Lecture 8

- Randomization and probabilistic analysis.
- Approximation algorithms.

In the few prior lectures, we discussed algorithm design strategies to arrive at correct algorithms that are guaranteed to terminate. In this lecture, we consider algorithms based on randomization, where the underlying analysis is probabilistic only. We consider also approximation algorithms which trade correctness for efficiency. As such, randomization and approximation are design strategies similar to the prior ones, but ones which trade-off along an axis (e.g., termination) to gain along another axis (e.g., time-efficiency in the expected-case).

We first discuss randomization and randomized algorithms. RandMedian from Lecture 1 is an example of a randomized algorithm for finding the median. Another example is 'Selection in expected linear time' from CLRS that was part of the lecture on Divide-n-conquer, in which randomization is employed in the choice of pivots for partition.

We now discuss two other examples of probabilistic analysis: randomly built binary search trees, and the hash table data structure with collisions resolved by chaining. In the context of the latter, we consider universal hashing and perfect hashing as well.

⋆ 12.4 Randomly built binary search trees

We have shown that all the basic operations on a binary search tree run in $O(h)$ time, where h is the height of the tree. The height of a binary search tree varies, however, as items are inserted and deleted. If, for example, the items are inserted in strictly increasing order, the tree will be a chain with height $n - 1$. On the other hand, Exercise B.5-4 shows that $h \geq \lfloor \lg n \rfloor$. As with quicksort, we can show that the behavior of the average case is much closer to the best case than the worst case.

Unfortunately, little is known about the average height of a binary search tree when both insertion and deletion are used to create it. When the tree is created by insertion alone, the analysis becomes more tractable. Let us therefore define a ***randomly built binary search tree*** on n keys as one that arises from inserting the keys in random order into an initially empty tree, where each of the $n!$ permutations of the input keys is equally likely. (Exercise 12.4-3 asks you to show that this notion is different from assuming that every binary search tree on n keys is equally likely.) In this section, we shall show that the expected height of a randomly built binary search tree on n keys is $O(\lg n)$. We assume that all keys are distinct.

We start by defining three random variables that help measure the height of a randomly built binary search tree. We denote the height of a randomly built binary search on n keys by X_n, and we define the ***exponential height*** $Y_n = 2^{X_n}$. When we build a binary search tree on n keys, we choose one key as that of the root, and we let R_n denote the random variable that holds this key's rank within the set of n keys. The value of R_n is equally likely to be any element of the set $\{1, 2, \ldots, n\}$. If $R_n = i$, then the left subtree of the root is a randomly built binary search tree on $i - 1$ keys, and the right subtree is a randomly built binary search tree on $n - i$ keys. Because the height of a binary tree is one more than the larger of the heights of the two subtrees of the root, the exponential height of a binary tree is twice the larger of the exponential heights of the two subtrees of the root. If we know that $R_n = i$, we therefore have that

$$Y_n = 2 \cdot \max(Y_{i-1}, Y_{n-i}) \ .$$

As base cases, we have $Y_1 = 1$, because the exponential height of a tree with 1 node is $2^0 = 1$ and, for convenience, we define $Y_0 = 0$.

Next we define indicator random variables $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$, where

$$Z_{n,i} = \mathrm{I}\{R_n = i\} \ .$$

Because R_n is equally likely to be any element of $\{1, 2, \ldots, n\}$, we have that $\Pr\{R_n = i\} = 1/n$ for $i = 1, 2, \ldots, n$, and hence, by Lemma 5.1,

$$\mathrm{E}[Z_{n,i}] = 1/n \ , \tag{12.1}$$

for $i = 1, 2, \ldots, n$. Because exactly one value of $Z_{n,i}$ is 1 and all others are 0, we also have

$$Y_n = \sum_{i=1}^{n} Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right) \ .$$

We will show that $\mathrm{E}[Y_n]$ is polynomial in n, which will ultimately imply that $\mathrm{E}[X_n] = O(\lg n)$.

The indicator random variable $Z_{n,i} = \mathrm{I}\{R_n = i\}$ is independent of the values of Y_{i-1} and Y_{n-i}. Having chosen $R_n = i$, the left subtree, whose exponential height is Y_{i-1}, is randomly built on the $i-1$ keys whose ranks are less than i. This subtree is just like any other randomly built binary search tree on $i-1$ keys. Other than the number of keys it contains, this subtree's structure is not affected at all by the choice of $R_n = i$; hence the random variables Y_{i-1} and $Z_{n,i}$ are independent. Likewise, the right subtree, whose exponential height is Y_{n-i}, is randomly built on the $n-i$ keys whose ranks are greater than i. Its structure is independent of the value of R_n, and so the random variables Y_{n-i} and $Z_{n,i}$ are independent. Hence,

$$\begin{aligned}
\mathrm{E}[Y_n] &= \mathrm{E}\left[\sum_{i=1}^{n} Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right)\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right)\right] && \text{(by linearity of expectation)} \\
&= \sum_{i=1}^{n} \mathrm{E}[Z_{n,i}]\, \mathrm{E}\left[2 \cdot \max(Y_{i-1}, Y_{n-i})\right] && \text{(by independence)} \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot \mathrm{E}\left[2 \cdot \max(Y_{i-1}, Y_{n-i})\right] && \text{(by equation (12.1))} \\
&= \frac{2}{n} \sum_{i=1}^{n} \mathrm{E}\left[\max(Y_{i-1}, Y_{n-i})\right] && \text{(by equation (C.21))} \\
&\le \frac{2}{n} \sum_{i=1}^{n} \left(\mathrm{E}[Y_{i-1}] + \mathrm{E}[Y_{n-i}]\right) && \text{(by Exercise C.3-4)} \ .
\end{aligned}$$

Each term $\mathrm{E}[Y_0], \mathrm{E}[Y_1], \ldots, \mathrm{E}[Y_{n-1}]$ appears twice in the last summation, once as $\mathrm{E}[Y_{i-1}]$ and once as $\mathrm{E}[Y_{n-i}]$, and so we have the recurrence

$$\mathrm{E}[Y_n] \le \frac{4}{n} \sum_{i=0}^{n-1} \mathrm{E}[Y_i] \ . \tag{12.2}$$

Using the substitution method, we will show that for all positive integers n, the recurrence (12.2) has the solution

$$\mathrm{E}[Y_n] \le \frac{1}{4} \binom{n+3}{3} \ .$$

In doing so, we will use the identity

$$\sum_{i=0}^{n-1} \binom{i+3}{3} = \binom{n+3}{4} . \tag{12.3}$$

(Exercise 12.4-1 asks you to prove this identity.)

For the base case, we verify that the bound

$$1 = Y_1 = \mathrm{E}[Y_1] \le \frac{1}{4}\binom{1+3}{3} = 1$$

holds. For the substitution, we have that

$$\begin{aligned}
\mathrm{E}[Y_n] &\le \frac{4}{n}\sum_{i=0}^{n-1} \mathrm{E}[Y_i] \\
&\le \frac{4}{n}\sum_{i=0}^{n-1} \frac{1}{4}\binom{i+3}{3} \quad \text{(by the inductive hypothesis)} \\
&= \frac{1}{n}\sum_{i=0}^{n-1} \binom{i+3}{3} \\
&= \frac{1}{n}\binom{n+3}{4} \quad \text{(by equation (12.3))} \\
&= \frac{1}{n}\cdot\frac{(n+3)!}{4!\,(n-1)!} \\
&= \frac{1}{4}\cdot\frac{(n+3)!}{3!\,n!} \\
&= \frac{1}{4}\binom{n+3}{3} .
\end{aligned}$$

We have bounded $\mathrm{E}[Y_n]$, but our ultimate goal is to bound $\mathrm{E}[X_n]$. As Exercise 12.4-4 asks you to show, the function $f(x) = 2^x$ is convex (see page 1109). Therefore, we can apply Jensen's inequality (C.25), which says that

$$2^{\mathrm{E}[X_n]} \le \mathrm{E}[2^{X_n}] = \mathrm{E}[Y_n] ,$$

to derive that

$$\begin{aligned}
2^{\mathrm{E}[X_n]} &\le \frac{1}{4}\binom{n+3}{3} \\
&= \frac{1}{4}\cdot\frac{(n+3)(n+2)(n+1)}{6} \\
&= \frac{n^3+6n^2+11n+6}{24} .
\end{aligned}$$

Taking logarithms of both sides gives $\mathrm{E}[X_n] = O(\lg n)$. Thus, we have proven the following:

Theorem 12.4

The expected height of a randomly built binary search tree on n keys is $O(\lg n)$. ■

11 Hash Tables

Many applications require a dynamic set that supports only the dictionary operations INSERT, SEARCH, and DELETE. For example, a compiler for a computer language maintains a symbol table, in which the keys of elements are arbitrary character strings that correspond to identifiers in the language. A hash table is an effective data structure for implementing dictionaries. Although searching for an element in a hash table can take as long as searching for an element in a linked list—$\Theta(n)$ time in the worst case—in practice, hashing performs extremely well. Under reasonable assumptions, the expected time to search for an element in a hash table is $O(1)$.

A hash table is a generalization of the simpler notion of an ordinary array. Directly addressing into an ordinary array makes effective use of our ability to examine an arbitrary position in an array in $O(1)$ time. Section 11.1 discusses direct addressing in more detail. Direct addressing is applicable when we can afford to allocate an array that has one position for every possible key.

