BRAC UNIVERSITY

Inspiring Excellence

PHY111 Spring '24

Faculty : TUM

Lectures: Mid to Final

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PREFACE

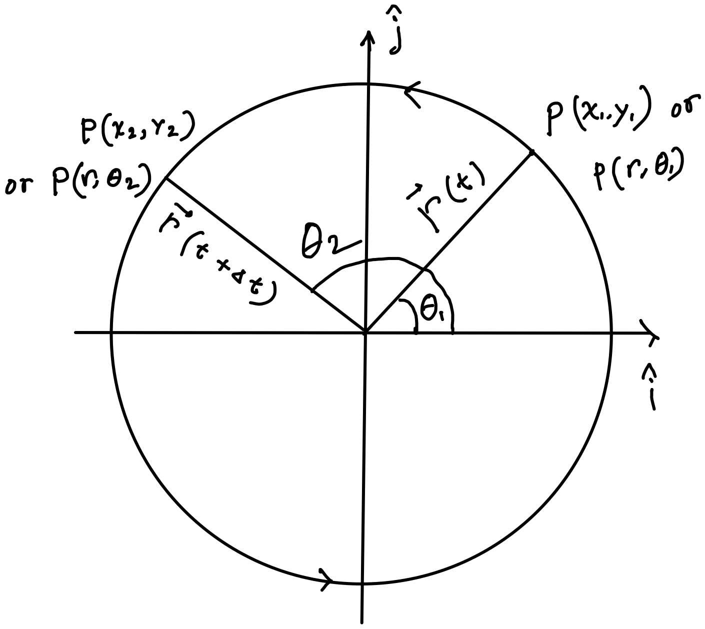
These lecture notes were taken in the course PHY111: Principles of Physics I, taught by Tushar Mitra as part of the BSc in CSE, MAT etc. program at BRAC University Spring 2024.

If you find any errors, please inform at: mh.turjoy@yahoo.com.

CIRCULAR MOTION

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{r}(t+\Delta t) = x(t+\Delta t)\,\hat{i} + y(t+\Delta t)\,\hat{j}$$



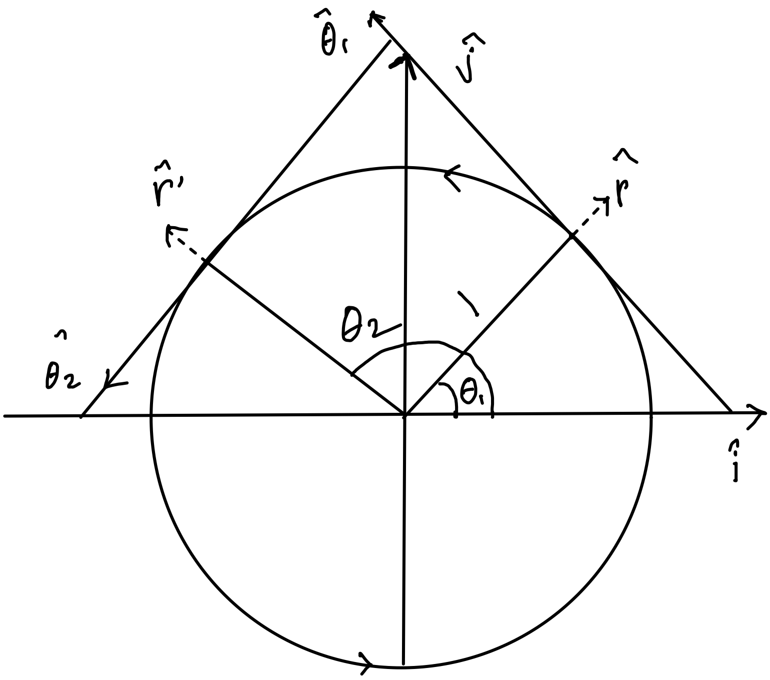
Direction of basis

$\hat{i} \longrightarrow x$ - axis

$\hat{j} \longrightarrow y$ - axis

$\hat{r} \longrightarrow$ center to outward

$\hat{\theta} \longrightarrow$ tangent

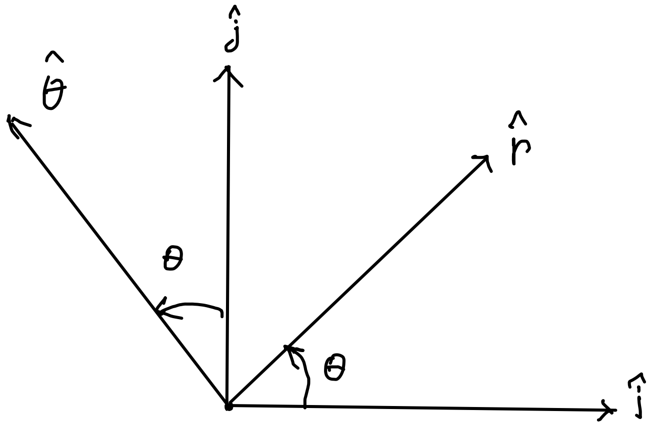


Relation between $\hat{i}, \hat{j}$ and $\hat{r}, \hat{\theta}$

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix}$$

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$



* $\theta = 0^\circ$, $\hat{r} = \hat{i}$, $\hat{\theta} = \hat{j}$

* $\theta = 90^\circ$, $\hat{r} = \hat{j}$, $\hat{\theta} = -\hat{i}$

$$\vec{r}(t) = r \cdot \hat{r}(t)$$

$$\Rightarrow \vec{r}(t) = r\left(\hat{i}\cos\theta(t) + \hat{j}\sin\theta(t)\right)$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left[r\left(\hat{i}\cos\theta(t) + \hat{j}\sin\theta(t)\right)\right]$$

$$= r\left[-\sin\theta(t)\frac{d\theta(t)}{dt}\hat{i} + \cos\theta(t)\frac{d\theta(t)}{dt}\hat{j}\right]$$

$$= r\frac{d\theta(t)}{dt}\left[-\sin\theta(t)\hat{i} + \cos\theta(t)\hat{j}\right]$$

$$= r\vec{\omega}(t)\hat{\theta}$$

Here, Angular Velocity $\vec{\omega}(t) = \dfrac{d\theta(t)}{dt}$

$$\therefore \vec{v}(t) = r\vec{\omega}(t)\hat{\theta}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left[r\vec{\omega}(t)\hat{\theta}\right]$$

$$= r\left[\frac{d}{dt}\vec{\omega}(t)\hat{\theta}\right]$$

$$= r\left[\hat{\theta}\frac{d}{dt}\vec{\omega}(t) + \vec{\omega}(t)\frac{d}{dt}\hat{\theta}\right]$$

$$= r\left[\hat{\theta}\frac{d}{dt}\vec{\omega}(t) + \vec{\omega}(t)\frac{d}{dt}\left(-\hat{i}\sin\theta(t) + \hat{j}\cos\theta(t)\right)\right]$$

$$= r\left[\hat{\theta}\,\frac{d}{dt}\vec{\omega}(t) + \vec{\omega}(t)\left(-\cos\theta(t)\,\hat{i} - \sin\theta(t)\,\hat{j}\right) \times \frac{d\theta(t)}{dt}\right]$$

$$= r.\hat{\theta}\alpha + r.\omega(t)\,[-\hat{r}(t)]\,\omega(t)$$

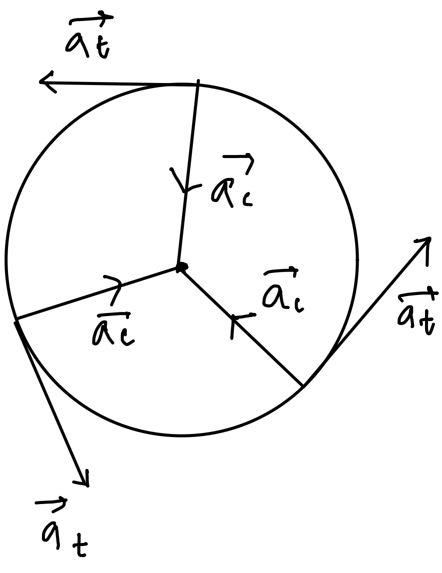
$a(t) = \vec{a_t} + \vec{a_c}$

Angular acceleration, $\alpha = \frac{d\omega}{dt}$

Centripetal acceleration, $\vec{a_c} = \omega^2 r.(-\hat{r})$

tangential acceleration, $\vec{a_t} = r\alpha\,\hat{\theta}$

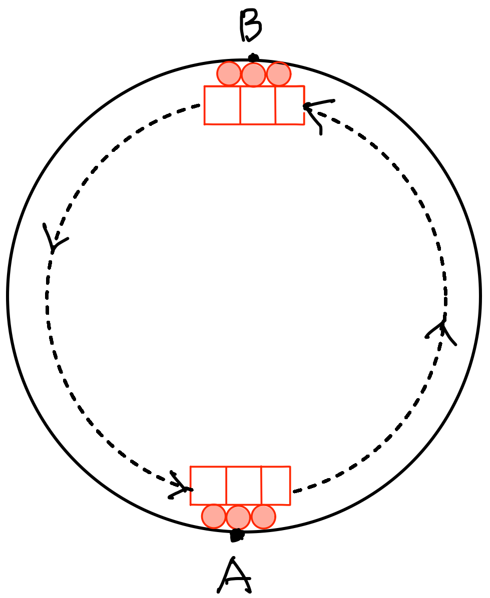
$\omega = \frac{v}{r}$ $\therefore a_c = \frac{v^2}{r}(-\hat{r})$

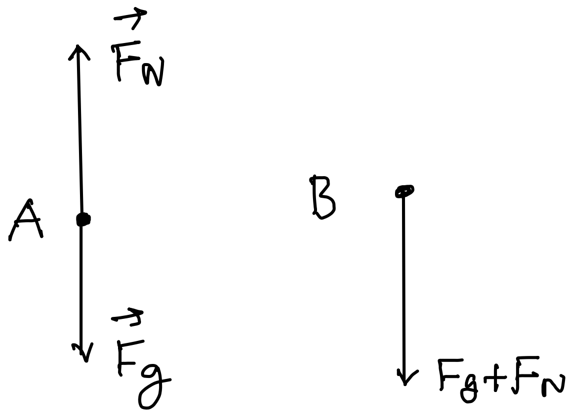


Uniform Circular motion

$\omega = \text{Constant}$, $\alpha = \frac{d\omega}{dt} = 0$

Designing a roller coaster





$$A: -F_g \hat{j} + F_N \hat{j} = m \vec{a_c}$$

$$\omega^2 R(-\hat{R})$$

$$\Rightarrow -F_g \hat{j} + F_N \hat{j} = m\omega^2 R \hat{j}$$

$$(-\hat{j})$$

$$\Rightarrow F_N = m\omega^2 R + F_g$$

$$B: -F_g \hat{j} - F_N \hat{j} = m \vec{a_c} (-\hat{j})$$

$$\Rightarrow -F_g \hat{j} - F_N \hat{j} = -m\omega^2 R \hat{j}$$

$$\Rightarrow F_N = m\omega^2 R - F_g$$

At the time of falling (ω_c)

$$F_N = 0$$

$$\Rightarrow m\omega_c^2 R - F_g = 0$$

$$\Rightarrow \omega_c^2 R - g = 0$$

$$\therefore \omega_c = \sqrt{\frac{g}{R}}$$

$$\omega_c = \frac{v_c}{R}$$

$$\frac{v_c}{R} = \sqrt{\frac{g}{R}}$$

$$\therefore v_c = \sqrt{gR}$$

KINEMATICS OF CIRCULAR MOTION

$$\alpha = \frac{d\omega}{dt}$$

$\alpha \rightarrow$ angular acceleration

$\omega \rightarrow$ angular velocity.

$$\Rightarrow \int_0^t \alpha(t)\, dt = \int_{\omega_0}^{\omega(t)} \frac{d\omega(t)}{dt}\, dt$$

$$\Rightarrow \int_0^t \alpha(t)\, dt = \int_{\omega_0}^{\omega(t)} d\omega(t)$$

$$\Rightarrow \int_0^t \alpha(t)\, dt = \omega(t) - \omega_0$$

$$\therefore \omega(t) = \omega_0 + \int_0^t \alpha(t)\, dt$$

If $\alpha = \text{constant}$, $\omega(t) = \omega_0 + \int_0^t \alpha\, dt$

$$\Rightarrow \omega(t) = \omega_0 + \alpha \int_0^t dt$$

$$\therefore \omega(t) = \omega_0 + \alpha t$$

Similar to linear velocity.

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\Rightarrow \int_0^t \omega(t)\, dt = \int_0^{\theta(t)} d\theta(t)$$

$$\Rightarrow \int_0^t \omega(t)\, dt = \theta(t) - \theta_0$$

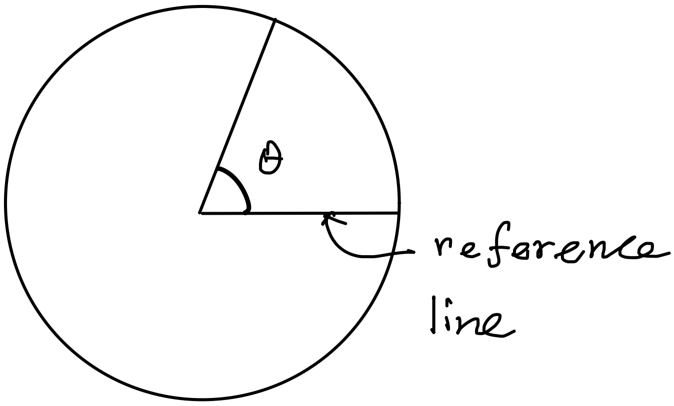
$$\therefore \theta(t) = \theta_0 + \int_0^t \omega(t)\, dt$$

Similar to $\vec{r}(t)$

if $\alpha = \text{constant}$, $\omega(t) = \omega_0 + \alpha t$

$$\theta(t) = \theta_0 + \int_0^t (\omega_0 + \alpha t)\, dt$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$



$\theta = +ve$, if counted counter clockwise

$\theta = -ve$, if counted clockwise

$\omega = +ve$, if rotation is counter clockwise

$\omega = -ve$ if rotation is clockwise.

$\alpha = +ve$, if it is increasing ω.

$\alpha = -ve$, if it is decreasing ω.

Direction of ω: Perpendicular to the plane of rotation using right hand rule.

Example The angular position of a particle was 0.5 rad w.r.t. a reference line. The angular acceleration is given by $\alpha(t) = (2-t)$ rad/s²

(i) If the particle was not moving initially, find –

(i) Angular velocity and position at time t = 3 s.

(ii) Centripetal and tangential force at t = 3 s.

(iii) How many complete rotations did the particle have during the 3 seconds.

Soln

(i)

$$\omega(t) = \omega_0 + \int_0^t \alpha(t)\,dt$$

$$= 0 + \int_0^t (2-t)\,dt$$

$$\therefore \omega(t) = \left(2t - \frac{t^2}{2}\right) \text{ rad/s}$$

$$\therefore \omega(3) = \left(2 \times 3 - \frac{3^2}{2}\right) = 1.5 \text{ rad/s}$$

$$\theta(t) = \theta_0 + \int_0^t \omega(t)\,dt$$

$$= 0.5 + \int_0^t \left(2t - \frac{t^2}{2}\right) dt$$

$$= 0.5 + \left(2 \cdot \frac{t^2}{2} - \frac{t^3}{3\times 2}\right)\Big|_0^t$$

$$\therefore \theta(t) = \left(0.5 + t^2 - \frac{t^3}{6}\right) \text{ rad}$$

$\therefore \theta(3) = 5$ rad.

