1)

a) $$\sum_{n=1}^{\infty} \log \frac{3n^2+1}{(n+2)^2}$$ $a_n > 0 \ \forall n \in \mathbb{N}$

To 1. $\log \frac{3n^2+1}{n^2+2n+4} \to \log 3$ $\Rightarrow$ diverge

b) $$\sum_{n=1}^{\infty} \frac{2n+1}{n+3}$$

$a_n \to 2$

$\Downarrow$

diverge

c) $$\sum_{n=1}^{\infty} \frac{4n+2}{(n+1)^5}$$

$a_n \to 0$

$a_n > 0 \quad \forall n \in \mathbb{N}$

$$\frac{4n+2}{(n+1)^5} < \frac{4n+2}{n^5}$$

$$\lim n^x \frac{4n+2}{n^5} = 4 > 0$$ $\Rightarrow a_n$ converge

$\downarrow$

$x = 4$

d) $$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3-6}$$

$a_n \to 0$

$\exists\, \nu \in \mathbb{N} : a_n > 0 \quad \forall n > \nu$

$$\lim n^x \frac{n^2+1}{n^3-6} = 1 > 0$$ $\Rightarrow a_n$ diverge

$x = 1$

$$\sum_{n=1}^{\infty} \frac{(n-1)^2}{(n+6)^8},$$

$a_n \to 0$

$$\frac{(n-1)^2}{(n+6)^8} < \frac{(n-1)^2}{n^8} = \frac{n^2-2n+1}{n^8}$$ è a termini non negativi per $n \in \mathbb{N}$

$$\lim n^x \cdot \frac{n^2\left(1-\frac{2}{n}+\frac{1}{n^2}\right)}{n^8} \underset{x=6}{=} 1 \Rightarrow a_n \to \text{converge}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{n^2+3}\right)}{n\sqrt{n}} \qquad 0<\frac{2n+1}{n^2+3}<1<\frac{\pi}{2} \Rightarrow \cos\left(\frac{2n+1}{n^2+3}\right)>0$$

$$\Downarrow$$

$$a_n>0 \quad \forall n\in\mathbb{N}$$

$$\frac{\cos\left(\frac{2n+1}{n^2+3}\right)}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}}$$

$$\lim n^x \frac{1}{n\sqrt{n}} \underset{x=\frac{3}{2}>1}{=} 1>0 \Rightarrow a_n \text{ converge}$$

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n-1}{n^2+3}\right)}{n^2+5}, \quad 0\leq\frac{n-1}{n^2+3}<1<\frac{\pi}{2} \Rightarrow \sin\left(\frac{n-1}{n^2+3}\right)\geq 0$$

a_n è a termini non negativi

$$a_n \overset{def}{=} \frac{\sin\left(\frac{n-1}{n^2+3}\right)}{n^2+5} < \frac{1}{n^2+5} \overset{def}{=} b_n \qquad a_n<b_n$$

$$\lim n^x \frac{1}{n^2+5} \underset{x=2>1}{=} 1>0 \Rightarrow b_n \text{ converge} \Rightarrow a_n \text{ converge}$$

$$\sum_{n=1}^{\infty} \frac{1-\cos\frac{2}{n^2}}{\frac{1}{n}},$$

$$\cos x \leq 1 \quad \forall x \in \mathbb{R}$$

$$\Downarrow$$

$$1-\cos\frac{2}{n^2} \geq 0$$

a_n è a termini non negativi

$$\cancel{\frac{1-\cos\frac{2}{n^2}}{\frac{1}{n}} \leq \frac{2}{\frac{1}{n}} = 2n}$$

Non posso dedurre nulla in quanto diverge

$$a_n = \underbrace{\frac{1-\cos\frac{2}{n^2}}{\frac{2}{n^2}}}_{\to 1} \cdot n \cdot \underbrace{\frac{2}{n^2}}_{\to 0} \to 0$$

$$1-\cos(t) \sim \frac{1}{2}t^2$$

$$a_n \sim \frac{\frac{1}{2}\cdot\left(\frac{2}{n^2}\right)^2}{\frac{1}{n}} = \frac{4}{n^3}$$ che converge

