

MAT422: Theory of Numbers

Notes taken by Mahmudul Hasan Turjoy
Based on the lectures of Arnab Chakraborty

Last Updated: February 12, 2025

Preface

These lecture notes were taken in the course *MAT422: Theory of Numbers* taught by *Arnab Chakraborty* at BRAC University as part of the BSc. in Mathematics program Spring 2025.

These notes are not endorsed by the lecturer, and I often modified them after lectures. They are not accurate representations of what was actually lectured, and in particular, all errors are surely mine. If you find anything that needs to be corrected or improved, please inform me: mh.turjoy@yahoo.com.

— Mahmudul Hasan Turjoy

Contents

1	Preliminaries	3
1.1	Number Systems	3
1.2	Well-Ordering Principle	3
1.3	Goals and Motivation for doing NT	3
2	Divisibility and Primes	4
2.1	Basics of Divisibility	4
2.2	Primes & Related Theorems	5
2.3	Distribution of Primes	8
3	References	9

1

Preliminaries

§ 1.1 Number Systems

The most natural and familiar number system is the set of *natural numbers*,

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

However, solving equations using only natural numbers is often challenging. To address this, we extend \mathbb{N} by introducing 0 and additive inverses, forming the set of *integers*,

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

\mathbb{Z} allows for systematic techniques in algebra, particularly in solving linear equations. In \mathbb{Z} , we can add, subtract, and multiply, but division is not always possible. To allow division, we extend \mathbb{Z} to the set of *rational numbers*,

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0, \gcd(p, q) = 1 \right\}.$$

However, \mathbb{Q} is still not sufficient to solve all equations. For example, $\sqrt{2}$ is not a rational number but a solution to the equation $x^2 = 2$. To include such numbers, we extend \mathbb{Q} to the set of *real numbers*, \mathbb{R} . Yet even \mathbb{R} is not enough for certain equations, such as $x^2 + 1 = 0$ have no solutions in \mathbb{R} . To resolve this, we introduce the *complex numbers*,

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}.$$

According to the *Fundamental Theorem of Algebra*, \mathbb{C} is *algebraically closed*, meaning every polynomial equation with complex coefficients has a solution in \mathbb{C} . Thus, \mathbb{C} is the most comprehensive number system we need for solving polynomial equations, and we do not extend it further in this context. Notice,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

§ 1.2 Well-Ordering Principle

In Peano Arithmetic, the *Principle of Mathematical Induction (PMI)* is assumed as an axiom. And from this axiom, we can derive a fundamental result known as the *Well-Ordering Principle*.

Well-Ordering Principle. Every non-empty subset of \mathbb{N} has a least element.

§ 1.3 Goals and Motivation for doing NT

To be written once I figure out myself. Hopefully, at the end of this course I will be able to write something here. :3

2

Divisibility and Primes

§ 2.1 Basics of Divisibility

Definition 2.1 (Divisibility). Let $a, b \in \mathbb{Z}$. We say that $a \neq 0$ divides b if there exists an integer c such that $b = ac$. We write $a \mid b$ if a divides b .

We say a is a divisor of b if $a \mid b$. The set of all divisors of b is denoted by $Div(b)$.

$$Div(b) := \{a \in \mathbb{Z} : a \mid b\}.$$

We can also say $a \mid b$ implies there is a solution in \mathbb{Z} to the equation $ax = b$.

Proposition 2.1. 1. For all $a \in \mathbb{Z} \setminus 0$, $a \mid 0$.
 2. $a \mid b$ and $b \mid c$ implies $a \mid c$.
 3. $a \mid b$ and $a \mid c$ implies $a \mid (bx + cy)$ for all $x, y \in \mathbb{Z}$.

Proof. 1. $a \mid 0$ since $0 = a \cdot 0$.
 2. $a \mid b$ implies $b = ax$ and $b \mid c$ implies $c = by = axy = a(xy)$ implies $a \mid c$.
 3. $a \mid b$ implies $b = ax$ and $a \mid c$ implies $c = ay$. So, $bx + cy = a(x^2 + y^2)$ implies $a \mid (bx + cy)$. □

Theorem 2.1 (Division Algorithm). Let $a, b \in \mathbb{Z}$ with $b > 0$. Then there exist unique integers q, r such that $a = qb + r$ and $0 \leq r < b$.

