Lecture 6

- The greedy strategy.

A property of the underlying problem that lends itself to a divide-n-conquer strategy can be called *optimal substructure*. By optimal substructure, we mean that solutions to subproblems can be efficiently composed to yield a solution to the original problem. (Of course, in this context "optimal" is being used rather loosely. For example, for the sorting problem, what we mean is that sorted sub-arrays can be composed efficiently to yield a sorted array.)

We now discuss the altogether different greedy strategy. The greedy strategy is to make a choice that is locally-optimal, in the hope that a sequence of such choices leads to a global optimum. As such a characterization of the strategy suggests, the greedy strategy is typically employed for optimization problems.

The greedy strategy does not always work. For it to work, the problem needs to possess, or satisfy, a so-called *greedy choice property*. The greedy choice property, as its name suggests, is that there is indeed a choice that is locally optimal, that leads to a globally optimal solution.

Fractional and 0-1 knapsack As an example, consider the following problem, which is called the fractional knapsack problem. A thief has a knapsack whose capacity is represented by some weight it can carry, W, a positive integer. He confronts several piles of gold dust. Each pile is a pair, $\langle w_i, v_i \rangle$, where w_i is the weight of the pile, and v_i is its total value. The thief can only carry as much total weight as his knapsack allows, W. And

he wants to maximise the value he steals.

The above fractional knapsack problem possesses the greedy choice property. The fractional value of each pile, v_i/w_i, is a greedy choice, which leads to a globally optimal solution. That is, the thief first fills his knapsack as much as possible with the pile of highest v_i/w_i value. Then moves on to the next pile of highest fractional value.

For example, given three piles, $p_1 = \langle 4, 8\rangle, p_2 = \langle 3, 5\rangle, p_3 = \langle 3, 4\rangle$, and a knapsack that can hold $W = 6$, the thief takes the entirety of pile p_1, as it has the highest fractional value of $8/4 = 2$. He also takes weight of 2 of the pile p_2, as it has the second highest fractional value of $5/3$.

Now consider a twist of the problem, which is called 0-1 knapsack. The thief is now confronted not by piles of gold dust, but by solid gold bars which cannot be subdivided. The "0-1" in this version reflects the fact that the thief must take either the entirety of a bar, or none of it.

This version does not possess the same greedy choice as fractional knapsack. Suppose, as in our example above, the thief is faced with three gold bars, $b_1 = \langle 4, 8\rangle, b_2 = \langle 3, 5\rangle, b_3 = \langle 3, 4\rangle$, and has a knapsack that can hold $W = 6$. Then, greedily picking b_1 is not wise, as the thief ends up with a total value of 8 only. Instead, the thief should pick b_2, b_3, which yields a total value of 9.

We emphasize that 0-1 knapsack does not posses the particular greedy choice that fractional knapsack does. It may possess some other greedy choice. That is, we are not categorically saying that 0-1 knapsack possesses no greedy choice. Indeed, if we can say that, then we will have resolved the most famous open problem on the planet, the question of whether $\mathbf{P} \neq \mathbf{NP}$. We revisit this later in the course.

Change making The 0-1 knapsack problem is similar to the problem of making change. In the change-making problem, our inputs are a vector, D, and an integer amount, A. We seek to make change for the amount A using the denominations of coins in D such that the fewest coins result. We assume that we have as many coins as we need of each denomination, and that the smallest denomination is $\textcircled{1}$; that way, we are guaranteed to be able to make change for any A.

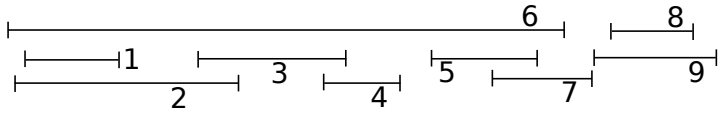


Figure 1: An example of an input to interval scheduling.

For example, for $D = \left\langle (1), (5), (10), (25) \right\rangle$, and $A = 123$, the solution is $4 \times (25) + 2 \times (10) + 0 \times (5) + 3 \times (1)$, for a total of 9 coins.

For some choices of D, change-making possesses a greedy choice. And that greedy choice is to always hand out as many coins of the highest denomination as possible, then the next highest, and so on. An example is expressed by the following claim.

Claim 44. *Suppose $D = \left\langle (1), (5), (10), (25) \right\rangle$. Then, change-making possesses the greedy choice of handing out most coins of the highest denomination first, then second-highest and so on.*

However, change-making does not possess such a greedy choice for every D. E.g., if $D = \left\langle (1), (3), (4) \right\rangle$, for $A = 6$, the greedy approach would be to hand out $1 \times (4) + 0 \times (3) + 2 \times (1)$, for a total of 3 coins. But we could instead hand out $0 \times (4) + 2 \times (3) + 0 \times (1)$.

Interval scheduling A third example of a problem that possesses a greedy choice property is a version of scheduling that is called interval scheduling. Our content here for this is from Kleinberg and Tardos, "Algorithm Design."

The problem is as follows. Given as input a set of meetings, where the i^{th} meeting, $m_i = \langle s_i, f_i \rangle$, where s_i is a start-time for the meeting, and f_i is the finish time, we want, as output, a maximum-sized subset of that set in which no two meetings are in conflict. Two meetings, $\langle s_i, f_i \rangle$ and $\langle s_j, f_j \rangle$ are said to conflict if $s_i < f_j$, and $f_i > s_j$; that is, one of them starts before the other finishes, and finishes after the other starts.

Figure 1 is an example of an input to interval scheduling. In the example in the figure, Meeting 1 conflicts with meetings 2 and 6, and Meeting 3 conflicts with meetings $2, 4$ and 6.

It may not be immediately obvious that interval scheduling possesses a greedy choice property. Indeed, some seemingly obvious greedy choices do not work. For example, picking a meeting with the earliest start time is not a valid greedy choice. This can be seen from the example in Figure 1; if we pick Meeting 6, then we can schedule only one other meeting, either 8 or 9. It turns out that some other possible choices, such as the meeting with the fewest conflicts, or the shortest meeting, does not work either.

What does work, i.e., is a valid greedy choice, however, is the earliest finish time. In the example in Figure 1, this would cause us to pick Meeting 1. Doing so forces us to eliminate Meetings 2 and 6 which conflict with it. We would then pick Meeting 3, which eliminates Meeting 4. Then, we would pick Meeting 5, which eliminates 7, and we would finally pick Meeting 8. What we mean when we say that the earliest finish time is a valid greedy choice is that we claim that the maximum number of meetings we can schedule for this input is four, and a specific set of such meetings is $\{1, 3, 5, 8\}$. Note that this set is not necessarily unique: we could have chosen Meeting 9 instead of 8, for example. However, the maximum number is unique; four, in this example.

We now prove that the earliest finish time is indeed a valid greedy choice. We do this by considering an output, $G = \{g_1, g_2, \ldots, g_k\}$ that this greedy choice results in, vs. an optimal set of meetings, $T = \{t_1, t_2, \ldots, t_m\}$. Suppose we represent the start time of a meeting as a function, $s(\cdot)$, and finish time as a function, $f(\cdot)$. We assume, without any loss of generality, that the meetings in the two sets are ordered by earliest finish time. Note that as the meetings in the respective sets are not in conflict, this implies that they are ordered by earliest start time as well.

We know, for the sets G and T, that $m \geq k$. Our objective is to prove that $k = m$. We do this in two stages. In the first, which we do in the following claim, we show that the greedy choice always causes us to "stay ahead" of any optimal choice.

Claim 45. *For every $i = 1, \ldots, k$, $f(g_i) \leq f(t_i)$.*

Proof. By induction on i. For the base case, consider $i = 1$. The greedy choice is to pick a meeting with the earliest finish time. This guarantees that $f(g_1) \leq f(t_1)$. For the step, assume that the assertion is true for $i =$

$1, \ldots, p-1$. For $i = p$, we know that $f(g_{p-1}) \leq f(t_{p-1}) \leq s(t_p)$. Thus, the meeting t_p does not conflict with g_{p-1}, and therefore, is available to be chosen after g_{p-1} is chosen. Thus, $f(g_p) \leq f(t_p)$ because we choose the meeting with the earliest finish time. □

Claim 46. $k = m$

Proof. Assume otherwise, for the purpose of contradiction, and that $m > k$. Then, there exists a meeting t_{k+1} in the set T. But, $f(g_k) \leq f(t_k)$ by Claim 45. Thus, the meeting t_{k+1} does not conflict with t_k, and therefore does not conflict with g_k, and is available to be chosen after g_k is chosen, which contradicts the claim that no more meetings are left that can be chosen after g_k is chosen. □

Single-source shortest paths for non-negative edge weights Yet another problem that possesses the greedy choice property is the problem of computing shortest distances and corresponding paths in directed or undirected graphs, when edge-weights are non-negative. Recall that we already discussed two algorithms for versions of this problem in the previous chapter: BFS and Bellman-Ford. It turns out that if we restrict edge-weights to non-negative reals only, then the problem possesses greedy choice. An algorithm is the well-known Dijkstra's algorithm. We reproduce content from CLRS on this. The algorithm employs a data structure called a *priority queue*. After the discussions on Dijkstra's algorithm, we include also content from CLRS on heaps, which provide a realization of priority queues.

24.3 Dijkstra's algorithm

Dijkstra's algorithm solves the single-source shortest-paths problem on a weighted, directed graph $G = (V, E)$ for the case in which all edge weights are nonnegative. In this section, therefore, we assume that $w(u, v) \geq 0$ for each edge $(u, v) \in E$. As we shall see, with a good implementation, the running time of Dijkstra's algorithm is lower than that of the Bellman-Ford algorithm.

