Classification of Constraints

- Holonomic : Conditions of constraint can be expressed as equations connecting the coordinates.

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, t) = 0$$

- Non-Holonomic : Constraints expressed as inequalities.
- Rheonomous : If the equation of the constraint contains time explicitly.
- Scleronomous : Not dependent on time explicitly.

Non holonomic example : Particles confined to move inside a container.

Difficulty with constraints

- The coordinates are no longer all independent, being connected via the constraint equations.
- Imposing such constraint equations on a system are a way of stating the existence of forces in the system the form of which cannot be directly specified. Rather the effects of these constraint forces on the motion of the system are indicated via such constraint equations.
- Forces of constraints are unknown, hence the need to formulate the mechanics (or equations of motion) such that the constraint forces disappear.
- Forces of constraints: Forces that restrict the movement of an object on a given surface for example!
- Constraint forces determine the object's displacement in the system, limiting it within a range. It eliminates all displacements in that direction, and hence any work done by such constraint forces is zero.
- Except, for one kind of constraint force. Can you guess?

Usually forces of constraints do NO work!

What if we now work with a formulation where the constraint equations may be utilized, so that our equations of motion need not depend on the forces of constraint(s)?

Let us only work with holonomic constraints for which the equation of constraint may be written as,

$$f(\vec{r}_1, \vec{r}_2, \ldots, t) = 0$$

In case of holonomic constraints, we can now introduce "generalized coordinates"

A system of N particles that are free of constraints can be expressed in terms of 3N Cartesian coordinates.

Now, if there exist 'k' number of constraint equations, then we may use these to eliminate 'k' of the 3N coordinates.

We are left with (3N-k) 'degrees of freedom'. $\Rightarrow$ Independent Coordinates

OR 'GENERALISED COORDINATES'

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots, q_{3N-k}, t)$$

$$\vdots$$

$$\vec{r}_N = \vec{r}_N(q_1, q_2, \ldots, q_{3N-k}, t)$$

Constraints :

- 3N DoF's for a system of N free, independent particles in 3D space.

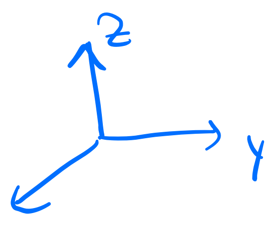
- No. of DoF's are reduced by certain constraints :

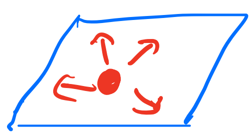
$$\boxed{\text{No. of DoF's} = \underbrace{3N}_{\text{N particles in 3D}} - \underbrace{k}_{\text{No. of constraints}}}$$

- Stick to holonomic constraints.

Examples :

(i) particle confined in 2d plane





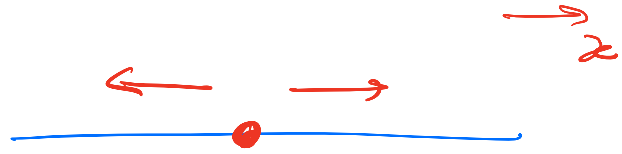
Constraint Equation

$z = 0$

$3 \times 1 - 1 = 2$ DoF

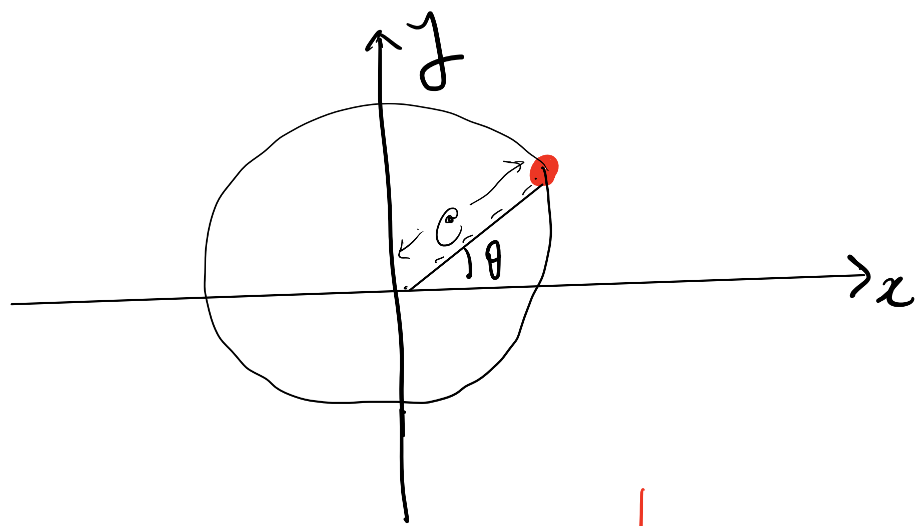
$\therefore q = \{x, y\}$ or $\{r, \theta\}$

