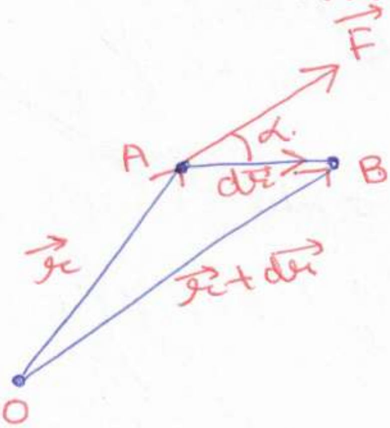
Principle of Virtual Work:

- If a particle, rigid body or a system of rigid bodies, which is in equilibrium under various forces is given an arbitrary displacement (virtually) from the position of equilibrium, the net work done by the external forces during that displacement is $\boxed{\text{zero}}$.
- Importance : It is an alternative method for solving problems involving the equilibrium of a particle, a rigid body, or a system of connected rigid bodies.

Recapitulation: Work of a Force.

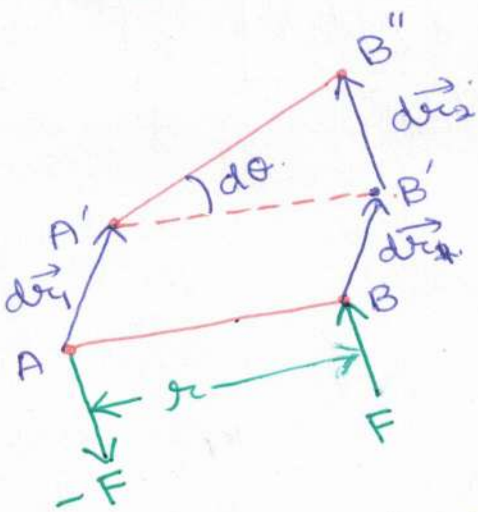


$dW = \vec{F}\cdot\vec{dr}$ = work of the force $\vec{F}$ corresponding to the displacement $\vec{dr}$.

$\Rightarrow dW = (F\cos\alpha)\,ds = F(ds\cos\alpha)$

$\Rightarrow dW = F\,ds\cos\alpha.$

Work of a Couple.



⊛ Couple : $M = Fr$.

- Small displacement of a rigid body:
  - rigid body translation of AB to A'B'.
  - rigid body rotation of B' about A' to B''.

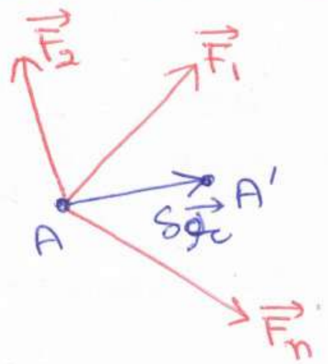
Using the concept of work of a force:

$dW = -\vec{F}\cdot\vec{dr_1} + \vec{F}\cdot(\vec{dr_1}+\vec{dr_2})$

$\Rightarrow dW = \vec{F}\cdot\vec{dr_2} = F\,ds_2 = F\,r\,d\theta$

$\Rightarrow dW = M\,d\theta$

Principle of Virtual Work (Derivation).



- Imagine a small virtual displacement of the particle, which is acted upon by several forces.
- The corresponding virtual work is given by:-

$$\delta W = \vec{F_1}\cdot\delta\vec{r} + \vec{F_2}\cdot\delta\vec{r} + \vec{F_3}\cdot\delta\vec{r} + \dots + \vec{F_n}\cdot\delta\vec{r}$$

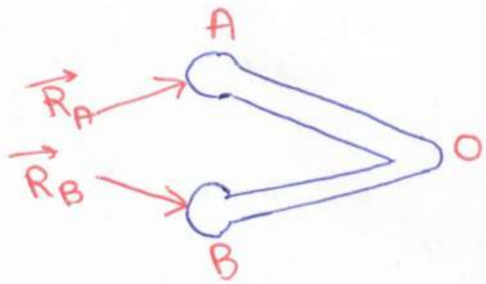
$$\Rightarrow \delta W = (\vec{F_1} + \vec{F_2} + \vec{F_3} + \dots + \vec{F_n})\cdot\delta\vec{r}$$

$$\Rightarrow \delta W = \vec{R}\cdot\delta\vec{r}.$$

(*) For a rigid body to be in equilibrium, the resultant force $\vec{R} = 0$.

$\therefore \boxed{\delta W = 0}$

(*) If a system of connected rigid bodies remain connected during the virtual displacement, only the work of external forces need to be considered.

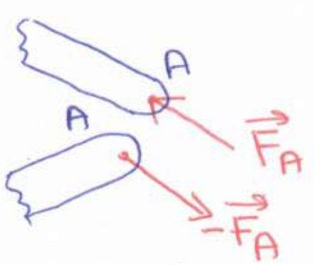


(*) Reactive forces which act at fixed support positions where no virtual displacement takes place in the direction of the force.

