Elementary Electromagnetism Using Python

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Preface

This book was developed as a textbook for the course Fys1120 - Electromagnetism at the University of Oslo, Norway from 2018 and onwards. The book was written to provide students with an integrated text that includes both analytical and computational methods in a seamless manner and that interfaces with how students learn to use computational methods in associated courses in mathematics.

I wrote the book as I was making a major transformation on both the content and learning methods used in the course, where I wanted to focus on student agency — teaching students to become active partipants in their own learning process and trying to push students towards higher levels in Bloom's taxonomy, towards creativity, exporation and expression. In course this was done by introducing students to computational essays. These are not part of the text, but you can find examples of computational essays here[1], you can find details about how the computational essays where implemented here[2], and you can find a set of research articles describing the project online. The computational essay and student agency part of the course was introduced and driven by Tor Ole Odden.

This book is written in the spirit of Hans Petter Langtangen, who was my great mentor in textbook writing and the integration of computing across the sciences. I have also been greatly inspired by Tor Ole Odden, when it comes to how to create good learning progressions, how to inspire student agency, on how to integrate our knowledge of best practice in

[1] `https://uio-ccse.github.io/computational-essay-showroom/intro`

[2] `https://www.uio.no/studier/emner/matnat/fys/FYS1120/h19/computationalessay/compessay-assignment-d`

physics education in our teaching, and, most importantly, when it comes to the computational essays. The first year I was teaching this course it was with the help of Henrik Anderson Sveinsson, and many of the exercises have been developed by him. The text is also inspired by the norwegian text written by Johannes Skaar, "Elektromagnetisme", and the lecture series by Johannes Skaar, which both are inspiringly concise and have inspired both exposition and examples in this book. This book has also benefited from several students who have helped develop examples and exercises, such as Rene Ask, Sigurd Rustad and Bror Hjemgaard, and from Learning Assistants who have helped improve text and examples, including Kine Ødegaard Hanssen and Even Nordhagen.

This book has been written using DocOnce, which allows me to write one book with both Python and Matlab code, compile it to LaTeX, Jupyter Notebooks and dynamic html, while keeping with the simple goals of LaTeX where focus is on structure and content. The companion website with learning material, exercises and solutions was also built using DocOnce.

April 2021 *Anders Malthe-Sørenssen*

Contents

0 Introduction

In this book we will introduce the fundamental laws of electromagnetism and learn to use them to understand nature and to develop and understand technology. The field of electromagnetism is rich and beautiful, and the theory of electromagnetism is one of the most beautiful theories we know of — decribed concisely by Maxwell's four equations. However, the theory by itself is of little use if we do not know how to harness it to understand and master nature. This text therefore focuses on making you a proficient user of electromagnetism by providing you with a solid understanding of the fundamentals, a good understanding of how to model physical systems with simplified physics-based models, and the tools to solve the models analytically and computationally. The book has a unique approach where the computational methods are deeply integrated into exposition, examples and exercises. The book has companion material in the form of computational essay examples and projects that help students become active participants in their own learning and to start asking their own questions, posing their own problems, and discovering how to pursue these problems, analyse them and communicate the results effectively. The book also has extensive sets of analytical and computational exercises with answers and solutions in a dynamic web-page setting that facilitate good learning. My goal is for you as a student to succeed by making electromagnetism a part of you, so that physics in general and electromagnetism in particular shape how you think, and help you think, discover, innovate and communicate better — letting physics become an extended part of you.

0.1 Electromagnetism

Electromagnetic interactions are everywhere. Electromagnetism is a fundamental theory of interactions that are both hidden and evident in the world around you. In mechanics, we have addressed gravitational forces and various types of contacts forces such as normal forces, friction and air resistance. All the macroscopic forces beyond gravity are electromagnetic forces in various forms. All contact forces are due to electromagnetic interactions. When we treat atoms classically, by for example integrating their equations of motion to discover the behavior of many-particle systems, the forces between atoms are typically modelled as various forms of electrostatic forces. The attractive interaction between surfaces due to van der Waals interactions is due to an induced dipole-dipole interaction — which we will discuss in detail in this book. Many aspects of cell mechanics, in particular for nerve cells, are best understood in terms of electromagnetic concepts, and of course modern technology depends on our mastery of electromagnetism for circuits, light, sensors, and communications. Your cellphone is a marvel of engineering, based on our understanding of electromagnetism as well as other aspects of physics.

