Memory-dependent signal processing in Josephson junctions: quantum and classical

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I. INTRODUCTION

Any real-time response to a time changing stimulus can be treated in the most general sense as a signal processing scenario. The later response patterns can depend on the previous experience of the system as it has been exposed to stimulus. Therefore, it is possible that the specifi signal processing function of the system is changed after a time period of being exposed to special stimulus. This is like learning from experience to acheive a particular kind of signal processing ability. The real-time response-stimulus relation put in a general signal processing scheme can thus be viewed as an example of learning in physical systems. As most analysis of learning is focused on statistics from a large amount of data and processed in a network, here we focus on a single node tha learns from input of time sequences.

Here we particularly inspect a generic Josephson junction, which has both "quantum" and "classical" aspects. So we are able to compare the learning behaviour of a "classical" Josephson junction with that of a "quantum" Josephson junction.

II. JOSEPHSON RELATIONS

Here we pedagogically derive the Josephson relation. The Josephson junction is made of two superconducting leads connected by a thin insulating space. The wavefunction on each lead is given by $\psi_i = \sqrt{n_i}e^{i\theta_i}$, for $i = 1, 2$, labeling each lead and n_i is real non-negative standing for the number of Cooper pairs on lead i and θ_i is the phase for this macroscopic wavefunction. The two complex wave amplitudes are subject to the Schroedinger equation,

$$i\hbar \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & k \\ k & \mu_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{1}$$

Here μ_i is the chemical potential on lead i and k is real coupling constant. The Josephson relation aims to relate the junction current,

$$I_J = 2e\dot{n}_1 = -2e\dot{n}_2, \tag{2}$$

to the phase difference

$$\theta \equiv \theta_2 - \theta_1. \tag{3}$$

The rate of change of Cooper pair number on lead i is

$$\begin{aligned} \dot{n}_i &= \dot{\psi}_i^* \psi_i + \psi_i^* \dot{\psi}_i = \frac{i}{\hbar} \left[\left(\mu_i \psi_i^* + k\psi_{\bar{i}}^* \right) \psi_i - \psi_i^* \left(\mu_i \psi_i + k\psi_{\bar{i}} \right) \right] \\ &= -\frac{2k}{\hbar} \sqrt{n_1 n_2} \sin\left(\theta_i - \theta_{\bar{i}}\right), \end{aligned}$$

leading to

$$I_J = 2e\dot{n}_1 = \frac{4ek}{\hbar} \sqrt{n_1 n_2} \sin(\theta) = -2e\dot{n}_2. \tag{4}$$

By defining

$$I_c = \frac{4ek}{\hbar} \sqrt{n_1 n_2}, \tag{5}$$

then we have

$$I_J(t) = I_c \sin(\theta(t)). \tag{6}$$

The voltage across the junction is given by

$$V = \frac{\mu_1 - \mu_2}{2e}. \tag{7}$$

Substituting $\psi_i = \sqrt{n_i} e^{i\theta_i}$ into the equation for $\dot{\psi}_i$, we obtain

$$i\hbar \left(\frac{\partial \sqrt{n_i}}{\partial t} + i\dot{\theta}_i \sqrt{n_i} \right) = \mu_i \sqrt{n_i} + k\sqrt{n_{\bar{i}}} e^{i(\theta_{\bar{i}} - \theta_i)}, \tag{8}$$

where $\bar{i}$ labels the opposite lead of i, namely, $\theta_{\bar{1}} = \theta_2$ and $\theta_{\bar{2}} = \theta_1$. Separating the real-part on both sides of the equation gives

$$\dot{\theta}_i = -\frac{\mu_i}{\hbar} - \frac{k}{\hbar}\sqrt{\frac{n_{\bar{i}}}{n_i}} \cos\left(\theta_{\bar{i}} - \theta_i\right). \tag{9}$$

Note that the two leads are separated by a very thin spacer such that the Coopair pair number does not change significantly across the spacer from the boundary one of lead to that of the other. We therefore assume that

$$n_1 \approx n_2, \tag{10}$$

at the boundaries of the spacer. This leads Eq. (9) to

$$\dot{\theta}\left(t\right) = \dot{\theta}_2 - \dot{\theta}_1 = -\frac{\mu_2 - \mu_1}{\hbar} = \frac{2e}{\hbar} V\left(t\right) = \frac{2\pi}{\Phi_0} V\left(t\right), \tag{11}$$

where

$$\Phi_0 = \frac{h}{2e}, \tag{12}$$

is the flux quantum. The results Eqs. (6) and (11) are called Josephson relations.

