Course Syllabus and Plan

Black body radiation: Classical approach (Rayleigh-Jeans theory and Wien's displacement law); ultraviolet catastrophe; Planks law of Black body radiation.

Reading:

- ♣ Concepts in Thermal Physics by Stephen J. Blundell and Katherine M. Blundell
- ♣ Fundamentals of Statistical and Thermal Physics by F. Reif

Disclaimer: This is not a text book. Rather, these lecture notes are written by **Dr. Ritwik Mondal**, and edited by Dr. Tusharkanti Dey, Department of Physics, IIT (ISM) Dhanbad. The contents are taken from several text books. Please read the above books for your reference. If you find any mistakes or typos, please report to Dr. Tusharkanti Dey.
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Thermodynamic Emission

Thermal radiation is the emission of electromagnetic waves from all matter that has a temperature greater than absolute zero. In such processes, the thermal energies are converted to the electromagnetic energies. Such emission creates a spectrum of radiation. Well, you know that everything in this universe is made of fundamental

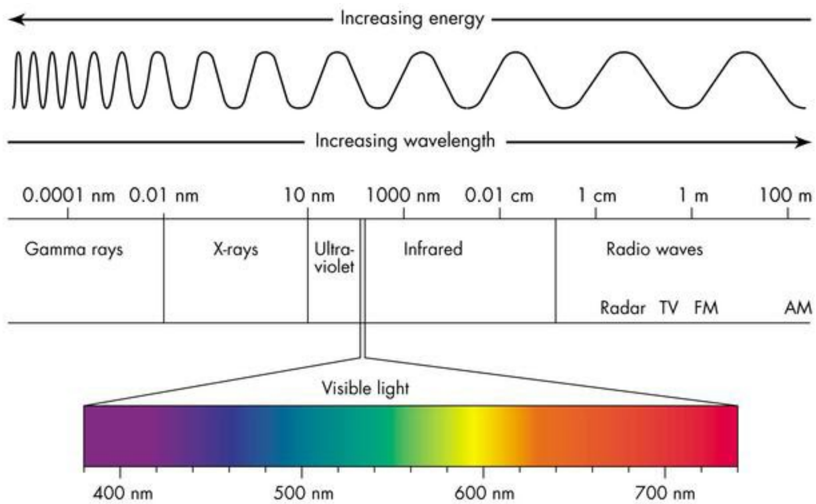

Figure 1: The electromagnetic spectrum.

atomic particles e.g., electrons, protons, neutrons. Of course, at $T \neq 0$, the fundamental particles will experience some thermal agitation which leads to the acceleration of the charged particles. Therefore, they emit some electromagnetic radiation which is very broad. All the thermodynamic processes happens at room temperature has radiation of energies that fall in the infrared regime. Thus, we can not see the radiation. However, if a system has higher energy and the radiation energy falls in the visible spectrum, then we can see the color. An example of such radiation is that when we heat up a piece of iron, we can not see anything by eye, however, after some time we see the radiation of red color.

Black Body Radiation

The first question is what is a black body and why do we need to study them?

Figure 2: The color of iron while it is heating up at high temperature.

To answer this question, let us understand first whether we can distinguish the behavior of radiation from a thermally agitated system. We know that when an incident light falls on a system, the energy can be absorbed, reflected, or transmitted. On top of this, the system can emit (radiate) some amount of energy. In almost every system such separation of radiation energy from the others is impossible, except a black body.

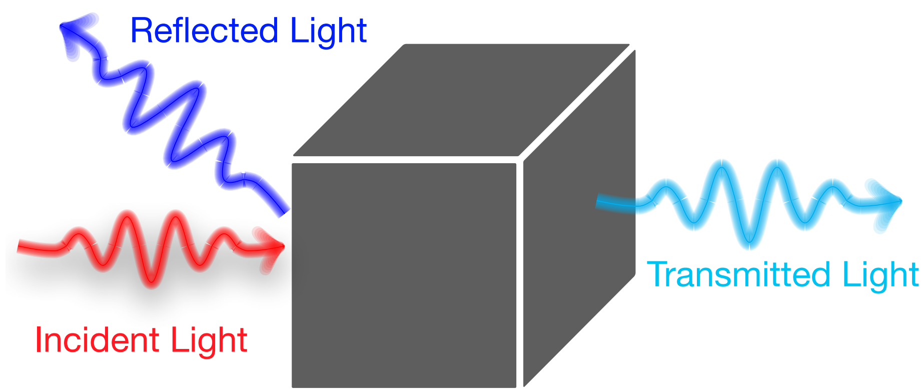

Figure 3: The incident, reflected and transmitted light. These are different than the radiation of light from an object.