When the number of keys actually stored is small relative to the total number of possible keys, hash tables become an effective alternative to directly addressing an array, since a hash table typically uses an array of size proportional to the number of keys actually stored. Instead of using the key as an array index directly, the array index is *computed* from the key. Section 11.2 presents the main ideas, focusing on "chaining" as a way to handle "collisions" in which more than one key maps to the same array index.

The bottom line is that hashing is an extremely effective and practical technique: the basic dictionary operations require only $O(1)$ time on the average. Section 11.5 explains how "perfect hashing" can support searches in $O(1)$ *worst-case* time, when the set of keys being stored is static (that is, when the set of keys never changes once stored).

11.1 Direct-address tables

Direct addressing is a simple technique that works well when the universe U of keys is reasonably small. Suppose that an application needs a dynamic set in which each element has a key drawn from the universe $U = \{0, 1, \ldots, m-1\}$, where m is not too large. We shall assume that no two elements have the same key.

To represent the dynamic set, we use an array, or ***direct-address table***, denoted by $T[0..m-1]$, in which each position, or ***slot***, corresponds to a key in the universe U. Figure 11.1 illustrates the approach; slot k points to an element in the set with key k. If the set contains no element with key k, then $T[k] = \text{NIL}$.

The dictionary operations are trivial to implement.

DIRECT-ADDRESS-SEARCH(T, k)
  return $T[k]$

DIRECT-ADDRESS-INSERT(T, x)
  $T[key[x]] \leftarrow x$

DIRECT-ADDRESS-DELETE(T, x)
  $T[key[x]] \leftarrow \text{NIL}$

Each of these operations is fast: only $O(1)$ time is required.

For some applications, the elements in the dynamic set can be stored in the direct-address table itself. That is, rather than storing an element's key and satellite data in an object external to the direct-address table, with a pointer from a slot in the table to the object, we can store the object in the slot itself, thus saving space. Moreover, it is often unnecessary to store the key field of the object, since if we have the index of an object in the table, we have its key. If keys are not stored, however, we must have some way to tell if the slot is empty.

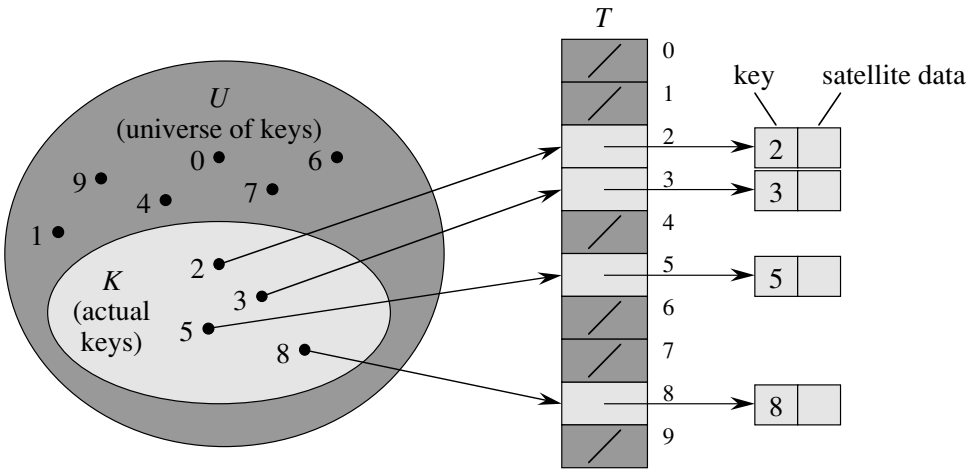

Figure 11.1 Implementing a dynamic set by a direct-address table T. Each key in the universe $U = \{0, 1, \ldots, 9\}$ corresponds to an index in the table. The set $K = \{2, 3, 5, 8\}$ of actual keys determines the slots in the table that contain pointers to elements. The other slots, heavily shaded, contain NIL.

11.2 Hash tables

The difficulty with direct addressing is obvious: if the universe U is large, storing a table T of size $|U|$ may be impractical, or even impossible, given the memory available on a typical computer. Furthermore, the set K of keys *actually stored* may be so small relative to U that most of the space allocated for T would be wasted.

When the set K of keys stored in a dictionary is much smaller than the universe U of all possible keys, a hash table requires much less storage than a direct-address table. Specifically, the storage requirements can be reduced to $\Theta(|K|)$ while we maintain the benefit that searching for an element in the hash table still requires only $O(1)$ time. The only catch is that this bound is for the *average time*, whereas for direct addressing it holds for the *worst-case time*.

With direct addressing, an element with key k is stored in slot k. With hashing, this element is stored in slot $h(k)$; that is, we use a ***hash function*** h to compute the slot from the key k. Here h maps the universe U of keys into the slots of a ***hash table*** $T[0\,..\,m-1]$:

$$h : U \to \{0, 1, \ldots, m-1\} \ .$$

We say that an element with key k ***hashes*** to slot $h(k)$; we also say that $h(k)$ is the ***hash value*** of key k. Figure 11.2 illustrates the basic idea. The point of the hash function is to reduce the range of array indices that need to be handled. Instead of $|U|$ values, we need to handle only m values. Storage requirements are correspondingly reduced.

There is one hitch: two keys may hash to the same slot. We call this situation a ***collision***. Fortunately, there are effective techniques for resolving the conflict created by collisions.

Of course, the ideal solution would be to avoid collisions altogether. We might try to achieve this goal by choosing a suitable hash function h. One idea is to make h appear to be "random," thus avoiding collisions or at least minimizing their number. The very term "to hash," evoking images of random mixing and chopping, captures the spirit of this approach. (Of course, a hash function h must be deterministic in that a given input k should always produce the same output $h(k)$.) Since $|U| > m$, however, there must be at least two keys that have the same hash value; avoiding collisions altogether is therefore impossible. Thus, while a well-designed, "random"-looking hash function can minimize the number of collisions, we still need a method for resolving the collisions that do occur.

The remainder of this section presents the simplest collision resolution technique, called chaining. Section 11.4 introduces an alternative method for resolving collisions, called open addressing.

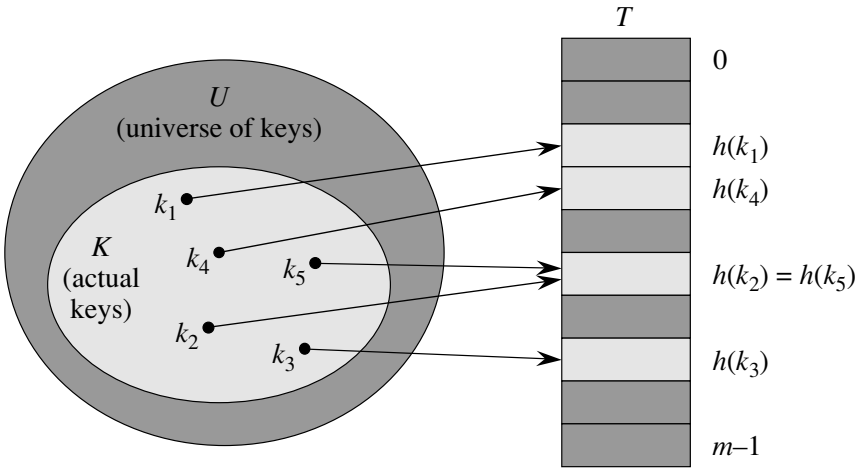

Figure 11.2 Using a hash function h to map keys to hash-table slots. Keys k_2 and k_5 map to the same slot, so they collide.

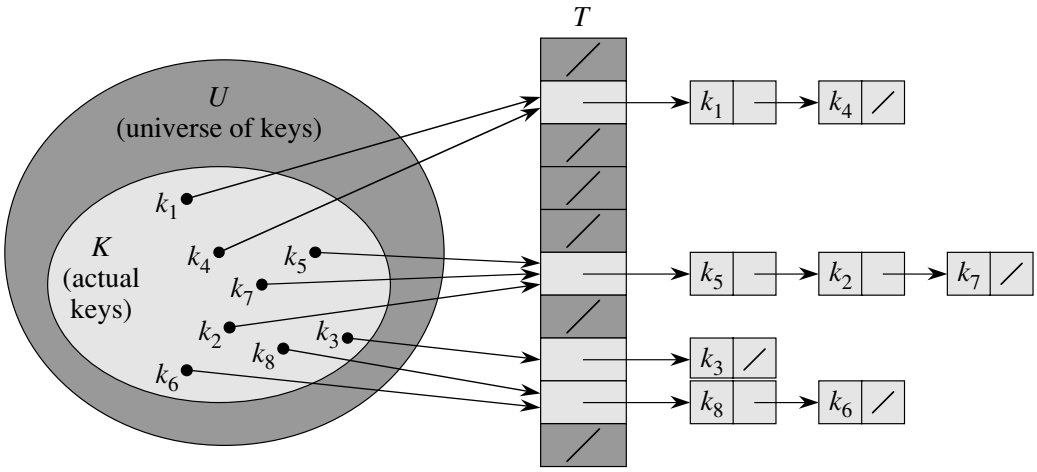

Figure 11.3 Collision resolution by chaining. Each hash-table slot $T[j]$ contains a linked list of all the keys whose hash value is j. For example, $h(k_1) = h(k_4)$ and $h(k_5) = h(k_2) = h(k_7)$.