An elegant theory. Electromagnetism is also an elegant theory. Maybe this is your first encounter with a theory that is so beautiful that you are only left to marvel. All the electromagnetic phenomena you have learned about in various contexts so far in your education will be covered by the four Mawell equations. However, this compactness and beauty relies on a well developed mathematical machinery. This book assumes that you have had an introductory course in vector calculus. However, it is fully possible to follow the text without this background knowledge. To help you if you are in this situation, we have provided you with a brief introduction to vector calculus.

A lasting theory. When you have read this book, you fully understand the last chapter, Maxwell's equations, and you have built your skills so that you can use them effectively, you have become a master of one of the cornerstones of our civilization. And you will have acquired lasting knowledge and skills. News, celebrities, human glory and suffering is transient, but the fundamental theory of electromagnetism remain unchanged. We understand this well.

A complete theory. However, you should not be fooled by the beauty of the theory. It also means that this part of physics is largely finished. We understand the fundamentals and have been able to formulate the laws concisely. My advise to you is to pursue areas where our understanding is yet lacking, where there is no beautiful theory — yet — and be the one to create this theory. But whatever it is you will pursue — if it is to understand our brain or develop new green technologies and energy solutions — I am sure you will need a solid understand of electromagnetism to succeed!

0.2 Computational methods

Power of physics. This book has been written to provide a textbook where computational methods are deeply integrated into the exposition, theory, examples and problems. I want you to be able to use computational methods as naturally as analytical methods when you learn and apply physics. And I want to help you build a set of tools that you can use to address any physics problem, not only the ones that are analytically tractable. As you read this book and work through the examples and problems, you should gradually get the feeling of the *power of physics*. That you learn methods that are powerful and that can be applied to any problem. That there are robust solution methods that can be applied and that allows you to study phenomena and processes you are intrigued by.

Worked examples. I provide you with complete worked examples, where all aspects are addressed: The problem is specified and a physics-based model is introduced, the model is solved computationally, sometimes we make a simplified theoretical model to understand the behavior we observe computationally. The full computer code is provided and you can find all the examples as Jupyter Notebooks so that you can change parameters, modify the code or add your own comments to the text.

Building and implementing models. You will also see that many of the same computational methods are applied over and over. This will help you see mathematical and methodological similarities between different aspects of electromagnetism. The book has a focus on modeling, which is the process of simplifying and abstracting a physical system into a model system that we can address using the theoretical and computational tools we have available. Learning to model and to simplify a problem down to its core components is a very important part of becoming a physicist — indeed it may be argued that this is the core of physics as a practice.

However, it is often difficult to learn these skills and to learn to trust your own intuition. I am sure you will often ask yourself if you made the model too simple or why you can ignore a particular aspect of a problem. Often the answer is in the pudding. You test your model to see if it reproduces the aspects you are interested in modeling, and thus learn to trust it. Learning modeling is a long process, which requires that you are challenged by your teachers, peers and yourself. But it should also be a fun process with a feeling of adventure as you learn to see the world in its simplest/simplified form.

0.3 Learning physics

Learning effectively. We have significant amounts of research-based knowledge on how to teach and how to learn physics. Most of educational research boils down to the following: To learn effectively you have to work on problems that are just beyond your current level of mastery, formulate your understanding, and receive constructive and immediate feedback. We know that learning is a social process, you learn better if you learn with others, and that it should be a streneous activity, you should be tired after trying to learn. We also know that it is good to alternate between being highly focused, which is the streneous activity, and defocused, where you allow you mind to wander more freely.

