Algorithms

Asymptotic Performance

Asymptotic Performance

- *Asymptotic performance*: How does algorithm behave as the problem size gets very large?
  - Running time
  - Memory/storage requirements
- Remember that we use the RAM model:
  - All memory equally expensive to access
  - No concurrent operations
  - All reasonable instructions take unit time
    - Except, of course, function calls
  - Constant word size
    - Unless we are explicitly manipulating bits

Running Time

- Number of primitive steps that are executed
  - Except for time of executing a function call most statements roughly require the same amount of time
  - We can be more exact if need be
- Worst case vs. average case

An Example: Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
           A[j+1] = A[j]
           j = j - 1
      }
      A[j+1] = key
  }
}
```

An Example: Insertion Sort

30	10	40	20
1	2	3	4

$i = \varnothing \quad j = \varnothing \quad key = \varnothing$

$A[j] = \varnothing \quad A[j+1] = \varnothing$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

30	*10*	*40*	*20*
1	2	3	4

$i = 2 \quad j = 1 \quad key = 10$

$A[j] = 30 \quad A[j+1] = 10$

```
InsertionSort(A, n) {
   for i = 2 to n {
         key = A[i]
         j = i - 1;
         while (j > 0) and (A[j] > key) {
               A[j+1] = A[j]
               j = j - 1
         }
         A[j+1] = key
   }
}
```

An Example: Insertion Sort

30	*30*	*40*	*20*
1	2	3	4

$i = 2 \quad j = 1 \quad key = 10$

$A[j] = 30 \quad A[j+1] = 30$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

1	2	3	4
30	*30*	*40*	*20*

$i = 2 \quad j = 1 \quad key = 10$

$A[j] = 30 \quad A[j+1] = 30$

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

An Example: Insertion Sort

1	2	3	4
30	*30*	*40*	*20*

$i = 2$ $j = 0$ $key = 10$

$A[j] = \varnothing$ $A[j+1] = 30$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

30	30	40	20
1	2	3	4

$i = 2$ $j = 0$ $key = 10$

$A[j] = \varnothing$ $A[j+1] = 30$

```
InsertionSort(A, n) {
  for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
  }
}
```

An Example: Insertion Sort

10	*30*	*40*	*20*
1	2	3	4

$i = 2 \quad j = 0 \quad key = 10$

$A[j] = \varnothing \quad A[j+1] = 10$

```
InsertionSort(A, n) {
   for i = 2 to n {
       key = A[i]
       j = i - 1;
       while (j > 0) and (A[j] > key) {
             A[j+1] = A[j]
             j = j - 1
       }
       A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

$i = 3 \quad j = 0 \quad key = 10$

$A[j] = \varnothing \quad A[j+1] = 10$

```
InsertionSort(A, n) {
   for i = 2 to n {
         key = A[i]
         j = i - 1;
         while (j > 0) and (A[j] > key) {
               A[j+1] = A[j]
               j = j - 1
         }
         A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*30*	*40*	*20*
1	2	3	4

$i = 3$ $j = 0$ $key = 40$

$A[j] = \varnothing$ $A[j+1] = 10$

```
InsertionSort(A, n) {
   for i = 2 to n {
         key = A[i]
         j = i - 1;
         while (j > 0) and (A[j] > key) {
               A[j+1] = A[j]
               j = j - 1
         }
         A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

$i = 3 \quad j = 0 \quad key = 40$

$A[j] = \varnothing \quad A[j+1] = 10$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*30*	*40*	*20*
1	2	3	4

i = 3 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
         key = A[i]
         j = i - 1;
         while (j > 0) and (A[j] > key) {
                A[j+1] = A[j]
                j = j - 1
         }
         A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*30*	*40*	*20*
1	2	3	4

$i = 3 \quad j = 2 \quad key = 40$

$A[j] = 30 \quad A[j+1] = 40$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

$i = 3$ $j = 2$ $key = 40$

$A[j] = 30$ $A[j+1] = 40$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4   j = 2   key = 40
A[j] = 30   A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
         A[j+1] = A[j]
         j = j - 1
      }
      A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