Thus, the reactive forces do no work during virtual displacement.

$\therefore \delta W = \vec{F}\cdot 0 = 0$

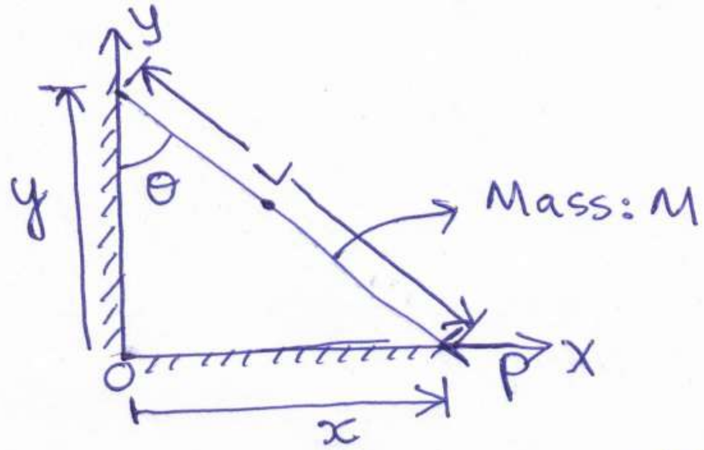
(*)



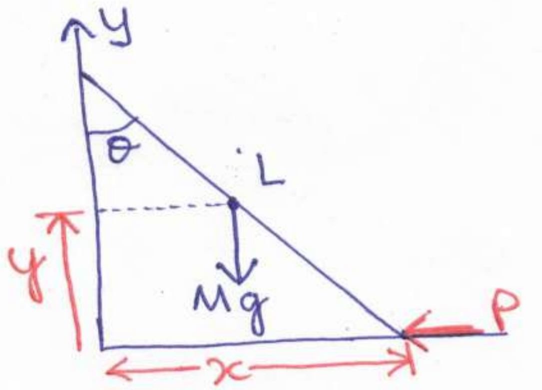
- Internal forces are forces in the member where the connections are established.
- During any possible movement of the system or its parts, the net work done by the internal forces at the connection is zero.

$\therefore \delta W = (-\vec{F} + \vec{F})\cdot\delta\vec{r} = 0.$

Eg: A homogeneous ladder of mass 'M' and length 'L' is held in equilibrium by a horizontal force 'P' as shown. Using the principle of virtual work, express 'P' in terms of 'M'.



Solution



- $x = L\sin\theta$

$\Rightarrow \delta x = L\cos\theta\,\delta\theta$

- $y = L\cos\theta - \frac{L}{2}\cos\theta$

$\Rightarrow y = \frac{L}{2}\cos\theta$

$\Rightarrow \delta y = \frac{L}{2}(-\sin\theta)\delta\theta$

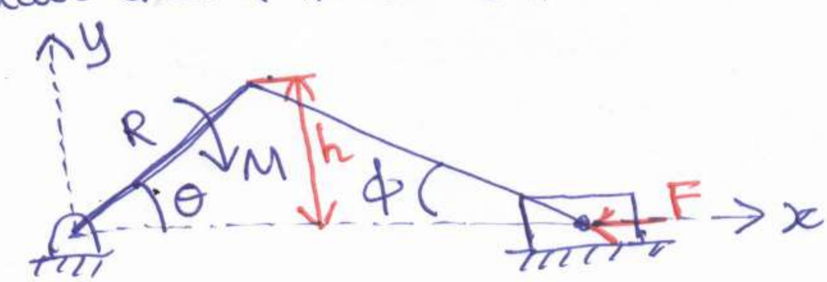
$\Rightarrow |\delta y| = \frac{L}{2}\sin\theta\,\delta\theta$

$$\delta W = P|\delta x| - Mg|\delta y| = 0$$

$$\Rightarrow PL\cos\theta\,\delta\theta = Mg\frac{L}{2}\sin\theta\,\delta\theta$$

$$\Rightarrow \boxed{P = \frac{Mg}{2}\tan\theta}$$

Eg. Using the principle of virtual work, determine the relationship between the applied moment 'M' to the crank 'R' and the force 'F' applied to crosshead of the slider-crank mechanism as shown below.



