MAT314: Complex Analysis

Summer 2025

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Preface

These lecture notes were taken in the course MAT314: Complex Analysis taught by Dr. Md. Shariful Islam at BRAC University as part of the BSc. in Mathematics program Summer 2025.
These notes are not endorsed by the lecturer, and I often modified them after lectures. They are not accurate representations of what was actually lectured, and in particular, all errors are surely mine.
If you find anything that needs to be corrected or improved, please inform me at: mh.turjoy@yahoo.com.

Prerequisites: Basic idea of Linear Algebra, Abstract Algebra and Topology.

Contents

Preliminaries **2**
- 1.1 Complex Numbers 2
- 1.2 Algebraic Structures on $\mathbb{C}$ 3
- 1.3 Geometric and Topological Structures on $\mathbb{C}$ 4
- 1.4 Argand Diagram 5
- 1.5 Complex Exponentiation and Polar Representation 6
- 1.6 Roots of Complex Numbers 7

1 Preliminaries

§ 1.1 Complex Numbers

We know the quadratic polynomial

$$ax^2 + bx + c,$$

has two roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac \geq 0$, then we have two roots to the quadratic (not necessarily distinct). But if we have $b^2 - 4ac < 0$, then we get something that involves square root of negative, which is not in $\mathbb{R}$.

But if we define a new mathematical object i such that $i^2 = -1$, then quadratics will always have a solution. We call any number of the form ai, where $a \in \mathbb{R}$, *imaginary number*. But what meaning can such numbers have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these 'imaginary' numbers started proving so useful (especially in the work of Cauchy) that the philosophical concerns ultimately became side-issues.

Definition 1.1 (Complex Number)**.** A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$ and i is a formal symbol with $i^2 = -1$.

We will usually denote a complex number by z. If $z = x + iy$, then x and y are called *real part* and *imaginary part* of z respectively. We write

$$\text{Re}(z) = x \text{ and } \text{If}(z) = y.$$

And $\bar{z} = x - iy$ is called the *complex conjugate* of z.

Note that a real number is 'essentially' just a complex number with zero imaginary part. So, $\mathbb{R} \subseteq \mathbb{C}$.

The set of all complex numbers is denoted by $\mathbb{C}$. So

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i^2 = -1\}.$$

§ 1.2 Algebraic Structures on $\mathbb{C}$

We can put various algebraic structures on the set $\mathbb{C}$ by introducing addition and multiplication in obvious sensible ways.

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$.

Define addition "+"on $\mathbb{C}$ point-wise, i.e.,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Then it can be easily seen that $(\mathbb{C}, +)$ forms an aeolian group.

Notice the addition is entirely comparable with adding two vectors in xy-plane. In fact, we can consider $\mathbb{C}$ as 2-dimensional real vector space where scalar multiplication is defined by

$$cz = cx + icy,$$

for $c \in \mathbb{R}, z \in \mathbb{C}$. Furthermore, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as vector space.

$$\mathbb{C} \cong \mathbb{R}^2.$$

Now, define multiplication "$\cdot$" on $\mathbb{C}$ as follows:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Proposition 1.2. $(\mathbb{C} \setminus \{0\}, \cdot)$ forms an aeolian group.

Proof. Multiplication on $\mathbb{C}$ is defined kind of similar to how we multiply real algebraic expressions. Hence, proof of closure **closure** and **associativity** is trivial.

For all $z = x + iy \in \mathbb{C}$, $(1 + i0)$ is the **identity element** because

$$(x + iy) \cdot (1 + i0) = (x - 0) + i(0 + y) = x + iy.$$

And

$$z^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2},$$

is the **inverse element** (tedious to verify!). ∎

It can be easily seen that $\cdot$ **distributes** over $+$. Hence,

Proposition 1.3. $(\mathbb{C}, +, \cdot)$ forms a field.

$\mathbb{C}$ is not an ordinary field, it is an *algebraic field*, i.e., every polynomial with coefficients in $\mathbb{C}$ has all the roots in $\mathbb{C}$. Moreover, $\mathbb{R}, \mathbb{C} \cong \mathbb{R}^2$ are the only Euclidean spaces that have the field structure!

But there is a difference between $\mathbb{R}$ and $\mathbb{C}$: we know $\mathbb{R}$ is an ordered field (definition 1.4) with usual addition, multiplication and ordering. But $\mathbb{C}$ can not be made into an ordered field. The reason is that for an ordered field we must have **(1)** $-1 < 0$ and **(2)** $x^2 \geq 0$ for every x. But in $\mathbb{C}$, $i^2 = -1$.

Definition 1.4 (Ordered Field)**.** An ordered field is a field F that is also an ordered set such that the field operations $(+, \cdot)$ are compatible with order relation, i.e.,

- $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
- $xy > 0$ if $x, y \in F$ and $x, y > 0$.

