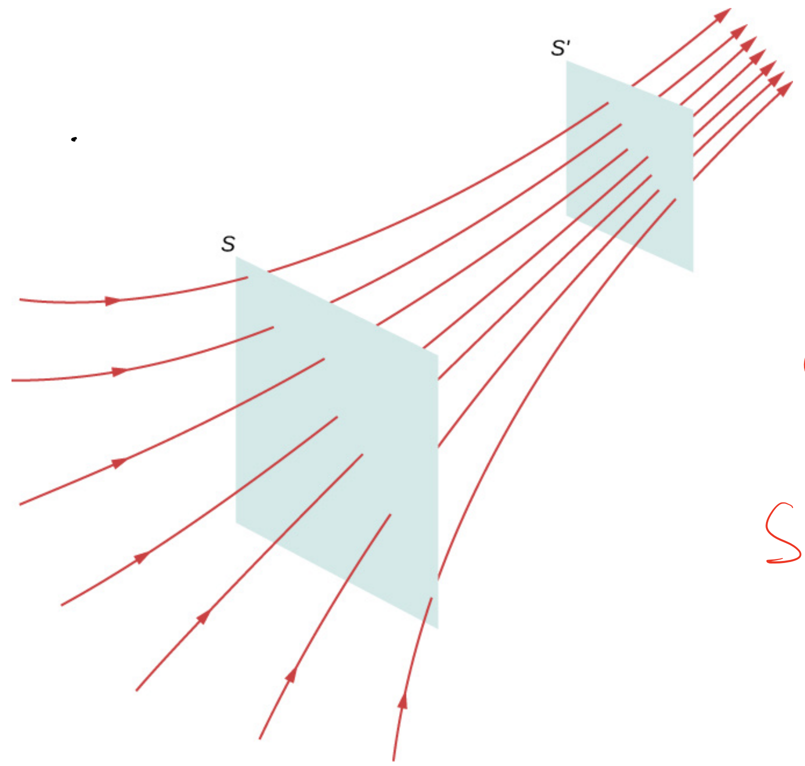
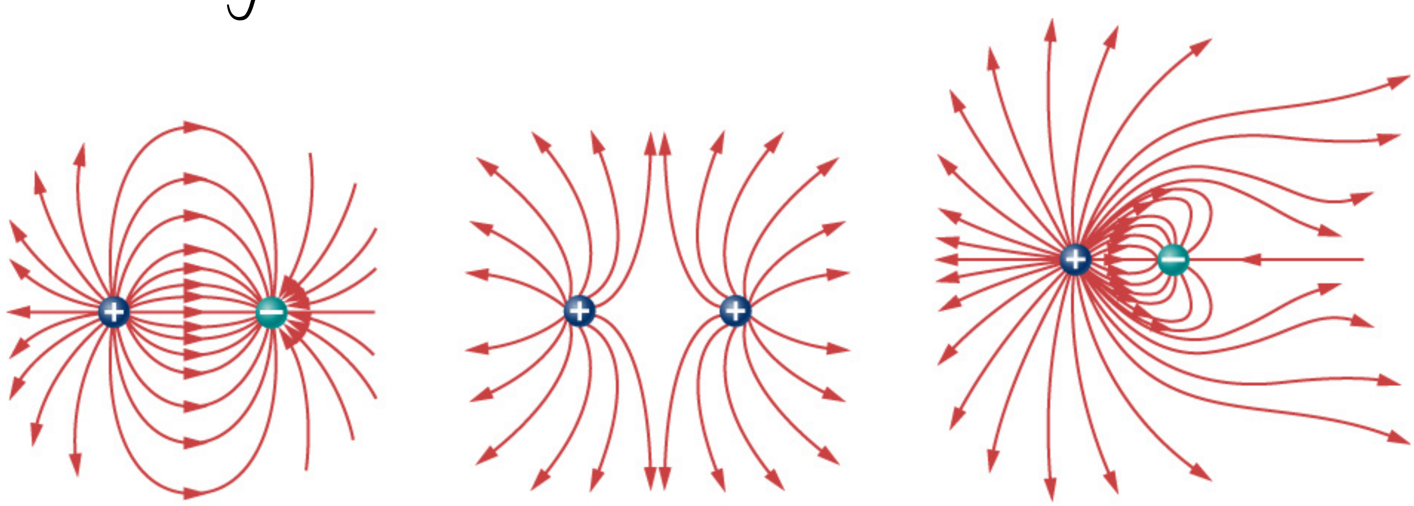
Electric field lines passing through imaginary areas. Since the number of lines passing through each area is the same, but the areas themselves are different, the field line density is different.