Using worked examples to learn. Research indiciates that you learn physics well by going through worked examples before you solve the first problem. Use examples actively and solutions carefully. Solutions to problems are like Kryptonite. If you come too close to the solution your powers of learning will weaken. I provide solutions to all problems in this text. Be careful when you unroll them. Try hard first. Then look. Use the examples actively and look them up when you are attempting to solve a problem — see if you can find a structure from the example that may help you solve the problem.

Methods are clearly described. I provide "Method" boxes throughout the text that provides you with step-by-step intructions extracted from the examples on how to solve a particular type of problem. Use these and learn them so that you can apply them when you need them, but also deviate from the methods when the situations requires you to.

0.4 Structure of the book

This book follows a traditional structure of the exposition. We start from electrostatics and magnetostatics, where charges and currents are stationary, before moving to electrodynamics, where charges can move and currents may change with time. Each chapter contains a few key worked examples and associated “Methods” sections. Many sections have been written to provide you with a solid toolbox to address problems: We introduce computational methods to find the electric and magnetic fields from any charge or current distribution, and we provide you with three tools to solve Laplace equation. The goal is not for you to learn all the details of the methods, but instead to feel confident in their use, and to learn their limitations so that you can evaluate the results you get in real situations.

1 Electric field

In this chapter we will introduce the fundamental law for the force between two charged bodies — Coulomb's law — and the corresponding field termed the electric field. We will address forces between point charges and continuous distributions of charges in space, on surfaces and along lines. Methods to calculate and visualize the electric field analytically and numerically will be discussed. We will also start to discuss modeling: How we simplify a physical situation and gradually build a detailed physical model of it.

Motivation.

1.1 Coulomb's law

If I rub a balloon against my hair and push it towards a wall, it will stick to the wall as illustrated in Fig. 1.1. Why? We know from mechanics that gravity will pull the balloon down. Since it does not slide down, there must be another force acting in the opposite direction of gravity. What is this force?

Forces between charged objects. The force that acts from the wall on the balloon is an electrostatic force. It is a fundamental force, just like gravity, acting between all *charged* objects. The balloon becomes charged when we rub it against our hair. Electrostatic forces are along with gravity one of the four fundamental forces of nature. In our everyday lives we mostly have experience from gravity and electrostatic forces. However, we often do not always realize that forces around us have an electrostatic origin, and we often given these forces other names. For example, the normal force between you and the ground you are standing on is due to electrostatic forces between the atoms on your surface and the surface of the ground. Indeed, all interatomic forces can be considered to be electrostatic forces that are due to the distribution of charges around the atoms.

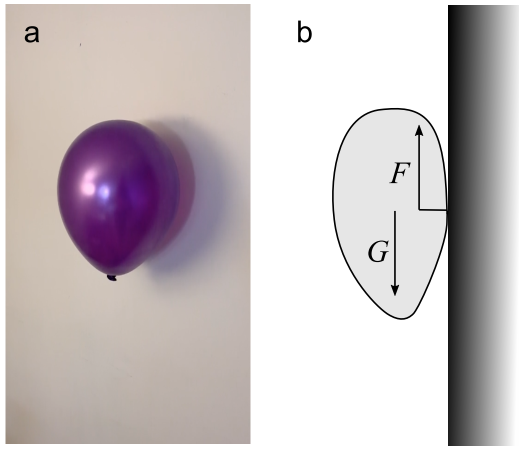


Fig. 1.1 (a) Illustration of a balloon hanging on a wall. (b) Force diagram for the balloon, indicating gravity, **G**, and the contact force, **F**, due to electrostatic interactions.

