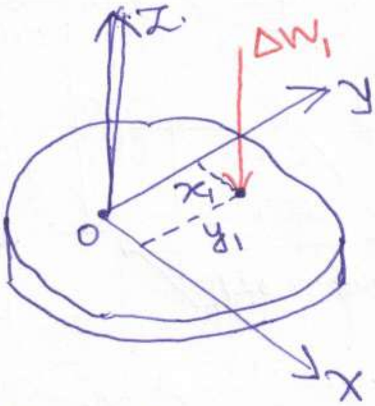
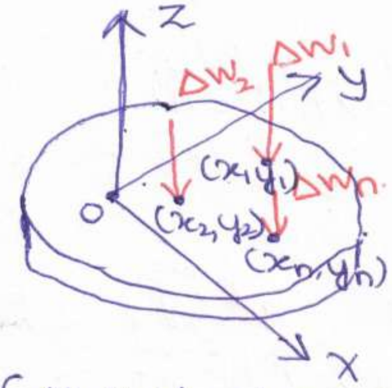
Center of Gravity of 2D Body

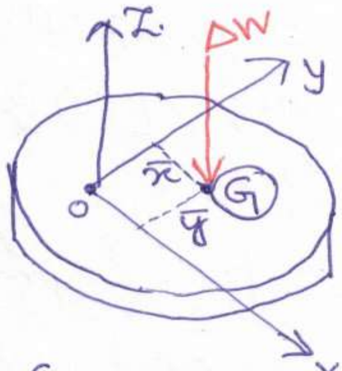
- A distributed system of forces can be replaced by a single resultant force acting at a specific point on an object. This specific point is called the object's "center of gravity".



(Single element of the plate)



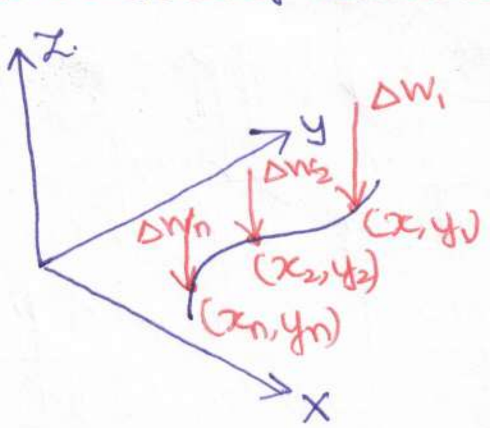
(Multiple elements of the plate)

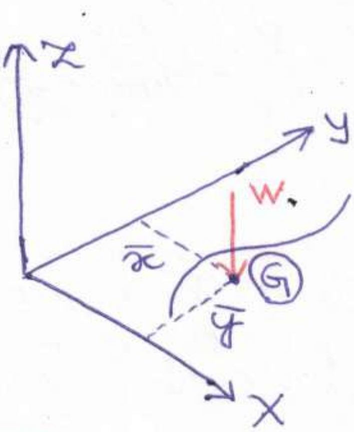


(Center of gravity).

- $\sum F_z: \; W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \dots + \Delta W_n$
- $\sum M_x: \; \bar{y} W = \sum y \Delta W = \int y \, dw \Rightarrow \boxed{\bar{y} = \dfrac{\int y \, dw}{W}}$
- $\sum M_y: \; \bar{x} W = \sum x \Delta W = \int x \, dw \Rightarrow \boxed{\bar{x} = \dfrac{\int x \, dw}{W}}$

Center of gravity of a wire

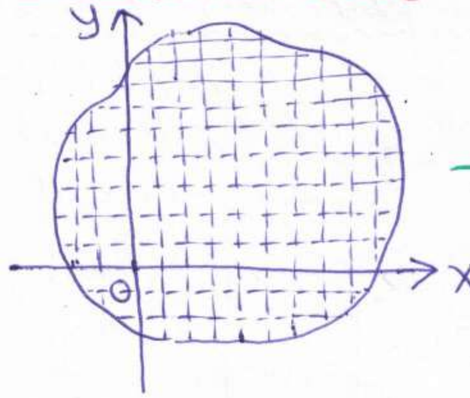


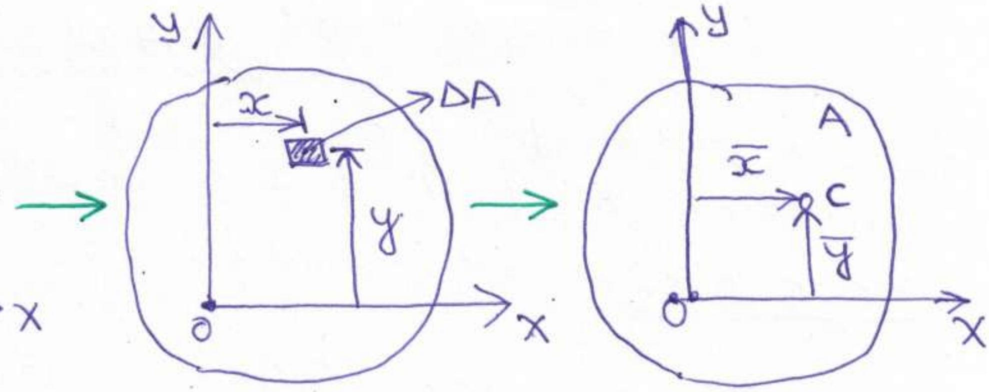


$$\boxed{\bar{x} = \frac{\int x \, dW}{W}} \qquad \boxed{\bar{y} = \frac{\int y \, dW}{W}}$$

(*) The center of gravity of the wire may or may not lie on the wire. It can be located outside the wire as well !!!

Centroid of Areas





$\bar{x} W = \int x \, dW$

- $W = \gamma V = \gamma t A$

  γ: weight/volume.

$\Rightarrow dW = \gamma t \, dA$.

