COMP9414/9814/3411: Artificial Intelligence

14. Uncertainty

Russell & Norvig, Chapter 13.

Outline

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time? Problems:

- partial observability, noisy sensors
- uncertainty in action outcomes (flat tire, etc.)
- immense complexity of modelling and predicting traffic

Hence a purely logical approach either
1) risks falsehood: "A_{25} will get me there on time", or
2) leads to conclusions that are too weak for decision making:
"A_{25} will get me there on time if there's no accident on the bridge
and it doesn't rain and my tires remain intact etc etc."
(A_{1440} might be safe but I'd have to stay overnight in the airport ...)

Methods for handling Uncertainty

Default or nonmonotonic logic:

Assume my car does not have a flat tire, etc.
Assume A_{25} works unless contradicted by evidence
Issues: What assumptions are reasonable? How to handle contradiction?

Probability

Given the available evidence,
A_{25} will get me there on time with probability 0.04
Mahaviracarya (9th C.), Cardamo (1565) theory of gambling

Probability

Probabilistic assertions summarize effects of

Laziness: failure to enumerate exceptions, qualifications, etc.

Ignorance: lack of relevant facts, initial conditions, etc.

Subjective or Bayesian probability:
Probabilities relate propositions to one's own state of knowledge
e.g. $P(A_{25}|\text{no reported accidents}) = 0.06$

These are not claims of a "probabilistic tendency" in the current situation (but might be learned from past experience of similar situations)

Probabilities of propositions change with new evidence:
e.g. $P(A_{25}|\text{no reported accidents}, \text{ 5 a.m.}) = 0.15$

(Analogous to logical entailment status $KB \models \alpha$, not absolute truth)

Making decisions under uncertainty

Suppose I believe the following:

$$\begin{aligned}
P(A_{25} \text{ gets me there on time}|\ldots) &= 0.04\\
P(A_{90} \text{ gets me there on time}|\ldots) &= 0.70\\
P(A_{120} \text{ gets me there on time}|\ldots) &= 0.95\\
P(A_{1440} \text{ gets me there on time}|\ldots) &= 0.9999
\end{aligned}$$

Which action to choose?

Depends on my preferences for missing flight vs. airport cuisine, etc.

Utility theory is used to represent and infer preferences

Decision theory = utility theory + probability theory

Probability basics

Begin with a set Ω – the sample space (e.g. 6 possible rolls of a die)

$\omega \in \Omega$ is a sample point/possible world/atomic event

A probability space or probability model is a sample space
with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.

$0 \leq P(\omega) \leq 1$

$\sum_\omega P(\omega) = 1$
e.g. $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$.

An event A is any subset of Ω

$$P(A) = \sum_{\{\omega \in A\}} P(\omega)$$

e.g. $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

Random variables

A random variable (r.v.) is a function from sample points to some range (e.g. the Reals or Booleans)

For example, $\texttt{Odd}(3) = \texttt{true}$.

P induces a probability distribution for any r.v. X:

$$P(X = x_i) = \sum_{\{\omega : X(\omega) = x_i\}} P(\omega)$$

e.g., $P(\texttt{Odd} = \texttt{true}) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

Propositions

Think of a proposition as the event (set of sample points)
where the proposition is true

Given Boolean random variables A and B:

event a = set of sample points where $A(\omega) = \texttt{true}$
event $\neg a$ = set of sample points where $A(\omega) = \texttt{false}$
event $a \wedge b$ = points where $A(\omega) = \texttt{true}$ and $B(\omega) = \texttt{true}$

With Boolean variables, sample point = propositional logic model
e.g., $A = \texttt{true}$, $B = \texttt{false}$, or $a \wedge \neg b$.
Proposition = disjunction of atomic events in which it is true

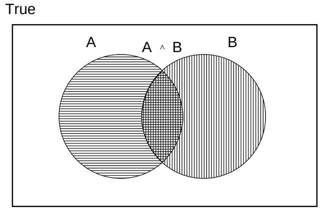
e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

$\rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Why use probability?

The definitions imply that certain logically related events must have related probabilities

For example, $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

Syntax for propositions

Propositional or Boolean random variables

e.g., `Cavity` (do I have a cavity?)

`Cavity = true` is a proposition, also written `Cavity`

Discrete random variables (finite or infinite)

e.g., `Weather` is one of $\langle$`sunny,rain,cloudy,snow`$\rangle$

`Weather = rain` is a proposition

Values must be exhaustive and mutually exclusive

Continuous random variables (bounded or unbounded)

e.g. `Temp` = 21.6; also allow, e.g. `Temp` < 22.0

Arbitrary Boolean combinations of basic propositions.

