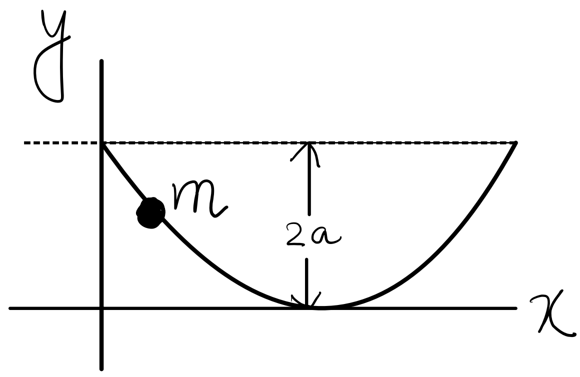
Q1.

5



Marks:

$2+2+\underline{1}+2=7$

Solution:

$$L = T - V$$

Kinetic energy $= T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$x = a(\theta - \sin\theta)$

$\Rightarrow \dot{x} = a(\dot{\theta} - \cos\theta\,\dot{\theta})$

$y = a(1 + \cos\theta)$

$= -a\sin\theta\,\dot{\theta}$

So,

$$T = \frac{1}{2}m\,a^2\dot{\theta}^2\left[(1-\cos\theta)^2 + \sin^2\theta\right]$$

$$\Rightarrow T = \frac{1}{2} m a^2 \dot{\theta}^2 \left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]$$

$$= \frac{1}{2} m a^2 \dot{\theta}^2 \, 2(1 - \cos\theta)$$

$$\therefore \boxed{T = m a^2 \dot{\theta}^2 (1 - \cos\theta)}$$

Potential Energy: $V = mgy = mga(1+\cos\theta)$

$$L = m a^2 \dot{\theta}^2 (1-\cos\theta) - mga(1+\cos\theta)$$

(b) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left[m a^2 (1-\cos\theta) \frac{\partial \dot{\theta}^2}{\partial \dot{\theta}}\right] = \frac{\partial}{\partial \theta}\left[m a^2 \dot{\theta}^2 (1-\cos\theta) - mga(1+\cos\theta)\right]$$

$$\Rightarrow \frac{d}{dt}\left(2 m a^2 \dot{\theta} (1-\cos\theta)\right) = m a^2 \dot{\theta}^2 \sin\theta + mga \sin\theta$$

$$\Rightarrow \frac{d}{dt}\left[\dot{\theta}(1-\cos\theta)\right] = \frac{1}{2}\sin\theta\,\dot{\theta}^2 + \frac{g}{2a}\sin\theta$$

$$\Rightarrow \ddot{\theta}(1-\cos\theta) + \dot{\theta}^2\sin\theta = \frac{1}{2}\sin\theta\,\dot{\theta}^2 + \frac{g}{2a}\sin\theta$$

$$\Rightarrow (1-\cos\theta)\ddot{\theta} + \frac{1}{2}\dot{\theta}^2\sin\theta - \frac{g}{2a}\sin\theta = 0$$

(c) Generalized momentum:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$= 2ma^2\,\dot{\theta}(1-\cos\theta)$$

(d) $H = p_\theta \dot{\theta} - L$

$$= 2ma^2\dot{\theta}^2(1-\cos\theta) - ma^2\dot{\theta}^2(1-\cos\theta) + mga(1+\cos\theta)$$

$$= ma^2\dot{\theta}^2(1-\cos\theta) + mga(1+\cos\theta)$$

$$= T + V$$

Q2. Show that the force field given by, $\vec{F} = x^2yz\hat{i} - xyz^2\hat{k}$ is non-conservative. Marks : 3

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 0 & -xyz^2 \end{vmatrix}$$

$$= \hat{i}\ \frac{\partial}{\partial y}(-xyz^2) - \hat{j}\left(\frac{\partial}{\partial x}(-xyz^2) - \frac{\partial}{\partial z}(x^2yz)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial y}(-x^2yz)\right)$$

$$= -xz^2\hat{i} + yz^2\hat{j} - \hat{k}\,x^2z$$
$$+ x^2y\,\hat{j}$$

$$= -xz^2\hat{i} + (x^2y + yz^2)\hat{j} - x^2z\hat{k}$$

Since $\vec{\nabla} \times \vec{F} \neq 0$, the field is non-conservative.