Dijkstra's algorithm maintains a set S of vertices whose final shortest-path weights from the source s have already been determined. The algorithm repeatedly selects the vertex $u \in V - S$ with the minimum shortest-path estimate, adds u to S, and relaxes all edges leaving u. In the following implementation, we use a min-priority queue Q of vertices, keyed by their d values.

```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S ← ∅
3  Q ← V[G]
4  while Q ≠ ∅
5       do u ← EXTRACT-MIN(Q)
6          S ← S ∪ {u}
7          for each vertex v ∈ Adj[u]
8               do RELAX(u, v, w)
```

Dijkstra's algorithm relaxes edges as shown in Figure 24.6. Line 1 performs the usual initialization of d and π values, and line 2 initializes the set S to the empty set. The algorithm maintains the invariant that $Q = V - S$ at the start of each iteration of the **while** loop of lines 4–8. Line 3 initializes the min-priority queue Q to contain all the vertices in V; since $S = \emptyset$ at that time, the invariant is true after line 3. Each time through the **while** loop of lines 4–8, a vertex u is extracted from $Q = V - S$ and added to set S, thereby maintaining the invariant. (The first time through this loop, $u = s$.) Vertex u, therefore, has the smallest shortest-path

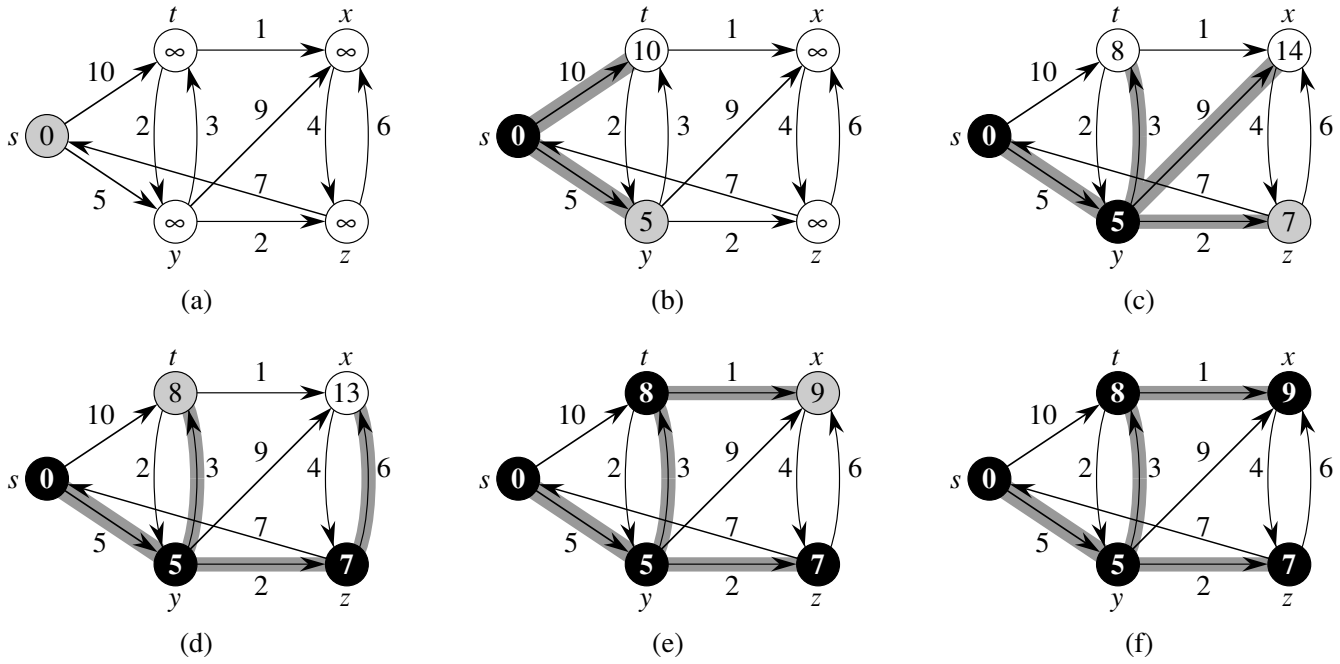


Figure 24.6 The execution of Dijkstra's algorithm. The source s is the leftmost vertex. The shortest-path estimates are shown within the vertices, and shaded edges indicate predecessor values. Black vertices are in the set S, and white vertices are in the min-priority queue $Q = V - S$. **(a)** The situation just before the first iteration of the **while** loop of lines 4–8. The shaded vertex has the minimum d value and is chosen as vertex u in line 5. **(b)–(f)** The situation after each successive iteration of the **while** loop. The shaded vertex in each part is chosen as vertex u in line 5 of the next iteration. The d and π values shown in part (f) are the final values.

estimate of any vertex in $V - S$. Then, lines 7–8 relax each edge (u, v) leaving u, thus updating the estimate $d[v]$ and the predecessor $\pi[v]$ if the shortest path to v can be improved by going through u. Observe that vertices are never inserted into Q after line 3 and that each vertex is extracted from Q and added to S exactly once, so that the **while** loop of lines 4–8 iterates exactly $|V|$ times.

Because Dijkstra's algorithm always chooses the "lightest" or "closest" vertex in $V - S$ to add to set S, we say that it uses a greedy strategy. Greedy strategies are presented in detail in Chapter 16, but you need not have read that chapter to understand Dijkstra's algorithm. Greedy strategies do not always yield optimal results in general, but as the following theorem and its corollary show, Dijkstra's algorithm does indeed compute shortest paths. The key is to show that each time a vertex u is added to set S, we have $d[u] = \delta(s, u)$.

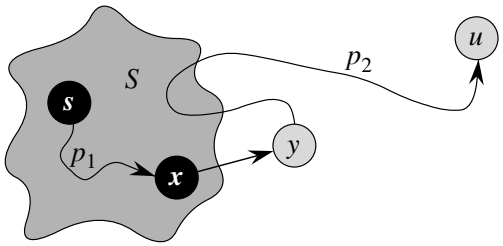


Figure 24.7 The proof of Theorem 24.6. Set S is nonempty just before vertex u is added to it. A shortest path p from source s to vertex u can be decomposed into $s \overset{p_1}{\rightsquigarrow} x \rightarrow y \overset{p_2}{\rightsquigarrow} u$, where y is the first vertex on the path that is not in S and $x \in S$ immediately precedes y. Vertices x and y are distinct, but we may have $s = x$ or $y = u$. Path p_2 may or may not reenter set S.

Theorem 24.6 (Correctness of Dijkstra's algorithm)
Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with nonnegative weight function w and source s, terminates with $d[u] = \delta(s, u)$ for all vertices $u \in V$.

Proof We use the following loop invariant:

> At the start of each iteration of the **while** loop of lines 4–8, $d[v] = \delta(s, v)$ for each vertex $v \in S$.

It suffices to show for each vertex $u \in V$, we have $d[u] = \delta(s, u)$ at the time when u is added to set S. Once we show that $d[u] = \delta(s, u)$, we rely on the upper-bound property to show that the equality holds at all times thereafter.

Initialization: Initially, $S = \emptyset$, and so the invariant is trivially true.

Maintenance: We wish to show that in each iteration, $d[u] = \delta(s, u)$ for the vertex added to set S. For the purpose of contradiction, let u be the first vertex for which $d[u] \neq \delta(s, u)$ when it is added to set S. We shall focus our attention on the situation at the beginning of the iteration of the **while** loop in which u is added to S and derive the contradiction that $d[u] = \delta(s, u)$ at that time by examining a shortest path from s to u. We must have $u \neq s$ because s is the first vertex added to set S and $d[s] = \delta(s, s) = 0$ at that time. Because $u \neq s$, we also have that $S \neq \emptyset$ just before u is added to S. There must be some path from s to u, for otherwise $d[u] = \delta(s, u) = \infty$ by the no-path property, which would violate our assumption that $d[u] \neq \delta(s, u)$. Because there is at least one path, there is a shortest path p from s to u. Prior to adding u to S, path p connects a vertex in S, namely s, to a vertex in $V - S$, namely u. Let us consider the first vertex y along p such that $y \in V - S$, and let $x \in S$ be y's predecessor. Thus, as shown in Figure 24.7, path p can be decomposed as $s \overset{p_1}{\rightsquigarrow} x \rightarrow y \overset{p_2}{\rightsquigarrow} u$. (Either of paths p_1 or p_2 may have no edges.)

We claim that $d[y] = \delta(s, y)$ when u is added to S. To prove this claim, observe that $x \in S$. Then, because u is chosen as the first vertex for which $d[u] \neq \delta(s, u)$ when it is added to S, we had $d[x] = \delta(s, x)$ when x was added to S. Edge (x, y) was relaxed at that time, so the claim follows from the convergence property.

We can now obtain a contradiction to prove that $d[u] = \delta(s, u)$. Because y occurs before u on a shortest path from s to u and all edge weights are nonnegative (notably those on path p_2), we have $\delta(s, y) \leq \delta(s, u)$, and thus

$$\begin{aligned} d[y] &= \delta(s, y) \\ &\leq \delta(s, u) \\ &\leq d[u] \qquad \text{(by the upper-bound property)} \ . \end{aligned} \tag{24.2}$$

But because both vertices u and y were in $V - S$ when u was chosen in line 5, we have $d[u] \leq d[y]$. Thus, the two inequalities in (24.2) are in fact equalities, giving

$$d[y] = \delta(s, y) = \delta(s, u) = d[u] \ .$$

Consequently, $d[u] = \delta(s, u)$, which contradicts our choice of u. We conclude that $d[u] = \delta(s, u)$ when u is added to S, and that this equality is maintained at all times thereafter.

Termination: At termination, $Q = \emptyset$ which, along with our earlier invariant that $Q = V - S$, implies that $S = V$. Thus, $d[u] = \delta(s, u)$ for all vertices $u \in V$. ■

Corollary 24.7

If we run Dijkstra's algorithm on a weighted, directed graph $G = (V, E)$ with nonnegative weight function w and source s, then at termination, the predecessor subgraph G_π is a shortest-paths tree rooted at s.

Proof Immediate from Theorem 24.6 and the predecessor-subgraph property. ■

Analysis

How fast is Dijkstra's algorithm? It maintains the min-priority queue Q by calling three priority-queue operations: INSERT (implicit in line 3), EXTRACT-MIN (line 5), and DECREASE-KEY (implicit in RELAX, which is called in line 8). INSERT is invoked once per vertex, as is EXTRACT-MIN. Because each vertex $v \in V$ is added to set S exactly once, each edge in the adjacency list $Adj[v]$ is examined in the **for** loop of lines 7–8 exactly once during the course of the algorithm. Since the total number of edges in all the adjacency lists is $|E|$, there are a total of $|E|$ iterations of this **for** loop, and thus a total of at most $|E|$ DECREASE-KEY operations. (Observe once again that we are using aggregate analysis.)

