Second order PDE

$$\rightarrow f(x, y, z, p, q, r, s, t) = 0$$

where $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$

Example $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} + xyz = 0 \Rightarrow s + q + xyz = 0$

Another notation $z_{xy} + z_y + xyz = 0$

Linear Partial differential equation of second order

$$Rr + Ss + Tt + Pp + Qq + Zz = F \quad \text{--- } \mathrm{I}$$

R, S, T, P, Q, Z, F are function x and y only

What are homogeneous second order PDEs?

- If in eq I, $F \equiv 0$, then it is called <u>homogeneous</u> otherwise <u>non-homogeneous</u>.

Classification in some domain D of xy plane

$$D = S^2 - 4RT$$

1. $D > 0$ Hyperbolic
2. $D = 0$ Parabolic
3. $D < 0$ Elliptic

Eg 1. $2\left(\dfrac{\partial^2 u}{\partial x^2}\right) + 4\dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 2$

$S = 4$

$R = 2$

$T = 3$

$S^2 - 4RT$

$= 16 - 24 < 0$

This is elliptic irrespective of values of x and y.

Eg 2 $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\quad u \to t, x$

↳ wave equation in one dimension

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad c \neq 0$$

$S = 0 \quad R = -c^2$

$T = 1$

$S^2 - 4RT$

$= 0 - 4(-c^2)$

$= 4c^2 > 0$

Hyperbolic $c \neq 0$

If $c = 0$, Parabolic

Q. $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

$R = xy \quad S = -(x^2 - y^2)$

$T = -xy$

$S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2 > 0$

$(x, y) \neq (0,0)$ Hyperboli

$x^4 + y^4 - 2x^2y^2 + 4x^2y^2$

$(x^2 + y^2)^2 \geq 0$

If $(x = 0$ and $y = 0$

Parabolic

Try! Classify the following:

a) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ b) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$

c) $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$

d) Laplace Equation: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Wave Equation

Displacement

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One dimension

time

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

One-dimension heat-flow

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One dimens

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Poisson's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Seperation of Variable :

① $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$

Let us assume u be of the following form

$\rightarrow u(x,t) = X(x)\,T(t)$

$$\boxed{\frac{\partial u}{\partial t} = X\frac{dT}{dt}}$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx}$$

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

$$\rightarrow X\frac{dT}{dt} = 4T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{4T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = c$$

What will happen if the above constt,

$c_1 = p^2$, $c_1 = 0$ and $c_1 = -p^2$

Positive

Negative

Case 1 Positive

$$\frac{1}{4T}\frac{dT}{dt} = p^2$$

$$\int \frac{dT}{T} = 4p^2 dt$$

$$\log(T) = 4p^2 t + C_1'$$

$$T = e^{4p^2 t + C_1'}$$

$$T(t) = C_1 e^{4p^2 t}$$

$$\frac{1}{X}\frac{\partial^2 X}{dx^2} = p^2$$

$$\frac{d^2 X}{dx^2} - p^2 X = 0$$

$$\frac{d^2 X}{dx^2} - p^2 X = 0$$

Complete !

Also try when $c = 0$ and $c = -p^2$