(ii) particle in 1D :



Constraint equations : $z=0$, $y=0$

(iii) Circular Motion :



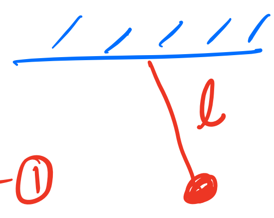
Constraint equation :
x, y are connected

$x^2 + y^2 = c^2$ (1)
$z = 0$ (2)
Motion in x-y plane

No. DOF : $3 \times 1 - 2 = 1$ DoF ; $q = \{\theta\}$

$x = c\cos\theta$, $y = c\sin\theta$.

Homework : Planar Pendulum :



Constraint Equations : $x^2+y^2=L^2$ —(1)
$z=0$ —(2)

FIND NUMBER OF DOF.

Generalised Coordinates

- Any set of independent coordinates, $\{q_i\}$, which can be used to specify the state of a system.
- Choice is NOT unique.
- 'n' generalised coordinates for 'n' DoF.
- Configuration space of a system is spanned by $\{q_i\}$.

Transformation Equations

Let, $\vec{r}_l = x_l \hat{i} + y_l \hat{j} + z_l \hat{k}$ be the position vector of the nth particle with respect to the Cartesian coordinate system.

$$x_l = x_l(q_1, q_2, \cdots, q_n, t)$$
$$y_l = y_l(q_1, q_2, \cdots, q_n, t)$$
$$z_l = z_l(q_1, q_2, \cdots, q_n, t)$$

where 't' is the time and $\{q_i\}$ are the 'n' generalised coordinates.

- These equations or functions are supposed to have continuous derivatives.
- The $\{q_i\}$'s are all independent of each other.

Q. DO THE E-L EQUATIONS REMAIN SAME WHEN WE GO TO ANOTHER COORDINATE SYSTEM?

Coordinate Invariance of the Euler-Lagrange equations:

Consider the set of coordinates,

$$\{x_i\} : (x_1, x_2, \ldots, x_n)$$

for example in case of a single particle moving in 3D, we have

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \quad (1 \leq i \leq n)$$

Considering a new set of variables,

$$\{q_i\} : (q_1, q_2, \ldots, q_l)$$

Show that the same equation can be applied,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m} \quad 1 \leq m \leq l$$

Of course we have to now write $L = T - V$ in terms of q_i's, $\dot{q}_i$'s.

[The coordinate transformation equations do not involve velocities].

Proof:

$$\frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_m}$$

Since,

$$x_i = x_i(q_1, q_2, \dots, q_l; t)$$

$$\dot{x}_i = \sum_{m=1}^{l} \frac{\partial x_i}{\partial q_m} \frac{dq_m}{dt} + \frac{\partial x_i}{\partial t}$$

$$= \sum_{m=1}^{l} \frac{\partial x_i}{\partial q_m} \dot{q}_m + \frac{\partial x_i}{\partial t}$$

$$\therefore \frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m}$$

Substituting in

$$\frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_m}$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} \frac{\partial x_i}{\partial q_m}$$

E-L eqn : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$

Show that : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \frac{\partial x_i}{\partial q_m} + \left(\frac{\partial L}{\partial \dot{x}_i}\right) \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right)\right]$$

$$= \sum_{i=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \frac{\partial x_i}{\partial q_m} + \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_m}\right)$$

Switch posn of $\frac{d}{dt}$ & $\frac{\partial}{\partial q_m}$

(Use E-L eqn in $\{x_i\}$ coordinates) (Allowed, can be proved)

$$= \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial q_m} + \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_m} \right)$$

$$= \frac{\partial L}{\partial q_m}$$

[So we get for $\{q_i\}$ coordinates,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$]

where we have used,

$$L = L\left(\{x_i\}, \{\dot{x}_i\}; t\right)$$

Ignore time dependence for now

Also,

$$L = L\left(\{q_m\}, \{\dot{q}_m\}\right)$$

But,

$$x_i = x_i\left(\{q_m\}\right)$$

$$\dot{x}_i = \dot{x}_i\left(\{q_m\}\right)$$

$$\frac{\partial L}{\partial q_m} = \sum_{i=1}^{n} \frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial q_m} + \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_m}$$

Define :

Generalized momentum :

(or sometimes denoted as P_i) $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ [Also known as Canonical momentum] or Conjugate momentum

** If 'T' has no explicit dependence on q_i, then :

Generalized force : ** $F_i = \dfrac{\partial L}{\partial q_i}$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \Rightarrow \boxed{F_i = \dot{p}_i} \Leftarrow \text{NL II in terms of generalised coordinates}$$

Euler-Lagrange Equation.

