Lecture - 4, 5

Section - A - 18/08/2025, 20/08/2025

Section - B - 18/08/2025, 25/08/2025

Section - C - 14/08/2025, 21/08/2025

Section - D - 19/08/2025, 21/08/2025

Lecture Plan : ① The Lagrangian Method

② Generalised Coordinates

③ Euler - Lagrange EQN

Ref. : ① Classical Mechanics by, Goldstein, Poole, Safko (Third Edition or later)

② Theoretical Mechanics, Murray R. Spiegel

In this lecture, we will learn a whole new way of looking at physics problems. In the beginning it may look quite non-intuitive, but the equations that we shall use can be derived through some generalised concepts. WE WILL HOWEVER START AS A DEFINITION AND DERIVE STUFF ONLY WHEN IT IS ABSOLUTELY NECESSARY.

THIS NEW METHOD IS IN FACT FAR SUPERIOR THAN THE NEWTONIAN METHOD.

We will first present this new method by first stating the rules (without any justification). If time permits we will give this method a proper justification.

Instead of using the equation $\vec{F} = m\vec{a}$, we will use the Euler-Lagrange equation.

Consider a seemingly silly combination of kinetic energy (T) & potential energy (V) given by,

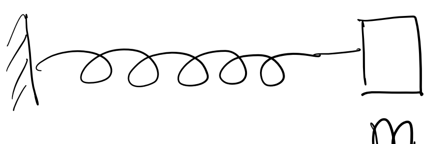
$$L = T - V$$

note the (-ve) sign

We call this the 'Lagrangian' of the system.

The Euler-Lagrange equation is given in terms of this quantity 'L' and a set of generalised coordinates in the system.

For example, in the problem of a mass at the end of a spring.



If 'x' is the displacement of the spring from the equilibrium position, then,

$$T = \frac{1}{2} m \dot{x}^2 \quad ; \quad V = \frac{1}{2} k x^2$$

So,

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Now write, (for each coordinate)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Euler-Lagrange equation (E-L)

$$\Rightarrow \frac{d}{dt}\left(m\dot{x}\right) + kx = 0$$

$$\Rightarrow \boxed{m\ddot{x} = -kx}$$

Equation of a spring for $F = ma$

In fact, if we now have an arbitrary potential, $V(x)$ such that,

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

Then by E-L equation, we have,

$$m\ddot{x} = \underbrace{-\frac{dV}{dx}}_{\text{force on the particle}}$$

Now let's look at more generalised setups where we may not work with Cartesian coordinates only.

START OF LECTURE - 5

In the following discussions we will introduce two new concepts:

1. Generalized Coordinates
2. Constraint forces

Ref. Goldstein Chapter 1

From the Newtonian approach of solving mechanics problems, we may have the notion that, all problems in mechanics are reduced to,

$$m_i \ddot{\vec{r}}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}$$

where,

$\vec{r}_i$: Coordinate of i-th particle

$\vec{F}_i^{(e)}$: External force on i-th particle

$\vec{F}_{ji}$: Internal force between the i-th and j-th particle

However our systems may be 'constrained' or their motions may be limited. At the same time we may not have any knowledge about the force of constraint.

In fact in most cases, one only knows the effect of these constraint forces with no knowledge of the force of constraint.

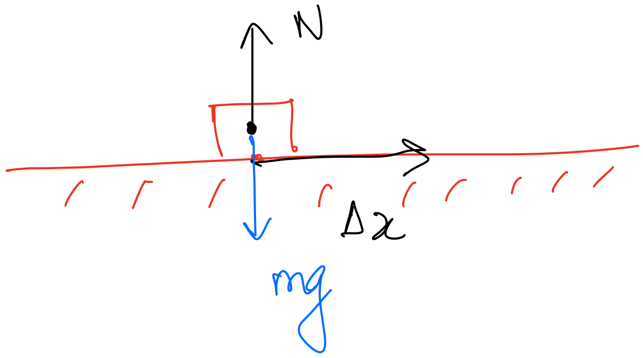
Examples of 'constrained' systems:

SEE NEXT PAGE

Examples :

QUESTION: WHAT ARE THE CONSTRAINT EQUATIONS?

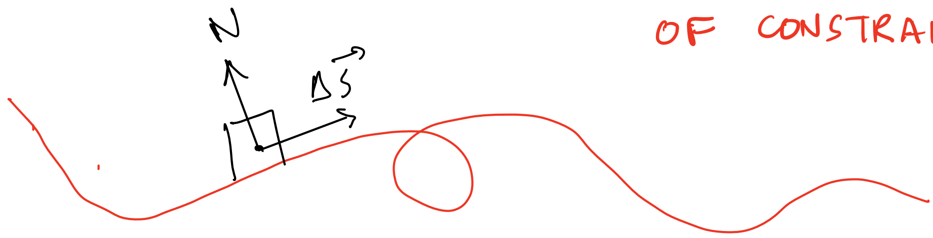
(1)



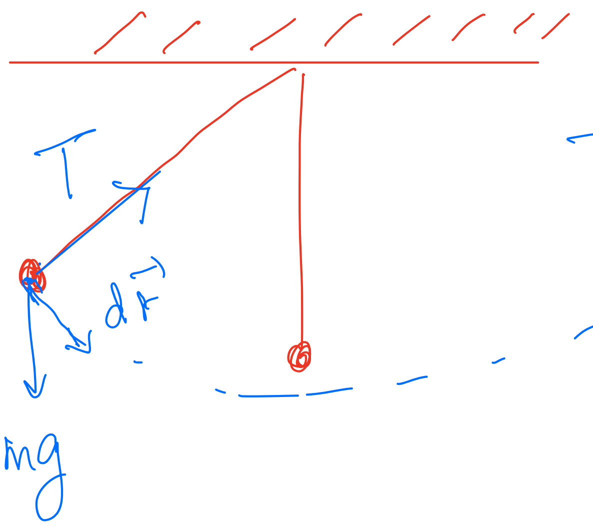
Normal 'contact' force is perpendicular to the displacement.

