JG

18/8/21

Ordinary Differential Equations (ODE's)

Equation ⇒ Relations

Algebraic equations ⇒ Only variables and their powers.

Ex:- $x^2 + x - y = 5$

Trancesent equations ⇒ Equation containing exponential function, trignometric function, etc.

Ex:- $x + \tan y = x^2 + 1$.

Ordinary Differential Equation (ODEs) ⇒

$$\frac{dy}{dx} + y = x^2 \qquad O = 1,\ d = 1$$

$$\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y = \tan x \qquad O = 2,\ d = 1$$

$$\frac{d^3y}{dx^3} = e^x \qquad O = 3,\ d = 1$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = (\ln x)^2 \qquad O = 2,\ d = 1$$

$\frac{dy}{dx}$ ⇒ derivative.

differentiation ⇒ of e^x in terms of x

$$\frac{d^2y}{dx^2} + 2y\sqrt{\frac{dy}{dx}} + y = \tan x \qquad O = 2,\ d = 2$$

make free from square roots.

$$\frac{d^3y}{dx^3} = \left(e^x + \frac{dy}{dx}\right)^{1/3}$$

O = 3
d = 3,

cube both sides to remove 1/3.

Formation of ordinary O.D.E's
Remove arbitary constants (RAC)

- $3x + 4y = a$ RAC

$3 + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} + \frac{3}{4} = 0$

- $y = ae^{3x} + be^{x}$

$\frac{dy}{dx} = 3ae^{3x} + be^{x}$ — (1)

$\cancel{\frac{d^2y}{dx^2} = 9ae^{3x} + be^{x}}$

$\cancel{\frac{d^2y}{dx^2} = 6ae^{3x} + \frac{dy}{dx}}$

$be^{x} = y - ae^{3x}$ in — (1)

$\frac{dy}{dx} = 3ae^{3x} + y - ae^{3x}$

$\frac{dy}{dx} = 2ae^{3x} + y \Rightarrow 6ae^{3x} = 3\frac{dy}{dx} - 3y$ — (2)

$\frac{d^2y}{dx^2} = 6ae^{3x} + \frac{dy}{dx}$

from — (2) $\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 3y + \frac{dy}{dx}$

Variable seperable method.

- $\int \frac{dx}{x} = \int \tan y \, dy$

order = 1.

$\frac{dy}{dx} = \frac{1}{x \tan y}$

$\ln x = \log(\sec y) + \ln c$

$\ln \frac{x}{\sec y} = \ln c$

$\ln x\cos y = \ln c$

$\boxed{x \cos y = c}$ one arbitary constant.

- $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\int e^{y} dy = \int (e^{x} + x^2)\, dx$

$e^{y} = e^{x} + \frac{x^3}{3} + c.$

Linear Differential Equation.

✓ $\frac{dy}{dx} + P(x)y = Q(x)$

Here, $\frac{dy}{dx}$ ← depend → y = dependent, x = independent

✗ $y\frac{dy}{dx} + y = x$ product

No product of dependent variable or $\frac{dy}{dx}$

✗ $\frac{dy}{dx} + P(y) = x^2$ Product

✗ $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + P(x)y = e^{x}$ Product

✗ $\frac{d^2y}{dx^2}\frac{dy}{dx} + \frac{dy}{dx} + P(x)y = e^{x}$ Product.

Order & degree of Linear Differential eq^n.

$$\frac{dy}{dx} + Py = Q$$

P & Q are function of x only

$$\frac{dx}{dy} + P'(y)x = Q'(y)$$

By multiplying integrating factor, we make it perfectly (exact) differential.

$$If = e^{\int P dx}$$

$$If . y = \int Q(x) \, If . dx + c$$

- $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$

$$\int e^x (x+1)^2 dx = I$$

$$\cancel{= e^x \frac{(x+1)^3}{3} -}$$

$$If = e^{\int -\frac{1}{1+x} dx}$$

$$\int e^x I \, dx$$

$$If = e^{-\ln(1+x)} = (1+x)^{-1} = \frac{1}{1+x}$$

$$\cancel{(1+x)^2} y = \int e^x (x+1)^2 dx \cancel{+ c}$$

$$(1+x)^{-1} y = \int e^x (x+1)^2 \frac{1}{\cancel{x+1}} dx + c$$

$$\frac{y}{1+x} = (x+1)e^x - \int e^x dx$$

$$y = e^x (x+1)[x+1-1] = e^x (x)(x+1)$$

1. $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

2. $y' + y \tan x = \cos x$, $y(0) = 0$

3. $1 + y^2 = (\tan^{-1} y - x)\frac{dy}{dx}$ $x = -\tan^{-1} y - 1 + ce^{\tan^{-1} y}$

2. $If = e^{\int \tan x} = e^{\log \sec x} = \sec x$

$$\sec x . y = \int \sec x . \cos x \, dx + c$$

$$\sec x . y = x + c$$

$y = 0$ at $x = 0$

$c = 0$

$$\sec x . y = x$$

$$y = x . \cos x.$$

4. $\frac{x}{xy}\frac{dy}{dx} + \frac{y}{xy}\log y = \frac{xy \, e^x}{xy}$

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x$$

$\log y = v$

$$\cancel{\frac{1}{y}\frac{dy}{dx}} \frac{dv}{dx} + \frac{1}{x} v = e^x$$

$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

$$If = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$x v = \int x e^x dx + c$$

$$x v = x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

$$x v = e^x (x-1) + c$$

$$x \log y = e^x (x-1) + c$$

1. $I.f = e^{\int 1/x\, dx} = e^{\ln x} = x$

$$y x = \int x(x^3 - 3)\, dx + c$$

$$xy = \frac{x^5}{5} - \frac{3x^2}{2} + c$$

3. $$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$\tan^{-1}y = t$

$\frac{1}{1+y^2} = \frac{dt}{dy}$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$If = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y \; e^{\tan^{-1}y}}{1+y^2} dy$$

$$x e^{\tan^{-1}y} = \int t e^t dt$$

$$x e^{\tan^{-1}y} = t e^t - \int e^t dt + c$$

$$x e^{\tan^{-1}y} = (t-1) e^t + c$$

$$x e^{\tan^{-1}y} = (\tan^{-1}y - 1) e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + c\, e^{-\tan^{-1}y}$$

6. $$-2\frac{y^4}{x^4} + \frac{2y^2}{x^2} - 2 = x^4 c^4$$

$$+2y^4 - 2x^2y^2 + 2x^4 + x^4c^4 = 0$$

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Homogenous equation.

Power of ~~variable~~ terms is same

↳ degree of ~~variable~~ terms

5. $(x+y)\,dy + (x-y)\,dx = 0$

5. $$\frac{dy}{dx} = -\frac{(x-y)}{x+y} = -\frac{x(1-y/x)}{x(1+y/x)}$$

$$\frac{y}{x} = v$$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{-1+v-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{1+v}$$

$$\int\left(\frac{1+v}{1+v^2}\right)dv = \int \frac{-1\,dx}{x}$$

$$\int\left(\frac{1}{1+v^2} + \frac{v}{1+v^2}\right)dv = -\ln x + \ln c$$

$$\tan^{-1}v + \frac{1}{2}\ln(1+v^2) = -\ln x + \ln c$$

$$\tan^{-1}y/x + \frac{1}{2}\ln(1+y^2/x^2) = -\ln x + \ln c$$

$$2\tan^{-1}y/x = \ln\frac{c'x^2}{x^2(x^2+y^2)}$$

6. $(2x^3 - xy^2)\,dx + (2y^3 - x^2y)\,dy = 0$

$(2y^3 - x^2y)\,dy = (xy^2 - 2x^3)\,dx$

$y/x = v$

$$\frac{dy}{dx} = \frac{xy^2 - 2x^3}{2y^3 - x^2y}$$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{xy^2}{2y^3 - x^2y} - \frac{2x^3}{2y^3 - x^2y}$$

$$\frac{dy}{dx} = \frac{xy^2}{xy^2(2y/x - x/y)} - \frac{2x^3}{}$$

$$\frac{dy}{dx} = \frac{x^2y\,(y/x)}{x^2y\,(2y^2/x^2 - 1)} - \frac{2x^3}{x^3(2y^3/x^3 - y/x)}$$

$$v + x\frac{dv}{dx} = \frac{v}{2v^2-1} - \frac{2}{2v^3 - v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 2 - 2v^4 + v^2}{2v^3 - v}$$

$$\int \frac{2v^3 - v}{-2v^4 + 2v^2 - 2}\,dv = \int \frac{1}{x}\,dx$$

$-2v^4 + 2v^2 - 2 = t$

$-8v^3 + 4v = \frac{dt}{dv} = 4(-2v^3 + v)$

$$\int \frac{1}{4}\cdot\frac{1}{t}\,dt = \ln x + \ln c$$

$\frac{1}{4}\ln(-2v^4 + 2v^2 - 2) = \ln xc$

$-2v^4 + 2v^2 - 2 = (xc)^4$

$\leftarrow$

Reducable to homogenous

7. $$\frac{dy}{dx} = \frac{10 + 2y - 2x}{3x - y - 9}$$

$y - 2x + 7 = c(x + y + 1)^4$ — (1)

7. $$\frac{dy}{dx} = \frac{2y - 2x + 10}{3x - y - 9}$$ degree of these 2 is 0.