$\therefore$ Displacement $= 5$ rad $- 0.5$ rad $= 4.5$ rad

(ii) number of rotations $= \dfrac{\theta(t) - \theta_0}{2\pi}$

$$= \frac{4.5}{2\pi} = 0.716$$

$\therefore$ complete rotation $= 0$.

(iii) $\vec{F_c} = m\vec{a_c} = m\omega^2 r(-\hat{r})$

$$= m(1.5)^2 r(-\hat{r})$$

$\omega(3) = 1.5$

$\vec{F_t} = m\vec{a_t} = m\alpha r\hat{\theta}$

$$= m(-1) r\hat{\theta}$$

$$= -mr\hat{\theta}$$

$\vec{a_t} = \alpha r\hat{\theta}$

$\alpha(3) = (2-3)$ rad/s

$= -1$ rad/s

$\alpha = (2-t)\ rad/s^2$

$\omega(t) = 2t - \frac{t^2}{2}\ rad/s$

$\theta(t) = \left(0.5 + t^2 - \frac{t^3}{6}\right)\ rad$

$\omega(t) = 0 \longrightarrow$ The moment when object stops.

$\Rightarrow 2t - \frac{t^2}{2} = 0$

$\therefore t = 0, 4.$

Coin on a rotating table

For the critical angular velocity,

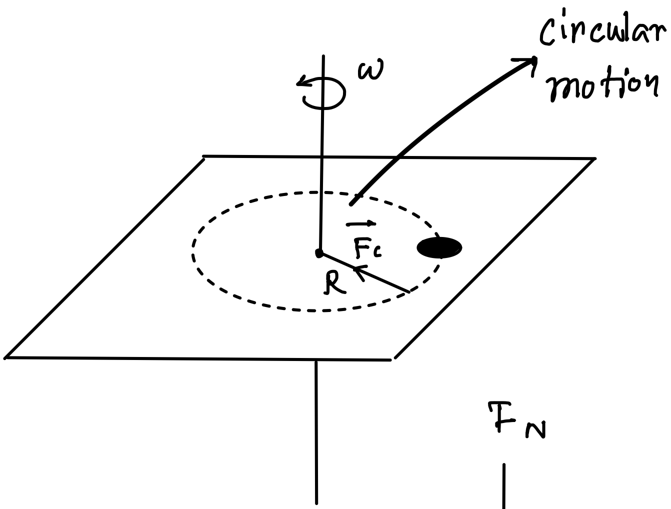
$f_{s,max} = m\omega_c^2 r$

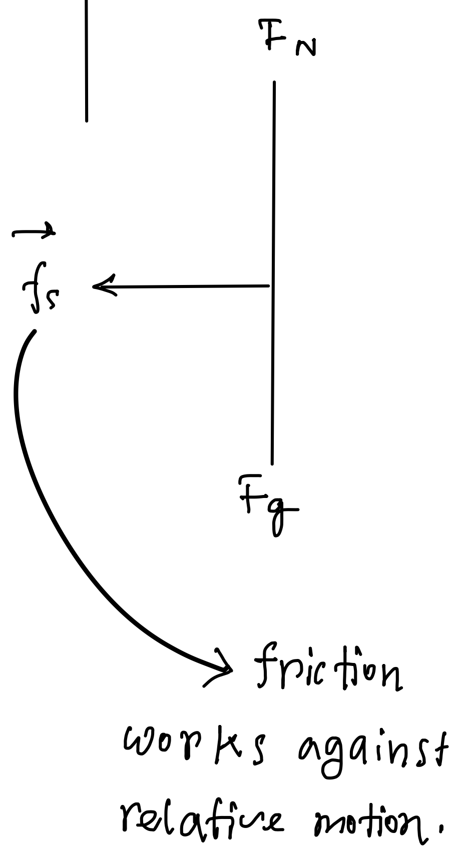
$\Rightarrow \mu_s F_N = m\omega_c^2 r$

$\Rightarrow \mu_s mg = m\omega_c^2 r$

$\therefore \omega_c = \sqrt{\frac{\mu_s mg}{mr}}$

$\therefore \omega_c \propto \frac{1}{\sqrt{r}}$



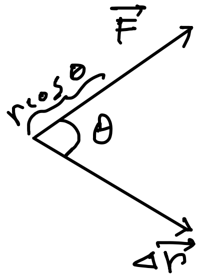


WORK AND ENERGY

Work done by constant force

$W = \vec{F} \cdot \Delta \vec{r}$

$W = F \Delta r \cos\theta$



If $0^\circ \leq \theta < 90^\circ$ $\quad W = (+)ve$

$\theta = 90^\circ$ $\quad W = 0$

$90^\circ < \theta \leq 180^\circ$ $\quad W = -ve$

Example:

$\vec{F} = (3\hat{i} + 4\hat{j} - 4\hat{k})\,N$

$\vec{r_i} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\vec{r_f} = -2\hat{i} + 7\hat{j} + 2\hat{k}$

$W = ?$

$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$

$= -5\hat{i} + 5\hat{j} + 3\hat{k}$

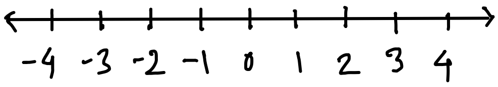
$\therefore \vec{F} \cdot \Delta\vec{r} = -15 + 20 - 12$

$= -7\hat{j}$

Work done by a variable force

1D

$\vec{F} = F(x)\hat{i}$



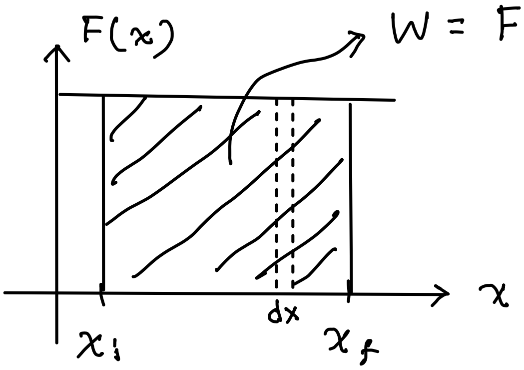
$\vec{x_i} = x_i\hat{i}$, $\vec{x}_f = x_f\hat{i}$

$W = F_x(x_f - x_i)$ $[\because \cos 0° = 1]$

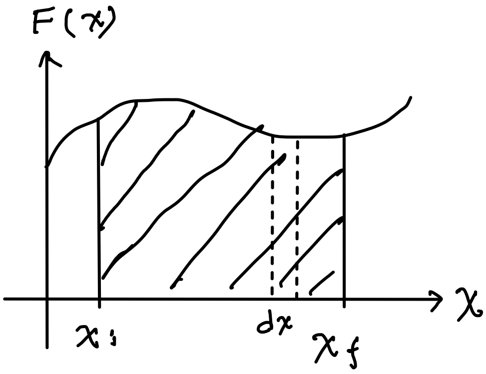
Graph

(i) For constant force, $F(x)$ vs x graph:

$W = F_x(x_f - x_i)$



In 1D work done is the area under the curve of $F(x)$ vs x.



$$W = \int_{x_i}^{x_f} \vec{F}(x) \cdot d\vec{x}$$

Example - 2 : $\vec{F}(x) = 2x\,\hat{i}$ $\quad x_i = 2\hat{i}\ ,\ x_f = 4\hat{i}$

$$W = \int_{2}^{4} 2x\hat{i} \cdot dx\hat{i}$$

$$= 2\int_{2}^{4} x\,dx = 2\left.\frac{x^2}{2}\right|_{2}^{4}$$

$$= 4^2 - 2^2 = 12\,J$$

Example - 3 : $x_i = 4\hat{i}\ ,\ x_f = 2\hat{i}$

$$W = \int_{4}^{2} 2x\hat{i} \cdot dx\hat{i}$$

$$= 2\int_{4}^{2} x\,dx = 2\left.\frac{x^2}{2}\right|_{4}^{2}$$

$$= 2^2 - 4^2 = -12\,J$$

In 2D

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r}) \cdot d\vec{r}$$

$d\vec{r} = d_x \hat{i} + d_y \hat{j}$

$\vec{F}(\vec{r}) = F_x(x,y)\hat{i} + F_y(x,y)\hat{j}$

Example :

$$\vec{F}(x,y) = xy\,\hat{i} + x^2y^2\,\hat{j}$$

$$\therefore W = \int (xy\,\hat{i} + x^2y^2\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{(x_i,y_i)}^{(x_f,y_f)} xy\,dx + \int_{(x_i,y_i)}^{(x_f,y_f)} x^2y^2\,dy$$

Kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}\cdot\vec{v})$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$$

$K \propto m$; if v is constant.

$K \propto v^2$; if m is constant.

A review of Kinematics

(i)

$$\int_{t_i}^{t_f} a_x\,dt = \int_{t_i}^{t_f} \frac{dv_x}{dt}\,dt = \int_{v_{x_i}}^{v_{x_f}} dv_x = v_{x_f} - v_{x_i}$$

(ii) $$\int_{x_i}^{x_f} a_x \, dx = \int_{x_i}^{x_f} \frac{dv_x}{dt} \, dx$$

$$= \int_{x_i}^{x_f} dv_x \, \frac{dx}{dt} = \int_{x_i}^{x_f} v_x \, dv_x$$

$$= \left. \frac{v_x^2}{2} \right|_{x_i}^{x_f} = \frac{v_{x_f}^2}{2} - \frac{v_{x_i}^2}{2} = \frac{1}{2}\left(v_{x_f}^2 - v_{x_i}^2\right)$$

$$\therefore \int_{x_i}^{x_f} a_x \, dx = \frac{1}{2}\left(v_{x_f}^2 - v_{x_i}^2\right)$$

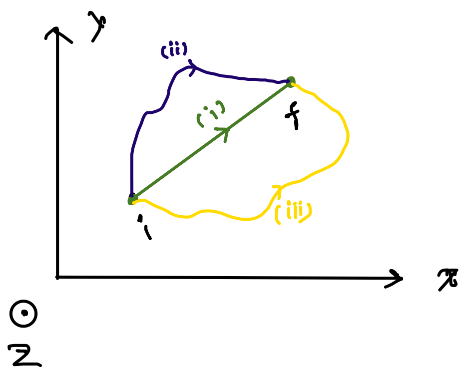
$$\Rightarrow \int_{x_i}^{x_f} m a_x \, dx = \frac{1}{2} m v_{x_f}^2 - \frac{1}{2} m v_{x_i}^2 \quad [\times m]$$

$$\Rightarrow W = K_f - K_i$$

KINETIC ENERGY-WORK THEOREM

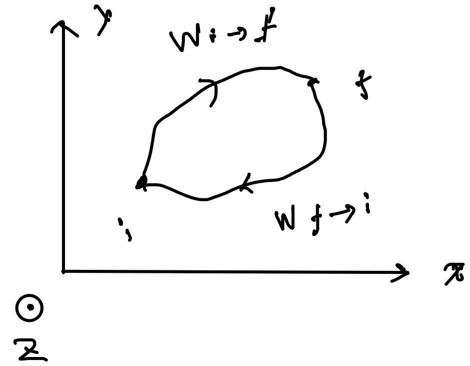
$$\sum W = \Delta K$$

Conservative Force



A force is conservative if the work done by the force does not depend on the specific path, but only depends on end points.

$$\therefore W^{(i)} = W^{(ii)} = W^{(iii)} = \cdots = W^{(\infty)} .$$



For a conservative force:

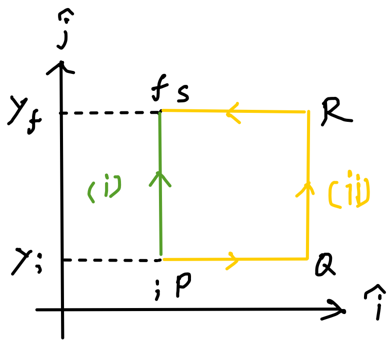
$$W_{\text{round trip}} = W_{i \to f} + W_{f \to i}$$

$$= W_{i \to f} - W_{i \to f}$$

$$= 0$$

Example of conservative forces:

(i) Gravitional force near earth:



$$\vec{F_g} = -mg\,\hat{j}$$

$P(x_i, y_i)$, $S(x_i, y_f)$

$Q(x_Q, y_i)$, $R(x_Q, y_f)$

$$W_{(i)} = \int_i^f \vec{F}\cdot d\vec{r} \qquad \Big|\ d\vec{r} = dy\,\hat{j}$$

$$= \int_{y_i}^{y_f} -mg\,\hat{j}\,dy\,\hat{j}$$

$$= -mg\int_{y_i}^{y_f} dy = -mg\,(y_f - y_i).$$

$$W_{(ii)} = \int_P^Q \vec{F_g}\cdot d\vec{r} + \int_Q^R \vec{F_g}\cdot d\vec{r} + \int_R^S \vec{F_g}\cdot d\vec{r}$$

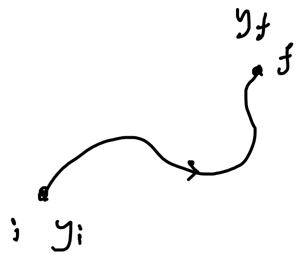
$$= -mg\,(y_f - y_i)$$

For a force to be conservative

$$\oint \vec{F}\cdot d\vec{r} = 0.$$

$W_g = \int_i^f \vec{F_g} \cdot d\vec{r}$

$$= \int_i^f -mg\,\hat{j} \cdot (dx\,\hat{i} + dy\,\hat{j})$$



$$= \int_{y_i}^{y_f} -mg\,dy$$

$$= -mg(y_f - y_i)$$

(ii) Spring Force :

$$\vec{F_s} = -K\vec{x}$$

$\vec{x}$ = displacement from the e

$$\Rightarrow \vec{F_s} = -K(x\,\hat{i})$$

$$\therefore \vec{F_s} = -Kx\,\hat{i}$$

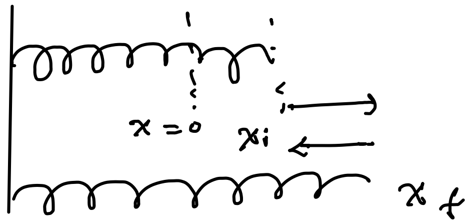
K =

$$\vec{F_s} = -K\vec{x}$$

$$\Rightarrow \vec{F_s} = -K(-x\,\hat{i})$$

$$\therefore \vec{F_s} = +Kx\,\hat{i}$$

$$W_{s_{i \to f}} = \int_i^f \vec{F_s} \cdot d\vec{r}$$



$$= \int_{x_i}^{x_f} -Kx\,\hat{i} \cdot dx\,\hat{i}$$

$$= -K \int_{x_i}^{x_f} x \, dx$$

$$= -K \cdot \frac{x^2}{2} \Big|_{x_i}^{x_f} = -\frac{K}{2} \left(x_f^2 - x_i^2 \right)$$

$$= -\left(\frac{1}{2} K x_f^2 - \frac{1}{2} K x_i^2 \right) = \frac{1}{2} K \left(x_f^2 - x_i^2 \right)$$

$$W_{s, f \to i} = \int_i^f \vec{F_s} \cdot d\vec{r}$$

$$= \int_i^f -K x \hat{i} \cdot dx \hat{i}$$

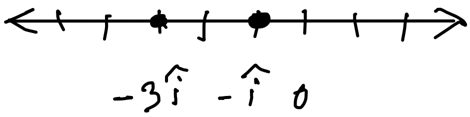
$$= -K \int_{x_f}^{x_i} x \, dx$$

$$= -K \frac{x^2}{2} \Big|_{x_f}^{x_i}$$

$$= -\frac{K}{2} \left(x_i^2 - x_f^2 \right)$$

* A spring of spring constant K moves from $-3\hat{i}$ to $-1\hat{i}$. Find the work done by the spring force.

$$W_{s\,i\to f} = \int_i^f \vec{F}_s \cdot d\vec{r}$$



$$= \int_i^f -K\vec{x} \cdot dx\hat{i}$$

$$= \int_{-3}^{-1} -Kx\hat{i} \cdot dx\hat{i}$$

$$= \int_{-3}^{-1} -Kx\hat{i} \cdot dx\hat{i}$$

$$= -K \frac{x^2}{2}\Big|_{-3}^{-1}$$

$$= -\frac{K}{2}\{(-1)^2 - (-3)^2\}$$

$$= -\frac{K}{2}(1-9)$$

$$= 4K.$$

Idea of potential energy

For any conservative force,

$$W_c = -\Delta U$$

$\rightarrow$ U has a unit of Joule

$\rightarrow$ U is some form of energy.