Proof. Consider the set

$$S = \{a - xb \mid x \in \mathbb{Z}, a - xb \geq 0\}.$$

The set is nonempty because $a - (-|a|)b = a + |a|b \geq a + |a| \geq 0$. According to the Well-Ordering Principle, S has a least element, say r . So, $r = a - qb$ for some $q \in \mathbb{Z}$. We will show that $0 \leq r < b$. $0 \leq r$ by the construction of S . We argue that $r < b$. If this were not the case, then $r \geq b \Rightarrow r - b \geq 0$. But

$$a - (q + 1)b = a - qb - b = r - b \geq 0.$$

Hence, $a - (q + 1)b = r - b \in S$ which contradicts the minimality of r . Hence, $0 \leq r < b$. This completes the proof of existence.

Now, we will show the uniqueness of q and r . Suppose there exist q_1, q_2, r_1, r_2 such that $a = q_1b + r_1$, $a = q_2b + r_2$, $0 \leq r_1, r_2 < b$. Subtracting the two equations, we get

$$(q_1 - q_2)b = r_2 - r_1.$$

Since $0 \leq r_1, r_2 < b$, we have $-b < r_2 - r_1 < b$, or,

$$|r_2 - r_1| < b.$$

Therefore, $|q_1 - q_2|b < b$, which implies,

$$0 \leq |q_1 - q_2| < 1$$

So, $r_2 - r_1 = 0 \Rightarrow r_1 = r_2$ and $q_1 = q_2$. This completes the proof of uniqueness. \square

§ 2.2 Primes & Related Theorems

Definition 2.2 (Prime Number). An element $p \in \mathbb{N}$ and $p > 1$ is called a prime number if for all $q \in \mathbb{N}$, if $q \mid p \Rightarrow q = \pm 1$ or $q = \pm p$.

Equivalently, $p > 1$ is prime if $Div(p) = \{\pm 1, \pm p\}$.

Example. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59 etc. are primes. \diamond

Example (Biggest Prime). As of now the biggest known prime number is

$$2^{136279841} - 1.$$

\diamond

Theorem 2.2 (Existence of Prime Factorization). Every positive integer greater than 1 can be written as a product of primes.

Proof. Assume for the sake of contradiction, that there exists a positive integer greater than 1 that cannot be written as a product of primes. By well-ordering principle, we will have a least n of this sort. If n is prime, then it is already a product of primes (just itself), which is a contradiction.

If n is composite, then it can be written as $n = a \cdot b$, for some $a, b \in \mathbb{N}$ where $1 < a, b < n$. Since $a, b < n$, they can be written as a product of primes. So, $n = a \cdot b$ can also be written as a product of primes, which is again a contradiction.

Hence, every positive integer greater than 1 can be written as a product of primes. \square

We can extend this concept to negative integers as well by including -1 as a factor. For any integer $n \neq 0$,

$$n = (-1)^{\varepsilon(n)} \prod_{i=1}^k p_i^{a(p_i)},$$

where $\varepsilon(n) = 0$ if $n > 0$ and 1 otherwise. And $a(p_i)$ is called *order of n at p_i* which is a non-negative integer such that $p^a \mid n$ but $p^{a+1} \nmid n$.

Next lemma requires a theorem which we will just state but not prove.

Theorem 2.3 (Bezout's Identity). Given two integers a and b , not both of which are zero, there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = \gcd(a, b).$$

Lemma 2.1. If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof. If $p \mid a$, then we are done. So, assume $p \nmid a$. Let $g = \gcd(a, p)$. Since, p is a prime $g = 1$ or $g = p$ (which is not possible). Hence, $g = 1$. By Bezout's Identity, there exist $x, y \in \mathbb{Z}$ such that $px + ay = 1$. Multiplying both sides by b , we get

$$bpx + aby = b.$$

In the equation, left side is divisible by p , so is the right side. Hence, $p \mid b$. \square

Corollary 2.1. If p is a prime and $p \mid a_1 a_2 \dots a_n$, then $p \mid a_i$ for some i .

