esercizi integrali 2023 24.pdf

1.a)

$$\int \frac{\arctan^2 x - \arctan x}{1+x^2}\,\mathrm{d}x$$

Metodo di sostituzione

$$\xrightarrow{\frac{1}{1+x^2}}$$

$$\int \frac{\arctan^2 x - \arctan x}{\cancel{1+x^2}} \cdot \cancel{(1+x^2)} \cdot \mathrm{d}[\arctan x]\,dx =$$

$$= \left(\int (y^2-y)\,dy\right)_{y=\arctan x}$$

Trovo $\int (y^2-y)\,dy =$

$$= \frac{y^3}{3} - \frac{y^2}{2} + c, \quad c \in \mathbb{R}$$

$$\left(\int (y^2+y)\,dy\right)_{y=\arctan x} = \frac{\arctan^3 x}{3} + \frac{\arctan^2 x}{2} + c, \quad c \in \mathbb{R}$$

1.b)

$$\int \frac{\log^2 x - 3\log x + 1}{x}dx,$$

$$\int f(x)^2 \cdot f'(x) = \frac{f(x)^{2+1}}{2+1} + c$$

$$\int \frac{\log^2 x}{x}\,dx - 3\int \frac{\log x}{x}\,dx + \int \frac{1}{x}\,dx =$$

$$\frac{\log^3|x|}{3} - \frac{3}{2}\log^2|x| + \log|x| + c, \quad c \in \mathbb{R}$$

1.c)

$$\int \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x,$$

Metodo di Sostituzione

$$\int \frac{\sin x}{1+\cos^2 x}\, dx = \int \frac{\cancel{\sin x}}{1+\cos^2 x} \cdot \frac{1}{\cancel{\sin x}}\, D[\cos x]\, dx =$$

$$= \int \frac{1}{1+\cos^2 x} \cdot D[\cos x]\, dx = \left(\int \frac{1}{1+y^2}\, dy \right)\Bigg|_{y=\cos x} =$$

$$= \left(\arctan y + c, \quad c \in \mathbb{R} \right)\Bigg|_{y=\cos x} = \arctan(\cos x) + c, \quad c \in \mathbb{R}$$

1.d)

$$\int x \arctan x \, dx,$$

Integrazione per parti

$$\int x \cdot \arctan x \, dx =$$

$f(x) = \arctan x \qquad f' = \dfrac{1}{1+x^2}$

$g'(x) = x \qquad g(x) = \dfrac{x^2}{2} + c$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{1+x^2}\, dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} x - \arctan x + c, \quad c \in \mathbb{R}$$

1.e)

$$\int x^3 \log x \, dx,$$

Integrazione per parti

$$\int \overset{g'(x)}{x^3}\, \overset{f(x)}{\log} x \, dx = \frac{x^4}{4} \cdot \log|x| - \int \frac{x^{\cancel{4}\,3}}{4} \cdot \frac{1}{\cancel{x}}\, dx = \frac{x^4}{4} \log|x| - \frac{x^4}{16} + c, \quad c \in \mathbb{R}$$

1.f)

$$\int x \sin x \, dx,$$

$$\int \overset{f(x)}{x} \, \overset{g'(x)}{\sin x} \, dx = x \cdot \cos x + \int \cos x \, dx = x \cos x + \sin x + c, \quad c \in \mathbb{R}$$

1.g)

$$\int (x+2)\cos x \, dx, = -(x+2)\sin x - \int \sin x \, dx = -(x+2)\sin x + \cos x + c, \quad c \in \mathbb{R}$$

1.h)

$$\int \frac{x+1}{x^3-6x^2+9x} dx, = \int \frac{x+1}{x(\underbrace{x^2-6x+9}_{\Delta = 36-4\cdot 9 = 0})} dx = \int \frac{x+1}{x(x-3)^2} dx$$

$$\frac{x+1}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2} = \frac{A(x-3)^2 + B \cdot x \cdot (x-3) + C \cdot x}{x(x-3)^2}$$

