## Idealization.

- Crutial step in engineering approach for the purpose of analysis.
- Actual behavior of system/systems are complex. Hence, considering all the features in a mathematical model is quite difficult or impossible
- Mathematically ideal model: simple to start with and analyze, yet exhibit the phenomenon under considerat
- Model developed is valid if the analytical solution matches reasonably well with the experimental observations/findings.

## ⊛ Rigid Body.

- One of the idealizations used in engineering mechanics
- "A rigid body is one in which all the particles remain at fixed distances from each other, irrespective of the forces that acts on the body" — It does not deform under the action of forces.
- In reality, no body is a rigid body. All the bodies/objects deform (even though slightly) when subjected to external loads.
- Rigid body idealization is used in the study of both statis & dynamics.

⊛ Physical systems and Physical actions are also idealized

Physical systems → represented by rigid body approximation

Physical actions → The force system are idealized either by means of a concentrated force or in some case by distributed force (which has a simpler mathematical formulation)

## ⊛ Continuum:

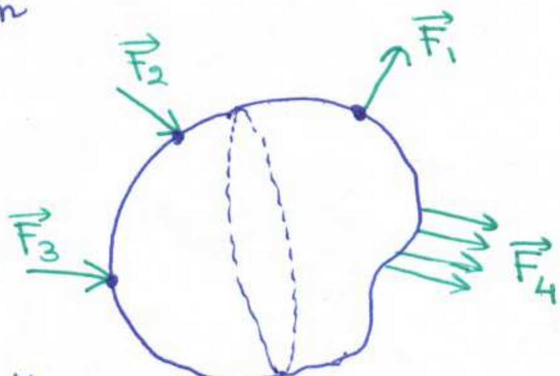
- Matter is assumed to be continuously distributed without any voids / empty spaces.

## Particle

- Mass of a body with negligible dimensions concentrated at a point, i.e, volume $\to 0$.
  - $\Rightarrow$ Entire mass is concentrated at a point.

## Concept of a Force

- Force is a simple and useful way of describing a complex physical interaction between bodies.
- "A force is the action of one body on another body which changes or tends to change the motion of the body acted on". (Force is an external agent capable of changing a body's state of rest or motion).
- Force is characterized by:
  - Point of application
  - Magnitude
  - Direction
- Force is represented by a vector.




## Newton's First Law:

- If the resultant force on the particle is zero, the body will remain at rest or continue to move in a straight line.

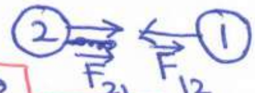
## Newton's Second Law:

- A body will have an acceleration proportional to a non-zero resultant applied force.

$$\boxed{\vec{F} = m\vec{a}}$$

## Newton's Third Law:

- The forces of action and reaction between two particles have magnitude and line of action with opposite direction.
- Action and reaction forces always act on different bodies.




$$\boxed{\vec{F}_{21} = -\vec{F}_{12}}$$

## Newton's Law of Gravitation.

- Bodies are attracted with equal & opposite forces.

$$\boxed{\vec{F} = G\frac{Mm}{R^2}}$$

# Force Interaction

- Direct Physical Contact
  - Eg ① Train wheels on Railway Track.
  - ② A body resting on a table.
  - ③ Many more examples around us.
- Physically separated
  - Eg. ① Gravitational force
  - ② Magnetic force
  - ③ Coulomb's force of static electricity, etc.

## Body Force:

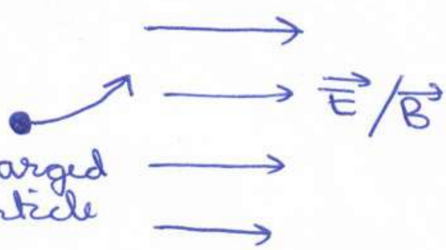
① Aircraft


Gravitation force is experienced by the aircraft which can be categorized as a body force.

②


- Weight of the body/object can also be categorized as a body force.

②

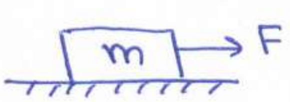


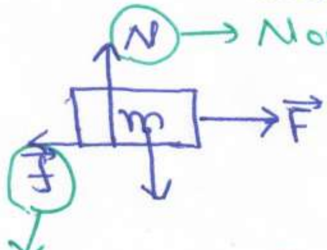

- A charged particle moving in an electrical/magnetic field also experiences electric/magnetic forces as body force.

## Surface Forces:

- Surface Forces act on each surface element of the body and is exerted by direct mechanical contact.

Eg.




N → Normal reaction ↳ surface force

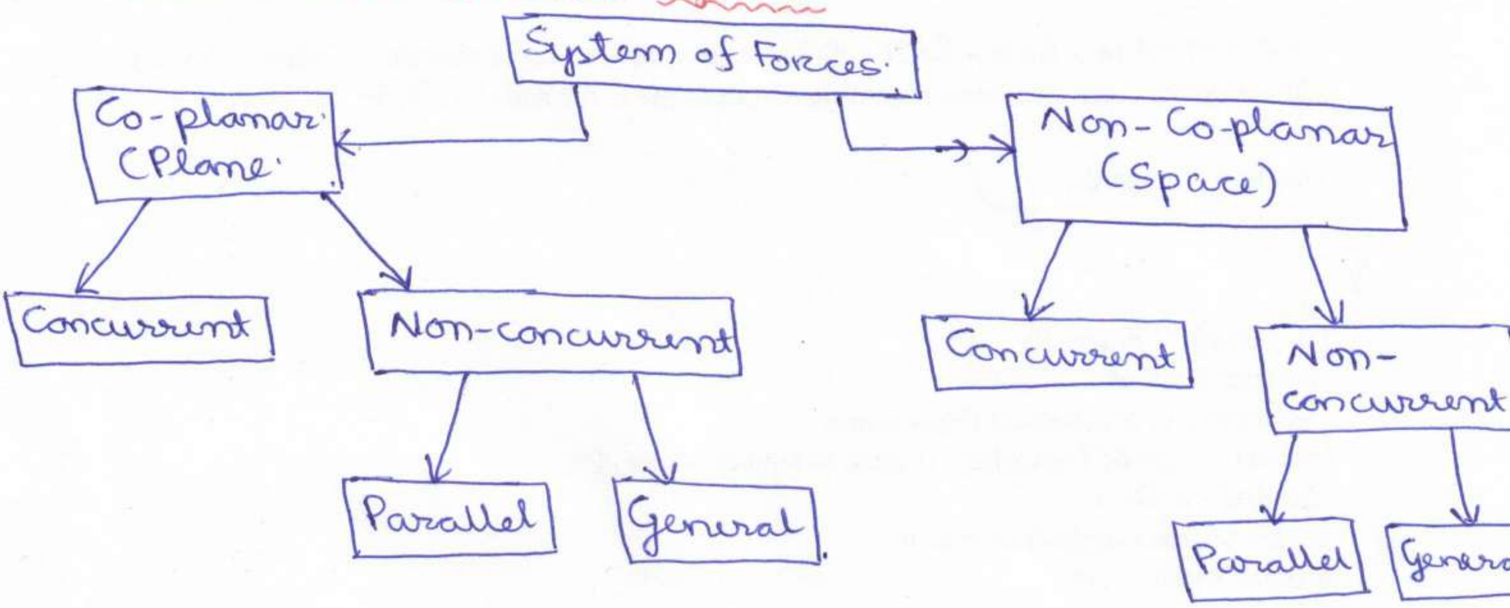
Frictional force → surface force

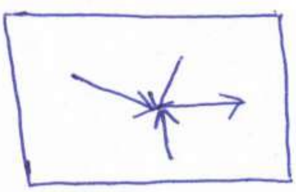

$R_A, R_B$ → Reaction forces → surface forces.

## Distributed and Concentrated Forces.

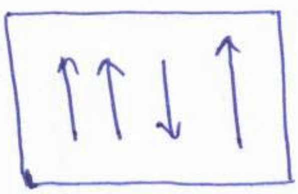
- Using composition of forces, a distributed force system can be reduced to an equivalent concentrated force for the calculation/simplification purpose.
- This equivalent force would have overall same effect at points away from the region of application of force. (Saint Venant's Principle).
- Representation of distributed forces by means of a concentrated force acting at specific points is just an abstraction, justified only because it greatly simplifies the analysis.

## Classification of Force System

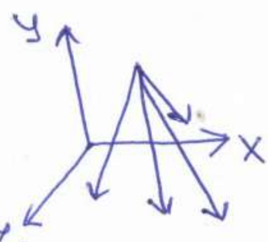


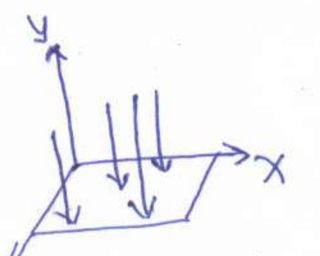
Coplanar Concurrent



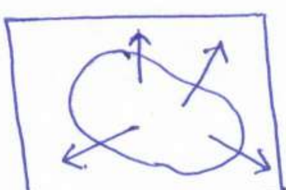
Coplanar Parallel.



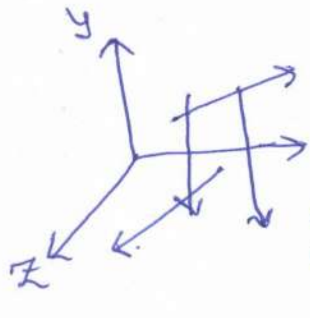

Non-coplanar concurrent




Non-coplanar parallel.



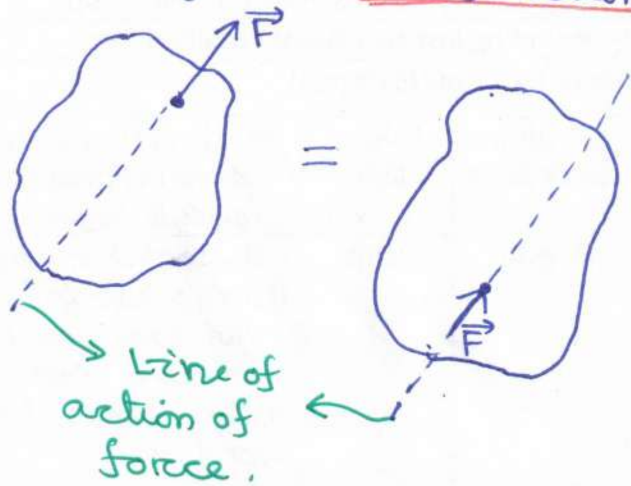
Coplanar non-concurrent.




Non-coplanar non-concurrent.

## Transmissibility of Force:

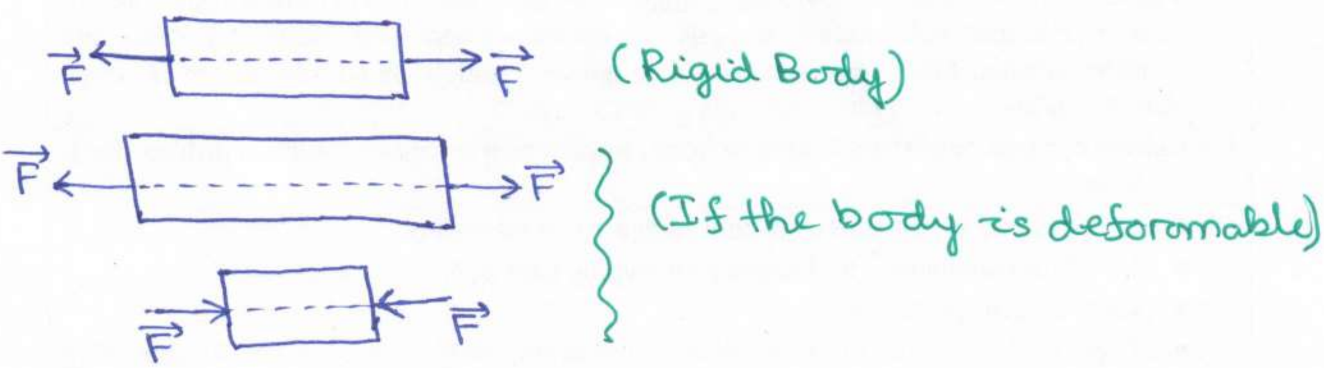
- Principle of transmissibility states that the external effect of the force are independent of the point of application of force along the line of action.




⇒ Pushing or Pulling produces the same result as long as the forces are applied along the line of action of the force (i.e., line of action of forces don't change).

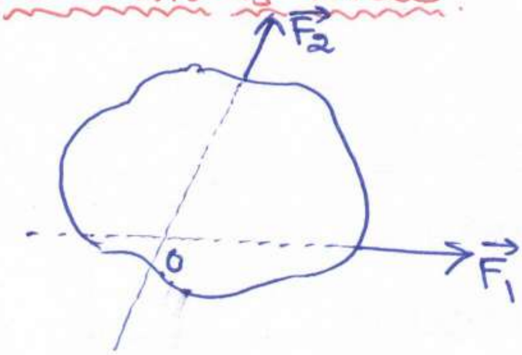
- The internal effects, however may vary greatly, as the forces move along the line of action of forces.


- External effect: Object is at rest !!! → both in cases(a), (b)
- Internal effect: Case-(a) → Under tension
  Case-(b) → Under compression

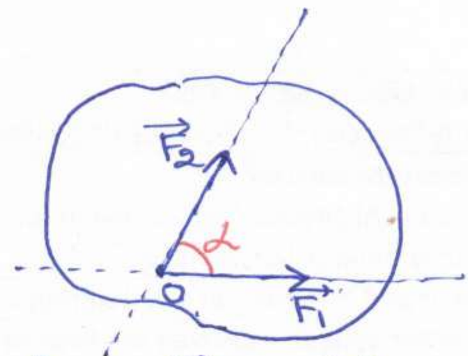



- Force is a sliding vector and it can be judiciously used when we are looking at the external effects of the force.