The running time of Dijkstra's algorithm depends on how the min-priority queue is implemented. Consider first the case in which we maintain the min-priority queue by taking advantage of the vertices being numbered 1 to $|V|$. We simply store $d[v]$ in the vth entry of an array. Each INSERT and DECREASE-KEY operation takes $O(1)$ time, and each EXTRACT-MIN operation takes $O(V)$ time (since we have to search through the entire array), for a total time of $O(V^2+E) = O(V^2)$.

If the graph is sufficiently sparse—in particular, $E = o(V^2/\lg V)$—it is practical to implement the min-priority queue with a binary min-heap. (As discussed in Section 6.5, an important implementation detail is that vertices and corresponding heap elements must maintain handles to each other.) Each EXTRACT-MIN operation then takes time $O(\lg V)$. As before, there are $|V|$ such operations. The time to build the binary min-heap is $O(V)$. Each DECREASE-KEY operation takes time $O(\lg V)$, and there are still at most $|E|$ such operations. The total running time is therefore $O((V + E)\lg V)$, which is $O(E \lg V)$ if all vertices are reachable from the source. This running time is an improvement over the straightforward $O(V^2)$-time implementation if $E = o(V^2/\lg V)$.

We can in fact achieve a running time of $O(V \lg V + E)$ by implementing the min-priority queue with a Fibonacci heap (see Chapter 20). The amortized cost of each of the $|V|$ EXTRACT-MIN operations is $O(\lg V)$, and each DECREASE-KEY call, of which there are at most $|E|$, takes only $O(1)$ amortized time. Historically, the development of Fibonacci heaps was motivated by the observation that in Dijkstra's algorithm there are typically many more DECREASE-KEY calls than EXTRACT-MIN calls, so any method of reducing the amortized time of each DECREASE-KEY operation to $o(\lg V)$ without increasing the amortized time of EXTRACT-MIN would yield an asymptotically faster implementation than with binary heaps.

Dijkstra's algorithm bears some similarity to both breadth-first search (see Section 22.2) and Prim's algorithm for computing minimum spanning trees (see Section 23.2). It is like breadth-first search in that set S corresponds to the set of black vertices in a breadth-first search; just as vertices in S have their final shortest-path weights, so do black vertices in a breadth-first search have their correct breadth-first distances. Dijkstra's algorithm is like Prim's algorithm in that both algorithms use a min-priority queue to find the "lightest" vertex outside a given set (the set S in Dijkstra's algorithm and the tree being grown in Prim's algorithm), add this vertex into the set, and adjust the weights of the remaining vertices outside the set accordingly.

6.1 Heaps

The ***(binary) heap*** data structure is an array object that we can view as a nearly complete binary tree (see Section B.5.3), as shown in Figure 6.1. Each node of the tree corresponds to an element of the array. The tree is completely filled on all levels except possibly the lowest, which is filled from the left up to a point. An array A that represents a heap is an object with two attributes: $A.length$, which (as usual) gives the number of elements in the array, and $A.heap\text{-}size$, which represents how many elements in the heap are stored within array A. That is, although $A[1..A.length]$ may contain numbers, only the elements in $A[1..A.heap\text{-}size]$, where $0 \leq A.heap\text{-}size \leq A.length$, are valid elements of the heap. The root of the tree is $A[1]$, and given the index i of a node, we can easily compute the indices of its parent, left child, and right child:

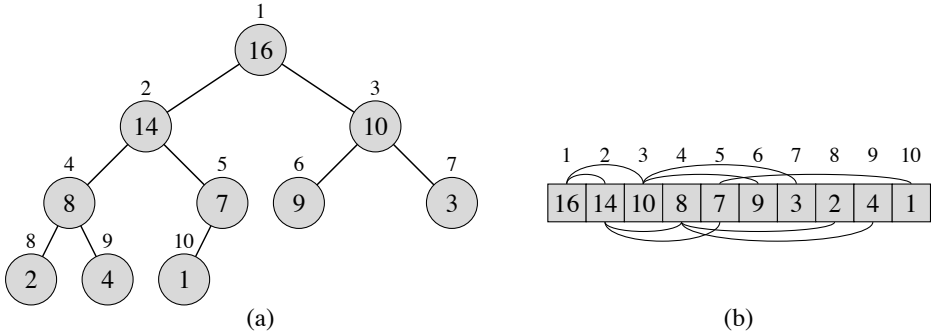


Figure 6.1 A max-heap viewed as **(a)** a binary tree and **(b)** an array. The number within the circle at each node in the tree is the value stored at that node. The number above a node is the corresponding index in the array. Above and below the array are lines showing parent-child relationships; parents are always to the left of their children. The tree has height three; the node at index 4 (with value 8) has height one.

```
PARENT(i)
1  return ⌊i/2⌋

LEFT(i)
1  return 2i

RIGHT(i)
1  return 2i + 1
```

On most computers, the LEFT procedure can compute $2i$ in one instruction by simply shifting the binary representation of i left by one bit position. Similarly, the RIGHT procedure can quickly compute $2i + 1$ by shifting the binary representation of i left by one bit position and then adding in a 1 as the low-order bit. The PARENT procedure can compute $\lfloor i/2 \rfloor$ by shifting i right one bit position. Good implementations of heapsort often implement these procedures as "macros" or "in-line" procedures.

There are two kinds of binary heaps: max-heaps and min-heaps. In both kinds, the values in the nodes satisfy a ***heap property***, the specifics of which depend on the kind of heap. In a ***max-heap***, the ***max-heap property*** is that for every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i] ,$$

that is, the value of a node is at most the value of its parent. Thus, the largest element in a max-heap is stored at the root, and the subtree rooted at a node contains

values no larger than that contained at the node itself. A ***min-heap*** is organized in the opposite way; the ***min-heap property*** is that for every node i other than the root,

$$A[\text{PARENT}(i)] \leq A[i] .$$

The smallest element in a min-heap is at the root.

For the heapsort algorithm, we use max-heaps. Min-heaps commonly implement priority queues, which we discuss in Section 6.5. We shall be precise in specifying whether we need a max-heap or a min-heap for any particular application, and when properties apply to either max-heaps or min-heaps, we just use the term "heap."

Viewing a heap as a tree, we define the ***height*** of a node in a heap to be the number of edges on the longest simple downward path from the node to a leaf, and we define the height of the heap to be the height of its root. Since a heap of n elements is based on a complete binary tree, its height is $\Theta(\lg n)$ (see Exercise 6.1-2). We shall see that the basic operations on heaps run in time at most proportional to the height of the tree and thus take $O(\lg n)$ time. The remainder of this chapter presents some basic procedures and shows how they are used in a sorting algorithm and a priority-queue data structure.

- The MAX-HEAPIFY procedure, which runs in $O(\lg n)$ time, is the key to maintaining the max-heap property.
- The BUILD-MAX-HEAP procedure, which runs in linear time, produces a max-heap from an unordered input array.
- The HEAPSORT procedure, which runs in $O(n \lg n)$ time, sorts an array in place.
- The MAX-HEAP-INSERT, HEAP-EXTRACT-MAX, HEAP-INCREASE-KEY, and HEAP-MAXIMUM procedures, which run in $O(\lg n)$ time, allow the heap data structure to implement a priority queue.

6.2 Maintaining the heap property

In order to maintain the max-heap property, we call the procedure MAX-HEAPIFY. Its inputs are an array A and an index i into the array. When it is called, MAX-HEAPIFY assumes that the binary trees rooted at LEFT(i) and RIGHT(i) are max-heaps, but that $A[i]$ might be smaller than its children, thus violating the max-heap property. MAX-HEAPIFY lets the value at $A[i]$ "float down" in the max-heap so that the subtree rooted at index i obeys the max-heap property.

```
MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
 2  r = RIGHT(i)
 3  if l ≤ A.heap-size and A[l] > A[i]
 4      largest = l
 5  else largest = i
 6  if r ≤ A.heap-size and A[r] > A[largest]
 7      largest = r
 8  if largest ≠ i
 9      exchange A[i] with A[largest]
10      MAX-HEAPIFY(A, largest)
```

Figure 6.2 illustrates the action of MAX-HEAPIFY. At each step, the largest of the elements $A[i]$, $A[\text{LEFT}(i)]$, and $A[\text{RIGHT}(i)]$ is determined, and its index is stored in *largest*. If $A[i]$ is largest, then the subtree rooted at node i is already a max-heap and the procedure terminates. Otherwise, one of the two children has the largest element, and $A[i]$ is swapped with $A[largest]$, which causes node i and its

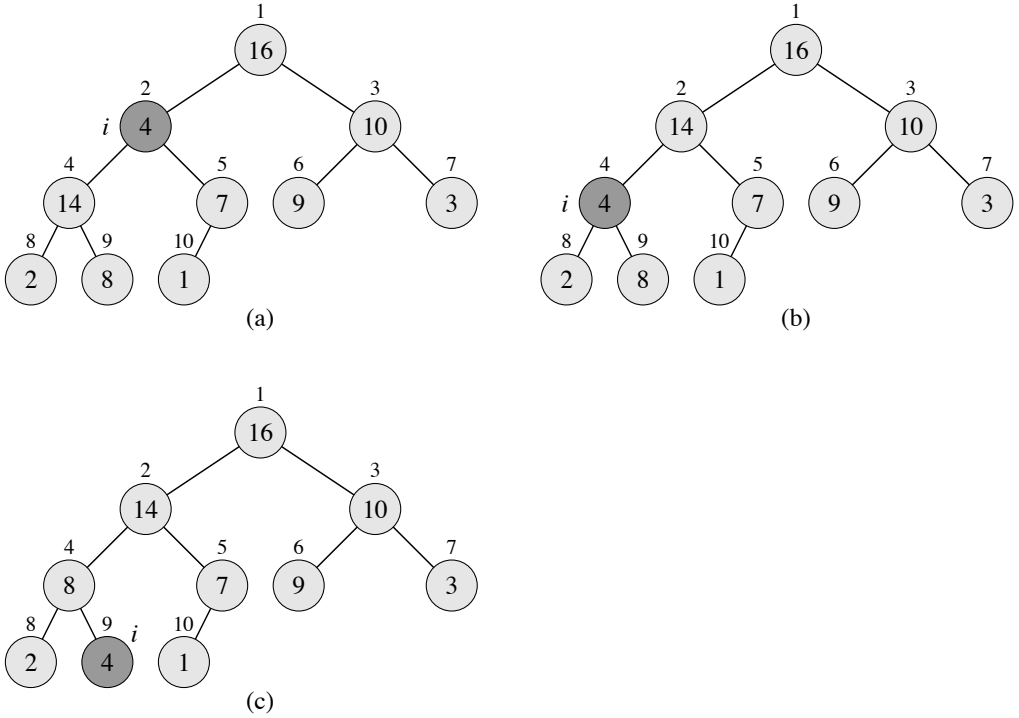


Figure 6.2 The action of MAX-HEAPIFY$(A, 2)$, where $A.heap\text{-}size = 10$. **(a)** The initial configuration, with $A[2]$ at node $i = 2$ violating the max-heap property since it is not larger than both children. The max-heap property is restored for node 2 in **(b)** by exchanging $A[2]$ with $A[4]$, which destroys the max-heap property for node 4. The recursive call MAX-HEAPIFY$(A, 4)$ now has $i = 4$. After swapping $A[4]$ with $A[9]$, as shown in **(c)**, node 4 is fixed up, and the recursive call MAX-HEAPIFY$(A, 9)$ yields no further change to the data structure.

children to satisfy the max-heap property. The node indexed by *largest*, however, now has the original value $A[i]$, and thus the subtree rooted at *largest* might violate the max-heap property. Consequently, we call MAX-HEAPIFY recursively on that subtree.