$\Rightarrow \bar{x} A = \int_A x \, dA$

= 1st area moment w.r.t y-axis

<u>Assumption</u>: homogeneous plate of uniform thickness

$$\Rightarrow \boxed{\bar{x} = \frac{\int_A x \, dA}{A}}$$

- Similarly, $\bar{y} A = \int_A y \, dA$.

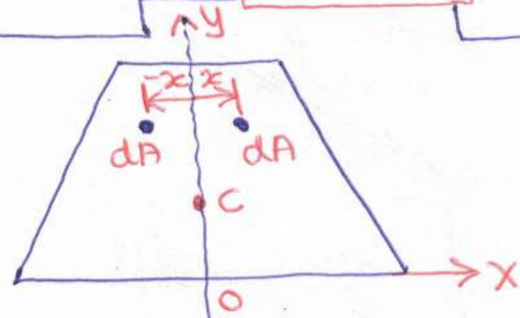
  = 1st area moment w.r.t X-axis.

$$\Rightarrow \boxed{\bar{y} = \frac{\int_A y \, dA}{A}}$$

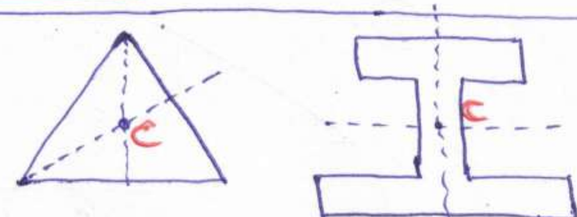
Centroid of a line:

<u>Assumption</u>: Homogeneous wire of uniform cross-section.

$$\boxed{\bar{x} = \frac{\int x \, dL}{L}} \, , \quad \boxed{\bar{y} = \frac{\int y \, dL}{L}}$$



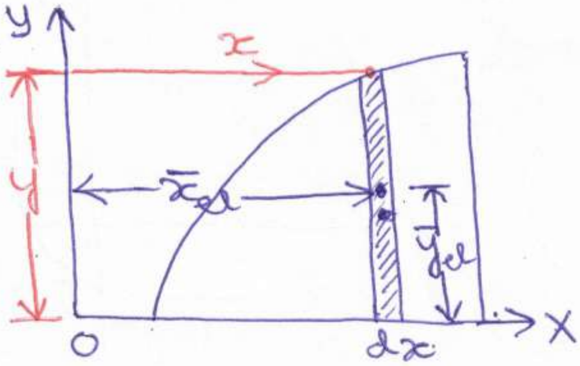
(*) If an area possesses an axis of symmetry, its centroid lies on that axis



Area possessing two axes of symmetry

(*) The centroid of the area coincides with the center of symmetry.

Determination of Centroids by Integration



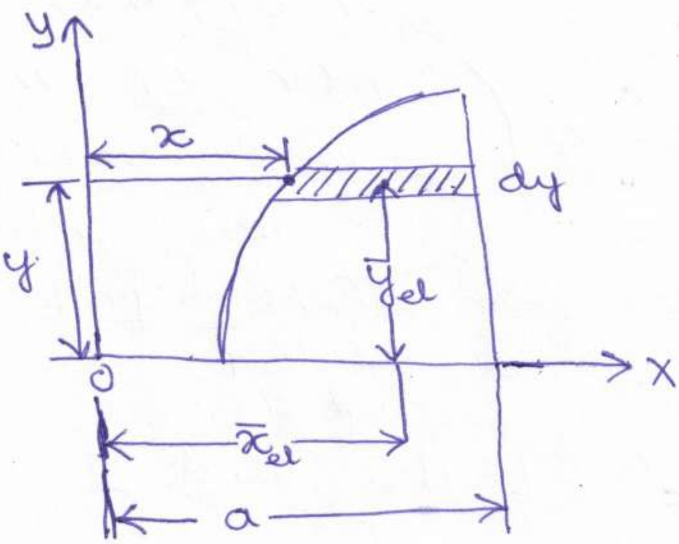
$\bar{x}_{el}, \bar{y}_{el}$: co-ordinates of the centroid of the element dA.

$\bar{x}_{el} = x$

$\bar{y}_{el} = \frac{y}{2}$

$dA = y\,dx$

- $\bar{x}A = \int \bar{x}_{el}\,dA = \int x(y\,dx)$
- $\bar{y}A = \int \bar{y}_{el}\,dA = \int\left(\frac{y}{2}\right)(y\,dx)$



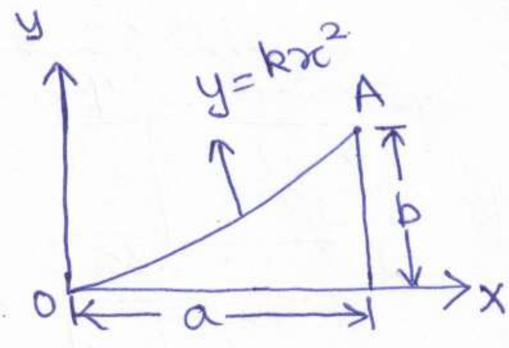
$\bar{x}_{el} = x + \frac{a-x}{2} = \frac{x+a}{2}$

$\bar{y}_{el} = y$

$dA = (a-x)\,dy$

- $\bar{x}A = \int \bar{x}_{el}\,dA = \int\left(\frac{x+a}{2}\right)(a-x)\,dy$
- $\bar{y}A = \int \bar{y}_{el}\,dA = \int y(a-x)\,dy$

Eg:



Determine by direct integration, the location of the centroid of a parabolic spandrel.

Soln.

Given: $y = kx^2$.