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$$ è a segni alterni

Non è assolutamente convergente

È crescente?

$$\frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} \geq 1 \ ?$$

$$\frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1} \Longleftrightarrow (n+1)(n^2+1) \leq n[(n^2+1)+1]$$

$$\underbrace{\lim a_n = 0 \quad \forall n \in \mathbb{N}}_{\Downarrow}$$

a_n converge

$$\sum_{n=1}^{\infty} 4^n \sin\frac{1}{2^n},$$

$$\frac{1}{2^n} < 1 < \frac{\pi}{2} \quad \forall n \in \mathbb{N}$$

$$\Downarrow$$

$$\sin\left(\frac{1}{2^n}\right) > 0 \quad \forall n \in \mathbb{N}$$

a_n è a termini positivi

$$a_n = 4^n \cdot \sin\frac{1}{2^n} = 2^{2n} \cdot \underbrace{\frac{\sin\left(\frac{1}{2^n}\right)}{\frac{1}{2^n}}}_{\to 1} \cdot \frac{1}{2^n} =$$

$$= 2^n \to +\infty$$

a_n diverge

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+5}{n^2+4},$$

a_n è a segni alterni

Studio la convergenza assoluta

$$\sum_{n=1}^{\infty} \frac{n+5}{n^2+4} \qquad \lim n^x \frac{n+5}{n^2+4} = 1 > 0$$

$$x = 1 \leq 1 \not\Rightarrow \text{diverge}$$

(non deduco nulla)

Studio la monotonia

$$a_n \overset{?}{\geq} a_{n+1}$$

$$\frac{n+5}{n^2+4} \geq \frac{n+6}{n^2+2n+5}$$ considero $n \in \mathbb{N}$

$$(n+5)(n^2+2n+5) \geq (n+6)(n^2+4)$$

$$n^3+2n^2+5n+5n^2+10n+25 \geq n^3+4n+6n^2+24$$

$$n^3+7n^2+15n+25 \geq n^3+6n^2+4n+24$$

$$n^2+11n+1 \geq 0 \quad \forall n \in \mathbb{N}$$

a_n è decrescente

$\lim a_n = 0$ $\Rightarrow$ a_n è convergente

$$\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{2n+8}{n+1}}_{a_n},$$

$\lim |a_n| = 2 \neq 0 \Rightarrow a_n$ è divergente

(non deduco nulla)

Studio monotonia

$$\lim \frac{2n+2+8}{n+2} \cdot \frac{n+1}{2n+8} = \lim \frac{2n^2+10n+2n+10}{2n^2+8n+4n+16} = 1$$

$$\frac{2n+10}{n+2} \overset{?}{\geq} \frac{2n+8}{n+1}$$

$$(2n+10)(n+1) \geq (2n+8)(n+2)$$

$$2n^2+2n+10n+10 \geq 2n^2+4n+8n+16$$

$$10 \geq 16$$

$$\nexists n \in \mathbb{N}$$

$$a_{n+1} \ngeq a_n$$

$$a_{n+1} \leq a_n$$

a_n è decrescente

$\lim a_n = 2$ $\Rightarrow$ a_n è oscillante

$$\underbrace{\sum_{n=1}^{\infty} (1-2x)^n}_{b_n}, = \underbrace{\sum_{n=1}^{\infty} (1-2x)^{n-1}}_{b_n} - \underset{\substack{\parallel \\ 1}}{a_1}$$

$\sum_{n=1}^{\infty} q^{n-1}$ — $q \geq 1$ div; $-1 < q < 1$ con; $q \leq -1$ irreg.

b_n:
- diverge per $1-2x \geq 1 \iff x \leq 0$
- converge a $\frac{1}{1-1+2x}$ per $-1 < 1-2x < 1 \iff 0 < x < 1$
- oscillante per $1-2x \leq -1 \iff x \geq 1$