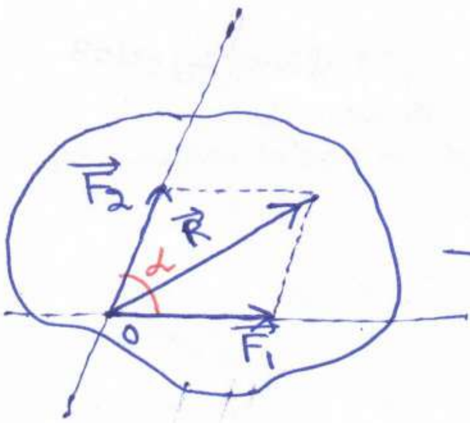
## Resultant of Forces:




$\Rightarrow$



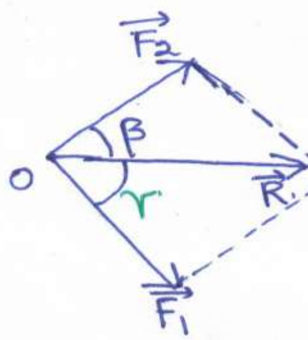

(The forces $\vec{F_1}$ and $\vec{F_2}$ could be moved to point 'O' using the principle of Transmissibility of Forces).



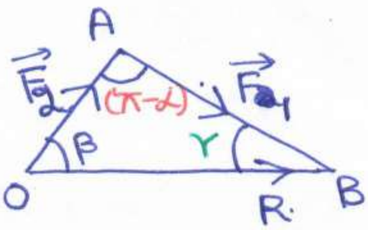

R ≡ Resultant of forces $\vec{F_1}$ and $\vec{F_2}$ at point 'O'.

→ Parallelogram Law of Forces.

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}.$$




$\Rightarrow$

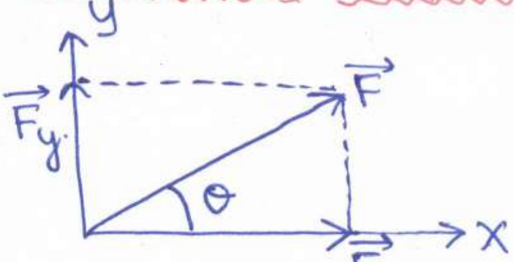



\# Angles '$\beta$' and '$\gamma$' can be calculated using Sine-Law of Triangle,

- $\dfrac{F_1}{\sin\beta} = \dfrac{R}{\sin(\pi-\alpha)} \Rightarrow \boxed{\dfrac{F_1}{\sin\beta} = \dfrac{R}{\sin\alpha}}$

- $\dfrac{F_2}{\sin\gamma} = \dfrac{R}{\sin(\pi-\alpha)} \Rightarrow \boxed{\dfrac{F_2}{\sin\gamma} = \dfrac{R}{\sin\alpha}}$

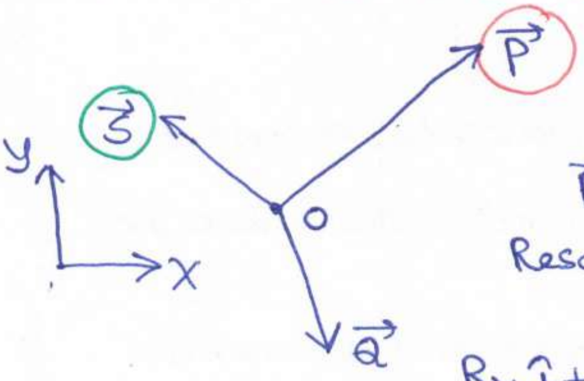
## Rectangular Components of a Force:




- $\vec{F} = \vec{F_x} + \vec{F_y}$
  $= F_x\hat{i} + F_y\hat{j}$ ($\hat{i}$ and $\hat{j}$ are unit vectors along X and Y axes, respectively)
- $(F_x, F_y) \rightarrow$ Scalar components of $\vec{F}$

- $\boxed{F_x = F\cos\theta}$, $\boxed{F_y = F\sin\theta}$, $\boxed{\tan\theta = F_y/F_x}$

# Addition of Forces by Summing 'X' and 'Y' components



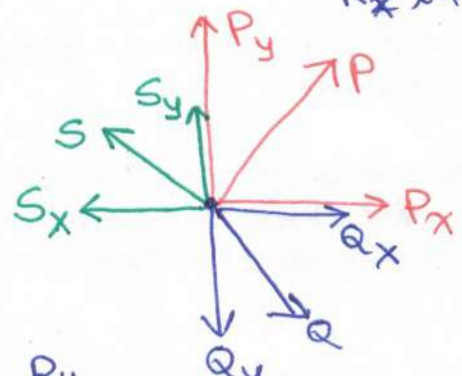

Resultant of the three forces $\vec{P}, \vec{Q}, \vec{S}$:-

$$\vec{R} = \vec{P} + \vec{Q} + \vec{S}$$

Resolving each force into its rectangular components,

$$R_x\hat{i} + R_y\hat{j} = (P_x\hat{i} + P_y\hat{j}) + (Q_x\hat{i} + Q_y\hat{j}) + (S_x\hat{i} + S_y\hat{j})$$

$\equiv$



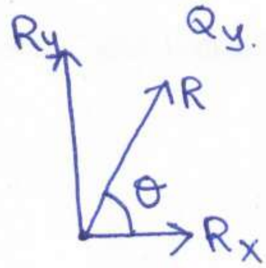

$$\Rightarrow R_x = (P_x + Q_x + S_x) = \sum F_x$$

$$R_y = (P_y + Q_y + S_y) = \sum F_y$$

$$\therefore R = R_x\hat{i} + R_y\hat{j}$$

$$\Rightarrow \boxed{R = \left(\sum F_x\right)\hat{i} + \left(\sum F_y\right)\hat{j}}$$

$\equiv$

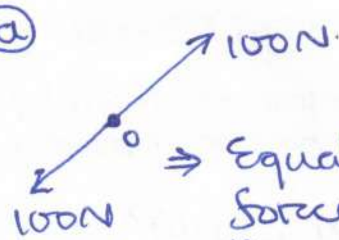



$$\boxed{\tan\theta = \frac{R_y}{R_x}}$$

## Equilibrium of a Particle

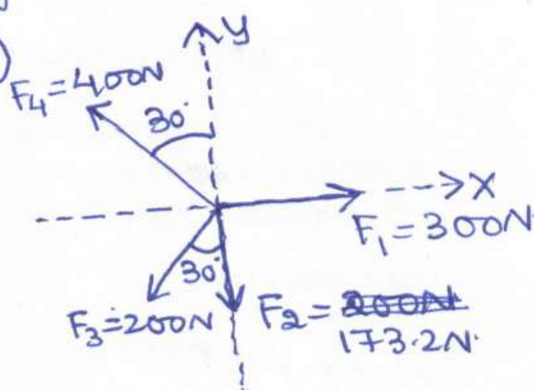
— "A particle is said to be in equilibrium, when the resultant of all the forces acting on the particle is zero".

(a)



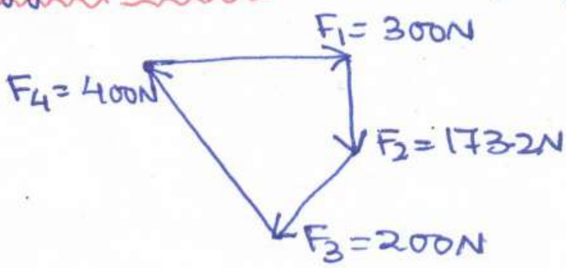

$\Rightarrow$ Equal and opposite forces acting along the same line of action of the forces.

(b)




$\Rightarrow \sum F_x = 0$ , $\sum F_y = 0$ (Resolving the force components along 'X' and 'Y' directions).

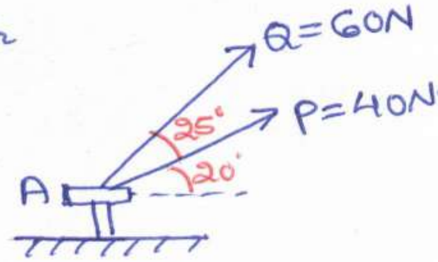
Force Polygon Method: Arranging the forces in a tip-to-tail fashion.



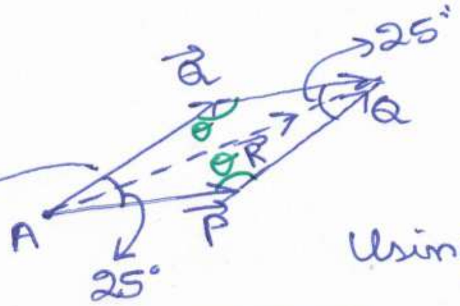

— Force polygon is closed for the particle in equilibrium, i.e.,

$$\sum F_x = \sum F_y = 0$$

Que) The forces $\vec{P}$ and $\vec{Q}$ act at bolt 'A' as shown in the figure. Determine their resultants and its orientation w.r.t the horizontal axis.



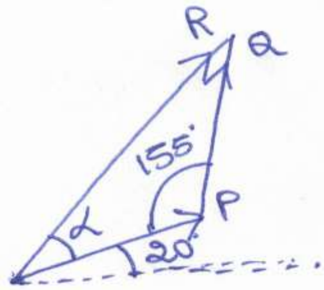

Solution




Using Parallelogram Law of Forces:-

$$|\vec{R}| = \sqrt{P^2+Q^2+2PQ\cos 25^\circ} = \sqrt{(40)^2+(60)^2+2(40)(60)\cos 25^\circ}$$

$$\Rightarrow |\vec{R}| = 97.73\text{N}$$




From this parallelogram -

$$25^\circ + 25^\circ + 2\theta = 360^\circ$$

$$\Rightarrow \theta = \frac{310^\circ}{2} = 155^\circ$$

Using sine law of triangles -

$$\frac{|\vec{Q}|}{\sin\alpha} = \frac{|\vec{R}|}{\sin 155^\circ} \Rightarrow \sin\alpha = \frac{|\vec{Q}|}{|\vec{R}|}\sin 155^\circ$$

$$\Rightarrow \sin\alpha = \frac{60}{97.73}\sin 155^\circ$$

$$\Rightarrow \alpha = 15.04^\circ$$

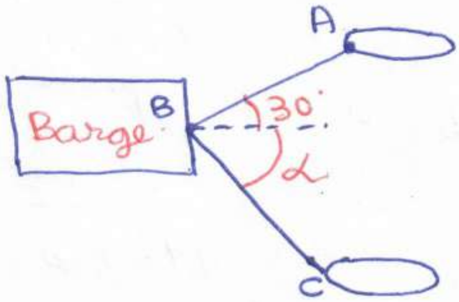
$\therefore$ Orientation of the resultant $\vec{R}$ w.r.t the horizontal

$$= \alpha + 20^\circ = 15.04^\circ + 20^\circ = \boxed{35.04^\circ}$$

Que) A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboats is 5000N directed along the axis of the barge, determine:

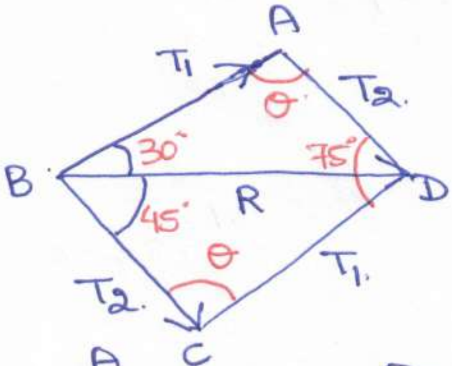
(a) tension in each ropes for $\alpha = 45°$

(b) the value of '$\alpha$' for which the tension in rope '2' is minimum.




Solution:

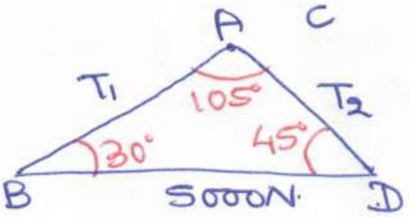
(a)




From the parallelogram,

$75° + 75° + 2\theta = 360°$

$\Rightarrow \boxed{\theta = 105°}$



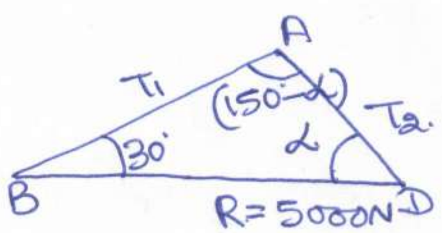

From the sine law of triangles -

$$\frac{\sin 105°}{5000} = \frac{\sin 45°}{T_1} = \frac{\sin 30°}{T_2}$$

$$\Rightarrow T_1 = \left(\frac{\sin 45°}{\sin 105°}\right) 5000 = \underline{\underline{3660.25N}}$$

$$T_2 = \left(\frac{\sin 30°}{\sin 105°}\right) 5000 = \underline{\underline{2588.19N.}}$$

(b)




From the sine law of triangles -

$$\frac{\sin(150-\alpha)}{5000} = \frac{\sin 30°}{T_2.}$$

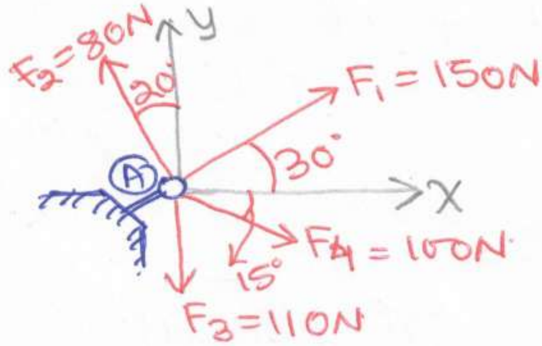
$$\Rightarrow T_2 = \left[\frac{\sin 30}{\sin(150-\alpha).}\right] * 5000.$$

$T_2$ is minimum, when $\sin(150-\alpha)$ is maximum.

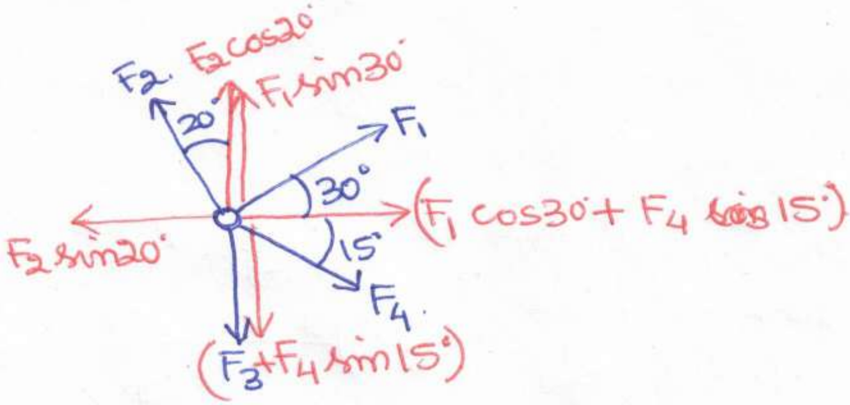
$\Rightarrow \sin(150-\alpha) = 1$

$\Rightarrow 150-\alpha = 90° \Rightarrow \boxed{\alpha = 60°}$

Que) Four forces act on the bolt 'A' as shown in the figure. Determine the resultant of the force on the bolt and its orientation w.r.t the horizontal.