§ 1.3 Geometric and Topological Structures on $\mathbb{C}$

Norm is a geometric property that in some (possibly abstract) sense describes the length, size, or extent of an object.

Definition 1.5 (Norm)**.** Given a vector space V over a subfield F of complex numbers, a **norm** on V is a function $V \longrightarrow \mathbb{R}$ that associates each $\mathbf{v} \in V$ a positive real number, denoted by $\|s\|$, which has the following properties:

1. **Definiteness:** $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. **Homogeneity:** $\|\alpha\mathbf{v}\| = |\alpha| \cdot \|\mathbf{v}\|, \forall \mathbf{v} \in V, \forall \alpha \in F$.
3. **Triangle Inequality:** $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|, \forall \mathbf{v}, \mathbf{w} \in V$.

Proposition 1.6 (Norm on $\mathbb{C}$)**.** Let $z = x + iy \in \mathbb{C}$. The function

$$\|\cdot\| : \mathbb{C} \longrightarrow \mathbb{R}, z \longmapsto \sqrt{x^2 + y^2},$$

defines a norm on $\mathbb{C}$ over $\mathbb{R}$.

Proof. To be written... ■

We will interpret the norm defined above pictorially shortly.

The norm defined above induces a topology on $\mathbb{C}$ where the basic open sets are of the form

$$B_r(x) = \{y \mid \|x - y\| < r\},$$

for all $x \in \mathbb{C}$.

Here are some examples of open sets in $\mathbb{C}$.

Example 1.7. 1. $A = \{z \mid \|z\| < r\}$,

2. $B = \{z \mid \mathrm{Im}(z) > 0\}$.

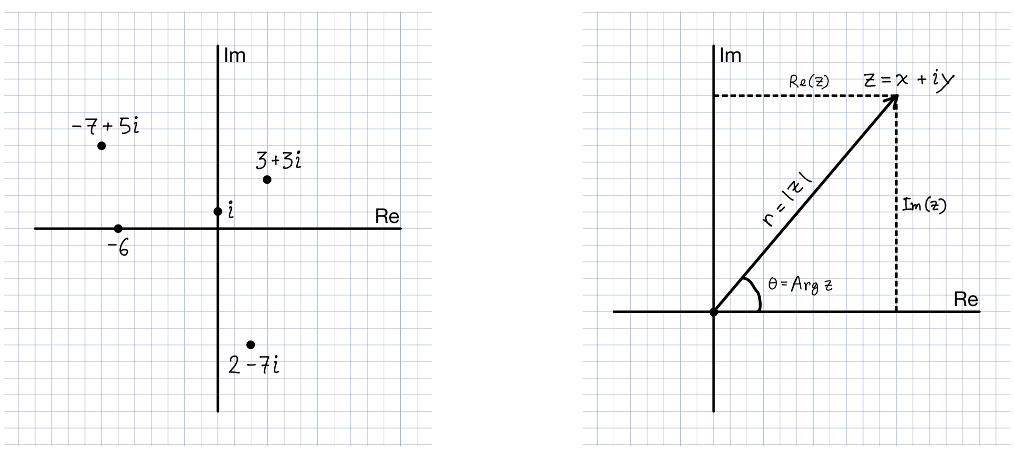
§ 1.4 Argand Diagram

Complex numbers can be represented on a plane same as elements of $\mathbb{R}^2$. Then we refer it as *complex plane* (horizontal and vertical axis is called *real* and *imaginary axis* respectively), where the point (x, y) represents the complex number $z = x + iy$. Diagram we get by representing complex numbers in this manner is called *Argand diagram*. We sometimes think of complex numbers as vector in complex plane instead of just a point for convenience. Then length of the vector is called *modulus* of z and denoted by $|z|$. By Pythagorean theorem,

$$|z| = \|z\|.$$

And the angle this vector creates with real axis, is called *argument* of z. Equivalently, argument of z is real number θ for which $x = r\cos\theta$ and $y = r\sin\theta$ are satisfied.