Quindi a_n:
- diverge per $x \leq 0$
- converge a $\frac{1}{2x} - 1$ per $0 < x < 1$

- oscilla per $x \geq 1$

$$\underbrace{\sum_{n=1}^{\infty} 3^{4nx-1}}_{a_n} = \sum_{n=1}^{\infty} \frac{3^{4nx}}{3} = \sum_{n=1}^{\infty} \frac{\left(3^{4x}\right)^n}{3} = \sum_{n=1}^{\infty} \left(\frac{3^{4x}}{\sqrt[n]{3}}\right)^n$$

$$\lim \sqrt[n]{\left(\frac{3^{4x}}{\sqrt[n]{3}}\right)^n} = \lim \frac{3^{4x}}{\underset{\to 1}{\sqrt[n]{3}}} = 3^{4x} > 1$$

a_n diverge se $3^{4x} > 1 \iff x > 0$

a_n converge se $3^{4x} < 1 \iff x < 0$

Quando $x = 0$?

$$\sum_{n=1}^{\infty} 3^{-1} = \frac{1}{3} \quad \text{converge}$$

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n\sqrt{n+1}}$$

Per $x \geq 0$ è a termini non negativi

Per $x < 0$ è a segni alterni

Per $x > \frac{1}{2}$ a_n diverge in quanto $\lim a_n = +\infty$

Per $0 \leq x < \frac{1}{2}$ a_n tende a 0: potrebbe convergere

$$\lim n \left(\frac{\cancel{(2x)^n}}{\cancel{n}} \cdot \frac{n+1}{\cancel{(2x)^n} \cdot 2x} - n \right) = \lim \frac{n+1-2xn}{2x} =$$

$1 - 2x > 0$

$x < \frac{1}{2}$

$$\lim \frac{n\left(1 + \frac{1}{n} - 2x\right)}{2x} = +\infty$$

$\Rightarrow$ Raabe a_n converge

Con $x = \frac{1}{2}$

$$\lim \frac{1}{2x} = \frac{1}{2x} \Rightarrow a_n \text{ converge}$$

Per $x<0$

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n\sqrt{n+1}} \qquad \text{Studio} \quad \sum_{n=1}^{\infty} \frac{(2|x|)^n}{n\sqrt{n+1}}$$

$\Downarrow$

converge per $-\frac{1}{2} \leq x < 0$

Studio la monotonia per $x < -\frac{1}{2}$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{2x \cdot n \cdot \sqrt{n+1}}{n\sqrt{n+2} + \sqrt{n+2}} \quad ; \quad \frac{2x \cdot n \cdot \sqrt{n+1}}{n\sqrt{n+2}} \Rightarrow 2x > 1$$

$|a_n|$ è crescente

$\Downarrow$

a_n oscillante

$$\sum_{n=1}^{\infty} \frac{(4x)^n}{(n+1)\sqrt{n}}$$

a_n è a termini non negativi per $x \geq 0$

a_n è a segni alterni per $x<0$

Per $x=0$ $a_n=0 \Rightarrow$ converge

$x>0$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(4x)^{n+1}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{(4x)^n} =$$

$$= \lim \frac{\cancel{(4x)^n} \cdot 4x}{(n+2)\sqrt{n+1}} \frac{(n+1)\sqrt{n}}{\cancel{(4x)^n}} = \lim \frac{[n\sqrt{n} + \sqrt{n}]\,4x}{n\sqrt{n+1} + 2\sqrt{n+1}} =$$

$$= \lim \frac{4x \cdot n\sqrt{n}}{n\sqrt{n+1}} = 4x > 1 \iff x > \frac{1}{4}$$

Per $x > \frac{1}{4}$ a_n diverge

Per $0 \leq x < \frac{1}{4}$ a_n converge

Con Raabe a_n converge per $x = \frac{1}{4}$

Studio a_n per $x<0$

Per $-\frac{1}{4} \leq x < 0$ a_n converge

Studio per $x < -\frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{(4x)^n}{(n+1)\sqrt{n}}$$

$$\lim \frac{(4|x|)^{n+1}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{(4x)^n} = \lim \frac{(4|x|)^n \cdot 4x \cdot (n+1)\sqrt{n}}{(n+2)\sqrt{n+1} \cdot (4x)^n} =$$