$$\oint_{\text{surface}} \vec{E} \cdot d\vec{a}$$

HW : Field lines of three groups of discrete charges are shown below. Can you tell from the diagram the relative magnitudes of the charge (larger, smaller, equal)?



(a) (b) (c)

Flux of $\vec{E}$ through a surface S:

$$\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$$

electric flux (Φ_E) $\equiv$ number of field lines penetrating surface.

There are two caveats you should keep in mind when you think of electric flux as the number of electric field lines penetrating a surface. The first is that field lines are only a convenient representation of the electric field, which is actually continuous in space. The number of field lines you

Flux is proportional to the no. of lines drawn.

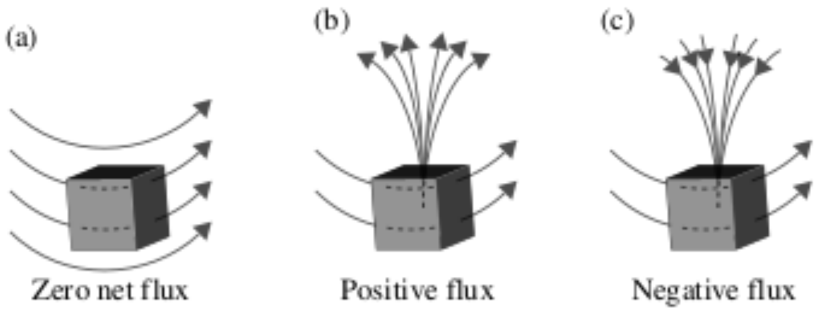


Figure 1.8 Flux lines penetrating closed surfaces.

choose to draw for a given field is up to you, so long as you maintain consistency between fields of different strengths – which means that fields that are twice as strong must be represented by twice as many field lines per unit area.

Some Useful Theorems :

The fundamental theorem for divergence :

Gauss' Theorem :

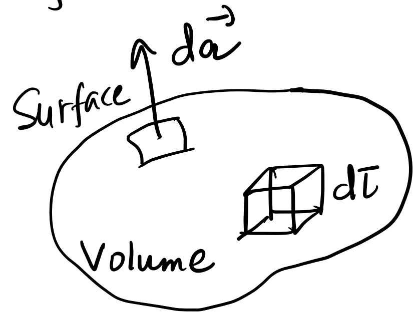
$$\int_{\text{Vol.}} (\vec{\nabla} \cdot \vec{E})\, d\tau = \oint_{\text{Surf.}} \vec{E} \cdot d\vec{a}$$

Flux of $\vec{E}$

Area element

Volume element

Integral of a derivative over a region (volume) is equal to the value of the function over the boundary (surface).

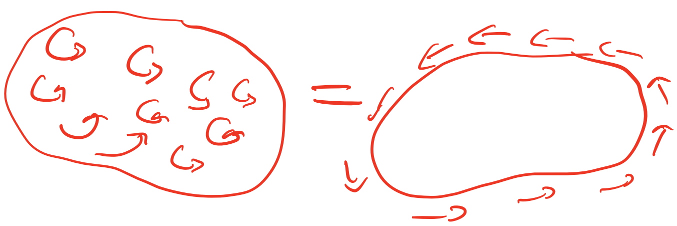


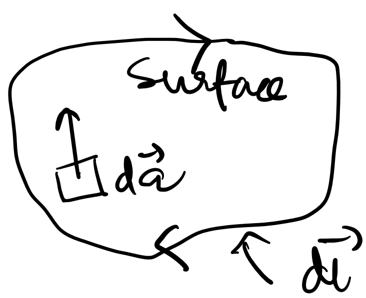
The fundamental theorem for Curl :

(Stokes' Theorem)

$$\int_{\text{Surf.}} (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint_{\text{Path}} \vec{E} \cdot d\vec{l}$$

RECALL: [Before Midsem] $\Downarrow$ Concept of scalar potential.