$$x+1 = Ax^2 - 6x + A9 + Bx^2 - B3x + Cx$$

$$x+1 = (A+B)x^2 + (-6 - B3 + C)x + 9A$$

$$\begin{cases} 0 = A+B \\ 1 = -6 - B3 + C \\ 1 = 9A \end{cases} \rightarrow \begin{cases} A = -B \\ 7 = -B3 + C \\ A = \frac{1}{9} \end{cases} \rightarrow \begin{cases} A = \frac{1}{9} \\ B = -\frac{1}{9} \\ C = 7 - \frac{1}{\cancel{9}_3}\cancel{3} = \frac{20}{3} \end{cases}$$

$$\int \frac{1+x}{x(x-3)^2} dx = \frac{1}{9}\int \frac{1}{x} dx - \frac{1}{9}\int \frac{1}{x-3} dx + \frac{20}{3}\int \frac{1}{(x-3)^2} dx =$$

$$= \frac{1}{9}\ln|x| - \frac{1}{9}\ln|x-3| - \frac{20}{3}\cdot\frac{1}{x-3} + c_1 \quad c \in \mathbb{R}$$

1.i)

$$\int \frac{x+4}{x^2-x-6}\,dx,$$

$\Delta = 1 + 6\cdot 4 = 25 > 0$ $\quad x = \frac{1 \pm 5}{2}$ $\quad 3$, $\quad 2$

$$x^2 - x - 6 = (x-2)(x-3)$$

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$$\frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$x+4 = A(x-3) + B(x-2)$$

$$x+4 = Ax - 3A + Bx - 2B$$

$$x+4 = (A+B)x + (-3A-2B)$$

$$\begin{cases} 1 = A + B \\ 4 = -3A - 2B \end{cases} \to \begin{cases} A = 1 - B \\ 4 = -3 + 3B - 2B \end{cases} \to \begin{cases} A = -6 \\ B = 7 \end{cases}$$

$$\int \frac{x+4}{(x-2)(x-3)}\,dx = -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx =$$

$$= -6\ln|x-2| + 7\ln|x-3| + c_1 \quad c \in \mathbb{R}$$

1.l)

$$\int \frac{x+1}{x^2+6x+10}\,dx, = \frac{1}{2}\int \frac{2x+2-1+1}{x^2+6x+10}\,dx = \frac{1}{2}\ln(x^2+6x+10) +$$

$$+ \int \frac{1}{x^2+6x+10}\,dx$$

$$x^2+6x+10 = x^2+6x+9+1 = (x+3)^2+1$$

$$\int 1\cdot\frac{1}{(x+3)^2+1}\,dx = \left(\arctan y\right)_{y=x+3} = \arctan(x+3)+c,\ c\in\mathbb{R}$$

$$D[x+3]=1$$

1.m)

$$\int \frac{2x-1}{x^2+x+4}\,\mathrm{d}x, = \int\frac{2x-1+2-2}{x^2+x+4}\,dx = \ln(x^2+x+4) - $$

$$-2\int\frac{1}{x^2+x+4}\,dx$$

$$x^2+x+4 = x^2\cdot 2\frac{x}{2}+\frac{1}{4}-\frac{1}{4}+4 = \left(x+\frac{1}{2}\right)^2+\frac{15}{4}$$

$$=\frac{(2x+1)^2}{4}+\frac{15}{4} = \frac{15}{4}\left(\frac{(2x+1)^2}{15}+1\right) =$$

$$=\frac{15}{4}\left(\left(\frac{2x+1}{\sqrt{15}}\right)^2+1\right)$$

$$\int\frac{1}{x^2+x+4}\,dx = \frac{4}{15}\int\frac{1}{\left(\frac{2x+1}{\sqrt{15}}\right)^2+1}\cdot\frac{\sqrt{15}}{2}\cdot D\left[\frac{2x+1}{\sqrt{15}}\right]dx =$$

$$=\frac{2\sqrt{15}}{15}\cdot\arctan\left(\frac{2x+1}{\sqrt{15}}\right)+c,\quad c\in\mathbb{R}$$