The electrostatic force law. This fundamental force law that acts between all charged objects is formulated in *Coulumb's* law. It is a fundamental law like gravity, and the force law has a similar form to that of gravity. The force between two point objects with charges q and Q at positions $\mathbf{r}_q$ and $\mathbf{r}_Q$ respectively is given Coulomb's law (see Fig. 1.2):

Coulomb's law

The force *on* a point-particle with charge q at a position $\mathbf{r}_q$ *from* a point-particle with charge Q at position $\mathbf{r}_Q$ is:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}}{R^2} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} , \tag{1.1}$$

where $\mathbf{R} = \mathbf{r}_q - \mathbf{r}_Q$, $\hat{\mathbf{R}} = \mathbf{R}/R$, and $R = |\mathbf{R}|$.

Notice the two different ways to write the law: Either with the unit vector $\hat{\mathbf{R}}$ divided by R^2 or by the vector $\mathbf{R}$ divided by R^3. The two ways are identical because $\hat{\mathbf{R}} = \mathbf{R}/R$.

Coulomb's law is an experimentally established law. It is correct as far as we can measure, up to a precision of 10^{-15}.

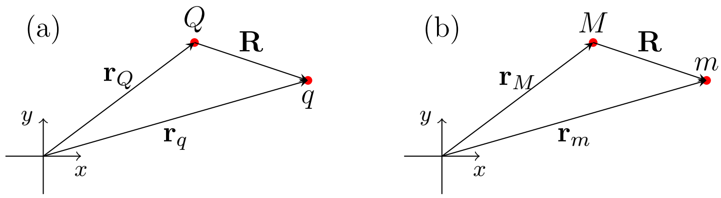


Fig. 1.2 (a) The force from a charge Q at $\mathbf{r}_Q$ on a charge q at $\mathbf{r}$. (b) The force from a mass M at $\mathbf{R}_M$ on a mass m at $\mathbf{r}_m$.

Similarity to gravity. Coulomb's law looks very much like Newton's law of gravity. We recall that the force on a mass m at position $\mathbf{r}_m$ from a mass M at a position $\mathbf{r}_M$ is

$$\mathbf{F}_G = -GmM\frac{\hat{\mathbf{R}}}{R^2} \,, \tag{1.2}$$

where $\mathbf{R} = \mathbf{r}_m - \mathbf{r}_M$, $\hat{\mathbf{R}} = \mathbf{R}/R$, and $R = |\mathbf{R}|$. You have a lot of experience with Newton's law: You know how to find its potential energy, you know how to solve the motion of objects affected by this force low, and you know how to solve such problems analytically and numerically. We will build on this intuition when we develop our understanding of Coulomb's law.

Charges versus masses. We see that Newton's law of gravity has the opposite sign of Coulomb's law: For gravity two objects always attract each other because the masses m and M are always positive. Whereas for Coloumb's law, two objects with repel each other if qQ is positive and attract each other if qQ is negative. You already have experience and intuition about masses. You know that mass is a fundamental property of matter and that masses only can be positive. What about charge q? We need to build a similar understanding and intuition for charge. Charge is also a fundamental property of matter. However, charges may be either positive or negative. This means that the product qQ can be either positive or negative. If the charges have the same sign, the objects will repel each other, but if they have different sign they will attract each other.

Electrons, protons and the conservation of charge. The fundamental building block of charges are the charges of electrons and protons: Electrons have a change of $-e$ and protons have a charge e. We will therefore often refer to the atomic model of matter when we build our intuition about charged objects. Charges are measured in units of Coulombs, where $e = 1.602 \times 10^{-19}$C is the charge of the proton. Charge is a *conserved quantity*. The net charge of a system does not change unless we add or remove charges from it. For example, a hydrogen atom has zero net charge. If it is split into an electron and a proton, the net charge is $-e + e = 0$. The net charge is still zero. In the example of the balloon above, we changed the net charge of the balloon when we rubbed it against our hair. Electrons were transferred from the hair to the balloon, making the balloon negatively charged. As a result the balloon had a net charge different from zero and a net force was acting between the wall and the balloon [1].