The running time of MAX-HEAPIFY on a subtree of size n rooted at a given node i is the $\Theta(1)$ time to fix up the relationships among the elements $A[i]$, $A[\text{LEFT}(i)]$, and $A[\text{RIGHT}(i)]$, plus the time to run MAX-HEAPIFY on a subtree rooted at one of the children of node i (assuming that the recursive call occurs). The children's subtrees each have size at most $2n/3$—the worst case occurs when the bottom level of the tree is exactly half full—and therefore we can describe the running time of MAX-HEAPIFY by the recurrence

$$T(n) \le T(2n/3) + \Theta(1) .$$

The solution to this recurrence, by case 2 of the master theorem (Theorem 4.1), is $T(n) = O(\lg n)$. Alternatively, we can characterize the running time of MAX-HEAPIFY on a node of height h as $O(h)$.

6.3 Building a heap

We can use the procedure MAX-HEAPIFY in a bottom-up manner to convert an array $A[1..n]$, where $n = A.length$, into a max-heap. By Exercise 6.1-7, the elements in the subarray $A[(\lfloor n/2 \rfloor + 1)..n]$ are all leaves of the tree, and so each is

a 1-element heap to begin with. The procedure BUILD-MAX-HEAP goes through the remaining nodes of the tree and runs MAX-HEAPIFY on each one.

```
BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

Figure 6.3 shows an example of the action of BUILD-MAX-HEAP.

To show why BUILD-MAX-HEAP works correctly, we use the following loop invariant:

> At the start of each iteration of the **for** loop of lines 2–3, each node $i + 1$, $i + 2, \ldots, n$ is the root of a max-heap.

We need to show that this invariant is true prior to the first loop iteration, that each iteration of the loop maintains the invariant, and that the invariant provides a useful property to show correctness when the loop terminates.

Initialization: Prior to the first iteration of the loop, $i = \lfloor n/2 \rfloor$. Each node $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$ is a leaf and is thus the root of a trivial max-heap.

Maintenance: To see that each iteration maintains the loop invariant, observe that the children of node i are numbered higher than i. By the loop invariant, therefore, they are both roots of max-heaps. This is precisely the condition required for the call MAX-HEAPIFY(A, i) to make node i a max-heap root. Moreover, the MAX-HEAPIFY call preserves the property that nodes $i + 1, i + 2, \ldots, n$ are all roots of max-heaps. Decrementing i in the **for** loop update reestablishes the loop invariant for the next iteration.

Termination: At termination, $i = 0$. By the loop invariant, each node $1, 2, \ldots, n$ is the root of a max-heap. In particular, node 1 is.

We can compute a simple upper bound on the running time of BUILD-MAX-HEAP as follows. Each call to MAX-HEAPIFY costs $O(\lg n)$ time, and BUILD-MAX-HEAP makes $O(n)$ such calls. Thus, the running time is $O(n \lg n)$. This upper bound, though correct, is not asymptotically tight.

We can derive a tighter bound by observing that the time for MAX-HEAPIFY to run at a node varies with the height of the node in the tree, and the heights of most nodes are small. Our tighter analysis relies on the properties that an n-element heap has height $\lfloor \lg n \rfloor$ (see Exercise 6.1-2) and at most $\lceil n/2^{h+1} \rceil$ nodes of any height h (see Exercise 6.3-3).

The time required by MAX-HEAPIFY when called on a node of height h is $O(h)$, and so we can express the total cost of BUILD-MAX-HEAP as being bounded from above by

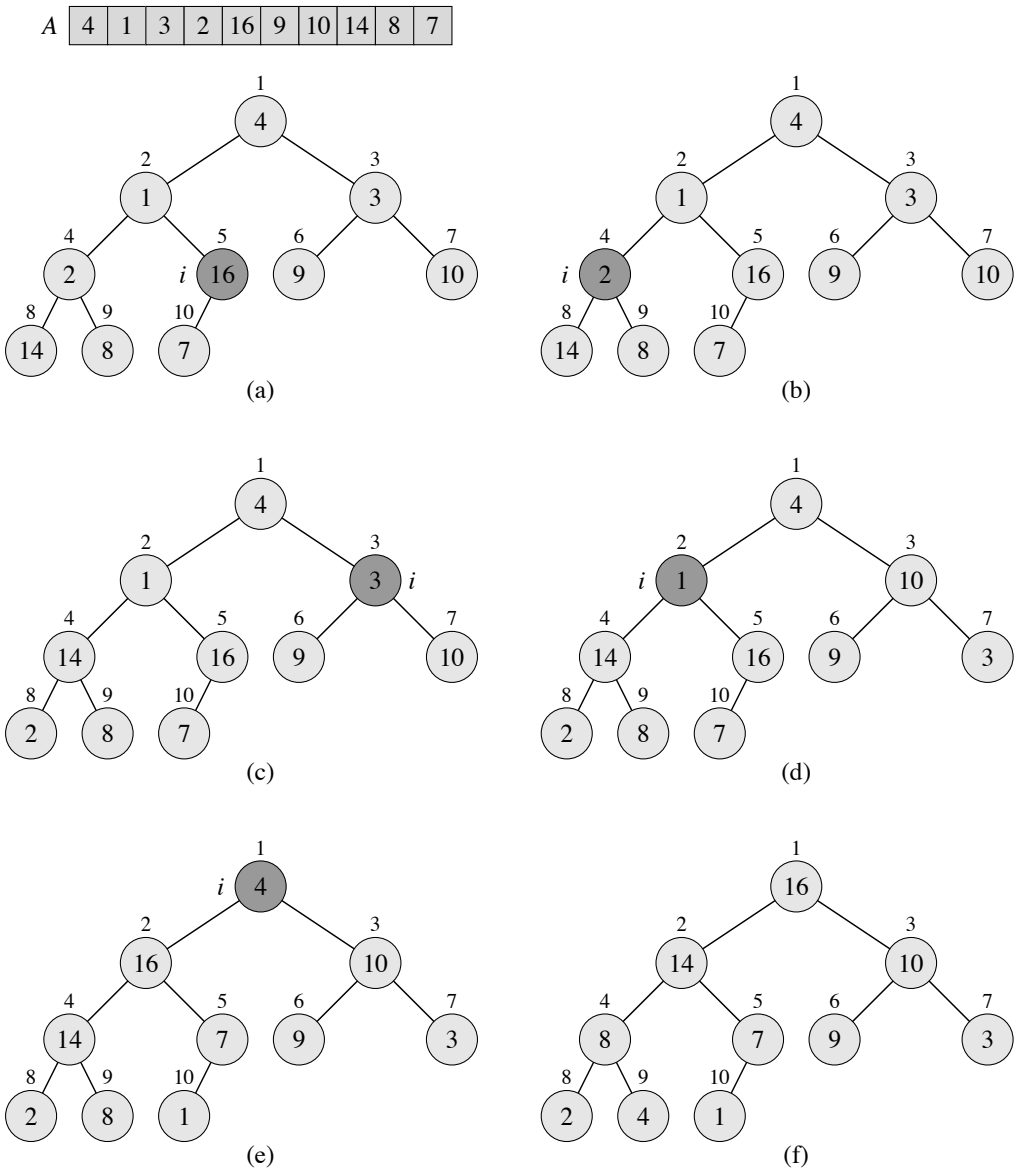


Figure 6.3 The operation of BUILD-MAX-HEAP, showing the data structure before the call to MAX-HEAPIFY in line 3 of BUILD-MAX-HEAP. **(a)** A 10-element input array A and the binary tree it represents. The figure shows that the loop index i refers to node 5 before the call MAX-HEAPIFY(A, i). **(b)** The data structure that results. The loop index i for the next iteration refers to node 4. **(c)–(e)** Subsequent iterations of the **for** loop in BUILD-MAX-HEAP. Observe that whenever MAX-HEAPIFY is called on a node, the two subtrees of that node are both max-heaps. **(f)** The max-heap after BUILD-MAX-HEAP finishes.

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) .$$

We evaluate the last summation by substituting $x = 1/2$ in the formula (A.8), yielding

$$\begin{aligned} \sum_{h=0}^{\infty} \frac{h}{2^h} &= \frac{1/2}{(1-1/2)^2} \\ &= 2 . \end{aligned}$$

Thus, we can bound the running time of BUILD-MAX-HEAP as

$$\begin{aligned} O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) &= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) \\ &= O(n) . \end{aligned}$$

Hence, we can build a max-heap from an unordered array in linear time.

We can build a min-heap by the procedure BUILD-MIN-HEAP, which is the same as BUILD-MAX-HEAP but with the call to MAX-HEAPIFY in line 3 replaced by a call to MIN-HEAPIFY (see Exercise 6.2-2). BUILD-MIN-HEAP produces a min-heap from an unordered linear array in linear time.

6.5 Priority queues

Heapsort is an excellent algorithm, but a good implementation of quicksort, presented in Chapter 7, usually beats it in practice. Nevertheless, the heap data structure itself has many uses. In this section, we present one of the most popular applications of a heap: as an efficient priority queue. As with heaps, priority queues come in two forms: max-priority queues and min-priority queues. We will focus here on how to implement max-priority queues, which are in turn based on max-heaps; Exercise 6.5-3 asks you to write the procedures for min-priority queues.

A ***priority queue*** is a data structure for maintaining a set S of elements, each with an associated value called a ***key***. A ***max-priority queue*** supports the following operations:

INSERT(S, x) inserts the element x into the set S, which is equivalent to the operation $S = S \cup \{x\}$.

MAXIMUM(S) returns the element of S with the largest key.

EXTRACT-MAX(S) removes and returns the element of S with the largest key.

INCREASE-KEY(S, x, k) increases the value of element x's key to the new value k, which is assumed to be at least as large as x's current key value.

Among their other applications, we can use max-priority queues to schedule jobs on a shared computer. The max-priority queue keeps track of the jobs to be performed and their relative priorities. When a job is finished or interrupted, the scheduler selects the highest-priority job from among those pending by calling EXTRACT-MAX. The scheduler can add a new job to the queue at any time by calling INSERT.

Alternatively, a ***min-priority queue*** supports the operations INSERT, MINIMUM, EXTRACT-MIN, and DECREASE-KEY. A min-priority queue can be used in an event-driven simulator. The items in the queue are events to be simulated, each with an associated time of occurrence that serves as its key. The events must be simulated in order of their time of occurrence, because the simulation of an event can cause other events to be simulated in the future. The simulation program calls EXTRACT-MIN at each step to choose the next event to simulate. As new events are produced, the simulator inserts them into the min-priority queue by calling INSERT.

We shall see other uses for min-priority queues, highlighting the DECREASE-KEY operation, in Chapters 23 and 24.

Not surprisingly, we can use a heap to implement a priority queue. In a given application, such as job scheduling or event-driven simulation, elements of a priority queue correspond to objects in the application. We often need to determine which application object corresponds to a given priority-queue element, and vice versa. When we use a heap to implement a priority queue, therefore, we often need to store a ***handle*** to the corresponding application object in each heap element. The exact makeup of the handle (such as a pointer or an integer) depends on the application. Similarly, we need to store a handle to the corresponding heap element in each application object. Here, the handle would typically be an array index. Because heap elements change locations within the array during heap operations, an actual implementation, upon relocating a heap element, would also have to update the array index in the corresponding application object. Because the details of accessing application objects depend heavily on the application and its implementation, we shall not pursue them here, other than noting that in practice, these handles do need to be correctly maintained.

Now we discuss how to implement the operations of a max-priority queue. The procedure HEAP-MAXIMUM implements the MAXIMUM operation in $\Theta(1)$ time.