Well, the black body is a body that absorbs all the electromagnetic radiation. It does not reflect or transmit any energy. Of course, it only radiates the absorbed energy. The radiation of energies that create the electromagnetic spectrum is very broad.

It means that incident visible light on a black body will turn black to a human eye.

A black body is purely a conventional object e.g., the Sun. The sun can absorb almost all the energies in the EM spectrum.
Definition: A black body is an ideal body that allows the whole of incident radiation to pass into itself without reflecting the energy and absorbs the whole incident radiation without passing on the energy. Such a property of a black body holds for any radiation energies at any angle. Therefore, a black body is a perfect absorber.

The best example of a black body is a cavity with a very small pin-hole. The radiation which goes through the pin-hole gets several reflections inside the wall of the cavity and it has negligible chance to get out of the cavity.

The spectral absorptivity $a(\lambda)d\lambda$ is the fraction of the incident radiation that is absorbed at wavelength λ.
The spectral emissive power $e(\lambda)$ of a surface is a function such that $e(\lambda)d\lambda$ is the power emitted per unit area by the electromagnetic radiation having wavelengths between λ and $\lambda + d\lambda$.
In order to be in thermal equilibrium, a body must keep a constant temperature. If a black body is kept at a temperature T, the black body has to emit all the energy it has absorbed. Kirchoff's law states that at thermal equilibrium, the power radiated by an object must be equal to the power absorbed. The ratio $e(\lambda)/a(\lambda)$ is a universal function of λ and T. Therefore, if you fix λ) and T, the ratio $e(\lambda)/a(\lambda)$ is fixed and hence $e(\lambda)$ is proportional to $a(\lambda)$. Thus, while a black body is a perfect absorber, it is also a perfect emitter.
However, this distinction is not possible in a simple object.

Such radiation in the form of energy again forms a spectrum at thermal equilibrium that looks like following.
Wien's Displacement Law: We can immediately see that the corresponding wavelength of maximum emission decreases as the temperature of the curve increases. It was Wien who first discovered that the wavelength of maximum emission (λ_{max}) must be inversely proportional to the temperature (T) such that

$$\begin{aligned} \lambda_{\mathrm{max}} &\propto \frac{1}{T} \\ \Rightarrow \lambda_{\mathrm{max}} T &= b \end{aligned} \tag{1}$$

From the set of data Wien found that $b = 2.898$ milimeter-Kelvin. This formula helps to determine the temperature of hot bodies e.g. the Sun.

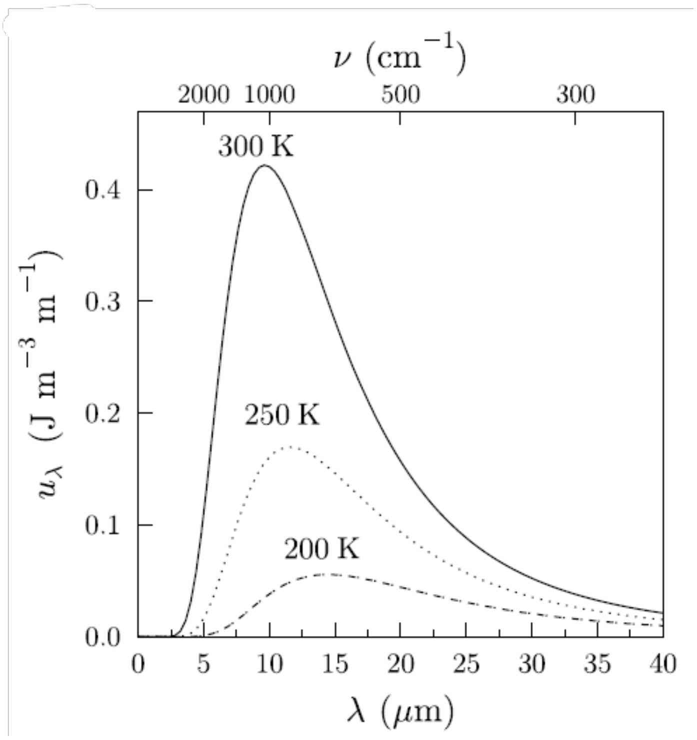

Figure 4: The distribution of spectral energy density of black body radiation for different temperatures as a function of wavelength. The upper scale shows the frequency in inverse centimetres.