Prior probability

Prior or unconditional probabilities of propositions

e.g. $P(\texttt{Cavity} = \texttt{true}) = 0.1$ and $P(\texttt{Weather} = \texttt{sunny}) = 0.72$
correspond to belief prior to arrival of any (new) evidence.

Probability distribution gives values for all possible assignments:

$P(\texttt{Weather}) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (normalized, i.e., sums to 1)

Joint probability

Joint probability distribution for a set of r.v.'s gives the probability of every atomic event on those r.v's (i.e., every sample point)

$P(\texttt{Weather}, \texttt{Cavity})$ is a 4×2 matrix of values:

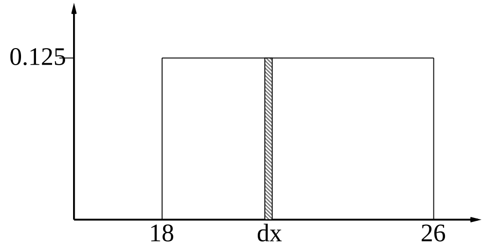
$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = \texttt{true}$	0.144	0.02	0.016	0.02
$Cavity = \texttt{false}$	0.576	0.08	0.064	0.08

Every question about a domain can be answered by the joint distribution because every event is a sum of sample points.

Probability for continuous variables

Express distribution as a parameterized function.

e.g. $P(X=x) = U[18,26](x)$ = uniform density between 18 and 26



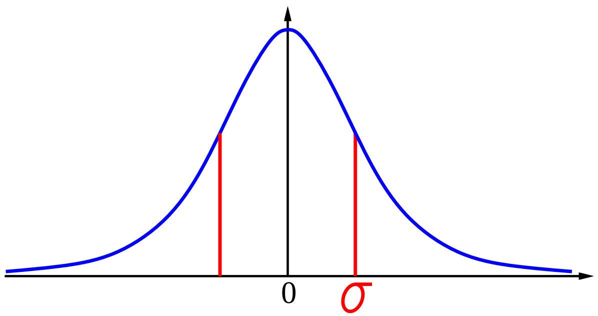
Here P is a density; integrates to 1.

$P(X=20.5) = 0.125$ really means

$$\lim_{dx \to 0} P(20.5 \le X \le 20.5 + dx)/dx = 0.125$$

Gaussian density

$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$



Conditional probability

Conditional or posterior probabilities

e.g., $P(\texttt{cavity}|\texttt{toothache}) = 0.8$

(Notation for conditional distributions:

$P(\texttt{Cavity}|\texttt{Toothache})$ = 2-element vector of 2-element vectors)

If we know more, e.g., `cavity` is also given, then we have

$P(\texttt{cavity}|\texttt{toothache}, \texttt{cavity}) = 1$

Note: the less specific belief remains valid after more evidence arrives, but is not always useful.

New evidence may be irrelevant, allowing simplification, e.g.,

$P(\texttt{cavity}|\texttt{toothache}, \texttt{49ersWin}) = P(\texttt{cavity}|\texttt{toothache}) = 0.8$

This kind of inference, sanctioned by domain knowledge, is crucial.

Conditional probability

Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

Alternative formulation: $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

A general version holds for whole distributions,
e.g. $P(\mathtt{Weather}, \mathtt{Cavity}) = P(\mathtt{Weather}|\mathtt{Cavity})P(\mathtt{Cavity})$
(View as a 4×2 set of equations, not matrix multiplication)

Chain rule is derived by successive application of product rule:
$P(X_1, \ldots, X_n) = P(X_1, \ldots, X_{n-1})\ P(X_n|X_1, \ldots, X_{n-1})$
$= P(X_1, \ldots, X_{n-2})\ P(X_{n-1}|X_1, \ldots, X_{n-2})\ P(X_n|X_1, \ldots, X_{n-1})$
$= \ldots = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1})$

Inference by enumeration

Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ *cavity*	**.016**	**.064**	**.144**	**.576**

For any proposition ϕ, sum the atomic events where it is true:
$P(\phi) = \sum_{\omega:\omega \models \phi} P(\omega)$

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$P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

Inference by enumeration

Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
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For any proposition ϕ, sum the atomic events where it is true:
$P(\phi) = \sum_{\omega:\omega \models \phi} P(\omega)$

$P(\mathtt{cavity} \vee \mathtt{toothache})$
$= 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

Inference by enumeration

Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

Can also compute conditional probabilities:

$$P(\neg cavity|toothache) = \frac{P(\neg \texttt{cavity} \wedge \texttt{toothache})}{P(\texttt{toothache})} = \frac{0.016+0.064}{0.108+0.012+0.016+0.064} = 0.4$$