CONSTRAINT EQUATIONS DEPICT THE EFFECT OF CONSTRAINT FORCES

(2)



(3)



Tension in the string keeps the pendulum bob in the circular path.

(4) WHAT ABOUT A RIGID BODY?

In a rigid body, the distance between any two particles remains constant. This is maintained by internal forces.

Now, what about the work done by constraint forces?

In all the examples discussed, constraint forces do not do any work.

Let's consider the rigid body example:

if $\vec{F}_{ik}$ denote the force on the i-th particle due to the k-th, we have work done in a displacement of the ith - particle as,

$$W_i = \sum_k \vec{F}_{ik} \cdot d\vec{r}_i$$

Total work done for all particles:

$$W = \sum_i W_i = \sum_i \sum_k \vec{F}_{ik} \cdot d\vec{r}_i$$

or, $$W = \sum_k W_k = \sum_k \sum_i \vec{F}_{ki} \cdot d\vec{r}_k$$

But,

$$\vec{F}_{ik} = -\vec{F}_{ki} \quad \text{(By Third law)}$$

So,

$$W = \frac{1}{2}\sum_i \sum_k \vec{F}_{ik} \cdot (d\vec{r}_i - d\vec{r}_k)$$

Equation of constraint for a rigid body?

$$|(\vec{r}_i - \vec{r}_k)|^2 = \text{constant}$$

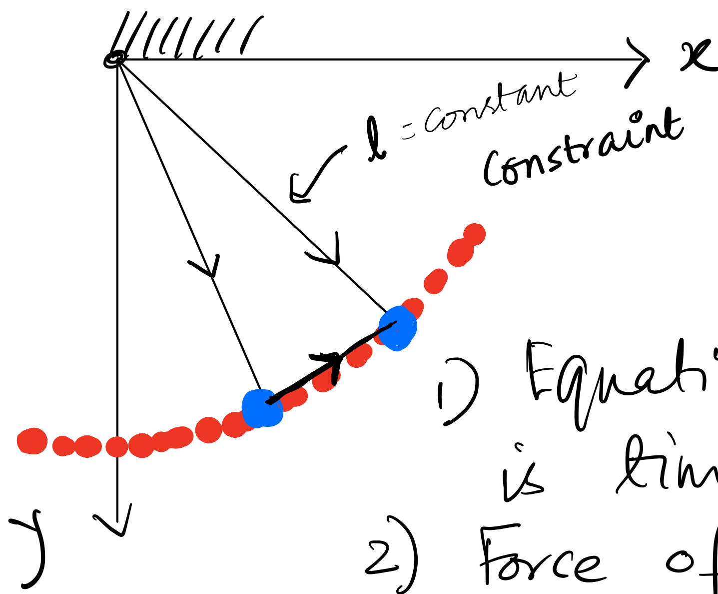
$$\Rightarrow \quad d|(\vec{r}_i - \vec{r}_k)|^2 = 0$$

$$\Rightarrow \quad 2(\vec{r}_i - \vec{r}_k)\cdot(d\vec{r}_i - d\vec{r}_k) = 0$$

Now, $\vec{F}_{ik}$ is directed along the vector $(\vec{r}_i - \vec{r}_k)$ so $\underline{\underline{W = 0}}$ in a rigid body due to the forces of constraints.

(5) WHAT ABOUT A PENDULUM WITH VARIABLE LENGTH?

Example:

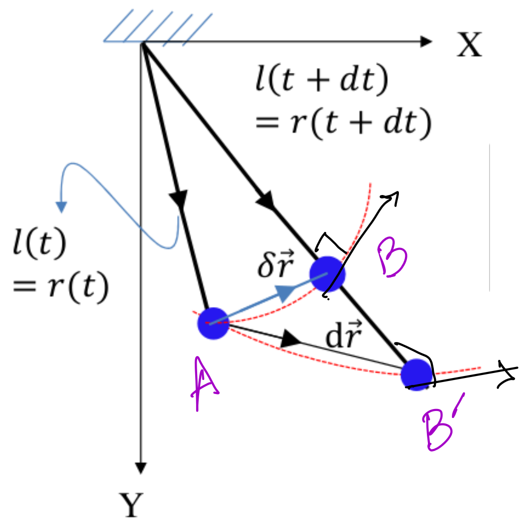


1) Equation of constraint is time independent.
2) Force of constraint does no work.

3) How do you classify this constraint?

Q. Classify according to the definitions discussed later.

What about this case?



Consider the following:

* Due to lengthening of pendulum string, the centre of the bob moves along AB′ instead of AB.