Stefan-Boltzmnn Law: At a particular temperature the area under a curve provides the total emissive power of the black body. This is defined as

$$Q = \int_0^\infty u(\lambda) d\lambda \tag{2}$$

This is found to be proportional to T^4 such that

$$Q = \sigma T^4 \tag{3}$$

where $\sigma = 5.67 \times 10^{-8}$ Watt-m^{-2}-K^{-4}.
At room temperature, the radiation spectrum peaks at infrared region. However, such peak of the spectrum moves to visible region at high temperature above 1000 K that can be seen below.
The above spectrum is just the experimental observations. It is important to understand the radiation curves. Can we explain these curves and derive a functional analytical form?

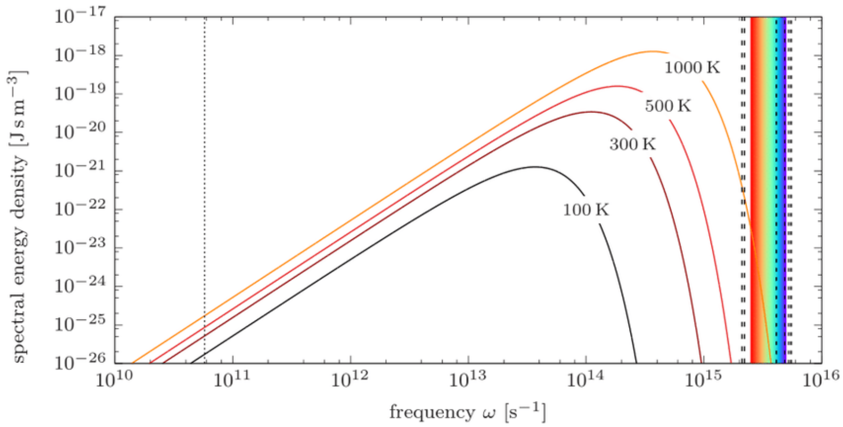

Figure 5: The black body radiation spectrum vs frequency at several low temperatures.

Rayleigh-Jeans Attempt:

Rayleigh and Jeans were the first to attempt toward a derivation of an analytical formula which can explain such observed curve. Even though we shall see that they were not successful in obtaining that.
From the classical physics perspective, Rayleigh and Jeans thought that the radiations inside a black body must be a standing wave. This is because there is hardly any energy loss while the waves are reflecting inside the cavity wall.

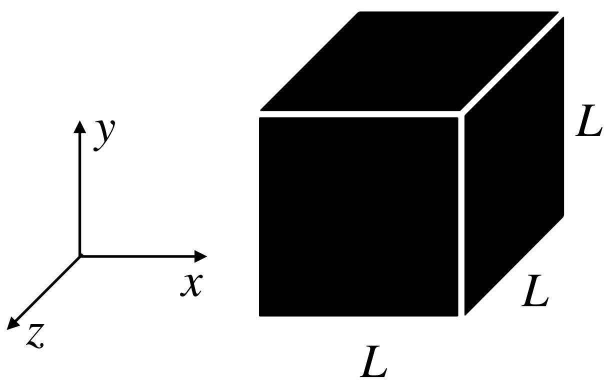

Figure 6: A cubic cavity with sides L.

Let us consider a cubic cavity with the length of wall L. The electromagnetic waves inside the cavity must follow the three-dimensional wave equation

$$\frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \tag{4}$$

where $\boldsymbol{E}(x, y, z, t)$ is the electric field vector of the standing wave inside the cavity. In order to solve such wave equation, we must have the following boundary conditions

$$\begin{aligned}
\boldsymbol{E}(x = 0, y, z, t) &= 0 \\
\boldsymbol{E}(x, y = 0, z, t) &= 0 \qquad (5)\\
\boldsymbol{E}(x, y, z = 0, t) &= 0 \qquad (6)\\
\boldsymbol{E}(x = L, y, z, t) &= 0 \qquad (7)\\
\boldsymbol{E}(x, y = L, z, t) &= 0 \qquad (8)\\
\boldsymbol{E}(x, y, z = L, t) &= 0 \qquad (9)
\end{aligned}$$

This wave equation reminds of the sound waves in one dimensional string attached in both sides. Essentially one has to solve the wave equation in one dimension but the sound wave has a velocity v. The sound wave equation is:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{10}$$