```
HEAP-MAXIMUM(A)
1  return A[1]
```

The procedure HEAP-EXTRACT-MAX implements the EXTRACT-MAX operation. It is similar to the **for** loop body (lines 3–5) of the HEAPSORT procedure.

```
HEAP-EXTRACT-MAX(A)
1  if A.heap-size < 1
2       error "heap underflow"
3  max = A[1]
4  A[1] = A[A.heap-size]
5  A.heap-size = A.heap-size − 1
6  MAX-HEAPIFY(A, 1)
7  return max
```

The running time of HEAP-EXTRACT-MAX is $O(\lg n)$, since it performs only a constant amount of work on top of the $O(\lg n)$ time for MAX-HEAPIFY.

The procedure HEAP-INCREASE-KEY implements the INCREASE-KEY operation. An index i into the array identifies the priority-queue element whose key we wish to increase. The procedure first updates the key of element $A[i]$ to its new value. Because increasing the key of $A[i]$ might violate the max-heap property,

the procedure then, in a manner reminiscent of the insertion loop (lines 5–7) of INSERTION-SORT from Section 2.1, traverses a simple path from this node toward the root to find a proper place for the newly increased key. As HEAP-INCREASE-KEY traverses this path, it repeatedly compares an element to its parent, exchanging their keys and continuing if the element's key is larger, and terminating if the element's key is smaller, since the max-heap property now holds. (See Exercise 6.5-5 for a precise loop invariant.)

```
HEAP-INCREASE-KEY(A, i, key)
1  if key < A[i]
2       error "new key is smaller than current key"
3  A[i] = key
4  while i > 1 and A[PARENT(i)] < A[i]
5       exchange A[i] with A[PARENT(i)]
6       i = PARENT(i)
```

Figure 6.5 shows an example of a HEAP-INCREASE-KEY operation. The running time of HEAP-INCREASE-KEY on an n-element heap is $O(\lg n)$, since the path traced from the node updated in line 3 to the root has length $O(\lg n)$.

The procedure MAX-HEAP-INSERT implements the INSERT operation. It takes as an input the key of the new element to be inserted into max-heap A. The procedure first expands the max-heap by adding to the tree a new leaf whose key is $-\infty$. Then it calls HEAP-INCREASE-KEY to set the key of this new node to its correct value and maintain the max-heap property.

```
MAX-HEAP-INSERT(A, key)
1  A.heap-size = A.heap-size + 1
2  A[A.heap-size] = −∞
3  HEAP-INCREASE-KEY(A, A.heap-size, key)
```

The running time of MAX-HEAP-INSERT on an n-element heap is $O(\lg n)$.

In summary, a heap can support any priority-queue operation on a set of size n in $O(\lg n)$ time.

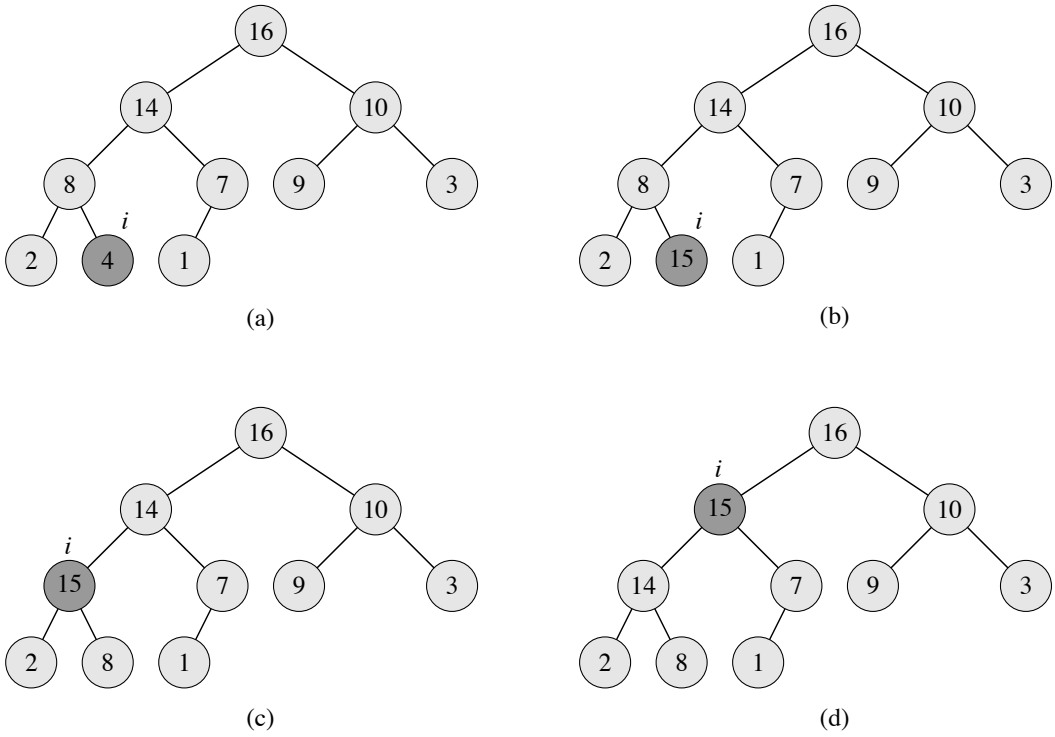


Figure 6.5 The operation of HEAP-INCREASE-KEY. **(a)** The max-heap of Figure 6.4(a) with a node whose index is i heavily shaded. **(b)** This node has its key increased to 15. **(c)** After one iteration of the **while** loop of lines 4–6, the node and its parent have exchanged keys, and the index i moves up to the parent. **(d)** The max-heap after one more iteration of the **while** loop. At this point, $A[\text{PARENT}(i)] \geq A[i]$. The max-heap property now holds and the procedure terminates.

Minimum spanning tree It turns out that a simple tweak to RELAX turns Dijkstra's algorithm in to a correct and efficient algorithm for the altogether different problem of determining a *minimum spanning tree* for a connected undirected graph. We now reproduce from CLRS background on the problem, and Kruskal's and Prim's algorithms, both of which are greedy. The latter is exactly Dijkstra's algorithm with a different version of RELAX. We revisit this after the content from CLRS.

23 Minimum Spanning Trees

In the design of electronic circuitry, it is often necessary to make the pins of several components electrically equivalent by wiring them together. To interconnect a set of n pins, we can use an arrangement of $n - 1$ wires, each connecting two pins. Of all such arrangements, the one that uses the least amount of wire is usually the most desirable.

We can model this wiring problem with a connected, undirected graph $G = (V, E)$, where V is the set of pins, E is the set of possible interconnections between pairs of pins, and for each edge $(u, v) \in E$, we have a weight $w(u, v)$ specifying the cost (amount of wire needed) to connect u and v. We then wish to find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

is minimized. Since T is acyclic and connects all of the vertices, it must form a tree, which we call a ***spanning tree*** since it "spans" the graph G. We call the problem of determining the tree T the ***minimum-spanning-tree problem***.[1] Figure 23.1 shows an example of a connected graph and its minimum spanning tree.

In this chapter, we shall examine two algorithms for solving the minimum-spanning-tree problem: Kruskal's algorithm and Prim's algorithm. Each can easily be made to run in time $O(E \lg V)$ using ordinary binary heaps. By using Fibonacci heaps, Prim's algorithm can be sped up to run in time $O(E + V \lg V)$, which is an improvement if $|V|$ is much smaller than $|E|$.

The two algorithms are greedy algorithms, as described in Chapter 16. At each step of an algorithm, one of several possible choices must be made. The greedy strategy advocates making the choice that is the best at the moment. Such a strategy is not generally guaranteed to find globally optimal solutions to problems. For

[1] The phrase "minimum spanning tree" is a shortened form of the phrase "minimum-weight spanning tree." We are not, for example, minimizing the number of edges in T, since all spanning trees have exactly $|V| - 1$ edges by Theorem B.2.

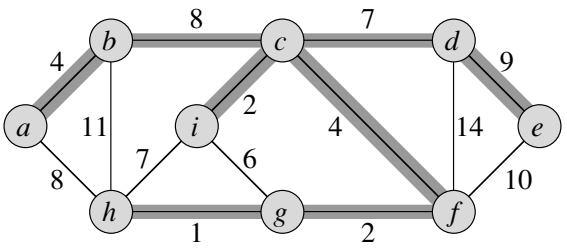


Figure 23.1 A minimum spanning tree for a connected graph. The weights on edges are shown, and the edges in a minimum spanning tree are shaded. The total weight of the tree shown is 37. This minimum spanning tree is not unique: removing the edge (b, c) and replacing it with the edge (a, h) yields another spanning tree with weight 37.

the minimum-spanning-tree problem, however, we can prove that certain greedy strategies do yield a spanning tree with minimum weight. Although the present chapter can be read independently of Chapter 16, the greedy methods presented here are a classic application of the theoretical notions introduced there.

Section 23.1 introduces a "generic" minimum-spanning-tree algorithm that grows a spanning tree by adding one edge at a time. Section 23.2 gives two ways to implement the generic algorithm. The first algorithm, due to Kruskal, is similar to the connected-components algorithm from Section 21.1. The second, due to Prim, is similar to Dijkstra's shortest-paths algorithm (Section 24.3).

23.1 Growing a minimum spanning tree

Assume that we have a connected, undirected graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}$, and we wish to find a minimum spanning tree for G. The two algorithms we consider in this chapter use a greedy approach to the problem, although they differ in how they apply this approach.

This greedy strategy is captured by the following "generic" algorithm, which grows the minimum spanning tree one edge at a time. The algorithm manages a set of edges A, maintaining the following loop invariant:

Prior to each iteration, A is a subset of some minimum spanning tree.

At each step, we determine an edge (u, v) that can be added to A without violating this invariant, in the sense that $A \cup \{(u, v)\}$ is also a subset of a minimum spanning tree. We call such an edge a ***safe edge*** for A, since it can be safely added to A while maintaining the invariant.