Gravitational potential energy

$$W_g = -\Delta U$$

$$\Rightarrow -mg\,y_f + mg\,y_i = -(U_f - U_i)$$

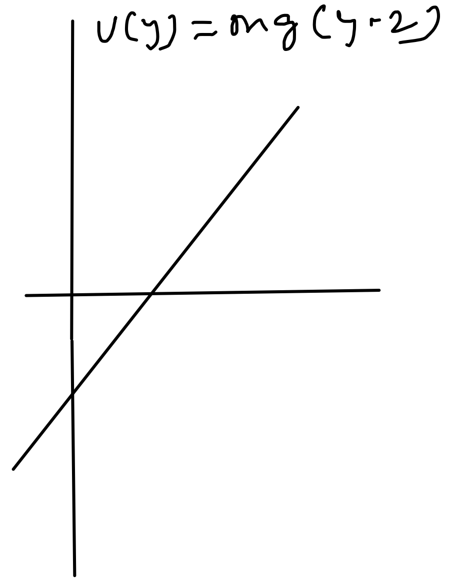
$$\therefore U_i - U_f = mg\,y_i - mg\,y_f$$

Say at $y_i = 0$, $U_i = 0$

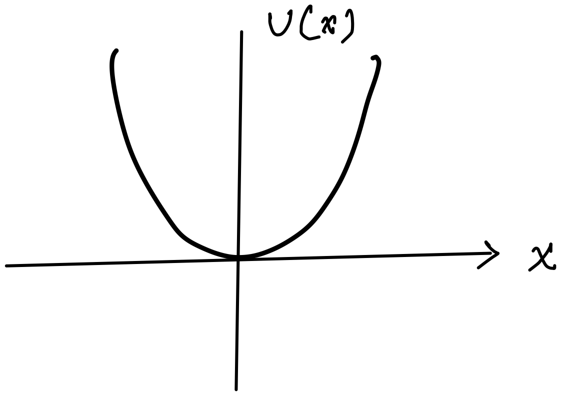
ref. point

$\therefore U_f = -mg\,y_f$ $\quad \therefore U(y_f) = -mg\,y_f$ $\quad \therefore U(y) = mgy$

$y_i = 2$, $U_i = 0$



$U_s = \frac{1}{2} K x^2$ | ref point $x = 0$



$W_c = -\Delta U$

$\Rightarrow W_c = -(U_f - U_i)$

$\Rightarrow K_f - K_i = -U_f + U_i$

$\Rightarrow U_i + K_i = U_f + K_f$

Mechanical energy

$E^{mech} = U + K$

$$\boxed{E_i^{mech} = E_f^{mech}}$$

theorem - 2

Conservation of mechanical energy'

If there are non conservative force as well

$\Sigma W = \Delta K$

$\Rightarrow W_c + W_{nc} = \Delta K$

$\Rightarrow -\Delta U + W_{nc} = \Delta K$

$\Rightarrow W_{nc} = \Delta K + \Delta U = \Delta (U + K)$

$\therefore \boxed{W_{nc} = \Delta E^{mech}}$ theorem - 3

Theorem:

1. Kinetic energy - work theorem : $\sum W = \Delta K$

2. Conservation of mechanical energy : $E_i^{mech} = E_f^{mech}$

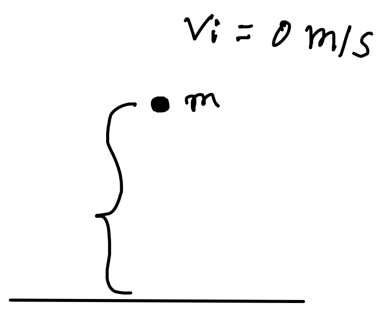
3. Non-conservative work theorem : $\sum W_{nc} = \Delta E^{mech}$

1 → applies for both conservative and non-c. forces.

2 → only applies for conservative forces.

3 → Do not apply for only conservative forces.

Problem-1:



Work energy approach :

Available forces → F_g
↓
conservative force

$$E_i^{mech} = E_f^{mech}$$

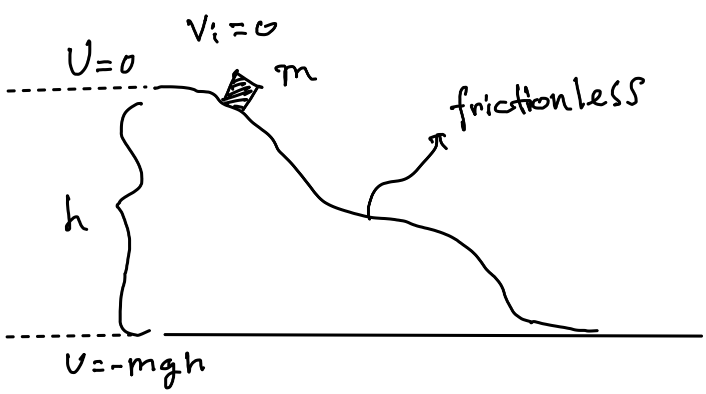
$$\Rightarrow K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 + mgh = \frac{1}{2} m v_f^2 + 0$$

$$\therefore v_f = \sqrt{2gh}$$

Problem-2:

Find the final speed when the block reaches the ground.



Soln: Forces $\rightarrow F_g, F_N \rightarrow$ could be avoided.
$\hookrightarrow$ conservative force

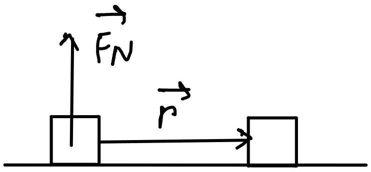
$$E_i^{mech} = E_f^{mech}$$

$$\Rightarrow U_i + K_i = U_f + K_f$$

$$\Rightarrow 0 + 0 = -mgh + \frac{1}{2} m v_f^2$$

$$\therefore v_f = \sqrt{2gh}$$

Work done by normal reaction force:



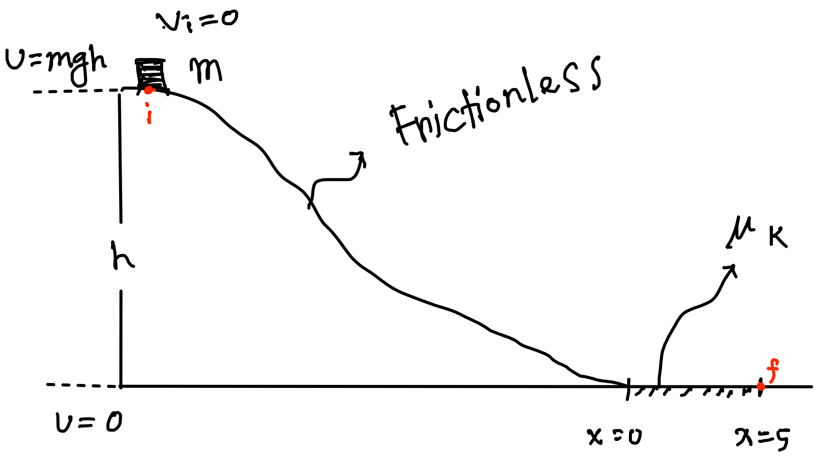
$$W_{F_N} = \int \vec{F_N} \cdot d\vec{r}$$

$$= \int F_N \hat{j} \cdot dx \hat{i}$$

$$= 0$$

Problem - 3

Find the speed of the particle after it crosses the friction path.



$$\vec{F_g} \rightarrow \hat{i}$$

Soln:

Forces $\rightarrow F_g, F_N, f_K$

$$\vec{f_K} = \mu_K F_N (-\hat{i})$$

$$= \mu_K mg (-\hat{i})$$

$$\sum W = \int_{x=0}^{x=5} \vec{f_K}\cdot dx\,\hat{i} = \frac{1}{2}mv_f^2 - mgh$$

$$\Rightarrow \int_{x=0}^{x=5} \mu_K mg(-\hat{i})\cdot dx\,\hat{i} = \frac{1}{2}mv_f^2 - mgh$$

$$\Rightarrow -\mu_K mg \int_{x=0}^{x=5} dx = \frac{1}{2}mv_f^2 - mgh$$

$$\Rightarrow -\mu_K \cancel{m}g\ 5 = \frac{1}{2}\cancel{m}v_f^2 - \cancel{m}gh$$

$$\therefore v_f = \sqrt{2gh - 10\mu_K g}$$

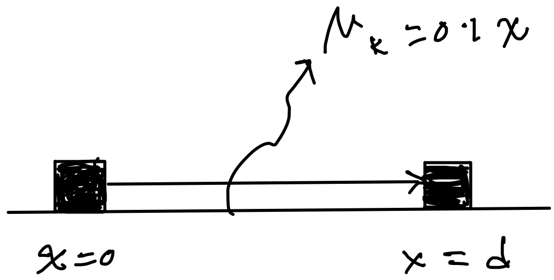
The condition the object moves past the friction path is:

$$v_f \geqslant 0$$

$$\Rightarrow 2gh - 10\mu_K g \geqslant 0$$

$$\therefore h > 5\mu_K$$

Problem - 4



$W_{\mu_k} = ?$

Sol^n:

$$W_{\mu_k} = \int_0^d mg\,\mu_k(-\hat{i}) \cdot dx\,\hat{i}$$

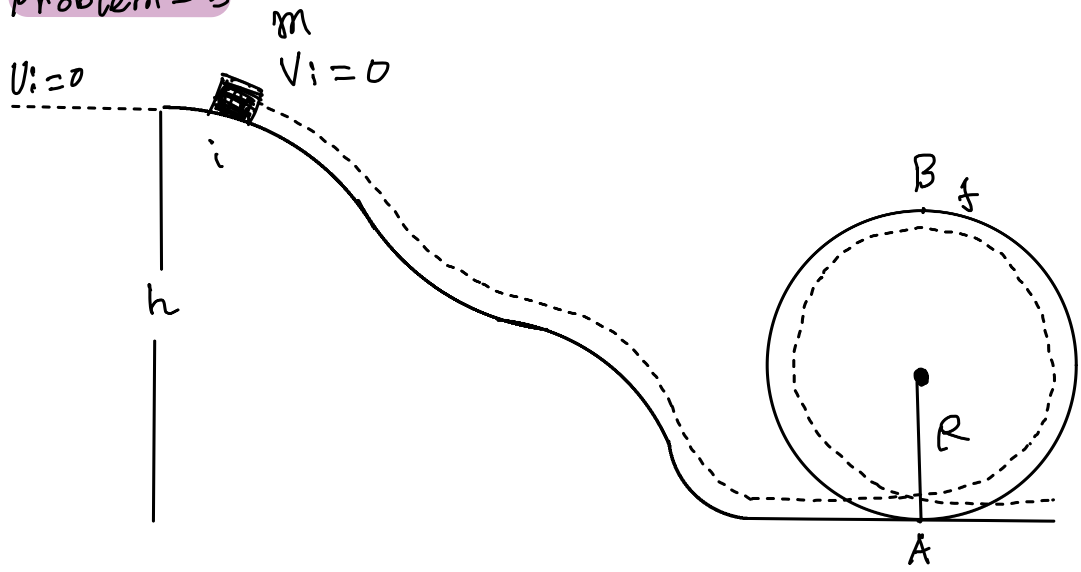
$$= \int_0^d -mg\;0.1x\;\hat{i} \cdot dx\,\hat{i}$$

$$= -mg(0.1)\int_0^d x \cdot dx$$

$$= -mg(0.1) \cdot \left(\frac{x^2}{2} \Big|_0^d \right)$$

$$= -mg\;0.1\;\frac{d^2}{2}$$

Problem - 5



Find the minimum height so that the block will complete a loop.

Soln: minimum speed required at point B is:

$$v_{min} = \sqrt{gR}$$

$$E_i^{mech} = E_f^{mech}$$

$$\Rightarrow U_i + K_i = U_f + K_f$$

$$\Rightarrow 0 + 0 = mg(h_{min} - 2R) + \frac{1}{2} m v_{min}^2$$

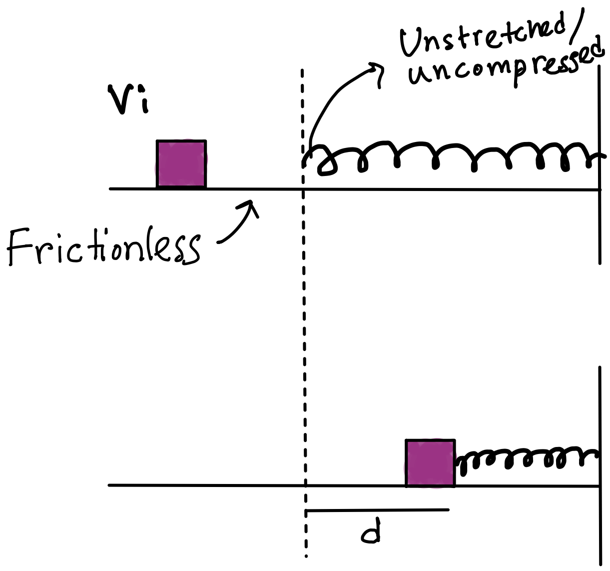
$$\Rightarrow 0 = gh_{min} - 2gR - \frac{1}{2} gR$$

$$\Rightarrow 0 = h_{min} - 2R - \frac{1}{2} R$$

$$\therefore h_{min} = \frac{5R}{2}$$

Problem - 6 :

Find the distance d by which the block will compress the spring before coming to rest.