Considering point 'A' of the curve,

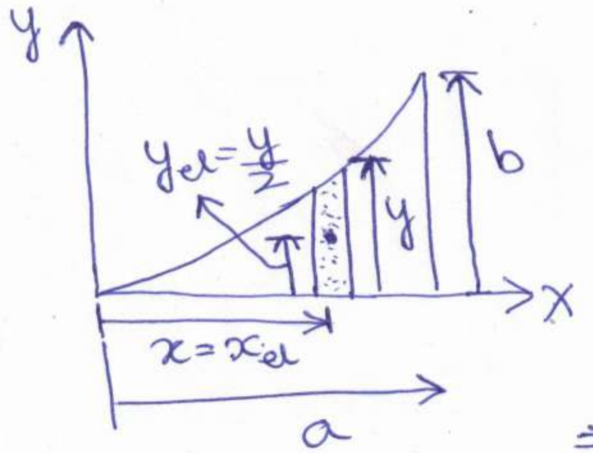
$b = k(a)^2 \Rightarrow \boxed{k = \frac{b}{a^2}}$

$\therefore y = \left(\frac{b}{a^2}\right)x^2$.

• Evaluating the total area:-

$$A=\int dA=\int_0^a y\,dx=\int_0^a \frac{b}{a^2}x^2\,dx=\frac{b}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$\Rightarrow \boxed{A=\frac{ab}{3}}$$



• $\bar{x}A=\int x_{el}\,dA$

$$\Rightarrow \bar{x}A=\int_0^a x\,y\,dx$$

$$\Rightarrow \bar{x}A=\int_0^a x\frac{b}{a^2}x^2\,dx$$

$$\Rightarrow \bar{x}\left(\frac{ab}{3}\right)=\left[\left(\frac{b}{a^2}\right)\cdot\left(\frac{x^4}{4}\right)\right]_0^a$$

$$\Rightarrow \bar{x}\,\frac{a\cancel{b}}{3}=\frac{\cancel{b}\cdot a^{\cancel{4}3}}{4\cancel{a^2}}$$

$$\Rightarrow \boxed{\bar{x}=\frac{3}{4}a}$$

• $\bar{y}A=\int y_{el}\,dA$

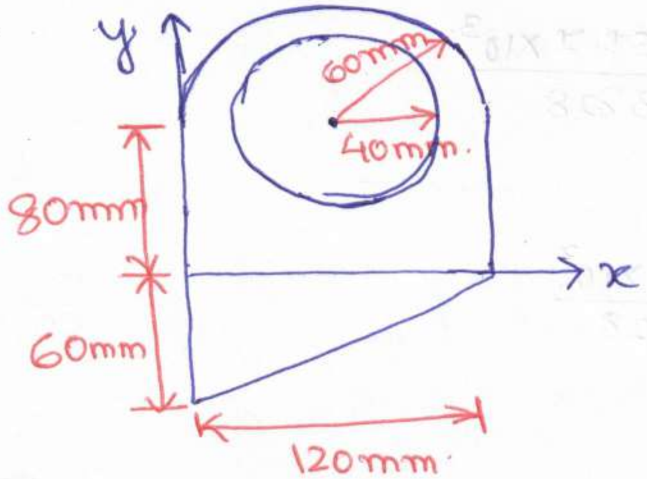
$$\Rightarrow \bar{y}\left(\frac{ab}{3}\right)=\int_0^a \frac{y}{2}\,\cancel{x}\,y\,dx=\frac{1}{2}\int_0^a \frac{b^2}{a^4}x^4\,dx$$

$$\Rightarrow \bar{y}\left(\frac{\cancel{a}\cancel{b}}{3}\right)=\frac{b^{\cancel{2}}}{2\cancel{a^2}\cancel{a^2}}\left(\frac{\cancel{a^5}}{5}\right)$$

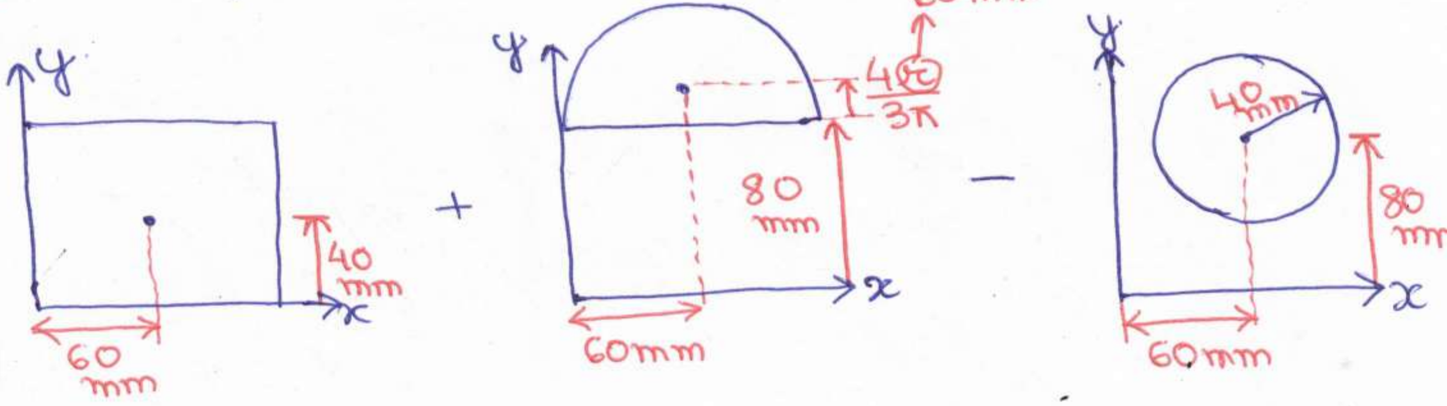
$$\Rightarrow \boxed{\bar{y}=\frac{3}{10}b}$$

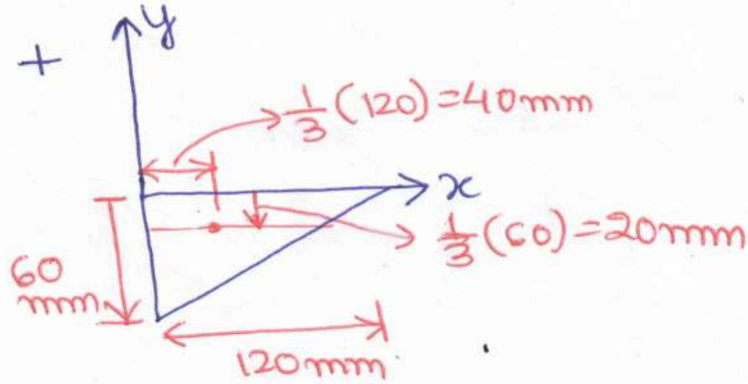
Centroid of a Composite Plate

Eg. For the plane area shown, determine the location of the centroid.