1.n)

$$\int\frac{\mathrm{d}x}{(x-1)(x^2+3)}$$

$$\frac{1}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

$$1 = Ax^2 + 3A + Bx^2 + Cx - B - C$$

$$\begin{cases} 0 = A+B \\ 0 = C \\ 1 = 3A - B - C \end{cases} \rightarrow \begin{cases} A = -B \\ 1 = 4A \\ C = 0 \end{cases} \rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = 0 \end{cases}$$

$$\int \frac{dx}{(x-1)(x^2+3)} = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\cdot\frac{1}{2}\int \frac{2x}{x^2+3}\, dx =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{8}\ln(x^2+3) + c, \quad c \in \mathbb{R}$$

2.a)

$$D[\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$\int \frac{1}{\sqrt{x}+1}\, dx, =$$

$$= \int \frac{1}{\sqrt{x}+1} \cdot 2\sqrt{x} \cdot D[\sqrt{x}]\, dx = \left(\int \frac{2y}{y+1}\, dy\right)_{y=\sqrt{x}}$$

$$\begin{array}{r|l} 2y & y+1 \\ \cline{2-2} 2y+2 & 2 \\ \hline // \quad -2 & \end{array}$$

$$\int \frac{2y}{y+1}\, dy = \int 2dx - \int \frac{2}{y+1}\, dx =$$

$$= 2y - 2\ln|y+1| + c, \quad c \in \mathbb{R}$$

$$\left(\int \frac{1}{\sqrt{x}+1}\, dx\right)_{y=\sqrt{x}} = 2\sqrt{x} - 2\ln(\sqrt{x}+1) + c, \quad c \in \mathbb{R}$$

2.b)

$$\int e^{\sqrt{x}}\, dx, = \int 2\sqrt{x}\, e^{\sqrt{x}} \cdot D[\sqrt{x}]\, dx = \left(2\int y e^y\, dy\right)_{y=\sqrt{x}}$$

$$\int y e^y\, dy = y e^y - \int e^y\, dy = y e^y - e^y \quad + c, \quad c \in \mathbb{R}$$

$$\left(2ye^y - 2e^y\right)_{y=\sqrt{x}} = 2\sqrt{x}e - 2e^{\sqrt{x}} + c, \quad c \in \mathbb{R}$$

2.c)

$$\int \cos(\log x)\, dx,$$

Integrazione per Parti

$f(x) = \cos(\log x)$

$g'(x) = 1$

$$\int \cos(\log x)\, dx = x \cdot \cos(\log x) - \int -\frac{1}{x} \operatorname{sen}(\log x) \cdot x\, dx =$$

$$= x \cdot \cos(\log x) + \int \operatorname{sen}(\log x)\, dx =$$

$$= x \cdot \cos(\log x) + x \cdot \operatorname{sen}(\log x) - \int \frac{1}{\cancel{x}} \cos(\log x) \cdot \cancel{x}\, dx$$

$$\int \cos(\log x)\, dx = x\cos(\log x) + x\operatorname{sen}(\log x) - \int \cos(\log x)\, dx$$

$$\int \cos(\log x)\, dx = \frac{x}{2}\left(\cos(\log x) + \operatorname{sen}(\log x)\right) + c, \quad c \in \mathbb{R}$$

2.d)

$$\int \frac{\log(1+x)}{x^2}\, dx, =$$

Int. per parti:

$f(x) = \log(1+x)$

$g'(x) = \frac{1}{x^2} \qquad g(x) = -\frac{1}{x}$

$$= -\frac{\log(1+x)}{x} - \int \frac{1}{1+x} \cdot \left(-\frac{1}{x}\right) dx =$$

$$= -\frac{\log(1+x)}{x} + \int \frac{1}{x(x+1)}\, dx$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx \qquad \begin{cases} 0 = A + B \\ 1 = A \end{cases} \rightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\int \frac{1}{x(x+1)}\, dx = \int \frac{1}{x}\, dx - \int \frac{1}{x+1}\, dx = \ln|x| - \ln|x+1| + c$$

$$I = \int \frac{\log(x+1)}{x^2}\, dx = -\frac{\log(x+1)}{x} + \ln|x| - \ln|x+1| + c, \quad c \in \mathbb{R}$$

