Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{--- (1)}$$

Stationary states:

For a given potential energy function ~~ψ(x)~~ $V(x)$ [From now on we will just say it "potential"] which is explicitly time independent,

We can find solution $\psi(x,t)$ for which the probability density is time independent. These states are called Stationary states.

For these states we can write the solution $\psi(x,t)$ in the following form

$$\psi(x,t) = \chi(x)\,\phi(t) \quad \text{--- (2)}$$

put in Eqn 1

$$i\hbar\,\chi(x)\frac{d\phi(t)}{dt} = -\frac{\hbar^2}{2m}\phi(t)\frac{d^2\chi(x)}{dx^2} + V\chi\phi \quad \text{--- (3)}$$

Devide eqn. 3 by $\chi\phi$

$$\underbrace{i\hbar \frac{1}{\phi(t)}\frac{d\phi(t)}{dt}}_{\text{Function of } t \text{ only}} = \underbrace{-\frac{\hbar^2}{2m}\frac{1}{\chi(x)}\frac{d^2\chi(x)}{dx^2} + V(x)}_{\text{Function of } x \text{ only}}$$

This can't be possible unless both sides are constant. Let us introduce the separation constant E

$$\boxed{i\hbar \frac{1}{\phi}\frac{d\phi}{dt} = E}$$

$$\frac{d\phi}{dt} = -\frac{iE}{\hbar}\phi$$

$$\int\frac{d\phi}{\phi} = -i\frac{E}{\hbar}\int dt$$

$$\phi = Ce^{-\frac{iEt}{\hbar}}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{1}{\chi}\frac{d^2\chi}{dx^2} + V(x) = E}$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{d^2\chi}{dx^2} + V\chi = E\chi}$$

Time independent Schrödinger equation (TISE)

$$\hat{H} \equiv \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)$$

Hamiltonian operator

TISE $\quad \hat{H}\chi = E\chi$

Steady state solution $\quad \psi = \chi\phi = \chi e^{-i\frac{Et}{\hbar}}$

$$|\psi|^2 = \psi\psi^* = \chi e^{-i\frac{Et}{\hbar}}\chi^* e^{+i\frac{Et}{\hbar}} = \chi\chi^* = |\chi|^2$$

Time indept

$\hat{H}\chi = E\chi$

In mathematical language → This equation is Eigenvalue equation for the operator $\hat{H}$.

The possible solutions χ_i are called eigen functions and corresponding E_i are eigenvalues

Each Solution χ_i represents a stationary state where the probability density is time independent.