Soln : The given area can be split up into simpler components





Component	A (mm²)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{x}A$ (mm³)	$\bar{y}A$ (mm³)
Rectangle	(120)(80) = 9600	60	40	576×10^3	384×10^3
Triangle	$\frac{1}{2}(120)(60) = 3600$	40	−20	144×10^3	-72×10^3
Semi Circle	$\frac{\pi(60)^2}{2} = 5655$	60	105.46	339.3×10^3	5964×10^3
Circle.	$-\pi(40)^2 = -5027$	60	80	-301.6×10^3	-4022×10
	$\Sigma A = 13828$			$\Sigma \bar{x}A = 757.7 \times 10^3$	$\Sigma \bar{y}A = 506.2 \times 10$

$$\therefore \bar{x} = \frac{\Sigma \bar{x}A}{A} = \frac{757.7 \times 10^3}{13828}$$

$$\Rightarrow \boxed{\bar{x} = 54.8\,\text{mm}}$$

$$\bar{y} = \frac{\Sigma \bar{y}A}{A} = \frac{506.2 \times 10^3}{13828}$$

$$\Rightarrow \boxed{\bar{y} = 36.6\,\text{mm}}$$

Area Moment of Inertia

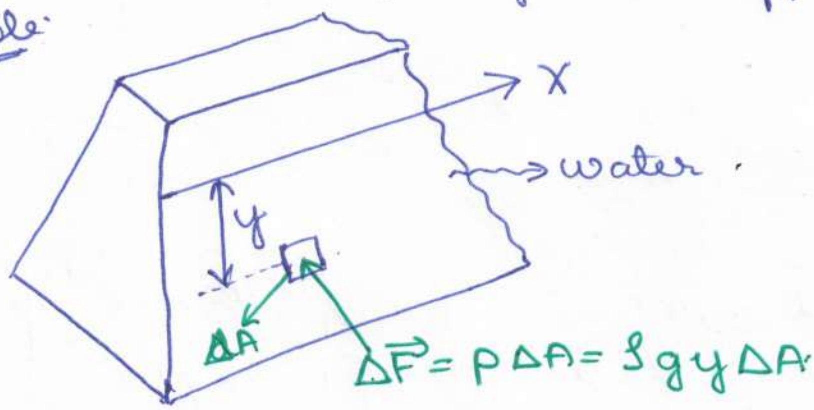
- When forces are distributed continuously over an area on which they act, it is often required to calculate the moment of these forces about some axis, either in or perpendicular to the plane of the area.
- In general, the intensity of the force (pressure / stress) is proportional to the distance of the line of action of force from the moment axis. Thus, the elemental force acting on an element of area, then, is proportional to distance times differential area, and elemental moment is proportional to distance squared times differential area.

$\therefore$ $\boxed{\text{Total moment} = \int_{\text{area}} (\text{distance})^2 \, d(\text{area})}$

This integral is known as second-moment of area or moment of inertia.

(*) This integral is a function of the geometry of the area and occurs frequently in the application of mechanics.

Example.



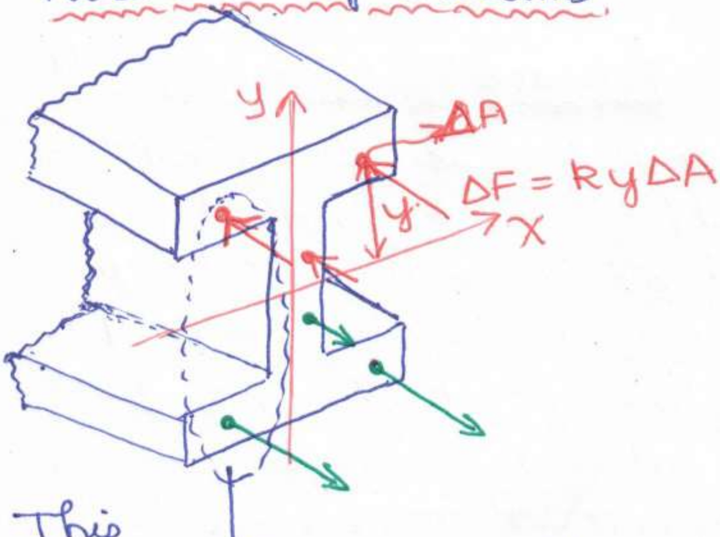
with $\Delta A \to 0$, $d\vec{F} = \rho g y \, dA$.

Moment of $d\vec{F}$ about X-axis,

$$M = \int dF \, y = \int \rho g y^2 \, dA$$

$$\Rightarrow M = \rho g \int y^2 \, dA$$

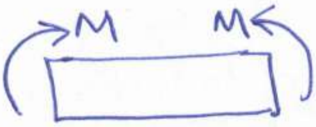
Pure Bending of Beams



The force intensity (force/area) is proportional to the linear variation of distance y (distance b/w elemental area and the line passing through the centroid of the section).

→ This can be proved using the concept of solid mechanics

This force system forms a couple.