$x = X + h$

$y = Y + k$

$$\frac{dy}{dx} = \frac{2Y + 2k - 2X - 2h + 10}{3X + 3h - Y - k - 9}$$

$$\frac{dy}{dx} = \frac{2Y - 2X + (2k - 2h + 10)}{3X - Y + (3h - k - 9)} \quad \text{— (2)}$$

$2k - 2h = -10$

$3h - k = 9$

$\Rightarrow$

$2k - 2h = -10$

$-2k + 6h = 18$

$4h = 8$

$h = 2$

$k = 3(2) - 9$

$k = 6 - 9 = -3$

Put $k = -3$ & $h = 2$ in — (2)

to check:

$$\frac{dy}{dx} = \frac{2(y+3) - 2(x-2)}{3(x-2) - (y+3)}$$

$X = x - 2$

$Y = y + 3$

$$\frac{dy}{dx} = \frac{2y + 6 - 2x + 4}{3x - 6 - y - 3} = \frac{2y - 2x + 10}{3x - y - 9}$$

$$\frac{dy}{dX} = \frac{2Y - 2X}{3X - Y}$$

$\frac{Y}{X} = v \qquad \frac{dY}{dX} = v + \frac{dv}{dX}$

$$v + \frac{dv}{dX} = \frac{X(2Y/X - 2)}{X(3 - Y/X)} = \frac{2v - 2}{3 - v}$$

$$x\frac{dv}{dx} = \frac{2v-2-3v+v^2}{3-v}$$

$$x\frac{dv}{dx} = \frac{v^2-v-2}{3-v}$$

$$\int\frac{3-v}{(v+1)(v-2)}dv = \int\frac{dx}{x}$$

$$\frac{A}{v+1} + \frac{B}{v-2}$$

$$A(v-2) + B(v+1) = 3-v$$

$$A + B = -1$$

$$-2A + B = 3$$

$$-3A = 4$$

$$A = -\frac{4}{3} \quad \& \quad B = -1 + \frac{4}{3} = \frac{1}{3}$$

$$-\frac{4}{3}\int\frac{1}{v+1}dv + \frac{1}{3}\int\frac{1}{v-2}dv = \int\frac{dx}{x}$$

$$-\frac{4}{3}\ln(v+1) + \frac{1}{3}\ln(v-2) = \ln x + \ln c$$

$$-\frac{4}{3}\ln(Y+X) + \cancel{\frac{4}{3}\ln X} + \frac{1}{3}\ln(Y-2X) - \cancel{\frac{1}{3}\ln X} = \cancel{\ln X} + \ln c$$

$$-\ln(y+3+x-2)^4 + \ln(y+3-2(x-2)) = \ln c^3$$

$$\frac{y-2x+7}{(y+x+1)^4} = c^3$$

$$y - 2x + 7 = c'(x+y+1)^4$$

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Exact differential equations.

Equations which we get ODE by ~~diff~~ directly ~~on~~ differentiating its primitive are exact differential eq.

Primitive ⇒ Solution of ODE.

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Condition for Exact differential eqⁿ.

Q. $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0$

$$\frac{\partial M}{\partial y} = \cancel{6x^2}\ 0 - 2xy - 2$$

$$\frac{\partial N}{\partial x} = \cancel{-2x^2} - 2xy - 2$$

Check :-

$$2x^3dx - (xy^2dx + x^2y\,dy) - 2(y\,dx + x\,dy) + 3dx = 0$$

or $$\int 2x^3dx - \frac{1}{2}d(x^2y^2) - 2d(xy) + 3dx = 0$$

$$\frac{2x^4}{4} - \frac{1}{2}x^2y^2 - 2xy + 3x = c$$

Formula $$\boxed{\int M\,dx + \int(\text{Non } x \text{ terms of } N)\,dy = c.}$$

$$\frac{2x^4}{4} - \frac{x^2y^2}{2} - 2xy + 3x = c.$$

$$x^4 - x^2y^2 - 4xy + 6x = c$$

9. $(y-x^3)\,dx + (x+y^3)\,dy = 0$

10. $(\sin x \tan y + 1)\,dx + (\cos x \cdot \sec^2 y)\,dy = 0$

9. Check for exact

$\frac{d(y-x^3)}{dy} = 1$ $\qquad \frac{d}{dx}(x+y^3) = 1$

$$\int (y-x^3)\,dx + \int y^3\,dy = c$$

$$xy - \frac{x^4}{4} + \frac{y^4}{4} = c.$$

10. $\frac{M_y - N_x}{N} = \sin x \sec^2 y - (-\sin x \sec^2 y) = \frac{2\sin x \sec^2 y}{\cos x \sec^2 y}$

$= 2\tan x$

$$IF = e^{2\int \tan x} = e^{2\ln \sec x} = \sec^2 x$$

$$\int (\sin x \sec^2 x \tan y + \sec^2 x)\,dx = c$$

$$\tan y \int (\sec x \tan x)\,dx + \int \sec^2 x\,dx = c$$

$$\tan y \sec x + \tan x = c$$

OR

$$\tan y \frac{1}{\cos x} + \frac{\sin x}{\cos x} = c$$

$$\sin x + \tan y = c \cdot \cos x$$

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Reduction of non-exact ODE to Exact ODE

- By Inspection

$x\,dy + y\,dx = x^2 y\,dx$ ← Non-exact

$\frac{x\,dy + y\,dx}{xy} = \frac{x^2 y}{xy}\,dx = x\,dx$ ← exact by multiplying $1/xy$.

- $M\,dx + N\,dy = 0$

when $\left(\frac{M_y - N_x}{N}\right) = g(x)$

$$IF = e^{\int g(x)\,dx}$$

Ex:- $\underbrace{y(2x^2 - xy + 1)}_{M}\,dx + \underbrace{(x-y)}_{N}\,dy = 0$

$M_y = \frac{\partial M}{\partial y} = 2x^2 - 2xy + 1$

$N_x = \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x^2 - 2xy = 2x(x-y)$

$\frac{M_y - N_x}{N} = \frac{2x(x-y)}{x-y} = 2x$

$IF = e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2} y(2x^2 - xy + 1)\,dx + e^{x^2}(x-y)\,dy = 0$. (Now solve)

Check for exact $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\cancel{2}e^{x^2}x - \cancel{2e^{x^2}xy} + e^{x^2}$

- $M\,dx + N\,dy = 0$

when $\left(\dfrac{M_y - N_x}{-M}\right) = g(y)$

$I_f = \int e^{\int g(y)\,dy}$

Ex:- $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

$M_y = \dfrac{\partial M}{\partial y} = 12x^2y^3 + 2x$

$N_x = \dfrac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\dfrac{M_y - N_x}{-M} = \dfrac{6x^2y^3 + 4x}{-(3x^2y^4 + 2xy)} = \dfrac{2(3x^2y^3 + 2x)}{-y(3x^2y^3 + 2x)} = \dfrac{-2}{y}$

$I_f = e^{\int -2/y\,dy} = e^{-2\ln y} = y^{-2} = \dfrac{1}{y^2}$

$\left(3x^2y^2 + \dfrac{2x}{y}\right)dx + \left(2x^3y - \dfrac{x^2}{y^2}\right)dy = 0$