Soln

$$E_i^{mech} = E_f^{mech}$$

$$\Rightarrow U_{i,s} + K_i = U_{f,s} + K_f$$

$$\Rightarrow 0 + \frac{1}{2} m v_i^2 = \frac{1}{2} k d^2 + 0$$

$$\therefore d = \sqrt{\frac{m v_i^2}{k}}$$

CENTER OF MASS

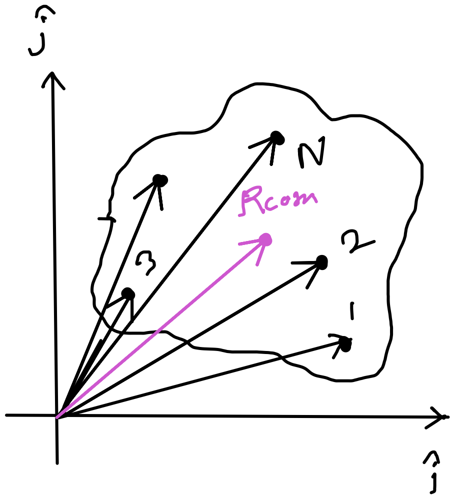
mass of particle 1 = m_1

mass of particle 2 = m_2

mass of particle 3 = m_3

$\vdots \qquad \vdots$

mass of particle N = m_N



Center of mass. $\vec{R}_{com} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots m_N\vec{r}_N}{m_1 + m_2 + \dots + m_n}$

$$\boxed{\vec{R}_{com} = \frac{\sum_{1}^{N} m_i \vec{r}_i}{\sum_{1}^{N} m_i}}$$

$\vec{R}_{com} = \vec{x}_{com} + \vec{y}_{com} + \vec{z}_{com}$

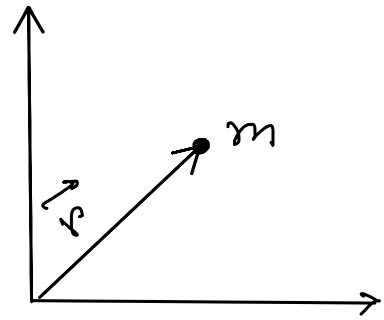
$$x_{com} = \frac{m_1x_1 + m_2x_2 + \dots + m_Nx_N}{m_1 + m_2 + \dots + m_N}$$

$$y_{com} = \frac{m_1y_1 + m_2y_2 + \dots + m_Ny_N}{m_1 + m_2 + \dots + m_N}$$

$$z_{com} = \frac{m_1y_1 + m_2y_2 + \dots + m_Ny_N}{m_1 + m_2 + \dots + m_N}$$

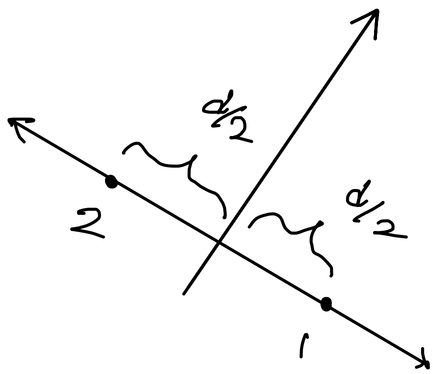
1 Particle System:

$$\vec{R}_{com} = \frac{m\vec{r}}{m} = \vec{r}$$



2 Particle System:

$$x_{com} = \frac{m_1 \frac{d}{2} - m_2 \frac{d}{2}}{m_1 + m_2}$$



$$y_{com} = \frac{m_1 0 + m_2 0}{m_1 + m_2}$$

$$= 0$$

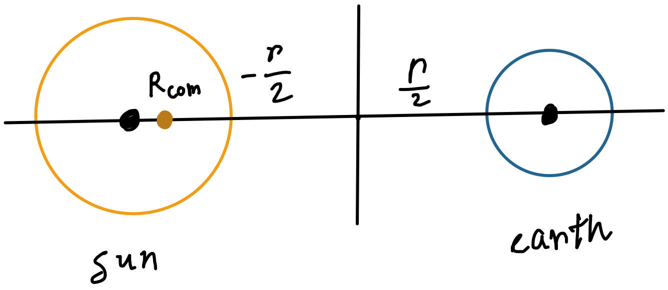
$$\therefore \vec{R}_{com} = \frac{\left(m_1 \frac{d}{2} - m_2 \frac{d}{2}\right)}{m_1 + m_2}\hat{i} + 0\hat{j}$$

$$= \frac{(m_1 - m_2)}{m_1 + m_2}\frac{d}{2}\hat{i} + 0\hat{j}$$

If $m_1 = m_2$, $\vec{R}_{com} = 0\hat{i} + 0\hat{j}$

If $m_1 >> m_2$, $\vec{R}_{com} \approx \frac{d}{2}\hat{i} + 0\hat{j}$

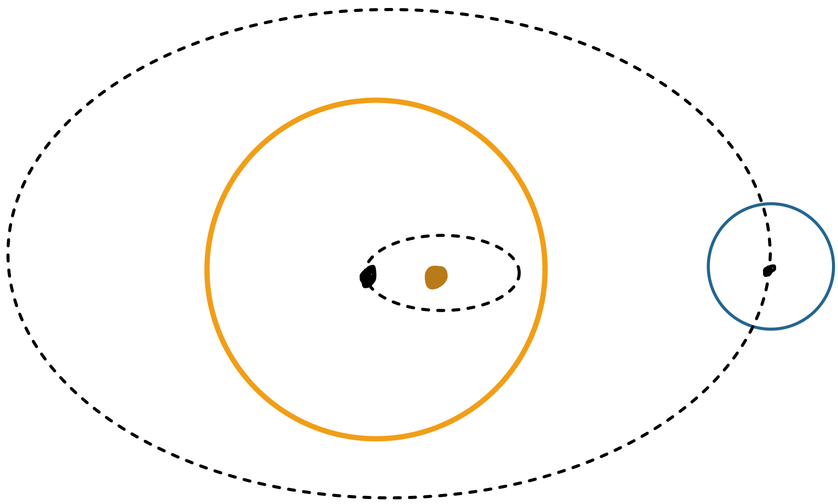
Sun and earth's center of mass



$m_s >> m_e$,

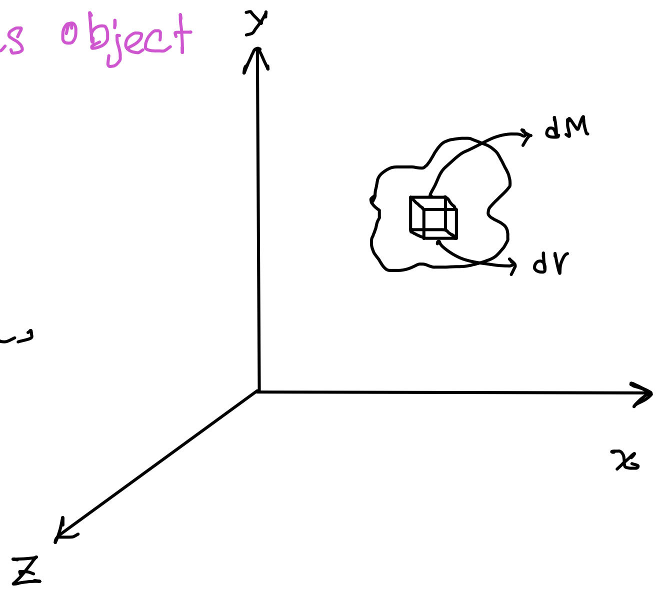
$\vec{R}_{com} \approx -\frac{r}{2}\hat{i}$.

Two objects rotate around their center of mass.



One planet system

Center of mass for continuous object



Density, $\rho = \dfrac{dM}{dV}$

If the object is uniform,

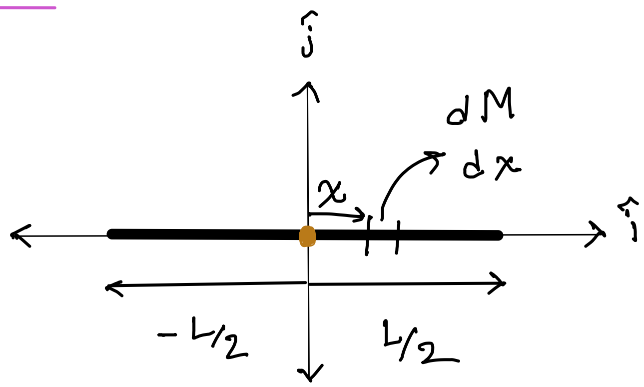
$$\rho = \frac{M}{V}$$

$$\vec{R}_{com} = \frac{\int \vec{r} \cdot dM}{\int dM}$$

$$\vec{R}_{com} = \frac{\sum r_i m_i}{\sum m_i}$$

$$\therefore \vec{R}_{com} = \frac{\int_{\text{whole obj}} \vec{r} \cdot dM}{M}$$

C.O.M. of an uniform 1D rod :



Density, $\lambda = \dfrac{M}{L}$

$$\vec{R}_{com} = \frac{\int_{-L/2}^{L/2} x \cdot dM \,\hat{i}}{M}$$

$$= \frac{\int_{-L/2}^{L/2} x \cdot \lambda \, dx}{M} \,\hat{i}$$

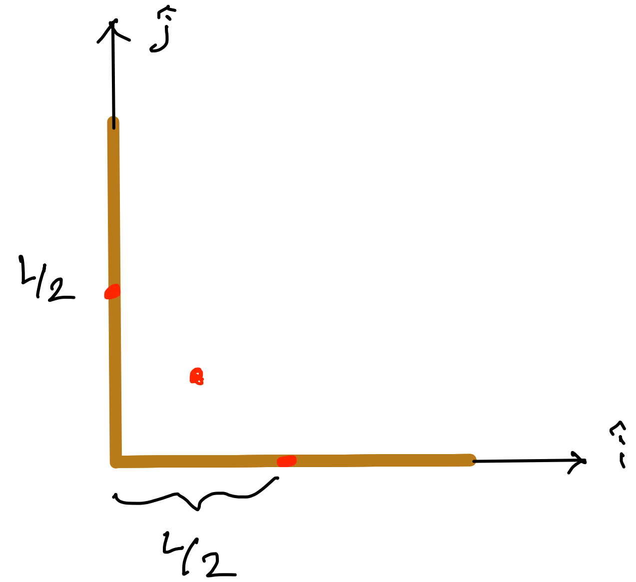
$$\lambda = \frac{dM}{dx}$$

$$= \frac{\lambda \left. \frac{x^2}{2} \right|_{-L/2}^{L/2}}{M} \hat{i}$$

$$= \frac{\lambda/2 \left[(L/2)^2 - (-L/2)^2 \right]}{M}$$

$$= 0 \hat{i}$$

Find the center of mass of the following object.



Sol^n

C.O.M. of horizontal rod, $\vec{R}_{com,h} = \frac{L}{2}\hat{i} + 0\hat{j}$

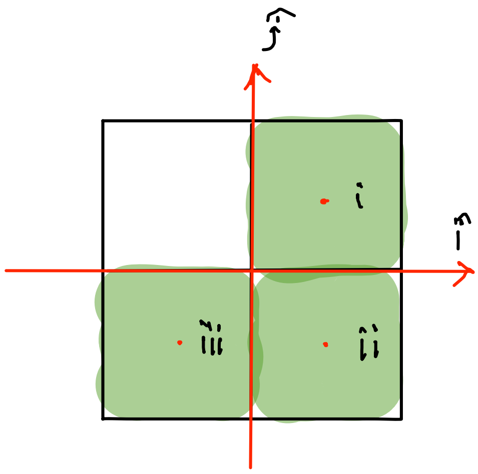
C.O.M. of verticle rod, $\vec{R}_{com,v} = 0\hat{i} + \frac{L}{2}\hat{j}$

$$\therefore \vec{R}_{com} = \frac{M_h \frac{L}{2}\hat{i} + M_v \frac{L}{2}\hat{j}}{M_h + M_v}$$

$$= \frac{M L/2 (\hat{i} + \hat{j})}{2M}$$ (since mass are same)

$$= \frac{L}{4}\hat{i} + \frac{L}{4}\hat{j}$$

Example Find the center of mass of the object.



Each side of the small square is = a unit and mass is = M

1st approach

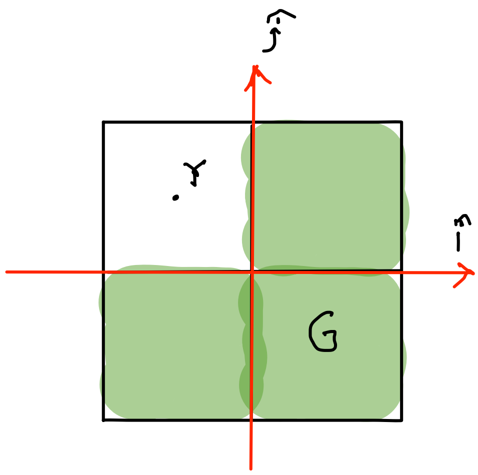
$$\vec{r_i} = \frac{a}{2}\hat{i} + \frac{a}{2}\hat{j}, \quad \vec{r_{ii}} = \frac{a}{2}\hat{i} - \frac{a}{2}\hat{j}, \quad \vec{r_{iii}} = -\frac{a}{2}\hat{i} - \frac{a}{2}\hat{j}$$

$$\vec{R}_{com} = \frac{M\vec{r_i} + M\vec{r_{ii}} + M\vec{r_{iii}}}{3M}$$

$$= \frac{\cancel{M}\left(\frac{a}{2}\hat{i} - \frac{a}{2}\hat{j}\right)}{3\cancel{M}}$$

$$= \frac{a}{6}\hat{i} - \frac{a}{6}\hat{j}$$

2nd approach



$$\vec{R}_{tot} = \frac{M_r\vec{r_r} + M_G \cdot \vec{r_G}}{M_r + M_G}$$

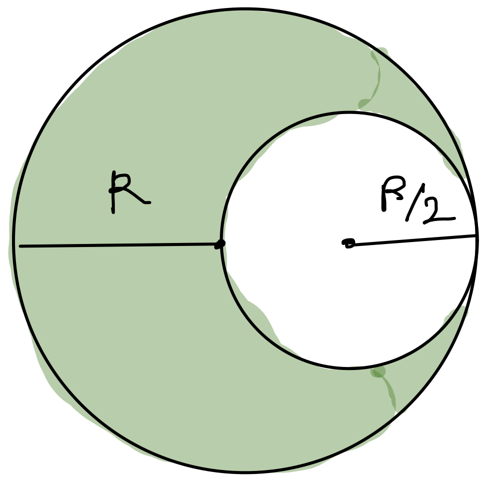
$$= \frac{M\left(-\frac{a}{2}\hat{i} + \frac{a}{2}\hat{j}\right) + 3M\,\vec{r_G}}{M + 3M}$$

$$\Rightarrow 0\hat{i} + 0\hat{j} = \frac{-\frac{a}{2}\hat{i} + \frac{a}{2}\hat{j} + 3\vec{r}_G}{4}$$

$$\Rightarrow \frac{a}{2}\hat{i} - \frac{a}{2}\hat{j} = 3\vec{r}_G$$

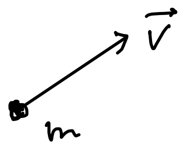
$$\Rightarrow \therefore \vec{r}_G = \frac{a}{6}\hat{i} - \frac{a}{6}\hat{j}$$

Example Find the C.O.M of the Green part.