Solution




- Resultant along X- direction:-

$$R_x = (F_1 \cos 30° + F_4 \cos 15° - F_2 \sin 20°)$$

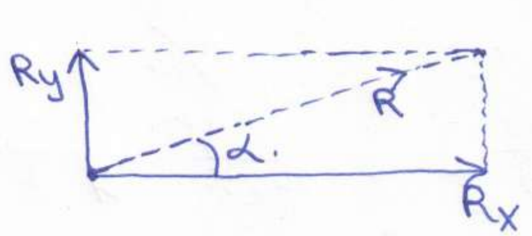
$$= (150 * \cos 30° + 100 * \cos 15° - 80 \sin 20°)$$

$$R_x = 199.13\ N$$

- Resultant along y- direction:-

$$R_y = (F_1 \sin 30° + F_2 \cos 20° - F_3 - F_4 \sin 15°)$$

$$\Rightarrow R_y = 14.29\ N$$




$$|R| = \sqrt{(R_x)^2 + (R_y)^2}$$

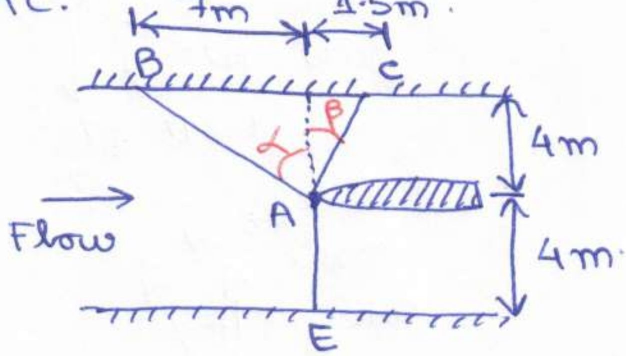
$$|R| = \sqrt{(199.13)^2 + (14.29)^2}$$

$$\Rightarrow \boxed{|R| = 199.64\ N}$$

$$\tan \alpha = \frac{R_y}{R_x} = \frac{14.29}{199.13}$$

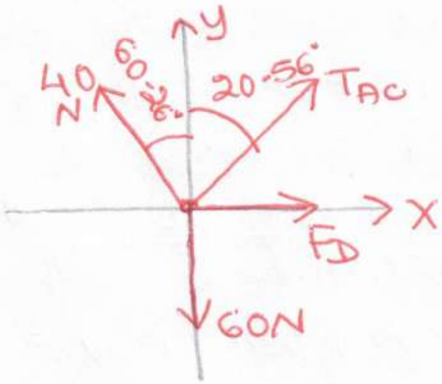
$$\Rightarrow \boxed{\alpha = 4.1°}$$

Que) It is desired to determine the drag ~~speed~~ force at a given speed on the prototype sailboat hull. The model is placed in a test channel and three cables are used to align it on the channel centreline. For a given speed, the tension in cables AB and AE are 40N and 60N, respectively. Determine the drag force exerted on the hull and tension in cable AC.




Solution

- $\tan\alpha = \frac{7}{4} \Rightarrow \boxed{\alpha = 60.26°}$
- $\tan\beta = \frac{1.5}{4} \Rightarrow \boxed{\beta = 20.56°}$




- For the equilibrium of the body, $\sum F_x = 0$ and $\sum F_y = 0$

- $\sum F_x = 0$

$\Rightarrow F_D + T_{AC} \sin 20.56° - 40 \sin 60.26° = 0$

$\Rightarrow F_D + 0.35\, T_{AC} = 34.73$ ...... (1)

- $\sum F_y = 0$

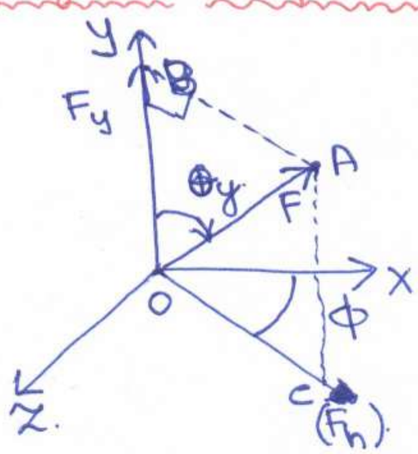
$\Rightarrow T_{AC} \cos 20.56° + 40 \cos 60.26° - 60 = 0$

$\Rightarrow \boxed{T_{AC} = 42.88N}$

$F_D + (42.88 * 0.35) = 34.73$

$\Rightarrow \boxed{F_D = 19.72 N}$

# Rectangular Components of a Force in Space




Scalar components of F on OBAC plane -

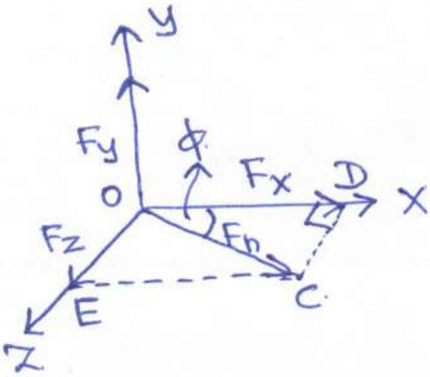
$F_y = F\cos\theta_y$

$F_h = F\sin\theta_y$

Using Pythagoreas Theorem in $\Delta OAB$

$OA^2 = OB^2 + AB^2 = OB^2 + OC^2$

$\Rightarrow F^2 = F_y^2 + F_h^2$




$F_x = F_h\cos\phi = F\sin\theta_y\cos\phi$

$F_z = F_h\sin\phi = F\sin\theta_y\sin\phi$

Using Pythagoreas Theorem in $\Delta ODC$

$OC^2 = OD^2 + CD^2 = OD^2 + OE^2$

$\Rightarrow F_h^2 = F_x^2 + F_z^2$

Now substituting $F_h^2$ in expression of $F^2$,

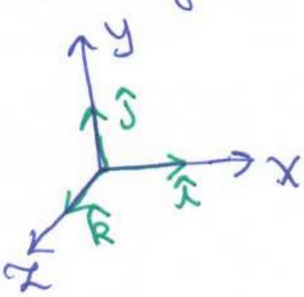
$$F^2 = F_x^2 + F_y^2 + F_z^2$$

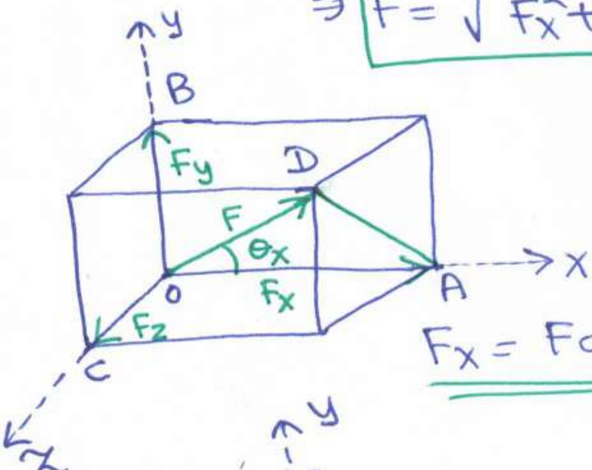
$$\Rightarrow \boxed{F = \sqrt{F_x^2 + F_y^2 + F_z^2}}$$

- Vectorial Representation

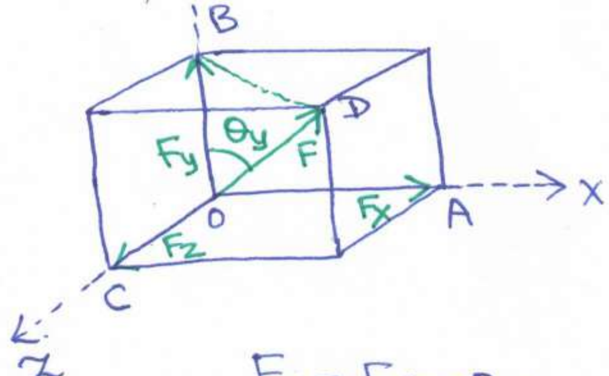
$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along X, y, Z directions, respectively.

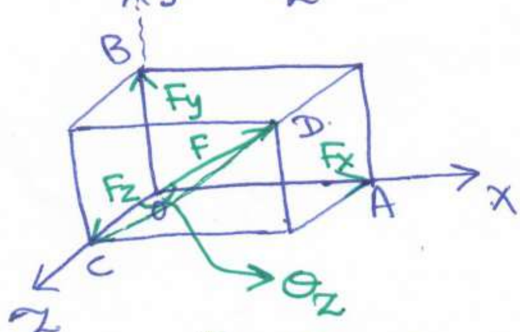



$F_x = F\cos\theta_x$




$F_y = F\cos\theta_y$




$F_z = F\cos\theta_z$

- cosines of $\theta_x$, $\theta_y$ and $\theta_z$ are called as direction cosines of force $\vec{F}$.

— Now, $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$.

$$\Rightarrow \vec{F} = F\cos\theta_x\hat{i} + F\cos\theta_y\hat{j} + F\cos\theta_z\hat{k}$$

$$\Rightarrow \vec{F} = F(\cos\theta_x\hat{i} + \cos\theta_y\hat{j} + \cos\theta_z\hat{k}).$$

$$\Rightarrow \boxed{\vec{F} = F\lambda}, \text{ where } \boxed{\lambda = \cos\theta_x\hat{i} + \cos\theta_y\hat{j} + \cos\theta_z\hat{k}}$$

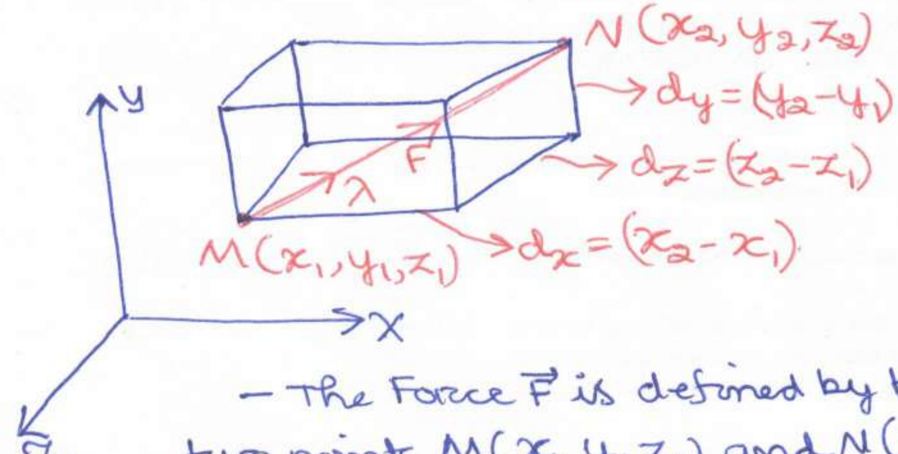
Unit vector along the line of action of force $\vec{F}$.

— $F^2 = F_x^2 + F_y^2 + F_z^2$

$$\Rightarrow F^2 = F^2(\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z)$$

$$\Rightarrow \boxed{\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1}$$ Relationship among the direction cosines.

## Force defined by its magnitude and two points on its line of action




— The Force $\vec{F}$ is defined by the co-ordinates of two points $M(x_1, y_1, z_1)$ and $N(x_2\, y_2\, z_2)$ located on its line of action.

$$\vec{MN} = d_x\hat{i} + d_y\hat{j} + d_z\hat{k}.$$

By definition, $\vec{F} = F\lambda$

$$\Rightarrow F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = F\frac{\vec{MN}}{|MN|} = F\frac{(d_x\hat{i} + d_y\hat{j} + d_z\hat{k})}{d}$$

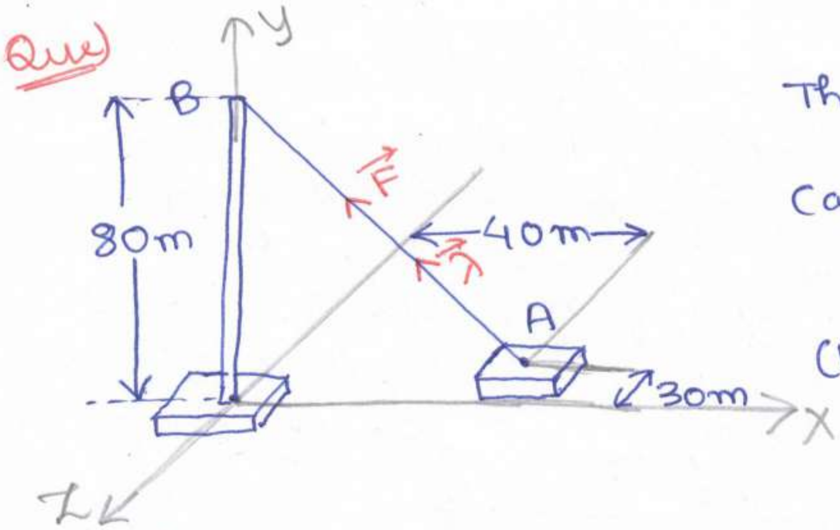
$$\therefore \boxed{\begin{aligned} F_x &= F\left(\frac{d_x}{d}\right) \\ F_y &= F\left(\frac{d_y}{d}\right) \\ F_z &= F\left(\frac{d_z}{d}\right) \end{aligned}}$$

$$d = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

$$\Rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Also, $$\boxed{\begin{aligned} \cos\theta_x &= \frac{d_x}{d} \\ \cos\theta_y &= \frac{d_y}{d} \\ \cos\theta_z &= \frac{d_z}{d} \end{aligned}}$$

Que)




The tension in the wire AB is 2500N. Determine:

(a) The components of the tension along X, y, and z directions

(b) The angle $\theta_x$, $\theta_y$, and $\theta_z$ defining the direction of the force.

