A QUESTION FOR MID SEMESTER EXAMINATION

Problem 1. Let $f(x) := \tan^{-1}(2x)$. Solve the following, showing all the steps.
(a). Find the Maclaurin series for f up to and including the term in x^3. **[4]**
(b). Use your answer to part (a), together with the value $x = 0.5$, to obtain an estimate for π. Show that this estimate is correct to only 1 significant figure. **[2]**
Solution (a): The Maclaurin series for a function f is given by

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots . \qquad [\mathbf{2}]$$

Given $f(x) = \tan^{-1}(2x)$. Then the first few derivatives of f are:

$$\begin{aligned} f'(x) &= \frac{2}{1+4x^2}, \\ f''(x) &= \frac{-16x}{(1+4x^2)^2}, \\ f'''(x) &= \frac{16(12x^2-1)}{(1+4x^2)^3}. \end{aligned}$$

Substituting $x = 0$, we obtain $f(0) = 0, f'(0) = 2, f''(0) = 0$, and $f'''(0) = -16$. **[1]**
Hence, the desired Maclaurin series of f is

$$\tan^{-1}(2x) = 2x - \frac{8}{3}x^3 + \cdots . \qquad [\mathbf{1}] \tag{0.1}$$

(b): Substituting $x = 0.5$ in (0.1), we get

$$\begin{aligned} &\tan^{-1}(2 \times 0.5) \approx 2 \times 0.5 - \frac{8}{3}(0.5)^3. \\ \Longrightarrow\ &\tan^{-1}(1) = \frac{\pi}{4} \approx 1 - \frac{1}{3}. \\ \Longrightarrow\ &\pi \approx 2.6667 \qquad [\mathbf{1}] \end{aligned}$$

and the actual value of π is $3.1416\ldots$. Now, rounding both values of π to one significant digit, we get 3 in both the cases, hence the claim is established. **[1]**

Aliter:
(a) We know that the Maclaurin series of $\tan^{-1}(x)$ is given by

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \cdots . \qquad [\mathbf{2}] \tag{0.2}$$

Replacing x by $2x$ in (0.2), we obtain the desired series

$$\tan^{-1}(2x) = 2x - \frac{8}{3}x^3 + \cdots . \tag{0.3}$$

[**2**]

(b) Same as above.

Question 2. Find the first four non-zero terms of the Taylor series expansion of

$$f(x) = \sqrt{3 + x^2}$$

about the point $x = -1$.

Solution: The Taylor series of a function $f(x)$ centered at $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)$$ 2

Hence

$$f(x) = (3 + x^2)^{1/2}, \quad a = -1$$

$$f'(x) = \frac{x}{\sqrt{3 + x^2}}$$

$$f''(x) = \frac{3}{(3 + x^2)^{3/2}}$$ 2

$$f^{(3)}(x) = \frac{-9x}{(3 + x^2)^{5/2}}$$

At $x = -1$

$$f(-1) = \sqrt{3+1} = \sqrt{4} = 2$$

$$f'(-1) = -1/2, \quad f''(-1) = 3/8$$ 2

$$f^{(3)}(-1) = 9/32,$$

Hence

$$
\begin{aligned}
f(x) &= f(-1) + (x+1)f'(-1) + \frac{(x+1)^2}{2!}f''(-1) + \frac{(x+1)^3}{3!}f^{(3)}(-1) \\
&\quad + \frac{(x+1)^4}{4!}f^{(4)}(-1) + \cdots \\
&= 2 - 1/2\,(x+1) + 3/16\,(x+1)^2 + 3/64\,(x+1)^3
\end{aligned}
$$

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The first four non-zero terms of the Taylor series of $\sqrt{3+x^2}$ at $x = -1$ are

$$f(x) = 2 - 1/2\,(x+1) + 3/16\,(x+1)^2 + 3/64\,(x+1)^3$$

Answer 3

$$f(x,y) = (2x+3y)^{(2x+3y)} = e^{(2x+3y)\ln(2x+3y)}$$

$$f(1,1) = 5^5 = A_1 \text{ (say)}$$

$$f_x = e^{(2x+3y)\ln(2x+3y)} \left(2\ln(2x+3y) + \frac{(2x+3y)}{(2x+3y)}\cdot 2\right)$$

