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46th International Mathematical Olympiad 2005

1. Six points are chosen on the sides of an equilateral triangle  $ABC$ . Points  $A_1$  and  $A_2$  lie on  $BC$ , points  $B_1$  and  $B_2$  lie on  $CA$ , and points  $C_1$  and  $C_2$  lie on  $AB$ , such that they are the vertices of a convex hexagon  $A_1A_2B_1B_2C_1C_2$  with equal side lengths. Prove that the lines  $A_1B_2$ ,  $B_1C_2$  and  $C_1A_2$  are concurrent.
2. Let  $a_1, a_2, a_3$  and further terms be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for every positive integer  $n$  the numbers  $a_1, a_2, a_3$  up to  $a_n$  leave  $n$  different remainders upon division by  $n$ . Prove that every integer occurs exactly once in the sequence.
3. Let  $x, y, z$  be three positive real numbers such that  $xyz \geq 1$ . Prove that  $\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$ .
4. Determine all positive integers relatively prime to all the terms of the infinite sequence  $a_n = 2^n + 3^n + 6^n - 1$ ,  $n \geq 1$ .
5. Let  $ABCD$  be a fixed convex quadrilateral with  $BC = DA$  and  $BC$  not parallel to  $DA$ . Let two variable points  $E$  and  $F$  lie on the sides  $BC$  and  $DA$  respectively and satisfy  $BE = DF$ . The lines  $AC$  and  $BD$  meet at  $P$ , the lines  $BD$  and  $EF$  meet at  $Q$ , and the lines  $EF$  and  $AC$  meet at  $R$ . Prove that the circumcircles of the triangles  $PQR$ , as  $E$  and  $F$  vary, have a common point different from  $P$ .
6. In a mathematical competition in which six problems were posed to the participants, every two of these problems were solved by more than the fraction  $\frac{2}{5}$  of the contestants. Moreover, no contestant solved all six problems. Show that there are at least two contestants who solved exactly five problems each.