CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair

HW 09

1 Symmetric Marbles

Note 14

A bag contains 4 red marbles and 4 blue marbles. Rachel and Brooke play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Rachel wins if there are more red than blue marbles, and Brooke wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?

(b) What is the probability that Rachel wins the game?

(c) Given that Rachel wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Rachel wins if the third marble is red, and Brooke wins if the third marble is blue. All other rules stay the same.

(d) What is the probability that the third marble is red?

(e) Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k.

(f) Given that the third marble is red, what is the probability that Rachel wins the game?

Solution:

(a) They are not independent; removing one red marble lowers the probability of the next marble being red.

(b) Let p be the probability that Rachel wins. Since there are an equal number of red and blue marbles, by symmetry, the probability that Rachel wins and the probability that Brooke wins is the same. Thus, the probability that there is a tie is $1 - p - p = 1 - 2p$.

We now compute the probability that there is a tie. For there to be a tie, two of the four marbles need to be red. There are $\binom{8}{4}$ ways to pick 4 marbles, and $\binom{4}{2}\binom{4}{2}$ to pick 2 red and blue marbles, respectively, giving a probability of

$$\frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \boxed{\frac{18}{35}}.$$

We conclude that $1-2p = \frac{18}{35}$. Solving for p gives $p = \boxed{\frac{17}{70}}$.

(c) Let A be the event that there are 3 red marbles drawn, and let B be the event that there are 4 red marbles drawn. We wish to compute

$$\mathbb{P}[B \mid (A \cup B)] = \frac{\mathbb{P}[B \cap (A \cup B)]}{\mathbb{P}[A \cup B]} = \frac{\mathbb{P}[B]}{\mathbb{P}[A] + \mathbb{P}[B]}.$$

Similar to the calculation in part (b), the probability that there are 3 red marbles drawn is $\frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} = \frac{16}{70}$, and the probability that there are 4 red marbles drawn is $\frac{\binom{4}{4}\binom{4}{0}}{\binom{8}{4}} = \frac{1}{70}$, giving a final answer of $\frac{\frac{1}{70}}{\frac{16}{70}+\frac{1}{70}} = \boxed{\frac{1}{17}}$.

(d) By symmetry, the probability that the third marble is red is the same as the probability that the first marble is red, or the same as any marble being red. One way to see this is to imagine drawing the four marbles in order, then moving the first marble drawn to the third position. This is another way to draw four marbles that yields the same distribution.

There are 8 red marbles, and 12 marbles in total. Thus, the probability that the third marble is red is $\frac{8}{12} = \boxed{\frac{2}{3}}$.

(e) We are given that there are k red marbles among the 4 drawn. By symmetry, each marble has the same probability of being red, so the probability that the third marble is red is $\boxed{\frac{k}{4}}$.

(f) The only way for Rachel to lose the game given that the third marble is red is if all the other marbles are blue. The probability that the third marble is red and all the other marbles are blue is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{495}$, and the probability that the third marble is red is $\frac{8}{12} = \frac{2}{3}$, so the probability that Rachel loses given that the third marble is red is $\frac{\frac{8}{495}}{\frac{2}{3}} = \frac{4}{165}$, and the probability that Rachel wins given that the third marble is red is $\boxed{\frac{161}{165}}$.

2 Man Speaks Truth

Note 14

Consider a man who speaks the truth with probability $\frac{3}{4}$.

(a) Suppose the man flips a biased coin that comes up heads $1/3$ of the time, and reports that it is heads.

(i) What is the probability that the coin actually landed on heads?

(ii) Unconvinced, you ask him if he just lied to you, to which he replies "no". What is the probability now that the coin actually landed on heads?

(iii) Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.

(b) Suppose the man rolls a fair 6-sided die. When you ask him if the die came up with a 6, he answers "yes".

(i) What is the probability that the die actually came up with a 6?

(ii) Skeptical, you also ask him whether the die came up with a 1, to which he replies "yes". What is the probability now that the die actually came up with a 6?

(iii) Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.

Solution:

(a) (i) Let S_H denote the event the man says heads, H be the event that the coin comes up heads, and T be the event that the coin comes up tails (note that H and T are complements of each other).

