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Problem 1.

Six points are chosen on the sides of an equilateral triangle ABC. Points A_1 and A_2 lie on BC, points B_1 and B_2 lie on CA, and points C_1 and C_2 lie on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2, and C_1A_2 are concurrent.

Problem 2.

Let a_1, a_2 and subsequent terms be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for every positive integer n, the numbers a_1, a_2 up to a_n leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

Problem 3.

Let x, y, z be positive real numbers such that $xyz >= 1$. Prove that

$$(x^5-x^2)/(x^5+y^2+z^2)+(y^5-y^2)/(x^2+y^5+z^2)+(z^5-z^2)/(x^2+y^2+z^5) >= 0.$$

Problem 4.

Determine all positive integers that are relatively prime to all terms of the infinite sequence defined by

$$a_n = 2^n + 3^n + 6^n - 1,$$

for all positive integers n.

Problem 5.

Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel to DA. Let two variable points E and F lie on the sides BC and DA respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, and the lines EF and AC meet at R. Prove that the circumcircles of triangles PQR, as E and F vary, have a common point different from P.

Problem 6.

In a mathematical competition with six problems, every two of the problems were solved by more than $5/2$ of the contestants. Moreover, no contestant solved all six problems. Show that there are at least two contestants who solved exactly five problems.