$\int\left(3x^2y^2 + \dfrac{2x}{y}\right)dx + \int = c.$

$3\dfrac{x^3}{3}y^2 + \dfrac{\cancel{2}x^2}{\cancel{2}y} = c$

$x^3y^3 + x^2 = cy$

12. $(y - x^3)\,dx + (x + y^3)\,dy = 0$ $\quad 4xy - x^4 + y^4 = c$

13. $2xy\,dy - (x^2 + y^2 + 1)\,dx = 0$ $\quad y^2 - x^2 + 1 = cx$

14. $2xy\,dx + (y^2 - x^2)\,dy = 0$, $y(2) = 1$. $\quad x^2 + y^2 = 5y$

12. $\dfrac{\partial M}{\partial y} = 1 \qquad \dfrac{\partial N}{\partial x} = 1$

$\int (y - x^3)\,dx + \int (y^3)\,dy = c$

$xy - \dfrac{x^4}{4} + \dfrac{y^4}{4} = c$

$4xy - x^4 + y^4 = c'$

13. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2y. \qquad \dfrac{\partial N}{\partial x} = -2y$

$\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{4y}{-2xy} = -\dfrac{2}{x}$

$I_f = e^{\int -2/x\,dx} = e^{-2\ln x} = \dfrac{1}{x^2}$

$\left(1 + \dfrac{y^2}{x^2} + \dfrac{1}{x^2}\right)dx - \dfrac{2y}{x}\,dy = 0.$

$\int\left(1 + \dfrac{y^2}{x^2} + \dfrac{1}{x^2}\right)dx = c$

$x - \dfrac{y^2}{x} - \dfrac{1}{x} = c$

$x^2 - y^2 - 1 = cx$

$y^2 - x^2 + 1 = c'x \qquad c' = -c$

14. $\frac{\partial M}{\partial y} = 2x$ $\qquad \frac{\partial N}{\partial x} = -2x$

$\frac{M_y - N_x}{-M} = \frac{4x}{-2xy} = \frac{-2}{y}$

$IF = e^{\int -2/y \, dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$

$\frac{2x}{y} dx + \left(1 - \frac{x^2}{y^2}\right) dy = 0$

$\int \frac{2x}{y} dx + \int 1 \, dy = c$

$\frac{2x^2}{2y} + y = c$

$x^2 + y^2 = cy$

for $x = 2$ & $y = 1$

$4 + 1 = c \Rightarrow c = 5$

$x^2 + y^2 = 5y$

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- $M\,dx + N\,dy = 0$

M & N are homogenous functions of same degree then,

$IF = \frac{1}{xM + yN}$; $xM + yN \neq 0$

15. $\underbrace{x^2y}_{\text{degree}=3} dx - \underbrace{(x^3+y^3)}_{\text{degree}=3} dy = 0$

$M = x^2y$

$N = -x^3 - y^3$

$IF = \frac{1}{x^3y - x^3y - y^4} = -\frac{1}{y^4}$

$-\frac{x^2}{y^3} dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0$

$\int -\frac{x^2}{y^3} dx + \int \frac{1}{y} dy = c$

$$\boxed{-\frac{x^3}{3y^3} + \ln y = c}$$

16. $(x^4 + y^4)\,dx - xy^3\,dy = 0$

$M = x^4 + y^4$ $\qquad N = -xy^3$ $\qquad$ degree = 4

$IF = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$

$\int \left(\frac{1}{x} + \frac{y^4}{x^5}\right) dx ~~+ \int 0~~ = c$

$\ln x + \frac{y^4 (x^{-5+1})}{-5+1} = c$

$\ln x - \frac{y^4}{4x^4} = c$

or

$4x^4 \ln x - y^4 = 4x^4 c$

- $y\, g(xy)\, dx + x\, h(xy)\, dy = 0$

$xM - yN \neq 0$

$$IF = \frac{1}{xM - yN}$$

17. $y(1+xy)\,dx + x(1-xy)\,dy = 0$

Check for exact $\Rightarrow$ $\frac{\partial M}{\partial y} = 1 + 2xy$

$\frac{\partial N}{\partial x} = 1 - 2xy$ Not exact.

$$IF = \frac{1}{xy + x^2y^2 - xy + x^2y^2} = \frac{1}{2x^2y^2}$$

$$= \frac{1}{2x^2y}(1+xy)\,dx + \frac{1}{2xy^2}(1-xy)\,dy = 0$$

$$= \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$\frac{-1}{2xy} + \frac{1}{2}\ln x - \frac{1}{2}\ln y = c$$

$$\frac{-1}{xy} + \ln x - \ln y = c$$

$$\ln\frac{x}{y} - \frac{1}{xy} = c$$

18. $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0$

$x \sec xy = cy$

$$IF = \frac{1}{x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin xy + xy \cos xy}$$

$$IF = \frac{1}{2xy \cos xy}$$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$$

$$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int \frac{-1}{2y}\,dy = \frac{2c}{C}$$

$$\frac{y \ln \sec xy}{y} + \ln x - \ln y = c$$

$$\boxed{x \sec xy = cy}$$

OR

$$\boxed{\frac{x}{y} \sec xy = c}$$

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- $M\,dx + N\,dy = 0$

$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$

$IF = x^h y^k$

19. doubt

$(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0$

multiply

$y(2y\,dx + 4x\,dy) + x^2(4y\,dx + 3x\,dy) = 0$

$IF = x^h y^k$

$(2x^h y^{k+2} + 4x^{h+2} y^{k+1})\,dx + (4x^{h+1}y^{k+1} + 3x^{3+h}y^{k})\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$2(k+2)x^h y^{k+1} + 4(k+1)x^{h+2}y^k = \frac{\partial M}{\partial y}$

$\frac{\partial N}{\partial x} = 4(h+1)y^{k+1}x^h + 3(h+3)x^{2+h}y^k$

$2(k+2) = 4(h+1)$

$2k+4 = 4h+4$

$4h - 2k = 0$ — (1)

$3(h+3) = 4(k+1)$

$3h+9 = 4k+4$

$3h - 4k = -5$

$-8h \mp 4k = 0$

$-5h = -5$

$h = 1$

$k = \frac{4h}{2} = \frac{4}{2} = 2.$

1.

2

$If = xy^2$

$(2xy^4 + 4x^3y^3)dx + (4x^2y^3 + 3x^4y^2)dy = 0$

$\int M dx + \int \text{non } x \text{ terms of } N = c$

$\frac{2x^2y^4}{2} + \frac{4x^4y^3}{4} = c$

$x^2y^4 + x^4y^3 = 0$

20. $x^3y^3(2y\,dx + x\,dy) - (5y\,dx + 7x\,dy) = 0$

$x^3y^3 + 2 = cx^{5/3}y^{7/3}$

$If = x^h y^k$

$2x^{3+h}y^{4+k}dx + x^{4+h}y^{k+3}dy - 5x^h y^{k+1}dx - 7x^{h+1}y^k dy = 0$

$(2x^{h+3}y^{k+4} - 5x^h y^{k+1})dx + (x^{h+4}y^{k+3} - 7x^{h+1}y^k)dy = 0.$

$\frac{\partial M}{\partial y} = 2(k+4)x^{h+3}y^{k+3} - 5(k+1)x^h y^k$

$\frac{\partial N}{\partial x} = (h+4)x^{h+3}y^{k+3} - 7(h+1)x^h y^k$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$2(k+4) = h+4$

$2k+8 = h+4$

$2k - h + 4 = 0$ — (1)

$+5(k+1) = +7(h+1)$

$5k+5 = 7h+7$

$5k - 7h - 2 = 0$ (2)

Multiply by 7

$5k - 7h - 2 = 0$

$14k - 7h + 28 = 0$

$-9k \quad -30 = 0$

$k = -\frac{30}{9} = \frac{-10}{3}$

$h = \frac{-60}{9} + 4 = \frac{-60+36}{9}$

$h = \frac{-24}{9} = \frac{-8}{3}.$

$(2x^{1/3}y^{2/3} - 5x^{-8/3}y^{-7/3})dx + (x^{4/3}y^{-1/3} - 7x^{-5/3}y^{-10/3})dy = 0$

$\int (2x^{1/3}y^{2/3} - 5x^{-8/3}y^{-7/3})dx = c$

$\frac{3(+2)x^{4/3}}{4}y^{2/3} + \frac{5\times(3)x^{-5/3}}{5}y^{-7/3} = c$

$\cancel{\frac{4}{3}\left(\frac{y}{x}\right)^{1/3} + \frac{55}{3x^{11/3}y^{7/3}} = c}$

