

LECTURE - 3

(Also see Tutorial 1)

SECTION - C

07/08/2025

SECTION - D

SECTION - B

SECTION - A

① Continuation from L2 about potential.

TOPICS :

① Beyond Second Law

Why alternative formulations are required?

② Example of Problem with a different coordinate system.

Potential Energy

Scalar potential $V(\vec{r})$ only defined up to a constant.

$$\vec{F} = -\vec{\nabla} V$$

Let's define,
a new potential

$$V_2(\vec{r}) = V(\vec{r}) + V_0$$

Then, $\vec{F} = -\vec{\nabla}(V_2(\vec{r}) - V_0)$

$$= -\vec{\nabla} V_2(\vec{r})$$

$$\therefore \vec{\nabla} V_0 = 0$$

→ Same force (and hence, same physics)
independent of added constant ' V_0 '.

** FOR INTERESTED STUDENTS

- This is like "GAUGE SYMMETRY".
(Unknown upto a gauge field)
- We will encounter this in
"ELECTRODYNAMICS" part.
- Absolute SCALAR POTENTIAL is ambiguous.

LET'S SEE HOW THIS WORKS IN

PRACTICE !

$$W_{12} = \oint_{①}^② \vec{F} \cdot d\vec{r} = - \oint_{①}^② \vec{\nabla}V \cdot d\vec{r} = V_1 - V_2$$

\int potential

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}; \quad \frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k}$$

Check this out in the 1 dimensional version :

1d version: $- \int_{x_1}^{x_2} \frac{d}{dx} V(x) dx = - [V(x_2) - V(x_1)]$

$$W_{12} = T_2 - T_1 = V_1 - V_2 \iff \text{Conservative force}$$

$$\Rightarrow V_1 + T_1 = V_2 + T_2$$

Total energy at the start of journey at pt. ① is equal to the total energy at the end of journey at pt. ②.

$$E = T + V = \text{constant}$$

\Rightarrow CONSERVATION OF ENERGY

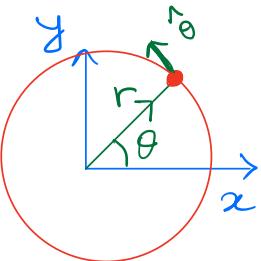
Examples of Newtonian Formulation

- ① In Coordinate systems other than Cartesian
- ② In moving reference frames ** FOR INTERESTED STUDENTS

Until now, we have been working with Cartesian coordinates. Note that the usefulness of the plane polar coordinates for certain problems is discussed in Tutorial 1.

- ① Let us find $\underline{\text{NL}}$ in **plane polar** coordinates (2-dimension).

\Rightarrow Single particle in some potential $V(\vec{r})$

$$\Rightarrow \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$\vec{r} = r \hat{r}$$

$$= \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \Rightarrow \hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\text{i.e. } \hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$$

Now,

$\underline{\text{NL}}$ in Cartesian Coordinates:

$$m\ddot{x} = -\frac{\partial V}{\partial x} \quad ; \quad m\ddot{y} = -\frac{\partial V}{\partial y}$$

NLII in polar coordinates:

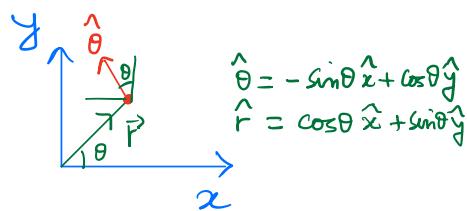
$$\text{Is it } m\ddot{r} = -\frac{\partial V}{\partial r} \text{ ? NO !}$$

Let us revisit the position vector $\vec{r} = r \hat{r}$

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d\hat{r}}{d\theta} \frac{d\theta}{dt} = \frac{d}{d\theta} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \frac{d\theta}{dt} \\ &= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \dot{\theta} = \dot{\theta} \hat{\theta}\end{aligned}$$

Recall:

$$\hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



$$\text{Now, } \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \frac{d\theta}{dt} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} \dot{\theta} = -\dot{\theta} \hat{\theta}$$

$$\begin{aligned}\dot{\vec{r}} &= \frac{d}{dt} (\vec{r}) = \frac{d}{dt} (r \hat{r}) = \frac{d}{dt} (r) \hat{r} + r \frac{d\hat{r}}{dt} \\ &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}} &= \frac{d}{dt} (\dot{\vec{r}}) = \frac{d}{dt} (\dot{r} \hat{r}) + \dot{r} \frac{d\hat{r}}{dt} + \frac{d}{dt} (r \dot{\theta} \hat{\theta}) \\ &\quad + r \frac{d}{dt} (\dot{\theta}) \hat{\theta} + r \dot{\theta} \frac{d}{dt} (\hat{\theta})\end{aligned}$$

Substitute all the terms :

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2) \hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\theta}$$

Also, $\vec{J} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \Rightarrow$ Given
[Can be proved by chain rule].

NLII in plane polar coordinates:

$$\vec{F} = m\vec{a} \Rightarrow -\vec{\nabla} V = m\ddot{\vec{r}} \quad \text{#} m\dot{r} = -\frac{\partial V}{\partial r}$$

along \hat{r} : $-\frac{\partial V}{\partial r} = m\ddot{r} - m r \dot{\theta}^2 \rightarrow$ Centrifugal force

along $\hat{\theta}$: $-\frac{1}{r} \frac{\partial V}{\partial \theta} = 2m\dot{r}\dot{\theta} + m\ddot{\theta}$

TAKE HOME MESSAGE :

$m\ddot{q} = F_q$: no such Newton equation of motion for a generalized coordinate q .

NLII works in Cartesian coordinate.
(when broken into components)

The second law changes if we change the coordinate system.