Adding masses and charges. Because charges can be positive and negative, the *net charge* of an object can be zero even if there is internal structure in the charges. For example, the net charge of a hydrogen atom is zero: It has one proton with charge e and one electron with charge $-e$. This is a significant difference from gravity, because masses only add up. We will gradually learn the consequences of this difference, but an important effect is that since most objects have approximately zero net charge, the effective electrostatic force falls off more rapidly with distance for electromagnetic forces than for gravity. Gravity forces always add up. The gravitational force from the Sun is the sum of all the masses in the Sun, which is non-negligigle. Because electrostatic forces both add and subtract, the electrostatic force from the Sun is the net sum of all the attractive and repulsive forces from the individual charges in the Sun, which is typically small since the net charge of the Sun is close to zero.

Comparing the magnitude of gravity and electrostatic forces. How large are the electrostatic forces compared to gravity? We can compare the electrostatic and gravitational forces between an electron and a proton. The mass of the electron is $m_e = 9.10938356 \times 10^{-31}$ kg, the mass of the

[1] However, the wall would typically have zero net charge before it comes in contact with the balloon. To understand what happened in this case, we need to develop our understanding of electrostatic systems further. We will return to this case when we have developed the necessary concepts.

proton is $m_p = 1.6726219 \times 10^{-27}$ kg, the gravitational constant is $G = 6.67430 \times 10^{-11}$ m^{3}kg$^{-1}s^{-2}$. The gravitational force is $F_G = Gm_em_p/R^2$ and the electrostatic force is $F_C = e^2/(4\pi\epsilon_0 R^2)$. Here ϵ_0 is called the permittivity of vacuum and its value is $\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2\mathrm{N}^-1\mathrm{m}^{-2}$. Both forces depend on R^2. If we look at the ratio, the Rs will cancel out:

$$\frac{F_G}{F_C} = \frac{Gm_em_p}{e^2/(4\pi\epsilon_0)} = \frac{Gm_em_p4\pi\epsilon_0}{e^2} \tag{1.3}$$

```
import numpy as np
e = 1.602e-19
me = 9.10938356e-31
mp = 1.6726219e-27
G = 6.67430e-11
epsilon0 = 8.85e-12
ratio = G*me*mp*4*np.pi*epsilon0/(e*e)
print("ratio = ",ratio)
```

```
ratio =  4.406772215446942e-40
```

The gravitational force between an electron and a proton is therefore very much smaller than the electrostatic force — irrespective of distance. If we for simplicity assume that all matter consists of electrons and protons (forgetting about neutrons for the argument), this should also hold for the Earth and the Sun. But we must then remember that the *net* charge often is approximately zero, as discussed above. Masses add up, but charges tend to (almost) cancel out.

The permittivity in vacuum. The quantity ϵ_0 is called the permittivity of vacuum. We have written the constant in front of Coulumb's law as $1/(4\pi\epsilon_0)$. This seems a strange choice. Why not simply call it $K = 1/(4\pi\epsilon_0)$ in analogy to the G in the gravitational law? It turns out that this way of writing it is practical, but the reason for this will first become apparent later (in Chapter).

Point charges and extended objects. We recall from mechanics that we started by describing objects as point objects before we moved on to address extended objects as sums of point objects. We will pursue a similar approach in electromagnetism. We will start by addressing *point charges*, which are charged bodies where the dimensions of the body is much smaller than typical distances between bodies of interest. Electrons and protons are considered point charges, but we will use this concept also as simplified models of macroscopic bodies. To find the net effect of many point charges, we need to apply the superposition principle of forces.

Superposition principle. The superposition principle for forces also holds for Coulomb's force law: The force *on* a point charge q at $\mathbf{r}$ *from* point charges Q_1 and Q_2 at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, is:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \frac{qQ_1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}_1}{R_1^2} + \frac{qQ_2}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}_2}{R_2^2}\,, \tag{1.4}$$

where $\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1$ and $\mathbf{R}_2 = \mathbf{r} - \mathbf{r}_2$. The superposition principle can be extended to any number of point charges. The principle will be used as both a theoretical and as a practical tool in this text — so spend some time to ensure you master the principle now.