Coordinate Systems : i) Spherical, 2) Polar, 3) Cylindrical

HOMEWORK: [For Interested Students]

Q1. Check the divergence theorem using the function,

$$\vec{v} = y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + (2yz)\,\hat{z}$$

and the unit cube situated at the origin.

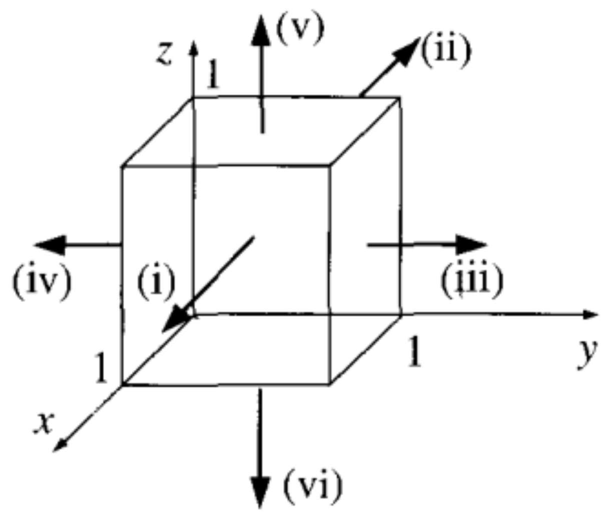


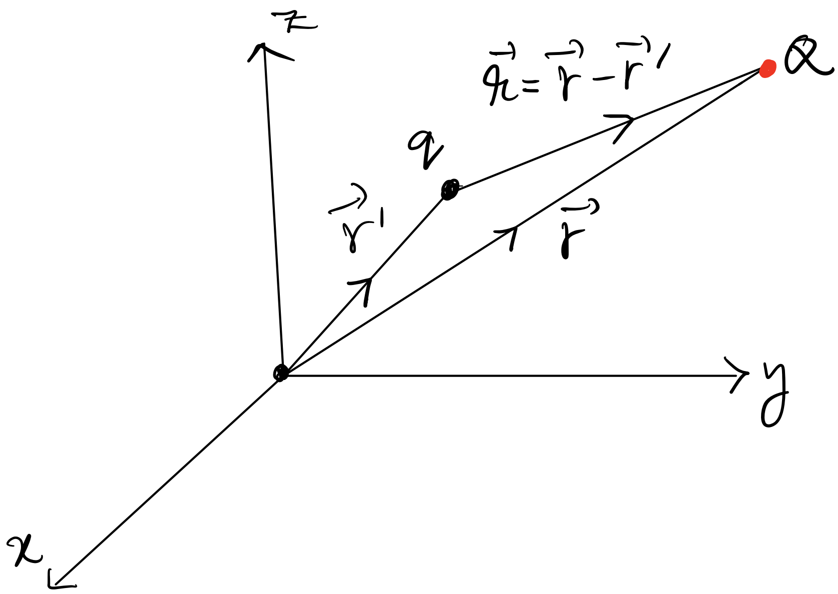
Figure 1.29

[From Griffiths 'Introduction to Electrodynamics']