We're given that the man says heads (i.e. that S_H occurs), and we want to find the probability that the coin comes up heads. Using Bayes' rule and total probability, we have

$$\mathbb{P}[H \mid S_H] = \frac{\mathbb{P}[S_H \cap H]}{\mathbb{P}[S_H]} = \frac{\mathbb{P}[S_H \mid H]\mathbb{P}[H]}{\mathbb{P}[S_H \mid H]\mathbb{P}[H] + \mathbb{P}[S_H \mid T]\mathbb{P}[T]}.$$

Here, we have $\mathbb{P}[H] = \frac{1}{3}$ and $\mathbb{P}[T] = \frac{2}{3}$. We know that the man tells the truth $\frac{3}{4}$ of the time, so we also have the conditional probabilities $\mathbb{P}[S_H \mid H] = \frac{3}{4}$ (since the man tells the truth here) and $\mathbb{P}[S_H \mid T] = \frac{1}{4}$ (since the man tells a lie here).

Plugging these probabilities in, we have

$$\mathbb{P}[H \mid S_H] = \frac{\frac{3}{4} \cdot \frac{1}{3}}{\frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3}} = \frac{3}{3+2} = \frac{3}{5}.$$

(ii) Suppose we define the following events:

- H is the event the coin lands heads, and T is the event the coin lands tails.
- S_H is the event the man says heads, and S_T is the event the man says tails.
- S_Y is the event the man says he lied (responded yes to the question), and S_N is the event the man says he told the truth (responded no to the question).

We're given that the man says heads (i.e. that S_H occurs), and that the man said he did not lie (i.e. that S_N occurs); we want to find the probability that the coin comes up heads. Using Bayes' rule with the product rule and total probability, we have

$$\begin{aligned}
\mathbb{P}[H \mid S_N \cap S_H] &= \frac{\mathbb{P}[S_N \cap S_H \cap H]}{\mathbb{P}[S_N \cap S_H]} \\
&= \frac{\mathbb{P}[S_N \cap S_H \cap H]}{\mathbb{P}[S_N \cap S_H \cap H] + \mathbb{P}[S_N \cap S_H \cap T]} \\
&= \frac{\mathbb{P}[S_N \mid S_H \cap H]\mathbb{P}[S_H \mid H]\mathbb{P}[H]}{\mathbb{P}[S_N \mid S_H \cap H]\mathbb{P}[S_H \mid H]\mathbb{P}[H] + \mathbb{P}[S_N \mid S_H \cap T]\mathbb{P}[S_H \mid T]\mathbb{P}[T]}
\end{aligned}$$

Here, we again have $\mathbb{P}[H] = \frac{1}{3}$ and $\mathbb{P}[T] = \frac{2}{3}$. We also have $\mathbb{P}[S_H \mid H] = \frac{3}{4}$ (since the man told the truth here) and $\mathbb{P}[S_H \mid T] = \frac{1}{4}$ (since the man lied here).

Further, we have $\mathbb{P}[S_N \mid S_H \cap H] = \frac{3}{4}$ (the man told the truth here; he did not lie when he said heads, since the coin was actually heads) and we also have $\mathbb{P}[S_N \mid S_H \cap T] = \frac{1}{4}$ (the man lied here; he had lied prior when he said heads, so his reply that he did not lie is *also* a lie).

Plugging these values in, we have

$$\mathbb{P}[H \mid S_N \cap S_H] = \frac{\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{3}}{\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{2}{3}} = \frac{9}{9+2} = \frac{9}{11}.$$

Intuitively, what we're doing here is taking into account the *order* in which the events occur through the product rule. We essentially have three points in time: (1) the man flips the coin, (2) the man states the outcome of the coin, and (3) the man answers whether he lied or not. The outcomes of the last two points in time are fixed, given the context of the problem (we *know* the man stated the coin is heads, and we *know* the man stated he did not lie), so the only thing that can possibly vary is the outcome of the coin (i.e. it can either be heads or tails). This gives the two branches in the denominator, and it is why we only look at one of these branches in the numerator (i.e. when the coin landed heads).

(iii) This probability went up with the new information. Intuitively, this is because the man tells the truth more often than he lies. The fact that he reaffirms that he did not lie means that there's a higher probability that the coin actually did land on heads.

(b) (i) Let R_6 be the event that he rolled a 6, let R_N be the event that he did not roll a 6, and let S_6 be the event that the man says it was a 6.