```
GENERIC-MST(G, w)
1  A ← ∅
2  while A does not form a spanning tree
3       do find an edge (u, v) that is safe for A
4          A ← A ∪ {(u, v)}
5  return A
```

We use the loop invariant as follows:

Initialization: After line 1, the set A trivially satisfies the loop invariant.

Maintenance: The loop in lines 2–4 maintains the invariant by adding only safe edges.

Termination: All edges added to A are in a minimum spanning tree, and so the set A is returned in line 5 must be a minimum spanning tree.

The tricky part is, of course, finding a safe edge in line 3. One must exist, since when line 3 is executed, the invariant dictates that there is a spanning tree T such that $A \subseteq T$. Within the **while** loop body, A must be a proper subset of T, and therefore there must be an edge $(u, v) \in T$ such that $(u, v) \notin A$ and (u, v) is safe for A.

In the remainder of this section, we provide a rule (Theorem 23.1) for recognizing safe edges. The next section describes two algorithms that use this rule to find safe edges efficiently.

We first need some definitions. A ***cut*** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V. Figure 23.2 illustrates this notion. We say that an edge $(u, v) \in E$ ***crosses*** the cut $(S, V - S)$ if one of its endpoints is in S and the other is in $V - S$. We say that a cut ***respects*** a set A of edges if no edge in A crosses the cut. An edge is a ***light edge*** crossing a cut if its weight is the minimum of any edge crossing the cut. Note that there can be more than one light edge crossing a cut in the case of ties. More generally, we say that an edge is a ***light edge*** satisfying a given property if its weight is the minimum of any edge satisfying the property.

Our rule for recognizing safe edges is given by the following theorem.

Theorem 23.1

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A.

Proof Let T be a minimum spanning tree that includes A, and assume that T does not contain the light edge (u, v), since if it does, we are done. We shall

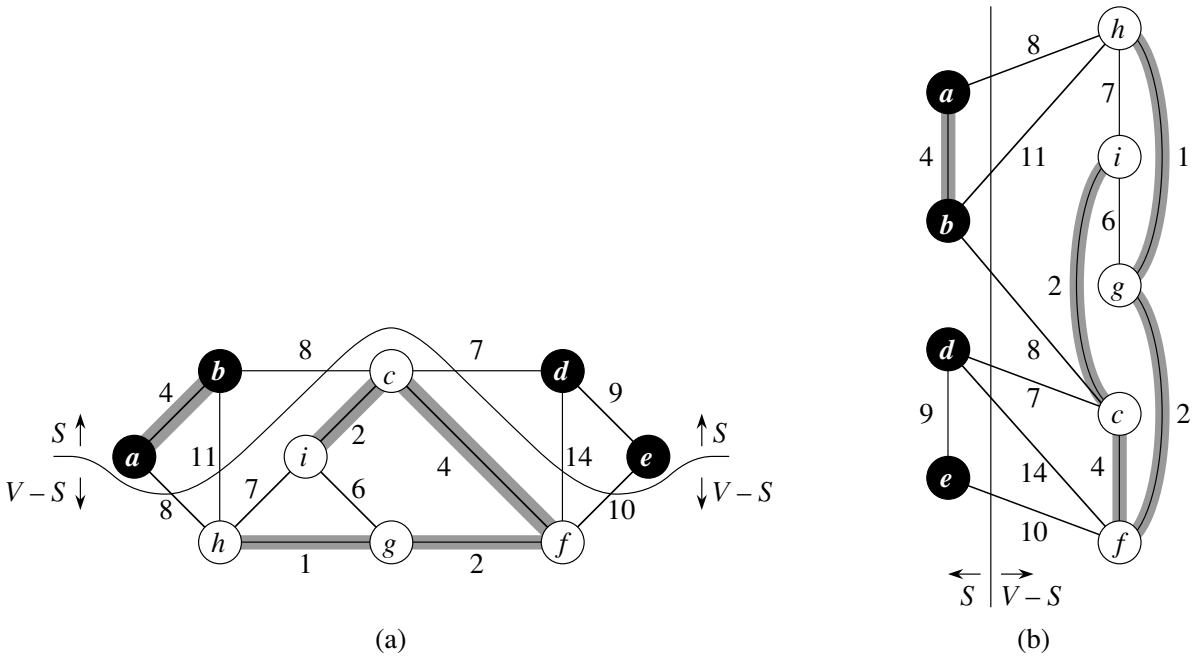


Figure 23.2 Two ways of viewing a cut $(S, V - S)$ of the graph from Figure 23.1. **(a)** The vertices in the set S are shown in black, and those in $V - S$ are shown in white. The edges crossing the cut are those connecting white vertices with black vertices. The edge (d, c) is the unique light edge crossing the cut. A subset A of the edges is shaded; note that the cut $(S, V - S)$ respects A, since no edge of A crosses the cut. **(b)** The same graph with the vertices in the set S on the left and the vertices in the set $V - S$ on the right. An edge crosses the cut if it connects a vertex on the left with a vertex on the right.

construct another minimum spanning tree T' that includes $A \cup \{(u, v)\}$ by using a cut-and-paste technique, thereby showing that (u, v) is a safe edge for A.

The edge (u, v) forms a cycle with the edges on the path p from u to v in T, as illustrated in Figure 23.3. Since u and v are on opposite sides of the cut $(S, V - S)$, there is at least one edge in T on the path p that also crosses the cut. Let (x, y) be any such edge. The edge (x, y) is not in A, because the cut respects A. Since (x, y) is on the unique path from u to v in T, removing (x, y) breaks T into two components. Adding (u, v) reconnects them to form a new spanning tree $T' = T - \{(x, y)\} \cup \{(u, v)\}$.

We next show that T' is a minimum spanning tree. Since (u, v) is a light edge crossing $(S, V - S)$ and (x, y) also crosses this cut, $w(u, v) \leq w(x, y)$. Therefore,

$$\begin{aligned} w(T') &= w(T) - w(x, y) + w(u, v) \\ &\leq w(T) . \end{aligned}$$

But T is a minimum spanning tree, so that $w(T) \leq w(T')$; thus, T' must be a minimum spanning tree also.

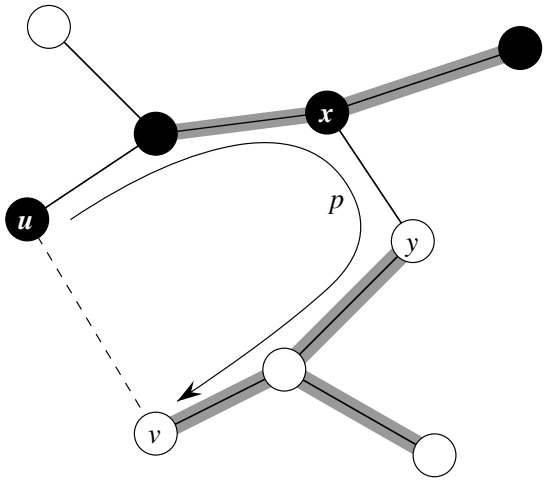


Figure 23.3 The proof of Theorem 23.1. The vertices in S are black, and the vertices in $V - S$ are white. The edges in the minimum spanning tree T are shown, but the edges in the graph G are not. The edges in A are shaded, and (u, v) is a light edge crossing the cut $(S, V - S)$. The edge (x, y) is an edge on the unique path p from u to v in T. A minimum spanning tree T' that contains (u, v) is formed by removing the edge (x, y) from T and adding the edge (u, v).

It remains to show that (u, v) is actually a safe edge for A. We have $A \subseteq T'$, since $A \subseteq T$ and $(x, y) \notin A$; thus, $A \cup \{(u, v)\} \subseteq T'$. Consequently, since T' is a minimum spanning tree, (u, v) is safe for A. ■

Theorem 23.1 gives us a better understanding of the workings of the GENERIC-MST algorithm on a connected graph $G = (V, E)$. As the algorithm proceeds, the set A is always acyclic; otherwise, a minimum spanning tree including A would contain a cycle, which is a contradiction. At any point in the execution of the algorithm, the graph $G_A = (V, A)$ is a forest, and each of the connected components of G_A is a tree. (Some of the trees may contain just one vertex, as is the case, for example, when the algorithm begins: A is empty and the forest contains $|V|$ trees, one for each vertex.) Moreover, any safe edge (u, v) for A connects distinct components of G_A, since $A \cup \{(u, v)\}$ must be acyclic.

The loop in lines 2–4 of GENERIC-MST is executed $|V| - 1$ times as each of the $|V| - 1$ edges of a minimum spanning tree is successively determined. Initially, when $A = \emptyset$, there are $|V|$ trees in G_A, and each iteration reduces that number by 1. When the forest contains only a single tree, the algorithm terminates.

The two algorithms in Section 23.2 use the following corollary to Theorem 23.1.

Corollary 23.2

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_C, E_C)$ be a connected component (tree) in

the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A.

Proof The cut $(V_C, V - V_C)$ respects A, and (u, v) is a light edge for this cut. Therefore, (u, v) is safe for A. ■

23.2 The algorithms of Kruskal and Prim

The two minimum-spanning-tree algorithms described in this section are elaborations of the generic algorithm. They each use a specific rule to determine a safe edge in line 3 of GENERIC-MST. In Kruskal's algorithm, the set A is a forest. The safe edge added to A is always a least-weight edge in the graph that connects two distinct components. In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree.

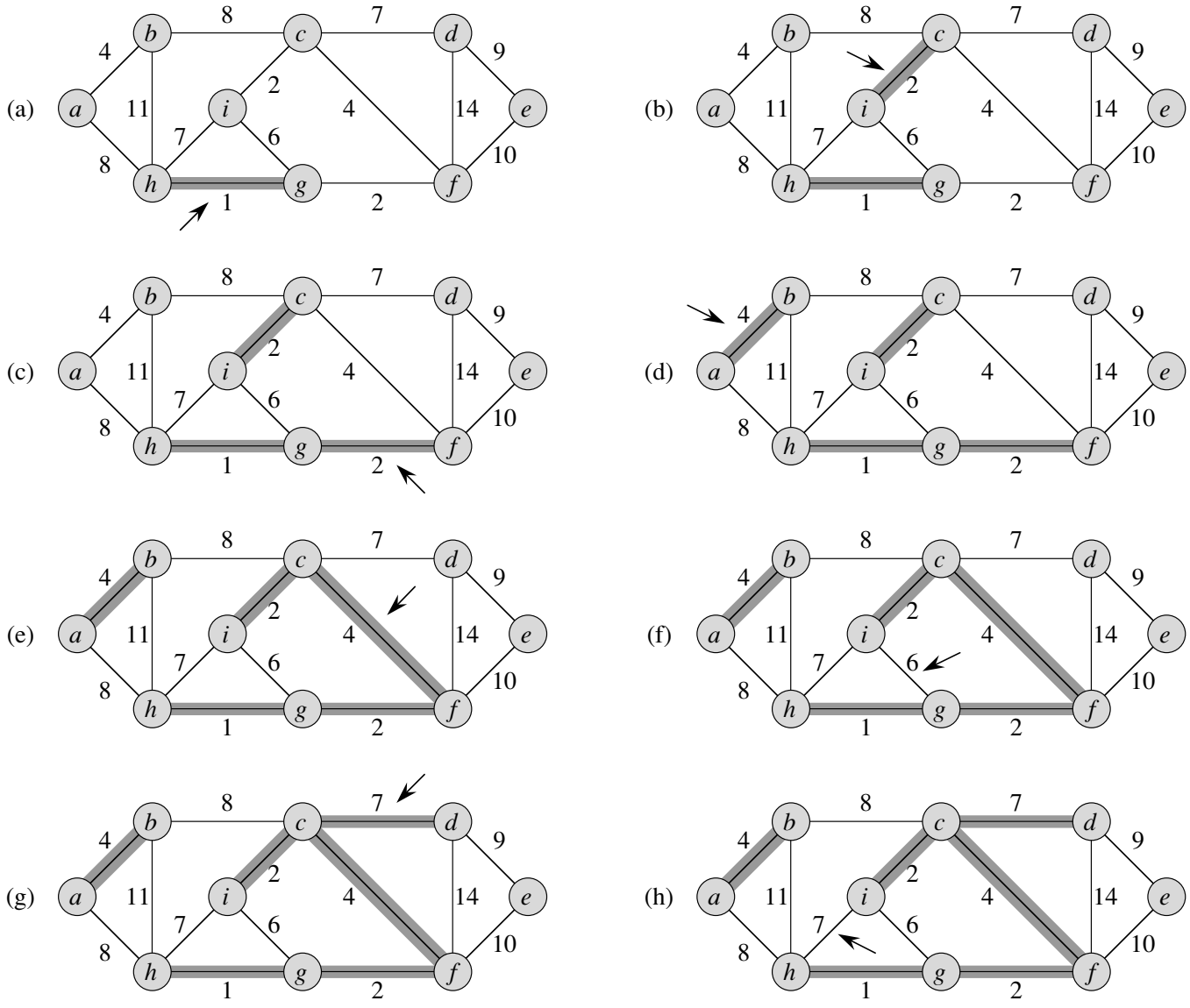
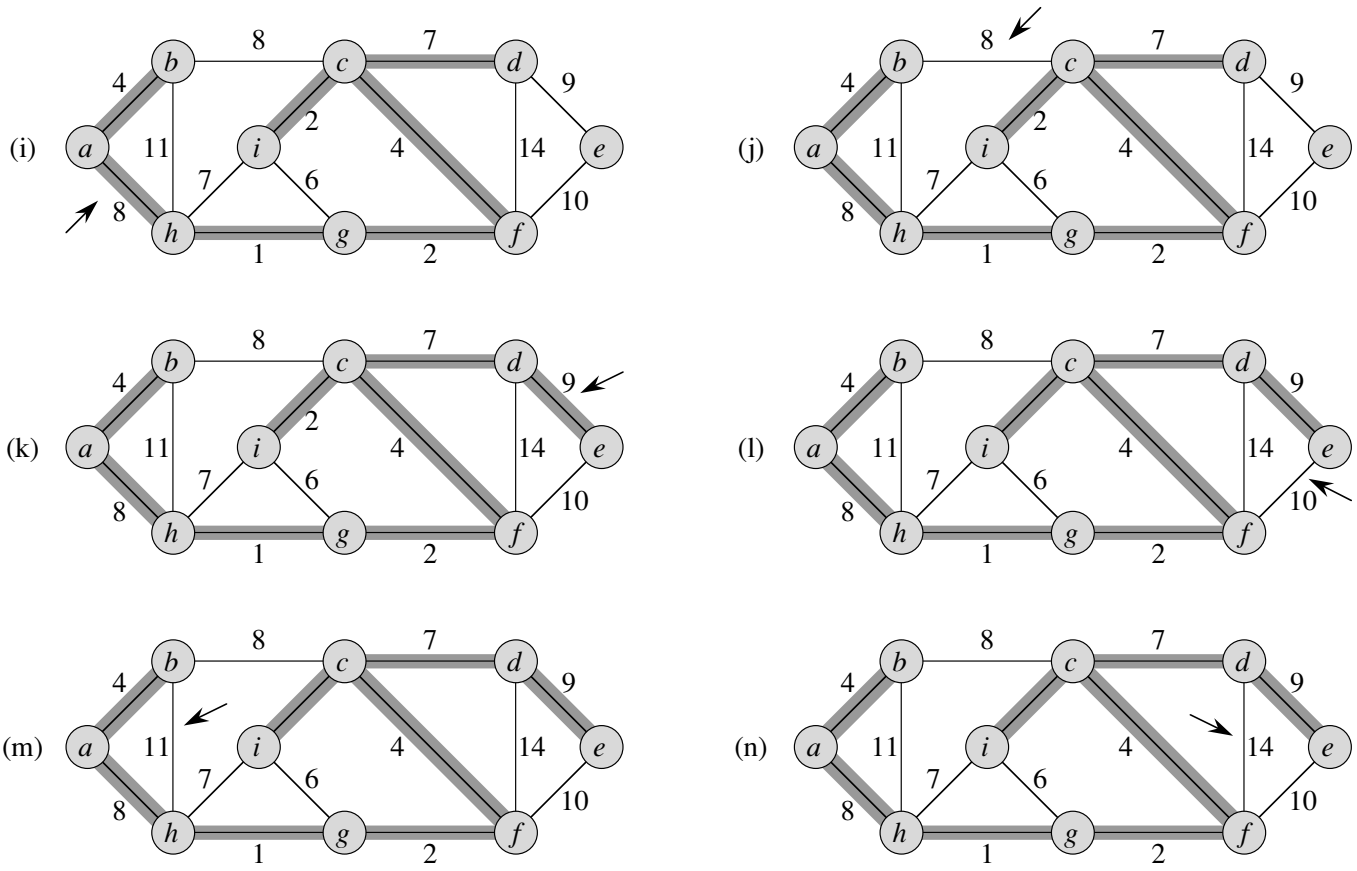


Figure 23.4 The execution of Kruskal's algorithm on the graph from Figure 23.1. Shaded edges belong to the forest A being grown. The edges are considered by the algorithm in sorted order by weight. An arrow points to the edge under consideration at each step of the algorithm. If the edge joins two distinct trees in the forest, it is added to the forest, thereby merging the two trees.

Kruskal's algorithm

Kruskal's algorithm is based directly on the generic minimum-spanning-tree algorithm given in Section 23.1. It finds a safe edge to add to the growing forest by finding, of all the edges that connect any two trees in the forest, an edge (u, v) of least weight. Let C_1 and C_2 denote the two trees that are connected by (u, v). Since (u, v) must be a light edge connecting C_1 to some other tree, Corollary 23.2



implies that (u, v) is a safe edge for C_1. Kruskal's algorithm is a greedy algorithm, because at each step it adds to the forest an edge of least possible weight.

Our implementation of Kruskal's algorithm is like the algorithm to compute connected components from Section 21.1. It uses a disjoint-set data structure to maintain several disjoint sets of elements. Each set contains the vertices in a tree of the current forest. The operation FIND-SET(u) returns a representative element from the set that contains u. Thus, we can determine whether two vertices u and v belong to the same tree by testing whether FIND-SET(u) equals FIND-SET(v). The combining of trees is accomplished by the UNION procedure.