No self interaction. Just like we have for the gravitational force from a single mass particle, a point charge does not interact with itself. However, just like for gravitational systems, the point charges making up an extended body may interact with other point charges in the same body, but not with themselves.

1.1.1 Example: Three ways to apply the superposition principle

Let us look at three ways to use the superposition principle to find the net force on a charge q in the origin. Fig. 1.3 shows a system of four charges Q_1, Q_2, Q_3 and Q_4 at positions $(a,0)$, $(-a,0)$, $(0,a)$, and $(0,-a)$ respectively.

Graphical vector addition. If all the charges are the same ($Q_i = Q$), we can use graphical vector addition to find the solution. As illustrated in Fig. 1.3b the sum of all the forces is zero.

Analytical vector addition. The forces in the x-direction *from* charge Q_1 *on* charge q depends on $\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1 = (-a,0)$, where we have used that $\mathbf{r} = \mathbf{0}$ and $\mathbf{r}_1 = (a,0)$:

$$\mathbf{F}_1 = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}_1}{R_1^3} = \frac{qQ}{4\pi\epsilon_0}\frac{(-a,0)}{\left((a^2+0^2)^{1/2}\right)^3} = -\frac{qQ}{4\pi\epsilon_0 a^2}\,\hat{\mathbf{x}}\,. \tag{1.5}$$

For charge Q_2 we have $\mathbf{R}_2 = \mathbf{0} - \mathbf{r}_2 = (a,0)$ and the force is:

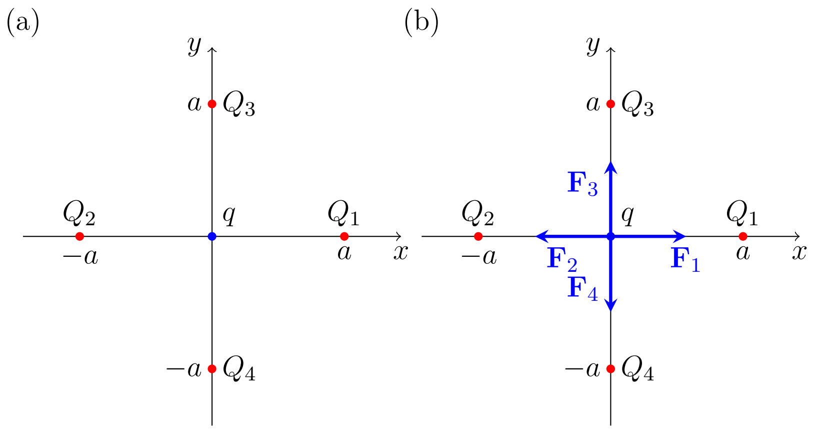


Fig. 1.3 (a) A system with four charges Q_1, Q_2, Q_3, Q_4 in symmetric positions around a charge q in the origin, (b) Forces acting on q are illustrated.

$$\mathbf{F}_2 = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}_2}{R_2^3} = \frac{qQ}{4\pi\epsilon_0}\frac{(a,0)}{(a^2+0^2)^{3/2}} = \frac{qQ}{4\pi\epsilon_0 a^2}\,\hat{\mathbf{x}}\ , \tag{1.6}$$

The net force in the x-direction is the sum, which we find by component-wise addition:

$$\mathbf{F}_x = \mathbf{F}_1 + \mathbf{F}_2 = -\frac{qQ}{4\pi\epsilon_0 a^2}\hat{\mathbf{x}} + \frac{qQ}{4\pi\epsilon_0 a^2}\hat{\mathbf{x}} = \frac{qQ}{4\pi\epsilon_0 a^2}\left(\hat{\mathbf{x}} - \hat{\mathbf{x}}\right) = \mathbf{0}\ . \tag{1.7}$$

Numerical vector addition. We can also calculate the sum numerically in Python, by first calculating the forces and adding them using vector addition. When we calculate Coulomb's law numerically, it is common to use the following form:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3}\ . \tag{1.8}$$