Collision resolution by chaining

In ***chaining***, we put all the elements that hash to the same slot in a linked list, as shown in Figure 11.3. Slot j contains a pointer to the head of the list of all stored elements that hash to j; if there are no such elements, slot j contains NIL.

The dictionary operations on a hash table T are easy to implement when collisions are resolved by chaining.

CHAINED-HASH-INSERT(T, x)
insert x at the head of list $T[h(key[x])]$

CHAINED-HASH-SEARCH(T, k)
search for an element with key k in list $T[h(k)]$

CHAINED-HASH-DELETE(T, x)
delete x from the list $T[h(key[x])]$

The worst-case running time for insertion is $O(1)$. The insertion procedure is fast in part because it assumes that the element x being inserted is not already present in the table; this assumption can be checked if necessary (at additional cost) by performing a search before insertion. For searching, the worst-case running time is proportional to the length of the list; we shall analyze this operation more closely below. Deletion of an element x can be accomplished in $O(1)$ time if the lists are doubly linked. (Note that CHAINED-HASH-DELETE takes as input an element x and not its key k, so we don't have to search for x first. If the lists were singly linked, it would not be of great help to take as input the element x rather than the key k. We would still have to find x in the list $T[h(key[x])]$, so that the *next* link of x's predecessor could be properly set to splice x out. In this case, deletion and searching would have essentially the same running time.)

Analysis of hashing with chaining

How well does hashing with chaining perform? In particular, how long does it take to search for an element with a given key?

Given a hash table T with m slots that stores n elements, we define the ***load factor*** α for T as n/m, that is, the average number of elements stored in a chain. Our analysis will be in terms of α, which can be less than, equal to, or greater than 1.

The worst-case behavior of hashing with chaining is terrible: all n keys hash to the same slot, creating a list of length n. The worst-case time for searching is thus $\Theta(n)$ plus the time to compute the hash function—no better than if we used one linked list for all the elements. Clearly, hash tables are not used for their worst-case performance. (Perfect hashing, described in Section 11.5, does however provide good worst-case performance when the set of keys is static.)

The average performance of hashing depends on how well the hash function h distributes the set of keys to be stored among the m slots, on the average. Section 11.3 discusses these issues, but for now we shall assume that any given element is equally likely to hash into any of the m slots, independently of where any other element has hashed to. We call this the assumption of ***simple uniform hashing***.

For $j = 0, 1, \ldots, m-1$, let us denote the length of the list $T[j]$ by n_j, so that

$$n = n_0 + n_1 + \cdots + n_{m-1} , \tag{11.1}$$

and the average value of n_j is $\mathrm{E}[n_j] = \alpha = n/m$.

We assume that the hash value $h(k)$ can be computed in $O(1)$ time, so that the time required to search for an element with key k depends linearly on the length $n_{h(k)}$ of the list $T[h(k)]$. Setting aside the $O(1)$ time required to compute the hash function and to access slot $h(k)$, let us consider the expected number of elements examined by the search algorithm, that is, the number of elements in the list $T[h(k)]$ that are checked to see if their keys are equal to k. We shall consider two cases. In the first, the search is unsuccessful: no element in the table has key k. In the second, the search successfully finds an element with key k.

Theorem 11.1
In a hash table in which collisions are resolved by chaining, an unsuccessful search takes expected time $\Theta(1 + \alpha)$, under the assumption of simple uniform hashing.

Proof Under the assumption of simple uniform hashing, any key k not already stored in the table is equally likely to hash to any of the m slots. The expected time to search unsuccessfully for a key k is the expected time to search to the end of list $T[h(k)]$, which has expected length $\mathrm{E}[n_{h(k)}] = \alpha$. Thus, the expected number of elements examined in an unsuccessful search is α, and the total time required (including the time for computing $h(k)$) is $\Theta(1 + \alpha)$. ■

The situation for a successful search is slightly different, since each list is not equally likely to be searched. Instead, the probability that a list is searched is proportional to the number of elements it contains. Nonetheless, the expected search time is still $\Theta(1 + \alpha)$.

Theorem 11.2
In a hash table in which collisions are resolved by chaining, a successful search takes time $\Theta(1 + \alpha)$, on the average, under the assumption of simple uniform hashing.

Proof We assume that the element being searched for is equally likely to be any of the n elements stored in the table. The number of elements examined during a successful search for an element x is 1 more than the number of elements that appear before x in x's list. Elements before x in the list were all inserted after x was inserted, because new elements are placed at the front of the list. To find the expected number of elements examined, we take the average, over the n elements x in the table, of 1 plus the expected number of elements added to x's list after x was added to the list. Let x_i denote the ith element inserted into the ta-

ble, for $i = 1, 2, \ldots, n$, and let $k_i = key[x_i]$. For keys k_i and k_j, we define the indicator random variable $X_{ij} = \mathrm{I}\{h(k_i) = h(k_j)\}$. Under the assumption of simple uniform hashing, we have $\Pr\{h(k_i) = h(k_j)\} = 1/m$, and so by Lemma 5.1, $\mathrm{E}[X_{ij}] = 1/m$. Thus, the expected number of elements examined in a successful search is

$$
\begin{aligned}
&\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] \\
&\quad = \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\mathrm{E}[X_{ij}]\right) \quad \text{(by linearity of expectation)} \\
&\quad = \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right) \\
&\quad = 1+\frac{1}{nm}\sum_{i=1}^{n}(n-i) \\
&\quad = 1+\frac{1}{nm}\left(\sum_{i=1}^{n}n-\sum_{i=1}^{n}i\right) \\
&\quad = 1+\frac{1}{nm}\left(n^2-\frac{n(n+1)}{2}\right) \quad \text{(by equation (A.1))} \\
&\quad = 1+\frac{n-1}{2m} \\
&\quad = 1+\frac{\alpha}{2}-\frac{\alpha}{2n}\,.
\end{aligned}
$$

Thus, the total time required for a successful search (including the time for computing the hash function) is $\Theta(2 + \alpha/2 - \alpha/2n) = \Theta(1 + \alpha)$. ■

What does this analysis mean? If the number of hash-table slots is at least proportional to the number of elements in the table, we have $n = O(m)$ and, consequently, $\alpha = n/m = O(m)/m = O(1)$. Thus, searching takes constant time on average. Since insertion takes $O(1)$ worst-case time and deletion takes $O(1)$ worst-case time when the lists are doubly linked, all dictionary operations can be supported in $O(1)$ time on average.

★ 11.3.3 Universal hashing

If a malicious adversary chooses the keys to be hashed by some fixed hash function, then he can choose n keys that all hash to the same slot, yielding an average retrieval time of $\Theta(n)$. Any fixed hash function is vulnerable to such terrible worst-case behavior; the only effective way to improve the situation is to choose the hash function *randomly* in a way that is *independent* of the keys that are actually going to be stored. This approach, called ***universal hashing***, can yield provably good performance on average, no matter what keys are chosen by the adversary.

The main idea behind universal hashing is to select the hash function at random from a carefully designed class of functions at the beginning of execution. As in the case of quicksort, randomization guarantees that no single input will always evoke worst-case behavior. Because of the randomization, the algorithm can behave differently on each execution, even for the same input, guaranteeing good average-case performance for any input. Returning to the example of a compiler's symbol table, we find that the programmer's choice of identifiers cannot now cause consistently poor hashing performance. Poor performance occurs only when the compiler chooses a random hash function that causes the set of identifiers to hash

poorly, but the probability of this situation occurring is small and is the same for any set of identifiers of the same size.

Let $\mathcal{H}$ be a finite collection of hash functions that map a given universe U of keys into the range $\{0, 1, \ldots, m-1\}$. Such a collection is said to be ***universal*** if for each pair of distinct keys $k, l \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k) = h(l)$ is at most $|\mathcal{H}|/m$. In other words, with a hash function randomly chosen from $\mathcal{H}$, the chance of a collision between distinct keys k and l is no more than the chance $1/m$ of a collision if $h(k)$ and $h(l)$ were randomly and independently chosen from the set $\{0, 1, \ldots, m-1\}$.

The following theorem shows that a universal class of hash functions gives good average-case behavior. Recall that n_i denotes the length of list $T[i]$.

Theorem 11.3

Suppose that a hash function h is chosen from a universal collection of hash functions and is used to hash n keys into a table T of size m, using chaining to resolve collisions. If key k is not in the table, then the expected length $\mathrm{E}[n_{h(k)}]$ of the list that key k hashes to is at most α. If key k is in the table, then the expected length $\mathrm{E}[n_{h(k)}]$ of the list containing key k is at most $1 + \alpha$.

Proof We note that the expectations here are over the choice of the hash function, and do not depend on any assumptions about the distribution of the keys. For each pair k and l of distinct keys, define the indicator random variable $X_{kl} = \mathrm{I}\{h(k) = h(l)\}$. Since by definition, a single pair of keys collides with probability at most $1/m$, we have $\Pr\{h(k) = h(l)\} \leq 1/m$, and so Lemma 5.1 implies that $\mathrm{E}[X_{kl}] \leq 1/m$.