Solution: $\delta W = F|\delta x| - M|\delta \theta| = 0.$

$$x = R\cos\theta + l\cos\phi. \qquad h = R\sin\theta = l\sin\phi.$$

$$\Rightarrow x = R\cos\theta + l\sqrt{1-\frac{R^2}{l^2}\sin^2\theta} \qquad \cos\phi = \sqrt{1-\sin^2\phi}$$

$$\Rightarrow \delta x = -R\sin\theta\,\delta\theta + \left[\frac{l(-2\sin\theta)\cos\theta\,\delta\theta\,\frac{R^2}{l^2}}{2\sqrt{1-\frac{R^2}{l^2}\sin^2\theta}}\right] \qquad \Rightarrow \cos\phi = \sqrt{1-\frac{R^2}{l^2}\sin^2\theta}$$

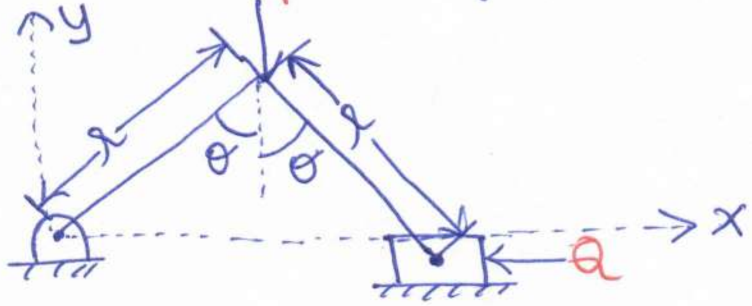
$$\Rightarrow \delta x = -\left(R\sin\theta + \frac{R^2}{l}\frac{\sin\theta\cos\theta}{\sqrt{1-\frac{R^2}{l^2}\sin^2\theta}}\right)\delta\theta.$$

$$\therefore F|\delta x| = M|\delta\theta|$$

$$\Rightarrow F\left(R\sin\theta + \frac{R^2}{l}\frac{\sin\theta\cos\theta}{\sqrt{1-\frac{R^2}{l^2}\sin^2\theta}}\right)\delta\theta = M\,\delta\theta$$

$$\Rightarrow \boxed{M = FR\sin\theta\left[1+\left(\frac{R}{l}\right)\frac{\cos\theta}{\sqrt{1-\frac{R^2}{l^2}\sin^2\theta}}\right]}$$

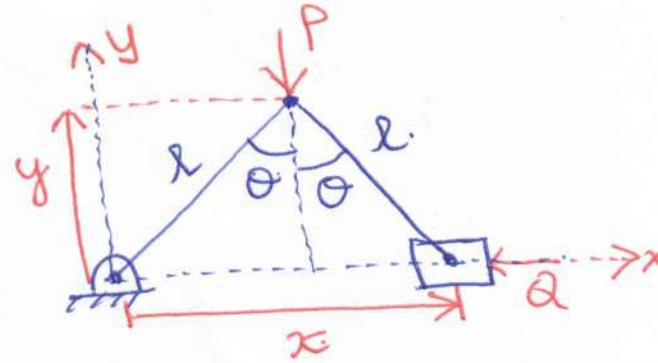
Eg: Determine the force of the vice on the block for a given force 'P', so that the system remains in equilibrium.



Solution

① Case 1: No friction b/w the block the base.

Let P displace by δy so that the block virtually displace by δx towards right.



$\delta W = +P|\delta y| - Q|\delta x| = 0$

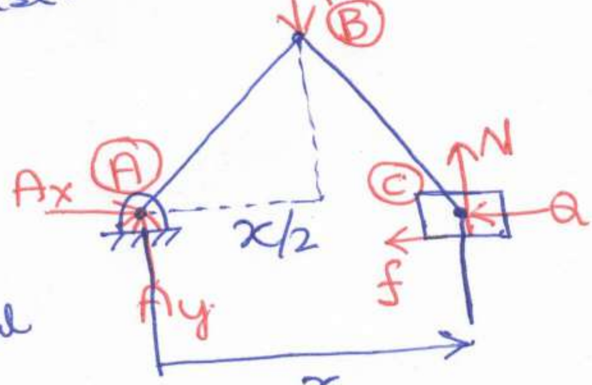
$\Rightarrow P|\delta y| = Q|\delta x|.$

$\Rightarrow P\, l \sin\theta\, \delta\theta = Q(2l\cos\theta)\delta\theta$

$\Rightarrow \boxed{Q = \frac{P}{2}\tan\theta}$

- $x = 2l\sin\theta$
  $\Rightarrow |\delta x| = 2l\cos\theta\,\delta\theta$
- $y = l\cos\theta$
  $\Rightarrow |\delta y| = l\sin\theta\,\delta\theta$

② Case 2: Assuming the friction to be present between the block and the base.