We guess the solution to be

$$y(x, t) = y_0 \sin(kx)\sin(2\pi f t) \tag{11}$$

where f and λ are the frequency and wavelength such that $f\lambda = v$. The angular frequency is defined as $2\pi f$. We have the boundary condition:

$$y(x = L, t) = 0 \Rightarrow \sin(kL) = 0 \Rightarrow kL = n\pi \tag{12}$$

where n is the positive integer. Applying the solution in the equation we obtain

$$k^2 - \frac{(2\pi f)^2}{v^2} = 0 \Rightarrow \left(\frac{n\pi}{L}\right)^2 = \frac{(2\pi f)^2}{v^2} \Rightarrow \left(\frac{2\pi}{\lambda}\right)^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \lambda = \frac{2L}{n} \tag{13}$$

Therefore the wave solution is

$$y(x, t) = y_0 \sin\left(\frac{n\pi x}{L}\right)\sin\left(2\pi f t\right) \tag{14}$$

Taking this solution to the three-dimensional case, we find the solution to be

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}_0 \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \sin(2\pi f t) \tag{15}$$

Inserting this solution into the wave equation we obtain

$$\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{L}\right)^2 - \left(\frac{2\pi f}{c}\right)^2 = 0$$

$$\Rightarrow n_x^2 + n_y^2 + n_z^2 = \left(\frac{2L}{\lambda}\right)^2 \tag{16}$$

The above equation reminds of a sphere with a radius $R = \frac{2L}{\lambda}$. As the values n_x, n_y and n_z are positive integers, we only have to consider the $\frac{1}{8}$ of the total volume of the sphere. Thus total number of modes available can be calculated as

$$N(\lambda) = \frac{1}{8} \times \frac{4\pi}{3} \left(\frac{2L}{\lambda}\right)^3 = \frac{4\pi L^3}{3\lambda^3} \tag{17}$$

Now there are two modes of electromagnetic polarization of $\boldsymbol{E}_0$ for a linearly polarised wave. Thus the correct no of modes is

$$N(\lambda) = \frac{8\pi L^3}{3\lambda^3} \tag{18}$$

The number of modes per unit wavelength can be calculated taking a differentiation

$$\frac{dN(\lambda)}{d\lambda} = -\frac{8\pi L^3}{\lambda^4} \tag{19}$$

The negative sign implies to the fact that while the wavelength λ is increasing, the no of modes will decrease. Dropping the negative sign, we obtain the density of modes per unit wavelength

$$D(\lambda) = \frac{8\pi}{\lambda^4} \tag{20}$$

Rayleigh and Jeans argued that different modes inside the cavity form some kind of oscillators. This is due to the fact that there is reflection in the wall of the cavity without losing any energy. Those oscillators must have some kinetic and potential energies. Now, the equipartition theorem assigns the kinetic energy to every mode as $k_B T/2$ and the potential energy to every mode as $k_B T/2$. Thus the average energy of every mode is $k_B T$. The energy density is obtaine as

$$u(\lambda) = \frac{8\pi k_B T}{\lambda^4} \tag{21}$$

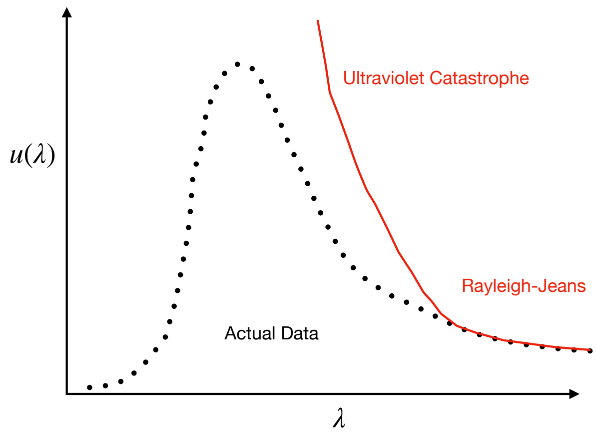

Figure 7: The theory of Rayleigh-Jeans and ultraviolet catastrophe.

This is what Rayleigh-Jeans calculated.
Ultraviolet Catastrophe:
The classical formula derived by Rayleigh and Jeans could only explain the black body radiation at higher wavelengths (low frequencies). However, the energy density increases very rapidly at lower wavelengths (high frequency) following $u(\lambda) \propto \lambda^{-4}$. In fact at $\lambda \to 0$, the energy density $u(\lambda) \to \infty$. Such phenomena never occurs in a black body radiation in an experimental data. Thus, the classical derivation of Rayleigh and Jeans does not explain the radiation spectrum completely and this is known as ultraviolet catastrophe.

