Problem Set 2

Charisios Grivas

cgrivas@math.aau.dk

Aalborg University

Comment: Exercises with * are a bit more challenging and hence can be considered out of scope. However, all students are invited to try them.

1. Let $Z_t : t = 1, 2, ...$ be a sequence of independent random variables such that $E(Z_t) = 0$ and $E(Z_t^2) = \sigma^2 < \infty$ for all t. Which of the following stochastic processes are weakly stationary? Obtain the mean and autocovariance function of each process.

a) $X_t = \alpha + \beta Z_1$ (α, β constants)

b) $X_t = Z_t Z_{t-1}$.

2. Consider the $MA(2)$ process

$$X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

a) Under what condition(s) is X_t weakly stationary and invertible?

b) Obtain the mean, variance and autocorrelation function of X_t.

3. Consider the $AR(2)$ process

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

a) Explain what is meant by the Wold Decomposition Theorem. Under what conditions does this theorem apply to X_t?

$b)$ Obtain the mean and variance of X_t.

4. Let $\varepsilon_t \sim WN(0, \sigma^2)$. Determine which of the following $ARMA$ processes are weakly stationary and/or invertible.

1. $X_t = -0.2X_{t-1} + 0.48X_{t-2} + \varepsilon_t$
2. $X_t = 0.6X_{t-1} + \varepsilon_t + 1.2\varepsilon_{t-1}$
3. $X_t = -1.9X_{t-1} - 0.88X_{t-2} + \varepsilon_t + 0.2\varepsilon_{t-1} + 0.7\varepsilon_{t-2}$

5. * Consider the $AR(2)$ model

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t \tag{1}$$

or in lag notation

$$(1 - \phi_1 L - \phi_2 L^2)y_t = c + \epsilon_t.$$

Further, assume that it is covariance-stationary and that the following holds:

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2)^2 = \psi_0 + \psi_1 L + \psi_2 L^2 + \dots$$

Multiplying both sides of (1) with the above, gives

$$Y_t = \psi(L)c + \psi(L)\epsilon_t$$

Show that,

$a)$ $\psi(L)c = \frac{c}{1-\phi_1-\phi_2}$

$b)$ $\sum_{j=0}^{\infty} |\psi_j| < \infty$

$c)$ Prove that absolute summability implies square summability i.e $\sum_{j=0}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=0}^{\infty} \psi_j^2 < \infty$. **Hint:** You might want to use the Cauchy criterion for convergence.