$\cancel{\Sigma F_y = 0}$

$\Rightarrow P = \cancel{N + A_y}$

- $\Sigma M_A = 0$
  $Nx = P(x/2)$
  $\Rightarrow \boxed{N = P/2}$

- $f = \mu N$ (limiting case)

$\Rightarrow f = \mu \frac{P}{2}$

Using principle of virtual work,

$\delta W = 0$

$\Rightarrow +P|\delta y| - f|\delta x| - Q|\delta x| = 0$

$\Rightarrow P|\delta y| = (f + Q)|\delta x|$

$\Rightarrow P\, l\sin\theta\,\delta\theta = \left(\frac{\mu P}{2} + Q\right)(2l\cos\theta\,\delta\theta)$

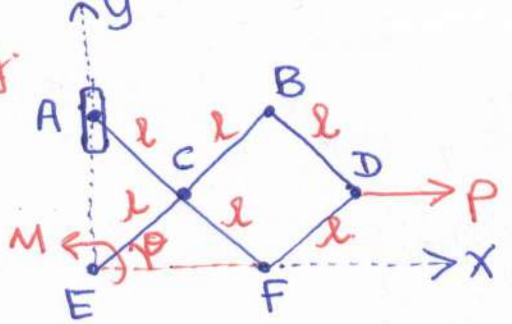
$\Rightarrow \boxed{Q = \frac{P}{2}(\tan\theta - \mu)}$

- $\eta = \frac{\text{Output work}}{\text{Input work}} = \frac{-f|\delta x| + Q|\delta x|}{P|\delta y|}$

$\Rightarrow \eta = \frac{\frac{P}{2}\tan\theta - \frac{\mu P}{2}}{P}\frac{(2l\cos\theta\,\delta\theta)}{l\sin\theta\,\delta\theta}$

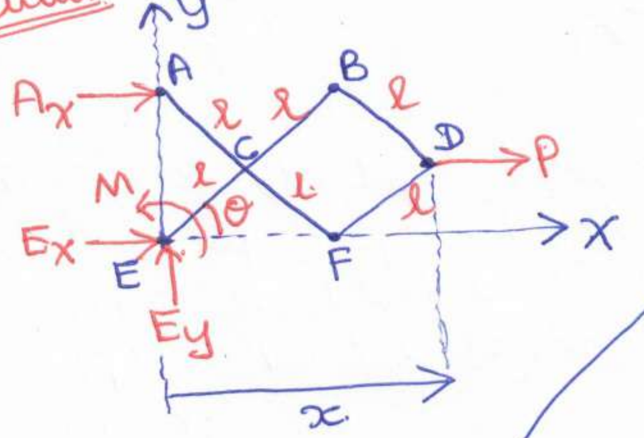
$\Rightarrow \boxed{\eta = 1 - \mu\cot\theta}$

Eg.



Determine the magnitude of the couple 'M' required to maintain the equilibrium of the mechanism.

Solution



$x = 3l\cos\theta$

$|\delta x| = 3l\sin\theta\,\delta\theta$

Invoking principle of virtual work

$\delta W = 0$

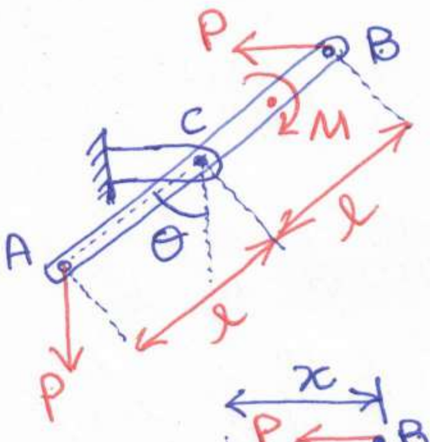
$\Rightarrow P|\delta x| - M\,\delta\theta = 0$

$\Rightarrow P|\delta x| = M\,\delta\theta$

$\Rightarrow P(3l\sin\theta)\,\delta\theta = M\,\delta\theta$

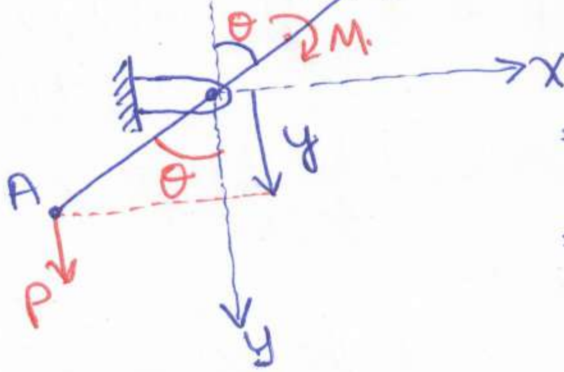
$\Rightarrow \boxed{M = 3Pl\sin\theta}$

Eg.



Derive the magnitude of the couple M required to maintain the equilibrium of the linkage shown.