Wien's Law:

Wien used the thermodynamics to show that the energy density

$$u(\lambda) = \frac{A}{\lambda^5} f(\lambda T) \tag{22}$$

To find the functional form of $f(\lambda T)$, Wien compared it to the Maxwell-Boltzmann distribution function such that

$$f(\lambda T) = e^{-\frac{a}{\lambda T}} \tag{23}$$

Therefore, Wien found the distribution law as

$$u(\lambda) = \frac{A}{\lambda^5} e^{-\frac{a}{\lambda T}} \tag{24}$$

where A and a are constants. We will not prove the Wien's distribution law, however, will understand how it can explain the observed black body radiation.
From the distribution law, we see that at low wavelength, the exponent $a/\lambda T$ is large. This means $e^{-\frac{a}{\lambda T}}$ will be smaller and will be dominating over λ^{-5}. The overall curve will still be increasing with λ. However, at higher λ the part λ^{-5} will dominate over the exponential function. Therefore, the overall curve will decrease with λ.

Unlike Rayleigh-Jeans law, the formula by Wien does not break down at smaller wavelength. However, it can be observed that such formula cannot explain the observed data at higher wavelengths.

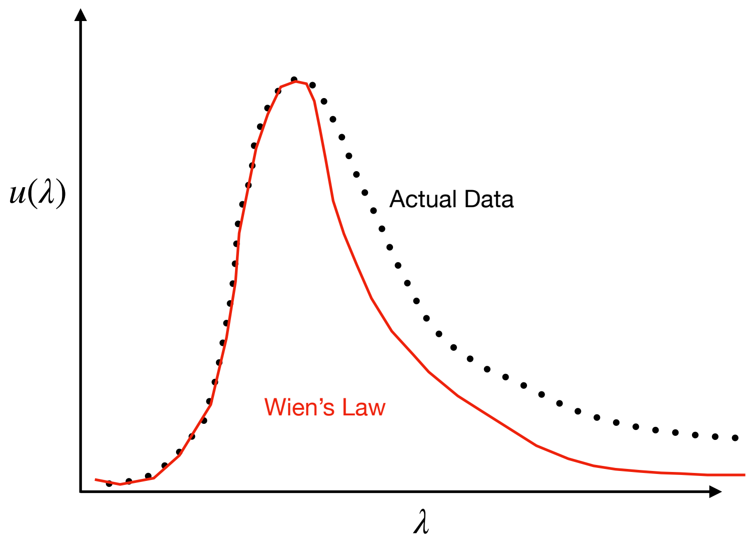

Figure 8: The Wien's distribution law holds at low wavelengths, but it fails at high wavelengths.

Planck's idea:
The explanation of the black body radiation curve was a real problem to the physicists towards the end of nineteenth century. Prof. Max Planck was a professor at Berlin University. He was also the advisor of the famous journal "Annalen der Physik" where most of the interesting research results got published. In the year 1894, Planck turned his attention to the problem of black-body radiation. The question were of course: how does the intensity of the electromagnetic radiation emitted by a black body (a perfect absorber, also known as a cavity radiator) depend on the frequency of the radiation (i.e., the color of the light) and the temperature of the black body?

Even though Max Planck was a brilliant scientist, he looked into the previous theories by Rayleigh-Jeans (explains the black body radiation only at high wavelengths) and Wien (explains the black body radiation only at low wavelengths). He did not continue to derive any formula for black body radiation. Rather, he fitted the radiation curve at a particular temperature.
Finally working on the night on 7th February 1900, he obtained the fitted formula:

$$u(\lambda) = \frac{A}{\lambda^5}\frac{1}{e^{\frac{a}{\lambda T}} - 1} \tag{25}$$

He then presented his formula to the scientific community. However, he himself had no idea about how the formula can be derived. In the next two weeks, he tried to derive the formula, but he did not believe himself. His derivation questioned the all time classical physics theories: (1) electromagnetic theory and (2) thermodynamics. Later, his invention led to the quantum revolution, however, he did not believe it.