III. SIGNAL PROCESSING WITH JOSEPHSON JUNCTIONS

A. Resistively capacitively shunted junction (RCSJ) model

The Josephson relations result in the energy

$$U_J = \int \mathrm{d}t' I_J\left(t'\right) V\left(t'\right) = \frac{\Phi_0}{2\pi} \int \mathrm{d}t' I_J\left(t'\right) \frac{\mathrm{d}\theta\left(t'\right)}{\mathrm{d}t'} = \frac{\Phi_0}{2\pi} \int \mathrm{d}\theta I_c \sin\left(\theta\right) = -\frac{\Phi_0}{2\pi} I_c \cos\left(\theta\right), \tag{13}$$

of the junction element. Consider a circuit in which the Josephson junction is in parallel to a capacitor with capacitance C and also in parallel to a resistor with resistance R. The capacitor and the resistor are in parallel to each other. The three elements thus share the same voltage drop. The capacitor is charged by an amount q and the voltage across the capacitor equals that across the Josephson junction, namely,

$$V = \frac{q}{C} = \frac{\Phi_0}{2\pi}\dot{\theta}. \tag{14}$$

The current through the capacitor is $I^C = \mathrm{d}q/\mathrm{d}t$ is thus

$$I^C = \frac{\Phi_0}{2\pi} C\ddot{\theta}. \tag{15}$$

The current through the resistor I^R is also derived using the paralell geometry, namely, the voltage across the capacitor is also that across the Josephson junction. That's

$$V = RI^R = \frac{\Phi_0}{2\pi}\dot{\theta}, \tag{16}$$

and

$$I^R = \frac{\Phi_0}{2\pi R}\dot{\theta}. \tag{17}$$

The total current I is given by summing up each element's current, namely,

$$I = I^C + I^R + I_J \Rightarrow \tag{18}$$

$$I = \frac{\Phi_0}{2\pi}C\ddot{\theta} + \frac{\Phi_0}{2\pi R}\dot{\theta} + I_c \sin(\theta). \tag{19}$$

Given the temporal profile of the driving current $I = I(t)$, one can thus solve Eq. (19) to get $\theta(t)$ given the initial values of θ and $\dot{\theta}$. The voltage $V(t)$ is expected to depend on the pulsing history of $I(t)$. The signal processing scheme is defined by using $I(t)$ as stimulus (input) and $V(t)$ as the response (output).

Two time scales can be identified from Eq. (19) via dividing by $\frac{\Phi_0}{2\pi}C$ from both sides of the equation and rewriting it into

$$\ddot{\theta} = -\frac{1}{\tau_{RC}}\dot{\theta} - \omega_p^2 \sin(\theta) + \omega_p^2\left(\frac{I}{I_c}\right), \tag{20}$$

where

$$\tau_{RC} = RC, \tag{21}$$

is the RC-relaxation time and

$$\omega_p^2 = 2\pi\frac{I_c}{\Phi_0 C}, \tag{22}$$

corresponds to the oscillation frequency as one expands $\sin(\theta)$ around $\theta = 0$ as $\sin(\theta) \approx \theta$, then Eq. (20) reads like $\ddot{\theta} \approx \frac{1}{\tau_{RC}}\dot{\theta} - \omega_p^2\theta$, in analogy to a damped harmonic oscillator with frequency ω_p. The quality factor of the damped oscillation is then measured by the dimensionless quantity $\beta \equiv \omega_p\tau_{RC}$.

B. Generic driven two-level system

Equation (1) can be put in a context not relating to Josephson junction at all. It is the Schroedinger equation for the amplitudes of the wavefunction of a two-state system. We now restore to Eq. (8) and examine the imaginary part of the both sides of the equation. This gives

$$\hbar\frac{\partial\sqrt{n_i}}{\partial t} = k\sqrt{n_{\bar{i}}}\sin(\zeta_i\theta), \tag{23}$$

where $\zeta_1 = 1$ and $\zeta_2 = -1$. It can be further rewritten to

$$\frac{\partial\sqrt{n_i}}{\partial t} = \frac{1}{2\sqrt{n_i}}\dot{n}_i = \frac{k}{\hbar}\sqrt{n_{\bar{i}}}\sin(\zeta_i\theta) \Rightarrow$$

$$\dot{n}_i = \zeta_i\frac{2k}{\hbar}\sqrt{n_1 n_2}\sin(\theta). \tag{24}$$

The phase dynamics is found from Eq. (9),

$$\dot{\theta} = \frac{\Delta\mu}{\hbar} + \frac{k(n_1 - n_2)}{\hbar\sqrt{n_1 n_2}}\cos(\theta), \tag{25}$$

where

$$\Delta\mu = \mu_1 - \mu_2. \tag{26}$$

For a closed two-state system, we have

$$n_1 + n_2 = 1. \tag{27}$$

Defining,

$$\Delta n = n_1 - n_2, \tag{28}$$

we have

$$n_1 = \frac{1 + \Delta n}{2}, \tag{29}$$

$$n_2 = \frac{1 - \Delta n}{2}, \tag{30}$$

and Eq. (24) becomes

$$\Delta \dot{n} = \frac{2k}{\hbar}\sqrt{1 - \Delta n^2}\sin(\theta), \tag{31}$$

while Eq. (25) becomes

$$\dot{\theta} = \frac{\Delta\mu}{\hbar} + \frac{2k\Delta n}{\hbar\sqrt{1 - \Delta n^2}}\cos(\theta). \tag{32}$$

Given the equations for the population dynamics and the phase dynamics of a generic two-state system, Eqs. (31) and (32) respectively, we now take $\Delta\mu = \Delta\mu(t)$ as the stimulus (input) and $\mathrm{d}\Delta n/\mathrm{d}t$, the population transfer rate, or say, the tunneling current, as the response (output). Since $\theta(t)$ depends on the driving history of $\Delta\mu$ via Eq. (32), the tunneling current $\mathrm{d}\Delta n/\mathrm{d}t = \Delta\dot{n}$ also depends on the driving history of $\Delta\mu$ due to Eq. (31).