Solution



- $x = l\sin\theta$

  $|\delta x| = l\cos\theta\,\delta\theta$
- $y = l\cos\theta$

  $|\delta y| = l\sin\theta\,\delta\theta$

- Let 'δx' virtual displacement at B.
- $\delta W = 0$

  $\Rightarrow P|\delta x| - M|\delta\theta| + P|\delta y| = 0$

$\Rightarrow Pl\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta = M\delta\theta$

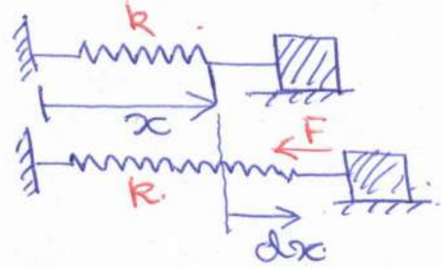
$\Rightarrow \boxed{M = Pl(\sin\theta + \cos\theta)}$

Conservative Forces:

- When a force does work, that depends only on the initial and final positions of the force, and is independent of the path it travels, then the force is referred as a conservative force.
- Example: Weight of the body and spring force.

$$dW = \vec{W} \cdot d\vec{r}$$
$$\overset{(-)}{=} W(dr\cos\theta)$$
$$= (-W)\,dy$$
$$\therefore \int_A^B dW = -\int_0^h W\,dy$$
$$\Rightarrow (W_B - W_A) = W = -Wh$$

↳ depends on initial & final position of the weight



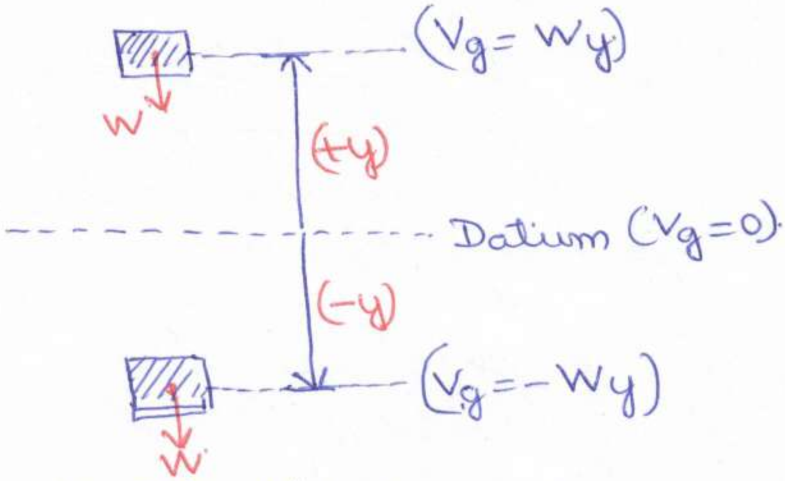
$$dW = -F\,dx$$
$$\Rightarrow \int_1^2 dW = -\int_1^2 kx\,dx$$
$$\Rightarrow W_2 - W_1 = W = -\frac{1}{2}k(x_2^2 - x_1^2)$$

↳ depends on the initial and final position of the spring.

Potential Energy:

- A conservative force can give the body the capacity to do work.
- This capacity, measured as "potential energy", depends on the location/position of the body measured relative to the fixed reference position or datum.

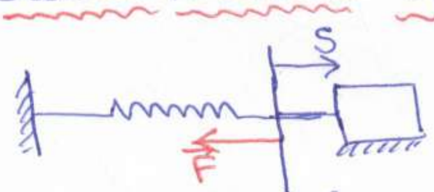
⊛ Gravitational Potential Energy (V_g)



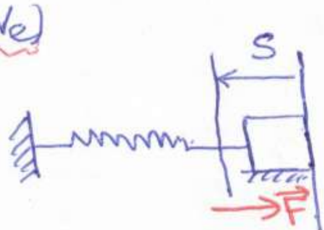
$\therefore \boxed{V_g = +Wy}$

considering 'y' as +ve upwards.

⊛ Elastic Potential Energy (V_e)



Undeformed position.



Undeformed position.

$$\boxed{V_e = \frac{1}{2} k s^2}$$

- V_e is always +ve since the spring force attached on the body does +ve work on the body as the force returns the body to the spring's unstretched position.

Potential Function

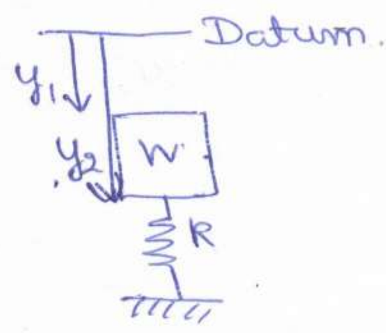
$$\boxed{V = V_g + V_e}$$

*If the vertical distance from the datum is defined as 'q', the potential function of the system (V) is defined as-

$V = V(q)$

* The work done by all the weights and spring forces acting on the system in moving it from q_1 to q_2 is measured by the difference in V.

$W_{1-2} = V(q_1) - V(q_2)$.



$$W_{1-2} = V(y_1) - V(y_2)$$

$$= \left(-Wy_1 + \frac{1}{2} ky_1^2\right) - \left(-Wy_2 + \frac{1}{2} ky_2^2\right)$$

$$\boxed{W_{1-2} = -W(y_1 - y_2) + \frac{1}{2}(k)(y_1^2 - y_2^2)}.$$

Potential Energy Criterion for Equilibrium.