Planck's Derivation:

Max Planck understood that in order to derive a formula that perfectly describes black body radiation, the following two assumptions have to be made:

1. The emission and absorption of electromagnetic radiation from the cavity black body must have discrete packet energies.
2. The energy associated to the each packet is proportional to the frequency of the cavity oscillator. This means $E = hf$ where f are the frequencies.

These two assumptions were clearly opposite from the classical theory where the energy values are continuous. According to Planck's assumption, if there are n oscillators with frequency f inside the cavity wall, the radiation energy would be $E = nhf$.
In fact Maxwell himself said that
"*after immense struggle, managed to derive the curve from the strange and unprecedented assumption that tiny "oscillators" in the cavity's wall could take on only a set of discrete energy values*".
He borrowed the density of modes from the calculation of Rayleigh and Jeans that is

$$D(\lambda) = \frac{8\pi}{\lambda^4} \tag{26}$$

However, the equipartition theorem is no longer valid as that is based on classical thermodynamics. We have to calculate the average energy corresponding to each oscillator with frequency f. To find the total no. of oscillator he used the Boltzmann distribution function that states the no. of electrons in a given energy level E at a temperature T must be given by

$$N(E) = N_0 e^{-\frac{E}{kT}} \tag{27}$$

where N_0 are the no of electrons in a ground state. The oscillators can have energies as $E_0 = 0$, $E_1 = hf$, $E_2 = 2hf$, $E_3 = 3hf$, $\ldots$, $E_n = nhf$. If the no. of oscillators in the ground state is N_0, we find the following:

$$N_1 = N_0 e^{-\frac{E_1}{kT}} = N_0 e^{-\frac{hf}{kT}} = N_0 x \text{ with } x = e^{-\frac{hf}{kT}} \tag{28}$$

$$N_2 = N_0 e^{-\frac{E_2}{kT}} = N_0 e^{-\frac{2hf}{kT}} = N_0 x^2 \tag{29}$$

$$N_3 = N_0 e^{-\frac{E_3}{kT}} = N_0 e^{-\frac{3hf}{kT}} = N_0 x^3 \tag{30}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \tag{31}$$

$$N_n = N_0 e^{-\frac{E_n}{kT}} = N_0 e^{-\frac{nhf}{kT}} = N_0 x^n \tag{32}$$

The average energy of each oscillator can be calculated as

$$\begin{aligned}
\langle E \rangle &= \frac{N_0 E_0 + N_1 E_1 + N_2 E_2 + \cdots + N_n E_n + \ldots \infty}{N_0 + N_1 + N_2 + \cdots + N_n + \ldots \infty} \\
&= \frac{N_0 \cdot 0 + N_0 x \cdot hf + N_0 x^2 \cdot 2hf + \ldots \infty}{N_0 + N_0 x + N_0 x^2 + \ldots \infty} \\
&= \frac{xhf + 2x^2 hf + \ldots \infty}{1 + x + x^2 + \ldots \infty} \\
&= xhf \left[\frac{1 + 2x + 3x^2 + \ldots \infty}{1 + x + x^2 + \ldots \infty} \right] \qquad (33) \\
&= xhf \left[\frac{S_1}{S_0} \right] \qquad (34)
\end{aligned}$$

We can easily calculate the sum of the infinite series with the fact that $x < 1$:

$$S_0 = 1 + x + x^2 + x^3 + \ldots \infty = \frac{1}{1-x} \tag{35}$$

$$S_1 = 1 + 2x + 3x^2 + \ldots \infty = \frac{d}{dx} S_0 = \frac{1}{(1-x)^2} \tag{36}$$

Now it is easy to calculate the ratio of these two sums

$$\frac{S_1}{S_0} = \frac{1}{1-x} \tag{37}$$

The average energy can thus be calculated as

$$\langle E\rangle = hf\frac{x}{1-x} = \frac{hf}{x^{-1}-1} = \frac{hf}{e^{\frac{hf}{kT}}-1} = \frac{hc}{\lambda}\frac{1}{e^{\frac{hc}{\lambda kT}}-1} \tag{38}$$

The total energy density will simply be given by the density of modes $D(\lambda)$ multiplied by the average energy as

$$u(\lambda) = D(\lambda)\cdot\langle E\rangle = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1} \tag{39}$$

This is derivation provided by Max Planck. However, he did not believe that the oscillators in the cavity wall could take discrete energies. Max Planck never touched upon the question of electromagnetic radiation because radiation was always a wave phenomena without a doubt!