Next we define, for each key k, the random variable Y_k that equals the number of keys other than k that hash to the same slot as k, so that

$$Y_k = \sum_{\substack{l \in T \\ l \neq k}} X_{kl} \ .$$

Thus we have

$$\begin{aligned} \mathrm{E}[Y_k] &= \mathrm{E}\left[\sum_{\substack{l \in T \\ l \neq k}} X_{kl}\right] \\ &= \sum_{\substack{l \in T \\ l \neq k}} \mathrm{E}[X_{kl}] \quad \text{(by linearity of expectation)} \\ &\leq \sum_{\substack{l \in T \\ l \neq k}} \frac{1}{m} \ . \end{aligned}$$

The remainder of the proof depends on whether key k is in table T.

- If $k \notin T$, then $n_{h(k)} = Y_k$ and $|\{l : l \in T \text{ and } l \neq k\}| = n$. Thus $\mathrm{E}[n_{h(k)}] = \mathrm{E}[Y_k] \leq n/m = \alpha$.
- If $k \in T$, then because key k appears in list $T[h(k)]$ and the count Y_k does not include key k, we have $n_{h(k)} = Y_k + 1$ and $|\{l : l \in T \text{ and } l \neq k\}| = n - 1$. Thus $\mathrm{E}[n_{h(k)}] = \mathrm{E}[Y_k] + 1 \leq (n-1)/m + 1 = 1 + \alpha - 1/m < 1 + \alpha$. ■

The following corollary says universal hashing provides the desired payoff: it is now impossible for an adversary to pick a sequence of operations that forces the worst-case running time. By cleverly randomizing the choice of hash function at run time, we guarantee that every sequence of operations can be handled with good expected running time.

Corollary 11.4
Using universal hashing and collision resolution by chaining in a table with m slots, it takes expected time $\Theta(n)$ to handle any sequence of n INSERT, SEARCH and DELETE operations containing $O(m)$ INSERT operations.

Proof Since the number of insertions is $O(m)$, we have $n = O(m)$ and so $\alpha = O(1)$. The INSERT and DELETE operations take constant time and, by Theorem 11.3, the expected time for each SEARCH operation is $O(1)$. By linearity of expectation, therefore, the expected time for the entire sequence of operations is $O(n)$. ■

Designing a universal class of hash functions

We begin by choosing a prime number p large enough so that every possible key k is in the range 0 to $p-1$, inclusive. Let $\mathbf{Z}_p$ denote the set $\{0, 1, \ldots, p-1\}$, and let $\mathbf{Z}_p^*$ denote the set $\{1, 2, \ldots, p-1\}$. Since p is prime, we can solve equations modulo p with the methods given in Chapter 31. Because we assume that the size of the universe of keys is greater than the number of slots in the hash table, we hav $p > m$.

We now define the hash function $h_{a,b}$ for any $a \in \mathbf{Z}_p^*$ and any $b \in \mathbf{Z}_p$ using a linear transformation followed by reductions modulo p and then modulo m:

$$h_{a,b}(k) = ((ak + b) \bmod p) \bmod m \,. \tag{11.3}$$

For example, with $p = 17$ and $m = 6$, we have $h_{3,4}(8) = 5$. The family of all such hash functions is

$$\mathcal{H}_{p,m} = \left\{h_{a,b} : a \in \mathbf{Z}_p^* \text{ and } b \in \mathbf{Z}_p\right\} \,. \tag{11.4}$$

Each hash function $h_{a,b}$ maps $\mathbf{Z}_p$ to $\mathbf{Z}_m$. This class of hash functions has the nice property that the size m of the output range is arbitrary—not necessarily prime—a feature which we shall use in Section 11.5. Since there are $p - 1$ choices for a and there are p choices for b, there are $p(p - 1)$ hash functions in $\mathcal{H}_{p,m}$.

Theorem 11.5

The class $\mathcal{H}_{p,m}$ of hash functions defined by equations (11.3) and (11.4) is universal.

(The proof is not part of the course.)

⋆ 11.5 Perfect hashing

Although hashing is most often used for its excellent expected performance, hashing can be used to obtain excellent *worst-case* performance when the set of keys is ***static***: once the keys are stored in the table, the set of keys never changes. Some applications naturally have static sets of keys: consider the set of reserved words in a programming language, or the set of file names on a CD-ROM. We call a hashing technique ***perfect hashing*** if the worst-case number of memory accesses required to perform a search is $O(1)$.

The basic idea to create a perfect hashing scheme is simple. We use a two-level hashing scheme with universal hashing at each level. Figure 11.6 illustrates the approach.

The first level is essentially the same as for hashing with chaining: the n keys are hashed into m slots using a hash function h carefully selected from a family of universal hash functions.

Instead of making a list of the keys hashing to slot j, however, we use a small ***secondary hash table*** S_j with an associated hash function h_j. By choosing the hash functions h_j carefully, we can guarantee that there are no collisions at the secondary level.

In order to guarantee that there are no collisions at the secondary level, however, we will need to let the size m_j of hash table S_j be the square of the number n_j of keys hashing to slot j. While having such a quadratic dependence of m_j on n_j may seem likely to cause the overall storage requirements to be excessive, we shall show that by choosing the first level hash function well, the expected total amount of space used is still $O(n)$.

We use hash functions chosen from the universal classes of hash functions of Section 11.3.3. The first-level hash function is chosen from the class $\mathcal{H}_{p,m}$, where as in Section 11.3.3, p is a prime number greater than any key value. Those keys

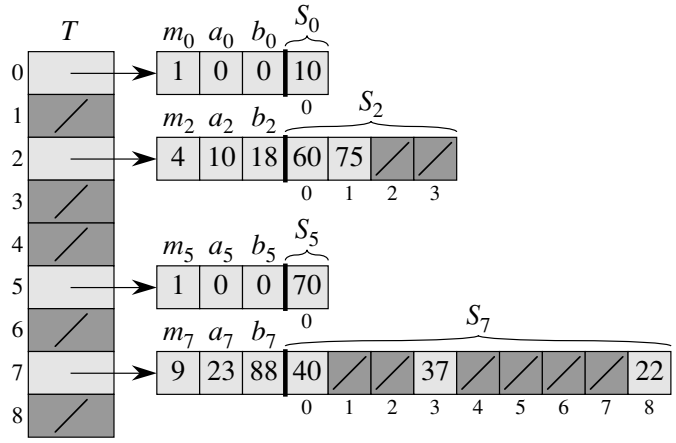

Figure 11.6 Using perfect hashing to store the set $K = \{10, 22, 37, 40, 60, 70, 75\}$. The outer hash function is $h(k) = ((ak + b) \bmod p) \bmod m$, where $a = 3$, $b = 42$, $p = 101$, and $m = 9$. For example, $h(75) = 2$, so key 75 hashes to slot 2 of table T. A secondary hash table S_j stores all keys hashing to slot j. The size of hash table S_j is m_j, and the associated hash function is $h_j(k) = ((a_j k + b_j) \bmod p) \bmod m_j$. Since $h_2(75) = 1$, key 75 is stored in slot 1 of secondary hash table S_2. There are no collisions in any of the secondary hash tables, and so searching takes constant time in the worst case.

hashing to slot j are re-hashed into a secondary hash table S_j of size m_j using a hash function h_j chosen from the class $\mathcal{H}_{p,m_j}$.[1]

We shall proceed in two steps. First, we shall determine how to ensure that the secondary tables have no collisions. Second, we shall show that the expected amount of memory used overall—for the primary hash table and all the secondary hash tables—is $O(n)$.

Theorem 11.9

If we store n keys in a hash table of size $m = n^2$ using a hash function h randomly chosen from a universal class of hash functions, then the probability of there being any collisions is less than $1/2$.

Proof There are $\binom{n}{2}$ pairs of keys that may collide; each pair collides with probability $1/m$ if h is chosen at random from a universal family $\mathcal{H}$ of hash functions. Let X be a random variable that counts the number of collisions. When $m = n^2$, the expected number of collisions is

[1]When $n_j = m_j = 1$, we don't really need a hash function for slot j; when we choose a hash function $h_{a,b}(k) = ((ak + b) \bmod p) \bmod m_j$ for such a slot, we just use $a = b = 0$.

$$\begin{aligned} \mathrm{E}[X] &= \binom{n}{2} \cdot \frac{1}{n^2} \\ &= \frac{n^2 - n}{2} \cdot \frac{1}{n^2} \\ &< 1/2 \,. \end{aligned}$$

(Note that this analysis is similar to the analysis of the birthday paradox in Section 5.4.1.) Applying Markov's inequality (C.29), $\Pr\{X \geq t\} \leq \mathrm{E}[X]/t$, with $t = 1$ completes the proof. ∎

In the situation described in Theorem 11.9, where $m = n^2$, it follows that a hash function h chosen at random from $\mathcal{H}$ is more likely than not to have *no* collisions. Given the set K of n keys to be hashed (remember that K is static), it is thus easy to find a collision-free hash function h with a few random trials.

When n is large, however, a hash table of size $m = n^2$ is excessive. Therefore, we adopt the two-level hashing approach, and we use the approach of Theorem 11.9 only to hash the entries within each slot. An outer, or first-level, hash function h is used to hash the keys into $m = n$ slots. Then, if n_j keys hash to slot j, a secondary hash table S_j of size $m_j = n_j^2$ is used to provide collision-free constant-time lookup.