But for generalised force & momentum (holds for any coordinate system)

$p_i \equiv$ Generalized momentum
OR
Conjugate momentum to coordinate q_i

Concept of Cyclic Coordinates.

If a particular coordinate does not appear in the Lagrangian it is called CYCLIC or IGNORABLE coordinates.

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad \text{E-L } \underline{\text{EQN}}$$

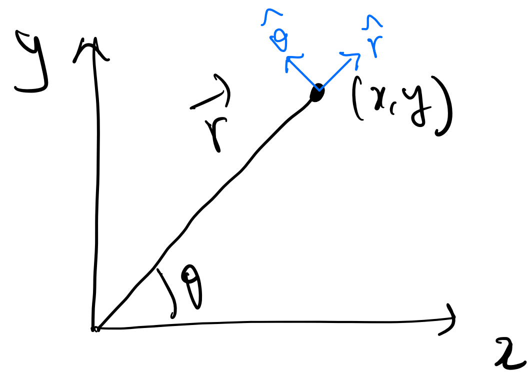
if q_i is not explicitly present in 'L', then $\frac{\partial L}{\partial q_i} = 0$

$$\dot{p}_i = 0 \Rightarrow p_i \text{ is CONSERVED}$$

$\Rightarrow$ SYMMETRY

Example: Euler-Lagrange equation in polar coordinates.

Generalised coordinates:
$q_1 = r$
$q_2 = \theta$



$\vec{r} = r\hat{r}$

Recall:
$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\dot{\vec{r}}^{\,2} = \dot{\vec{r}}\cdot\dot{\vec{r}} = \dot{r}^2 + r^2\dot{\theta}^2$$

$\hat{r}\cdot\hat{r} = 1$
$\hat{\theta}\cdot\hat{\theta} = 1$
$\hat{r}\cdot\hat{\theta} = 0$

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^{\,2} - V(r,\theta)$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - V(r,\theta)$$

(Check next page for derivation)

$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) - V(r,\theta)$$

Generalised momentum: $\frac{\partial L}{\partial \dot{q}_i} = P_i \Rightarrow P_r = \frac{\partial L}{\partial \dot{r}}$

$P_\theta = \frac{\partial L}{\partial \dot{\theta}}$

$\Rightarrow P_r = m\dot{r}$; $P_\theta = mr^2\dot{\theta}$

Euler-Lagrange equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$

$\Rightarrow m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = -\frac{\partial V}{\partial \theta}$

$$\Rightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\frac{\partial V}{\partial \theta}$$

$$\Rightarrow \quad -\frac{1}{r}\frac{\partial V}{\partial \theta} = mr\ddot{\theta} + 2m\dot{r}\dot{\theta}$$

In polar coordinates, the equations of motion are:

$$-\frac{\partial V}{\partial r} = m\ddot{r} - mr\dot{\theta}^2$$

$$-\frac{1}{r}\frac{\partial V}{\partial \theta} = mr\ddot{\theta} + 2m\dot{r}\dot{\theta}$$

$$\dot{x}^2 + \dot{y}^2$$

$$= \left[\frac{d}{dt}(r\cos\theta)\right]^2 + \left[\frac{d}{dt}(r\sin\theta)\right]^2$$

$$= (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2 + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2$$

$$= \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 - \cancel{2r\dot{r}\dot{\theta}\cos\theta\sin\theta}$$

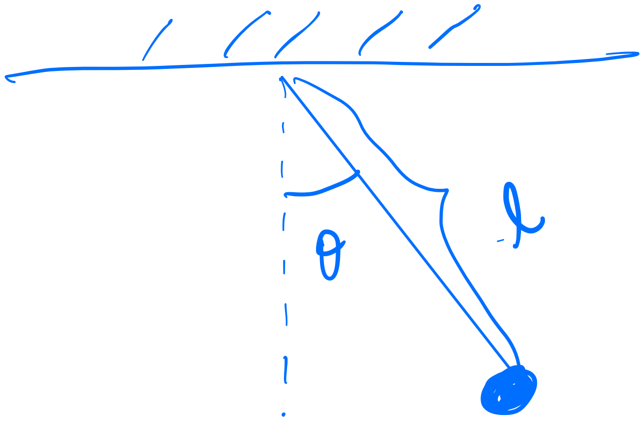
$$+ \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

So, $\quad KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

HOMEWORK : Use the above strategy to find out the equations of motion of a simple pendulum as shown below :



The length of the string is 'l'.
At any instant of time, the angle made by this string is θ.