sin^{\frac{2}{5}-\frac{3}{5}}\theta \, d\theta$

$= \frac{2}{5}\int_0^{\pi/2} \sin^{-\frac{1}{5}}\theta \, d\theta$

$= \frac{2}{5}\int_0^{\pi/2} \sin^{\left(2\cdot\frac{2}{5}-1\right)}\theta \cos^{(2\cdot\frac{1}{2}-1)}\theta \, d\theta$

$= \frac{1}{5}B\left(\frac{2}{5}, \frac{1}{2}\right)$ (proved).

(ii)

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)}$$

Solution: Beta Function:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0, n > 0$$

Take

$$x = \sin^{2/4}\theta \Rightarrow \quad x^4 = \sin^2\theta$$

$$\Rightarrow x = \sin^{\frac{1}{2}}\theta \quad \Rightarrow dx = \frac{1}{2}\sin^{-1/2}\theta \cos\theta d\theta$$

Therefore,

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{1}{\cos\theta} \cdot \frac{1}{2} \sin^{-1/2}\theta \cos\theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{1}{\sin^{1/2}\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^{(2\times\frac{1}{4}-1)}\theta \cos^{(2\times\frac{1}{2}-1)}\theta d\theta$$

$$= \frac{1}{4} \times 2 \int_0^{\pi/2} \sin^{2m-1} \cos^{2n-1}\theta d\theta$$

where $m = \frac{1}{4}, n = \frac{1}{2}$

$$= \frac{1}{4} \times B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4} \times \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)} \quad \text{(proved)}$$

12. Show that

(i)

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta = \frac{1}{2} \frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)}$$

Solution:

$$I = \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta = \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(\cos^2\theta)^{m+n}(b + a\tan^2\theta)^{m+n}} d\theta = \int_0^{\pi/2} \frac{\tan^{2m-1}\theta \sec^2\theta}{(b + a\tan^2\theta)^{m+n}} d\theta$$

$put \tan\theta = t \implies \sec^2\theta\, d\theta = dt,$

$\theta = 0 \implies t = 0$ and $\theta = \pi/2 \implies t = \infty$

$$\implies I = \int_0^{\infty} \frac{t^{2m-1}}{(b+at^2)^{m+n}} dt = \int_0^{\infty} \frac{(t^2)^{m-1}t}{(b+at^2)^{m+n}} dt$$

Substitute $y = \frac{at^2}{b}$

$$I = \int_0^{\infty} \frac{(by/a)^{m-1}b/2a}{(b+by)^{m+n}} dy = \frac{1}{2a^m b^n} \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy = \frac{1}{2a^m b^n} B(m,n) = \frac{1}{2a^m b^n} \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

(ii)

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

Solution: Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

put $x = 1/1+y$, we get

$$\beta(m,n) = \int_\infty^0 \frac{-y^{n-1}}{(1+y)^{m+n}}\, dy = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy = I_1 + I_2$$

putting $z = 1/y$ in I_2 we get

$$I_2 = \int_1^0 \frac{(1/z)^{n-1}(-1/z^2)}{(1+1/z)^{m+n}}\, dz = \int_0^1 \frac{z^{m-1}}{(z+1)^{m+n}}\, dz = \int_0^1 \frac{y^{m-1}}{(y+1)^{m+n}}\, dy$$

Using I_1 and I_2 and changing variable from y to x we get

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

(iii)

$$\beta(m,1/2) = 2^{2m-1}\beta(m,m)$$

Solution:

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} \implies \beta(m,1/2) = \frac{\Gamma m \Gamma 1/2}{\Gamma(m+1/2)}$$

Using duplicate formula

$$\Gamma m \Gamma(m+1/2) = \frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}} \implies \Gamma(m+1/2) = \frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}\Gamma m}$$

using $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ we get

$$\beta(m,1/2)\frac{\sqrt{\pi}\Gamma 2m}{2^{2m-1}\Gamma m} = \Gamma m \Gamma 1/2 \implies \beta(m,1/2) = \frac{\Gamma m \Gamma m}{\Gamma 2m} 2^{2m-1} = \beta(m,m)2^{2m-1}$$

(iv)

$$\beta(n,n) = \frac{\sqrt{\pi}\Gamma n}{2^{2n-1}\Gamma(n+1/2)}$$

Solution:

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \implies \beta(n,n) = 2\int_0^{\pi/2} \sin^{n-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta$$

$$\implies \frac{(\Gamma n)^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\int_0^\pi \sin^{2n-1}\phi\, d\phi \quad (take 2\theta = \phi) - (1)$$

As $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ put m = 1/2 we get $\frac{\Gamma n \Gamma 1/2}{\Gamma(n+1/2)} = 2\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta$ - (2)

From (1) and (2) we get

$$\frac{(\Gamma n)^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\frac{\Gamma n \Gamma 1/2}{\Gamma(n+1/2)} \implies \beta(n,n) = \frac{1}{2^{2n-1}}\frac{\sqrt{\pi}\Gamma n}{\Gamma(n+1/2)}$$

13. For $n > -1$, $m < 1$. We have to show

$$\frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + \frac{m(m+1)(m+2)}{3!(n+4)} + \cdots = \beta(n+1, 1-m).$$

Now, $n > -1$

$$\Rightarrow n+1 > 0$$

and $m < 1 \Rightarrow 1 - m > 0$.

Let $p = n+1$, $q = 1-m$. So, the Beta function is:

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$$= \int_0^1 x^n (1-x)^{-m}\,dx$$

Now,

$$(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} (-x)^k = \sum_{k=0}^{\infty} \frac{m(m+1)\cdots(m+k-1)}{k!} x^k$$

$$therefore \beta(p,q) = \beta(n+1, 1-m)$$

$$= \int_0^1 x^n \left(\sum_{k=0}^{\infty} \frac{m(m+1)\cdots(m+k-1)}{k!} x^k \right) dx$$

$$= \sum_{k=0}^{\infty} \left(\frac{m(m+1)\cdots(m+k-1)}{k!} \int_0^1 x^{n+k}\,dx \right)$$

Since,

$$\int_0^1 x^{n+k}\,dx = \left[\frac{x^{n+k+1}}{n+k+1} \right]_0^1 = \frac{1}{n+k+1}$$

$$therefore \beta(n+1, 1-m) = \sum_{k=0}^{\infty} \frac{m(m+1)\cdots(m+k-1)}{k!(n+k+1)}$$

$$= \frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + \frac{m(m+1)(m+2)}{3!(n+4)} + \cdots$$

Hence, proved.