We're given that the man says it was a 6 (i.e. S_6 occurs), and we want to find the probability that he actually rolled a 6. Using Bayes' rule and total probability, we have

$$\mathbb{P}[R_6 \mid S_6] = \frac{\mathbb{P}[R_6 \cap S_6]}{\mathbb{P}[S_6]} = \frac{\mathbb{P}[S_6 \mid R_6]\mathbb{P}[R_6]}{\mathbb{P}[S_6 \mid R_6]\mathbb{P}[R_6] + \mathbb{P}[S_6 \mid R_N]\mathbb{P}[R_N]}.$$

Here, since the die is fair, we have $\mathbb{P}[R_6] = \frac{1}{6}$ and $\mathbb{P}[R_N] = \frac{5}{6}$. We also have $\mathbb{P}[S_6 \mid R_6] = \frac{3}{4}$ (since the man told the truth here) and $\mathbb{P}[S_6 \mid R_N] = \frac{1}{4}$ (since the man lied here).

Plugging these values in, we have

$$\mathbb{P}[R_6 \mid S_6] = \frac{\frac{3}{4} \cdot \frac{1}{6}}{\frac{3}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{5}{6}} = \frac{3}{3+5} = \frac{3}{8}.$$

(ii) Suppose we define the following events:

- R_6 is the event he rolled a 6, R_1 is the event he rolled a 1, and R_N is the event he rolled neither 1 nor 6.
- S_6 is the event he says the die was a 6, S_1 is the event he says the die was a 1.

We're given that the man says the die was a 6 (i.e. S_6 occurred) *and* he says the die was a 1 (i.e. S_1 occurred); we want to find the probability that the die actually was a 6. Using

Bayes' rule and total probability (note that we have a partition of *three* events here), we have

$$\begin{aligned}
&\mathbb{P}[R_6 \mid S_1 \cap S_6] \\
&= \frac{\mathbb{P}[S_1 \cap S_6 \cap R_6]}{\mathbb{P}[S_1 \cap S_6]} \\
&= \frac{\mathbb{P}[S_1 \cap S_6 \cap R_6]}{\mathbb{P}[S_1 \cap S_6 \cap R_1] + \mathbb{P}[S_1 \cap S_6 \cap R_6] + \mathbb{P}[S_1 \cap S_6 \cap R_N]} \\
&= \frac{\mathbb{P}[S_1 \mid S_6 \cap R_6]\mathbb{P}[S_6 \mid R_6]\mathbb{P}[R_6]}{\mathbb{P}[S_1 \mid S_6 \cap R_6]\mathbb{P}[S_6 \mid R_6]\mathbb{P}[R_6] + \mathbb{P}[S_1 \mid S_6 \cap R_1]\mathbb{P}[S_6 \mid R_1]\mathbb{P}[R_1] + \mathbb{P}[S_1 \mid S_6 \cap R_N]\mathbb{P}[S_6 \mid R_N]\mathbb{P}[R_N]}
\end{aligned}$$

Here, since the die is fair, we have $\mathbb{P}[R_1] = \mathbb{P}[R_6] = \frac{1}{6}$, while $\mathbb{P}[R_N] = \frac{4}{6}$.
For each of the three possible events for the outcome of the die, we have the conditional probabilities

- $\mathbb{P}[S_6 \mid R_1] = \frac{1}{4}$, since the man lied
- $\mathbb{P}[S_6 \mid R_6] = \frac{3}{4}$, since the man told the truth
- $\mathbb{P}[S_6 \mid R_N] = \frac{1}{4}$, since the man lied

Notice that if we condition on the event that the die had a specific outcome (ex. conditioned on the event the die rolled a 6), the event that the man says it was a 6 is actually independent of the event the man says it was a 1. Intuitively, this is because we already know the outcome of the die, so the fact the man says it was a 6 doesn't impact the probability he says it is also a 1—it only encapsulates whether the man lied or not.
This means that we also have the following conditional probabilities:

- $\mathbb{P}[S_1 \mid S_6 \cap R_1] = \mathbb{P}[S_1 \mid R_1] = \frac{3}{4}$, since the man told the truth
- $\mathbb{P}[S_1 \mid S_6 \cap R_6] = \mathbb{P}[S_1 \mid R_6] = \frac{1}{4}$, since the man lied
- $\mathbb{P}[S_1 \mid S_6 \cap R_N] = \mathbb{P}[S_1 \mid R_N] = \frac{1}{4}$, since the man lied

Plugging these all in, we have

$$\mathbb{P}[R_6 \mid S_1 \cap S_6] = \frac{\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{6}}{\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{6} + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{4}{6}} = \frac{3}{3+3+4} = \frac{3}{10}.$$