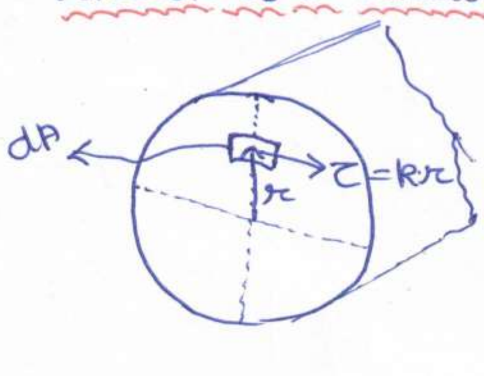
$dM = \Delta F y$

With $\Delta A \to 0$, the total moment (bending) acting on the entire cross section is given by -

$$\int dM = \int dF y$$

$$\Rightarrow M = R\int y^2 dA$$

Torsion of a circular shaft:



The shear flow distribution in a circular shaft subjected to torsional load is proportional to the radial distance r. (From solid mechanics)

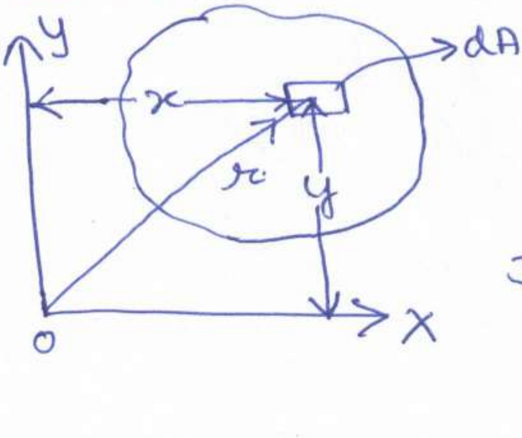
$\therefore \tau = kr$

$\Rightarrow$ The moment about the central axis is given by -

$$\int dM = \int (\tau dA) r$$

$$\Rightarrow M = k\int r^2 dA$$

Determination of Area Moment of Inertia by Integration



$$\boxed{I_{xx} = \int_A y^2 dA}$$

$$\boxed{I_{yy} = \int_A x^2 dA}$$

Rectangular M.I

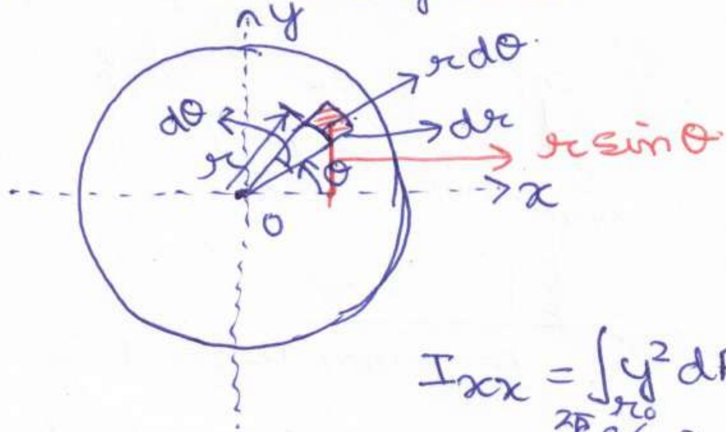
$$J_0 = I_{zz} = \int_A r^2 dA = \int_A (x^2 + y^2) dA$$

$$= \int_A x^2 dA + \int_A y^2 dA$$

$$\Rightarrow \boxed{I_{zz} = I_{yy} + I_{xx} = J_0}$$ → Polar M-I

Que) Find the polar moment of inertia of a circular area by method of integration.

Sol^n.



$$I_{xx} = \int y^2\, dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{r_0} (r^2 \sin^2\theta)(r\,d\theta)\,dr$$

$$\Rightarrow I_{xx} = \int_0^{r_0} r^3\,dr \int_0^{2\pi} \frac{(1-\cos 2\theta)}{2}\,d\theta$$

$$= \left(\frac{r_0^4}{4}\right) \times \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\pi r_0^4}{4} = \frac{\pi d_0^4}{64}$$

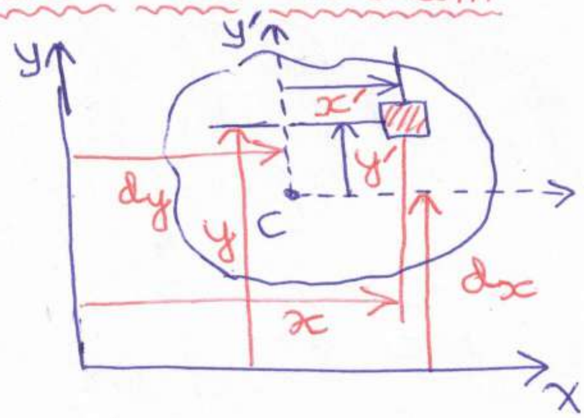
Similarly, $I_{yy} = \dfrac{\pi d_0^4}{64}$

$$\therefore J_0 = I_{xx} + I_{yy} = 2 I_{xx}$$

$$\Rightarrow \boxed{J_0 = \frac{\pi d_0^4}{32}}$$

Parallel-Axis Theorem:

$(x', y') \rightarrow$ centroidal co-ordinate system



• $I_{xx} = \int_A y^2\,dA$

$= \int_A (y' + d_x)^2\,dA$

$= \int_A (y')^2\,dA + d_x^2 \int_A dA + 2d_x \underbrace{\int_A y'\,dA}_{0}$

$\left(\because \bar{y} = \dfrac{\int_A y'\,dA}{\int_A dA} \text{ and } \bar{y} = 0 \text{ as centroid is on X-axis}\right)$

$$\Rightarrow \boxed{I_{xx} = \bar{I}_{xx} + A d_x^2}$$

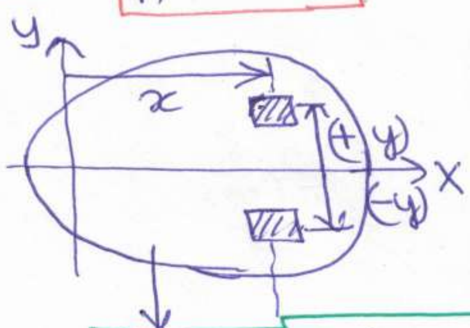
• Similarly, it can be proved that,

$$\boxed{I_{yy} = \bar{I}_{yy} + A d_y^2}$$

⊛ $(\bar{I}_{xx}, \bar{I}_{yy})$: MI about the centroidal X and Y axes, respectively.