$$f_x(1,1) = 5^5(2\ln 5 + 2) = A_2 \text{ (say)}$$

$$f_y(1,1) = 5^5(3\ln 5 + 3) = A_3 \text{ (say)} \quad 2$$

$$f_{xx} = e^{(2x+3y)\ln(2x+3y)} (2\ln(2x+3y)+2)^2 + e^{(2x+3y)\ln(2x+3y)}\left(\frac{4}{2x+3y}\right)$$

$$f_{xx}(1,1) = 5^5\left((2\ln 5+2)^2 + \frac{4}{5}\right) = A_4 \text{ (say)}$$

$$f_{yy}(1,1) = 5^5\left((3\ln 5+3)^2 + \frac{9}{5}\right) = A_5 \text{ (say)}$$

$$f_{xy}(1,1) = 5^5\left((2\ln 5+2)(3\ln 5+3) + \frac{2\cdot 3}{5}\right) = A_6 \text{ (say)} \quad 3$$

Second degree Taylor's Expansion of $f(x,y)$ is

$$= A_1 + \left((x-1)A_2 + (y-1)A_3\right) + \frac{1}{2!}\left((x-1)^2 A_4 \right. \quad 1$$

4 $$\left. + 2(x-1)(y-1)A_6 + (y-1)^2 A_5\right).$$

Now Maximum Error $= \frac{B}{3!}(\delta_1+\delta_2)^3$ 1

$= \frac{B}{3!} 3^3$ where $B = \max(|f_{xxx}|, |f_{xxy}|, |f_{yyx}|, |f_{yyy}|)$.
$= |f_{yyy}|$

$$|f_{yyy}| = 13^{13}\left((3\ln 13+3)^3 + \frac{27}{13}\left(3\ln 13+3\right) + \frac{27}{13}\right)$$

1

Que: $w = \sqrt{x^2+y^2+z^2}$ $\quad x = u\cos v$, $y = u\sin v$ $\quad z = uv$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{+u\cos v}{\sqrt{u^2+u^2v^2}} = \frac{+\cos v}{\sqrt{1+v^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{\sin v}{\sqrt{1+v^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{v}{\sqrt{1+v^2}}$$

$$\frac{\partial w}{\partial u} = \frac{\cos v}{\sqrt{1+v^2}}\cdot\cos v + \frac{\sin v}{\sqrt{1+v^2}}\cdot\sin v + \frac{v}{\sqrt{1+v^2}}\cdot v$$ (2)

$$= \frac{1}{\sqrt{1+v^2}} + \frac{v^2}{\sqrt{1+v^2}} = \sqrt{1+v^2}$$

$$\frac{\partial w}{\partial v} = \frac{\cos v}{\sqrt{1+v^2}}(-u\sin v) + \frac{\sin v}{\sqrt{1+v^2}}(u\cos v) + \frac{v}{\sqrt{1+v^2}}\cdot u \quad \text{—— (2)}$$

$$= \frac{uv}{\sqrt{1+v^2}}$$

$$\Rightarrow u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = u\sqrt{1+v^2} - \frac{uv^2}{\sqrt{1+v^2}} = u\frac{(1+v^2-v^2)}{\sqrt{1+v^2}} = \frac{u}{\sqrt{1+v^2}} \quad \text{—— (1)}$$

(b). $$\frac{\partial(w,z)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ v & u \end{pmatrix} \quad \text{—— (1)}$$

$$= u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}} \quad (\text{by (a)})$$

5 (a) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

$\underline{\text{sol}^n}$ let $\epsilon > 0$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}}$$

$$(x-y)^2 \geq 0$$
$$x^2+y^2-2xy \geq 0$$
$$\frac{x^2+y^2}{2} \geq xy$$

$$\leq \frac{1}{2}\frac{(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$= \frac{\sqrt{x^2+y^2}}{2} < \frac{\delta}{2}$$

So if we choose $\frac{\delta}{2} \leq \epsilon$ then above inequality will be in terms of limit def^n

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \frac{\delta}{2} \leq \epsilon \qquad \left\{ \delta = 2\epsilon \right.$$

then it will become

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon \qquad \forall\ (x,y)\quad 0 < \sqrt{x^2+y^2} < \delta$$

hence 0 is the limit

5 (b)

$$f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+3y^4} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

<u>Sol^n</u> for path $y = mx$

$$f(x,y) = f(x, mx) = \frac{4m^2 x}{1 + 3m^4 x^2} \rightarrow 0 \text{ as } x \rightarrow 0$$

for path $x = y^2$

$$f(x,y) = f(y^2, y) = \frac{4y^4}{y^4 + 3y^4} = 1$$

$\therefore$ different limits for different paths

function is not continuous at $(0,0)$.