An Example: Insertion Sort

10	30	40	20
1	2	3	4

i = 4   j = 2   key = 20

A[j] = 30   A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
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        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*30*	*40*	*20*
1	2	3	4

i = 4 j = 3 key = 20

A[j] = 40 A[j+1] = 20

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

1	2	3	4
10	*30*	*40*	*20*

i = 4 j = 3 key = 20
A[j] = 40 A[j+1] = 20

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 3 key = 20
A[j] = 40 A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
         A[j+1] = A[j]
         j = j - 1
      }
      A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 3 key = 20

A[j] = 40 A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	40
1	2	3	4

i = 4 j = 3 key = 20
A[j] = 40 A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
         A[j+1] = A[j]
         j = j - 1
      }
      A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*30*	*40*	*40*
1	2	3	4

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	40	40
1	2	3	4

$i = 4$ $j = 2$ $key = 20$
$A[j] = 30$ $A[j+1] = 40$

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

1	2	3	4
10	*30*	*30*	*40*

i = 4 j = 2 key = 20
A[j] = 30 A[j+1] = 30

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4 j = 2	key = 20
A[j] = 30	A[j+1] = 30

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4   j = 1   key = 20
A[j] = 10   A[j+1] = 30

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

1	2	3	4
10	*30*	*30*	*40*

i = 4   j = 1   key = 20
A[j] = 10   A[j+1] = 30

```
InsertionSort(A, n) {
   for i = 2 to n {
         key = A[i]
         j = i - 1;
         while (j > 0) and (A[j] > key) {
                  A[j+1] = A[j]
                  j = j - 1
         }
         A[j+1] = key
   }
}
```

An Example: Insertion Sort

1	2	3	4
10	*20*	*30*	*40*

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 20

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

An Example: Insertion Sort

10	*20*	*30*	*40*
1	2	3	4

i = 4   j = 1   key = 20
A[j] = 10   A[j+1] = 20

```
InsertionSort(A, n) {
   for i = 2 to n {
        key = A[i]
        j = i - 1;
        while (j > 0) and (A[j] > key) {
              A[j+1] = A[j]
              j = j - 1
        }
        A[j+1] = key
   }
}
```

Done!

Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

What is the precondition for this loop?

Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
      }
      A[j+1] = key
  }
}
```

How many times will this loop execute?

Insertion Sort

Statement	Effort
`InsertionSort(A, n) {`	
`for i = 2 to n {`	$c_1 n$
`key = A[i]`	$c_2(n-1)$
`j = i - 1;`	$c_3(n-1)$
`while (j > 0) and (A[j] > key) {`	$c_4 T$
`A[j+1] = A[j]`	$c_5(T-(n-1))$
`j = j - 1`	$c_6(T-(n-1))$
`}`	0
`A[j+1] = key`	$c_7(n-1)$
`}`	0
`}`	

$T = t_2 + t_3 + \ldots + t_n$ where t_i is number of while expression evaluations for the i^{th} for loop iteration

Analyzing Insertion Sort

- $T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 T + c_5(T - (n-1)) + c_6(T - (n-1)) + c_7(n-1)$

  $= c_8 T + c_9 n + c_{10}$
- What can T be?
  - Best case -- inner loop body never executed
    - $t_i = 1$ ➔ T(n) is a linear function
  - Worst case -- inner loop body executed for all previous elements
    - $t_i = i$ ➔ T(n) is a quadratic function
  - Average case
    - ???

Analysis

- Simplifications
  - Ignore actual and abstract statement costs
  - *Order of growth* is the interesting measure:
    - Highest-order term is what counts
      - Remember, we are doing asymptotic analysis
      - As the input size grows larger it is the high order term that dominates

Upper Bound Notation

- We say InsertionSort's run time is *$O(n^2)$*
  - Properly we should say run time is *in* $O(n^2)$
  - Read O as "Big-O" (you'll also hear it as "order")
- In general a function
  - f(n) is O(g(n)) if there exist positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$