Product Moment of Inertia:

$\boxed{I_{xy} = \int_A xy\,dA} \rightarrow$ can be (+ve), (−ve) or zero.



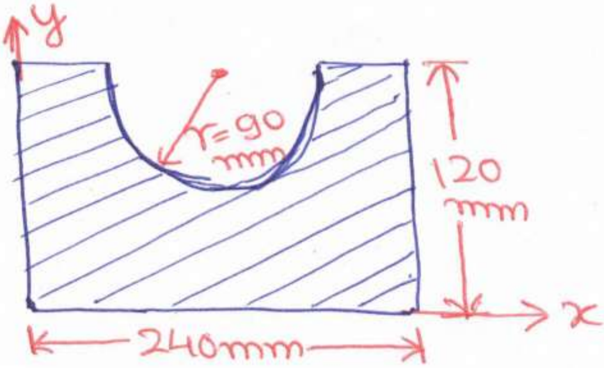
$\boxed{\text{Symmetrical cross-section} \Rightarrow I_{xy} = 0}$

* $I_{xy} = \int_A xy\,dA = \int_A (x' + d_y)(y' + d_x)\,dA$

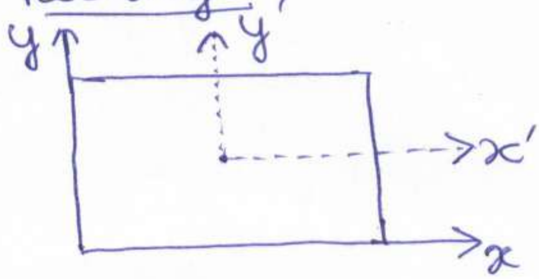
$= \int_A x'y'\,dA + d_x \underbrace{\int_A x'\,dA}_{0} + d_y \underbrace{\int_A y'\,dA}_{0} + d_x d_y \int_A dA$

$$\Rightarrow \boxed{I_{xy} = \bar{I}_{xy} + d_x d_y A}$$

Eg: Determine the moment of inertia of the shaded area w.r.t the x-axis.



Solⁿ. (1) Rectangle,

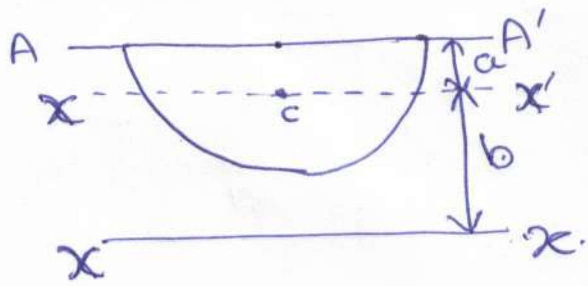


$$I_{xx} = I_{x'x'} + A\bar{y}^2$$
$$= \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2$$
$$= \frac{bh^3}{12} + \frac{bh^3}{4} = \frac{bh^3}{3}$$

Here $b = 240$ mm
$h = 120$ mm.

$$\therefore (I_{xx})_1 = \frac{(240)(120)^3}{3} = 138.2 \times 10^6 \text{ mm}^4.$$

(2) Semi-Circle.



$$I_{AA'} = \frac{\pi d^4}{64} \times \frac{1}{2}$$
$$= \frac{\pi r^4 (16)}{2 \times 64} = \frac{\pi r^4}{8} = 25.76 \times 10^6 \text{ mm}^4$$

$$a = \frac{4r}{3\pi} = \frac{4(90)}{3\pi} = 38.2 \text{ mm}$$

$$I_{xx'} + Aa^2 = I_{AA'}$$

$$\Rightarrow I_{xx'} = I_{AA'} - Aa^2.$$

$b = 120 - a = 81.8$ mm

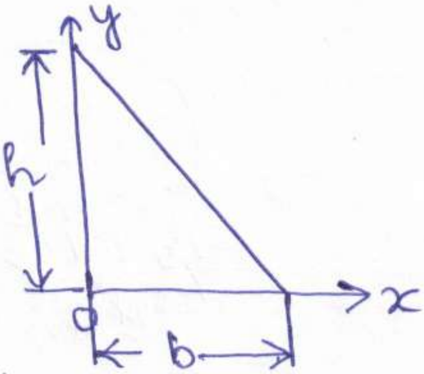
Similarly, $(I_{xx})_2 = I_{xx'} + Ab^2$.

$$= I_{AA} + A(b^2 - a^2) = 92.3 \times 10^6 \text{ mm}^4.$$

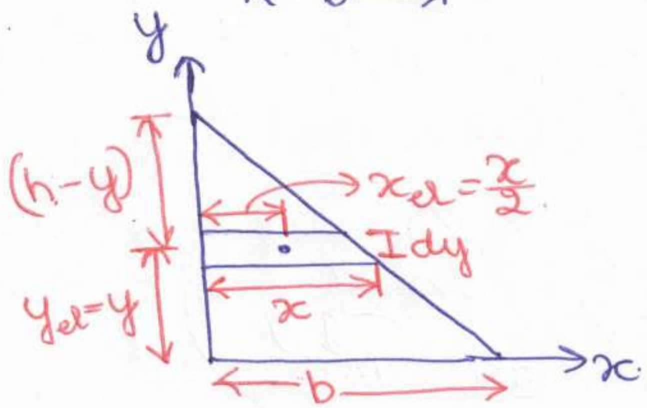
Now, $I_{xx} = (I_{xx})_1 - (I_{xx})_2 = (138.2 - 92.3) \times 10^6 \text{ mm}^4$.

$$\Rightarrow I_{xx} = 45.9 \times 10^6 \text{ mm}^4 \text{ (Ans)}.$$

Eg. Determine the product of inertia of the right triangle
(a) w.r.t x and y axes.
(b) w.r.t the centroidal axes parallel to x and y axes.