Ques 6 . $f(x,y) = \sqrt{|xy|}$ at $(0,0)$

Soln $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \; (= \alpha$

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$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0 \; (= \beta$

If function is differentiable at $(0,0)$ then

$$f(h,k) - f(0,0) = 0h + 0k + h\phi + k\psi$$

where ϕ, ψ are funct of h & k and $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

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putting $h = \rho\cos\theta$, $k = \rho\sin\theta$ and dividing by ρ we get

$$|\cos\theta \sin\theta|^{1/2} = \phi\cos\theta + \psi\sin\theta$$

for arbitrary θ, $\rho \to 0 \Rightarrow (h,k) \to (0,0)$

taking limit as $\rho \to 0$ we get

$$|\cos\theta \sin\theta|^{1/2} = 0$$

which is not possible for all arbitrary θ .

$\therefore$ f is not diffn at $(0,0)$

2 marks for f_x, f_y

OR

If funct is diff^n at $(0,0)$ then by def^n

$$f(h,k) - f(0,0) = \alpha h + \beta k + \sqrt{h^2+k^2}\ \phi(h,k)$$

then $\phi \to 0$ as $(h,k) \to (0,0)$

$$\Rightarrow \sqrt{|hk|} - 0 = 0 + 0 + \sqrt{h^2+k^2}\ \phi(h,k)$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$$

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for path $k=0$

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} = 0$$

for path $h=k$

$$\lim_{h\to 0} \frac{h}{h\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\therefore\ \phi(h,k) \not\to 0$ as $(h,k) \to (0,0)$

$\therefore\ f(x,y)$ is not diff^n at $(0,0)$

MARKING SCHEME OF MID SEM EXAM QUESTION NO 7

SNEHAJIT MISRA

Problem on Maxima and Minima

Marks 3+5=8

Question: Analyze the following functions for local maxima, local minima and saddle points :

(1) $f(x, y) = x^2 + 2y^2 - x$ for $(x, y) \in \mathbb{R}^2$.
(2) $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ for $(x, y) \in \mathbb{R}^2$.

Answer:

(1) **Step 1: Finding the critical points. (Marks : 1.5)**

We have

$$f(x, y) = x^2 + 2y^2 - x.$$

First, compute the first-order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = 2x - 1, \qquad f_y = \frac{\partial f}{\partial y} = 4y.$$

Setting them equal to zero for critical points:

$$2x - 1 = 0 \quad \Rightarrow \quad x = \tfrac{1}{2}, \qquad 4y = 0 \quad \Rightarrow \quad y = 0.$$

Thus, the only critical point is

$$\left(\tfrac{1}{2}, 0\right).$$

Step 2 : Analyzing the critical points using second order partial derivative test. (1.5 marks)

Now, we compute the second order partial derivatives:

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = f_{yx} = 0.$$

The Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2)(4) - 0^2 = 8 > 0.$$

Since $f_{xx} = 2 > 0$, the point $\left(\tfrac{1}{2}, 0\right)$ is a **local minimum**.

(2) **Step 1: Finding the critical points. (Marks : 1.5 marks)**

Given

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$

First-order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3 = 3(x^2 - 1), \qquad f_y = \frac{\partial f}{\partial y} = 3y^2 - 12 = 3(y^2 - 4).$$

Set $f_x = f_y = 0$ to find critical points:

$$x^2 = 1 \Rightarrow x = \pm 1, \qquad y^2 = 4 \Rightarrow y = \pm 2.$$

Thus the critical points are

$$(1,2), \quad (1,-2), \quad (-1,2), \quad (-1,-2).$$

Step 2 : Analyzing the critical points using second order partial derivative test. (3.5 marks)

Second-order partial derivatives and Hessian:

$$f_{xx} = 6x, \qquad f_{yy} = 6y, \qquad f_{xy} = f_{yx} = 0,$$

so the Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 36xy.$$

Now we classify each critical point using D and f_{xx}.