$+\frac{3}{2}x^{4/3}y^{2/3} + \frac{3}{x^{5/3}y^{7/3}} = c$

$$+\frac{1}{2} x^{\frac{4+5}{3}} y^{\frac{2+7}{3}} + 1 = \frac{c}{3} x^{7/3} y^{7/3}$$

$$+ x^3 y^3 + 2 = C x^{5/3} y^{7/3}$$

$$\boxed{x^3 y^3 + 2 = C x^{5/3} y^{7/3}}$$

→ 21. $y^2 dx + (x^2 - xy - y^2) dy = 0$ $\quad (x-y) y^2 = c (x+y)$

22. $2xy\, dy - (x^2+y^2+1) dx = 0$ $\quad y^2 - x^2 + 1 = cx$

→ 23. $(x^3y^3 + x^2y^2 + xy + 1) y\, dx + (x^3y^3 - x^2y^2 + xy ...) x\, dy = ...$ $\quad x^2y^2 - 2xy \ln cy = 1$

$\cancel{\frac{\partial M}{\partial y} = 2x}$ $\quad \cancel{\frac{\partial N}{\partial}}$

22. $\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2y$

$$\frac{M_y - N_x}{N} = \frac{-4y}{2yx} = -\frac{2}{x}$$

$$If = \frac{1}{x^2}$$

$$\left(\frac{2y}{x}\right) dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = 0$$

$$-\int \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = c$$

$$-\left(x + \frac{y^2 x^{-2+1}}{-2+1} + \frac{x^{-2+1}}{-2+1}\right) = c$$

$$= -\left(x - \frac{y^2}{x} - \frac{1}{x}\right) = c$$

$$= -x^2 + y^2 + 1 = cx$$

21. $If = \frac{1}{xy^2 + x^2y - xy^2 - y^3} = \frac{1}{y(x^2-y^2)}$

N has no x terms.

$$\int \frac{y^2}{y(x^2-y^2)} dx = c$$

$$y \int \frac{1}{x^2-y^2} = c$$

$$y\left[\frac{1}{2y} \ln \frac{x-y}{x+y}\right] = \ln c$$

$$x - y = c(x+y).$$

23. $If = \frac{1}{x^4y^4 + x^3y^3 + x^2y^2 + xy - x^4y^4 + x^3y^3 + x^2y^2 + xy}$

$$If = \frac{1}{2x^3y^3}$$

$$\frac{1}{2}\int \left[y + \frac{1}{x} + \frac{1}{x^2y} + \frac{1}{x^3y^2}\right] dx + \frac{1}{2}\int \frac{1}{y} dy = c$$

$$xy + \ln x - \frac{1}{xy} - \frac{1}{2x^2y^2} + \ln y = c.$$

$$x^2y^2 - xy - 1 + x^2y^2 \ln x + x^2y^2 \ln y = cx^2y^2$$

$$x^2y^2 - xy + x^2y^2 (\ln xy\, c) \cdot 1$$

2nd order ODE's

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + q(x)\,y = R(x)$$

Higher order ODE's

$$\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n(x)\,y = b(x)$$

Higher order ODE's with constant coeff.

$$\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1\frac{dy}{dx} + a_0 y = Q(x)$$

where $a_0, a_1, a_2, \dots a_{n-1}$ are constant.

Examples :-

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = x^2$$

$$2\frac{d^2y}{dx^2} + 8y = \sin x$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4 = x^2$$

$$2\frac{d^2y}{dx^2} = \sin x - 8x$$

$$\frac{dy}{dx} = x$$

free from constant.

$$y = \underbrace{\frac{x^2}{2}}_{\text{Particular integral (P.I.)}} + \underbrace{c}_{\text{arbitrary constant}} \rightarrow \text{complementary function}$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x$$

if ~~x=0~~ Right hand side $=0$.

$y =$ complementary func. (C.F.) + $\underbrace{\text{P.I.}}_{=0}$

* P.I. dependents on right hand side function.

Process → To find C.F. we write auxiliary eq^n / helping eq^n.

$$\boxed{y \Rightarrow 1, \quad \frac{dy}{dx} \rightarrow m, \quad \frac{d^2y}{dx^2} \rightarrow m^2}$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0$

$m = -2, -2$

$CF = (C_1 + C_2 x)\,e^{-2x}$

$y = CF + \underbrace{PI}_{\to 0}$

$$\frac{d^2y}{dx^2} - 4y = 0$$

A.E :- $m^2 - 4 = 0 \Rightarrow m = 2, -2.$

$$\boxed{C.f. :- C_1 e^{m_1 x} + C_2 e^{m_2 x}}$$

$C.f. \Rightarrow C_1 e^{2x} + C_2 e^{-2x}$

$y = C.f. + \underbrace{P.I.}_{\to 0}$

$y = C_1 e^{2x} + C_2 e^{-2x}$

arbitrary const

24. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$(m-3)(m-2) = 0$

$m = 2, 3.$

$C.F. = C_1 e^{3x} + C_2 e^{2x}$

$P.I. = 0$

$\therefore y = C_1 e^{3x} + C_2 e^{2x}$

25. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 11y = 0$ $C_1 e^{(-3x+\sqrt{2}i)x} + C_2 e^{(-3x-(\sqrt{2}}$

$m^2 + 6m + 11 = 0$

$m = \frac{-6 \pm \sqrt{36-44}}{2} = \frac{-6 \pm 2\sqrt{2}i}{2}$

$m = -3 \pm \sqrt{2}i$

$C.F. = C_1 e^{(-3+\sqrt{2}i)x} + C_2 e^{(-3-\sqrt{2}i)x}$

OR

more → appropriate $C.F = e^{-3x}(A\cos\sqrt{2}x + B\sin\sqrt{2}x)$

26. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0.$

$m^3 - 4m^2 + m + 6 = 0$

~~$m^3 - 3m + 6 = 0$~~

$m =$

$$\begin{array}{r|l} & m^2 - 5m + 6 \\ m+1 & m^3 - 4m^2 + m + 6 \\ & -m^3 \pm m^2 \\ & -5m^2 + m \\ & \mp 5m^2 \mp 5m \\ & 6m + 6 \\ & 6m + 6 \\ & \times \end{array}$$

$(m+1)(m^2 - 5m + 6) = 0.$

$(m+1)(m-2)(m-3) = 0.$

$m = 2, 3, -1$

$C.f. = C_1 e^{2x} + C_2 e^{3x} + C_3 e^{-x}$

$P.I = 0.$

* If roots are same i.e; $m = m_1, m_1$

$C.F. = (C_1 + C_2 x) e^{m_1 x}$

28. $\frac{d^3y}{dx^3} - 27y = 0$

$m^3 - 27 = 0$

$m^3 - 3^3 = 0$

$(m-3)(m^2 + 3m + 9) = 0$

$m = 3, \left(\frac{-3}{2} + \frac{5}{2} i\right), \left(\frac{-3}{2} - \frac{5}{2} i\right)$

$C.F = C_1 e^{3x} + e^{-3/2 x}\left(C_1 \cos\frac{5}{2}x + C_2 \sin\frac{5}{2}x\right)$

29. $y'''' - y''' = 0$

$m^4 - m^3 = 0$

$m^3(m-1) = 0$

$m = 1, 0, 0, 0$

$C.F. = C_1 e^x + (C_2 + C_3 x + C_4 x^2) e^{0x}$

$C.F. = C_1 e^x + C_2 + C_3 x + C_4 x^2$

30. $y'''' + 6y''' + 9y'' = 0.$

$m^4 + 6m^3 + 9m^2 = 0$

$m^2(m^2 + 6m + 9) = 0$

$m^2(m^2 + 3m + 3m + 9) = 0$

$m^2(m(m+3) + 3(m+3)) = 0$

$m = 0, 0, -3, -3.$

$C.F. = (C_1 + C_2 x) e^{0x} + (C_3 + C_4 x) e^{-3x}$

$C.F = C_1 + C_2 x + (C_3 + C_4 x) e^{-3x}$

31. $y'' + 4y = 0$

$m^2 + 4 = 0$

$m = \pm 2i = 2i, -2i$

* If $m = a + bi,\ a - bi$ then; $CF = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

