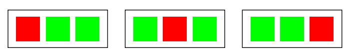
Notes, 1(d)

ECE 606

Counting and Discrete Probability

By "counting" we mean answering a question such as:

> I have three books, two of which are identical to one another. How many different ways exist for me to arrange them left to right on my shelf?
>
> The answer is three: 

"Or" $\equiv +$

> I have five different pairs of shorts and four different pairs of pants. How many different choices do I have to choose what I wear?
>
> Answer: $5 + 4 = 9$.

"And" $\equiv \times$

> I have five different shirts and 4 different pairs of pants. How many different choices of outfits do I have?
>
> Answer: $5 \times 4 = 20$.

Ordered, with replacement	string	n^k
Ordered, without replacement	permutation	$P(n,k)$
Unordered with replacement	multichoose	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$
Unordered without replacement	combination	$\binom{n}{k}$

credit: `https://mathworld.wolfram.com/BallPicking.html`

1. How many base-8 strings exist of length 5, allowing leading 0's?

   Answer: 8^5

2. How many ways to pick a chair, vice chair and treasurer from 10 people such that no one may serve in more than one role?

   Answer: $P(10,3) = 10!/7!$

3. I have 9 bags of different powdered spice. In how many ways can I choose 5 pinches of spice from amongst those 9, with the same spice allowed in multiple pinches?

   Answer: $\left(\!\binom{9}{5}\!\right)$. Reason: think of 8 "dividers", that yield 9 slots, represented by 1's, and a pinch of spice represented by a 0. So now we are asking: how many binary strings exist with exactly five 0's and eight 1's.

   For example, three pinches of the second spice, one of the fourth, and one of the eight is represented as: 1000110111101. Two pinches of the first spice and three pinches of the ninth spice is: 0011111111000.

4. How many ways to pick four office holders from 10 people, such that one person may occupy at most one slot only?

   Answer: $\displaystyle\binom{10}{4} = \frac{10!}{6!\,4!}$

   One way to think of this: first think of each office slot as distinct, i.e., ordered selection. Then "factor out" the duplicates by dividing by 4!.

   E.g., How many ways to rearrange the letters "TORONTO"?

   Answer: $\displaystyle\frac{7!}{2!\,3!\,1!\,1!}$

Probability

Experiment, e.g., toss a two-sided coin

Elementary event = experiment + an outcome

Sample space, S = set of elementary events, e.g., $\{H, T\}$.

Event, $E \subseteq S$, e.g., roll a 6-sided die and it lands an odd number.

A probability distribution, $\Pr\{\}$, on a sample space S is a function from events of S to $\mathbf{R}$, i.e., $\Pr\colon 2^S \to \mathbf{R}$, such that the probability axioms are satisfied:

1. $\Pr\{E\} \geq 0$ for every $E \subseteq S$
2. $\Pr\{S\} = 1$
3. $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$ if $A \cap B = \emptyset$.

Can prove many claims from the above basic definitions, e.g.,

- $\Pr\{\emptyset\} = 0$.
- $\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}$.

A discrete probability distribution: S is countable.

Countable can mean finite, or countable inifnite. In this course, it would refer to as finite only.

- In this course, all our S's are finite.
- Also, for every $E \subseteq S$, $\Pr\{E\} = \sum_{e \in E} \Pr\{e\}$
- Uniform distribution for finite S: for every $e \in S$, $\Pr\{e\} = 1/|S|$.

E.g., toss a fair coin n times; what is $\Pr\{\text{exactly } k \text{ heads}\}$?

- Each member of S is a binary string of length n.
- So $|S| = 2^n$. And this is a uniform distribution.
- Number of strings with exactly k 1's (or exactly k 0's): $\binom{n}{k}$.
- So solution: $\binom{n}{k}/2^n$

Conditional probability, $\Pr\{A \mid B\} = \dfrac{\Pr\{A \cap B\}}{\Pr\{B\}}$

Events A, B said to be independent if:

- $\Pr\{A \cap B\} = \Pr\{A\} \times \Pr\{B\}$.
- Equivalently, $\Pr\{A \mid B\} = \Pr\{A\}$.