We now turn to the issue of ensuring that the overall memory used is $O(n)$. Since the size m_j of the jth secondary hash table grows quadratically with the number n_j of keys stored, there is a risk that the overall amount of storage could be excessive.

If the first-level table size is $m = n$, then the amount of memory used is $O(n)$ for the primary hash table, for the storage of the sizes m_j of the secondary hash tables, and for the storage of the parameters a_j and b_j defining the secondary hash functions h_j drawn from the class $\mathcal{H}_{p,m_j}$ of Section 11.3.3 (except when $n_j = 1$ and we use $a = b = 0$). The following theorem and a corollary provide a bound on the expected combined sizes of all the secondary hash tables. A second corollary bounds the probability that the combined size of all the secondary hash tables is superlinear.

Theorem 11.10

If we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions, then

$$\mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right] < 2n \,,$$

where n_j is the number of keys hashing to slot j.

Proof We start with the following identity, which holds for any nonnegative integer a:

$$a^2 = a + 2\binom{a}{2}. \tag{11.6}$$

We have

$$\begin{aligned}
&\mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right] \\
&\quad = \mathrm{E}\left[\sum_{j=0}^{m-1}\left(n_j + 2\binom{n_j}{2}\right)\right] && \text{(by equation (11.6))} \\
&\quad = \mathrm{E}\left[\sum_{j=0}^{m-1} n_j\right] + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] && \text{(by linearity of expectation)} \\
&\quad = \mathrm{E}[n] + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] && \text{(by equation (11.1))} \\
&\quad = n + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] && \text{(since } n \text{ is not a random variable)}\ .
\end{aligned}$$

To evaluate the summation $\sum_{j=0}^{m-1}\binom{n_j}{2}$, we observe that it is just the total number of collisions. By the properties of universal hashing, the expected value of this summation is at most

$$\binom{n}{2}\frac{1}{m} = \frac{n(n-1)}{2m} = \frac{n-1}{2}\ ,$$

since $m = n$. Thus,

$$\begin{aligned}
\mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right] &\le n + 2\,\frac{n-1}{2} \\
&= 2n - 1 \\
&< 2n\ .
\end{aligned}$$

■

Corollary 11.11

If we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions and we set the size of each secondary hash table to $m_j = n_j^2$ for $j = 0, 1, \ldots, m-1$, then the expected amount of storage required for all secondary hash tables in a perfect hashing scheme is less than $2n$.

Proof Since $m_j = n_j^2$ for $j = 0, 1, \ldots, m-1$, Theorem 11.10 gives

$$\mathrm{E}\left[\sum_{j=0}^{m-1} m_j\right] = \mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right] < 2n \ , \tag{11.7}$$

which completes the proof. ■

Corollary 11.12
If we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions and we set the size of each secondary hash table to $m_j = n_j^2$ for $j = 0, 1, \ldots, m-1$, then the probability that the total storage used for secondary hash tables exceeds $4n$ is less than $1/2$.

Proof Again we apply Markov's inequality (C.29), $\Pr\{X \geq t\} \leq \mathrm{E}[X]/t$, this time to inequality (11.7), with $X = \sum_{j=0}^{m-1} m_j$ and $t = 4n$:

$$\begin{aligned} \Pr\left\{\sum_{j=0}^{m-1} m_j \geq 4n\right\} &\leq \frac{\mathrm{E}\left[\sum_{j=0}^{m-1} m_j\right]}{4n} \\ &< \frac{2n}{4n} \\ &= 1/2 \ . \end{aligned}$$

■

From Corollary 11.12, we see that testing a few randomly chosen hash functions from the universal family will quickly yield one that uses a reasonable amount of storage.

We now include, from CLRS, some of their material on approximation algorithms. We have discussed one already — an approximation algorithm for the vertex cover problem in Lecture 2. The following portions from CLRS refer to "**NP**-complete" and "**NP**-hard" which we have not discussed yet. For the purposes of the discussions in this lecture, simply assume that both those mean that the problem is computationally hard, which motivates the investigation into approximation algorithms for them.

We discuss three algorithms: one for travelling salesman, one for set cover, and one for subset sum.

NOTE: the approximation algorithm for subset sum is not part of the course. I include it in the textbook only because the class of algorithms to which that one belongs, FPRAS, is a particularly important one. But you can skip it for the course, and look at it out of your own interest only.

35 Approximation Algorithms

Performance ratios for approximation algorithms

Suppose that we are working on an optimization problem in which each potential solution has a positive cost, and we wish to find a near-optimal solution. Depending on the problem, we may define an optimal solution as one with maximum possible cost or one with minimum possible cost; that is, the problem may be either a maximization or a minimization problem.

We say that an algorithm for a problem has an ***approximation ratio*** of $\rho(n)$ if, for any input of size n, the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n) \ . \tag{35.1}$$

If an algorithm achieves an approximation ratio of $\rho(n)$, we call it a ***$\rho(n)$-approximation algorithm***. The definitions of the approximation ratio and of a $\rho(n)$-approximation algorithm apply to both minimization and maximization problems. For a maximization problem, $0 < C \leq C^*$, and the ratio C^*/C gives the factor by which the cost of an optimal solution is larger than the cost of the approximate

solution. Similarly, for a minimization problem, $0 < C^* \leq C$, and the ratio C/C^* gives the factor by which the cost of the approximate solution is larger than the cost of an optimal solution. Because we assume that all solutions have positive cost, these ratios are always well defined. The approximation ratio of an approximation algorithm is never less than 1, since $C/C^* \leq 1$ implies $C^*/C \geq 1$. Therefore, a 1-approximation algorithm[1] produces an optimal solution, and an approximation algorithm with a large approximation ratio may return a solution that is much worse than optimal.

For many problems, we have polynomial-time approximation algorithms with small constant approximation ratios, although for other problems, the best known polynomial-time approximation algorithms have approximation ratios that grow as functions of the input size n. An example of such a problem is the set-cover problem presented in Section 35.3.

Some NP-complete problems allow polynomial-time approximation algorithms that can achieve increasingly better approximation ratios by using more and more computation time. That is, we can trade computation time for the quality of the approximation. An example is the subset-sum problem studied in Section 35.5. This situation is important enough to deserve a name of its own.

An ***approximation scheme*** for an optimization problem is an approximation algorithm that takes as input not only an instance of the problem, but also a value $\epsilon > 0$ such that for any fixed ϵ, the scheme is a $(1 + \epsilon)$-approximation algorithm. We say that an approximation scheme is a ***polynomial-time approximation scheme*** if for any fixed $\epsilon > 0$, the scheme runs in time polynomial in the size n of its input instance.

The running time of a polynomial-time approximation scheme can increase very rapidly as ϵ decreases. For example, the running time of a polynomial-time approximation scheme might be $O(n^{2/\epsilon})$. Ideally, if ϵ decreases by a constant factor, the running time to achieve the desired approximation should not increase by more than a constant factor (though not necessarily the same constant factor by which ϵ decreased).

We say that an approximation scheme is a ***fully polynomial-time approximation scheme*** if it is an approximation scheme and its running time is polynomial in both $1/\epsilon$ and the size n of the input instance. For example, the scheme might have a running time of $O((1/\epsilon)^2 n^3)$. With such a scheme, any constant-factor decrease in ϵ comes with a corresponding constant-factor increase in the running time.

---

[1] When the approximation ratio is independent of n, we use the terms "approximation ratio of ρ" and "ρ-approximation algorithm," indicating no dependence on n.

35.2 The traveling-salesman problem

In the traveling-salesman problem , we are given a complete undirected graph $G = (V, E)$ that has a nonnegative integer cost $c(u, v)$ associated with each edge $(u, v) \in E$, and we must find a hamiltonian cycle (a tour) of G with minimum cost. As an extension of our notation, let $c(A)$ denote the total cost of the edges in the subset $A \subseteq E$:

$$c(A) = \sum_{(u,v)\in A} c(u,v) .$$

In many practical situations, the least costly way to go from a place u to a place w is to go directly, with no intermediate steps. Put another way, cutting out an intermediate stop never increases the cost. We formalize this notion by saying that the cost function c satisfies the ***triangle inequality*** if, for all vertices $u, v, w \in V$,

$$c(u,w) \le c(u,v) + c(v,w) .$$

The triangle inequality seems as though it should naturally hold, and it is automatically satisfied in several applications. For example, if the vertices of the graph are points in the plane and the cost of traveling between two vertices is the ordinary euclidean distance between them, then the triangle inequality is satisfied. Furthermore, many cost functions other than euclidean distance satisfy the triangle inequality.

35.2.1 The traveling-salesman problem with the triangle inequality

we shall first compute a structure—a minimum spanning tree—whose weight gives a lower bound on the length of an optimal traveling-salesman tour. We shall then use the minimum spanning tree to create a tour whose cost is no more than twice that of the minimum spanning tree's weight, as long as the cost function satisfies the triangle inequality. The following algorithm implements this approach, calling the minimum-spanning-tree algorithm MST-PRIM from Section 23.2 as a subroutine. The parameter G is a complete undirected graph, and the cost function c satisfies the triangle inequality.