- Formally
  - $O(g(n)) = \{ f(n)$: $\exists$ positive constants c and n_0 such that $f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0$

Insertion Sort Is O(n^2)

- Proof
  - Suppose runtime is $an^2 + bn + c$
    - If any of a, b, and c are less than 0 replace the constant with its absolute value
  - $an^2 + bn + c \le (a + b + c)n^2 + (a + b + c)n + (a + b + c)$
  - $\le 3(a + b + c)n^2$ for $n \ge 1$
  - Let $c' = 3(a + b + c)$ and let $n_0 = 1$
- Question
  - Is InsertionSort O(n^3)?
  - Is InsertionSort O(n)?

Big O Fact

- A polynomial of degree k is $O(n^k)$
- Proof:
  - Suppose $f(n) = b_k n^k + b_{k-1} n^{k-1} + \ldots + b_1 n + b_0$
    - Let $a_i = | b_i |$
  - $f(n) \leq a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$

$$\leq \quad n^k \sum a_i \frac{n^i}{n^k} \quad \leq \quad n^k \sum a_i \quad \leq \quad cn^k$$

Lower Bound Notation

- We say InsertionSort's run time is $\Omega(n)$
- In general a function
  - f(n) is $\Omega(g(n))$ if $\exists$ positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n) \quad \forall\, n \geq n_0$
- Proof:
  - Suppose run time is $an + b$
    - Assume a and b are positive (what if b is negative?)
  - $an \leq an + b$

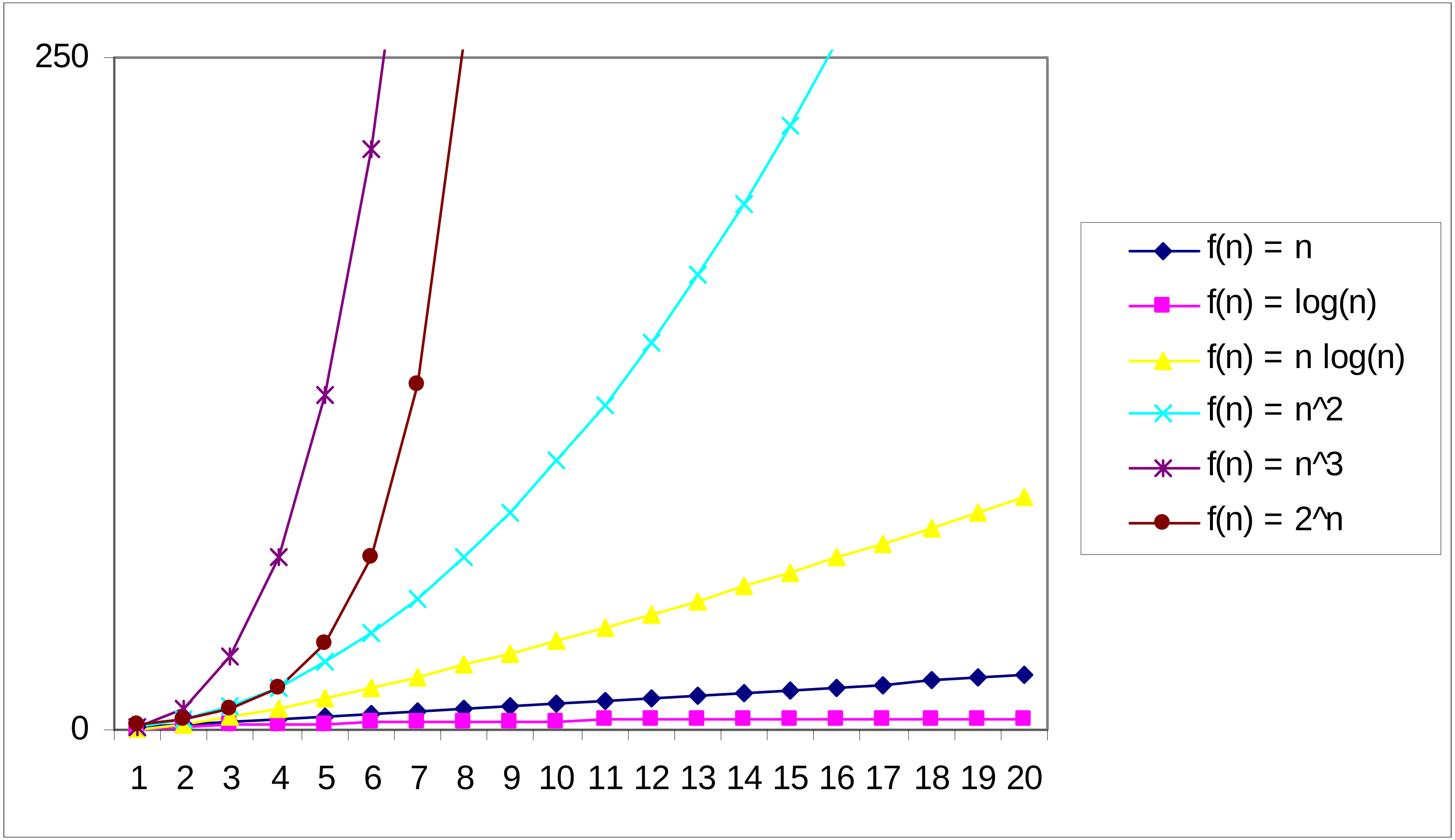
Asymptotic Tight Bound

- A function f(n) is $\Theta(g(n))$ if $\exists$ positive constants c_1, c_2, and n_0 such that

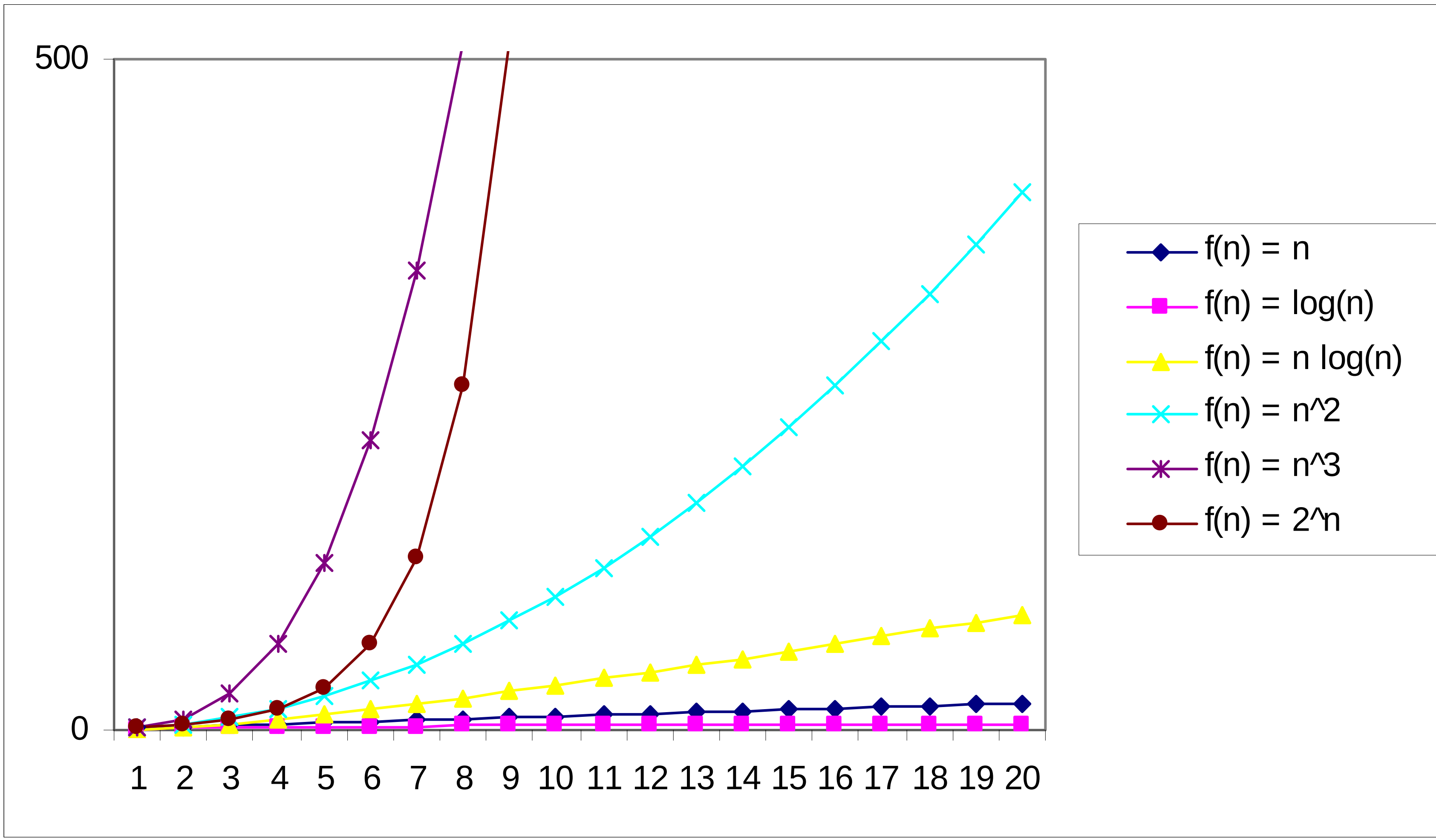
$$c_1\, g(n) \le f(n) \le c_2\, g(n)\ \forall\ n \ge n_0$$

- Theorem