Proof. We will prove this by induction on n . For $n = 1$, the statement is trivial. When $n = 2$, the result is the content of Lemma 2.1. Assume the result is true for $n = k$. We will show that it is true for $n = k + 1$. By Lemma 2.1, $p \mid a_1 a_2 \dots a_k$ or $p \mid a_{k+1}$. Let $p \mid a_1 a_2 \dots a_{k+1}$. By induction hypothesis, $p \mid a_1 a_2 \dots a_k$ implies $p \mid a_i$ for some i and we are done. Otherwise, $p \mid a_{k+1}$. This completes the proof. \square

Theorem 2.4 (Fundamental Theorem of Arithmetic). Every positive integer greater than 1 can uniquely be written as a product of primes (upto reordering).

Proof. Existence is already proved in theorem 2.2. We will now show uniqueness. Assume for the sake of contradiction that there exists a positive integer greater than 1 that can be written as a product of primes in two different ways. Let n be the smallest such number. Let $n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_l$ where p_i, q_i are primes. Since $p_1 \mid q_1 q_2 \dots q_l$, by Corollary 2.1, $p_1 \mid q_i$ for some i . WLOG, let $p_1 \mid q_1$. Since q_1 is prime, $p_1 = q_1$. Let, $n' = p_2 p_3 \dots p_k = q_2 q_3 \dots q_l$. Clearly, $n > n'$ which contradicts the minimality of n . Hence, every positive integer greater than 1 can uniquely be written as a product of primes. \square

Theorem 2.5. There are infinitely many primes.

Proof. Assume for the sake of contradiction that there are finitely many primes, say p_1, p_2, \dots, p_n . Consider the number $N = p_1 p_2 \dots p_n + 1$. Since $N > 1$, it can be written as a product of primes. But none of p_1, p_2, \dots, p_n divides N as N leaves a remainder of 1 when divided by any of p_i . Hence, N is a prime which is not in the list p_1, p_2, \dots, p_n . This contradicts the assumption that there are finitely many primes. \square

Definition 2.3 (Riemann Zeta Function). The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where the function is defined for $s \in \mathbb{R}$ (in this context) and $s > 1$.

Theorem 2.6 (Euler's Product Formula). For $s > 1$, we have

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Proof. The geometric series formula gives us,

$$\frac{1}{1 - p^{-s}} = \sum_{m=0}^{\infty} \frac{1}{p^{ms}}.$$

So,

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} \sum_{m=0}^{\infty} \frac{1}{p^{ms}} = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}.$$

By Fundamental Theorem of Arithmetic, $a_n = 1$ for all n . Hence, the product is equal to $\zeta(s)$. \square

Now, set $s = 1$ in Euler's Product Formula to get

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-1}}.$$

Left side is the harmonic series which diverges to ∞ . So, the product on the right side must also diverge to ∞ . This implies that there are infinitely many primes.

§ 2.3 Distribution of Primes

Gauss conjectured that the density of primes is approximately given by:

$$\frac{1}{\log x}$$

Let $\pi(x)$ denote the number of primes less than or equal to x . Then, an asymptotic estimate for $\pi(x)$ is:

$$\pi(x) \sim \frac{x}{\log x}$$

In asymptotic notation, if $f(x) \sim g(x)$, it means:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1.$$

A more refined approximation is given by the logarithmic integral:

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

The **Prime Number Theorem (PNT)** states that:

$$\pi(x) \sim \text{Li}(x),$$

which was proven in 1896.

Von Mangoldt's Explicit Formula: Von Mangoldt's explicit formula connects the sum of the von Mangoldt function $\Lambda(n)$ to the nontrivial zeros of the Riemann zeta function. It is a key result in analytic number theory, particularly in understanding the distribution of prime numbers.

Define the von Mangoldt function $\Lambda(n)$ as:

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

The Chebyshev function $\psi(x)$ is given by:

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Von Mangoldt's explicit formula states:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}),$$

where the sum runs over all the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$.

3

References

1. An Introduction to the Theory of Numbers by Ivan Niven, Herbert S. Zuckerman, Hugh L. Montgomery,
2. A Classical Introduction to Modern Number Theory by Kenneth F. Ireland and Michael Wayne Rosen.