2.e)

$$I=\int \overset{f(x)}{x^3}\,\overset{g'(x)}{\operatorname{sen}(9x)}\,dx, = -9x^3\cdot\cos(9x)-\int 3x^2\cdot(-9)\cdot\cos(9x)\,dx$$

$$= -9x^3\cos(9x)+27\int x^2\cos(9x)\,dx$$

$$\int x^2\cos(9x)\,dx = 9x^2\operatorname{sen}(9x)-\int 2x\cdot 9\cdot \operatorname{sen}(9x)\,dx =$$

$$= 9x^2\operatorname{sen}(9x)-18\int x\operatorname{sen}(9x)\,dx =$$

$$\int x\operatorname{sen}(9x)\,dx = -9x\cos(9x)-\int -9\cos(9x)\,dx =$$

$$= -9x\cos(9x)+\operatorname{sen}(9x)+c,\ c\in\mathbb{R}$$

$$I = -9x^3\cos(9x)+27\cdot 9x^2\operatorname{sen}(9x)-27\cdot 18\cdot(-9)x\cos(9x)+$$

$$+27\cdot 18\operatorname{sen}(9x)+c,\quad c\in\mathbb{R}$$

2.f)

$$\int \frac{1}{x\log x(1+\log^2 x)}\,dx = \left(\int \frac{1}{y\cdot(1+y^2)}\,dy\right)_{y=\log x}$$

$$\frac{1}{y(y^2+1)} = \frac{A}{y}+\frac{By+C}{y^2+1}$$

$$1 = Ay^2+A+By^2+Cy$$

$$1 = (A+B)y^2 + Cy + (A+B)$$

$$\begin{cases} 0 = A + B \\ 0 = C \\ 1 = A \end{cases} \rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\int \frac{1}{y(y^2+1)}\,dy = \int \frac{1}{y}\,dy - \frac{1}{2}\int \frac{2y}{y^2+1}\,dy =$$

$$= \ln|y| - \frac{1}{2}\ln|y^2+1| + c, \quad c \in \mathbb{R}$$

$$= \ln|\ln x| - \frac{1}{2}\ln|\ln^2 x + 1| + c, \quad c \in \mathbb{R}$$

1.g)

$$\int \frac{x^5}{x^4-1}\,\mathrm{d}x,$$

$$\begin{array}{ll|l} x^5 & & x^4-1 \\ \hline x^5 & -x & x \\ // & -x & \end{array}$$

$$\int \frac{x^5}{x^4-1}\,dx = \frac{x^2}{2} - \int \frac{x}{x^4-1}\,dx$$

$$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\frac{x}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$$x = Ax^3 + Ax + Ax^2 + A + Bx^3 + Bx - Bx^2 - B + Cx^3 - Cx + Dx^2 - D$$

$$\begin{cases} 0 = A+B+C \\ 0 = A-B+D \\ 1 = A+B-C \\ 0 = A-B-D \end{cases} \Rightarrow \begin{cases} A = -B-C \\ -2B-C+D = 0 \\ C = -\frac{1}{2} \\ -2B-C-D = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} - B \\ -2B+\frac{1}{2}+D = 0 \\ C = -\frac{1}{2} \\ D = -2B+\frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = 1 \\ C = -\frac{1}{2} \\ D = -\frac{3}{4} \end{cases}$$

$$\int \frac{x}{x^4-1}\,dx = -\frac{1}{2}\int \frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{4\left(\frac{1}{2}x + \frac{3}{4}\right)}{x^2+1}\,dx =$$

$$= -\frac{1}{2}\ln|x-1| + \ln|x+1| - \frac{1}{4}\int \frac{2x+3}{x^2+1}\,dx$$

$$\int \frac{2x+3}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx = \ln|x^2+1| + 3\arctan x + c$$

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2.h)

$$\int \frac{x}{\sqrt{1-x^4}}\,\mathrm{d}x,$$

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2.i)

$$\int \sqrt{x}\arctan\sqrt{x}\,\mathrm{d}x. =$$

1ª Formula di sostituzione

$$= \int \cancel{\sqrt{x}} \cdot \arctan\sqrt{x} \cdot \frac{1}{2\cancel{\sqrt{x}}} \cdot D[\sqrt{x}]\, dx =$$

$$= \frac{1}{2}\left(\int \arctan y \, dy\right)_{y=\sqrt{x}}$$

Per parti: $g'(x)=1$, $f(x)=\arctan x$

$$\int \arctan y \, dy = y \arctan y - \frac{1}{2}\int 2y \frac{1}{1+y^2}\, dy =$$

$$= y \arctan y - \ln|1+y^2| + c, \quad c \in \mathbb{R}$$

$$I = \frac{1}{2}\sqrt{x} \cdot \arctan\sqrt{x} - \ln|1+x| + c, \quad c \in \mathbb{R}$$