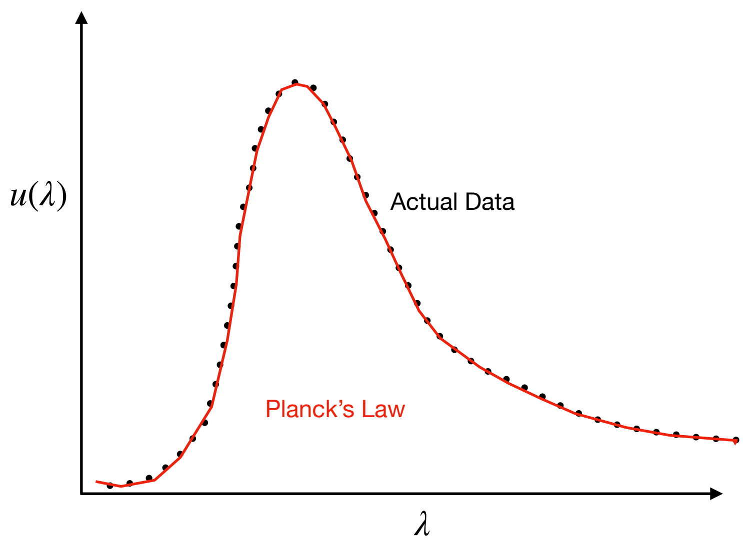

Figure 9: The Planck's distribution law holds at low wavelengths, and at high wavelengths.

In the year 1905, Albert Einstein was a class III technical officer at Bern, Switzerland. He himself got into the black body radiation. He said
"*it strikes me that the recent observations of black radiation of photo-luminescence of the production of cathode rays by ultraviolet light, and the other types of phenomena that involve the production and transformation of light might better be understood through the notion that the energy carried by light is distributed discontinuously in space.*"

Einstein rejected the formula by Rayleigh and Jeans because of ultraviolet catastrophe. He also did not understand the formula given by Max Planck. He looked into the formula by Wien

$$u(\lambda)\Big|_{\text{Wien}} = \frac{A}{\lambda^5}e^{-\frac{a}{\lambda T}} \tag{40}$$

He was reminded of another similar but very well-known formula of Maxwell-Boltzmann energy distribution formula for an ideal gas. This has the following formula

$$u(\text{K.E.})\Big|_{\text{M-B}} = \text{K.E.}e^{-\frac{\text{K.E.}}{kT}} \tag{41}$$

From the analogy, Einstein concluded that kinetic energy and $a/\lambda = af/c = hf/k$ (consider $a/c = h/k$) must play very similar role in physics.
At this point, Einstein calculated the change in entropy in an ideal gas (with N no of gas molecules) contraction using the M-B energy distribution. He found if the initial and final volumes of the gas are V and V' respectively, the change in entropy

$$\Delta S\Big|_{\text{M-B}} = k\log\left(\frac{V}{V'}\right)^N \tag{42}$$

Then Einstein calculated the same change of entropy when a black body is contracted from the volume V to V'. He found the change in entropy in this case

$$\Delta S\Big|_{\text{Black-Body}} = k\log\left(\frac{V}{V'}\right)^{\frac{E}{hf}} \tag{43}$$

Einstein came to the conclusion that both the entropy change should be similar as suggested from the formula by Wien. Therefore, a natural mapping of N with E/hf is possible meaning

$$N \sim \frac{E}{hf} \tag{44}$$

The left side is the number of gas molecules and thus the right side has to be the number of "light molecules".
In 1905, the work of Einstein was sent to the journal Annalen der Physik where Max Planck was already the advisor. He did not publish at first place because he did not believe Einstein. Apart from the science, Einstein was merely a technical office, he was not known in science community at all. But eventually, the work of Einstein got published. In 1913, Maxwell said about Einstein

"He may occasionally have gone a bit overboard in his speculations, – as, for instance, in his hypothesis of light quanta – should not be held too much against him, for even in the exact sciences, if one never takes any risks, no genuine innovations can be accomplished."

More on Planck's formula:

The energy density of a black body radiation is

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \tag{45}$$

As $\lambda \to 0$, $e^{\frac{hc}{\lambda kT}} \to \infty$ meaning $u(\lambda) \sim e^{-\frac{hc}{\lambda kT}} \to 0$. This means the ultraviolet catastrophe does not exist.

1. Show that Planck's radiation law reduces to Rayleigh-Jeans law at longer wavelengths.
2. Show that Planck's radiation law reduces to Wien's law at shorter wavelengths.
3. Calculate the total energy density using Planck's law.