Soln.



$\bar{x}_{el} = \frac{x}{2}$

$y_{el} = y$

From similarity of triangles,

$$\frac{h-y}{x} = \frac{h}{b}$$

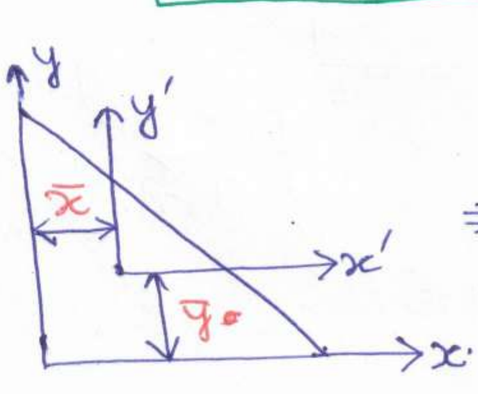
$$\Rightarrow x = \frac{b}{h}(h-y).$$

$$I_{xy} = \iint x_{el}\, y_{el}\, dA = \int \underbrace{\frac{b}{h}(h-y)\frac{1}{2}}_{\left(\frac{x}{2}\right)} y\, x\, dy$$

$$= \frac{1}{2}\int x^2 y\, dy$$

$$= \frac{1}{2}\cdot\frac{b^2}{h^2}\int_0^h (h-y)^2 y\, dy = \frac{h^2}{2b^2}\left[\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3}\right]$$

$$\Rightarrow \boxed{I_{xy} = \frac{b^2h^2}{24}}$$



$I_{xy} = I_{x'y'} + \bar{x}\bar{y}A$ | Here $\bar{x} = \frac{b}{3}$, $\bar{y} = \frac{h}{3}$

$$\Rightarrow \frac{b^2h^2}{24} = I_{x'y'} + \frac{bh}{9}\cdot\frac{1}{2}(b)(h)$$

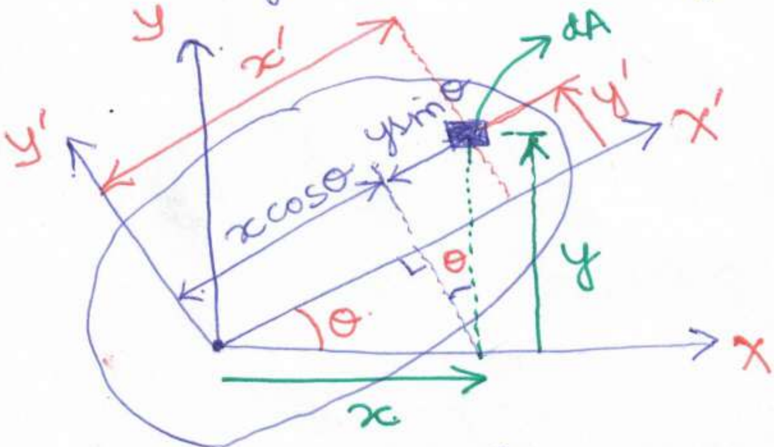
$$\Rightarrow \boxed{I_{x'y'} = -\frac{b^2h^2}{72}}$$

Principal Moment of Inertia.

We know that,

$$I_x = \int_A y^2 dA \ , \ I_y = \int_A x^2 dA \ , \ I_{xy} = \int_A xy\, dA$$

We wish to determine the moment of inertia and product moment of inertia with respect to new axes X' and Y', rotating axes about origin through angle θ.



$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

$$\Rightarrow \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$I_{x'} = \int_A (y')^2 dA = \int_A (y\cos\theta - x\sin\theta)^2 dA$$

$$\Rightarrow I_{x'x'} = \cos^2\theta \int_A y^2 dA + \sin^2\theta \int_A x^2 dA - 2\sin\theta\cos\theta \int_A xy\, dA$$

$$\Rightarrow I_{x'} = I_{xx}\cos^2\theta - 2I_{xy}\cos\theta\sin\theta + I_{yy}\sin^2\theta$$

$$= I_{xx}\left(\frac{1+\cos2\theta}{2}\right) - I_{xy}\sin2\theta + I_{yy}\left(\frac{1-\cos2\theta}{2}\right)$$

$$\Rightarrow I_{x'x'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos2\theta - I_{xy}\sin2\theta \quad \text{—} \ (1)$$

Similarly, it can be proved that -

- $$I_{y'y'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) - \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos2\theta + I_{xy}\sin2\theta \quad \text{—} \ (2)$$
- $$I_{x'y'} = \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin2\theta + I_{xy}\cos2\theta \quad \text{—} \ (3)$$

Adding (1) and (2), we get -

$$\boxed{I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy}}$$

Equation ① can be written as -

$$\left(I_{x'x'} - \frac{I_{xx}+I_{yy}}{2}\right) = \frac{I_{xx}-I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta \quad \text{—} ④$$

