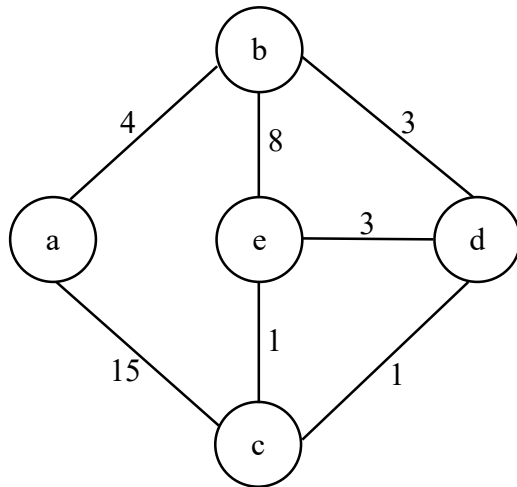A counterexample for Alice's claim would be a graph G like following:

In this graph, BFS with a as source yields $\pi[c]$ = a, while the path a→b→e→c is a shorter path.

BFS with b as source yields $\pi[e]$ = b, while the path b→d→e is a shorter path.

BFS with e as source yields $\pi[b]$ = e, while the path e→d→b is a shorter path.

BFS with c as source yields $\pi[a]$ = c, while the path c→e→b→a is a shorter path.

BFS with d as source yields $\pi[e]$ = d, while the path d→c→e is a shorter path.

Since BFS with any vertex as source could not produce the right result based on Alice's claim, this would be a counterexample to disprove her claim.