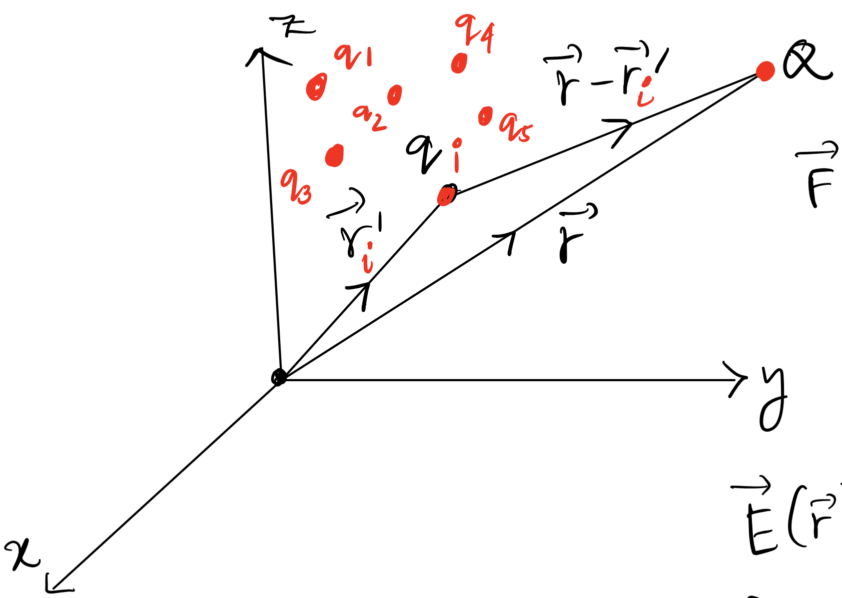
To be consistent let us write down the Coulomb's law for a system of particles:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathfrak{r}^2} \hat{\mathfrak{r}} \; ; \; \epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$$

→ Force on a test charge Q due to a single point charge q



Force on test charge 'Q' due to a collection of test charges:



$$\vec{F}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i Q}{\mathfrak{r}_i^2} \hat{\mathfrak{r}}_i$$

$$= Q E(\vec{r}) \; ;$$

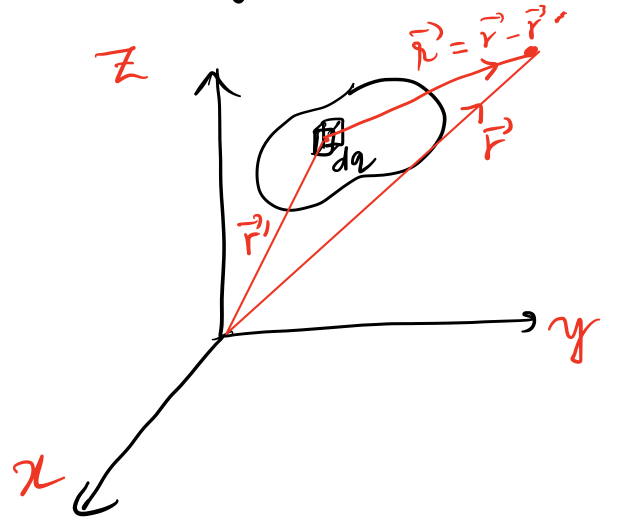
$$\vec{\mathfrak{r}}_i = \vec{r} - \vec{r}_i'$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1}{\mathfrak{r}_1^2} \hat{\mathfrak{r}}_1 + \frac{q_2}{\mathfrak{r}_2^2} \hat{\mathfrak{r}}_2 + \cdots \right]$$

Electric field, a vector that varies as a function in space.

Electric field due to a continuous charge distribution:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$



For line charge : $dq = \lambda(\vec{r}')\,dl'$

" surface " : $dq = \sigma(\vec{r}')\,da'$

" volume " : $dq = \rho(\vec{r}')\,d\tau'$

dl': infinitesimal line element

da': infinitesimal area element

$d\tau'$: infinitesimal volume element.

Aim: Calculation of fields efficiently using tricks.

$$dq = \int \rho\, d\tau$$

Electric flux and Gauss' law :

$$\boxed{\oint_{surf} \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}}$$

This is the Gauss' law in integral form.