MOMENTUM

Momentum, $\vec{p} = m\vec{v}$



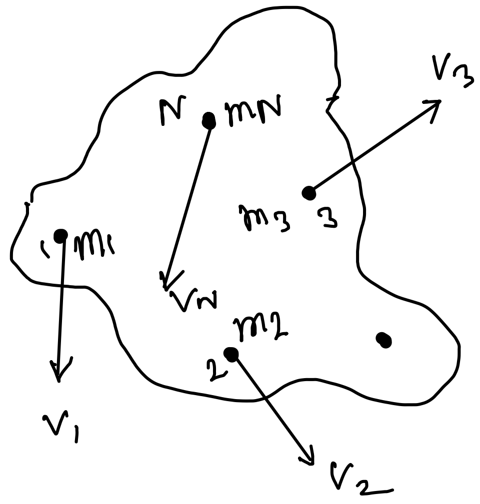
$$\vec{P}_{sys} = \vec{P}_1 + \vec{P}_2 + \dots + \vec{P}_N$$

$$= m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_N\vec{v}_N$$

$$= m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \dots + m_N\frac{d\vec{r}_N}{dt}$$

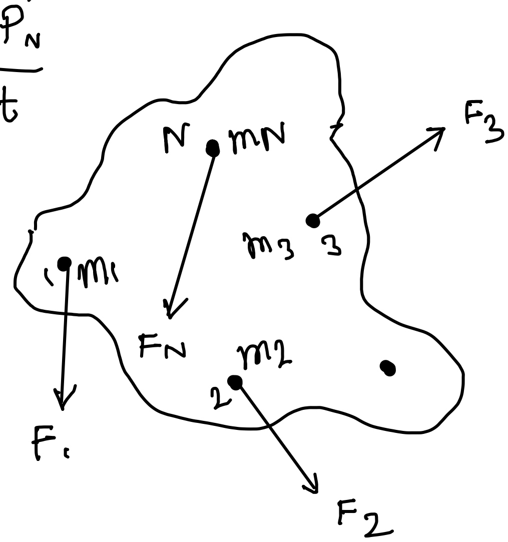
$$= \frac{d}{dt}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_N\vec{r}_N\right)$$

$$= \frac{d}{dt}\left(\vec{R}_{com} \times M\right)$$



System

$$F_1 = \frac{d\vec{P}_1}{dt}, \quad F_2 = \frac{d\vec{P}_2}{dt} \quad \dots, \quad F_N = \frac{d\vec{P}_N}{dt}$$



$$F_1 + F_2 + \dots + F_N = \frac{d}{dt}\left(\vec{P}_1 + \vec{P}_2 + \dots + \vec{P}_3\right)$$

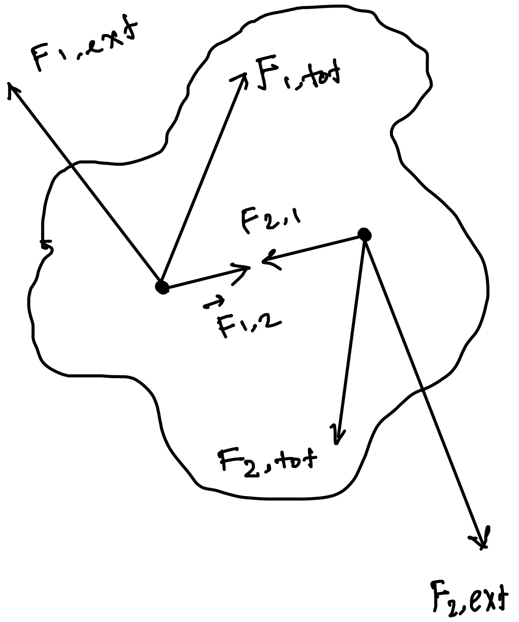
$$\therefore \vec{F}_{tot,sys} = \frac{d\vec{P}_{sys}}{dt}$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(\vec{R}_{coM} \times M\right)\right)$$

$$= M\frac{d^2}{dt^2}\left(\vec{R}_{coM}\right)$$

$$\boxed{\vec{F}_{tot,sys} = M\vec{a}_{com}}$$

Conservation of Linear Momentum



$$\vec{F}_{1,tot} = \vec{F}_{1,ext} + \vec{F}_{1,2}$$

$$\vec{F}_{2,tot} = \vec{F}_{2,ext} + \vec{F}_{2,1}$$

$$\vec{F}_{1,tot} + \vec{F}_{2,tot} = \vec{F}_{1,ext} + \vec{F}_{1,2} + \vec{F}_{2,ext} + \vec{F}_{2,1}$$

$$= \vec{F}_{1,ext} + \vec{F}_{2,ext}$$

$$\vec{F}_{tot,sys} = \vec{F}_{tot,ext}$$

$$\vec{F}_{tot,sys} = \frac{d\vec{P}_{sys}}{dt}$$

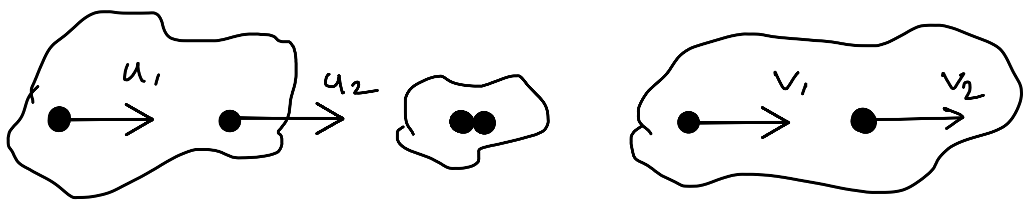
$$\Rightarrow \vec{F}_{tot,ext} = \frac{d\vec{P}_{sys}}{dt}$$

So, if $\vec{F}_{tot,ext} = 0$, $\frac{d\vec{P}_{sys}}{dt} = 0$

$$\therefore \vec{P}_{sys} = \text{constant}$$

If the external force on a system is zero, then the momentum of the system is conserved.

$U_1 > U_2$



$\vec{F}_{tot,ext} = 0$, $\vec{P}_{sys} = 0$, $\vec{P}_{sys,i} = \vec{P}_{sys,f}$

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

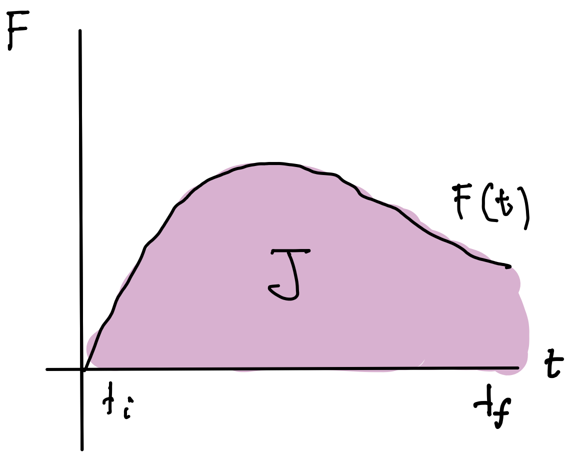
IMPULSE

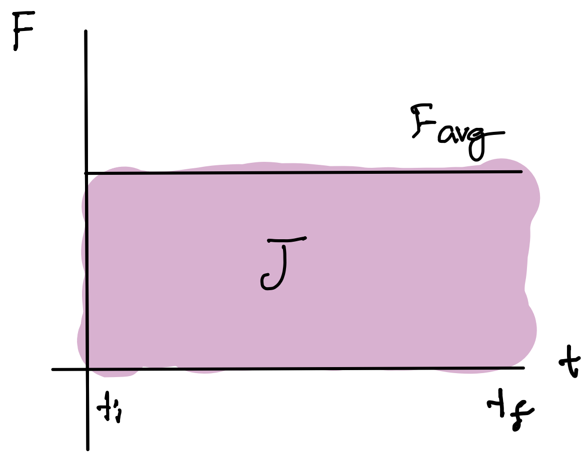
$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \int_{p_i}^{p_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}\, dt$$

$$\Rightarrow \vec{p_f} - \vec{p_i} = \int_{t_i}^{t_f} \vec{F}\, dt$$

$$\therefore \boxed{\text{impulse, } \vec{J} = \Delta \vec{P} = \int_{t_i}^{t_f} \vec{F}\, dt}$$

1D





$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\, dt$$

$$\vec{J} = \vec{F}_{avg}(t_f - t_i)$$
$$= \vec{F}_{avg}\, \Delta t$$

$$\vec{F}_{avg} = \frac{\vec{J}}{\Delta t} = \frac{\int_{t_i}^{t_f} \vec{F}\, dt}{\Delta t}$$

Example $\vec{v}(t) = 2t\,\hat{i}$, $m = 5\,kg$.

Find the impulse between $t = 2s$ and $t = 5s$.

Sol^n.1: $\vec{J} = P_f - P_i = m\vec{v}(5) - m\vec{v}(2)$

$$= m\left(2\times 5\,\hat{i} - 2\times 2\,\hat{i}\right)$$

$$= 5\left(10\hat{i} - 4\hat{i}\right)$$

$$= 30\ kg\,m/s\ \hat{i}$$

Sol^n.2 $\vec{J} = \int_2^5 \vec{F}\,dt = \int_2^5 m\vec{a}\,dt = \int_2^5 m\frac{d\vec{v}}{dt}\,dt$

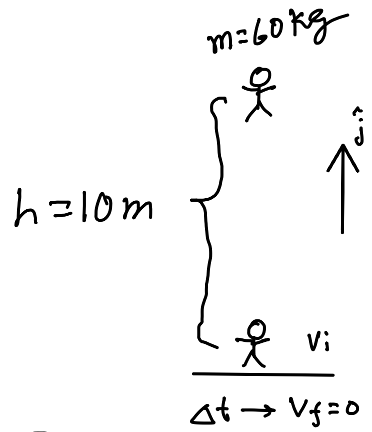
$$= m\int_2^5 \frac{d}{dt}(2t\,\hat{i})\cdot dt = m\int_2^5 2\cdot dt$$

$$= 5\times 2\,\hat{i}\,(5-2) = 30\,\hat{i}$$

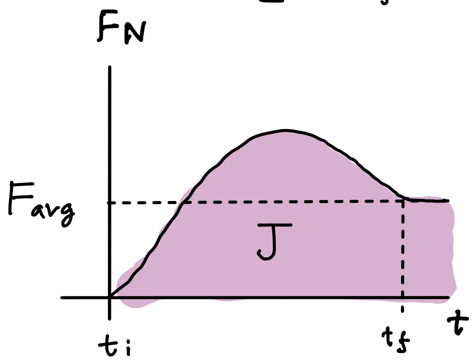
Example $v_i = \sqrt{2gh} = \sqrt{60\times10\times9.8}$

$= 14\ m/s$

$v_f = 0$.



At the time of hitting the ground, $F_N = 0$. After hitting the ground $F_N > mg$.



Impulse, $\vec{J} = m(v_f - v_i)$

$= 60 \times (0 - (-14\hat{j}))$

$= 840\ \hat{j}$

$\therefore \vec{F}_{avg} = \dfrac{840\hat{j}}{\Delta t}$

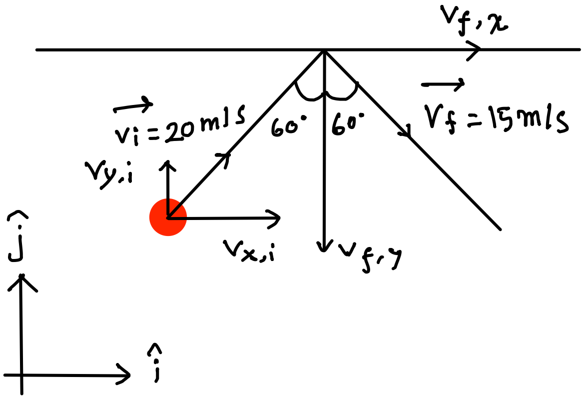
(i) On hard surface, $\Delta t = 0.05\ s$,

$\therefore \vec{F}_{avg} = 16800\ \hat{j}\ N$

(ii) On matrass, $\Delta t = 0.5\ s$

$\therefore F_{avg} = 1680\ N$

Example Find the impulse.
If the collision time is 0.01 s, find average force.



Soln

$\therefore \vec{J} = m[\vec{v}_f - \vec{v}_i]$

$= m[-20\hat{j}]$

$\therefore \vec{F}_{avg} = \dfrac{\vec{J}}{\Delta t}$

$= \dfrac{-20m\hat{j}}{0.01}$

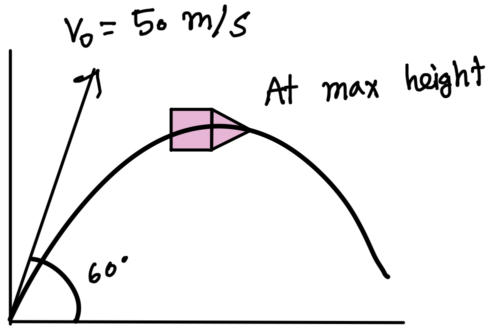
$\vec{v}_i = v_i \cos 30°\,\hat{i} + v_i \sin 30°\,\hat{j}$

$= 17.32\,\hat{i} + 10\,\hat{j}$

$\vec{v}_f = v_f \sin 60°\,\hat{i} - v_f \cos 60°\,\hat{j}$

$= 17.32\,\hat{i} - 10\,\hat{j}$

Example Find the velocity of ② immedietely after collision.



Soln At collision,

$\square \rightarrow \vec{v}_1 = 0\hat{i} + 0\hat{j}$
$\rightarrow m_1 = 1\,kg$

$\triangle \rightarrow m_2 = 3\,kg$
$\rightarrow \vec{v}_2 = 25\hat{i} + 0\hat{j}$

$P_{i,sys} = (m_1 + m_2)\vec{v}$

$= (1+3)(25\hat{i} + 0\hat{j})$

$= 100\,\hat{i}$

At max height

$\vec{y} = 0\,\hat{j}$

$\vec{x} = V_0 \cos\theta_0$

$= 50 \cos 60°$

$= 25\,\hat{i}$

$$P_{f,sys} = m_1 \vec{v_1} + m_2 \vec{v_2}$$

$$= \vec{0} + m_2 V_{2,x}\,\hat{i} + m_2 V_{2,y}\,\hat{j}$$

$$P_{i,sys,x} = P_{f,sys,x}$$

$$\Rightarrow 100\,\hat{i} = m_2 V_{2,x}\,\hat{i}$$

$$\therefore V_{2,x} = \frac{100}{m_2} = \frac{100}{3} = 33.33$$

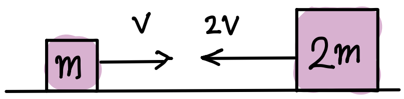
$$P_{i,sys,y} = P_{f,sys,y}$$

$$\Rightarrow 0\,\hat{j} = m_2 V_{2,y}\,\hat{j}$$

$$\therefore V_{2,y} = 0$$

$$\therefore V_2 = 33.33\,\hat{i} + 0\,\hat{j}$$

Example After collision two objects are glued together. Find the final velocity.