Bayes's theorem: if $\Pr\{A\} \neq 0, \Pr\{B\} \neq 0$, then:

$$\Pr\{A \mid B\} = \frac{\Pr\{A\} \Pr\{B \mid A\}}{\Pr\{B\}}$$

Another useful version, replace $\Pr\{B\}$ with:

$$\Pr\{B\} = \Pr\{A\} \Pr\{B \mid A\} + \Pr\{\overline{A}\} \Pr\{B \mid \overline{A}\}$$

We only consider Discrete random variables and finite sample spaces.

(Discrete) random variables and Expectation

Suppose S is a finite sample space. Then, a discrete random variable, X, is a function with domain S to the real numbers, i.e., $X\colon S \to \mathbf{R}$.

So – kind of like a probability distribution, but more permissive.

E.g., toss a two-sided coin, win \$5 if H, lose \$2 if T.
$X\colon \{H, T\} \to \mathbf{R}$ such that $X(H) = 5, X(T) = -2$.

For random variable X on sample space S and $y \in \mathbf{R}$, define the event "$X = y$" as the event: $\{e \in S \mid X(e) = y\}$.
Then, $\Pr\{X = y\} = \sum_{\{e \in S \mid X(e) = y\}} \Pr\{e\}$

E.g., roll a 6-sided die, win \$2 for an even number, \$1 for an odd number. Then, if X is a random variable that is the winning from a toss:

$$\begin{aligned}
&\Pr\{X = 2\} = 1/6 + 1/6 + 1/6 = \Pr\{X = 1\} \\
&\Pr\{X = y\} = 0 \quad \text{for all } y \notin \{1, 2\}
\end{aligned}$$

The expectation of a discrete random variable X is:

$$E[X] = \sum_y y \Pr\{X = y\}$$

E.g., Alice offers Bob the following bet. She tosses a 2-sided coin and pays him \$5 if it lands H. He pays her \$2 if it lands T. Should Bob take Alice up on the bet?

Answer: depends on what $\Pr\{H\}$, and therefore $\Pr\{T\}$ is. Suppose X is a random variable that is Bob's winning. If it is a fair coin, then:

$$E[X] = 1/2 \times 5 + 1/2 \times (-2) = 1.5$$

So if the coin is fair, Bob expects to make money and should take Alice up on the bet. On the other hand, if $\Pr\{H\} = 1/4$, then:

$$E[X] = 1/4 \times 5 + 3/4 \times (-2) < 0$$

Particularly useful fact about expectation: linearity.

$$E[X+Y] = E[X] + E[Y]$$

Final thing I want to highlight in this lecture: Bernoulli trial.

A Bernoulli trial has one of two outcomes only: success or failure.

Interesting observation about a Bernoulli trial: suppose $\Pr\{\text{success}\} = p$, then # trials before we can expect success $= 1/p$.

E.g., consider the following algorithm to find the median in an array $A[1, \ldots, n]$ of distinct integers where n is odd. Uniformly pick an index $i \in \{1, \ldots, n\}$. Check if it is the median. If it is not, repeat.

Probability of success in each trial $= 1/n$. So # trials before we expect to successfully find the median $= n$.

Another example: suppose we have a possibly biased coin with $\Pr\{H\} = p \in (0, 1)$. How can we get an unbiased coin-toss?

Possible solution (algorithm): toss the coin twice. If the outcome of the two tosses is different, you're done: output the first of those two tosses. Otherwise, repeat.

This is unbiased. Because $\Pr\{\text{we output } H\} = p\,(1-p) = \Pr\{\text{we output } T\}$.

pairs of tosses before we expect to have an output: think of each pair of tosses as a Bernoulli trial, with "success" meaning we return a result. We "succeed" with probability $2p(1-p)$.

So # pairs of tosses before we expect to return $= \dfrac{1}{2p(1-p)}$

Sanity check: expectation increases as coin is more biased.