Solution:

$$\vec{r}_{AB} = \vec{r}_B - \vec{r}_A.$$
$$= (80\hat{j}) - (40\hat{i} - 30\hat{k}).$$
$$\Rightarrow \vec{r}_{AB} = -40\hat{i} + 80\hat{j} + 30\hat{k}.$$

$$|\vec{r}_{AB}| = \sqrt{(-40)^2 + (80)^2 + (+30)^2}$$
$$\Rightarrow |\vec{r}_{AB}| = 94.34N.$$

$$\vec{\lambda} = \frac{\vec{r}_{AB}}{|r_{AB}|} = \left(\frac{-40}{94.34}\right)\hat{i} + \left(\frac{80}{94.34}\right)\hat{j} + \left(\frac{+30}{94.34}\right)\hat{k}.$$

$$\Rightarrow \vec{\lambda} = -0.42\hat{i} + 0.85\hat{j} + 0.32\hat{k}.$$

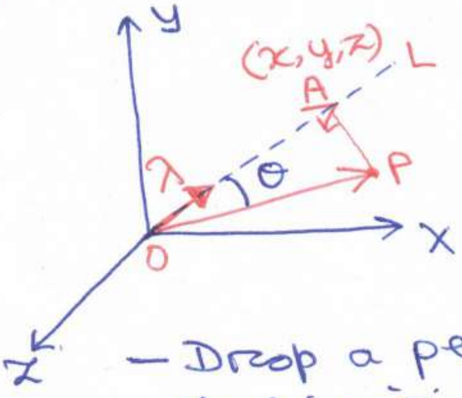
$$\therefore \vec{F} = F\vec{\lambda} = 2500(-0.42\hat{i} + 0.85\hat{j} + 0.32\hat{k}).$$
$$\Rightarrow \vec{F} = (-1050)\hat{i} + 2125\hat{j} + 800\hat{k}$$
$$\therefore \boxed{F_x = -1050N}, \boxed{F_y = 2125N} \text{ and } \boxed{F_z = 800N}$$

⊛ The direction cosines $\theta_x$, $\theta_y$ and $\theta_z$ can be evaluated from $\vec{\lambda}$ which can be written as—

$$\cos\theta_x\hat{i} + \cos\theta_y\hat{j} + \cos\theta_z\hat{k} = \lambda = -0.42\hat{i} + 0.85\hat{j} + 0.32\hat{k}$$

$$\Rightarrow \boxed{\begin{aligned} \theta_x &= \cos^{-1}(-0.42) = 114.83^\circ \\ \theta_y &= \cos^{-1}(0.85) = 31.78^\circ \\ \theta_z &= \cos^{-1}(0.32) = 71.33^\circ \end{aligned}}$$

# Projection of a Force on a Line.




We have to find the projection of force $\vec{P}$ along the line OL.

- Angle b/w $\vec{P}$ & $\vec{OL}$ is $\theta$.

— Drop a perpendicular from the tip of 'P' onto the line 'OL'.

— This $\perp^r$ intersects the line $\vec{OL}$ at A.

— $|OA| = P\cos\theta \equiv$ Force component of $\vec{P}$ along OA.

— Now, the magnitude of the force along OL is known $= P\cos\theta$.

— The direction of force along OL is established by evaluating the unit vector along OA (i.e., $\lambda$).

$$\therefore \vec{P}_{OL} = (P\cos\theta)\lambda.$$

$$= (P\cos\theta)\left(\frac{x}{d}\hat{i} + \frac{y}{d}\hat{j} + \frac{z}{d}\hat{k}\right), \text{ where } d = \sqrt{x^2+y^2+z^2}.$$

$$\Rightarrow \vec{P}_{OL} = \left(\frac{Px}{d}\cos\theta\,\hat{i} + \frac{Py}{d}\cos\theta\,\hat{j} + \frac{Pz}{d}\cos\theta\,\hat{k}\right)$$

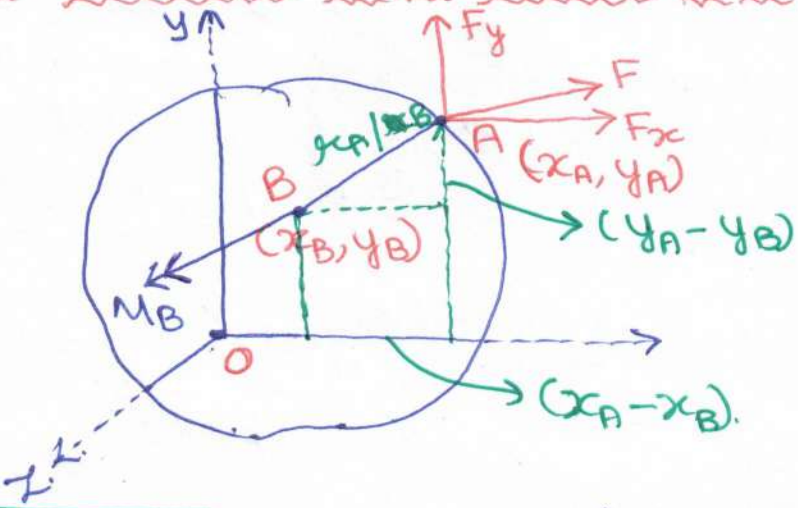
— Furthermore,

$$\vec{P}\cdot\vec{\lambda} = |P||\lambda|\cos\theta \quad (\because \text{Angle between } \vec{P} \text{ and } \vec{\lambda} \text{ is } \theta)$$

$$\Rightarrow \vec{P}\cdot\vec{\lambda} = P\cos\theta$$

$$\Rightarrow \boxed{\vec{P}\cdot\vec{\lambda} = |P_{OL}|}$$

# Rectangular components of the Moment of a Force About a Point




- $\boxed{\vec{M}_B = \vec{r}_{A/B} \times \vec{F}} = (\vec{r}_A - \vec{r}_B) \times \vec{F}$

$$= \left[ (x_A - x_B)\hat{i} + (y_A - y_B)\hat{j} \right] \times (F_x \hat{i} + F_y \hat{j}).$$

$$\Rightarrow (M_B)_z \hat{k} = \left[ (x_A - x_B) F_y - (y_A - y_B) F_x \right] \hat{k}.$$

- For the three-dimensional case,

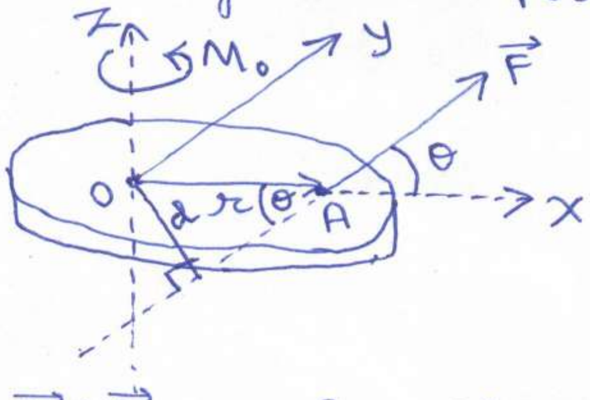
$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} (M_B)_x &= (y_A - y_B) F_z - (z_A - z_B) F_y \\ (M_B)_y &= (z_A - z_B) F_x - (x_A - x_B) F_z \\ (M_B)_z &= (x_A - x_B) F_y - (y_A - y_B) F_x \end{aligned}}$$

## Moment of a Force About a Point

- Moment of a Force is the measure of the tendency of the force to rotate the body about the point of interest 'O'.




$\vec{M_o} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector joining 'O' to 'A'.

$= rF\sin\theta\,\hat{k}$

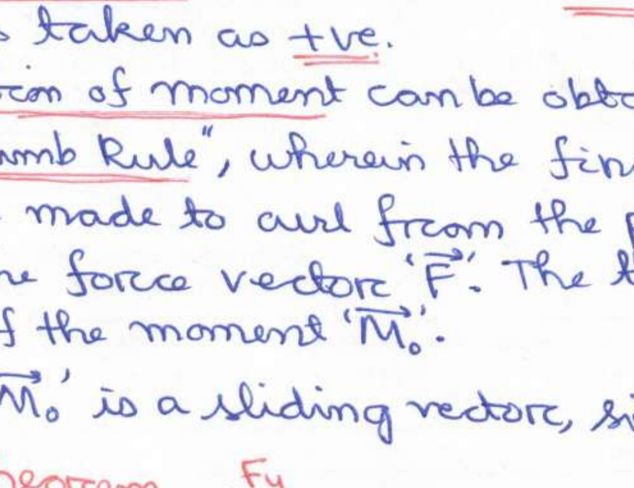
$= (r\sin\theta)F(\hat{k})$

$\Rightarrow \boxed{M_o = Fd}$ ($\because$ Angle b/w 'F' and 'd' is $\pi/2$)

magnitude

- clockwise moment is taken as -ve and anti-clockwise moment is taken as +ve.
- The direction of moment can be obtained using "Right-Hand Thumb Rule", wherein the fingers of the right hand are made to curl from the position vector '$\vec{r}$' towards the force vector '$\vec{F}$'. The thumb points to the direction of the moment '$\vec{M_o}$'.
- Moment '$\vec{M_o}$' is a sliding vector, similar to force $\vec{F}$.

## Varignon's Theorem




$\vec{M_o} = \vec{r} \times \vec{F} = (x\hat{i} + y\hat{j}) \times (F_x\hat{i} + F_y\hat{j})$.

$\Rightarrow \vec{M_o} = (xF_y - yF_x)\hat{k}$

- It is seen that the magnitude of moment is given by the algebraic sum of the magnitudes of the moments of the force components about 'O'.
- In many case, calculation of 'd' may be cumbersome, & hence, Varignon's Theorem may be used to find $M_o$.

Que) A 100N-vertical force is applied at the end of a lever which is attached to the shaft at 'O'. Determine:

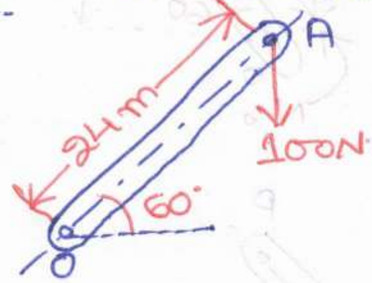
(a) moment about 'O'

(b) horizontal force at 'A' which creates the same moment

(c) smallest force at 'A' which produces the same moment.

(d) location for a 240-N vertical force to produce the same moment.

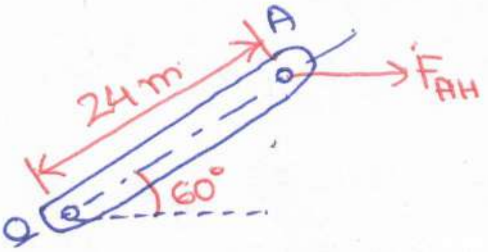
(e) whether any of the forces from b, c, and d is equivalent to the original force.




Solution

(a) $M_o = (100)(24\cos 60^\circ) = 100 \times 24 \times 0.5 = \underline{1200\,N\text{-}m}$

(b)

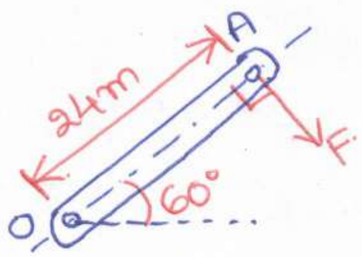



• $F_{AH}(24\sin 60^\circ) = 1200$

$\Rightarrow F_{AH} = \dfrac{1200 \times 2}{24 * \sqrt{3}} = \dfrac{100}{\sqrt{3}}\,N$

$= \underline{57.74N}$

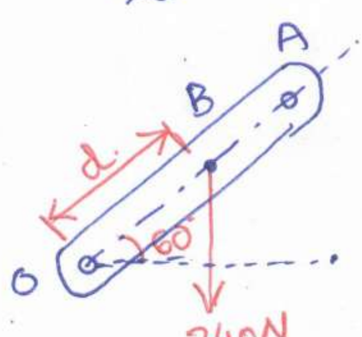
(c) The smallest force at 'A' to produce the same moment about 'O' occurs when the perpendicular distance is max$^m$. or $F \perp OA$.