Or,
$$\int_{vol.} (\vec{\nabla}\cdot\vec{E})\,d\tau' = \frac{Q_{enc}}{\epsilon_0}$$

Using: $\int_{vol} (\vec{\nabla}\cdot\vec{E})\,d\tau = \oint_{surf} \vec{E}\cdot d\vec{a}$

Or,
$$\int_{vol} (\vec{\nabla}\cdot\vec{E})\,d\tau' = \int \frac{1}{\epsilon_0}\rho(\vec{r}')\,d\tau'$$

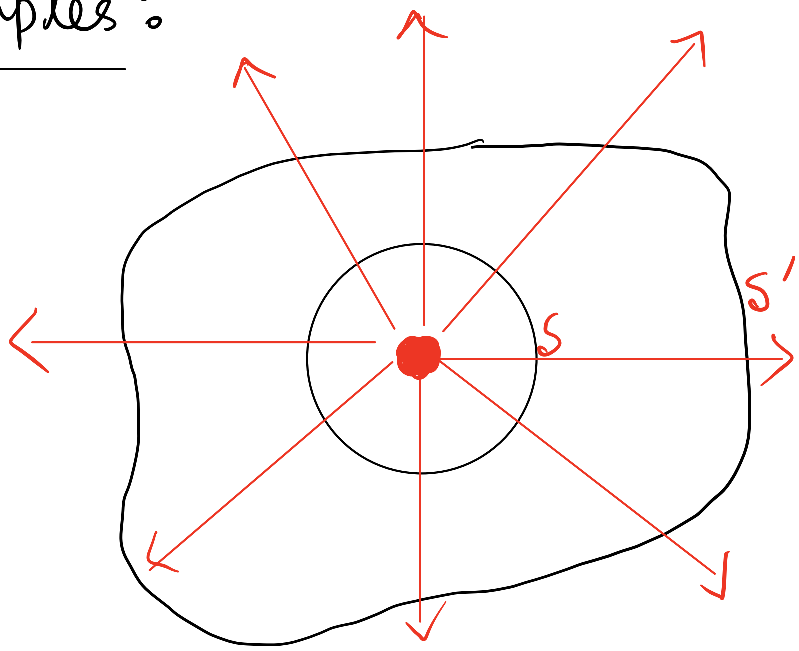
For volume charge density :

$Q_{enc} = \int \rho(\vec{r}')\,d\tau$

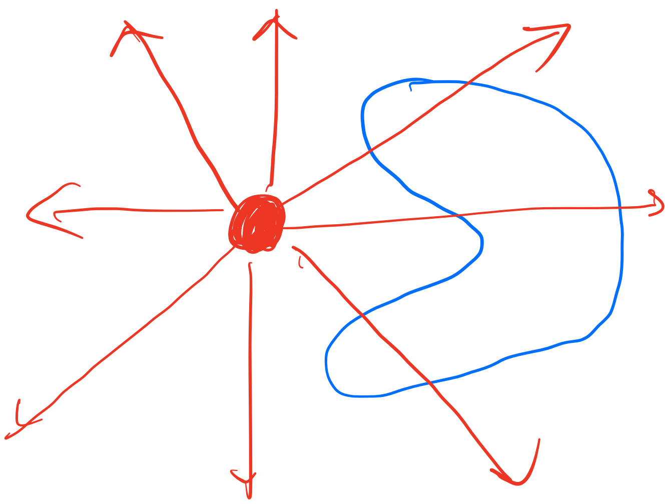
$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0}$$

Gauss' Law in differential form

Examples:



Flux through 'S' & 'S'' are same



Flux through S vanishes.
(Surface does not surround the entire charge)

The integral of the flux does not depend on the geometry of the surface & it must surround some charge to have some flux.

How can you determine the charge enclosed by a surface? In some problems, you're free to choose a surface that surrounds a known amount of charge, as in the situations shown in Figure 1.9. In each of these cases, the total charge within the selected surface can be easily determined from geometric considerations.

For problems involving groups of discrete charges enclosed by surfaces of any shape, finding the total charge is simply a matter of adding the individual charges.

$$\text{Total enclosed charge} = \sum_i q_i.$$

While small numbers of discrete charges may appear in physics and engineering problems, in the real world you're far more likely to encounter charged objects containing billions of charge carriers lined along a wire, slathered over a surface, or arrayed throughout a volume. In such cases, counting the individual charges is not practical – but you can determine the total charge if you know the charge density. Charge density may be specified in one, two, or three dimensions (1-, 2-, or 3-D).