- If a frictionless connected system has one DOF and its pos^n. is defined by the coordinate 'q', then if it displaces from 'q' to $(q+dq)$, we have

$$dW = V(q) - V(q+dq).$$

$$\Rightarrow \boxed{dW = -dV}$$

- If the system is in eqlb^m. and undergoes a virtual displacement δq (instead of the actual displacement dq), the above equation can be written as-

$$\delta W = -\delta V$$

From the principle of virtual work: $\delta U = 0$.

$$\therefore \delta V = 0$$

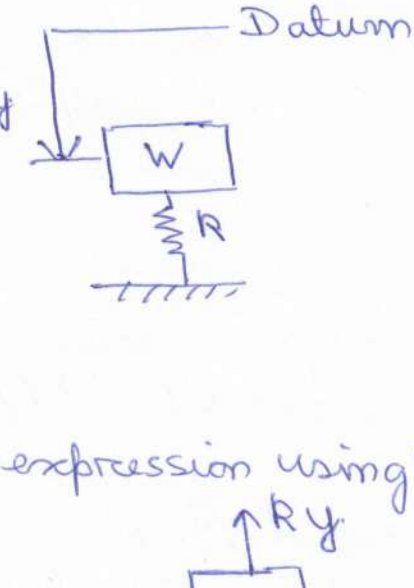
$$\Rightarrow \left(\frac{dV}{dq}\right)\delta q = 0$$

As $\delta q \neq 0$; so $\boxed{\frac{dV}{dq} = 0}$

$$V = \cancel{V(e)+} V(y).$$

$$= -Wy + \frac{1}{2}ky^2$$

Here $q = y$



(*) $\frac{dV}{dy} = 0$

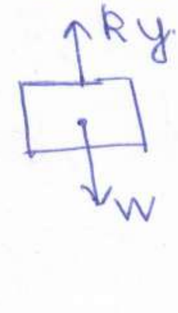
$$\Rightarrow -W + ky = 0$$

$$\Rightarrow y = \frac{W}{k} = y_{eq}.$$

↳ One obtains the same expression using $\Sigma F_y = 0$

$$\Rightarrow W = ky$$

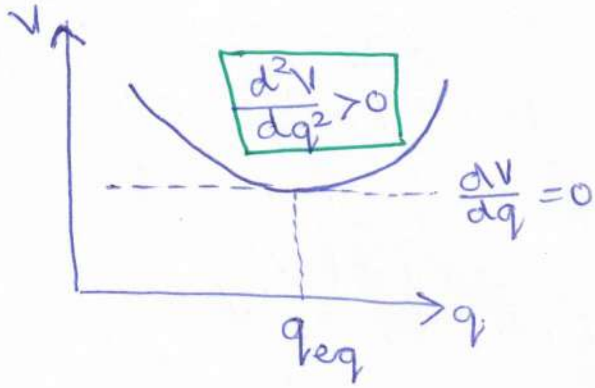
$$\Rightarrow \boxed{y = W/k}$$

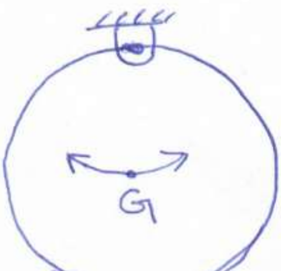


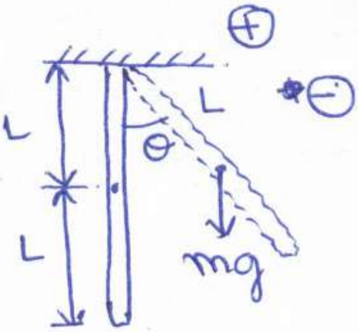
Stability of Equilibrium Configuration

Stable Equilibrium

- The tendency to return to its original posn. when a small displacement is given to the system
- In this case the potential energy of the system is at its minimum.







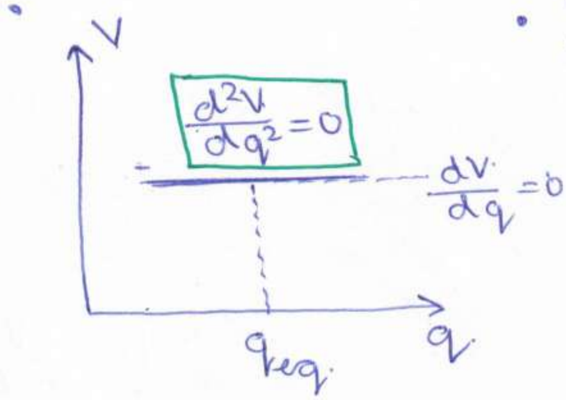
$$V = -MgL\cos\theta$$

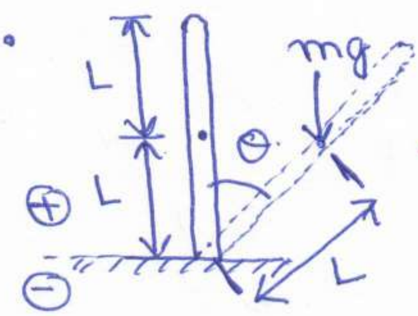
$$\Rightarrow \frac{dV}{d\theta} = MgL\sin\theta$$