$\therefore F(24) = 1200\,N\text{-}m$

$\Rightarrow F = \dfrac{1200}{24} = \underline{50N}$

(d)

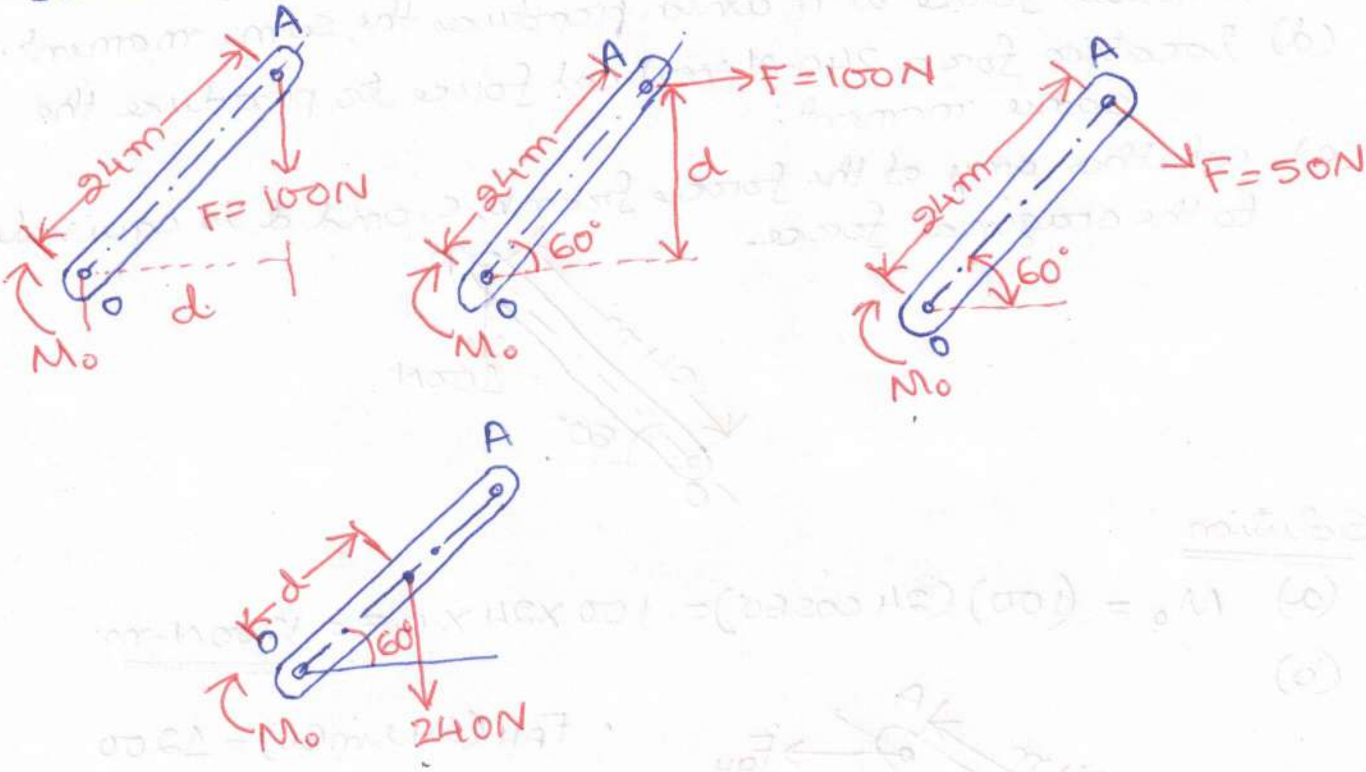



• $M_o = 240(d\cos 60^\circ)$

$\Rightarrow 1200 = 240 * \frac{1}{2} d$

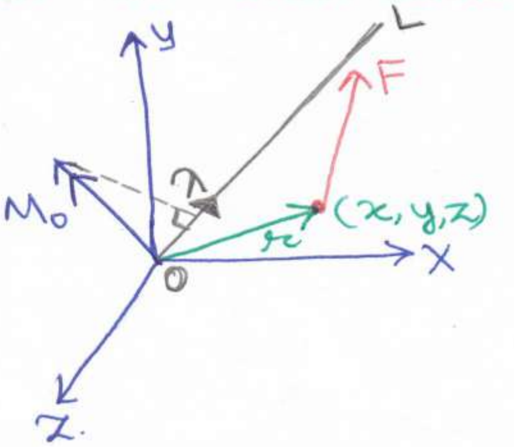
$\Rightarrow d = OB = \underline{10m}$

(e) Although each of the forces in parts (b), (c), and (d) produce the same moment ~~of~~ as 100N force, none are of the same magnitude and sense, or on the same line of action. None of the force systems are equivalent to the 100N force.



# Moment of a Force About a Given Axis

⊛

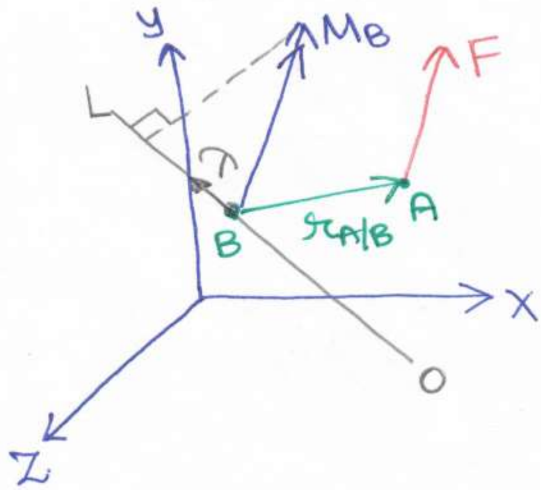



$$(M_0)_{OL} = \vec{\lambda} \cdot \vec{M}_0$$

$$= \vec{\lambda} \cdot (\vec{r} \times \vec{F})$$

$$= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

⊛




$$(M_B)_{OL} = \vec{\lambda} \cdot \vec{M}_B$$

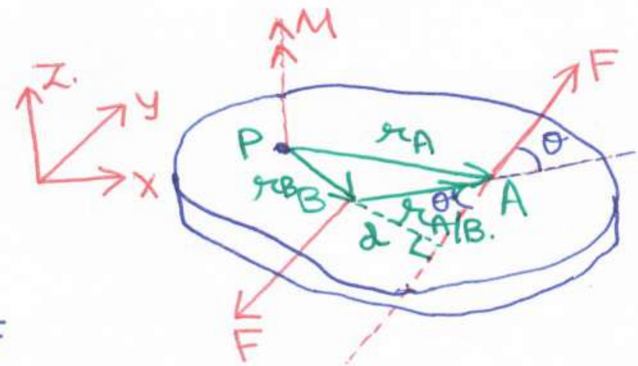
$$= \vec{\lambda} \cdot (\vec{r}_{A/B} \times \vec{F})$$

$$= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix}$$

## Moment of a Couple.

- The force system consisting of two equal & parallel but non-collinear forces with opposite sense is called a couple.

$$\vec{M} = \vec{r}_A \times \vec{F} - \vec{r}_B \times \vec{F}$$




$$= (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

$$= \vec{r}_{A/B} \times \vec{F}$$

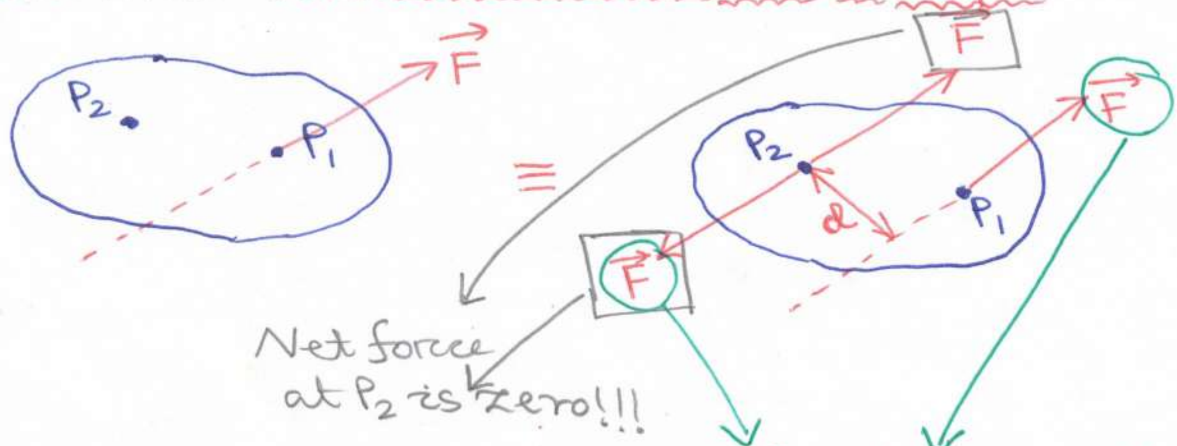
$$= r_{A/B} F \sin\theta$$

$$= (r_{A/B} \sin\theta) F = dF$$

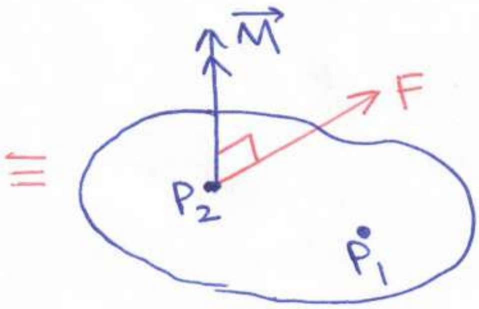
⇒ Point 'P' has no consequence and the moment $\vec{M}$ depends only on the distance between the two force vectors 'F'.

- For any point, the value of M and its direction is same
- Couple is a free vector.

# Resolution of a Force into a Force and a Couple

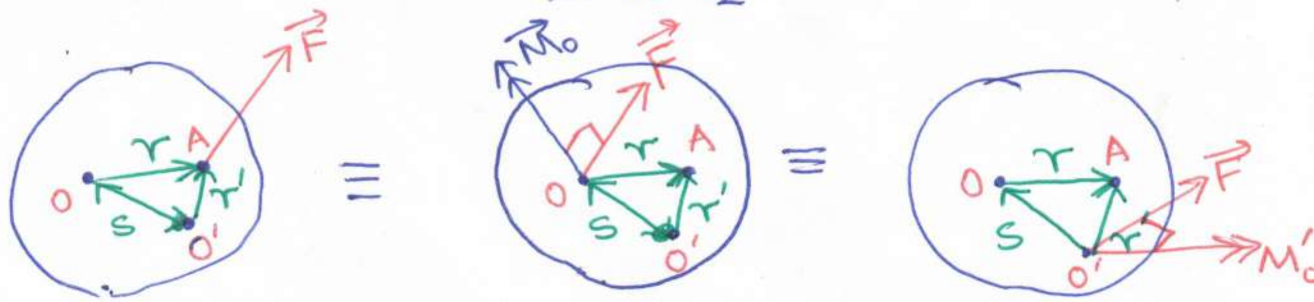



These two parallel, equal and non-collinear forces form a couple.




⊛ Since couple $\vec{M}$ is a free vector its value and direction is same at all the points under consideration in a plane. Hence, it is convinient to shift it to $P_2$.

⊛




$$\vec{M}_{o'} = (\vec{r}\,' \times \vec{F})$$

$$= (\vec{s} + \vec{r}) \times \vec{F}$$

$$= \vec{s} \times \vec{F} + \vec{r} \times \vec{F}$$

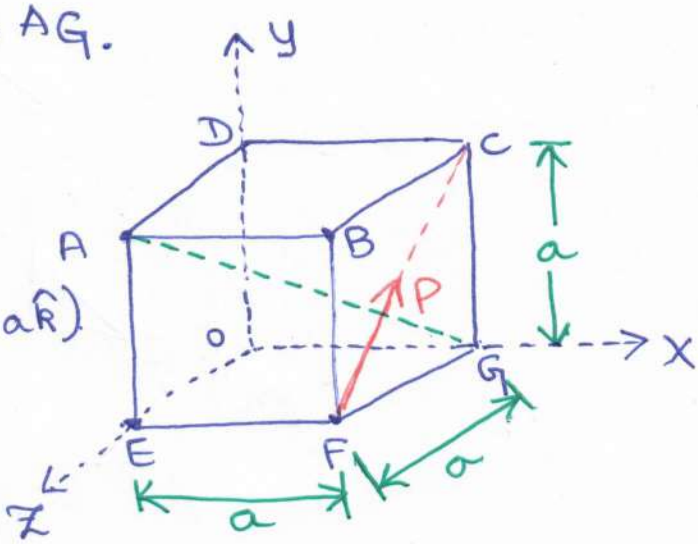
$$\Rightarrow \boxed{\vec{M}_{o'} = \vec{s} \times \vec{F} + \vec{M}_o}$$

**Que)** A cube is acted on by a force $\vec{P}$ as shown in the figure. Determine the moment of $\vec{P}$:

(a) about 'A'

(b) about the edge AB.

(c) about the diagonal AG.




**Soln.**

(a) $\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$

• $\vec{r}_{F/A} = \vec{r}_F - \vec{r}_A$

$= (a\hat{i} + a\hat{k}) - (a\hat{j} + a\hat{k})$

$\Rightarrow \vec{r}_{F/A} = a(\hat{i} - \hat{j})$.

• $\vec{P} = P\vec{\lambda}$

$= \dfrac{P\,\vec{r}_{C/F}}{|\vec{r}_{C/F}|} = \dfrac{P\,a(\hat{j}-\hat{k})}{\sqrt{2}\,a}$.

$\Rightarrow \vec{P} = \dfrac{P}{\sqrt{2}}(\hat{j} - \hat{k})$.

$\vec{r}_{C/F} = \vec{r}_C - \vec{r}_F$

$= (a\hat{i} + a\hat{j}) - (a\hat{i} + a\hat{k})$

$= a(\hat{j} - \hat{k})$

$|\vec{r}_{C/F}| = a(\sqrt{2})$

$\therefore \vec{M}_A = a(\hat{i} - \hat{j}) \times \dfrac{P}{\sqrt{2}}(\hat{j} - \hat{k})$.

$$\Rightarrow \boxed{\vec{M}_A = \frac{Pa}{\sqrt{2}}(\hat{i} + \hat{j} + \hat{k})}$$

(b) $M_{AB} = \vec{M}_A \cdot \dfrac{\vec{AB}}{|AB|}$

$= \dfrac{Pa}{\sqrt{2}}(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{i}$

$$\Rightarrow \boxed{M_{AB} = \frac{Pa}{\sqrt{2}}}$$

$\vec{AB} = \dfrac{\vec{r}_B - \vec{r}_A}{|AB|}$

$\Rightarrow \vec{AB} = \dfrac{a(\hat{i} + \hat{j} + \hat{k}) - (\hat{j} + \hat{k})a}{a}$

$\Rightarrow \vec{AB} = \hat{i}$.

(c) $M_{AG} = \vec{M}_A \cdot \dfrac{\vec{AG}}{|\vec{AG}|}$

$= \dfrac{Pa}{\sqrt{2}}(\hat{i} + \hat{j} + \hat{k}) \cdot \dfrac{\cancel{a}(\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}\,\cancel{a}}$

$\vec{AG} = \vec{r}_{GA} = \vec{r}_G - \vec{r}_A$

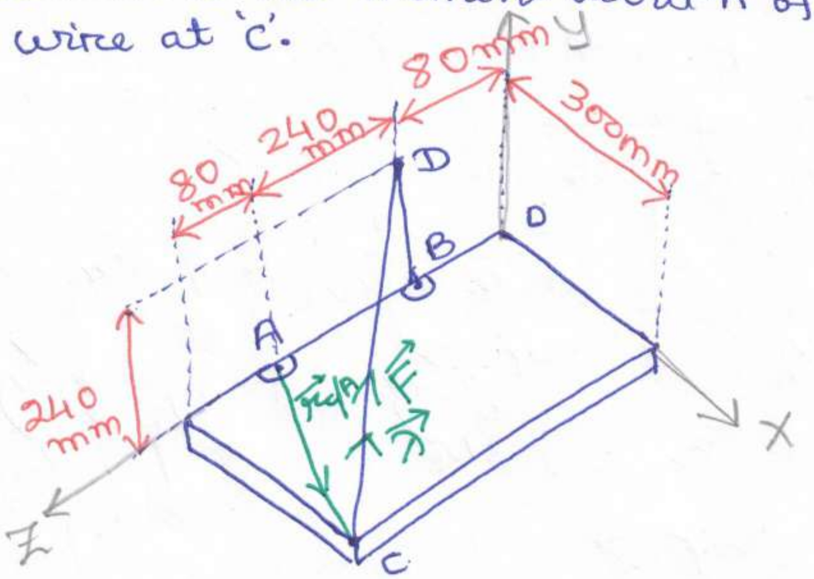
$\Rightarrow \vec{AG} = (a\hat{i} - a\hat{j} - a\hat{k})$

$|\vec{AG}| = a(\sqrt{3})$.