```
MST-KRUSKAL(G, w)
1  A ← ∅
2  for each vertex v ∈ V[G]
3       do MAKE-SET(v)
4  sort the edges of E into nondecreasing order by weight w
5  for each edge (u, v) ∈ E, taken in nondecreasing order by weight
6       do if FIND-SET(u) ≠ FIND-SET(v)
7            then A ← A ∪ {(u, v)}
8                 UNION(u, v)
9  return A
```

Kruskal's algorithm works as shown in Figure 23.4. Lines 1–3 initialize the set A to the empty set and create $|V|$ trees, one containing each vertex. The edges in E are sorted into nondecreasing order by weight in line 4. The **for** loop in lines 5–8 checks, for each edge (u, v), whether the endpoints u and v belong to the same tree. If they do, then the edge (u, v) cannot be added to the forest without creating a cycle, and the edge is discarded. Otherwise, the two vertices belong to different trees. In this case, the edge (u, v) is added to A in line 7, and the vertices in the two trees are merged in line 8.

The running time of Kruskal's algorithm for a graph $G = (V, E)$ depends on the implementation of the disjoint-set data structure. We shall assume the disjoint-set-forest implementation of Section 21.3 with the union-by-rank and path-compression heuristics, since it is the asymptotically fastest implementation known. Initializing the set A in line 1 takes $O(1)$ time, and the time to sort the edges in line 4 is $O(E \lg E)$. (We will account for the cost of the $|V|$ MAKE-SET operations in the **for** loop of lines 2–3 in a moment.) The **for** loop of lines 5–8 performs $O(E)$ FIND-SET and UNION operations on the disjoint-set forest. Along with the $|V|$ MAKE-SET operations, these take a total of $O((V + E)\,\alpha(V))$ time, where α is the very slowly growing function defined in Section 21.4. Because G is assumed to be connected, we have $|E| \geq |V| - 1$, and so the disjoint-set operations take $O(E\,\alpha(V))$ time. Moreover, since $\alpha(|V|) = O(\lg V) = O(\lg E)$, the total running time of Kruskal's algorithm is $O(E \lg E)$. Observing that $|E| < |V|^2$, we have $\lg |E| = O(\lg V)$, and so we can restate the running time of Kruskal's algorithm as $O(E \lg V)$.

Prim's algorithm

Like Kruskal's algorithm, Prim's algorithm is a special case of the generic minimum-spanning-tree algorithm from Section 23.1. Prim's algorithm operates much like Dijkstra's algorithm for finding shortest paths in a graph, which we shall see in Section 24.3. Prim's algorithm has the property that the edges in the set A always form a single tree. As is illustrated in Figure 23.5, the tree starts from an arbitrary root vertex r and grows until the tree spans all the vertices in V. At each step, a light edge is added to the tree A that connects A to an isolated vertex of $G_A = (V, A)$. By Corollary 23.2, this rule adds only edges that are safe for A; therefore, when the algorithm terminates, the edges in A form a minimum spanning tree. This strategy is greedy since the tree is augmented at each step with an edge that contributes the minimum amount possible to the tree's weight.

The key to implementing Prim's algorithm efficiently is to make it easy to select a new edge to be added to the tree formed by the edges in A. In the pseudocode below, the connected graph G and the root r of the minimum spanning tree to be grown are inputs to the algorithm. During execution of the algorithm, all vertices

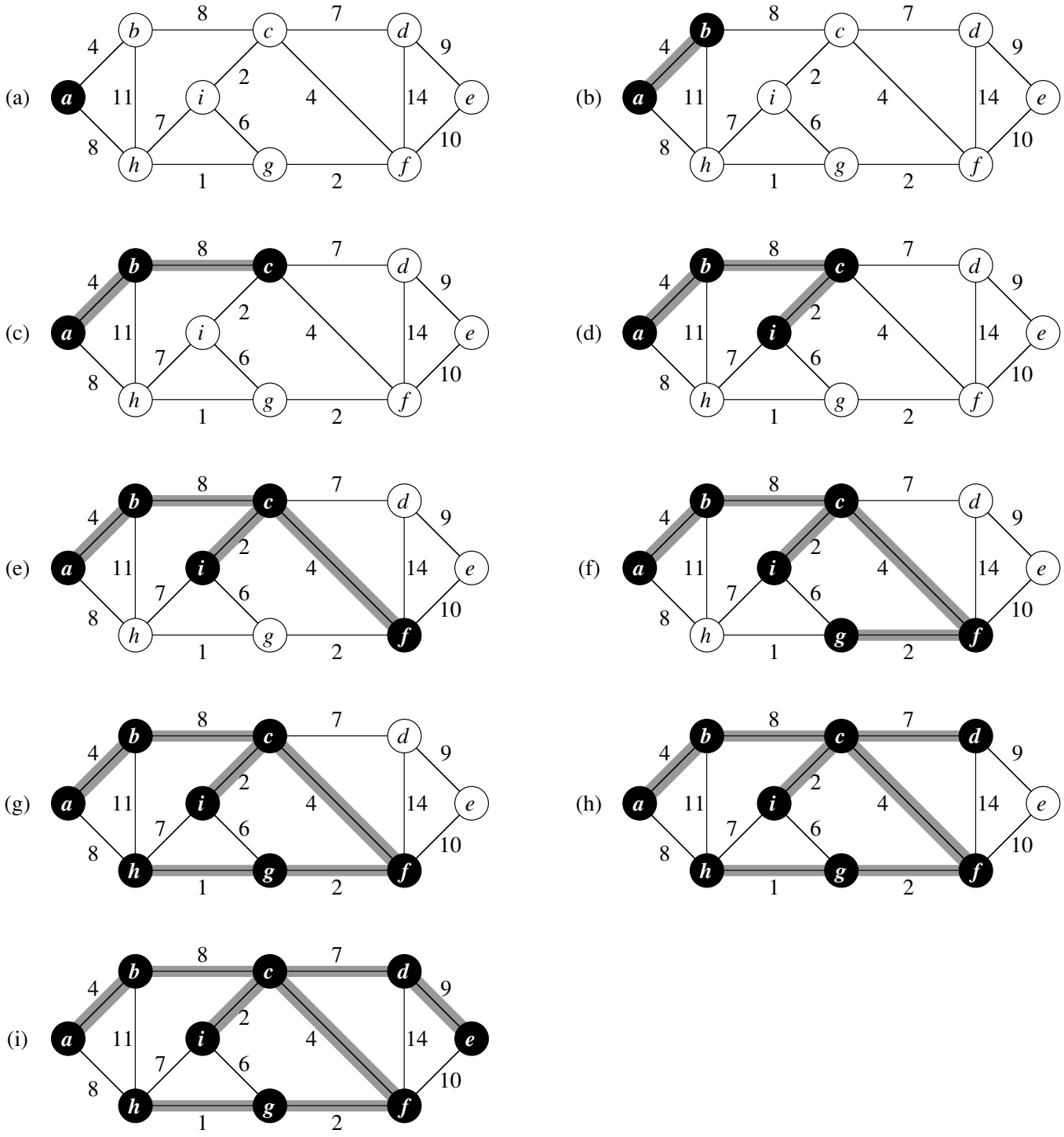


Figure 23.5 The execution of Prim's algorithm on the graph from Figure 23.1. The root vertex is a. Shaded edges are in the tree being grown, and the vertices in the tree are shown in black. At each step of the algorithm, the vertices in the tree determine a cut of the graph, and a light edge crossing the cut is added to the tree. In the second step, for example, the algorithm has a choice of adding either edge (b, c) or edge (a, h) to the tree since both are light edges crossing the cut.

that are *not* in the tree reside in a min-priority queue Q based on a *key* field. For each vertex v, $key[v]$ is the minimum weight of any edge connecting v to a vertex in the tree; by convention, $key[v] = \infty$ if there is no such edge. The field $\pi[v]$ names the parent of v in the tree. During the algorithm, the set A from GENERIC-MST is kept implicitly as

$$A = \{(v, \pi[v]) : v \in V - \{r\} - Q\} \ .$$

When the algorithm terminates, the min-priority queue Q is empty; the minimum spanning tree A for G is thus

$$A = \{(v, \pi[v]) : v \in V - \{r\}\} \ .$$