Normalization

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

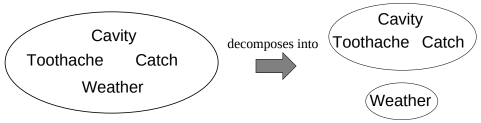
Denominator can be viewed as a normalization constant α

$$\begin{aligned} P(\texttt{Cavity}|\texttt{toothache}) & \alpha P(\texttt{Cavity},\texttt{toothache}) \\ &= \alpha [P(\texttt{Cavity},\texttt{toothache},\texttt{catch}) + P(\texttt{Cavity},\texttt{toothache},\neg\texttt{catch})] \\ &= \alpha [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle] \\ &= \alpha \langle 0.12, 0.08\rangle = \langle 0.6, 0.4\rangle \end{aligned}$$

General idea: compute distribution on query variable
by fixing evidence variables and summing over hidden variables

Independence

A and B are independent iff
$P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A,B) = P(A)P(B)$



$P(\texttt{Toothache},\texttt{Catch},\texttt{Cavity},\texttt{Weather})$
$= P(\texttt{Toothache},\texttt{Catch},\texttt{Cavity})P(\texttt{Weather})$

32 entries reduced to 12; for n independent biased coins, $2^n \to n$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables,
none of which are independent. What to do?

Conditional independence

$P(\texttt{Toothache},\texttt{Cavity},\texttt{Catch})$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
(1) $P(\texttt{Catch}|\texttt{Toothache},\texttt{cavity}) = P(\texttt{Catch}|\texttt{cavity})$

The same independence holds if I haven't got a cavity:
(2) $P(\texttt{Catch}|\texttt{Toothache},\neg\texttt{cavity}) = P(\texttt{Catch}|\neg\texttt{cavity})$

Catch is conditionally independent of Toothache given Cavity:
$P(\texttt{Catch}|\texttt{Toothache},\texttt{Cavity}) = P(\texttt{Catch}|\texttt{Cavity})$

Equivalent statements: $P(\texttt{Toothache}|\texttt{Catch},\texttt{Cavity}) = P(\texttt{Toothache}|\texttt{Cavity})$
$P(\texttt{Toothache},\texttt{Catch}|\texttt{Cavity}) = P(\texttt{Toothache}|\texttt{Cavity})P(\texttt{Catch}|\texttt{Cavity})$

Conditional independence contd.

Write out full joint distribution using chain rule:

$P(\texttt{Toothache}, \texttt{Catch}, \texttt{Cavity})$

$= P(\texttt{Toothache}|\texttt{Catch}, \texttt{Cavity})P(\texttt{Catch}, \texttt{Cavity})$

$= P(\texttt{Toothache}|\texttt{Catch}, \texttt{Cavity})P(\texttt{Catch}|\texttt{Cavity})P(\texttt{Cavity})$

$= P(\texttt{Toothache}|\texttt{Cavity})P(\texttt{Catch}|\texttt{Cavity})P(\texttt{Cavity})$

I.e., 2 + 2 + 1 = 5 independent numbers (equations 1 and 2 remove 2)

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\rightarrow \text{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Useful for assessing diagnostic probability from causal probability:

$$P(\text{Cause}|\text{Effect}) = \frac{P(\text{Effect}|\text{Cause})P(\text{Cause})}{P(\text{Effect})}$$

e.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

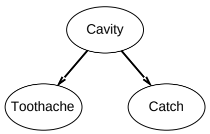
Note: posterior probability of meningitis still very small!

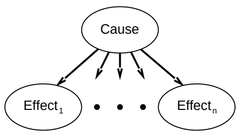
Bayes' Rule and conditional independence

$$\begin{aligned} & P(\texttt{Cavity}|\texttt{Toothache} \wedge \texttt{Catch}) \\ = \; & \alpha P(\texttt{Toothache} \wedge \texttt{Catch}|\texttt{Cavity})P(\texttt{Cavity}) \\ = \; & \alpha P(\texttt{Toothache}|\texttt{Cavity})P(\texttt{Catch}|\texttt{Cavity})P(\texttt{Cavity}) \end{aligned}$$

This is an example of a naive Bayes model:

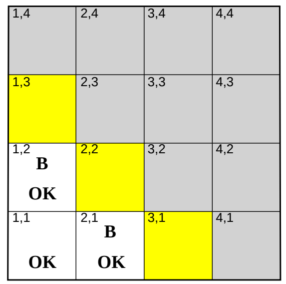
$$P(\text{Cause}, \text{Effect}_1, \ldots, \text{Effect}_n) = P(\text{Cause}) \prod_i P(\text{Effect}_i|\text{Cause})$$





Total number of parameters is linear in n

Wumpus World



$P_{ij} = \texttt{true}$ iff $[i, j]$ contains a pit

$B_{ij} = \texttt{true}$ iff $[i, j]$ is breezy

Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model.