$= 4|x| > 1 \iff x > \frac{1}{4} \Rightarrow |a_n|$ è crescente

$\Downarrow$

a_n oscilla per $x < -\frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\sqrt{n^2+2n}}$$

a_n è a termini non negativi per $x \geq 2$

" " segni alterni per $x < 2$

$x \geq 2$

Criterio di Raabe

$$\lim n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim n\left(\frac{\cancel{(x-2)^n}(n+1)\sqrt{(n+1)^2+2(n+1)}}{n\sqrt{n^2+2n}\cdot \cancel{(x-2)^n}(x-2)} - 1\right)$$

$$= \lim \frac{n\sqrt{(n+1)^2+2(n+1)}}{\sqrt{n^2+2n}\,(x-2)} - n = \lim \frac{n(1-x-2)}{x-2} = -\infty$$

$x < -1$

a_n diverge per $\forall x > 2$ Per $x=2$ $a_n = 0$

$\Downarrow$

converge

Studio per $x < 2$

$$\lim \frac{a_{n+1}}{a_n} = |x| - 2$$

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\sqrt{n^2+2n}}$$

$|x| - 2 > 1 \iff |x| > 3 \Rightarrow x < -3 \Rightarrow |a_n|$ è crescente

$\Downarrow$

a_n è oscillante

Per $-3 < x < 2$ a_n è decrescente

$\lim |a_n| = 0 \quad \forall x \in [1, 2[$

$\lim |a_n| = +\infty \quad \forall x \in [-3, 1[$

$\Rightarrow$ a_n converge $x \in [1, 2[$

a_n oscilla $x \in [-3, 1[$

$$\sum_{n=1}^{+\infty} \frac{1}{\sqrt[3]{n}\log(n+1)};$$

a_n é a termini positivi

$a_n \to 0$

$$a_n = \frac{1}{\sqrt[3]{n}\log(n+1)} \qquad b_n = \frac{1}{n} \qquad \lim \frac{a_n}{b_n} = \lim \frac{n}{\sqrt[3]{n}\cdot\log(n+1)} = +\infty$$

$\downarrow$ diverge

$\lim \frac{a_n}{b_n} = +\infty$

b_n diverge

$\Longrightarrow a_n$ diverge

2) Dire quali delle seguenti serie sono a segni alterni e stabilire il carattere di una a scelta fra le serie A), D), E).

A) $\sum_{n=1}^{\infty} \frac{2n^6-1}{4n^4+5}$ NO $\to$ diverge

B) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+3}$ SI ?

C) $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$ NO $\to$ diverge

D) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+4}$ SI $\to$ converge

E) $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{\sqrt{n}}$ SI $\to$ converge

F) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{(-1)^{n+1}}$ NO

D) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+4}$$

$$a_{n+1} \geq a_n$$

$$\frac{1}{\cancel{n^2}+2n+5} \geq \frac{1}{\cancel{n^2}+4}$$

$$2n+1 \leq 0 \iff n \leq -\frac{1}{2}$$

$|a_n|$ é decrescente

$\lim |a_n| = 0 \Longrightarrow a_n$ converge

E) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{\sqrt{n}}$$

$\sum_{n=1}^{\infty} |a_n|$ diverge

$$\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n}}$$

$$\sqrt{n+1} \leq \sqrt{n}$$

$\exists n \in \mathbb{N}$

$\Downarrow$

a_n è decrescente

$\lim |a_n| = 0 \Rightarrow a_n$ converge

$$\sum_{n=1}^{+\infty} \frac{3^n}{n^2(n-1)!}; \leq \frac{3^n}{(n-1)!}$$

serie resto di posto 1 della serie esponenziale, che converge quindi converge

$\Downarrow$

a_n converge

OPPURE criterio del rapporto

$$\lim \frac{3^{n+1}}{(n+1)^2 \cdot n!} \frac{n^2(n-1)!}{3^n} =$$

$$= \lim \frac{3 \cdot \cancel{3^n}}{(n+1)^2 \cdot n \cdot \cancel{(n-1)!}} \frac{n^2 \cancel{(n-1)!}}{\cancel{3^n}} = 0 < 1 \Rightarrow a_n \text{ converge}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)\sqrt{n}}(x-1)^n$$

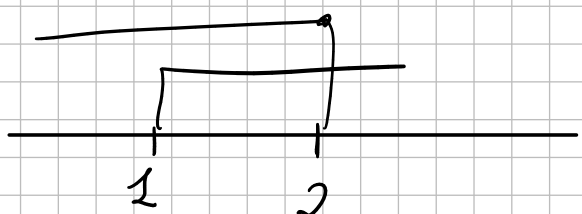
a_n è a termini non negativi se $x \geq 1$

// segni alterni se $x < 1$

$x \geq 1$ Raabe

$$\lim n \left(\frac{n\cancel{(x-1)^n}}{(n^2+1)\sqrt{n}} \cdot \frac{(x^2+2n+2)\sqrt{n+1}}{(n+1)\cancel{(x-1)^n}(x-1)} - 1 \right) =$$

$$= \lim n \frac{1-x+1}{x-1} = \lim n \frac{2-x}{x-1} =$$



$$= \begin{cases} +\infty & 1 < x < 2 \\ -\infty & x > 2 \\ 0 & x \in \{1, 2\} \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)\sqrt{n}}(x-1)^n$$

Rapporto

$$\lim \frac{(n+1)(x-1)^{n+1}}{(n^2+2n+2)\sqrt{n+1}} \frac{(n^2+1)\sqrt{n}}{n(x-1)^n} =$$

$$N.\ (n+1)(x-1)(n^2+1)\sqrt{n} = (nx - n + x - 1)(n^2+1)\sqrt{n}$$

$$= (n^3(x-1) + n^2 x - 1 + nx - n + x - 1)\sqrt{n} \sim (x-1)n^3\sqrt{n}$$

$$D. \sim n^3\sqrt{n}$$

$$= x - 1 \begin{cases} > 1 \iff x > 2 \Rightarrow \text{DIVERGE} \\ = 1 \iff x = 2 \Rightarrow \text{NON DEDUCO NULLA} \\ < 1 \iff x < 2 \Rightarrow \text{CONVERGE} \end{cases}$$

Per $x=2$ $\quad a_n = \dfrac{n}{(n^2+1)\sqrt{n}} \sim \dfrac{n}{n^2\sqrt{n}} \sim \dfrac{1}{n\sqrt{n}}$ che converge

Quindi a_n converge per $x=2$

Per $x<1$

Studio $\Sigma |a_n|$.

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)\sqrt{n}}(x-1)^n$$

Converge per $-2\leq x\leq -1$

$$\lim \frac{|a_{n+1}|}{|a_n|} = |x-1|>1 \iff x>2 \vee x<0$$

a_n cresce per $x<0$

$\Downarrow$

oscilla

Invece per $0<x<1$ $|a_n|$ decresce e siccome $\lim a_n = 0$ $\forall x \in \mathbb{R}$ a_n converge per $0<x<1$