$CF = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$

$C.F = A\cos 2x + B \sin 2x$

$y = A \cos 2x + B \sin 2x.$

Class Practice Question

7. $(x^4 + y^4)dx - xy^3 dy = 0$

$\frac{\partial M}{\partial y} = 4y^3 \qquad \frac{\partial N}{\partial x} = -y^3$

$\frac{M_y - N_x}{N} = \frac{5y^3}{-xy^3} = \frac{-5}{x}$

$IF = e^{-5\int 1/x\, dx} = e^{-5\ln x} = x^{-5} = \frac{1}{x^5}$

$\left[\frac{1}{x} + \frac{y^4}{x^5}\right]dx - \frac{y^3}{x^4} dy = 0$

$\int M dx + \int (\text{Non } x \text{ terms of } N) dy = C.$

$\ln x + \left(\frac{-y^4 x^{-4}}{4}\right) + 0 = c$

$-\frac{y^4}{x^4} + 4\ln x = c$

$y^4 = 4x^4 \ln x + c x^4$

* $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$

$If = x^h y^k$

$(4x^{h+1}y^{k+1} + 3x^h y^{k+4})\,dx + (2x^{h+2}y^{k+3} + 5x^{h+1}y^{k+3})\,dy$

$\frac{\partial M}{\partial y} = 4(k+1)x^{h+1}y^k + 3(k+4)x^h y^{k+3}$

$\frac{\partial N}{\partial x} = 2(h+2)x^{h+1}y^k + 5(h+1)x^h y^k$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$2(h+2) = 4(k+1)$ $\quad$ $5(h+1) = 3(k+4)$

$2h+4 = 4k+4$ $\quad$ $5h+5 = 3k+12$

$h - 2k = 0$ $\quad$ $5h - 3k = 7$

$5h - 10k = 0$

$5h - 3k = 7$ $\quad$ $k = 1$

$-7k = -7$ $\quad$ $h = 2$

$(4x^3y^2 + 3x^2y^5)\,dx + (2x^4y^{4} + 5x^3y^4)\,dy = 0$

$\int M\,dx + \int N(\text{non } x \text{ terms}) = c$

$\frac{4x^4y^2}{4} + \frac{3x^3y^5}{3} = c$

$x^4y^2 + x^3y^5 = c$

19. $2xy\,dx + (y^2 - x^2)\,dy = 0$, $y(2) = 1$

$\frac{\partial M}{\partial y} = 2x$ $\quad$ $\frac{\partial M}{\partial x} = -2x$

$\frac{M_y - N_x}{-M} = \frac{4x}{-2xy} = \frac{-2}{y}$

$If = e^{-2\int 1/y\,dy} = e^{-2\ln y} = \frac{1}{y^2}$

$\frac{2x}{y}\,dx + \left(1 - \frac{x^2}{y^2}\right)dy = 0.$

$\frac{\partial M}{\partial y} = 2x \ln y$ $\quad$ $\frac{\partial N}{\partial x} = -x^2 y^{-2-1}$ $= +2x^2$

$\frac{\partial N}{\partial x} = -\frac{1}{y^2}(2x) = \frac{-2x}{y^2}$

Not exact.

$\frac{dx}{dy} = \left(1 - \frac{x^2}{y^2}\right) \div \frac{2x}{y}$

$\frac{x}{y} = v.$ $\quad$ $x = vy$

$\frac{dx}{dy} = v + \frac{dv}{dy}y.$

$v + \frac{dv}{dy}y = \frac{1-v^2}{2v}$

$y\frac{dv}{dy} = \frac{1 - v^2 - 2v^2}{2v} = \frac{1 - 3v^2}{2v}$

$\frac{d}{dx}(e^{x^2})$ $\quad D \equiv \frac{d}{dx}$ $\quad Dy = \frac{dy}{dx}$ $\quad (D-2)(D-3)=0$

$D = 2, 3$ X

$x^2 + y^2 = yc$ $\quad$ At $x = 2, y = 1$.

$4 + 1 = c$

$\therefore c = 5$

$\therefore x^2 + y^2 = 5y$.

10 $(xy + y^2)\, dx + (x + 2y - 1)\, dy = 0$

$\frac{\partial M}{\partial y} = x + 2y$ $\qquad \frac{\partial M}{\partial x} = 1$

$\frac{M_y - N_x}{-N} = \frac{x + 2y - 1}{-(x + 2y - 1)} = -1$

$(D^2 - 5D + 6)y = \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y =$

32. $(D-2)(D-3)y = e^{x+2} = e^x \cdot e^2$ if

Auxilary eq^n :- $(m-2)(m-3) = 0$

$m = 2$ & $m = 3$

$Cf = C_1 e^{2x} + C_2 e^{3x}$

$PI = \frac{1}{(1-2)(1-3)} e^x = \frac{e^x}{2}$

$\therefore y = (C_1 e^{2x} + C_2 e^{3x}) + \frac{e^x}{2} e^2$

$\frac{d}{dx}(3x^2) = 3\frac{d}{dx}x^2$

$\int 3e^x = 3\int e^x$

* $F(D)y = e^x$

$y = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Solution = $y = cf + P.I.$

33. $\left(EI\frac{d^2y}{dx^2} + Py = a+1\right. = \left(D^2 + \frac{P}{EI}\right)y = \ldots$

$\Rightarrow EI\, m^2 + P = \cancel{\ldots}\ 0$

$m^2 + \frac{P}{EI} = \frac{\cancel{0}}{\cancel{EI}}\ 0$

~~Auxiliary~~ $CF = C_1 \cos\left(\frac{P}{EI}\right)^{\frac{1}{2}} x + C_2 \sin\left(\frac{P}{EI}\right)\ldots$

$PI = \frac{1}{0 + P/EI}\ (a+1)/EI \cdot e^{0x}$

$PI = \frac{EI}{P}\frac{(a+1)}{EI} \cdot 1 = \frac{a+1}{P}$

1. $y = C_1 \cos\sqrt{P/EI}\, x + C_2 \sin\sqrt{P/EI}\, x + \frac{a+1}{P}$

* $F(D)y = Q(x)$

2. $y = \frac{1}{F(D)} Q(x)$

when this is 0.

3. in case of exponential

$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$

34. $(D-2)(D-1)y = e^{2x}$

$P.I. = \frac{1}{(D-2)(D-1)} e^{2x} = \frac{1}{(D-2)(2-1)} e^{2x}$

$PI = \frac{x^1 e^{2x}}{1!} = x e^{2x}$

35. $(D-2)(D-4)y = \cosh(3x)$

36. $(D-3)(D-4)y = \sinh(4x)$

Particular Integral.

$f(D)y = Q(x)$.

$y = \frac{Q(x)}{f(D)}$

$(D-a)y = Q(x)$

Linear factor

$y = \frac{1}{D-a} Q(x)$

$PI = e^{ax}\int e^{-ax} Q(x)\, dx$

37. $(D-2)y = e^{3x}$

$PI = e^{2x}\int e^{-2x} e^{3x}\, dx$

$PI = e^{2x}\int e^{x}\, dx = e^{3x}$

38. $(D^2-4)y = x^2$

$PI = \frac{1}{D^2-4} x^2 = \frac{1}{(D-2)(D+2)} x^2$

1st method operate $\frac{1}{D+2} x^2$ then operate its result with $\frac{1}{D-2}$.