APPROX-TSP-TOUR(G, c)

1 select a vertex $r \in G.V$ to be a "root" vertex
2 compute a minimum spanning tree T for G from root r
  using MST-PRIM(G, c, r)
3 let H be a list of vertices, ordered according to when they are first visited
  in a preorder tree walk of T
4 **return** the hamiltonian cycle H

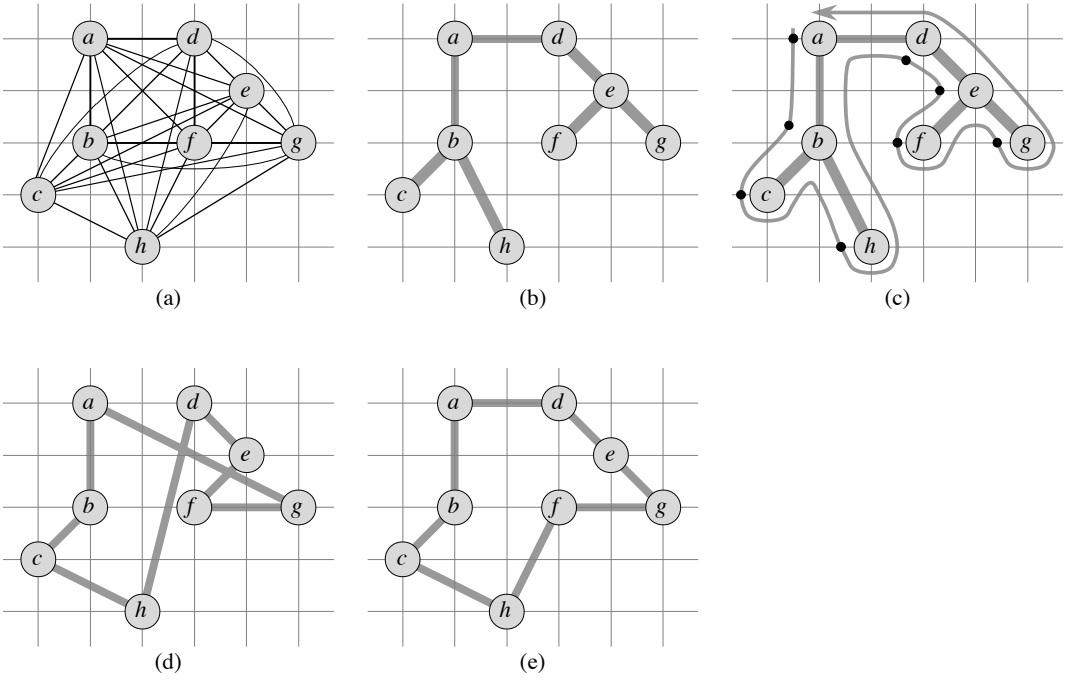

Figure 35.2 The operation of APPROX-TSP-TOUR. **(a)** A complete undirected graph. Vertices lie on intersections of integer grid lines. For example, f is one unit to the right and two units up from h. The cost function between two points is the ordinary euclidean distance. **(b)** A minimum spanning tree T of the complete graph, as computed by MST-PRIM. Vertex a is the root vertex. Only edges in the minimum spanning tree are shown. The vertices happen to be labeled in such a way that they are added to the main tree by MST-PRIM in alphabetical order. **(c)** A walk of T, starting at a. A full walk of the tree visits the vertices in the order $a, b, c, b, h, b, a, d, e, f, e, g, e, d, a$. A preorder walk of T lists a vertex just when it is first encountered, as indicated by the dot next to each vertex, yielding the ordering a, b, c, h, d, e, f, g. **(d)** A tour obtained by visiting the vertices in the order given by the preorder walk, which is the tour H returned by APPROX-TSP-TOUR. Its total cost is approximately 19.074. **(e)** An optimal tour H^* for the original complete graph. Its total cost is approximately 14.715.

Recall from Section 12.1 that a preorder tree walk recursively visits every vertex in the tree, listing a vertex when it is first encountered, before visiting any of its children.

Figure 35.2 illustrates the operation of APPROX-TSP-TOUR. Part (a) of the figure shows a complete undirected graph, and part (b) shows the minimum spanning tree T grown from root vertex a by MST-PRIM. Part (c) shows how a preorder walk of T visits the vertices, and part (d) displays the corresponding tour, which is the tour returned by APPROX-TSP-TOUR. Part (e) displays an optimal tour, which is about 23% shorter.

By Exercise 23.2-2, even with a simple implementation of MST-PRIM, the running time of APPROX-TSP-TOUR is $\Theta(V^2)$. We now show that if the cost function for an instance of the traveling-salesman problem satisfies the triangle inequality, then APPROX-TSP-TOUR returns a tour whose cost is not more than twice the cost of an optimal tour.

Theorem 35.2
APPROX-TSP-TOUR is a polynomial-time 2-approximation algorithm for the traveling-salesman problem with the triangle inequality.

Proof We have already seen that APPROX-TSP-TOUR runs in polynomial time.

Let H^* denote an optimal tour for the given set of vertices. We obtain a spanning tree by deleting any edge from a tour, and each edge cost is nonnegative. Therefore, the weight of the minimum spanning tree T computed in line 2 of APPROX-TSP-TOUR provides a lower bound on the cost of an optimal tour:

$$c(T) \leq c(H^*) . \tag{35.4}$$

A ***full walk*** of T lists the vertices when they are first visited and also whenever they are returned to after a visit to a subtree. Let us call this full walk W. The full walk of our example gives the order

$$a, b, c, b, h, b, a, d, e, f, e, g, e, d, a .$$

Since the full walk traverses every edge of T exactly twice, we have (extending our definition of the cost c in the natural manner to handle multisets of edges)

$$c(W) = 2c(T) . \tag{35.5}$$

Inequality (35.4) and equation (35.5) imply that

$$c(W) \leq 2c(H^*) , \tag{35.6}$$

and so the cost of W is within a factor of 2 of the cost of an optimal tour.

Unfortunately, the full walk W is generally not a tour, since it visits some vertices more than once. By the triangle inequality, however, we can delete a visit to any vertex from W and the cost does not increase. (If we delete a vertex v from W between visits to u and w, the resulting ordering specifies going directly from u to w.) By repeatedly applying this operation, we can remove from W all but the first visit to each vertex. In our example, this leaves the ordering

$$a, b, c, h, d, e, f, g .$$

This ordering is the same as that obtained by a preorder walk of the tree T. Let H be the cycle corresponding to this preorder walk. It is a hamiltonian cycle, since ev-

ery vertex is visited exactly once, and in fact it is the cycle computed by APPROX-TSP-TOUR. Since H is obtained by deleting vertices from the full walk W, we have

$$c(H) \le c(W) . \tag{35.7}$$

Combining inequalities (35.6) and (35.7) gives $c(H) \le 2c(H^*)$, which completes the proof. ■

In spite of the nice approximation ratio provided by Theorem 35.2, APPROX-TSP-TOUR is usually not the best practical choice for this problem. There are other approximation algorithms that typically perform much better in practice. (See the references at the end of this chapter.)

35.3 The set-covering problem

The set-covering problem is an optimization problem that models many problems that require resources to be allocated. Its corresponding decision problem generalizes the NP-complete vertex-cover problem and is therefore also NP-hard. The approximation algorithm developed to handle the vertex-cover problem doesn't apply here, however, and so we need to try other approaches. We shall examine a simple greedy heuristic with a logarithmic approximation ratio. That is, as the size of the instance gets larger, the size of the approximate solution may grow, relative to the size of an optimal solution. Because the logarithm function grows rather slowly, however, this approximation algorithm may nonetheless give useful results.

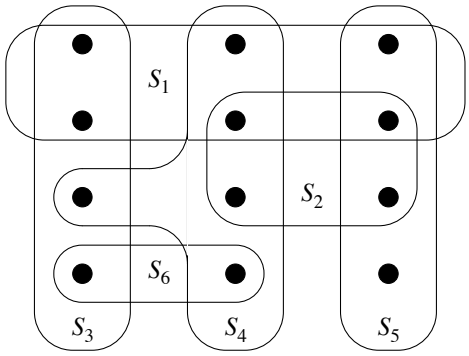

Figure 35.3 An instance $(X, \mathcal{F})$ of the set-covering problem, where X consists of the 12 black points and $\mathcal{F} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$. A minimum-size set cover is $\mathcal{C} = \{S_3, S_4, S_5\}$, with size 3. The greedy algorithm produces a cover of size 4 by selecting either the sets S_1, S_4, S_5, and S_3 or the sets S_1, S_4, S_5, and S_6, in order.

An instance $(X, \mathcal{F})$ of the ***set-covering problem*** consists of a finite set X and a family $\mathcal{F}$ of subsets of X, such that every element of X belongs to at least one subset in $\mathcal{F}$:

$$X = \bigcup_{S \in \mathcal{F}} S \ .$$

We say that a subset $S \in \mathcal{F}$ ***covers*** its elements. The problem is to find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$ whose members cover all of X:

$$X = \bigcup_{S \in \mathcal{C}} S \ . \tag{35.8}$$

We say that any $\mathcal{C}$ satisfying equation (35.8) ***covers*** X. Figure 35.3 illustrates the set-covering problem. The size of $\mathcal{C}$ is the number of sets it contains, rather than the number of individual elements in these sets, since every subset $\mathcal{C}$ that covers X must contain all $|X|$ individual elements. In Figure 35.3, the minimum set cover has size 3.