$$\Rightarrow \frac{d^2V}{d\theta^2} = MgL\cos\theta$$

Neutral Equilibrium

- The system stays in eqlbm., when the system is given a small displace-ment away from its original posn.
- In this case, the potential energy of the system is constant.





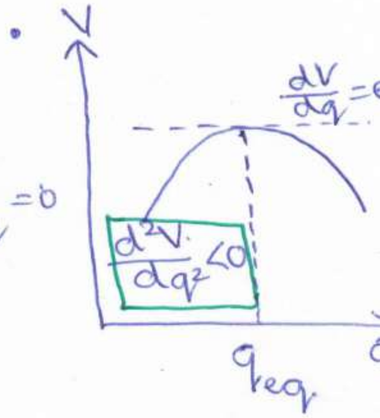
$$V = MgL\cos\theta$$

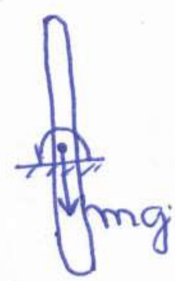
$$\frac{dV}{d\theta} = -MgL\sin\theta$$

$$\Rightarrow \frac{d^2V}{d\theta^2} = -MgL\cos\theta$$

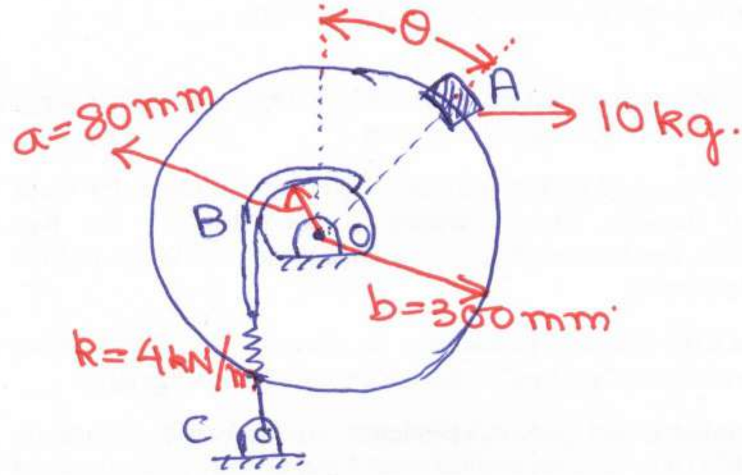
Unstable Equilibr-ium

- Tendency to displaced furth away from its original eqlbm. posn, when give a small displaceme
- In this case, the potential energy of the system is maximum.





Eg: Knowing that the spring BC is unstretched when $\theta = 0$; determine the position or positions of equilibrium and state whether the equilibrium is stable, unstable or neutral.

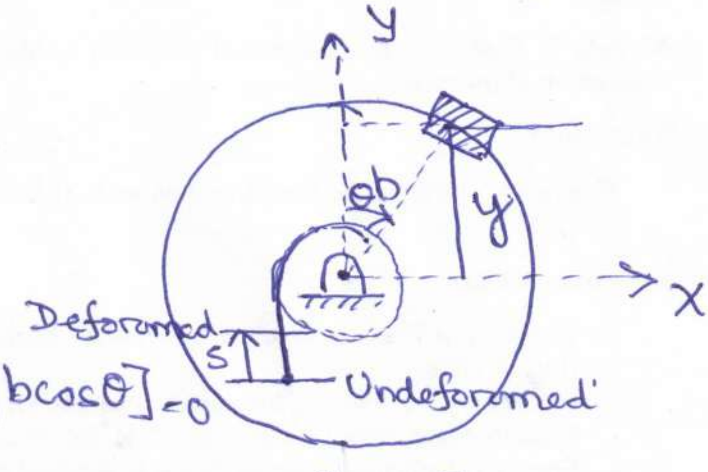


Solution:

Considering the total potential energy of the system:-

$V = V_g + V_e$

$\Rightarrow V = \frac{1}{2}ks^2 + mgb\cos\theta$



$S = a\theta$

$y = b\cos\theta$

Now, $\frac{dV}{d\theta} = 0$

$\Rightarrow \frac{d}{d\theta}\left[\frac{1}{2}ka^2\theta^2 + mgb\cos\theta\right] = 0$