Let us assume that the charges Q_1 and Q_3 are 1μC, Q_2 and Q_4 are 2μC and that $a = 1$cm.

```
import numpy as np
import scipy.constants as sc
epsilon0 = sc.epsilon_0
q = 1e-6 # in units of C
a = 1e-2 # in units of meters
r = np.array([0,0])
Q1 = 1e-6 # in units of C
R1 = r - np.array([a,0])
F1 = (q*Q1)/(4*np.pi*epsilon0)*R1/np.linalg.norm(R1)**3
```

```
Q2 = 2e-6 # in units of C
R2 = r - np.array([-a,0])
F2 = (q*Q2)/(4*np.pi*epsilon0)*R2/np.linalg.norm(R2)**3
Q3 = 1e-6 # in units of C
R3 = r - np.array([0,a])
F3 = (q*Q3)/(4*np.pi*epsilon0)*R3/np.linalg.norm(R3)**3
Q4 = 2e-6 # in units of C
R4 = r - np.array([0,-a])
F4 = (q*Q4)/(4*np.pi*epsilon0)*R4/np.linalg.norm(R4)**3
F = F1+F2+F3+F4
print("F = ",F)
```

```
F =  [ 89.87551787 89.87551787 ]
```

Notice that we use `np.linalg.norm` to find the length of the $\mathbf{R}_i$ vectors. Also notice the use of `scipy.constants` for physical constants. The result is a force, which is measured in units of Newtons ($\mathrm{N} = \mathrm{kgms}^{-2}$).

1.2 Electric field

Coulomb's law allows us to find the force acting on a charge q in a point $\mathbf{r}$ from a set of charges Q_i in positions $\mathbf{r}_i$ illustrated in Fig. 1.4 by summing the contributions from the individual forces

$$\mathbf{F} = \sum_i \frac{qQ_i}{4\pi\epsilon_0} \frac{\mathbf{R}_i}{R_i^3} \; . \tag{1.9}$$

First, we notice that the charge q is a factor in all the forces, and can be placed outside the sum:

$$\sum_i \frac{qQ_i}{4\pi\epsilon_0} \frac{\mathbf{R}_i}{R_i^3} = q \underbrace{\left(\sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R}_i}{R_i^3} \right)}_{\mathbf{E}} = q\mathbf{E} \; . \tag{1.10}$$

We call the term in brackets $\mathbf{E}(\mathbf{r}; Q_i, \mathbf{r}_i)$. It does not depend on the charge q, only on the charges Q_i, their positions $\mathbf{r}_i$ and the position $\mathbf{r}$ of the charge q. If the charge distribution is given and does not change — that is the set of charges Q_i and their positions $\mathbf{r}_i$ are given — the sum $\mathbf{E}$ depends only the position $\mathbf{r}$. We call such as function a *vector field*, a vector function that depends on the position $\mathbf{r}$ in space. The vector field $\mathbf{E}$ tells us what the forces from all the charges Q_i would be on a charge q if we place it in a position $\mathbf{r}$. We can calculate and study this vector field independently of the charge q. And we can also use the vector field to find the force on a charge q in a position $\mathbf{r}$ simply as $\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$. We call the vector field $\mathbf{E}(\mathbf{r})$ the *electric field*. Electrostatics, which is the subject we are now starting to study, is largely about the properties of electric fields from charge distributions. So if you can calculate and understand the electric field for a given charge distribution, you have come a far distance towards mastering electrostatics.

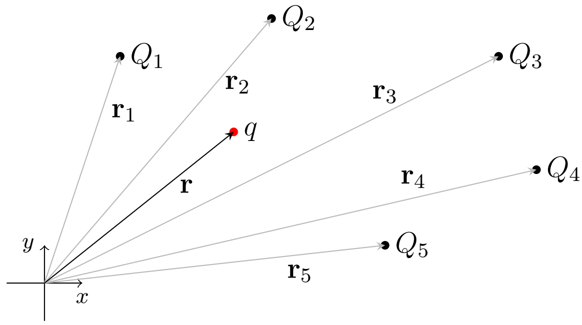


Fig. 1.4 Illustation of a charge distribution Q_i. The force on a charge q depends on the position $\mathbf{r}$ of the charge.