Manca $x=0$. Studio la monotonia

$$\frac{(n+1)}{(n^2+2n+2)\sqrt{n+1}} \geq \frac{n}{(n^2+1)\sqrt{n}}$$

$$(n+1)(n^2+1)\sqrt{n} \geq n(n^2+2n+2)\sqrt{n+1}$$

$$(n^3+n^2+n+1)\sqrt{n} \geq (n^3+2n+2n+2)\sqrt{n+1}$$

$\nexists n \in \mathbb{N}$ $|a_n|$ è decrescente $\Rightarrow a_n$ converge

$$\sum_{n=1}^{\infty} \frac{n \sin \frac{\sqrt{2}}{n!}}{n! + \log n},$$

$\sqrt{2} \approx 1,41$ $\frac{\pi}{2} \approx 1,57$

$\frac{\sqrt{2}}{n!} < \frac{\pi}{2}$ $\forall n \in \mathbb{N}$, $\log n \geq 0 \; \forall n \in \mathbb{N}$

a_n è a Termini non negativi

$$\frac{n \cdot \sin\left(\frac{\sqrt{2}}{n!}\right)}{n! + \log n} \leq \frac{n}{n! + \log(n)} = b_n$$

$$\lim \frac{n+1}{(n+1)! + \log(n+1)} \cdot \frac{n! + \log(n)}{n} =$$

$$= \lim \frac{(n+1) \cdot [n! + \log(n)]}{[(n+1) \cdot n! + \log(n)] \, n} = \lim \frac{(n+1) \cdot n! \quad (n+1) \cdot \log(n)}{n \cdot (n+1) \cdot n! + n \cdot \log(n)}$$

$= 0 < 1 \rightarrow b_n$ converge

Siccome $a_n < b_n \Longrightarrow a_n$ converge

$$\sum_{n=1}^{\infty} \frac{n! \sin \frac{\sqrt{2}}{n!}}{n! + \log n}. \leq \quad \frac{n!}{n!+\log n} \to 1 \to \text{diverge (non deduco nulla)}$$

$$\frac{n! \sin\left(\frac{\sqrt{2}}{n!}\right)}{n! + \log n} \sim \frac{\cancel{n!} \cdot \frac{\sqrt{2}}{\cancel{n!}}}{n! + \log(n)} \leq \frac{\sqrt{2}}{n!}$$

$$\frac{\cancel{\sqrt{2}}}{(n+1)\cdot \cancel{n!}} \frac{\cancel{n!}}{\cancel{\sqrt{2}}} \to 0 < 1 \text{ converge}$$

$\downarrow$

a_n converge

$$\sum_{n=1}^{\infty} \frac{1-n!}{n^n},$$

$$a_n = \overset{b_n}{\frac{1}{n^n}} - \overset{c_n}{\frac{n!}{n^n}}$$

$\frac{1}{n^n}$ $\downarrow$ diverge

$$\lim \frac{(n+1)\cdot \cancel{n!} \quad n^n}{(n+1)^{n+1} \quad \cancel{n!}}$$

$$\lim \frac{\cancel{(n+1)}\, n^n}{\cancel{(n+1)}\, (n+1)^n} =$$

$$= \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

$\downarrow$

c_n converge

$$\sum_{n=1}^{\infty} \frac{2^{nx^2}}{nx},\ x \neq 0.$$

a_n è a termini non negativi per $x>0$
// segni alterni per $x<0$

$x>0$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{2^{(n+1)x^2}}{(n+1)x} \cdot \frac{nx}{2^{nx^2}} =$$

$$= \lim \frac{\cancel{2^{nx^2}} \cdot 2^{(x^2)} \cdot nx}{(nx+x)\ \cancel{2^{nx^2}}} = \lim \frac{2^{(x^2)} \cdot nx}{nx+x} = 2^{(x^2)} = l$$

$$l>1 \iff 2^{(x^2)} > 1 \iff x^2 > 0 \iff \forall x \in \mathbb{R} \setminus \{0\}$$

a_n diverge per $x>0$

Studio $x<0$

$\lim \frac{|a_{n+1}|}{|a_n|}$ (l'ho calcolato prima)

$\Downarrow$

a_n è crescente

$\Downarrow$

a_n oscilla