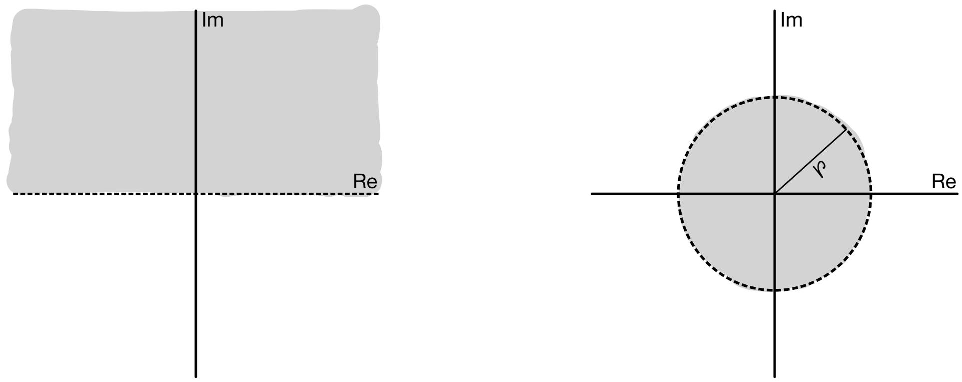
The *principle argument*, denoted by $\mathrm{Arg}(z)$, is an argument that lies in $(-\pi, \pi]$.



(a) Argand Diagram (b) Complex Numbers as Vectors

Figure 1: Geometric Representation of Complex Numbers

So open sets A and B in example 1.7 is just the upper half plane (figure 2a) and an open disc (figure 2b) respectively on the complex plane.



(a) Upper Half Plane

(b) Open Disc

Figure 2: Geometric Pictures of Set A and B

§ 1.5 Complex Exponentiation and Polar Representation

Let $z = x + iy \in \mathbb{C}$ and $\theta = \operatorname{Arg}(z)$ and $r = |z|$. Then

$$z = r(\cos\theta + i\sin\theta).$$

The exponential function e^z and trigonometric functions $\sin z$ and $\cos z$ is defined by the series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Theorem 1.8 (Euler's Identity)**.** For all $z \in \mathbb{C}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Here key observation is that the sequence i^k, where $k \geq 0$, equals $1, i, -1, -i, 1, \cdots$ repeating with period 4.

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(\theta)^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{(\theta)^{2k+1}}{(2k+1)!} \end{aligned}$$

$$= \cos\theta + i\sin\theta.$$

∎

Hence,

$$z = re^{i\theta},$$

which is called *polar representation* of complex number z.

Lemma 1.9. $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Proof.

$$\begin{aligned} e^{i\theta_1} \cdot e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= e^{i(\theta_1+\theta_2)}. \end{aligned}$$

∎

Theorem 1.10 (De Moivre's Theorem)**.** For a non-negative positive integer n

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof.

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

∎

Remark 1.11. We can prove De Moivre's theorem for $n \in \mathbb{Q}$ easily. Then using the fact that sin and cos are continuous functions and $\mathbb{Q}$ is dense in $\mathbb{R}$, De Moivre's theorem is true for all real numbers.

§ 1.6 Roots of Complex Numbers

Say we want to find n-th root of a comlex numebr c. Algebraically this means solving the equation

$$z^n = c,$$

for $z \in \mathbb{C}$.

Let $c = Re^{i\phi}, z = re^{i\theta}$. We need to find r and θ.

$$\begin{aligned}
&(re^{i\theta})^n = Re^{i\phi} \\
\implies &r^n e^{in\theta} = Re^{i\phi} \\
\implies &r = R^{\frac{1}{n}} \text{ and } n\theta = \phi + 2k\pi, k \in \mathbb{Z} \\
\implies &r = R^{\frac{1}{n}} \text{ and } \theta \in \left\{\frac{\phi}{n} + \frac{2k\pi}{n} \mid k \in \mathbb{Z}\right\}.
\end{aligned}$$

In this way can easily compute n-th root of a complex number z. When $c = 1$, the roots are called *roots of unity*. In that case, we have $R = 1$ and $\phi = 0$. Therefore, $r = 1$ and

$$\theta \in \left\{\frac{2k\pi}{n} \mid 0 \leq k \leq n-1\right\}.$$

So all the n-th roots of unity are of the form $e^{i\frac{2k\pi}{n}}$ and are evenly spaced points on the unit circle.

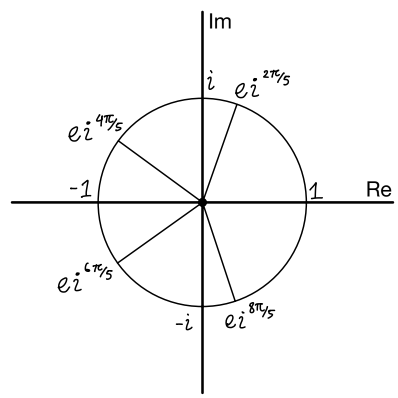


Figure 3: 5-th roots of unity

Example 1.12. 1. Find all those complex numbers z that satisfy $z^2 = i$.

2. Find all those complex numbers z that satisfy $z^2 = 1 + i$.

Solution. Coming in the next update (-_-). ♣

References

[1] Chowdhury, A. R. *Complex Analysis Lecutre Notes*. Retrieved from https://atonurc.github.io/assets/MAT314_CA.pdf

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