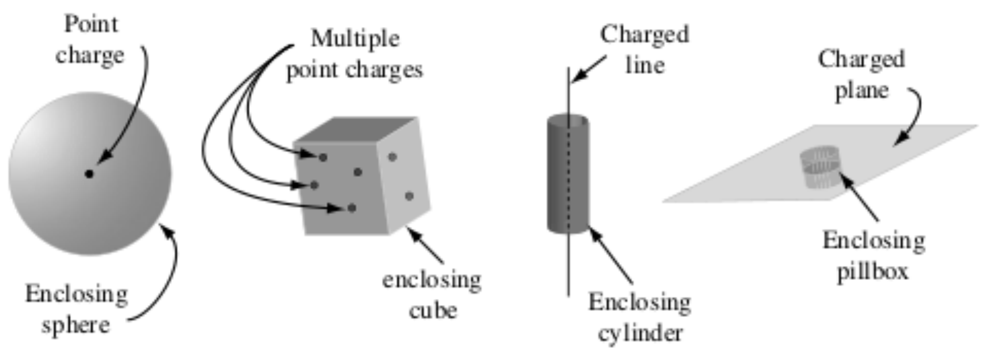


Figure 1.9 Surface enclosing known charges.

Dimensions	Terminology	Symbol	Units
1	Linear charge density	λ	C/m
2	Area charge density	σ	C/m^2
3	Volume charge density	ρ	C/m^3

If these quantities are constant over the length, area, or volume under consideration, finding the enclosed charge requires only a single multiplication:

$$\text{1-D}: \quad q_{enc} = \lambda L \quad (L = \text{enclosed length of charged line}), \tag{1.12}$$

$$\text{2-D}: \quad q_{enc} = \sigma A \quad (A = \text{enclosed area of charged surface}), \tag{1.13}$$

$$\text{3-D}: \quad q_{enc} = \rho V \quad (V = \text{enclosed portion of charged volume}). \tag{1.14}$$

You are also likely to encounter situations in which the charge density is not constant over the line, surface, or volume of interest. In such cases, the integration techniques described in the "Surface Integral" section of this chapter must be used. Thus,

$$\text{1-D}: \quad q_{enc} = \int_L \lambda \, dl \text{ where } \lambda \text{ varies along a line}, \tag{1.15}$$

$$\text{2-D}: \quad q_{enc} = \int_S \sigma \, da \text{ where } \sigma \text{ varies over a surface}, \tag{1.16}$$

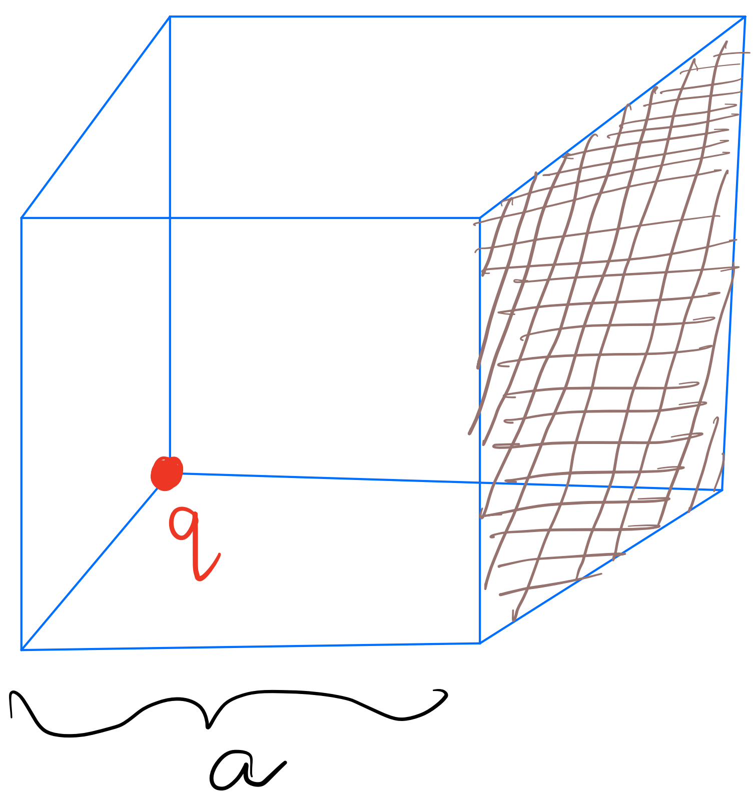
$$\text{3-D}: \quad q_{enc} = \int_V \rho \, dV \text{ where } \rho \text{ varies over a volume}. \tag{1.17}$$