Similar to the previous part, what we're doing here is essentially taking into account the order in which the events occur. However, here, it's equally valid to condition on S_1 first, and then compute the probability of S_6 (i.e. swapping S_1 and S_6 above), due to their independence conditioned on R_1, R_6, or R_N.
The three points in time we're considering now are (1) the man rolls the die, (2) the man states whether it was a 6, and (3) the man states whether it was a 1. Again, the latter two outcomes are fixed due to the context in the problem, and the first outcome is free to vary. This is why we need three branches in the denominator, one for each of R_1, R_6, and R_N. We also can't just divide it up into two events, since the response to "did the die come up with a 1" depends on whether the die actually landed on 1, and the response to "did the die come up with a 6" depends on whether the die actually landed on 6; it's easiest to consider each case separately.

(iii) The probability went down with the new information. Since it's impossible for the man to have told the truth in both situations (i.e. the die can't possibly have rolled both a 1 and a 6), it must be the case that one of the statements is a lie. This should lower the probability that the first statement is actually true, i.e. that the die actually landed on a 6.

3 Cliques in Random Graphs

Note 13
Note 14

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e)).

(e) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$. *Hint:* Use the union bound.

Solution:

(a) Between every pair of vertices, there is either an edge or there isn't. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) E_{V_1} and E_{V_2} are independent if and only if V_1 and V_2 share at most one vertex: If V_1 and V_2 share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}
\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \mathbb{P}[\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}] \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\
&= \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].
\end{aligned}$$

Conversely, if V_1 and V_2 share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, so we have

$$\begin{aligned}\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}-\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}-\binom{|V_3|}{2}} \\ &= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}+\binom{|V_2|}{2}-\binom{|V_3|}{2}} \neq \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].\end{aligned}$$

(d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\begin{aligned}\binom{n}{k} = \frac{n!}{(n-k)!k!} &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \\ &\leq n \cdot (n-1) \cdots (n-k+1) \\ &\leq n^k\end{aligned}$$

(e) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \leq \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \leq \frac{n^k}{\left(2^{(4\log n+1-1)/2}\right)^k} = \frac{n^k}{(2^{2\log n})^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}.$$

4 Combined Head Count

Note 19 Suppose you flip a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y?

(d) Compute the conditional probabilities $\mathbb{P}[Y = i \mid X = j]$ for all i, j.

(e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z.

Solution:

(a) $\{(T,T),(H,T),(T,H),(H,H)\}$.

(b)

$$X = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .5 \\ 2 & \text{w.p.} \quad .25 \end{cases}$$

(c)

$$Y = \begin{cases} 0 & \text{w.p.} \quad .75 \\ 1 & \text{w.p.} \quad .25 \end{cases}$$

(d)
- $\mathbb{P}[Y = 0 \mid X = 0]$: Since $X = 0$, we have no heads; therefore, there is no chance that the first coin is heads, so Y must be 0. So $\mathbb{P}[Y = 0 \mid X = 0] = 1$.
- $\mathbb{P}[Y = 1 \mid X = 0] = 0$ as $\mathbb{P}[Y = 1 \mid X = 0] = 1 - \mathbb{P}[Y = 0 \mid X = 0] = 1 - 1 = 0$.
- $\mathbb{P}[Y = 0 \mid X = 1]$: If we have one head, then we have one of two outcomes, (H,T) or (T,H), and since this is a fair coin, both outcomes happen with equal probability. Only (T,H) makes $Y = 0$; thus $\mathbb{P}[Y = 0 \mid X = 1] = \frac{1}{2}$.
- $\mathbb{P}[Y = 1 \mid X = 1] = 0$ as $\mathbb{P}[Y = 1 \mid X = 1] = 1 - \mathbb{P}[Y = 0 \mid X = 1] = 1 - \frac{1}{2} = \frac{1}{2}$.
- $\mathbb{P}[Y = 0 \mid X = 2]$: Since $X = 0$, we have no tails; therefore, there is no chance that the second coin is tails, so Y must be 0. So $\mathbb{P}[Y = 0 \mid X = 2] = 1$.
- $\mathbb{P}[Y = 1 \mid X = 2] = 0$ as $\mathbb{P}[Y = 1 \mid X = 2] = 1 - \mathbb{P}[Y = 0 \mid X = 2] = 1 - 1 = 0$.

