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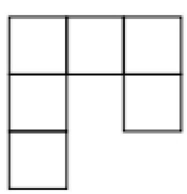
1. Let ABC be an acute-angled triangle with AB not equal to AC. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2. Find all polynomials f with real coefficients such that for all real numbers a, b, c with $a+b+c=0$ we have $f(a-b)+f(b-c)+f(c-a) = 2f(a+b+c)$.

3. Define a “hook” to be a figure made up of six unit squares as shown in the picture below, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all m by n rectangles that can be covered without gaps and without overlaps with hooks such that

The rectangle is covered without gaps and without overlaps. No part of a hook covers area outside the rectangle.

4. Let n be an integer greater than or equal to 3. Let t_1, t_2, t_3, t_4, t_5 be positive real numbers such that

   $n^2 + 1$ is greater than $\left(\sum_{i=1}^{n} t_i\right) \left(\sum_{i=1}^{n} \frac{1}{t_i}\right)$.

   Show that t_i, t_j, t_k are side lengths of a triangle for all indices i, j, k satisfying i less than j and j less than k, and each index at most n.

5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P inside $ABCD$ satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

   Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.