  - f(n) is $\Theta(g(n))$ iff f(n) is both $O(g(n))$ and $\Omega(g(n))$
  - Proof: someday

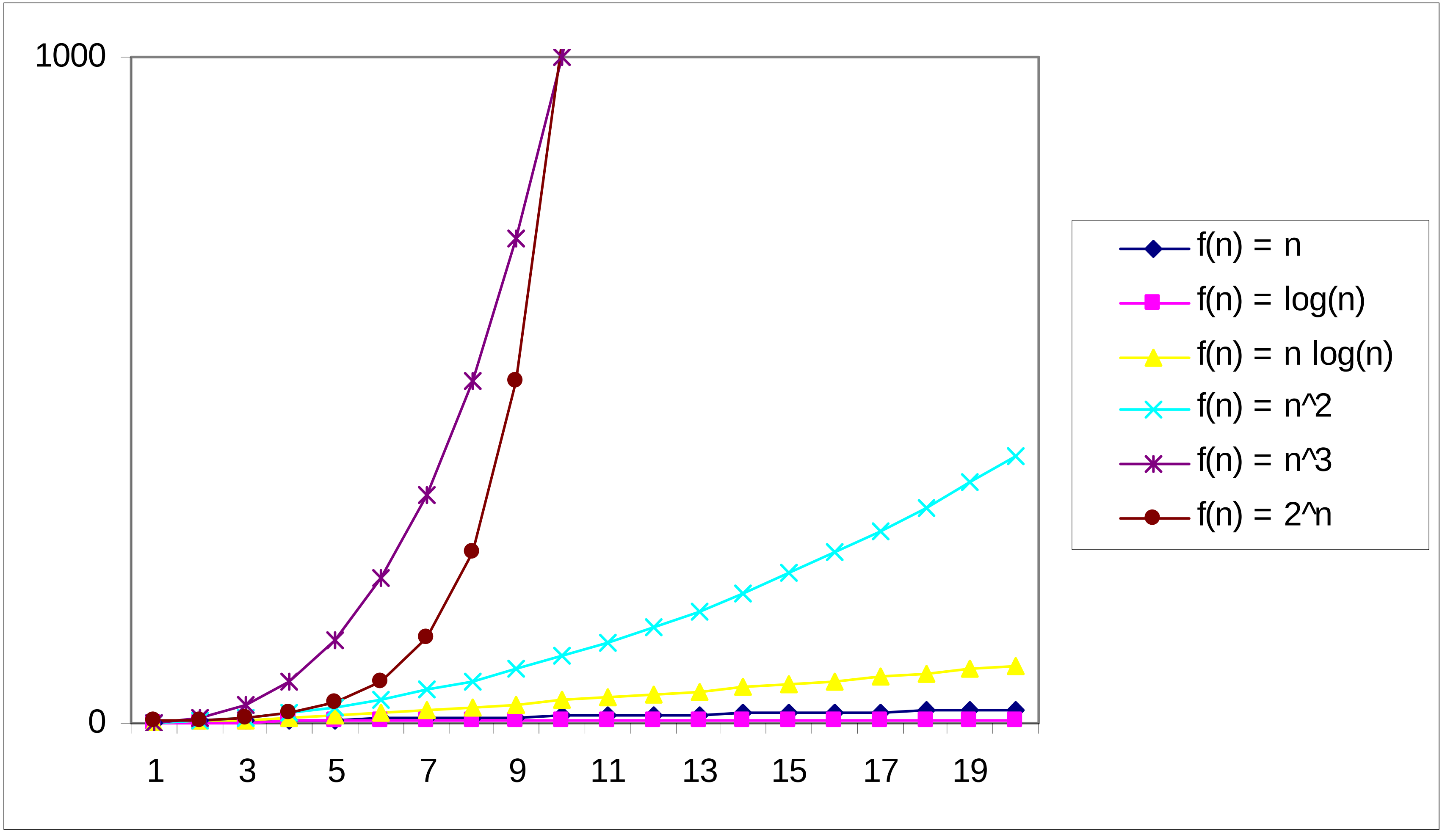
Practical Complexity



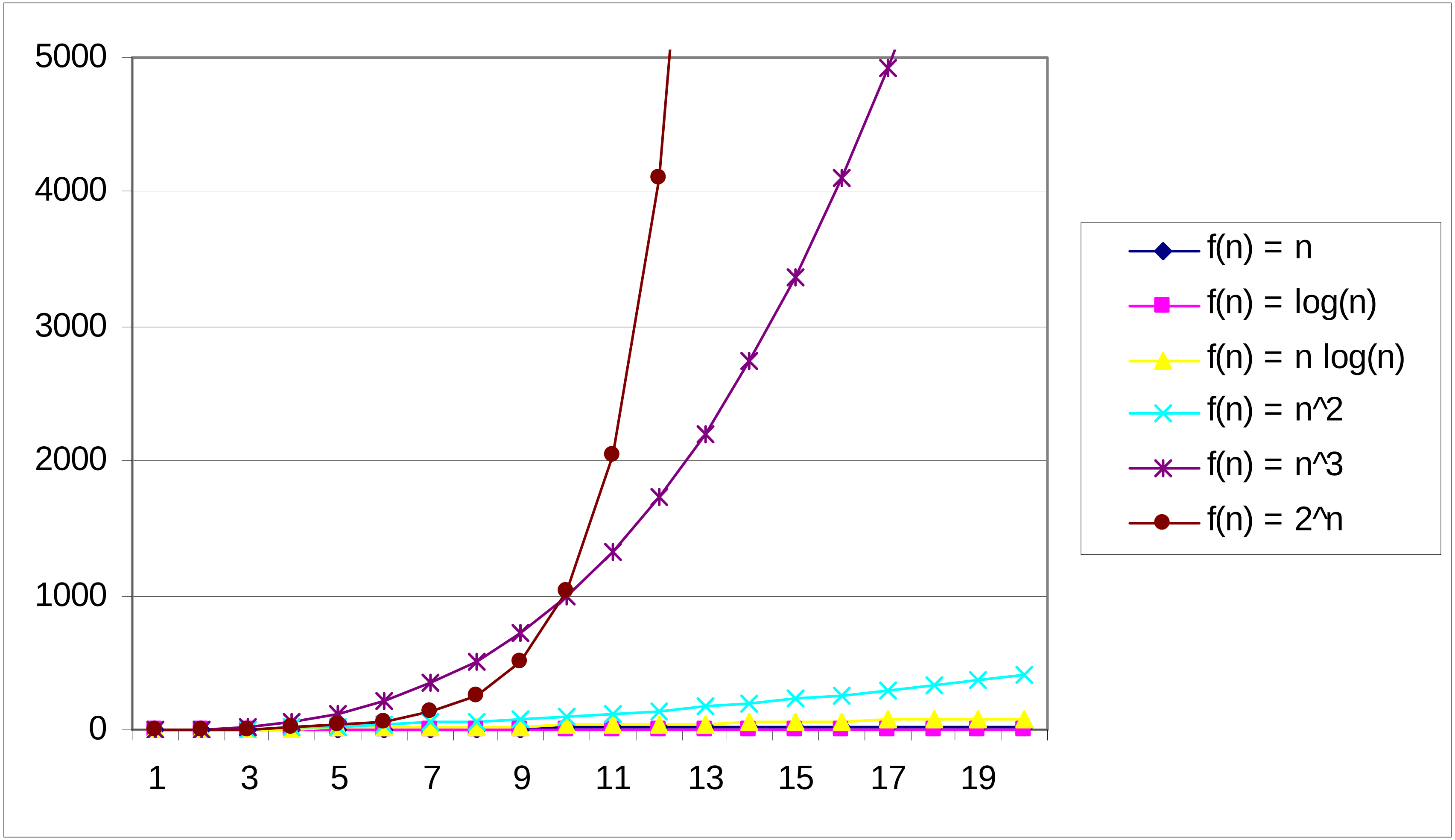
Practical Complexity



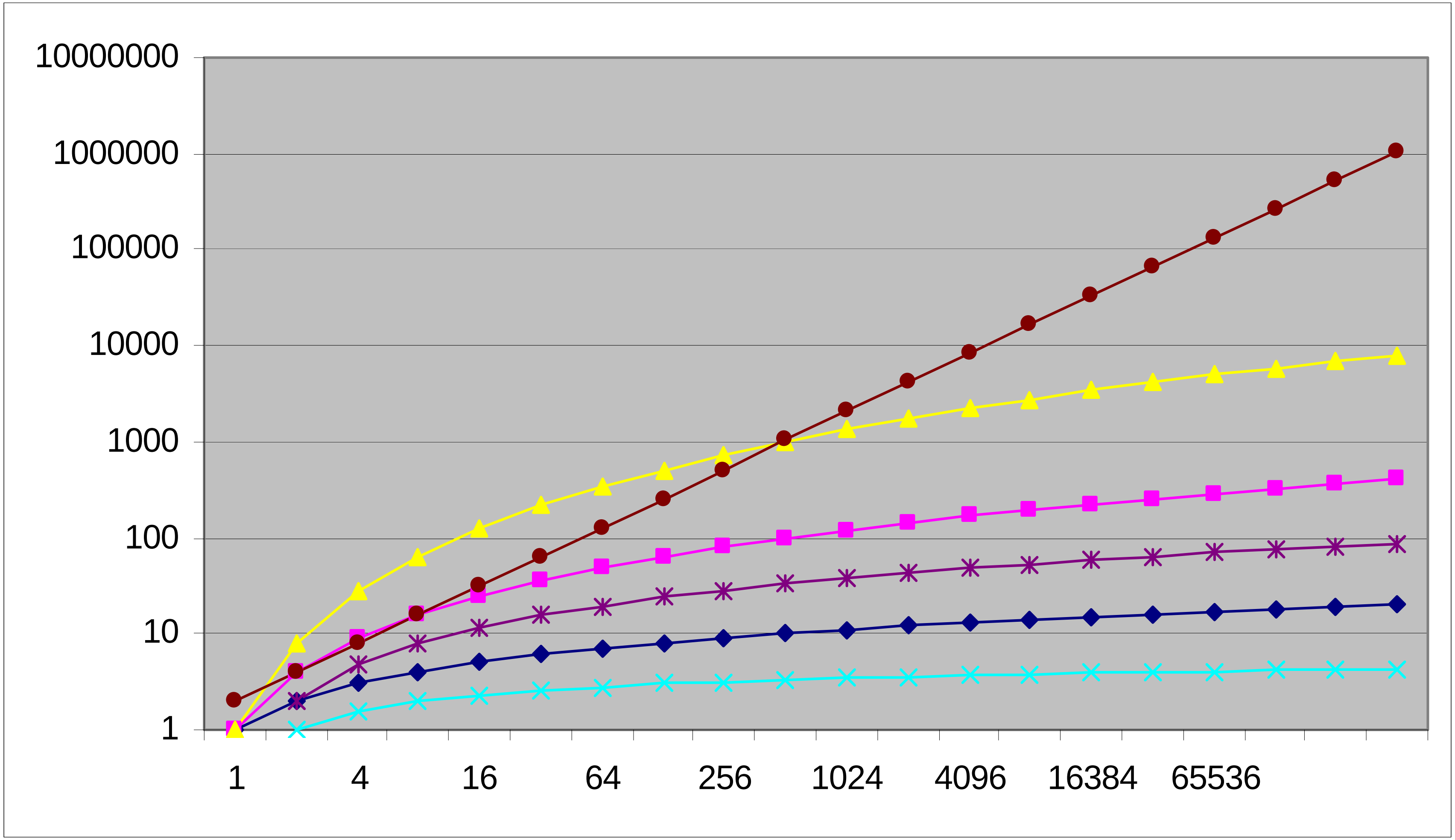
Practical Complexity



Practical Complexity



Practical Complexity



Other Asymptotic Notations

- A function f(n) is o(g(n)) if $\exists$ positive constants c and n_0 such that
  $$f(n) < c\, g(n) \;\forall\; n \geq n_0$$
- A function f(n) is ω(g(n)) if $\exists$ positive constants c and n_0 such that
  $$c\, g(n) < f(n) \;\forall\; n \geq n_0$$
- Intuitively,
  - o() is like <
  - ω() is like >
  - Θ() is like =
  - O() is like ≤
  - Ω() is like ≥

Up Next

- Solving recurrences
  - Substitution method
  - Master theorem