(e) Let's determine the values Z can take and the corresponding probabilities:

- $Z = 0$: $\mathbb{P}(Z = 0) = \mathbb{P}(X = 0 \cap Y = 0) = \mathbb{P}(X = 0) \cdot \mathbb{P}(Y = 0 \mid X = 0) = .25 \cdot 1 = .25$
- $Z = 1$:

$$\begin{aligned} \mathbb{P}(Z = 1) &= \mathbb{P}(X = 0 \cap Y = 1) + \mathbb{P}(X = 1 \cap Y = 0) \\ &= \mathbb{P}(X = 0) \cdot \mathbb{P}(Y = 1 \mid X = 0) + \mathbb{P}(X = 1) \cdot \mathbb{P}(Y = 0 \mid X = 1) \\ &= .25 \cdot 0 + .5 \cdot .5 = .25 \end{aligned} \tag{1}$$

- $Z = 2$:

$$\begin{aligned} \mathbb{P}(Z = 2) &= \mathbb{P}(X = 1 \cap Y = 1) + \mathbb{P}(X = 2 \cap Y = 0) \\ &= \mathbb{P}(X = 1) \cdot \mathbb{P}(Y = 1 \mid X = 1) + \mathbb{P}(X = 2) \cdot \mathbb{P}(Y = 0 \mid X = 2) \\ &= .5 \cdot 5 + .25 \cdot 1 = .5 \end{aligned} \tag{2}$$

- $Z = 3$: $\mathbb{P}(Z = 3) = \mathbb{P}(X = 2 \cap Y = 1) = \mathbb{P}(X = 2) \cdot \mathbb{P}(Y = 1 \mid X = 2) = .25 \cdot 0 = 0$

$$Z = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .25 \\ 2 & \text{w.p.} \quad .5 \end{cases}$$

5 Max/Min Dice Rolls

Note 15 Yining rolls three fair six-sided dice.

(a) Let X denote the maximum of the three values rolled. What is the distribution of X (that is, $\mathbb{P}[X=x]$ for $x=1,2,3,4,5,6$)? You can leave your final answer in terms of "x". [*Hint*: Try to first compute $\mathbb{P}[X \leq x]$ for $x=1,2,3,4,5,6$]. If you want to check your answer, you can solve this problem using counting and make sure it matches with the formula you derived.

(b) Let Y denote the minimum of the three values rolled. What is the distribution of Y?

Solution:

(a) Let X denote the maximum of the three values rolled. We are interested in $\mathbb{P}(X=x)$, where $x=1,2,3,4,5,6$. First, define X_1, X_2, X_3 to be the values rolled by the first, second, and third dice. These random variables are i.i.d. and uniformly distributed between 1 and 6 inclusive.

Following the hint we first compute $\mathbb{P}[X \leq x]$ for $x=1,2,3,4,5,6$:

$$\mathbb{P}[X \leq x] = \mathbb{P}[X_1 \leq x]\mathbb{P}[X_2 \leq x]\mathbb{P}[X_3 \leq x] = \left(\frac{x}{6}\right)\left(\frac{x}{6}\right)\left(\frac{x}{6}\right) = \left(\frac{x}{6}\right)^3$$

Then, observing that $\mathbb{P}[X=x] = \mathbb{P}[X \leq x] - \mathbb{P}[X \leq x-1]$:

$$\mathbb{P}(X=x) = \left(\frac{x}{6}\right)^3 - \left(\frac{x-1}{6}\right)^3 = \frac{3x^2-3x+1}{216} = \begin{cases} \frac{1}{216}, & x=1 \\ \frac{7}{216}, & x=2 \\ \frac{19}{216}, & x=3 \\ \frac{37}{216}, & x=4 \\ \frac{61}{216}, & x=5 \\ \frac{91}{216}, & x=6 \end{cases}$$

One can confirm that $\sum_{x=1}^{6} \mathbb{P}[X=x] = 1$.

(b) Similarly to the previous part, we first compute $\mathbb{P}[Y \geq y]$.

$$\mathbb{P}[Y \geq y] = \mathbb{P}[X_1 \geq y]\mathbb{P}[X_2 \geq y]\mathbb{P}[X_3 \geq y] = \left(\frac{6-(y-1)}{6}\right)\left(\frac{6-(y-1)}{6}\right)\left(\frac{6-(y-1)}{6}\right) = \left(\frac{7-y}{6}\right)^3 .$$

Then, observing that $\mathbb{P}[Y=y] = \mathbb{P}[Y \geq y] - \mathbb{P}[Y \geq y+1]$:

$$\mathbb{P}[Y=y] = \left(\frac{7-y}{6}\right)^3 - \left(\frac{6-y}{6}\right)^3 .$$