$\Rightarrow ka^2\theta - mgb\sin\theta = 0$

$\Rightarrow ka^2\theta - mgb\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = 0$

Neglecting 5th order terms-

$ka^2\theta = mgb\,\theta\left(1 - \frac{\theta^2}{6}\right)$

$\Rightarrow \boxed{\theta = 0^\circ}$ and $\frac{ka^2}{mgb} = 1 - \frac{\theta^2}{6}$

$\Rightarrow \theta^2 = \left(1 - \frac{ka^2}{mgb}\right)6$

$\Rightarrow \theta = 0.886\, rad = \underline{50.76^\circ}$

Now $\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$.

- Substituting $\theta = 0°$,

$$\frac{d^2V}{d\theta^2} = (4000)(0.08)^2 - (10)(9.81)(0.3) = -3.83$$

$\Rightarrow$ Unstable at $\theta = 0°$

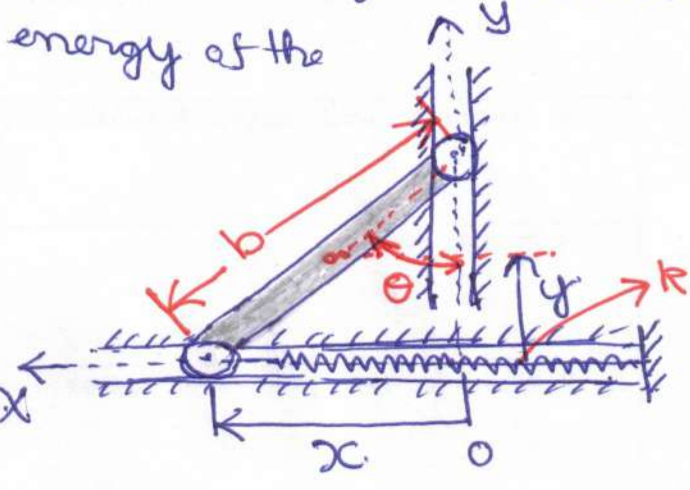
- Substituting $\theta = 50.76°$

$$\frac{d^2V}{d\theta^2} = (4000)(0.08)^2 - (10)(9.81)(0.3)\cos(50.76)$$

$= 5.03 \Rightarrow$ Stable at $\theta = 50.76°$

Eg. The ends of the uniform bar of mass 'm' slides freely in the horizontal and vertical guides. Examine the stability conditions for the positions of equilibrium. The spring of stiffness 'k' is undeformed when $x=0$.

<ins>Solution</ins>: Total potential energy of the system:-



$V = V_e + V_g$

$= \frac{1}{2}kx^2 + mgy$

$\Rightarrow V = \frac{1}{2}k(b\sin\theta)^2 + mg\frac{b}{2}\cos\theta$

Equilibrium occurs when $\frac{dV}{d\theta} = 0$

$\Rightarrow kb^2\sin\theta\cos\theta - \frac{mgb}{2}\sin\theta = 0$

$\Rightarrow \sin\theta\left(kb^2\cos\theta - \frac{mgb}{2}\right) = 0$

$\Rightarrow \underline{\sin\theta = 0}$ | $\cos\theta = \frac{mgb}{2kb^2}$

$\Rightarrow \underline{\cos\theta = \frac{mg}{2kb}}$

Now, $\frac{d^2V}{d\theta^2} = kb^2(\cos^2\theta - \sin^2\theta) - \frac{mgb}{2}\cos\theta.$

$\Rightarrow \frac{d^2V}{d\theta^2} = kb^2(2\cos^2\theta - 1) - \frac{mgb}{2}\cos\theta.$

• Case I: For $\theta = 0$

$\Rightarrow kb^2 - \frac{mgb}{2} = \frac{d^2V}{d\theta^2}.$

$\Rightarrow kb^2\left(1 - \frac{mg}{2kb}\right) = \frac{d^2V}{d\theta^2}$

$\Rightarrow \frac{d^2V}{d\theta^2} =$ +ve (stable), if $k > \frac{mg}{2b}$

$=$ −ve (unstable), if $k < \frac{mg}{2b}$.

• Case-II: For $\cos\theta = \frac{mg}{2kb}$.

$$\frac{d^2V}{d\theta^2} = kb^2\left[2\left(\frac{mg}{2kb}\right)^2 - 1\right] - \frac{mgb}{2}\left(\frac{mg}{2kb}\right)$$

$$= kb^2\left[\left(\frac{mg}{2kb}\right)^2 - 1\right]$$

As $\cos\theta < 1$, the solution is limited to the case, when $k > \frac{mg}{2b}$, which makes $\frac{d^2V}{d\theta^2} < 0$

$\Rightarrow$ Unstable configuration.