You should note that the enclosed charge in Gauss's law for electric fields is the *total* charge, including both free and bound charge. You can read about bound charge in the next section, and you'll find a version of Gauss's law that depends only on free charge in the Appendix.

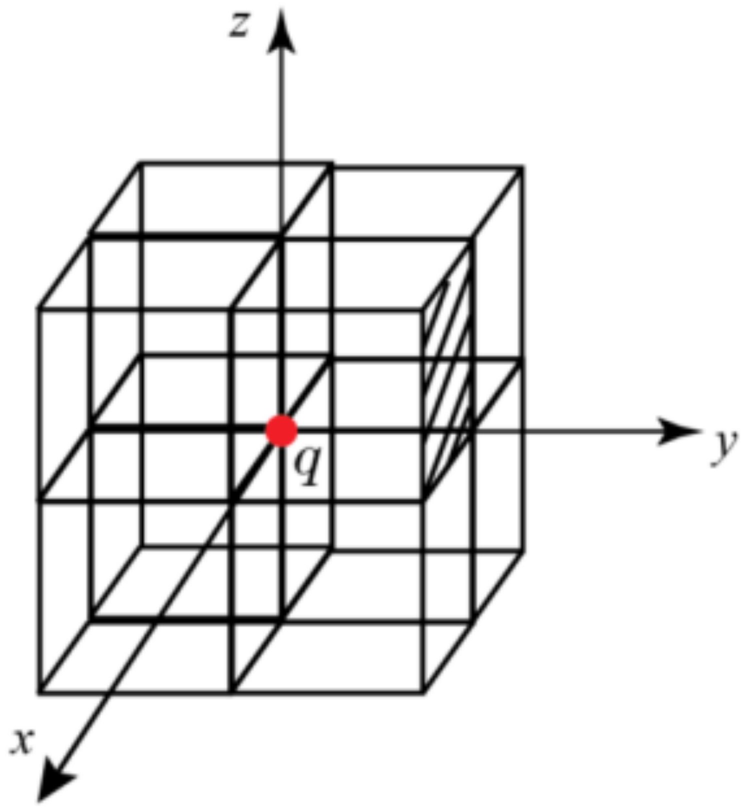
Once you've determined the charge enclosed by a surface of any size and shape, it is very easy to find the flux through that surface; simply divide the enclosed charge by ε_0, the permittivity of free space. The

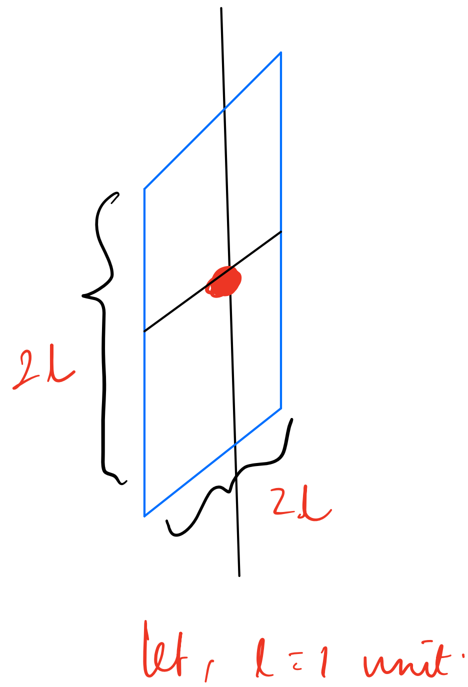
The choice of 'S' is called the Gaussian surface. A smart choice makes the problem simpler.

EXAMPLE: (1.0) A charge 'q' sits at the back corner of a cube, as shown below. What is the flux of $\vec{E}$ through the shaded side?