Derivation of Wien's displacement law:

If $u(\lambda)$ is maximum at a wavelength λ_{max}, we find

$$\begin{aligned}
\left.\frac{du(\lambda)}{d\lambda}\right|_{\lambda=\lambda_{\text{max}}} &= 0 \\
\Rightarrow \frac{d}{d\lambda}\left[\frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}\right]_{\lambda=\lambda_{\text{max}}} &= 0 \\
\Rightarrow -\frac{5}{\lambda^6}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1} + \frac{1}{\lambda^5}\frac{e^{\frac{hc}{\lambda kT}}}{\left(e^{\frac{hc}{\lambda kT}} - 1\right)^2} \times \frac{hc}{\lambda^2 kT} &= 0 \\
\Rightarrow \frac{hc}{\lambda kT}\frac{e^{\frac{hc}{\lambda kT}}}{e^{\frac{hc}{\lambda kT}} - 1} &= 5
\end{aligned} \tag{46}$$

Considering $x = \frac{hc}{\lambda kT}$ the above equation reduces to

$$x\frac{e^x}{e^x - 1} = 5 \Rightarrow xe^x = 5e^x - 5 \Rightarrow e^x = \frac{5}{5 - x} \tag{47}$$

This is an equation we can solve, even though this is a non algebric equation. The solution is approximately

$$x \approx 4.965 \tag{48}$$

We find

$$\frac{hc}{\lambda kT} = 4.965$$
$$\Rightarrow \lambda_{\max} T = \frac{hc}{4.965 \times k} = 2.989 \times 10^{-3}\text{meter} - \text{K} \tag{49}$$

This is exactly the Wien's displacement law.

Thermodynamics of Black Body Raiation

Let us create a black body such that there is a piston on it. With the help of piston the work can be done on the black body system or some work can be extracted from the black body. The combined form of first and second law of thermodynamics tells you

$$dU = TdS - PdV \tag{50}$$

So far we have derived the energy density and of course, the total energy in thermodynamics is $U = uV$ where V is the volume. Pressure exerted by isotropic radiation is $P = u/3$. We are not calculating this at the moment but can be proved. Using the first and second law of thermodynamics, we can write

$$d(uV) = TdS - \frac{u}{3}dV$$
$$\Rightarrow TdS = Vdu + \frac{4u}{3}dV$$
$$\Rightarrow dS = \frac{V}{T}du + \frac{4u}{3T}dV \tag{51}$$

Now we know that entropy S is an exact differential such that we can write

$$dS = \left(\frac{\partial S}{\partial u}\right)_V du + \left(\frac{\partial S}{\partial V}\right)_u dV \tag{52}$$

And we have

$$\left(\frac{\partial S}{\partial u}\right)_V = \frac{V}{T}; \quad \left(\frac{\partial S}{\partial V}\right)_u = \frac{4u}{3T} \tag{53}$$

From the Maxwells relation we must have

$$\begin{aligned}
\frac{\partial}{\partial V}\left[\left(\frac{\partial S}{\partial u}\right)_V\right]_u &= \frac{\partial}{\partial u}\left[\left(\frac{\partial S}{\partial V}\right)_u\right]_V \\
\Rightarrow \frac{\partial}{\partial V}\left(\frac{V}{T}\right) &= \frac{\partial}{\partial u}\left(\frac{4u}{3T}\right) \\
\Rightarrow \frac{1}{T} &= \frac{4}{3T} + \frac{4u}{3}\frac{\partial}{\partial u}\left(\frac{1}{T}\right) \\
\Rightarrow \frac{1}{T} &= \frac{4}{3T} - \frac{4u}{3T^2}\frac{\partial T}{\partial u} \\
\Rightarrow \frac{4u}{3T^2}\frac{\partial T}{\partial u} &= \frac{1}{3T} \\
\Rightarrow \frac{4u}{T}\frac{\partial T}{\partial u} &= 1 \\
\Rightarrow \frac{4}{T}dT &= \frac{du}{u}
\end{aligned} \tag{54}$$

Now, integrating on both sides we obtain

$$\ln u = 4\ln T + \ln\alpha(const.) \Rightarrow u = \alpha T^4 \tag{55}$$

The total emissive power (Q) of the black body is related to the energy density u. Hence, we get the Stefan-Boltzmann law

$$Q = \sigma T^4 \tag{56}$$