Recall the statement of second law; it states "acceleration of

an object is directly proportional to net force acting on it and inversely proportional to its mass."

Clearly, for the central force problem in plane polar coordinates this is not the case, unless we introduce the concept of some fictitious forces.

In other words, $F = m \frac{d^2x}{dt^2}$ in 1D

But now if we replace x by some coordinate θ where x is a function of θ , then, $\Rightarrow \frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt}$

$$F = m \frac{d^2x}{dt^2} = m \frac{d}{dt} \left(\frac{dx}{dt} \right) = m \frac{d}{dt} \left(\frac{dx}{d\theta} \dot{\theta} \right)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{dx}{d\theta} \right) &= \frac{d}{d\theta} \left(\frac{dx}{d\theta} \right) \ddot{\theta} \\ &= \frac{d^2x}{d\theta^2} \ddot{\theta} \end{aligned} \quad = m \left(\frac{dx}{d\theta} \ddot{\theta} + \frac{d^2x}{d\theta^2} \dot{\theta}^2 \right)$$

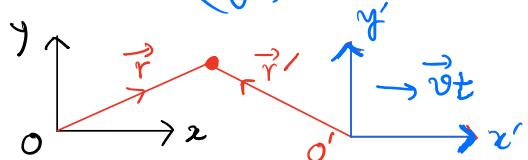
SUMMARY :

- Casting Newton's laws in non-inertial frames and for generalized coordinates becomes extremely complicated.
In other words Newton's equation of motion is not invariant as we change coordinate systems or work in non-inertial frames.
- Complications will increase as we increase the number of particles in the system.
- But there are alternative formulations of Classical Mechanics where these issues can be overcome.
- Lagrangian and Hamiltonian formulations are required to build connection with Quantum Mechanics.
- Moving Reference frames Γ_{inertial}

⇒ example of an accelerated frame of reference.

Original coordinates : $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ Cartesian

NLI_{II} : $-\vec{\nabla}V = m\ddot{\vec{r}}$



New coordinates : MOVING REFERENCE FRAME

$$\vec{r}' = \vec{r} - \vec{\vartheta} t$$

NOW, $\vec{\vartheta} = \text{constant}$, then, $\ddot{\vec{r}}' = \ddot{\vec{r}}$

NLI_{II} : $-\vec{\nabla}V = m\ddot{\vec{r}} = m\ddot{\vec{r}}'$

[SAME EXPRESSION!]

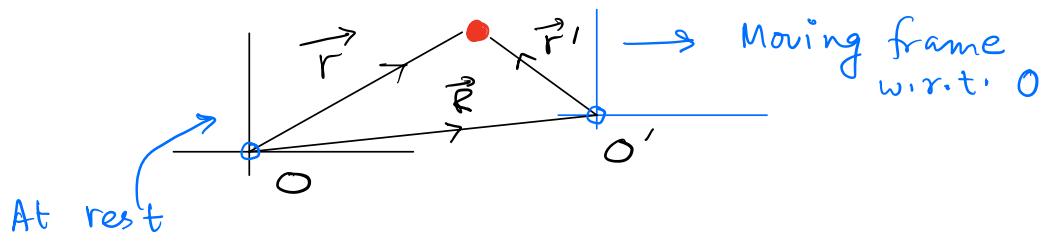
If $\vec{\vartheta}$ is NOT constant, [Non-Inertial]

NLI_{II} :

$$\begin{matrix} ? \\ o \end{matrix} \quad \begin{matrix} ? \\ o \end{matrix}$$

Applications
⇒ Weather events on Earth.

a. Ex. Linearly accelerating reference frames:



- ① At $t=0$, the origin of O & O' coincide. Then O' moves w.r.t. O . Let $\vec{R}(t)$ denote the position vector of the origin of O' as seen by observer in frame O .
- ② The velocity of O' is $\frac{d\vec{R}(t)}{dt} = \vec{v}_{O'}(t)$ and acceleration of O' is $\frac{d\vec{v}_{O'}(t)}{dt} = \vec{a}_{O'}(t)$ with respect to O frame.
- ③ Now position vector $\vec{r}(t)$ of an object in O is related to the position vector $\vec{r}'(t)$ of the object in O' :

$$\rightarrow r = \vec{r}' + \vec{r}_{O'}$$

$$\vec{r}(t) = \vec{r}(t) + \vec{r}(t)$$

① Differentiating the above :

$$\dot{\vec{r}}(t) = \dot{\vec{r}}'(t) + \dot{\vec{R}}(t)$$

Since O' is moving w.r.t. O !

Similarly,

$$\Rightarrow \ddot{\vec{r}}(t) = \ddot{\vec{r}}'(t) + \ddot{\vec{R}}(t)$$

Recall! In an inertial frame, O

$$m\ddot{\vec{r}} = \vec{F}_{\text{inertial}}$$

In the non-inertial frame NLI of object needs to be fixed :

$$m\ddot{\vec{r}}(t) = m\ddot{\vec{r}}'(t) + m\ddot{\vec{R}}(t)$$

$$\Rightarrow m\ddot{\vec{r}}'(t) = m\ddot{\vec{r}}(t) - m\ddot{\vec{R}}(t)$$

$$\Rightarrow m\vec{a}' = \vec{ma} - \vec{mA}$$

$$\Rightarrow m\vec{a}' = \vec{F}_{\text{inertial}} - \vec{mA}$$

Define: $-\vec{mA} = \vec{F}_{\text{fictitious}}$

$$\Rightarrow m\vec{a}' = \vec{F}_{\text{inertial}} + \vec{F}_{\text{fictitious}}$$

MODIFIED NEWTON'S SECOND LAW
IN NON-INERTIAL FRAME

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