2.l)

$$\int \frac{\log x + 1}{x(\log^2 x + 3)}\, dx,$$

$$= \int \frac{\log x + 1}{\cancel{x}(\log^2 x + 3)} \cdot \cancel{x}\, D[\log x]\, dx = \left(\int \frac{y+1}{y^2+3}\, dy\right)_{y=\ln x}$$

$$\frac{1}{2}\int \frac{2(y+1)}{y^2+3}\, dy = \frac{1}{2}\int \frac{2y+2}{y^2+3}\, dy = \frac{1}{2}\ln|y^2+3| + \int \frac{1}{y^2+3}\, dy$$

$$y^2+3 = 3\left(\frac{y^2}{3}+1\right) = 3\left[\left(\frac{y}{\sqrt{3}}\right)^2+1\right]$$

$$\int \frac{1}{y^2+3}\, dy = \frac{1}{3}\int \frac{1}{\left(\frac{y}{\sqrt{3}}\right)^2+1} \cdot \sqrt{3} \cdot D\left[\frac{y}{\sqrt{3}}\right] dy = \frac{\sqrt{3}}{3}\arctan\left(\frac{y}{\sqrt{3}}\right) + c$$

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3.a)

$$\int (\sin^3 x)(\cos^4 x)\,dx, = \int \sin^2 x \cdot \sin x \cdot \cos^4 x \, dx = \int (1-\cos^2 x)\cdot \cos^4 x \cdot \sin x \, dx =$$

$$= \int (\cos^4 x - \cos^6 x) \sin x \, dx = \int \cos^4 x \cdot \sin x \, dx - \int \cos^6 x \cdot \sin x \, dx =$$

$D[\cos x] = -\sin x$

$$= -\int \cos^4 x \cdot D[\cos x]\, dx + \int \cos^6 x \cdot D[\cos x]\, dx =$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^6 x}{6} + c, \quad c \in \mathbb{R}$$

3.b)

$$\int (\sin^4 x)(\cos^4 x)\,dx, = \int (\sin x \cdot \cos x)^4 \, dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx =$$

$$= \frac{1}{16}\int \sin^4(2x)\, dx = \frac{1}{16}\int \left[\sin^2(2x)\right]^2 dx =$$

$$= \frac{1}{16}\int \left[\frac{1-\cos(4x)}{2}\right]^2 dx = \frac{1}{16}\int \left[\frac{1}{2} - \frac{\cos(4x)}{2}\right]^2 dx =$$

$$= \frac{1}{16}\int \left[\frac{1}{4} + \frac{\cos^2 x}{4} - \frac{\cos(4x)}{2}\right] dx =$$

$$= \frac{1}{64}x + \frac{1}{64}\int \cos^2 x \, dx - \frac{1}{4\cdot 32}\int 4\cos(4x)\, dx =$$

$$= \frac{1}{64}x + \frac{1}{64}\int \frac{1+\cos(2x)}{2}\, dx - \frac{1}{128}\sin(4x)$$

$$= \frac{1}{64}x + \frac{1}{128}x + \frac{1}{256}\sin(2x) - \frac{1}{128}\sin(4x) + c, \quad c \in \mathbb{R}$$

3.c)

$$\int \cos^2 x \ \sin^4 \ x \, dx, = \int (1-\sin^2 x)\sin^4 x \ dx = \int (\sin^4 x - \sin^6 x)\, dx =$$

$$= \int \sin^4 x \ dx - \int \sin^6 x \ dx = \frac{1}{4}\int \left[1-\cos(2x)\right]^2 dx - \frac{1}{8}\int \left[1-\cos(2x)\right]^3 dx =$$

$$= \frac{1}{4}\int \left[1+\cos^2(2x) - 2\cos(2x)\right] - \frac{1}{8}\int \left[1-\cos^3(2x) - 3\cos(2x) + 3\cos^2(2x)\right] dx$$

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