$\Rightarrow M_{AG} = \dfrac{Pa}{\sqrt{6}}(\cancel{1} - \cancel{1} - 1)$

$$\Rightarrow \boxed{M_{AG} = -\frac{Pa}{\sqrt{6}}}$$

**Que)** The rectangular plate is supported by the brackets at A & B, and by a wire CD. Knowing the tension in the wire is 200N, determine the moment about 'A' of the force exerted by the wire at 'C'.




**Sol$^n$.**

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A$$
$$= (0.3\hat{i} + 0.4\hat{k})m - (0.32\hat{k})m$$
$$= (0.3\hat{i} + 0.08\hat{k})m$$

$\circledast$ $\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$.

We need to determine $\vec{F}$. We know that -

$$\vec{F} = F\vec{\lambda}$$
$$\Rightarrow \vec{F} = (200)\frac{\vec{r}_{D/C}}{|r_{D/C}|} = 200\frac{(\vec{r}_D - \vec{r}_C)}{|\vec{r}_{D/C}|}$$

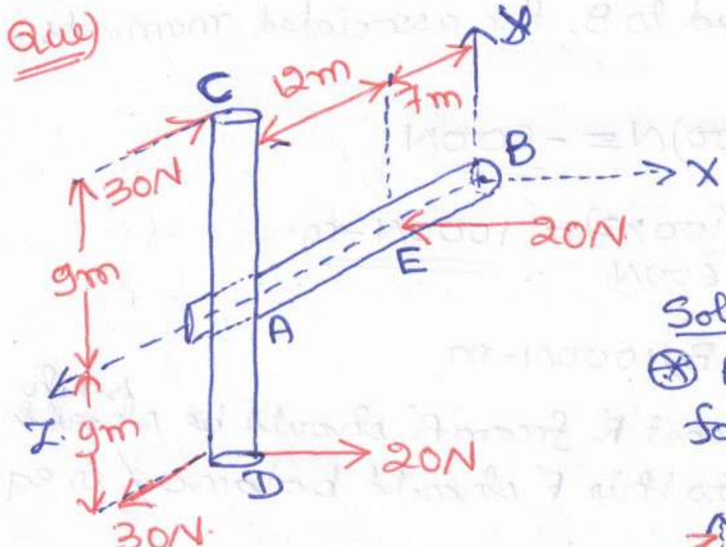
$$\vec{r}_{D/C} = \vec{r}_D - \vec{r}_C = (0.08\hat{k} + 0.24\hat{j}) - (0.3\hat{i} + 0.4\hat{k})$$
$$\Rightarrow \vec{r}_{D/C} = (-0.3\hat{i} + 0.24\hat{j} - 0.32\hat{k})m$$
$$|\vec{r}_{D/C}| = \sqrt{(-0.3)^2 + (0.24)^2 + (-0.32)^2} = 0.5m$$

$$\therefore \vec{F} = 200\frac{(-0.3\hat{i} + 0.24\hat{j} - 0.32\hat{k})}{0.5} = (-120\hat{i} + 96\hat{j} - 128\hat{k})N.$$

$$\therefore \vec{M}_A = (\cancel{-}0.3\hat{i} + \cancel{0.24\hat{j}} \overset{+0.08\hat{k}}{\cancel{+0.08\hat{k}}}) \times (-120\hat{i} + 96\hat{j} - 128\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.3 & 0 & 0.08 \\ (-120) & (96) & (-128) \end{vmatrix} = \boxed{(-7.68\hat{i} + 28.8\hat{j} + 28.8\hat{k})N\text{-}m = \vec{M}_A}$$

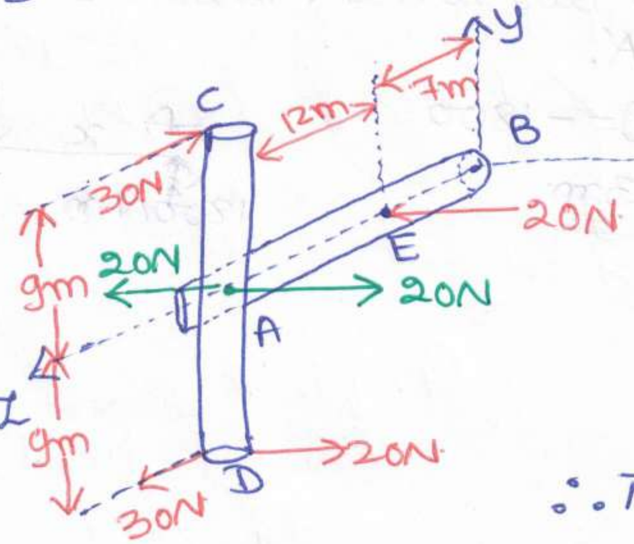
Que)




Determine the components of the single couple equivalent to the couples shown in the figure.

Solution.

⊛ Attach equal and opposite 20N force at 'A' in the ± X-directions.



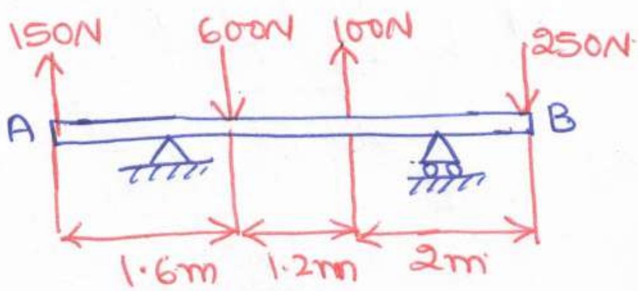

$\vec{M}_x = -(30 \times 18) = -540\,N\text{-}m$

$M_y = (20 \times 12) = +240\,N\text{-}m$

$M_z = (20 \times 9) = +180\,N\text{-}m$

$\therefore \vec{M} = (-540\hat{i} + 240\hat{j} + 180\hat{k})\ N\text{-}m.$

Que)




For the beam shown in the figure, reduce the system of forces to :-

(a) an equivalent force-couple system at A

(b) an equivalent force-couple system at B.

(c) a single resultant/force.

Solution.

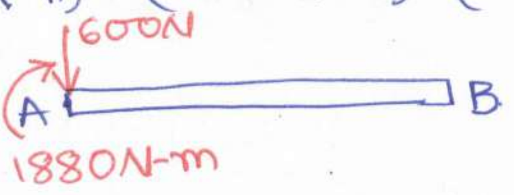
(a) when the forces will be shifted to A, the associated moments should also be shifted to A:

$\therefore\ \Sigma(R)_A = +150N - 600N + 100N - 250N$

$= -600N.$

$\Sigma(M_A) = (-600 \times 1.6) + (100 \times 2.8) - (250 \times 4.8) = (-1880\,N\text{-}m).$

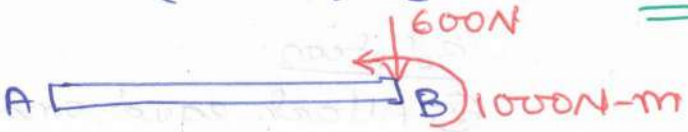
(Anti-clockwise moment = +ve)



(b) When the forces will be shifted to B, the associated moments will also be shifted to B.

$\cancel{\Sigma} R_B = (-250 + 150 - 600 + 100) N = \underline{-600N}$

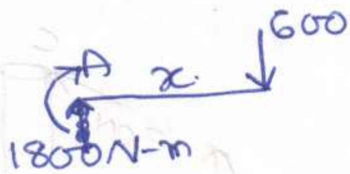
$\Sigma M_B = (150 \times 4.8) + (600 \times 3.2) - (100 \times 2) = \underline{1000N\text{-}m}$




(c) The distance of the resultant 'R' from 'A' should be ~~should~~ such that, the moment due to this 'R' should balance/ is equ to the moment at 'A'.

$(x)(600) = -1800$

$\Rightarrow \underline{x = -3m}$



# Equilibrium of Rigid Bodies.

- For a rigid body in static equilibrium, the external forces and moments are balanced and will impart no translation or rotation to the body.

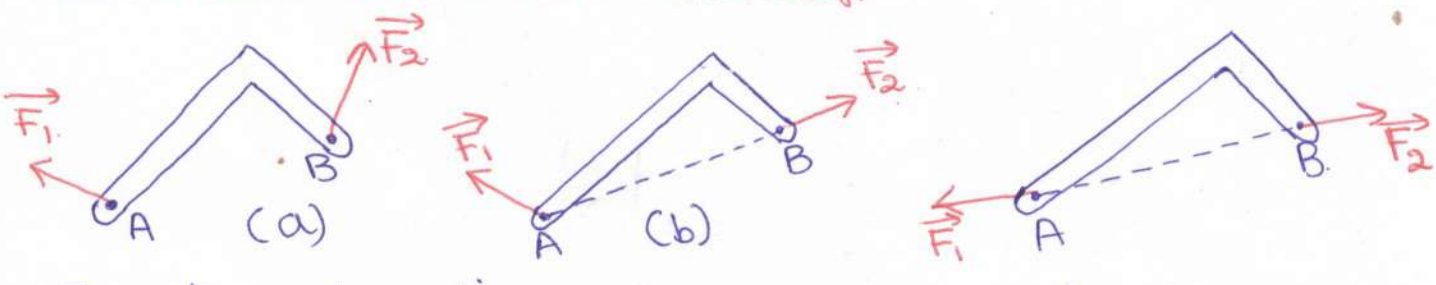
- The necessary and sufficient condition for the static equilibrium of a body are that the resultant force and couple from all external forces form a system equivalent to zero.

$$\Rightarrow \boxed{\sum \vec{F} = 0}\ ,\ \boxed{\sum \vec{M}_o = (\vec{r} \times \vec{F}) = 0}$$

- Resolving each force and moment into its rectangular components lead to 6 scalar equations, which also expresses the condition of static equilibrium.

$$\boxed{\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum F_z &= 0 \end{aligned}} \quad \boxed{\begin{aligned} \sum (M_o)_x &= 0 \\ \sum (M_o)_y &= 0 \\ \sum (M_o)_z &= 0 \end{aligned}}$$

## Equilibrium of a Two-Force Body.




- Consider a L-section subjected to forces $\vec{F_1}$ and $\vec{F_2}$ at A and B

- For the static eqlb$^m$., the sum of moments about 'A' must be zero. So, the moment of $\vec{F_2}$ must be zero.
  - $\Rightarrow$ Line of action of $\vec{F_2}$ must pass through A (Case-b)

- Similarly, the line of action of $\vec{F_1}$ must pass through 'B' for the sum of moments about 'B' to be zero (Case-c)

→ So, requiring the sum of moments to be zero, the $\boxed{\text{forces must be collinear}}$.
Also, for the eqlb$^m$. of forces, $\vec{F_1}$ and $\vec{F_2}$ must be equal in magnitude but opposite in sense.

## Free-Body Diagram (FBD):

— A free-body diagram is a sketch of a selected system consisting of a body, a portion of the body, or a collection of interconnected bodies completely isolated or free from all other bodies, showing the interaction of all other bodies by forces on the one being considered.

Three essential characteristics of an FBD:—

(i) It is a sketch/ diagram of the selected system.

(ii) The system is shown completely isolated from all other bodies including foundations, supports, etc.

(iii) The interaction on the system by each body removed in the isolating process is shown as a force(s) on the diagram.

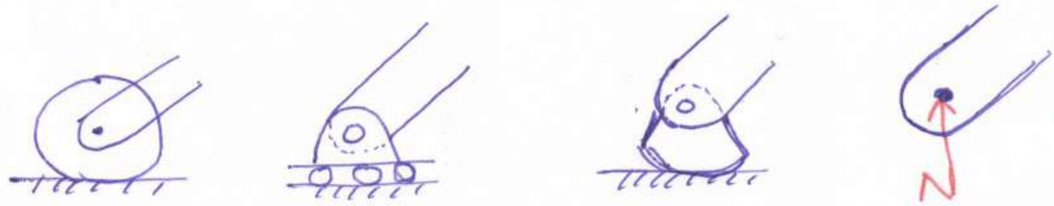
— Surface forces are applied through direct mechanical contact, whereas Body forces (gravitational/ magnetic/ electric) are applied through remote action.

### Reactions at Supports/ Connections

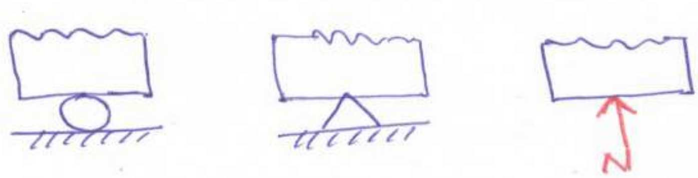
CLASS-I: ⊛ Support condition restrains translation in one-direction.

⊛ Only one unknown :- Magnitude of reaction force normal to the contact surface.

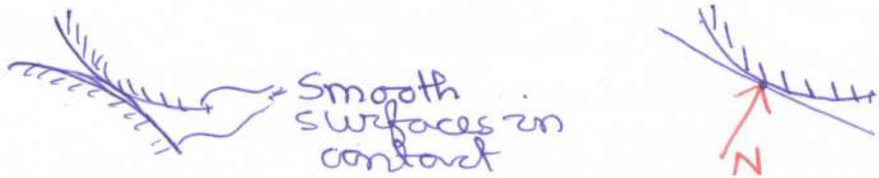
⊛ Example: Roller support, frictionless surfaces, Rockers, Wall edges, etc.

