For this question, in order to get the tight upper bound, need to find the upper-bound in O(·) and lower-bound in Ω(·).

Let u and v be the 2 vertices that we want to find the shortest paths of. So every possible vertex other than u and v is an intermediate vertex on a shortest path. When there are n vertices, suppose the shortest path length u → v has (n-1) edges, then in this case, an upper-bound for the number of shortest paths from u to v would be (n-2)! and this would be a loose upper-bound O((n-2)!).

For the lower-bound, consider that we have partitions of vertices, call them $V_0, V_2, \ldots, V_{k-1}$, where each $|V_i|$ = some m, and there are 4 cases.

(i) There are no edges between any vertices in V_i

(ii) There are edges from u to each $v \in V_0$

(iii) there are edges from each $v \in V_{i-1}$ to every $w \in V_i$

(iv) there are edges from each $w \in V_{k-1}$ to v

For example, for (i), when having no edges between any vertices in V_i, it means we have 0 edges, and so there exists 0 shortest path. For (iii), when having edges from each $v \in V_{i-1}$ to every $w \in V_i$, then this means each and every vertex is connected with each other, and the shortest distance is 1, hence there exists only 1 shortest path. So, A shortest path u ⇝ v now must be $u \to v_0 \to v_1 \to \ldots \to v_{k-1} \to v$, where each $v_i \in V_i$. And the number of shortest paths could be represented as a function of m.

By having the vertices of k partitions with each partition containing m vertices, it's obvious to see the that way we partition the vertices and the number of vertices within each partition could determine the max number of shortest paths we get, hence need to maximize the equation $m^k = (\frac{|n|-2}{k})^k$. We can find the max/min of a function by taking its derivative and let it equal to 0.

$$\frac{d}{dk}(\frac{|n|-2}{k})^k = \left(\ln\left(\frac{|n|-2}{k}\right) - 1\right)(\frac{|n|-2}{k})^k = 0$$

Then $(\frac{|n|-2}{k})^k = 0$ or $\left(\ln\left(\frac{|n|-2}{k}\right) - 1\right) = 0$

$$\left(\frac{|n|-2}{k}\right)^k = 0 \qquad \left(\ln\left(\frac{|n|-2}{k}\right) - 1\right) = 0$$

$$(\frac{|n|-2}{k}) = 0 \qquad \ln\left(\frac{|n|-2}{k}\right) = 1$$

$$\left(\frac{|n|-2}{k}\right) = e$$

$m = \begin{cases} 0 \\ e \end{cases}$ Since m represent the number of vertices in each partition, m cannot be 0, so m = e. And m needs to be an integer, so m = 2 or m = 3, k = $\left(\frac{|n|-2}{m}\right)$.

So, the number of shortest paths that can exists are $\lfloor m^k \rfloor = \left\lfloor e^{\frac{n-2}{e}} \right\rfloor$ and hence, the number of shortest paths u ⇝ v is $\Theta(e^n)$.