Specifying the probability model

The full joint distribution is $\mathbf{P}(P_{1,1},\ldots,P_{4,4},B_{1,1},B_{1,2},B_{2,1})$

Apply product rule: $\mathbf{P}(B_{1,1},B_{1,2},B_{2,1}\,|\,P_{1,1},\ldots,P_{4,4})\mathbf{P}(P_{1,1},\ldots,P_{4,4})$

(Do it this way to get $P(Effect|Cause)$.)

First term: 1 if pits are adjacent to breezes, 0 otherwise

Second term: pits are placed randomly, probability 0.2 per square:

$$\mathbf{P}(P_{1,1},\ldots,P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

Observations and query

We know the following facts:

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

Query is $\mathbf{P}(P_{1,3}|\mathtt{known},b)$

Define $Unknown$ = P_{ij}s other than $P_{1,3}$ and $Known$

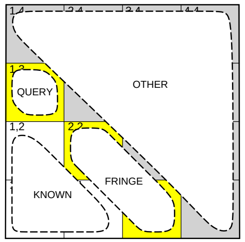
For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|\mathtt{known},b) = \alpha \sum_{\mathtt{unknown}} \mathbf{P}(P_{1,3},\mathtt{unknown},\mathtt{known},b)$$

Grows exponentially with number of squares!

Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



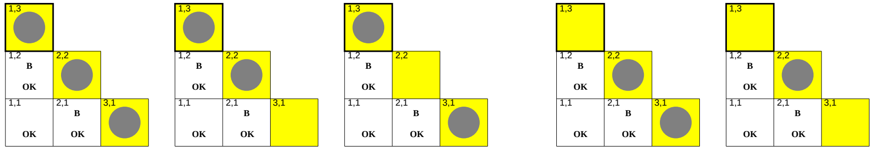
Define $Unknown = Fringe \cup Other$
$\mathbf{P}(b|P_{1,3},Known,Unknown) = \mathbf{P}(b|P_{1,3},Known,Fringe)$

Manipulate query into a form where we can use this!

Using conditional independence contd.

$$\begin{aligned}
\mathbf{P}(P_{1,3}|\mathtt{known},b) &= \alpha \sum_{\mathtt{unknown}} \mathbf{P}(P_{1,3},\mathtt{unknown},\mathtt{known},b) \\
&= \alpha \sum_{\mathtt{unknown}} \mathbf{P}(b|P_{1,3},\mathtt{known},\mathtt{unknown})\mathbf{P}(P_{1,3},\mathtt{known},\mathtt{unknown}) \\
&= \alpha \sum_{\mathtt{fringe}} \sum_{\mathtt{other}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe},\mathtt{other})\mathbf{P}(P_{1,3},\mathtt{known},\mathtt{fringe},\mathtt{other}) \\
&= \alpha \sum_{\mathtt{fringe}} \sum_{\mathtt{other}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe})\mathbf{P}(P_{1,3},\mathtt{known},\mathtt{fringe},\mathtt{other}) \\
&= \alpha \sum_{\mathtt{fringe}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe}) \sum_{\mathtt{other}} \mathbf{P}(P_{1,3},\mathtt{known},\mathtt{fringe},\mathtt{other}) \\
&= \alpha \sum_{\mathtt{fringe}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe}) \sum_{\mathtt{other}} \mathbf{P}(P_{1,3})P(\mathtt{known})P(\mathtt{fringe})P(\mathtt{other}) \\
&= \alpha P(\mathtt{known})\mathbf{P}(P_{1,3}) \sum_{\mathtt{fringe}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe})P(\mathtt{fringe}) \sum_{\mathtt{other}} P(\mathtt{other}) \\
&= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{\mathtt{fringe}} \mathbf{P}(b|\mathtt{known},P_{1,3},\mathtt{fringe})P(\mathtt{fringe})
\end{aligned}$$

Using conditional independence contd.



0.2 x 0.2 = 0.04   0.2 x 0.8 = 0.16   0.8 x 0.2 = 0.16   0.2 x 0.2 = 0.04   0.2 x 0.8 = 0.16

$$\begin{aligned} P(P_{1,3}|\texttt{Known}, b) &= \alpha' \langle 0.2(0.04+0.16+0.16),\ 0.8(0.04+0.16) \rangle \\ &\approx \langle 0.31, 0.69 \rangle \end{aligned}$$

$$P(P_{2,2}|\texttt{Known}, b) \approx \langle 0.86, 0.14 \rangle$$

Summary

Probability is a rigorous formalism for uncertain knowledge

Joint probability distribution specifies probability of every atomic event

Queries can be answered by summing over atomic events

For nontrivial domains, we must find a way to reduce the joint size

Independence and conditional independence provide the tools