Notes, 1(b)

ECE 606

Propositional logic and a bit beyond

1. Proposition: a statement with which we can associate true or false.
   (a) E.g., "The Earth is flat."
   (b) E.g., "This statement is true."
   (c) Not propositions:
      i. "Hey, you!"
      ii. "This statement is false."
      iii. "The value in the variable x is positive."

c2: If this is true, then it is false. But if it's false, then it's true, so we have a contradiction.

c3: The truth value of this statement depends on the value that x takes, so it's not a proposition.

2. We will sometimes refer to propositions abstractly. E.g., "Let p and q be propositions..."

This means p and q either takes the value ture, or false, but not both.

3. Atomic proposition: cannot be broken down further in to propositions.
4. Compound proposition: constructed from other propositions. Given propositions p, q, we will say that the following as also propositions:

   (p), $\neg p$, $p \wedge q$, $p \vee q$, $p \implies q$, $p \impliedby q$, $p \iff q$, $p \oplus q$, ...
5. Important point with logic (and other "languages"): syntax vs. semantics
6. The above list of items specify syntax only. We have not yet said, for example, what the truth-value of (p) is, if p is true.
   (a) I am hinting at a truth-table based semantics.
   (b) There are other ways to specify semantics: beyond the scope of this course.

Truth-table based semantics:

1. What is the truth-value of a proposition given the truth-values of its constituent propositions?

2. E.g.,

p	(p)
true	true
false	false

3. E.g.,

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

4. A particularly interesting one: implication.

p	q	$p \implies q$
true	true	true
true	false	false
false	true	true
false	false	true

   (a) E.g., "if the Earth is flat, then an emu can fly."
   E.g., "if $1 = 2$, then I am the pope."
   Both the above statements are true.

5. Note: no gray areas. A proposition is either true or false.

1. Suppose we use "≡" to mean "equivalent"; that is, $a \equiv b$ for propositions a, b means that both a and b always have the same truth-value.

   Claim 1. $p \implies q \equiv \neg p \vee q$

   Proof. By truth-table.

   identical

p	q	$p \implies q$	$\neg p$	$\neg p \vee q$
true	true	true	false	true
true	false	false	false	false
false	true	true	true	true
false	false	true	true	true

   □

2. Turns out we need $(\,), \neg, \wedge, \vee$ only. The others, e.g., $\implies, \oplus, \ldots$ can be specified in terms of those.

3. E.g., $\oplus$:

p	q	$p \oplus q$
true	true	false
false	true	true
true	false	true
false	false	false

$$
\begin{aligned}
p \oplus q &\equiv (p \vee q) \wedge \neg(p \wedge q) && \\
&\equiv (p \vee q) \wedge (\neg p \vee \neg q) && \because \text{De Morgan's} \\
&\equiv (p \wedge (\neg p \vee \neg q)) \vee (q \wedge (\neg p \vee \neg q)) && \because \wedge \text{ distributes over } \vee \\
&\equiv ((p \wedge \neg p) \vee (p \wedge \neg q)) \vee ((q \wedge \neg p) \vee (q \wedge \neg q)) && \because \vee \text{ distributes over } \wedge \\
&\equiv (p \wedge \neg q) \vee (q \wedge \neg p) && \because a \wedge \neg a \equiv \mathsf{false}, \mathsf{false} \vee a \equiv a
\end{aligned}
$$

4. Sets and propositions are related closely.

   $\cup \approx \vee, \cap \approx \wedge, \emptyset \approx \mathsf{false}, \mathbb{U} \approx \mathsf{true}, \ldots$

We need to go a bit beyond propositional logic: two quantifiers.

1. "$\exists$": existential quantifier
2. "$\forall$": universal quantifier

E.g., For $\mathbf{Q}$ the set of rationals, "$\sqrt{2} \notin \mathbf{Q}$" $\equiv$ "$\forall y \in \mathbf{Q}, y^2 \neq 2$".

Rules for negation: suppose $p(x)$ is a statement about the variable x.

1. $\neg(\exists x, p(x)) \equiv \forall x, \neg p(x)$
2. $\neg(\forall x, p(x)) \equiv \exists x, \neg p(x)$

E.g., "no mammal lays eggs."
Suppose M is the set of mammals. This is $\equiv$ to, "$\forall x \in M, x$ does not lay eggs".

It turns out that the above statement is false. So its negation is true. So what is true is:

$$\begin{aligned} \neg(\forall x \in M, x \text{ does not lay eggs}) &\equiv \exists x \in M, \neg(x \text{ does not lay eggs}) \\ &\equiv \exists x \in M, x \text{ lays eggs} \end{aligned}$$