2nd. $\frac{1}{4}\frac{(D+2)-(D-2)}{(D-2)(D+2)} x^2$

$= \frac{1}{4}\left[\frac{1}{D-2} - \frac{1}{D+2}\right] x^2$

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

* $f(D)y = x^n$

$PI = f(D)^{-1} x^n$

$(D^2-4)y = x^2$

$PI = \frac{1}{D^2-4} x^2$

$PI = \frac{1}{-4}\left(\frac{1}{1-D^2/4}\right) x^2$

1. $PI = \frac{1}{-4}\left(1-\frac{D^2}{4}\right)^{-1} x^2$

$PI = \frac{1}{-4}\left(1+\frac{D^2}{4}+\frac{D^4}{16}+ \cdots\right) x^2$

2. $PI = \frac{1}{-4}\left(x^2 + \frac{1}{4} D^2(x^2)\right)$

$= -\frac{1}{4}\left(x^2 + \frac{1}{4} 2\right) = -\frac{1}{4}\left(x^2+\right.$

3. $= -\left(\frac{2x^2+1}{8}\right)$

$f(D^2)y = \sin ax / \cos ax$

$PI = \frac{1}{f(D^2)} \sin ax / \cos ax$

$= \frac{1}{f(-a^2)} \sin ax / \cos ax$

39. $(D^2+5)y = \sin 2x$

$PI = \frac{1}{D^2+5} \sin 2x = \frac{1}{-4+5} \sin 2x$

40. $(D^2+D+5)y = \cos 2x$

$PI = \frac{1}{D^2+D+5} \cos 2x = \frac{1}{-4+D+5} \cos 2x$

$= \frac{1}{D+1} \cos 2x = \frac{D-1}{D^2-1} \cos 2x.$

$= \frac{D-1}{-4-1} \cos 2x = -\frac{1}{5}(D-1)\cos 2x$

$= \frac{+1}{5}\left[2\sin 2x + \cos 2x\right]$

When denomenator '0'.

41. $(D^2+a^2)y = \sin ax$

$PI = \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$

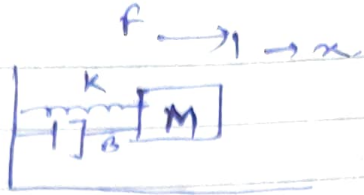
$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax.$

42. $(D^2-D+1)y = \sinh x$

$PI = \frac{1}{2} e^x - \frac{1}{6} e^{-x}$

HF sir

1.



Mechanical eqⁿ :- $F = M\frac{d^2x}{dt^2} + kx + B\frac{dx}{dt}$ or f = M

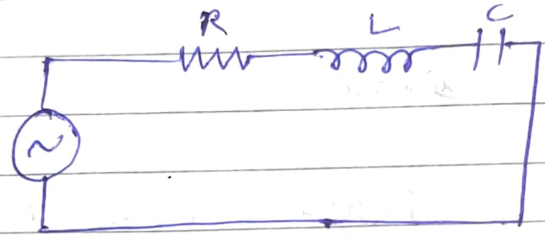
Voltage equation :- $V = L\frac{di}{dt} + \frac{1}{C}\int i\,dt$
$+ Ri$

Current equation :- $i = C\frac{dv}{dt} + \frac{1}{L}\int v\,dt +$

$I = I_1 + I_2 + \ldots$

1.

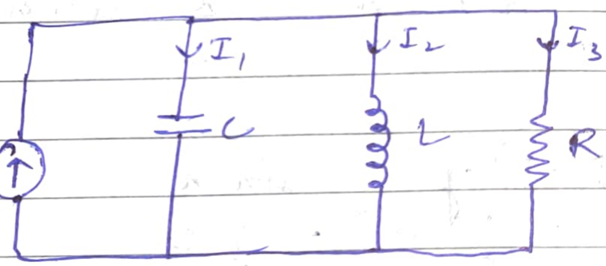
Electrical equivalent circuit.



2

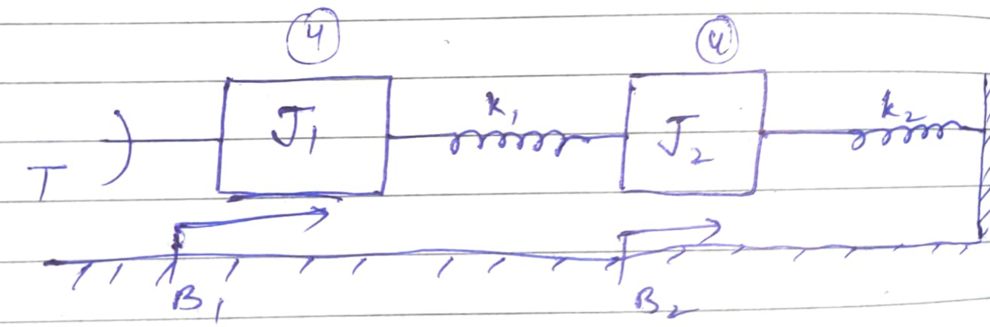
3

Current equivalent circuit.



L

2.



$$T = J_1\frac{d^2\theta_1}{dt^2} + k_1(\theta_1 - \theta_2) + B_1\frac{d\theta}{dt}$$

$$0 = J_2\frac{d^2\theta_2}{dt^2} + k_1(\theta_2 - \theta_1) + k_2\theta_2 + B_2$$

$x_0 = 0$ y_0 h k_1 k_2 P01BU k_4

$x_1 = 0.1$

$x_2 = 0.2$

Numerical solution of ODE's.

Runge kutta Method of 4th order.

$$\frac{dy}{dx} = f(x, y)$$

x_0, y_0 intial point

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $k_1 = h\, f(x_0, y_0)$

$k_2 = h\, f(x_0 + h/2,\ y_0 + k_1/2)$

$k_3 = h\, f(x_0 + h/2,\ y_0 + k_2/2)$

$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$

$$\frac{dy}{dx} = x + y^2 \qquad y(0) = 1 \quad \text{find } y \text{ for } x = 0.2.$$

$x_0 = 0,\ y_0 = 1$

$k_1 = 0.1(x_0 + y_0^2) = 0.1$

$k_2 = 0.1\left(0 + \frac{0.1}{2} + \left(1 + \frac{0.1}{2}\right)^2\right) = 0.11525$

$k_3 = 0.1\left(0 + \frac{0.1}{2} + (1 + 0.11525)^2\right) = 0.11685$

$\cancel{k_3 = 0.1(0.05 + 1.24378) = 0.129378}$ 0.116

$k_4 = 0.13474.$

$\Delta y = 0.11649.$

$y_1 = y_0 + \Delta y = 1.11649.$

Laplace transform

Integral Transforms.

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ → kernel of Laplace Transform

parameter of transformation

$L\{t\} = \int_0^\infty e^{-st}\, t\, dt$

$= \frac{1}{s^2}$

1. $L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \dots$

$L\{t^\alpha\} = \frac{\gamma(\alpha+1)}{s^{\alpha+1}}$

2. $\gamma(n+1) = n!$

$\gamma(6) = 5!$

3. $\gamma(n+1) = n(n-1)(n-2)\dots 1$ $\qquad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\gamma(n+1) = n\gamma(n)$

4. $L\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma(3/2)}{s^{3/2}}$

$= \frac{1}{2}\frac{\gamma(1/2)}{s^{3/2}} = \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}$

$\Gamma(n);\ n \neq 0, -1, -2, \dots$

$L\{e^{-at}\} = \frac{1}{s+a}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{10\} = L\{10t^0\} = \frac{10}{s}$

Properties :-

1. Linearity

$L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$

$= F(s) + G(s)$

Q1 $L\{e^t + t^e + e^e\} = L\{e^t\} + L\{t^e\} + L\{e^e\}$.

$= \frac{1}{s-1} + \frac{\Gamma(e+1)}{s^{e+1}} + \frac{e^e}{s}$

2. Shifting Property.

$L\{f(t)\} = F(s)$

$L\{e^{at} f(t)\} = F(s-a)$

$L\{t e^{3t}\} = \frac{1}{(s-3)^2}$ $\qquad L\{t\} = \frac{1}{s^2}$

Q2 $L\{\cos 4t \sinh t\} = L\{\cos 4t\left(\frac{e^t - e^{-t}}{2}\right)\}$

$= \frac{1}{2} L\{e^t \cos 4t\} - \frac{1}{2} L\{e^{-t}\cos 4t\}$

$= \frac{1}{2} \frac{s-1}{(s-1)^2+4^2} - \frac{1}{2} \frac{s+1}{(s+1)^2+4^2}$

* Laplace Transform of derivates.

If $L\{f(t)\} = F(s)$

$L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

1. $L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$

2. $L\{y'' + 2y' + y\}$ $\quad y(0) = 0,\ y'(0) = 1$

$= L\{y''\} + L\{2y'\} + L\{y\}$

$= s^2 Y(s) - 1 + 2(sY(s)) + Y(s)$

$= s^2 Y(s) + 2sY(s) + Y(s) - 1$

$= Y(s)(s^2 + 2s + 1) - 1$

3. $L\{t^{7/2} e^{3t}\}$

4. $L\{e^{t-2}\}$

5. $L\{\cos(t+1)\}$

6. $L\{(t+2)^2\}$

7. $L\{\sin^2 t\}$

8. $L\{(e^t - e^{-4t})^2\}$

3. $L\{t^{7/2}\} = \frac{\Gamma(7/2 + 1)}{s^{7/2+1}}$

$= \frac{\frac{7}{2}\Gamma(\frac{7}{2})}{s^{9/2}-3} = \frac{105\sqrt{\pi}}{16(s^{9/2}-3)}$.