The set-covering problem abstracts many commonly arising combinatorial problems. As a simple example, suppose that X represents a set of skills that are needed to solve a problem and that we have a given set of people available to work on the problem. We wish to form a committee, containing as few people as possible, such that for every requisite skill in X, at least one member of the committee has that skill. In the decision version of the set-covering problem, we ask whether a covering exists with size at most k, where k is an additional parameter specified in the problem instance. The decision version of the problem is NP-complete, as Exercise 35.3-2 asks you to show.

A greedy approximation algorithm

The greedy method works by picking, at each stage, the set S that covers the greatest number of remaining elements that are uncovered.

```
GREEDY-SET-COVER(X, F)
1  U = X
2  C = ∅
3  while U ≠ ∅
4      select an S ∈ F that maximizes |S ∩ U|
5      U = U − S
6      C = C ∪ {S}
7  return C
```

In the example of Figure 35.3, GREEDY-SET-COVER adds to $\mathcal{C}$, in order, the sets S_1, S_4, and S_5, followed by either S_3 or S_6.

The algorithm works as follows. The set U contains, at each stage, the set of remaining uncovered elements. The set $\mathcal{C}$ contains the cover being constructed. Line 4 is the greedy decision-making step, choosing a subset S that covers as many uncovered elements as possible (breaking ties arbitrarily). After S is selected, line 5 removes its elements from U, and line 6 places S into $\mathcal{C}$. When the algorithm terminates, the set $\mathcal{C}$ contains a subfamily of $\mathcal{F}$ that covers X.

We can easily implement GREEDY-SET-COVER to run in time polynomial in $|X|$ and $|\mathcal{F}|$. Since the number of iterations of the loop on lines 3–6 is bounded from above by $\min(|X|, |\mathcal{F}|)$, and we can implement the loop body to run in time $O(|X||\mathcal{F}|)$, a simple implementation runs in time $O(|X||\mathcal{F}|\min(|X|, |\mathcal{F}|))$. Exercise 35.3-3 asks for a linear-time algorithm.

Analysis

We now show that the greedy algorithm returns a set cover that is not too much larger than an optimal set cover. For convenience, in this chapter we denote the dth harmonic number $H_d = \sum_{i=1}^{d} 1/i$ (see Section A.1) by $H(d)$. As a boundary condition, we define $H(0) = 0$.

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-approximation algorithm, where

$$\rho(n) = H(\max\{|S| : S \in \mathcal{F}\}) .$$

Proof We have already shown that GREEDY-SET-COVER runs in polynomial time.

To show that GREEDY-SET-COVER is a $\rho(n)$-approximation algorithm, we assign a cost of 1 to each set selected by the algorithm, distribute this cost over the elements covered for the first time, and then use these costs to derive the desired relationship between the size of an optimal set cover $\mathcal{C}^*$ and the size of the set cover $\mathcal{C}$ returned by the algorithm. Let S_i denote the ith subset selected by GREEDY-SET-COVER; the algorithm incurs a cost of 1 when it adds S_i to $\mathcal{C}$. We spread this cost of selecting S_i evenly among the elements covered for the first time by S_i. Let c_x denote the cost allocated to element x, for each $x \in X$. Each element is assigned a cost only once, when it is covered for the first time. If x is covered for the first time by S_i, then

$$c_x = \frac{1}{|S_i - (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|} .$$

Each step of the algorithm assigns 1 unit of cost, and so

$$|\mathcal{C}| = \sum_{x \in X} c_x . \tag{35.9}$$

Each element $x \in X$ is in at least one set in the optimal cover $\mathcal{C}^*$, and so we have

$$\sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \geq \sum_{x \in X} c_x . \tag{35.10}$$

Combining equation (35.9) and inequality (35.10), we have that

$$|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x . \tag{35.11}$$

The remainder of the proof rests on the following key inequality, which we shall prove shortly. For any set S belonging to the family $\mathcal{F}$,

$$\sum_{x \in S} c_x \leq H(|S|) . \tag{35.12}$$

From inequalities (35.11) and (35.12), it follows that

$$\begin{aligned} |\mathcal{C}| &\leq \sum_{S \in \mathcal{C}^*} H(|S|) \\ &\leq |\mathcal{C}^*| \cdot H(\max\{|S| : S \in \mathcal{F}\}) , \end{aligned}$$

thus proving the theorem.

All that remains is to prove inequality (35.12). Consider any set $S \in \mathcal{F}$ and any $i = 1, 2, \ldots, |\mathcal{C}|$, and let

$$u_i = |S - (S_1 \cup S_2 \cup \cdots \cup S_i)|$$

be the number of elements in S that remain uncovered after the algorithm has selected sets $S_1, S_2, \ldots, S_i$. We define $u_0 = |S|$ to be the number of elements

of S, which are all initially uncovered. Let k be the least index such that $u_k = 0$, so that every element in S is covered by at least one of the sets $S_1, S_2, \ldots, S_k$ and some element in S is uncovered by $S_1 \cup S_2 \cup \cdots \cup S_{k-1}$. Then, $u_{i-1} \geq u_i$, and $u_{i-1} - u_i$ elements of S are covered for the first time by S_i, for $i = 1, 2, \ldots, k$. Thus,

$$\sum_{x \in S} c_x = \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{|S_i - (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|} .$$

Observe that

$$\begin{aligned} |S_i - (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| &\geq |S - (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| \\ &= u_{i-1} , \end{aligned}$$

because the greedy choice of S_i guarantees that S cannot cover more new elements than S_i does (otherwise, the algorithm would have chosen S instead of S_i). Consequently, we obtain

$$\sum_{x \in S} c_x \leq \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{u_{i-1}} .$$

We now bound this quantity as follows:

$$\begin{aligned} \sum_{x \in S} c_x &\leq \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{u_{i-1}} \\ &= \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{u_{i-1}} \\ &\leq \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{j} && \text{(because } j \leq u_{i-1}\text{)} \\ &= \sum_{i=1}^{k} \left(\sum_{j=1}^{u_{i-1}} \frac{1}{j} - \sum_{j=1}^{u_i} \frac{1}{j} \right) \\ &= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) \\ &= H(u_0) - H(u_k) && \text{(because the sum telescopes)} \\ &= H(u_0) - H(0) \\ &= H(u_0) && \text{(because } H(0) = 0\text{)} \\ &= H(|S|) , \end{aligned}$$

which completes the proof of inequality (35.12). ■

Corollary 35.5
GREEDY-SET-COVER is a polynomial-time $(\ln|X|+1)$-approximation algorithm.

Proof Use inequality (A.14) and Theorem 35.4. ■

In some applications, $\max\{|S| : S \in \mathcal{F}\}$ is a small constant, and so the solution returned by GREEDY-SET-COVER is at most a small constant times larger than optimal. One such application occurs when this heuristic finds an approximate vertex cover for a graph whose vertices have degree at most 3. In this case, the solution found by GREEDY-SET-COVER is not more than $H(3) = 11/6$ times as large as an optimal solution,

NOTE: the following approximation algorithm for subset sum is not part of the course. I include it in the textbook only because the class of algorithms to which that one belongs, FPRAS, is a particularly important one. But you can skip it for the course, and look at it out of your own interest only.

35.5 The subset-sum problem

Recall from Section 34.5.5 that an instance of the subset-sum problem is a pair (S, t), where S is a set $\{x_1, x_2, \ldots, x_n\}$ of positive integers and t is a positive integer. This decision problem asks whether there exists a subset of S that adds up exactly to the target value t. As we saw in Section 34.5.5, this problem is NP-complete.

The optimization problem associated with this decision problem arises in practical applications. In the optimization problem, we wish to find a subset of $\{x_1, x_2, \ldots, x_n\}$ whose sum is as large as possible but not larger than t. For example, we may have a truck that can carry no more than t pounds, and n different boxes to ship, the ith of which weighs x_i pounds. We wish to fill the truck with as heavy a load as possible without exceeding the given weight limit.

In this section, we present an exponential-time algorithm that computes the optimal value for this optimization problem, and then we show how to modify the algorithm so that it becomes a fully polynomial-time approximation scheme. (Recall that a fully polynomial-time approximation scheme has a running time that is polynomial in $1/\epsilon$ as well as in the size of the input.)

An exponential-time exact algorithm

Suppose that we computed, for each subset S' of S, the sum of the elements in S', and then we selected, among the subsets whose sum does not exceed t, the one whose sum was closest to t. Clearly this algorithm would return the optimal solution, but it could take exponential time. To implement this algorithm, we could use an iterative procedure that, in iteration i, computes the sums of all subsets of $\{x_1, x_2, \ldots, x_i\}$, using as a starting point the sums of all subsets of $\{x_1, x_2, \ldots, x_{i-1}\}$. In doing so, we would realize that once a particular subset S' had a sum exceeding t, there would be no reason to maintain it, since no superset of S' could be the optimal solution. We now give an implementation of this strategy.

The procedure EXACT-SUBSET-SUM takes an input set $S = \{x_1, x_2, \ldots, x_n\}$ and a target value t; we'll see its pseudocode in a moment. This procedure it-

eratively computes L_i, the list of sums of all subsets of $\{x_1, \ldots, x_i\}$ that do not exceed t, and then it returns the maximum value in L_n.