Sol^n

$$P_{i,sys} = mV\hat{i} + 2m\,2V(-\hat{i})$$

$$= -3mv\,\hat{i}$$

$$P_{f,sys} = (m+2m)\vec{V_f}$$

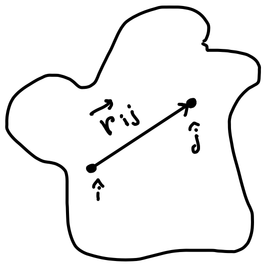
$$= 3m\vec{V_f}$$

$$\therefore P_{i,sys} = P_{f,sys} \Rightarrow -3mv\,\hat{i} = 3m\vec{V_f} \qquad \therefore V_f = V(-\hat{i})$$

Rotational Dynamics

Aka Rigid Body Rotation

Rigid body $\longrightarrow$

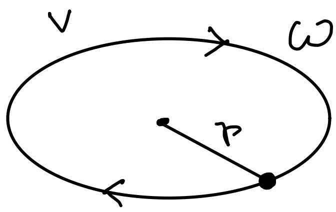


$|\vec{r}_{ij}|$ = The distance between ith and jth particle.

In a body, if $|\vec{r}_{ij}|$ is constant for all i and j, then it is called a rigid body.

Kinetic Energy of Rotation.

$K = \frac{1}{2} m v^2 \quad | \ v = \omega r$



$= \frac{1}{2} m \omega^2 r^2$

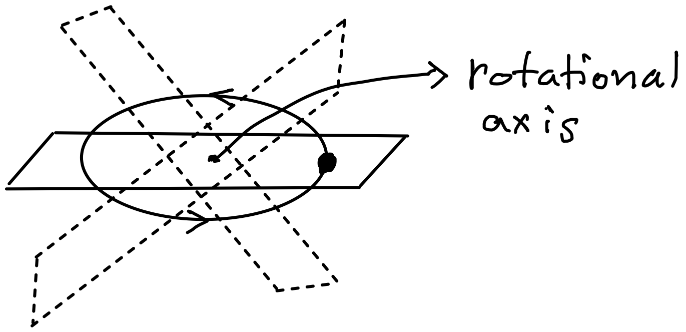
$= \frac{1}{2} (m \omega^2) r^2$

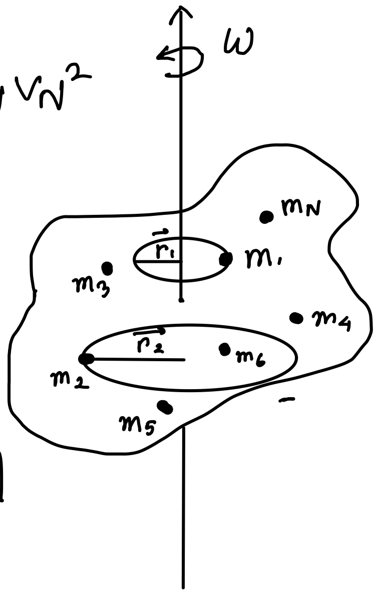
$\frac{1}{2} m v^2$ → Inertia term for translation

$\frac{1}{2} (m \omega^2) r^2$ → Inertia term for rotation →

$$\boxed{\text{Moment of Inertia, } I = m \omega^2}$$

In case of a bar





$$K_{rot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots + \frac{1}{2} m_N v_N^2$$

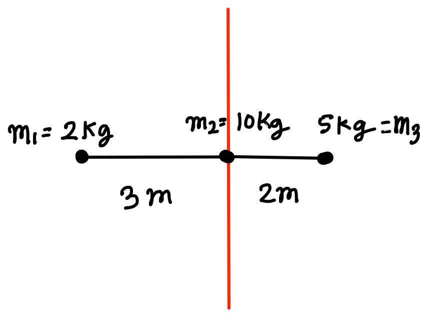
$$= \frac{1}{2} m_1 (\omega r_1)^2 + \frac{1}{2} m_2 (\omega r_2)^2 + \dots + \frac{1}{2} m_N (\omega r_N)^2$$

$$= \frac{1}{2} \omega^2 \left[m_1 r_1^2 + m_2 r_2^2 + \dots + m_N r_N^2 \right]$$

$$= \frac{1}{2} \omega^2 \sum_{i=1}^{N} m_i r_i^2$$

$$\boxed{\therefore I = \sum_{i=1}^{N} m_i r_i^2} \leftarrow \text{discrete case}$$

Example



$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

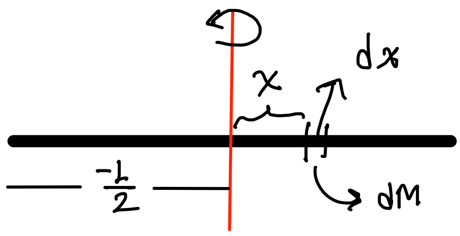
$$= 2 \times 3^2 + 10 \times 0^2 + 5 \times 2^2$$

$$= 58 \ \text{kg m}^2$$

For continuous case:

$$\boxed{I = \int_{\text{whole object}} r^2 \, dm}$$

Moment of inertia for an 1D rod



$$I = \int_{\text{whole rod}} x^2 \, dM$$

$$\lambda = \frac{M}{L}$$

$$\lambda = \frac{dM}{dx}$$

$$\Rightarrow dM = \lambda \, dx$$

$$= \int_{-L/2}^{L/2} x^2 \lambda \, dx$$

$$= \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} \cdot \lambda$$

$$= \left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3 \right] \cdot \frac{M}{L} \cdot \frac{1}{3}$$

$$= \left[\frac{L^3}{8} + \frac{L^3}{8} \right] \cdot \frac{M}{3L}$$

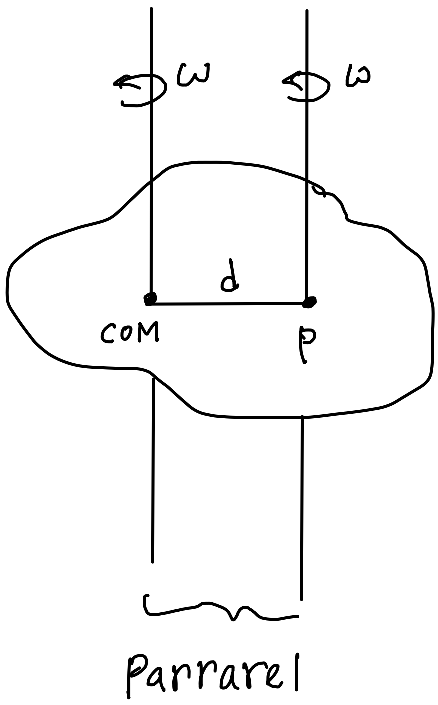
$$= \frac{L^3}{4} \cdot \frac{M}{3L}$$

$$I = \frac{1}{12} ML^2$$ ← Moment of inertia of an 1D rod when rotation axis passes through the mid point

Moment of inertia of an 1D rod when rotation axis passes through one of the edges is $I = \frac{1}{3}ML^2$

Parrarel axis theorem

$$\boxed{I_P = I_{com} + Md^2}$$



Following are Moment of inertia of some objects when the rotation axis passes through the center of mass.

* Solid Disk , $I = \frac{1}{2}MR^2$
* Solid Sphere , $I = \frac{2}{5}MR^2$
* hollow sphere , $I = \frac{2}{3}MR^2$
* Solid cylinder. $I = \frac{1}{2}MR^2$

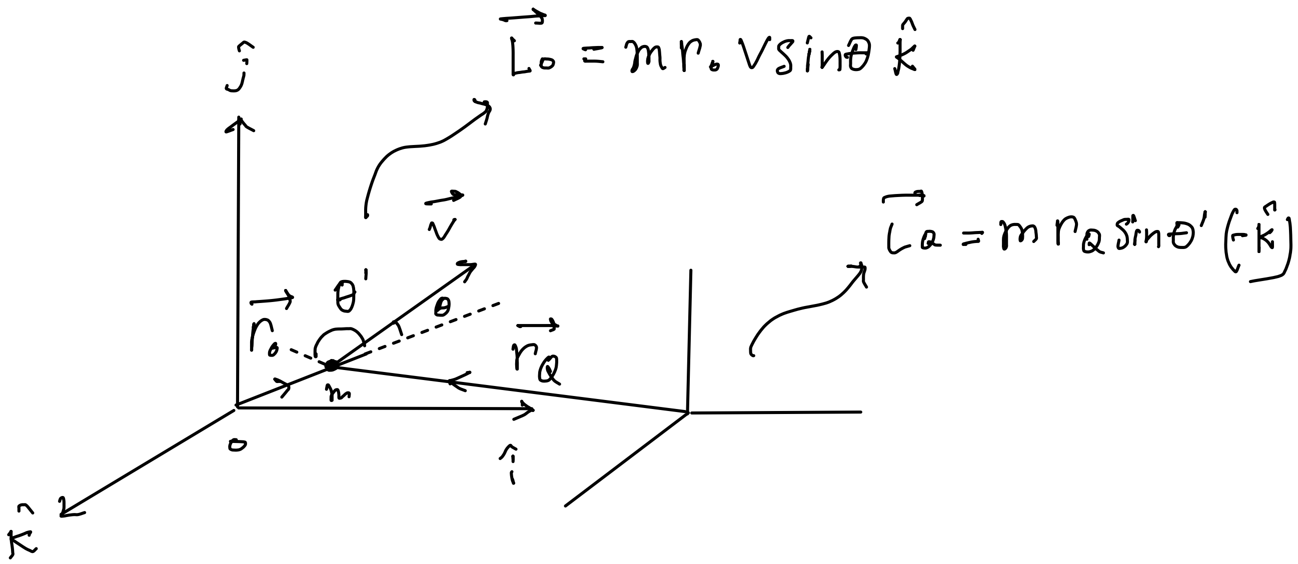
* No need to memorise

Angular momentum

Linear momentum, $\vec{p} = m\vec{v}$

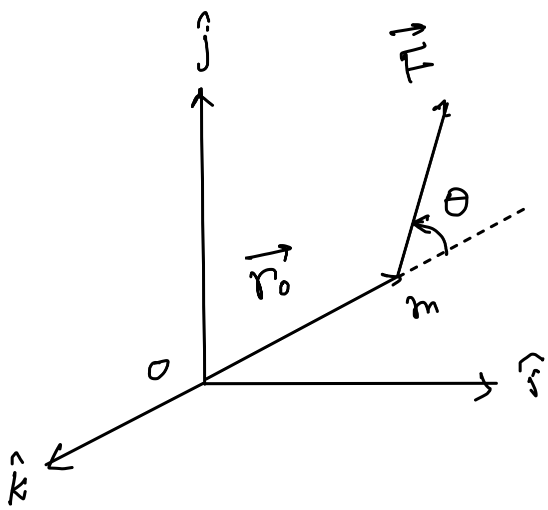
Angular Momentum,

$$\vec{L_o} = \vec{r_o} \times \vec{p} = \vec{r_o} \times m\vec{v} \quad , \quad \vec{L_Q} = \vec{r_Q} \times \vec{p} = \vec{r_Q} \times m\vec{v}$$
$$= m(\vec{r_Q} \times \vec{v})$$



Torque

$$\vec{\tau_o} = \vec{r_o} \times \vec{F}$$
$$= r_o F \sin\theta \, \hat{n}$$



$$\vec{L_o} = \vec{r_o} \times m\vec{v}$$

$$\frac{dL_o}{dt} = m\frac{d}{dt}(\vec{r_o} \times \vec{v}) = m\left[\frac{d\vec{r_o}}{dt} \times \vec{v} + r_o \times \frac{d\vec{v}}{dt}\right]$$

$$= m\left[\vec{v} \times \vec{v} + \vec{r}_o \times \vec{a}\right]$$

$$= m\vec{r}_o \times \vec{a} = \vec{r}_o \times (m\vec{a}) = \vec{r}_o \times \vec{F} = \vec{\tau}_o$$

Relation :

$\vec{L}_o \longrightarrow \vec{P}$

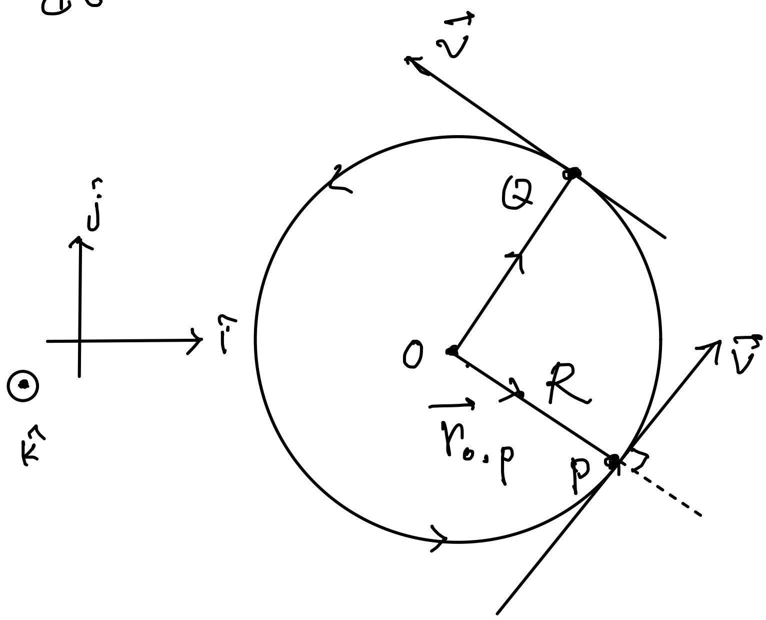
$\vec{\tau} \longrightarrow \vec{F}$

$\boxed{\vec{\tau}_o = \dfrac{d\vec{L}_o}{dt}} \longrightarrow \vec{F} = \dfrac{d\vec{P}}{dt}$

Uniform Circular Motion

$|\vec{V}| = v \longleftarrow$ constant

$V = \omega r$.



$$\vec{L}_{o,p} = \vec{r}_{o,p} \times m\vec{V}_p$$

$$= m|\vec{r}_{o,p}|\,|\vec{V}_p| \sin 90^\circ \hat{k}$$

$$= mRv\hat{k}$$

$$\vec{L}_{o,Q} = \vec{r}_{o,Q} \times m\vec{V}_Q$$

$$= m|\vec{r}_{o,Q}|\,|\vec{V}_p| \sin 90^\circ \hat{k}$$

$$= mRv\hat{k}$$

$\vec{\tau}_o = \dfrac{d}{dt}(\vec{L}_o) = 0$; since L^o is constant.