4. $L\{e^{t-2}\} = L\{e^t . e^{-2}\} = \frac{e^{-2}}{s-1}$

5. $L\{\cos(t+1)\} = L\{\cos t \cos 1 - \sin t \sin 1\}$.

$= L\{\cos 1 \cos t\} - L\{\sin 1 \sin t\}$.

$= \frac{s(\cos 1)}{s^2+1} - \frac{(\sin 1)}{s^2+1}$

6. $L\{t^2 + 4 + 4t\} = L\{t^2\} + L\{4t\} + L\{4\}$

$= \frac{2!}{s^{2+1}} + \frac{4}{s^2} + \frac{4}{s} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

7. $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$.

$\frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

8. $L\{(e^t - e^{-4t})^2\} = L\{e^{2t}\} - 2L\{e^{t-4t}\} + L\{e^{-8t}\}$

$= \frac{1}{s-2} - \frac{1}{s-8} - \frac{2.1}{s+3}$

change of

$L\{f(t)\} = F(s)$

$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

L.T. of integrals.

$L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

1. where $F(s) = L\{f(t)\}$

Division by T.

2 $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

If $L\{f(t)\} = F(s)$

$L\{t f(t)\} = -\frac{d}{ds} F(s)$

$L\{t^2 f(t)\} = \frac{d^2}{ds^2} F(s)$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

2nd shifting property

unit step function $f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$f(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$

2nd shifting property / 2nd Transformation property

$f(t) = \begin{cases} f_1(t) & t < a \\ f_2(t) & t > a \end{cases}$

$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}\, u(t-a)$

$f(t) = \begin{cases} f_1(t) & 0 < t < a_1 \\ f_2(t) & a_1 < t < a_2 \\ f_3(t) & a_2 < t \end{cases}$

$f(t) = f_1(t) + (f_2(t) - f_1(t))\, u(t-a_1) + (f_3(t) - f_2(t))\, u(t-a_2)$

If $L\{f(t)\} = F(s)$

$L\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$

$g(t) = \begin{cases} 0; & 0 < t < 5 \\ (t-5)+2; & t > 5 \end{cases}$

$g(t) = 0 + \{(t-5)+2\}\, u(t-5)$

~~$g(t) = u(t-5)\{t-3\}$~~

$g(t) = \{t-3\}\, u(t-5)$

$L\{(t-3)\, u(t-5)\}$

$$g(t) = \begin{cases} 0 & 0 < t \le 1 \\ t & 1 < t \le 2 \\ \sin t & 2 < t \end{cases}$$

$g(t) = 0\, u(t-0) + (t-0)\, u(\ldots$
$(\sin t - t)\, u(t-\ldots$

$g(t) = t\, u(t-1) + (\sin t - t)\, u(\ldots$

1. $L\{t\, u(t-1)\} = e^{-1s} L\{t+1\}$
$= e^{-1s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

2. $L\{(\sin t - t)\, u(t-2)\} = e^{-2s} L\{\ldots$
$-(t\ldots$

~~formula~~

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$

$= \int_0^1 0 + \int_1^2 t e^{-st}\, dt + \int_2^\infty \sin t\, e^{-\ldots}$

Impulse function

Example $\Rightarrow$ Earthquake

$\epsilon \Rightarrow$ denotes a very small quantity.

$\epsilon \to 0$

$$f_\epsilon(x) = \begin{cases} 0 & ; \; x < a \\ 1/\epsilon & ; \; a < x < a+\epsilon \\ 0 & ; \; x > a+\epsilon \end{cases}$$

$\int_a f_\epsilon(x)\, dx = 1$

Q $L^{-1}\left\{\dfrac{1}{(s-2)^2(s^2+9)}\right\} =$

$\Rightarrow \dfrac{A}{s-2} + \dfrac{B}{(s-2)^2} + \dfrac{Cs+D}{s^2+9}$

$As - 2A + s^2A + 9A + s^2B + 9B +$
$s^3C + 4sC - 4s^2C + Ds^2 + 4D - 4sD = 1$

$C = 0 \quad \text{——} \; (1)$

$A + B - C + D = 0$

$A + B + D = 0 \quad \text{——} \; (2)$

$A + 4C - 4D = 0$

$A - 4D = 0. \quad \text{——} \; (3)$

$A = 4D$

$4D + B + D = 0$

$5D + B = 0. \quad \text{——} \; (4)$

$B = 5D$

$\cancel{-2A}\quad 7A + 9B + 4D = 1.$

$7 \times 4D + 5 \times 9D + 4D = 1$

$28D + 45D + 4D = 1$

$77D = 1$

$D = \dfrac{1}{77}$

$A = \dfrac{4}{77}, \quad B = \dfrac{5}{77}, \quad C = 0, \quad D = \dfrac{1}{77}$

$\therefore \dfrac{1}{77}.\; L^{-1}\left\{\dfrac{4}{s-2} + \dfrac{5}{(s-2)^2} + \dfrac{1}{s^2+9}\right\}$

$= \dfrac{1}{77} \quad 4e^{2t} + 5te^{2t} +$

Convolution theorem

If $L\{f(t)\} = F(s)$ and
$L\{g(t)\} = G(s)$, then.

$$L^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

Ex:- $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$. $\int_0^t \underbrace{t}_{g(t)}\ \underbrace{\frac{\sin at}{a}}_{f(t)}$

1. $= \int_0^t \frac{1}{a} \sin au \cdot (t-u)\, du$ u variable, t constant

$= \frac{1}{a}\int_0 (t \sin au - \underset{I}{u}\ \underset{II}{\sin au})\, du$

2. $= \frac{1}{a}\left[\left[t(-\cos au)\right]_0^t\right] + \left[\frac{u\cos au}{a} + \frac{\sin au}{a^2}\right]_0^t$

$= \frac{1}{a}\left[\left[t(-\cos at) - t\right] + \left[\left(\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right)\right]\right]$

$u \sin au\, du = u\int \sin au\, du - \int\left(1 \cdot \frac{-\cos au}{a}\right)$

$= -u\frac{\cos au}{a} + \frac{\sin au}{a^2}$

$= \frac{1}{a}\left(-t\cos at - t + \frac{t\cos at}{a} + \frac{\sin at}{a^2}\right)$

$= -\frac{t}{a}\left[\cos at - 1\right) - \frac{1}{a}\left[t\cos at - \frac{\sin at}{a}\right.$

Q $L^{-1}\left\{\underbrace{\frac{s}{(s^2+4)}}_{F(s)}\ \underbrace{\frac{1}{(s^2+9)}}_{G(s)}\right\}$

$f(t) = \cos 2t \qquad g(t) = \frac{1}{3}\sin 3t$

$= \frac{1}{3}\int_0^t \sin 2u \sin 3(t-u)\, du$

$= \frac{1}{3}$

If $L\{f(t)\} = F(s)$

$L\{e^{at} f(t)\} = F(s-a)$

If $L^{-1}\{F(s)\} = f(t)$

$L^{-1}\{F(s-a)\} = e^{at} f(t)$

$L^{-1}\left\{\frac{1}{(s-2)^2}\right\} = e^{2t}\, t \qquad L^{-1}\left\{\frac{1}{s^2}\right\} = t$

Q. $L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$

$= \frac{1}{2} e^{-t} \sin 2t$

$L\{t f(t)\} = -\frac{d}{ds} F(s)$

$L^{-1}\left\{\frac{d}{ds} F(s)\right\} = t f(t)$

$L\{f'(t)\} = sF(s) - f(0)$

$L^{-1}\{sF(s)\} = f'(t)$

$L\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s} = \frac{1}{s} F(s)$

$\frac{1}{s(s^2+a^2)} = \frac{1}{s} \cdot \frac{1}{s^2+a^2}$

$y'' + 4y' + 5y = e^t, \quad y(0) = 0, y'(0) = 0$

Take L.T. to both sides.