If L is a list of positive integers and x is another positive integer, then we let $L + x$ denote the list of integers derived from L by increasing each element of L by x. For example, if $L = \langle 1, 2, 3, 5, 9 \rangle$, then $L + 2 = \langle 3, 4, 5, 7, 11 \rangle$. We also use this notation for sets, so that

$$S + x = \{s + x : s \in S\} \ .$$

We also use an auxiliary procedure MERGE-LISTS(L, L'), which returns the sorted list that is the merge of its two sorted input lists L and L' with duplicate values removed. Like the MERGE procedure we used in merge sort (Section 2.3.1), MERGE-LISTS runs in time $O(|L| + |L'|)$. We omit the pseudocode for MERGE-LISTS.

```
EXACT-SUBSET-SUM(S, t)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4       L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5       remove from L_i every element that is greater than t
6  return the largest element in L_n
```

To see how EXACT-SUBSET-SUM works, let P_i denote the set of all values obtained by selecting a (possibly empty) subset of $\{x_1, x_2, \ldots, x_i\}$ and summing its members. For example, if $S = \{1, 4, 5\}$, then

$$\begin{aligned} P_1 &= \{0, 1\} \ , \\ P_2 &= \{0, 1, 4, 5\} \ , \\ P_3 &= \{0, 1, 4, 5, 6, 9, 10\} \ . \end{aligned}$$

Given the identity

$$P_i = P_{i-1} \cup (P_{i-1} + x_i) \ , \tag{35.23}$$

we can prove by induction on i (see Exercise 35.5-1) that the list L_i is a sorted list containing every element of P_i whose value is not more than t. Since the length of L_i can be as much as 2^i, EXACT-SUBSET-SUM is an exponential-time algorithm in general, although it is a polynomial-time algorithm in the special cases in which t is polynomial in $|S|$ or all the numbers in S are bounded by a polynomial in $|S|$.

A fully polynomial-time approximation scheme

We can derive a fully polynomial-time approximation scheme for the subset-sum problem by "trimming" each list L_i after it is created. The idea behind trimming is

that if two values in L are close to each other, then since we want just an approximate solution, we do not need to maintain both of them explicitly. More precisely, we use a trimming parameter δ such that $0 < \delta < 1$. When we ***trim*** a list L by δ, we remove as many elements from L as possible, in such a way that if L' is the result of trimming L, then for every element y that was removed from L, there is an element z still in L' that approximates y, that is,

$$\frac{y}{1+\delta} \le z \le y \ . \tag{35.24}$$

We can think of such a z as "representing" y in the new list L'. Each removed element y is represented by a remaining element z satisfying inequality (35.24). For example, if $\delta = 0.1$ and

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle \ ,$$

then we can trim L to obtain

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle \ ,$$

where the deleted value 11 is represented by 10, the deleted values 21 and 22 are represented by 20, and the deleted value 24 is represented by 23. Because every element of the trimmed version of the list is also an element of the original version of the list, trimming can dramatically decrease the number of elements kept while keeping a close (and slightly smaller) representative value in the list for each deleted element.

The following procedure trims list $L = \langle y_1, y_2, \ldots, y_m \rangle$ in time $\Theta(m)$, given L and δ, and assuming that L is sorted into monotonically increasing order. The output of the procedure is a trimmed, sorted list.

```
TRIM(L, δ)
1  let m be the length of L
2  L' = ⟨y_1⟩
3  last = y_1
4  for i = 2 to m
5      if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
6          append y_i onto the end of L'
7          last = y_i
8  return L'
```

The procedure scans the elements of L in monotonically increasing order. A number is appended onto the returned list L' only if it is the first element of L or if it cannot be represented by the most recent number placed into L'.

Given the procedure TRIM, we can construct our approximation scheme as follows. This procedure takes as input a set $S = \{x_1, x_2, \ldots, x_n\}$ of n integers (in arbitrary order), a target integer t, and an "approximation parameter" ϵ, where

$$0 < \epsilon < 1 . \tag{35.25}$$

It returns a value z whose value is within a $1 + \epsilon$ factor of the optimal solution.

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4       L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5       L_i = TRIM(L_i, ε/2n)
6       remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

Line 2 initializes the list L_0 to be the list containing just the element 0. The **for** loop in lines 3–6 computes L_i as a sorted list containing a suitably trimmed version of the set P_i, with all elements larger than t removed. Since we create L_i from L_{i-1}, we must ensure that the repeated trimming doesn't introduce too much compounded inaccuracy. In a moment, we shall see that APPROX-SUBSET-SUM returns a correct approximation if one exists.

As an example, suppose we have the instance

$$S = \langle 104, 102, 201, 101 \rangle$$

with $t = 308$ and $\epsilon = 0.40$. The trimming parameter δ is $\epsilon/8 = 0.05$. APPROX-SUBSET-SUM computes the following values on the indicated lines:

line 2: $L_0 = \langle 0 \rangle$,

line 4: $L_1 = \langle 0, 104 \rangle$,
line 5: $L_1 = \langle 0, 104 \rangle$,
line 6: $L_1 = \langle 0, 104 \rangle$,

line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$,
line 5: $L_2 = \langle 0, 102, 206 \rangle$,
line 6: $L_2 = \langle 0, 102, 206 \rangle$,

line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$,
line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$,
line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$,

line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$,
line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$,
line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$.

The algorithm returns $z^* = 302$ as its answer, which is well within $\epsilon = 40\%$ of the optimal answer $307 = 104 + 102 + 101$; in fact, it is within 2%.

Theorem 35.8
APPROX-SUBSET-SUM is a fully polynomial-time approximation scheme for the subset-sum problem.

Proof The operations of trimming L_i in line 5 and removing from L_i every element that is greater than t maintain the property that every element of L_i is also a member of P_i. Therefore, the value z^* returned in line 8 is indeed the sum of some subset of S. Let $y^* \in P_n$ denote an optimal solution to the subset-sum problem. Then, from line 6, we know that $z^* \le y^*$. By inequality (35.1), we need to show that $y^*/z^* \le 1 + \epsilon$. We must also show that the running time of this algorithm is polynomial in both $1/\epsilon$ and the size of the input.

As Exercise 35.5-2 asks you to show, for every element y in P_i that is at most t, there exists an element $z \in L_i$ such that

$$\frac{y}{(1+\epsilon/2n)^i} \le z \le y \ . \tag{35.26}$$

Inequality (35.26) must hold for $y^* \in P_n$, and therefore there exists an element $z \in L_n$ such that

$$\frac{y^*}{(1+\epsilon/2n)^n} \le z \le y^* \ ,$$

and thus

$$\frac{y^*}{z} \le \left(1 + \frac{\epsilon}{2n}\right)^n \ . \tag{35.27}$$

Since there exists an element $z \in L_n$ fulfilling inequality (35.27), the inequality must hold for z^*, which is the largest value in L_n; that is,

$$\frac{y^*}{z^*} \le \left(1 + \frac{\epsilon}{2n}\right)^n \ . \tag{35.28}$$

Now, we show that $y^*/z^* \le 1 + \epsilon$. We do so by showing that $(1 + \epsilon/2n)^n \le 1 + \epsilon$. By equation (3.14), we have $\lim_{n\to\infty}(1 + \epsilon/2n)^n = e^{\epsilon/2}$. Exercise 35.5-3 asks you to show that

$$\frac{d}{dn}\left(1 + \frac{\epsilon}{2n}\right)^n > 0 \ . \tag{35.29}$$

Therefore, the function $(1 + \epsilon/2n)^n$ increases with n as it approaches its limit of $e^{\epsilon/2}$, and we have

$$\begin{aligned}\left(1+\frac{\epsilon}{2n}\right)^n &\le e^{\epsilon/2} \\ &\le 1+\epsilon/2+(\epsilon/2)^2 \quad \text{(by inequality (3.13))} \\ &\le 1+\epsilon \quad \text{(by inequality (35.25))} \ . \end{aligned} \tag{35.30}$$

Combining inequalities (35.28) and (35.30) completes the analysis of the approximation ratio.

To show that APPROX-SUBSET-SUM is a fully polynomial-time approximation scheme, we derive a bound on the length of L_i. After trimming, successive elements z and z' of L_i must have the relationship $z'/z > 1+\epsilon/2n$. That is, they must differ by a factor of at least $1+\epsilon/2n$. Each list, therefore, contains the value 0, possibly the value 1, and up to $\lfloor \log_{1+\epsilon/2n} t \rfloor$ additional values. The number of elements in each list L_i is at most

$$\begin{aligned}\log_{1+\epsilon/2n} t + 2 &= \frac{\ln t}{\ln(1+\epsilon/2n)} + 2 \\ &\le \frac{2n(1+\epsilon/2n)\ln t}{\epsilon} + 2 \quad \text{(by inequality (3.17))} \\ &< \frac{3n \ln t}{\epsilon} + 2 \quad \text{(by inequality (35.25))} \ . \end{aligned}$$

This bound is polynomial in the size of the input—which is the number of bits $\lg t$ needed to represent t plus the number of bits needed to represent the set S, which is in turn polynomial in n—and in $1/\epsilon$. Since the running time of APPROX-SUBSET-SUM is polynomial in the lengths of the L_i, we conclude that APPROX-SUBSET-SUM is a fully polynomial-time approximation scheme. ■