⊛ 


No. of unknowns = 1

⊛ 

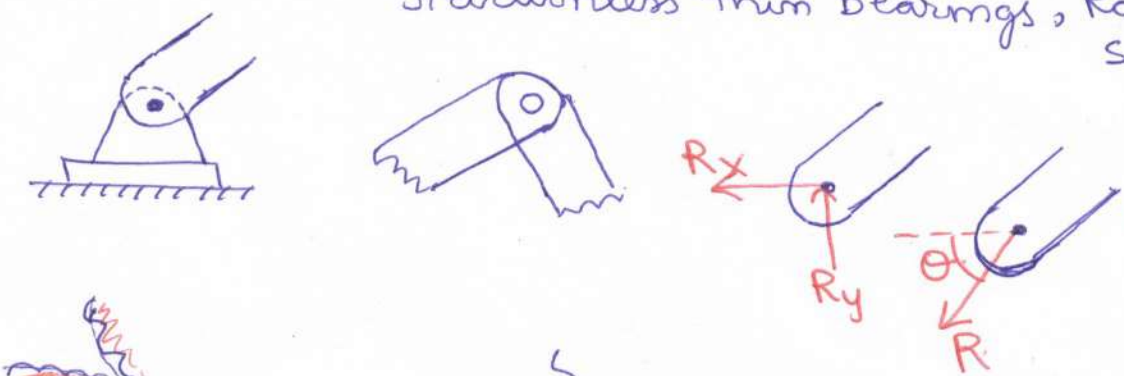

No. of unknowns = 1

⊛ 

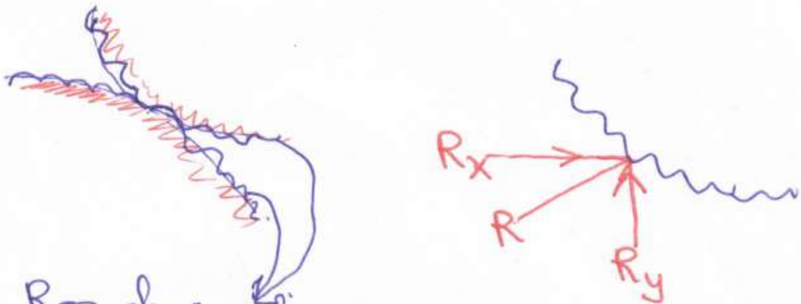

No. of unknowns = 1

CLASS-II :- Translation is restrained, but rotation is allowed.

- 2 unknowns: Magnitude and direction of force at support.
- Examples: Hinges, frictionless pin connections, frictionless thin bearings, Rough surfaces.




No. of unknowns = 2

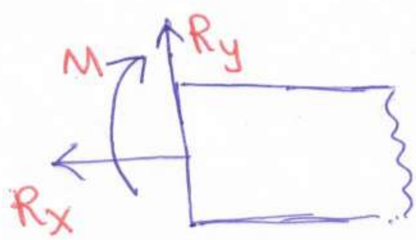



Rough surfaces in contact.

No. of unknowns = 2

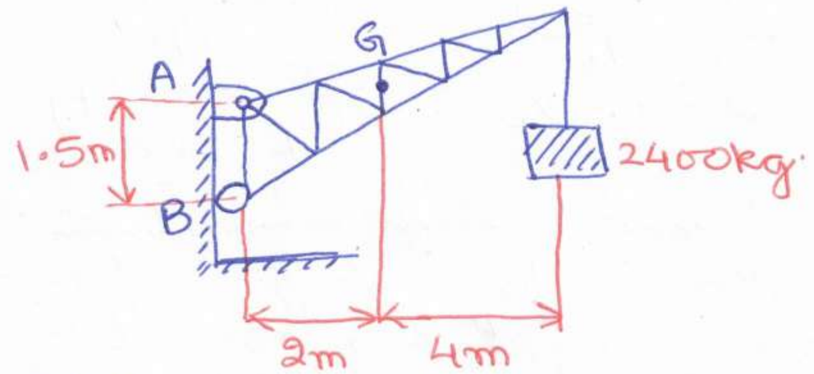
CLASS-III : - Translation and Rotation are totally restricted.

- Unknowns: (3) ⇒ Reduced to a force & couple.
- Examples: Fixed supports, welded ends, 2 closely spaced pinned joints

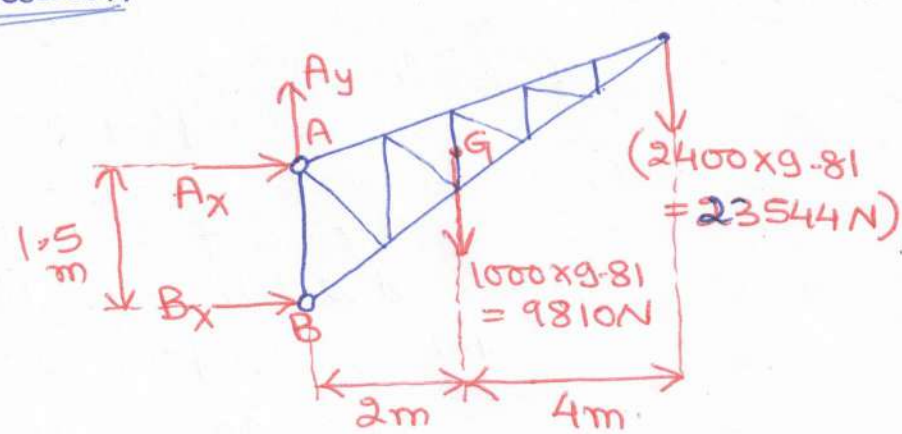




No. of unknowns = 3.

Que) A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held by a pin at A and a roller support at B. The center of gravity of the crane is located at 'G'. Determine the reactions at 'A' and 'B'.




Solution.




For the crane to be in equilibrium,

- $\sum \vec{F_y} = 0$

$\Rightarrow A_y = 9810 + 23544 = \underline{33354 N}$

- $\sum \vec{F_x} = 0$

$\Rightarrow A_x = -B_x$

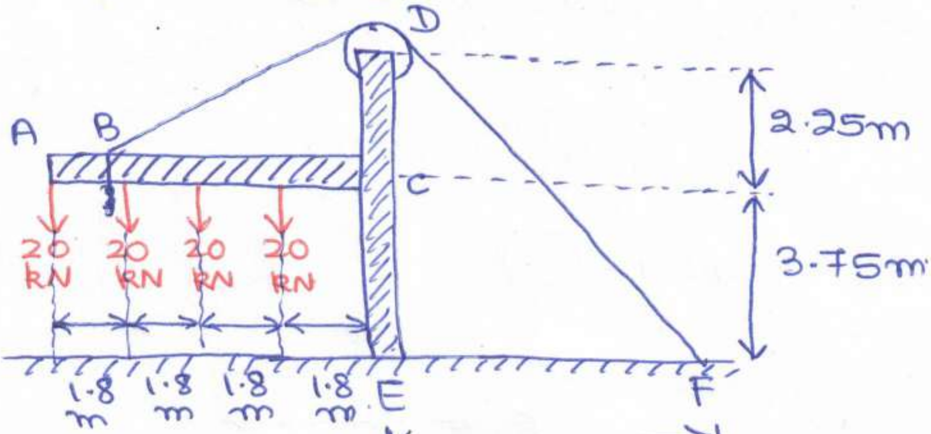
- $\sum \vec{M_A} = 0$

$\Rightarrow (B_x)(1.5) = (9810 \times 2) + (23544 \times 6)$

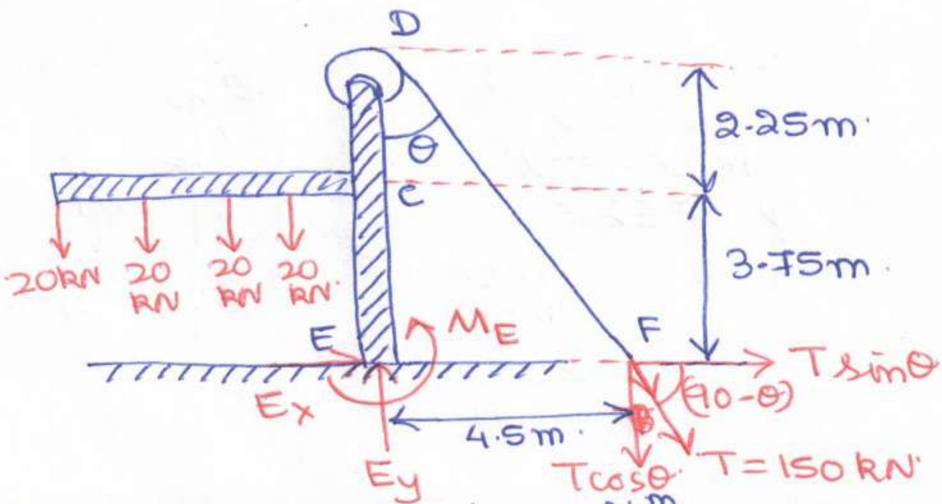
$\Rightarrow \boxed{B_x = 107256 N}$

$\Rightarrow \boxed{A_x = -107256 N}$

Que) The frame supports part of the roof of a small building. The tension in the cable is 150kN. Determine the reaction at fixed end E.




Solution




$DF = \sqrt{(4.5)^2+(6)^2}$

$\Rightarrow DF = 7.5m$

$\cos\theta = \frac{6}{7.5}$

$\sin\theta = \frac{4.5}{7.5}$

For the structure to be in eqlb$^m$.,

- $\sum \vec{F_x} = 0$

$\Rightarrow E_x + T\sin\theta = 0$

$\Rightarrow E_x = -150 \times \frac{EF}{DF} = -150 \times \frac{4.5}{\sqrt{(4.5)^2+(6)^2}} = -150 \times \frac{4.5}{7.5}$

$\Rightarrow \boxed{E_x = -90\,kN}$

- $\sum \vec{F_y} = 0$

$\Rightarrow E_y - 4(20) - T\cos\theta = 0$

$\Rightarrow E_y = 80 + 150 \times \frac{6}{7.5}$

$\Rightarrow \boxed{E_y = 200\,kN}$

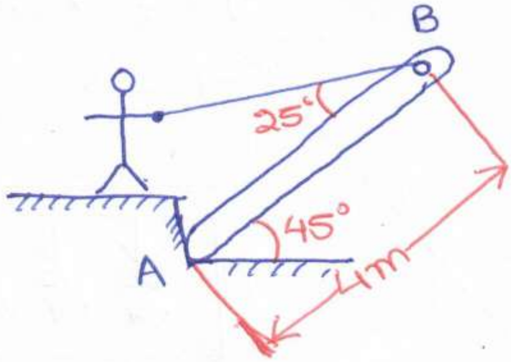
- $\sum \vec{M_E} = 0$

$\Rightarrow 20(7.2) + 20(5.4) + 20(3.6) + 20(1.8) + M_E - (T\cos\theta)(4.5) = 0$

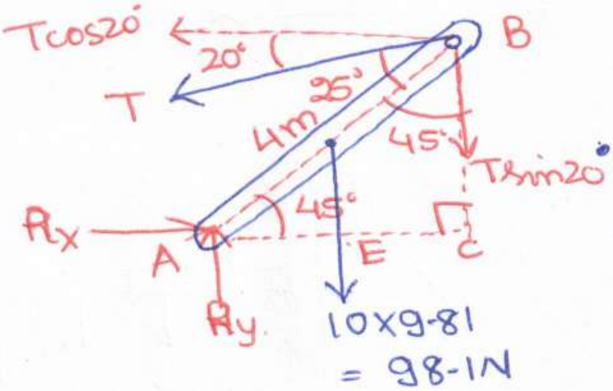
$\Rightarrow M_E = 150 \times \frac{6}{7.5}(4.5) - 20(1.8+3.6+5.4+7.2)$

$\Rightarrow \boxed{M_E = 180\,kN\text{-}m}$

Que) A man raises a 10 kg joist of length 4 m, by pulling a rope. Find the tension in the rope and the reaction at A:




Solution




For the joist to be in equilibrium –

• $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$

• $\Sigma(\vec{M_A}) = 0$

$\Rightarrow (T\cos 20)\,BC - (T\sin 20)(AC) - (98.1)(AE) = 0$

$\Rightarrow (T\cos 20)(2\sqrt{2}) - (T\sin 20)(2\sqrt{2}) = (98.1)(\sqrt{2})$

$\Rightarrow T = \dfrac{98.1}{2(\cos 20° - \sin 20°)} = 82.06N$

• $\Sigma \vec{F_x} = 0$

$\Rightarrow A_x = T\cos 20 = 82.06 \cos 20°$

$\Rightarrow \boxed{A_x = 77.11N}$

• $\Sigma \vec{F_y} = 0$

$\Rightarrow A_y = T\sin 20° + 98.1$

$\Rightarrow A_y = 82.06 \sin 20° + 98.1$

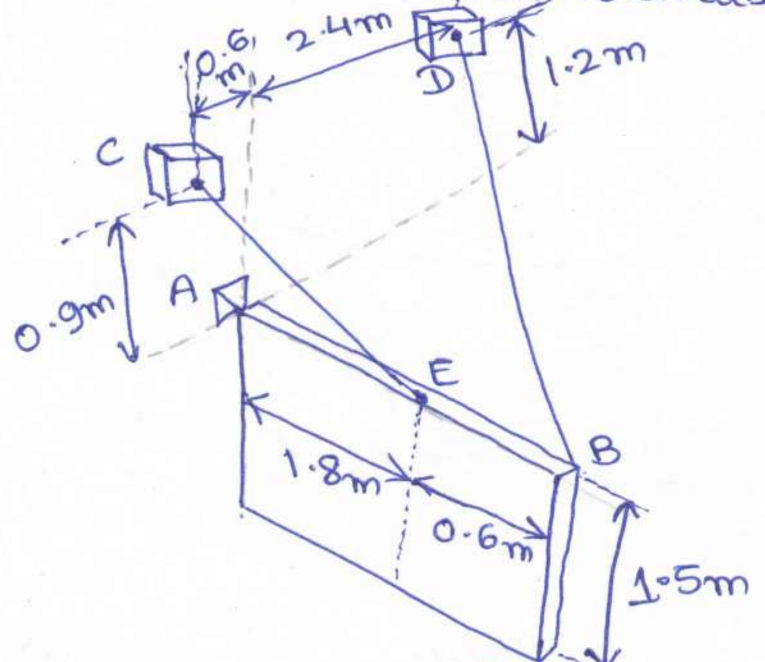
$\Rightarrow \boxed{A_y = 126.17N}$

$\therefore |A| = \sqrt{(A_x)^2 + (A_y)^2}$

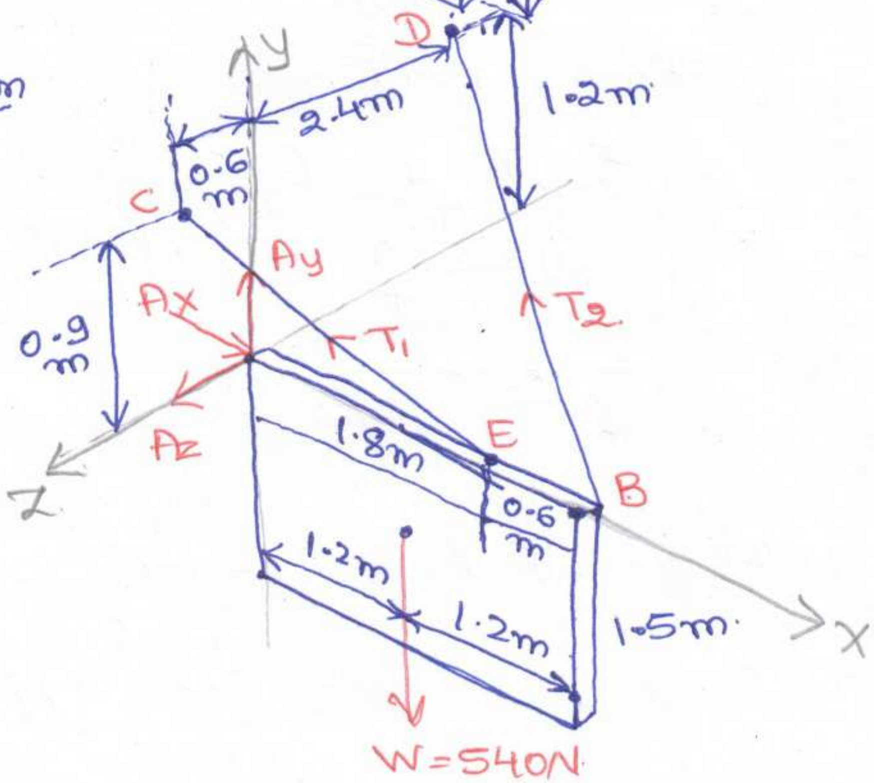
$= \sqrt{(77.11)^2 + (126.17)^2}$

$\Rightarrow |A| = 147.87N$

Que) A board of uniform density weighs 540N and is supported by a ball and socket joint at 'A' and by two cables. Determine the tension in each cable and the reaction at 'A'.




Solution




(*) $\vec{T_2} = T_2 \cdot \dfrac{\vec{r}_{DB}}{|\vec{r}_{DB}|}$

$\Rightarrow \vec{T_2} = T_2 \dfrac{(-2.4\hat{i}+1.2\hat{j}-2.4\hat{k})}{3.6}$

$\Rightarrow \vec{T_2} = T_2\left(-\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}-\frac{2}{3}\hat{k}\right)$

$\vec{r}_{DB} = \vec{r}_D - \vec{r}_B$

$= (1.2\hat{j}-2.4\hat{k})-(2.4\hat{i})$

$= -2.4\hat{i}+1.2\hat{j}-2.4\hat{k}$

$|\vec{r}_{DB}| = \sqrt{(-2.4)^2+(1.2)^2+(-2.4)^2}$

$= 3.6\,m.$

$$\circledast \ \vec{T_1} = T_1 \frac{\vec{r}_{CE}}{|\vec{r}_{CE}|} = T_1 \frac{(\vec{r}_C - \vec{r}_E)}{|\vec{r}_{CE}|}$$

$$= T_1\left(\frac{-1.8}{2.1}\hat{i} + \frac{0.9}{2.1}\hat{j} + \frac{0.6}{2.1}\hat{k}\right)$$

$$\Rightarrow \vec{T_1} = T_1\left(-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right)$$

$$\vec{r}_C - \vec{r}_E = (0.9\hat{j} + 0.6\hat{k}) - (1.8\hat{i}) = -1.8\hat{i} + 0.9\hat{j} + 0.6\hat{k}$$

$$|\vec{r}_{CE}| = \sqrt{(-1.8)^2 + (0.9)^2 + (0.6)^2} = 2.1\,m$$

- $\Sigma \vec{M}_A = 0$

$$\Rightarrow (\vec{r}_{EA} \times \vec{T_1}) + (\vec{r}_{BA} \times \vec{T_2}) + \cancel{(\times -540\hat{j})}\ (1.2\hat{i}) \times (-540\hat{j})$$

$$\Rightarrow 1.8\hat{i} \times T_1\left(\frac{-6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) + 2.4\hat{i} \times T_2\left(-\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) - 648\hat{k} = 0.$$

Equating '$\hat{j}$' and '$\hat{k}$' co-efficients on both the sides of the eqn.

$\hat{j}$: $1.6T_2 - 0.514T_1 = 0$

$\hat{k}$: $0.8T_2 + 0.771T_1 = 648$

Solving the above two eqns. we get

$$\boxed{T_1 = 630.3N}$$
$$\boxed{T_2 = 202.6N}$$

- $\Sigma \vec{F} = 0$

$$\Rightarrow \vec{A} + \vec{T_1} + \vec{T_2} = 0$$

$\Rightarrow \hat{i}$: $A_x - \frac{2}{3}T_2 - \frac{6}{7}T_1 = 0$

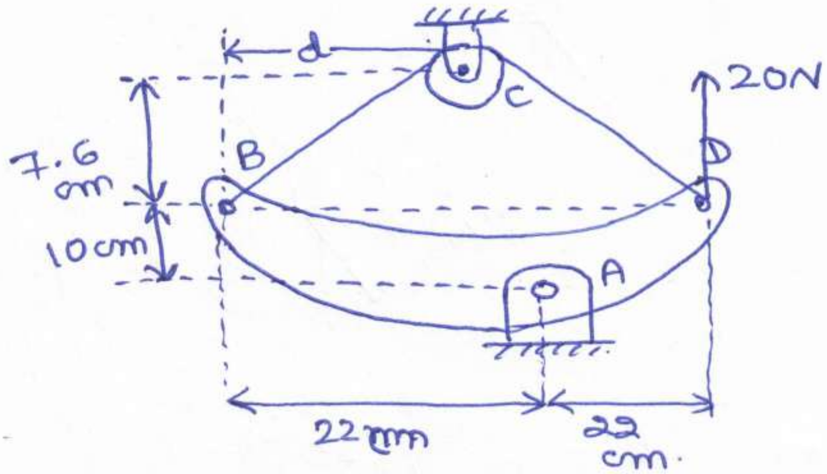
$\hat{j}$: $A_y + \frac{1}{3}T_2 + \frac{3}{7}T_1 = 0$

$\hat{k}$: $A_z - \frac{2}{3}T_2 + \frac{2}{7}T_1 = 0$

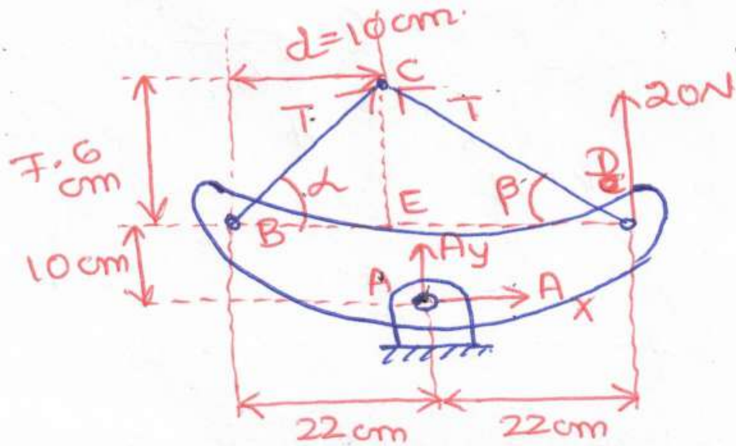
Substituting $T_1$ and $T_2$ in the above eqns, we have

$$\vec{A} = [(675.3)\hat{i} + (202.3)\hat{j} - (45.02)\hat{k}]N$$

Que) Neglecting the friction and radius of the pulley, determine the tension in the cable BCD and the reaction at the support 'A' when $d = 10\,cm$.




Solution




$\tan\alpha = \frac{7.6}{10}$

$\Rightarrow \alpha = 37.23°$

$\tan\beta = \frac{CE}{DE}$

$\Rightarrow \tan\beta = \frac{CE}{BD-BE} = \frac{7.6}{44-10}$

$\Rightarrow \beta = 12.6°$

- $\sum \vec{M}_A = 0$

$\Rightarrow (-T\cos\alpha)(10) - (T\sin\alpha)(22) + 20(22) + (T\sin\beta)(22) + (T\cos\beta)(10) = 0$

$\Rightarrow T(10\cos\alpha + 22\sin\alpha - 22\sin\beta - 10\cos\beta) = 440$

$\Rightarrow \boxed{T = 65.53\,N}$

- $\sum \vec{F}_x = 0$

$\Rightarrow T\cos\alpha + A_x = T\cos\beta$

$\Rightarrow A_x = 65.53(\cos 12.6° - \cos 37.23°)$

$\Rightarrow \boxed{A_x = 11.78\,N}$

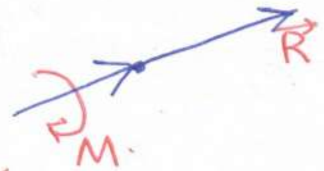
- $\sum \vec{F}_y = 0$

$\Rightarrow A_y + 20 + T\sin\alpha + T\sin\beta = 0$

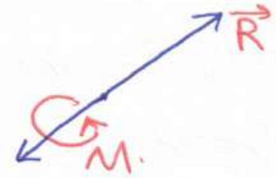
$\Rightarrow A_y = -20 - 65.53(\sin 37.23° + \sin 12.6°)$

$\Rightarrow \boxed{A_y = -73.94\,N}$

Wrench – The resultant couple $\vec{M}$ and the resultant force $\vec{R}$ act along the same line of action.

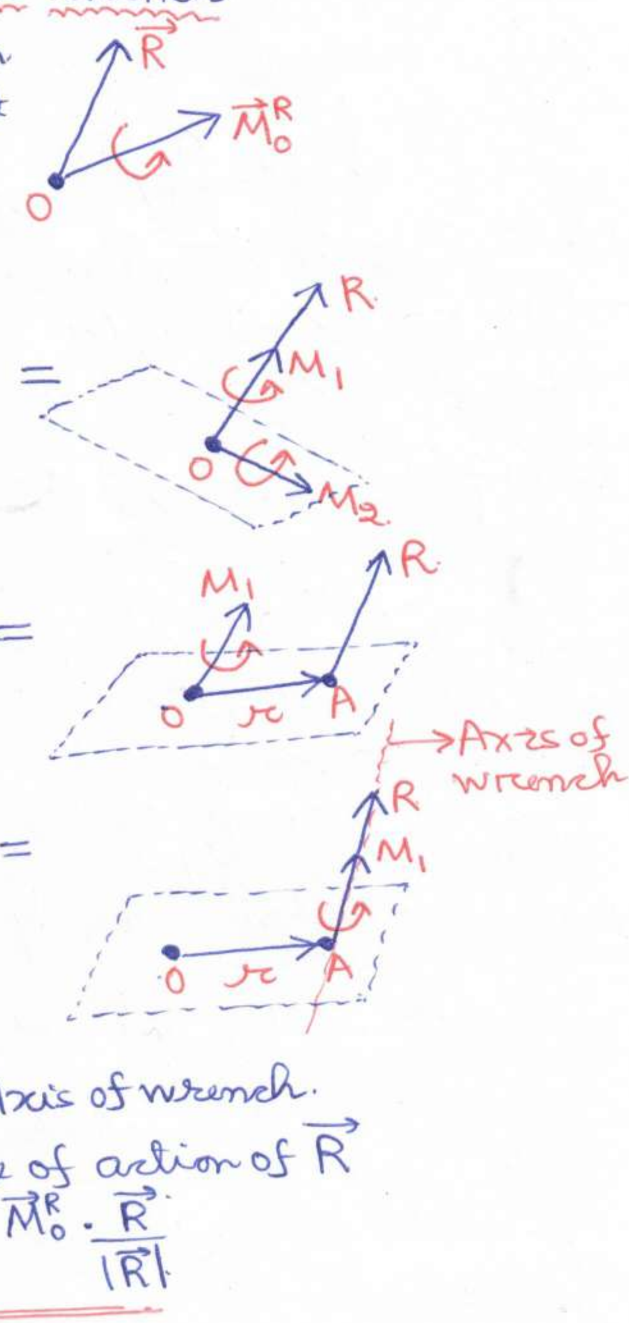



(Positive Wrench)




(Negative Wrench).

Reduction of a system of force to wrench.




- In the general case of the system of forces in space, the equivalent force-couple system at O consists of a force $\vec{R}$, and a couple vector $M_0^R$ which is not perpendicular.
- Couple $M_0^R$ is resolved into two components:
  - $M_1$: parallel to R.
  - $M_2$: perpendicular to R.
- $M_2 = \vec{r} \times \vec{R}$
- The couple vector $M_1$ and the resultant force vector $\vec{R}$ is shifted to a new-line of action of force

$$\therefore \vec{M}_0^R = \vec{M}_1 + (\vec{r} \times \vec{R})$$

- Pitch of wrench $= p = \dfrac{M_1}{|R|}$
- Line of action of wrench $\equiv$ Axis of wrench.
- Projection of $\vec{M}_0^R$ on the line of action of $\vec{R}$

$$\Rightarrow M_1 = \vec{M}_0^R \cdot \lambda = \vec{M}_0^R \cdot \frac{\vec{R}}{|\vec{R}|}$$

$$\therefore p = \frac{M_1}{|R|} = \frac{\vec{M}_0^R \cdot \vec{R}}{|R|^2}$$