More examples for Type I

(a) $pq = 1 \rightarrow z = ax + by + c$ is a sol^n

(b) $p^2x + q^2y = z$ (Hint: Reduce in Type I form first)

(c) $z^2 = pqxy$ (")

Type II: Clairaut Equation

$$\boxed{z = px + qy + f(p,q)}$$

Sol^n (complete integral)

$\rightarrow z = ax + by + f(a,b)$ where a, b are arbitrary cnts.

(a) $z = px + qy + pq$

$f(p,q) = pq$

$f(a,b) = ab$

$z = ax + by + ab$.

(b) $z = px + qy - 2\sqrt{pq}$

$f(p,q) = -2\sqrt{pq}$

$z = ax + by - 2\sqrt{ab}$

(c) $pqz = p^2(xq+p^2)+q^2(yp+q^2)$

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

dividing both side by pq

$$z = \frac{p}{q}(xq+p^2) + \frac{q}{p}(yp+q^2)$$

$$z = px + \frac{p^3}{q} + qy + \frac{q^3}{p}$$

$$z = px+qy+\left(\frac{p^3}{q}+\frac{q^3}{p}\right)$$

$$\rightarrow z = ax+by+\frac{a^3}{b}+\frac{b^3}{a}$$

__Type III__ Equation of the type $f(z,p,q)=0$.

To solve this we assume

$\boxed{z}$ is function of u

where $\boxed{u = x+ay}$ $\rightarrow$ $\frac{\partial u}{\partial x}=1$, $\frac{\partial u}{\partial y}=a$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

$$p = \frac{dz}{du} \qquad q = a\frac{dz}{du}$$

$$f(p,q,z) \rightarrow f\left(\frac{dz}{du}, a\frac{dz}{du}, z\right)$$

converted ODE $\rightarrow$ Then solve.

ⓐ $p(1+q) = qz \leftarrow f(p,q,z)$ → III

$z = f(u)$

$u = x+ay$

$p = \frac{dz}{du}$

$q = a\frac{dz}{du}$

$p(1+q) - qz = 0$

$\rightarrow \frac{dz}{du}\left(1+a\frac{dz}{du}\right) - az\frac{dz}{du} = 0$

$\rightarrow 1 + a\frac{dz}{du} = az$

$\Rightarrow a\frac{dz}{du} = az - 1$

$\Rightarrow \frac{dz}{du} = \frac{az-1}{a}$

$\Rightarrow \int \frac{dz}{az-1} = \int \frac{du}{a}$

$\Rightarrow \frac{\log(az-1)}{a} = \frac{u}{a} + c$

$u = \log(az-1) + c_1$

$$\boxed{x+ay = \log(az-1) + c_1}$$

b) $p^2 = qz$

$\left(\frac{dz}{du}\right)^2 = a\frac{dz}{du}(z) \Rightarrow \frac{dz}{du} = az$

$$\frac{dz}{z} = a\,du \Rightarrow \log z = au + \log c_1$$
$$\Rightarrow z = c_1 e^{au}$$
$$\Rightarrow z = c_1 e^{a(x+ay)}$$

c) $pz = 1 + q$

Type IV : Equation of the form

$$f_1(x,p) = f_2(y,q) = \underline{a}$$

Method :

Here, we let

$$\underline{f_1(x,p) = a} = f_2(y,q)$$

solve for p (in terms of x)

$p = F_1(x)$

solve for q (in terms of y)

$q = F_2(y)$

Since

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\frac{\partial z}{\partial x} \to p$, $\frac{\partial z}{\partial y} \to q$

$$\int dz = \int F_1(x)\,dx + \int F_2(y)\,dy$$

$$z = \int F_1(x)\,dx + \int F_2(y)\,dy + c$$

(a) $p - x^2 = q + y^2 \iff f_1(p,x) = f_2(p,y)$

$f_1(p,x)$: $p - x^2 = a \Rightarrow p = a + x^2$

$f_2(q,y)$: $q + y^2 = a \Rightarrow q = a - y^2$

$dz = (a+x^2)\,dx + (a-y^2)\,dy$

$$\boxed{z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + C}$$

(b) $py + qx + pq = 0$

$f_1(p,x) = f_2(p,x)$

$q(x+p) = -py$

$\frac{q}{-y} = \frac{p}{x+p}$

$\Rightarrow \frac{p}{x} = -\frac{q}{y+q}$

$p(y+q) = -qx$

$\frac{p}{x} = a \Rightarrow p = ax$

$\frac{-q}{y+q} = a \Rightarrow -q = ya + qa$

$\Rightarrow -q(1+a) = ya$

$\Rightarrow q = -\frac{ya}{1+a}$

$$\int dz = \int (ax)\,dx - \int \left(\frac{ya}{1+a}\right) dy$$

$$\boxed{z = \frac{ax^2}{2} - \frac{ay^2}{2(1+a)} + c}$$

Try!

(c) $z^2(p^2+q^2) = x^2+y^2$