```
MST-PRIM(G, w, r)
 1  for each u ∈ V[G]
 2       do key[u] ← ∞
 3          π[u] ← NIL
 4  key[r] ← 0
 5  Q ← V[G]
 6  while Q ≠ ∅
 7       do u ← EXTRACT-MIN(Q)
 8          for each v ∈ Adj[u]
 9               do if v ∈ Q and w(u, v) < key[v]
10                    then π[v] ← u
11                         key[v] ← w(u, v)
```

Prim's algorithm works as shown in Figure 23.5. Lines 1–5 set the key of each vertex to ∞ (except for the root r, whose key is set to 0 so that it will be the first vertex processed), set the parent of each vertex to NIL, and initialize the min-priority queue Q to contain all the vertices. The algorithm maintains the following three-part loop invariant:

Prior to each iteration of the **while** loop of lines 6–11,

1. $A = \{(v, \pi[v]) : v \in V - \{r\} - Q\}$.
2. The vertices already placed into the minimum spanning tree are those in $V - Q$.
3. For all vertices $v \in Q$, if $\pi[v] \neq$ NIL, then $key[v] < \infty$ and $key[v]$ is the weight of a light edge $(v, \pi[v])$ connecting v to some vertex already placed into the minimum spanning tree.

Line 7 identifies a vertex $u \in Q$ incident on a light edge crossing the cut $(V - Q, Q)$ (with the exception of the first iteration, in which $u = r$ due to line 4). Removing u

Thus, Prim's algorithm is exactly Dikstra's algorithm, but with the following version of RELAX.

```
RELAX_MST(u, v, w)
1 if v.d > w(u, v) then
2     v.d ← w(u, v)
3     v.π ← u
```

That is, rather than adding in $w(u, v)$, we simply set $v.d$ to the weight of the edge that is incident on it. The idea is that $v.d$ is no longer a distance estimate, but the cheapest estimate, so far, of the "cost" of including v in a minimum spanning tree. As soon as we find an edge from a vertex that is already in the tree that is cheaper than the current cost of adding v to the tree, we adopt this new edge.

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