Using definition of torque,

$$\vec{\tau}_{o,p} = \vec{r}_{o,p} \times \vec{F_c}$$

$$= |\vec{r}_{o,p}|\,|\vec{F_c}|\,\sin 180^\circ\,\hat{n}$$

$$= \bar{0}$$

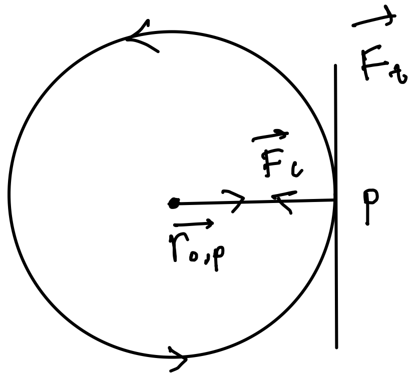
Torque for non-uniform circular motion.

$$\vec{\tau}_{o,p} = \vec{r}_{o,p} \times (\vec{F_c} + \vec{F_t})$$

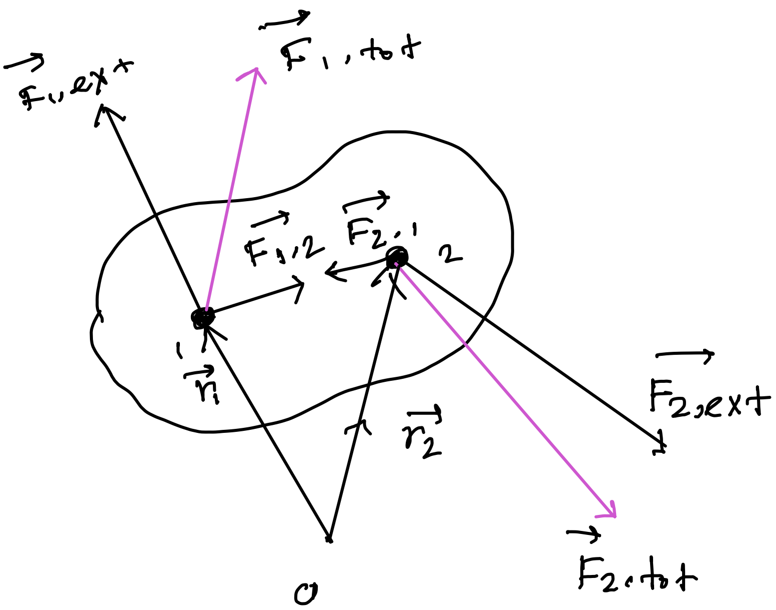
$$= \vec{r}_{o,p} \times \vec{F_c} + \vec{r}_{o,p} \times \vec{F_t}$$

$$= |\vec{r}_{o,p}|\,|\vec{F_t}|\,\sin 90^\circ\,\hat{n}$$

$$= R F_t \hat{n}$$



For a system of particles:



$$\vec{\tau}_{tot} = \vec{r}_1 \times \vec{F}_{1,tot} + \vec{r}_2 \times \vec{F}_{2,tot}$$

$$= \vec{r}_1 \times (\vec{F}_{1,ext} + \vec{F}_{1,2}) + \vec{r}_2 \times (F_{2,ext} + \vec{F}_{2,1})$$

$$= \vec{r}_1 \times \vec{F}_{1,ext} + \vec{r}_2 \times \vec{F}_{2,ext} + (\vec{r}_1 \times \vec{F}_{1,2} + \vec{r}_2 \times \vec{F}_{2,1})$$

$$= \vec{\tau}_{ext} + (\vec{r}_1 \times \vec{F}_{1,2} + \vec{r}_2 \times (-\vec{F}_{1,2}))$$

$$= \vec{\tau}_{ext} + (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{1,2} = \vec{\tau}_{ext}$$

If $\vec{\tau}_{ext,sys} = \vec{0}$

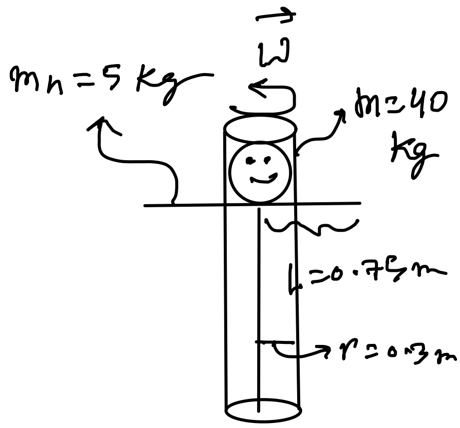
$$\frac{d\vec{L}_{sys}}{dt} = \vec{0}$$

$$\therefore \boxed{\vec{L}_{sys} = \text{constant}}$$

Conservation of Angular momentum.

Examples:

Ballet Dancer:



Body $\longrightarrow$ Solid Cylinder $\rightarrow \frac{1}{2}MR^2$

Hand $\longrightarrow$ 1D rod $\longrightarrow \frac{1}{3}M_h L^2$

For both rods, $I_{hand} = 2 \times \frac{1}{3} M_h l^2$

If the rotation axis passes through C.O.M.

$$L = I\omega \quad \text{w.r.t com}$$

$$L = mvr = m(\omega r)r = m\omega r^2 = (mr^2)\omega$$
$$= I\omega$$

Before pulling her hands close to the body, ω_i

After " " " " " " " , ω_f

Total ext torque $= 0$

$$L_i = L_f$$
$$\Rightarrow I_i\omega_i = I_f\omega_f$$
$$\therefore \omega_f = \frac{I_i}{I_f}\omega_i$$

$$I_i = \frac{1}{2} MR^2 + 2\,\frac{1}{3} M_u L^2$$

$$= \frac{1}{2} 40 \times (0.3)^2 + \frac{2}{3} \times 5 \times (0.75)^2 = 3.675 \text{ kgm}^2$$

$$I_f = \frac{1}{2} M'R^2 = \frac{1}{2} (40 + 10) \times (0.3)^2$$

$$= 2.25 \text{ Kg-m}^2$$

$$\omega_f = \frac{3.675}{2.25} \omega_i$$

$$\omega_f = 1.63\, \omega_i$$

Fate of a Star:

White dwarf $\longrightarrow 10^3$ km

Neutron Star $\longrightarrow 10$ km

Black hole ~ 0 km

$T_i = 27$ days

$\omega_i = \frac{2\pi}{T} = 0.233 \text{ day}^{-1}$

$L_i = L_f$

$\Rightarrow I_i\omega_i = I_f\omega_f$

$\therefore \omega_f = \frac{I_i}{I_f}\omega_i$

$$= \frac{\frac{2}{5}M_iR_i^2}{\frac{2}{5}M_fR_f^2}\omega_i = \left(\frac{R_i}{R_f}\right)^2\omega_i$$

$$= \left(\frac{7\times10^5}{10^3}\right)^2\omega_i$$

$$= 490000\,\omega_i$$

$$= 114170 \text{ day}^{-1}$$

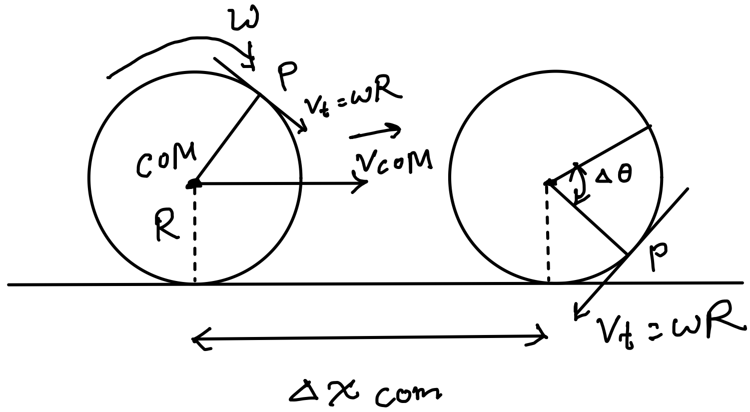
$$T_f = \frac{2\pi}{\omega_f} = \frac{2\pi}{114170} \text{ day}$$

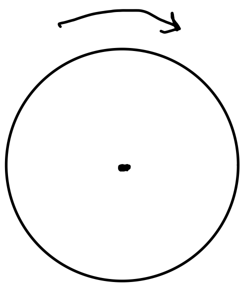
$$= 5.50\times10^{-5} \text{ day}$$

$$= 4.75 \text{ s}.$$

ROLLING MOTION

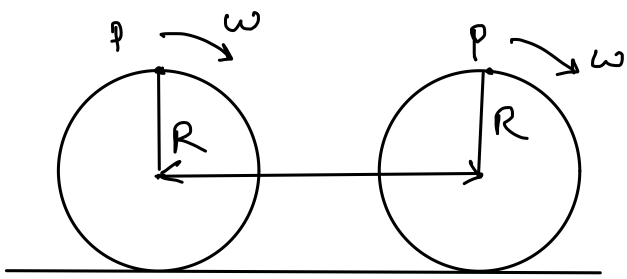
At time t: At time t+Δt:





→ Rotation $K_{rot} = \frac{1}{2} I \omega^2$

→ Translation : $K_{trans} = \frac{1}{2} m v^2$



After one complete rotation,

$\Delta x_{com} = 2\pi R$

If it rotates by $\Delta\theta$ angle.

$$\Delta x_{com} = R\Delta\theta$$

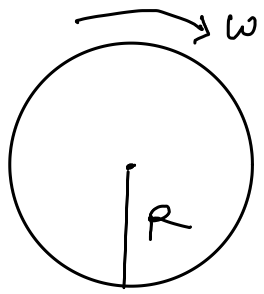
$$\Rightarrow \frac{\Delta x_{com}}{\Delta t} = R\frac{\Delta\theta}{\Delta t}$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x_{com}}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\therefore V_{com} = R\omega \qquad \frac{dV_{com}}{dt} = R\frac{d\omega}{dt}$$

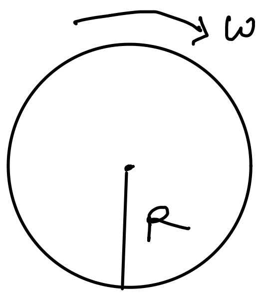
$$\therefore V_{com} = V_t \qquad \therefore a_{com} = R\alpha$$

Slipping

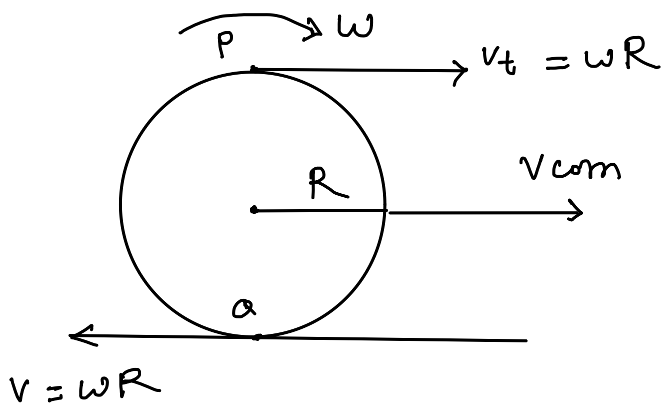


$$\omega R > V_{com}$$

Skidding



$$\omega R < V_{com}$$



$$\vec{v}_{P,G} = \vec{V}_{P,t} + \vec{V}_{com} = \omega R\,\hat{i} + V_{com}\,\hat{i}$$

$$\vec{v}_{Q,G} = \vec{V}_{Q,t} + \vec{V}_{com} = -\omega R\,\hat{i} + V_{com}\,\hat{i}$$

For rolling without slipping,

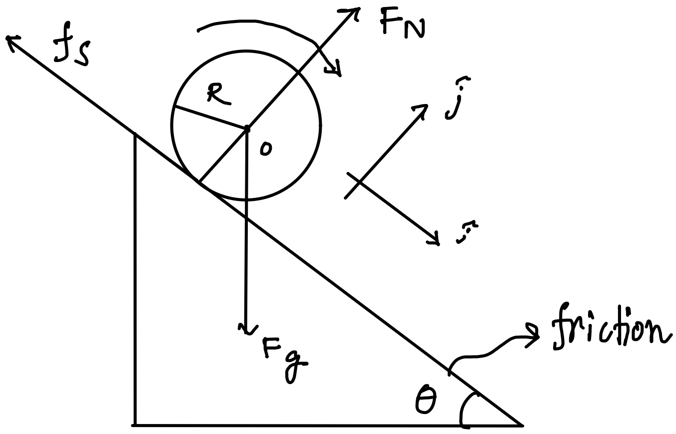
$$\vec{V}_{P,G} = \omega R\,\hat{i} + \omega R\,\hat{i} = 2\omega R\,\hat{i}$$

$$\vec{V}_{Q,G} = -\omega R\,\hat{i} + \omega R\,\hat{i} = \vec{0}$$

The point of contact with the ground is instantaneously at rest. So it will only experience static friction.

Example

sol^n



$$\vec{\tau}_o = \vec{r}_o \times \vec{F}$$

$$= \vec{r}_{o,F_g} \times \vec{F_g} + \vec{r}_{o,f_s} \times \vec{f_s} + \vec{r}_{o,F_N} \times \vec{F_N}$$

$$= \vec{0} \times \vec{F_g} + R(-\hat{j}) \times f_s(-\hat{i}) + R(-\hat{j}) \times F_N \hat{j}$$

$$= \vec{0} + Rf_s(-\hat{k}) + \vec{0}$$

$$\therefore \vec{\tau}_o = Rf_s(-\hat{k})$$

$$\vec{\tau}_{com} = \alpha_{com} I_{com}$$

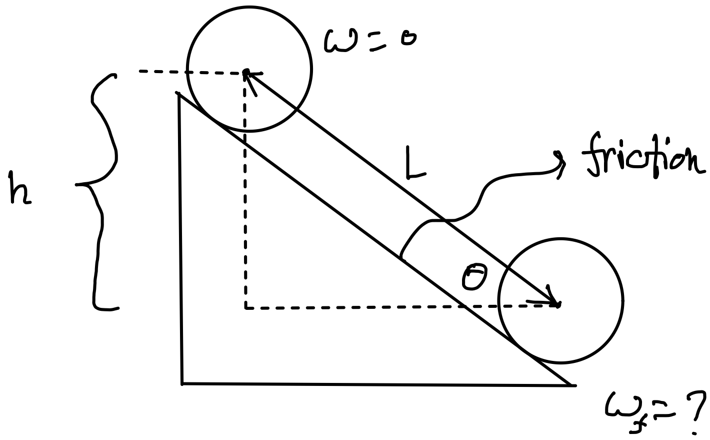
$$\vec{\tau}_{com} = 0 , \quad \alpha_{com} = \vec{0}$$

In our case,

$$Rf_s(-\hat{k}) = \alpha_{com} I_{com}$$

$$\therefore \alpha_{com} = \frac{Rf_s}{I_{com}}(-\hat{k})$$

Example Finding the speed at the end of the trajectory.



Soln

$$W_{nc} = \Delta E^{mech}$$

$$\Rightarrow 0 = \left(0 + \frac{1}{2} I \omega_f^2 + \frac{1}{2} m v_{com}^2\right) - (mgh + 0)$$

$$\Rightarrow \frac{1}{2} I \omega_f^2 + \frac{1}{2} m (\omega_f R)^2 = mgh$$

$$\Rightarrow \frac{1}{2} \omega_f^2 (I + mR^2) = mgh$$

$$\therefore \omega_f = \sqrt{\frac{2mgh}{I + mR^2}}$$

If the object is a solid sphere, then,

$$I = \frac{2}{5} mR^2$$

$$\therefore \omega_f = \sqrt{\frac{2mgh}{\frac{2}{5} mR^2 + mR^2}} = \sqrt{\frac{2mgh}{\frac{7mR^2}{5}}}$$

$$\therefore \omega_f = \sqrt{\frac{10 mgh}{7 mR^2}}$$

$$V_f = \omega_f R = \sqrt{\frac{10mgh}{7mR^2}} \times R$$

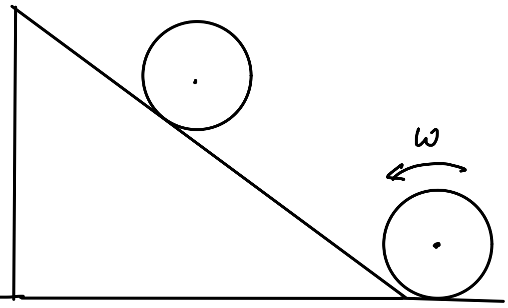
$$= \sqrt{\frac{10mgh}{7mR^2} \times R}$$

$$= \sqrt{\frac{10gh}{7}}$$

Example Find the final angular speed.