- At $(1,2)$: $f_{xx} = 6 > 0,\ D = 36 \cdot 1 \cdot 2 = 72 > 0.$
  Hence $(1,2)$ is a **local minimum**.

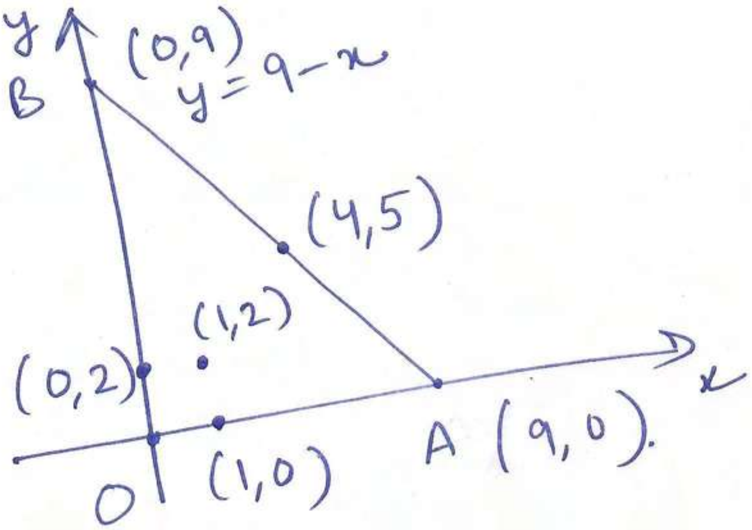
- At $(1,-2)$: $f_{xx} = 6 > 0,\ D = 36 \cdot 1 \cdot (-2) = -72 < 0.$
  Hence $(1,-2)$ is a **saddle point**.

- At $(-1,2)$: $f_{xx} = 6(-1) = -6 < 0,\ D = 36 \cdot (-1) \cdot 2 = -72 < 0.$
  Hence $(-1,2)$ is a **saddle point**.

- At $(-1,-2)$: $f_{xx} = -6 < 0,\ D = 36 \cdot (-1) \cdot (-2) = 72 > 0.$
  Since $D > 0$ and $f_{xx} < 0$, $(-1,-2)$ is a **local maximum**.

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

Sol^n The only places where f can assume these extreme values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary



— ①

a) Interior points

$f_x = 2 - 2x = 0$, $f_y = 4 - 2y = 0$

$\Rightarrow$ $(x,y) = (1,2)$ is a critical point.

Also $f(1,2) = 7$ — ②

b) Boundary Points

Line Segment OA

Here $y = 0$

$\therefore$ Let $g(x) = f(x,0) = 2 + 2x - x^2$, $0 \le x \le 9$

Its extreme value may occur at point where $g'(x) = 0$

$\therefore$ $g'(x) = 2 - 2x = 0 \Rightarrow x = 1$

$\therefore$ $g(1) = f(1,0) = 3$ — ③

Line Segment OB

Here $x = 0$

Let $h(y) = f(0,y) = 2 + 4y - y^2$

For extreme points $h'(y) = 0$

$\Rightarrow 4 - 2y = 0$

$\Rightarrow y = 2$

Now, $h(2) = f(0,2) = 6$. — 4.

Line Segment AB

Here $y = 9 - x$

$\therefore$ Let $k(x) = f(x, 9-x)$

$= 2 + 2x + 4(9-x) - x^2 - (9-x)^2$

$= -43 + 16x - 2x^2$

Now $k'(x) = 0$

$\Rightarrow 16 - 4x = 0 \Rightarrow x = 4$

Also $y = 9 - x = 5$

$\therefore$ ~~kt~~ The extremum point is $(4,5)$

& $f(4,5) = -11$

-5

Vertices

The extremum may also occur at the vertices

So, $f(0,0) = 2$

$f(9,0) = -61$

$f(0,9) = -43$

— (6)

Summary

∴ Absolute maximum and minimum out of the candidates : 7, 3, 6, −11, 2, −61, −43 are as follows :

Absolute Maxima is 7 at point (1,2)

Absolute Minima is −61 at point (9, 0).

— (8).