$= L\{y'' + 4y' + 5y\} = L\{e^t\}$

$= L\{y''\} + L\{4y'\} + L\{5y\} = L\{e^t\}$

Let $L\{y\} = Y(s) \Rightarrow L\{y'\} = sY(s) - y(0)$

$L\{y''\} = s^2 Y(s) - s y(0) - y'(0)$

$\Rightarrow s^2 Y(s) + 4s Y(s) + 5 Y(s) = \frac{1}{s-1}$

$= (s^2 + 4s + 5) Y(s) = \frac{1}{s-1}$

$= Y(s) = \frac{1}{(s-1)(s^2+4s+5)}$

Taking inverse Laplace Transform.

$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s-1)(s^2+4s+5)}\right\}$

$y = L^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{s^2+4s+5}\right\}$ $\quad e^t \qquad e^{-t}\sin \qquad \frac{1}{(s+2)^2+}$

$y = \int_0^t e^{-u} \sin u \; e^{t-u}\, du.$

$y = \int_0^t \underset{II}{e^{t-2u}} \; \underset{I}{\sin u}\, du.$

$y = \sin u \int_0^t e^{t-2u}\, du - \int_0^t \left(\cos u \int_0^t e^{t-2u}\, du\right)$

$y = \sin u \; \frac{e^{t-2u}}{-2} + \frac{1}{2}\int_0^t \cos u \; e^{t-2u}\, du$

$\int \cos u \, e \; du$

$\int \left(-\sin u \int e^{t-\cancel{2u}} \, du\right) du$

$= \cos u \dfrac{e^{t-2u}}{-2} - y$

$y = \sin u \left[\dfrac{e^{t-2u}}{-2}\right]_0^t + \dfrac{1}{2}\left[\cos u \dfrac{e^{t-2u}}{-2}\right.$

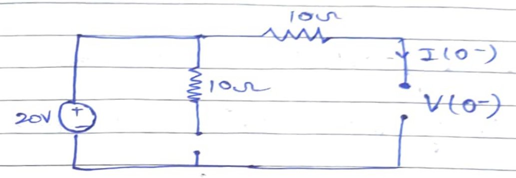
$y = \sin u \left(\dfrac{e^{-t}}{-2} - \dfrac{e^{t}}{-2}\right) + \dfrac{1}{2}\left(\cos t \dfrac{e^{-t}}{-2}\right.$

$\dfrac{1}{2}\left(\dfrac{e^{t}}{-2}\right) - y$

$2y = \sin u \dfrac{e^{-t}}{-2} - \sin u \dfrac{e^{t}}{-2} - \dfrac{1}{4}\cos t$

$+ \dfrac{1}{4} e^{t}$

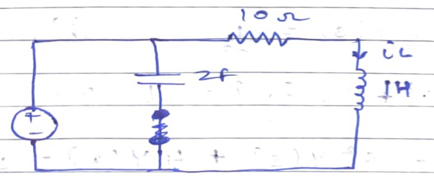
4. Steady state circuit ~~after~~ before switching.



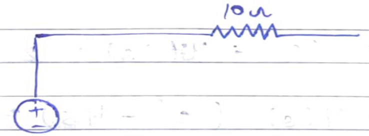
$I(0^-) = \dfrac{20}{10} = 2A$

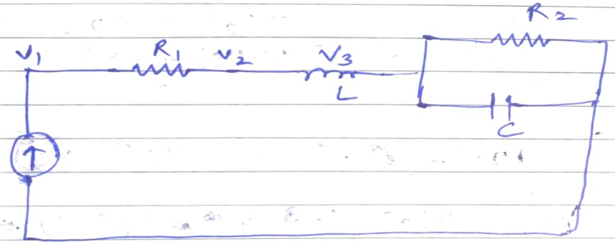
$V(0^-) = 0$

5.



7.





To solve system of ODE's

1. $x'' + 5y - x = t$

2. $2x' - y'' + 4y = 2$

$x = 0,\ y = 0$ for t

$\left.\frac{dy}{dt}\right|_{t=0} = 0$

$x = f_1(t),\ y = f_2(t)$

$\left.\frac{dx}{dt}\right|_{t=0} = 0$

Take L.T. of both eq^n.

Let

$L\{x(t)\} = X(s)\quad,\quad L\{y(t)\} = Y(s)$

$L\{x''\} + 5L\{y\} - L\{x\} = L\{t\}$

$s^2 X(s) - \cancel{s(0)} - \cancel{\frac{dx}{dt}(at\,0)} + 5Y(s) - X(s) = \frac{1}{s^2}$

$(s^2-1)X(s) + 5Y(s) = \frac{1}{s^2}$ —— (1)

$2L\{x'\} - L\{y''\} + 4L\{y\} = L\{2\}$

$2(sX(s) - 0) - s^2 Y(s) + 4Y(s) = \frac{2}{s}$

$2s^2 X(s) - s^3 Y(s) + 4sY(s) = 2$

$2s^2 X(s) - Y(s)(s^3 - 4s) = 2$ —— (2)

$2s^2(s^2-1)X(s) + 10s^2 Y(s) = 2$

$2s^2(s^2-1)X(s) - (s^3-4s)(s^2-1)Y(s) = (s$

$-\quad +\quad -$

$10s^2 - [s^5 - s^3 - 4s^3 + 4s]\,Y(s) = 2 - (2s$

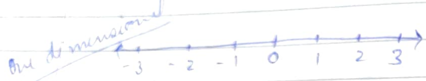
$10s^2\ Y(s)[(s^3-4s)(s^2-1)] = 10s^2 - 2 + 2s^2$

Complex Analysis

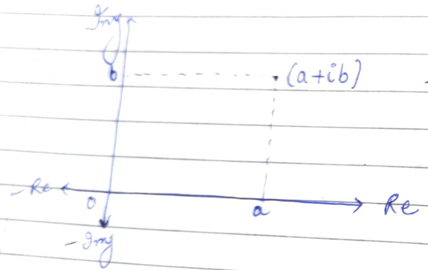
Complex number & Complex functions

Complex no. $a+ib$ a = real & b = imag

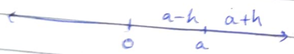
Complex variable $x+iy = z$



Any no. which can be plotted on this real no. line is a real no.



Argand plane.



$h \to 0$

$|z| = \sqrt{x^2+y^2}$

$\arg z = \tan^{-1} \frac{y}{x}$

conjugate of $(x+iy) = x - iy$.

Transpose $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$|z - z_0| = r$ (Standard eq^n of circle)

circumference

(x_0, y_0) r $z_0 = x_0 + iy_0$

area
- $|z - z_0| < r$ all points inside the circle. Open disc.
- $|z - z_0| \le r$ Closed disc   inside and boundary points

Neighbourhood of z_0

Group of points

$|z - z_0| < \delta$ $\delta \to$ arbitrary small

A point where a function is not analytic is singular point.

$\sin\left(\frac{1}{z}\right) \Rightarrow z = 0$ is singularity
essential singularity.

$\frac{1-e^z}{z}$ singularity but removable singularity

becox $\lim_{z\to 0} \frac{1-e^z}{z}$

$\frac{z}{(z-1)^5 (z-2)^2} \Rightarrow z = 1$ is a pole of order 5.
$z = 2$ is " " " " 2

$\frac{g(z)}{(z-z_0)^n (z-z_1)^m (z-z_2)^l}$ Pole at $z = z_0$ of order n
$z = z_1$ m
$z = z_2$ l

$$f(z) = \frac{z}{z^2+3z+1}$$

Poles

$$z^2+3z+1=0$$

$$z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}.$$

Zeros :-

$$f(z)=0$$

Q $f(z) = \frac{z^2+8}{z^3(z^2+1)}$

Poles $\Rightarrow$ $z=0$ of order 3

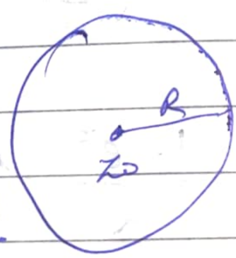
$z=\pm i$ of " 1 or simple poles.

Zeros $\Rightarrow$ $z = \pm 2\sqrt{2}\,i$

Taylor's Theorem.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{f^n(z_0)}{n!}$



$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z-z_0)^n$$

Maclaurin's Series

Taylor series about origin.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z)\big|_{z=0}}{n!} z^n$$