Example

Find h?



$$0 = E_f^{mech} - E_i^{mech}$$

$$= mgh + \frac{1}{2} I \omega_f^2 + \frac{1}{2} m v_f^2$$

$$- 0 - \frac{1}{2} I \omega_i^2 - \frac{1}{2} m v_{com}^2$$

$$\Rightarrow 0 = mgh + 0 - \frac{1}{2} I \omega_i^2 - \frac{1}{2} m v_{i\,com}^2$$

$$\therefore h = \frac{\frac{1}{2}(I\omega_i^2 + m\omega_i^2 R^2)}{mg}$$

$$= \frac{\omega_i^2 (I + mR^2)}{2mg}$$

Simple Harmonic Motion

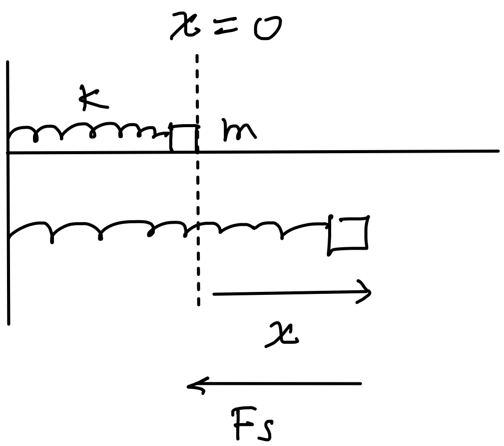
Periodic motion → If the same trajectory is repeated after some particular time.

Some particular time → Time period, T.

Simple harmonic motion

1. It is periodic motion.
2. There is a restoring force, $\vec{F} \propto -\vec{x}$

Spring-mass system.



$F_s = -K\vec{x}$

$\hat{i}: \sum \vec{F} = m\vec{a}$

$\Rightarrow -Kx\,\hat{i} = ma\,\hat{i}$

$\Rightarrow -Kx = m\dfrac{d^2x}{dt^2}$

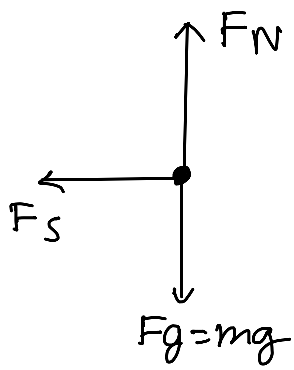
$\Rightarrow \dfrac{d^2x}{dt^2} = -\dfrac{K}{m}x$

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

SMH Differential Eqn

$$\boxed{\omega = \sqrt{\frac{K}{m}}}$$

Defn



Ansatz: $x(t) = e^{rt}$

Let's plug this in the SHM DE $\Rightarrow$

$$\frac{d^2}{dt^2}(e^{rt}) = -\omega^2 e^{rt}$$

$$\Rightarrow r^2 e^{rt} = -\omega^2 e^{rt}$$

$$\Rightarrow r^2 = -\omega^2$$

$$\therefore r = \pm i\omega$$

$$x_1 = e^{i\omega t}, \quad x_2 = e^{-i\omega t}$$

$$\frac{d^2 x}{dt^2} = -\omega^2 x$$

Order : 2 $\longrightarrow$ Second order

Homogenity $\longrightarrow$ Homogenous

Linearity $\longrightarrow$ Linear.

$\longrightarrow$ Second order, linear, homogenous differential equation.

If x_1, x_2 are solutions of a linear, homogenous DE, then any linear combination of them is a solution.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

If x_1 is a solution

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = 0$$

If x_2 is a solution

$$\frac{d^2x_2}{dt^2} + \omega^2 x_2 = 0$$

Now, we shall check whether $Ax_1 + Bx_2$ is a solution or not.

$$\frac{d^2}{dt^2}(Ax_1 + Bx_2) + \omega^2 (Ax_1 + Bx_2)$$

$$= A\frac{d^2x_1}{dt^2} + B\frac{d^2x_2}{dt^2} + A\omega^2 x_1 + B\omega^2 x_2$$

$$= A\left(\frac{d^2x_1}{dt^2} + \omega^2 x_1\right) + B\left(\frac{d^2x_2}{dt^2} + \omega^2 x_2\right)$$

$$= A \times 0 + B \times 0$$

$$= 0.$$

So, general solution : $Ax_1 + Bx_2$

General solution for SMH : $Ae^{i\omega t} + Be^{-i\omega t}$

$$\therefore x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

Euler identity: $e^{\pm i\theta} = \cos\theta \pm \sin\theta$

$$x(t) = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)$$

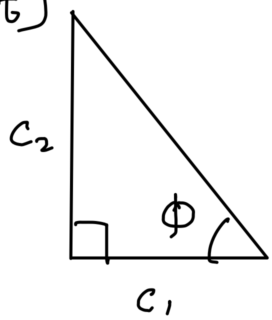
$$\Rightarrow x(t) = (A+B)\cos\omega t + (A-B)\, i \sin\omega t$$

So, $A+B$ = Real, $A-B$ = Imaginary.

$x(t) = c_1\cos\omega t + c_2\cos\omega t$. $\quad | \; c_1, c_2 \in \mathbb{R}$

$$= \sqrt{c_1^2+c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\cos\omega t\right]$$

$$= \sqrt{c_1^2+c_2^2}\,(\cos\phi\cos\omega t + \sin\phi\cos\omega t)$$

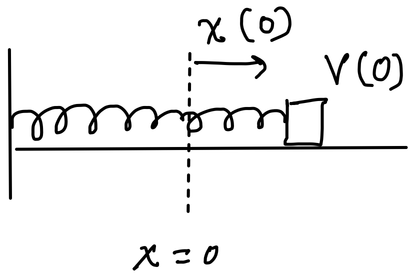


$$= A\cos(\omega t - \phi)$$

$A, \phi \longrightarrow$ undetermined constants

Initial condition : Position at time $t = 0 \rightarrow x(0)$

Velocity at time $t = 0 \rightarrow V(0)$



$$x(t) = A \cos(\omega t - \phi)$$

$$x(0) = A \cos(\omega \times 0 - \phi)$$

$$x(0) = A \cos(-\phi)$$

$$\therefore x(0) = A \cos \phi \quad \cdots (i)$$

$$V = \frac{dx}{dt}$$

$$= -A \sin(\omega t - \phi) \times \omega$$

$$= -\omega A \sin(\omega t - \phi)$$

$$V(0) = -\omega A \sin(-\phi)$$

$$\therefore V(0) = \omega A \sin \phi \quad \cdots (ii)$$

$$x(0) = A \cos \phi$$

$$\Rightarrow \frac{x(0)}{A} = \cos \phi \quad \cdots (iii)$$

$$V(0) = \omega A \sin \phi$$

$$\therefore \frac{V(0)}{\omega A} = \sin \phi \quad \cdots (iv)$$

$(iii)^2 + (iv)^2 \Rightarrow$

$$\frac{x(0)^2}{A^2} + \frac{V(0)^2}{\omega^2 A^2} = \cos^2 \phi + \sin^2 \phi = 1$$

$$\Rightarrow \frac{1}{A^2}\left(x(0)^2 + \frac{V(0)^2}{\omega^2 A^2}\right) = 1$$

$$\therefore \boxed{A = \sqrt{x(0)^2 + \frac{v(0)^2}{\omega^2 A^2}}}$$

(iv) $\div$ (iii) $\Rightarrow$

$$\frac{v(0)}{\omega \cancel{A}} \times \frac{\cancel{A}}{x(0)} = \tan \phi$$

$$\therefore \boxed{\phi = \tan^{-1}\left(\sqrt{\frac{v(0)}{\omega\, x(0)}}\right)}$$

Example An object of mass 0.5 Kg is connected to a massless spring which has spring constant 100 N/m. It was stretched by a distance of 0.5m to the right and was released with a speed of 2 m/s to the right. Assume the solution is given by $x(t) = A\sin(\omega t - \phi)$.

(i) Find A and ϕ.

(ii) Draw the graphs of x vs t, v vs t and a vs t

Soln

(i) $x(0) = A\sin(0 \times t - \phi)$

$\Rightarrow x(0) = A\sin(-\phi)$

$\Rightarrow x(0) = -A\sin\phi$

$\Rightarrow -\sin\phi = \dfrac{x(0)}{A} \quad \ldots (i)$

$v(t) = A\cos(\omega t - \phi)\cdot\omega$

$\therefore v(t) = \omega A\cos(\omega t - \phi)$

$\therefore v(0) = \omega A\cos(-\phi) = \omega A\cos\phi$

$\Rightarrow \cos\phi = \dfrac{v(0)}{\omega A} \quad \ldots (ii)$

$(i)^2 + (ii)^2 \Rightarrow$

$$\frac{x(0)^2}{A^2} + \frac{v(0)^2}{\omega^2 A^2} = \sin^2\phi + \cos^2\phi = 1$$

$$\Rightarrow \frac{1}{A^2}\left(x(0)^2 + \frac{v(0)^2}{\omega^2}\right) = 1$$

$$\therefore A = \sqrt{x(0)^2 + \frac{v(0)^2}{\omega^2}}$$

$(i) \div (ii) \Rightarrow$

$$\frac{-\sin\phi}{\cos\phi} = \frac{x(0)}{A} \times \frac{\omega A}{v(0)}$$

$$\Rightarrow -\tan\phi = \frac{x(0)\,\omega}{v(0)}$$

$$\therefore \phi = \tan^{-1}\left(-\frac{x(0)\,\omega}{v(0)}\right)$$

$x(0) = 0.5\,m$

$v(0) = 2\,m/s$

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{100}{0.5}} = \sqrt{200}\ rad\,s^{-1}$$

$$\therefore A = \sqrt{(0.5)^2 + \frac{2^2}{200}} = 0.52\ m$$

$$\therefore \phi = \tan^{-1}\left(-\frac{0.5 \times \sqrt{200}}{2}\right)$$

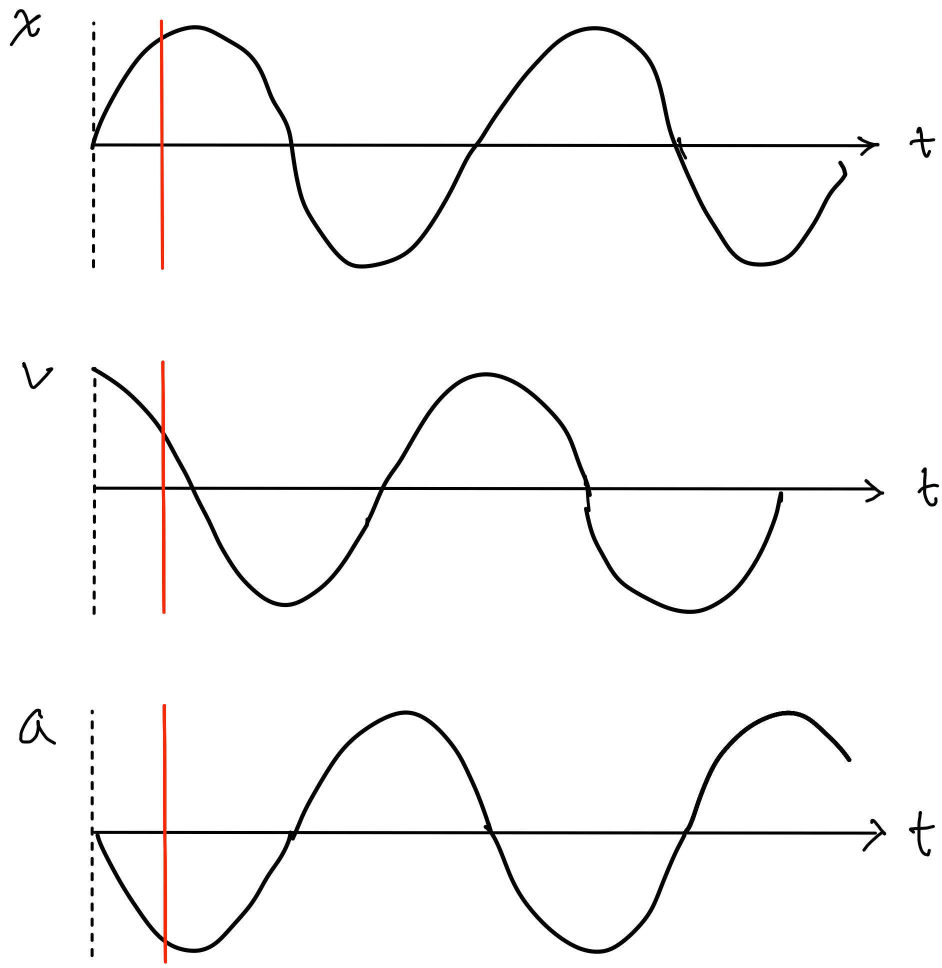
$$= -1.29 \text{ rad}$$

$$= -74.2^\circ$$

(ii) $x(t) = (0.52)\sin(\omega t + 1.29)$

$$v(t) = \frac{dx}{dt} = \omega(0.52)\sin(\omega t + 1.29)$$

$$a(t) = \frac{dv}{dt} = -\omega^2(0.52)\sin(\omega t + 1.29)$$



The origin is shifted 74.2°

Energy in SHM

Potential energy $= \frac{1}{2}Kx^2$

Kinetic energy $= \frac{1}{2}mv^2$

If $x = A\cos(\omega t - \phi)$, then $v = -\omega A \sin(\omega t - \phi)$

$$U = \frac{1}{2}KA^2\cos^2(\omega t - \phi)$$

$$K = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \phi)$$

$$\boxed{\begin{gathered}\omega = \sqrt{\frac{K}{m}} \\ \Rightarrow K = m\omega^2\end{gathered}}$$

$$\begin{aligned}
E^{mech} &= U + K \\
&= \frac{1}{2}A^2\left[K\cos^2(\omega t - \phi) + m\omega^2\sin^2(\omega t - \phi)\right] \\
&= \frac{1}{2}A^2\left[K\cos^2(\omega t - \phi) + K\sin^2(\omega t - \phi)\right] \\
&= \frac{1}{2}A^2 K
\end{aligned}$$

$$\therefore E^{mech} = \frac{1}{2}A^2K$$

H.W Find the potential and Kinetic energy of the oscillator in the previous problem at $t = 2s$, and then calculate the total mechanical energy.

Final Exam Mark Distro

1 Question → 15 marks (mandatory)

3 Questions → 10 marks each

→ Have to answer any two.