Surround the charge by similar surfaces. Enclose the charge inside.





The most symmetric way to do this is to keep the charge at the center.

There are six such surfaces of which the shaded area is just a part.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow \oint \vec{E} \cdot 24\, d\vec{a}$$

So,

$$6 \times 4 \underset{\text{Shaded Surface}}{\oint} \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow \underset{\text{Surface}}{\oint} \vec{E} \cdot d\vec{a} = \frac{1}{24} \frac{q}{\epsilon_0}$$

Q2.

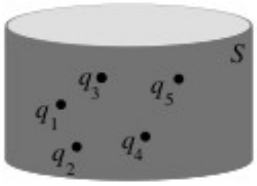
Derivation of Gauss' law [Maxwell's first Equation] from Coulomb's law.

You can use the concept of solid angle or we can use the concept of the Dirac - Delta function. [Next class].

Follow lecture 10 or 11.

Example 1.1: Given a charge distribution, find the flux through a closed surface surrounding that charge.

Problem: Five point charges are enclosed in a cylindrical surface S. If the values of the charges are $q_1 = +3$ nC, $q_2 = -2$ nC, $q_3 = +2$ nC, $q_4 = +4$ nC, and $q_5 = -1$ nC, find the total flux through S.



Solution: From Gauss's law,

$$\Phi_E = \oint_S \vec{E} \circ \hat{n}\, da = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

For discrete charges, you know that the total charge is just the sum of the individual charges. Thus,

$$\begin{aligned} q_{\text{enc}} = \text{Total enclosed charge} &= \sum\nolimits_i q_i \\ &= (3 - 2 + 2 + 4 - 1) \times 10^{-9}\,\text{C} \\ &= 6 \times 10^{-9}\,\text{C} \end{aligned}$$

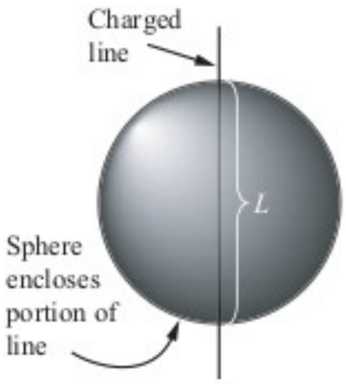
and

$$\Phi_E = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{6 \times 10^{-9}\,\text{C}}{8.85 \times 10^{-12}\,\text{C/Vm}} = 678\ \text{Vm}.$$

This is the total flux through *any* closed surface surrounding this group of charges.

Example 1.2: Given the flux through a closed surface, find the enclosed charge.

Problem: A line charge with linear charge density $\lambda = 10^{-12}$ C/m passes through the center of a sphere. If the flux through the surface of the sphere is 1.13×10^{-3} Vm, what is the radius R of the sphere?



Solution: The charge on a line charge of length L is given by $q = \lambda L$. Thus,

$$\Phi_E = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0},$$

and

$$L = \frac{\Phi_E \varepsilon_0}{\lambda}.$$

Since L is twice the radius of the sphere, this means

$$2R = \frac{\Phi_E \varepsilon_0}{\lambda} \quad \text{or} \quad R = \frac{\Phi_E \varepsilon_0}{2\lambda}.$$

Inserting the values for Φ_E, ε_0 and λ, you will find that $R = 5 \times 10^{-3}$ m.

$$= \frac{1.13 \times 10^{-3}\ \text{V m} \times 8.854 \times 10^{-12}\ C^2 Kg^{-1} m^{-3} s^2}{2 \times 10^{-11}\ C/m}$$

$$= \frac{1.13 \times 10^{-3} \times 8.854 \times 10^{-12}}{2 \times 10^{-12}} \cdot \frac{\cancel{Kg}\,\cancel{m^2}\,\cancel{s^{-2}}\,\cancel{m}\,\cancel{C^2}\,\cancel{Kg^{-1}}\,\cancel{m^{-3}}\,\cancel{s^2}}{C^2 m^{-1}}$$

$$= 5 \times 10^{-3}\ m$$