Squaring and adding eqns ④ and ③ on both the sides, we have

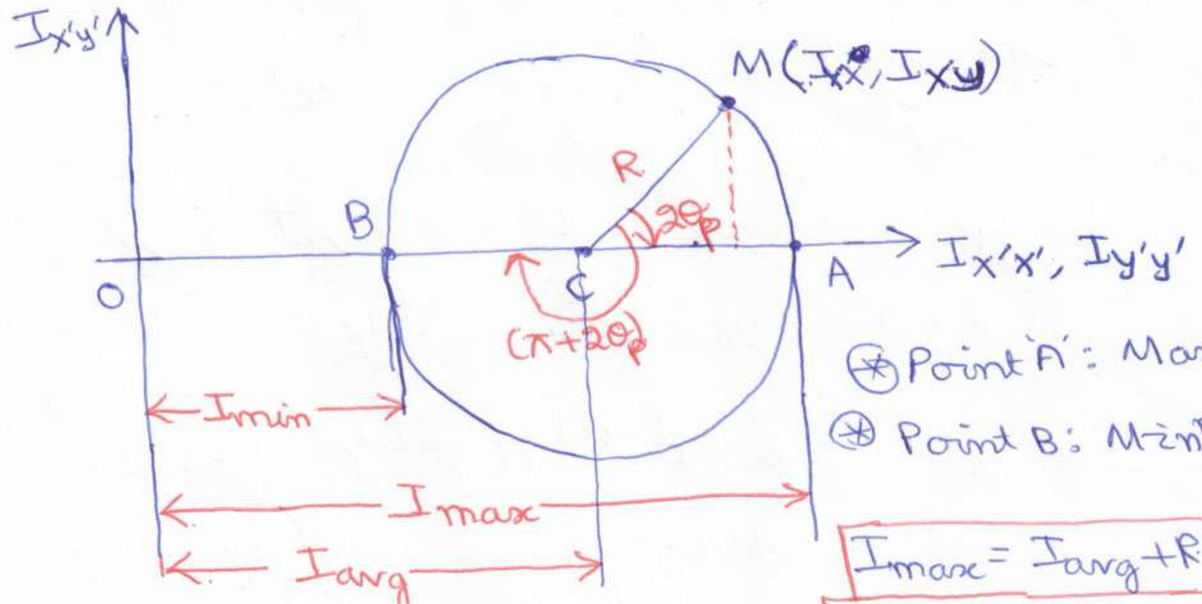
$$\left(I_{x'x'} - \frac{I_{xx}+I_{yy}}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_{xx}-I_{yy}}{2}\right)^2\cos^2 2\theta + I_{xy}^2\sin^2 2\theta$$
$$-\cancel{(I_{xx}-I_{yy})\sin 2\theta\cos 2\theta \cdot I_{xy}} + \left(\frac{I_{xx}-I_{yy}}{2}\right)^2\sin^2 2\theta$$
$$+ I_{xy}^2\cos^2 2\theta + \cancel{(I_{xx}-I_{yy})I_{xy}\sin 2\theta\cos 2\theta}$$

$$\Rightarrow \boxed{\left(I_{x'x'} - \frac{I_{xx}+I_{yy}}{2}\right)^2 + \left(I_{x'y'}\right)^2 = \left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$\Rightarrow \boxed{\left(I_{x'x'} - I_{avg}\right)^2 + \left(I_{x'y'}\right)^2 = R^2}, \quad \text{where } I_{avg} = \left(\frac{I_{xx}+I_{yy}}{2}\right)$$

$$R = \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

↳ Equation of a circle.



✱ Point 'A': Max^m Area MI (I_{ma}

✱ Point B: Min^m. Area MI (I_{min})

$$\boxed{I_{max} = I_{avg} + R}$$
$$\boxed{I_{min} = I_{avg} - R}$$

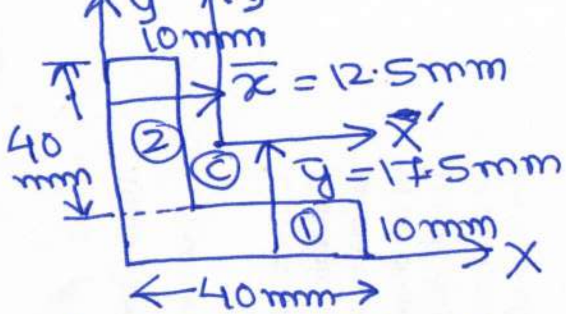
At points A and B, $I_{x'y'} = 0$

$$\Rightarrow \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta_p + I_{xy}\cos 2\theta_p = 0$$

$$\Rightarrow \boxed{\tan 2\theta_p = -\left(\frac{2I_{xy}}{I_{xx}-I_{yy}}\right)}$$

⇒ The maximum and minimum principal axes are separated by 180° apart from each other

* Principal Axes: Axes where $I_{x'y'} = 0$.



- $\bar{y} = \frac{A_i y_i}{\Sigma A_i} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$

$$= \frac{(10\times40)(5) + (10\times40)\left(\frac{40}{2}+10\right)}{2(10\times40)}$$

$$\Rightarrow \boxed{\bar{y} = 17.5\,mm}$$

- $\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{(10\times40)\left(\frac{40}{2}\right) + (10\times40)\left(\frac{10}{2}\right)}{2(40\times10)}$

$$\Rightarrow \boxed{\bar{x} = 12.5\,mm}$$

~~Ixx=~~

	A	x_{el}	y_{el}	$d_{x'}$	$d_{y'}$
1	400	20	5	-12.5	(5)
2	400	5	30	12.5	(-7.5)

- $I_{x'x'} = \frac{(40)(10^3)}{12} + (400)(12.5)^2 + \frac{(10)(40)^3}{12} + (400)(7.5)^2$

$$\Rightarrow \underline{I_{x'x'} = 18.17\times10^4\,mm^4}$$

- $I_{y'y'} = \frac{(10)(40)^3}{12} + (400)(7.5)^2 + \frac{(40)(10)^3}{12} + (400)(-7.5)^2$

$$\Rightarrow \underline{I_{y'y'} = 18.17\times10^4\,mm^4}$$

- $I_{x'y'} = A_1(d_x)_1(d_y)_1 + A_2(d_x)_2(d_y)_2$

$$= 400(-12.5\times7.5) + 400(12.5\times\bar{7}.5)$$

$$= \underline{-7.5\times10^4\,mm^4}$$

$$\tan 2\theta_p = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$

$$\Rightarrow \boxed{\theta_p = 30.96^\circ}$$

- $I_{max} = \frac{I_{x'x'} + I_{y'y'}}{2} + \sqrt{\left(\frac{I_{x'x'} - I_{y'y'}}{2}\right)^2 + I_{x'y'}^2}$

$$= (14.17 \pm 8.5)\times10^4\,mm^4$$

$$= 22.67\times10^4\,mm^4$$

- $I_{min} = (14.17 - 8.5)\times10^4\,mm^4$

$$= (5.67\times10^4)\,mm^4$$