Electric field. This motivates the introduction of the *electric field*:

Electric field

The electric field $\mathbf{E}(\mathbf{r})$ is defined as:

$$\mathbf{E} = \frac{\mathbf{F}_q}{q} \quad (q \to 0) \ , \tag{1.11}$$

where $\mathbf{F}_q$ is the force on test charge q from other charges in space. We assume q is a small[a] test charge.

The force from the electric field $\mathbf{E}$ on a charge q is

$$\mathbf{F} = q\mathbf{E} \ . \tag{1.12}$$

[a] Why do we say that the test charge is small? This is to make clear that the test charge q does not affect the distribution of charges, Q_i, that set up the electric field. The test charge is so small, that we do not have to think about how q would affect any other charge in the system.

The electric field is a vector field. The electric field varies with the position $\mathbf{r}$ in space. It is a vector field or a vector function: For each position $\mathbf{r}$, the fucntion returns a vector value, $\mathbf{E}(\mathbf{r})$. We can calculate the vector function for a given set of charges Q_i and position $\mathbf{r}_i$. The function may become complicated, but it is in principle straight forward to calculate it and to visualize it in space.

The electric field is analoguous to the gravitational field. You already have intuition about a similar concept: the gravitational field. You may recall that the gravitational force on an object of mass m is $m\mathbf{g}$. When the object is close to the Earth's surface, we know that $\mathbf{g}$ is approximately constant. The same relation is also valid for an astroid in the Solar system, but then $\mathbf{g}(\mathbf{r})$ depends on the position of the mass (astroid) relative to all the other masses in the Solar system. Just like the gravitational field is set up by the distribution of masses in space, the electric field is set up the distribution of charges in space.

1.2.1 Example: The electric field from a point charge

What is the electric field from a single point charge Q_1? The force from Q_1 on q can be found from Coulomb's law. We assume the charge Q_1 is in $\mathbf{r}_1$. According to Coulomb's law, the force on a charge q in $\mathbf{r}$ is:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} \ , \tag{1.13}$$

where $\mathbf{R}$ is the vector *from* the charge Q_1 at $\mathbf{r}_1$ *to* the charge q at $\mathbf{r}$: $\mathbf{R} = \mathbf{r} - \mathbf{r}_1$:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} \ , \tag{1.14}$$

and the electric field is:

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} \ . \tag{1.15}$$

Electric field in Cartesian coordinates. Let us make this more specific. Assume that the charge Q is a $(0,0,a)$. What is the electric field in the position $\mathbf{r} = (x,y,z)$? We insert these values in $\mathbf{R} = \mathbf{r} - \mathbf{r}_1$, getting $\mathbf{R} = (x,y,z) - (0,0,a) = (x,y,z-a)$ and $R = (x^2+y^2+(z-a)^2)^{1/2}$. We again insert this in (1.15), and find

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{(x,y,z-a)}{\left(x^2+y^2+(z-a)^2\right)^{3/2}} \ . \tag{1.16}$$

Notice that the electric field indeed depends on the position (x, y, z) in space. The formula is rather complex, even though this problem is about the simplest problem we could think of. We therefore need to develop effective methods to calculate electric fields — both analytically and numerically.

Visualization of the electric field. It is customary to visualize the electric field by drawing small arrows to illustrate the vector field. The arrows indicate the magnitude and direction of the field in a given point in space, which usually corresponds to where the arrow is starting. We draw a sufficient number of arrows to provide a good illustration of the field. (What you believe to be a sufficient number of arrow is up to your judgement). We have illustrated the field from a single change in $(0, 0, a)$ in Fig. 1.5.

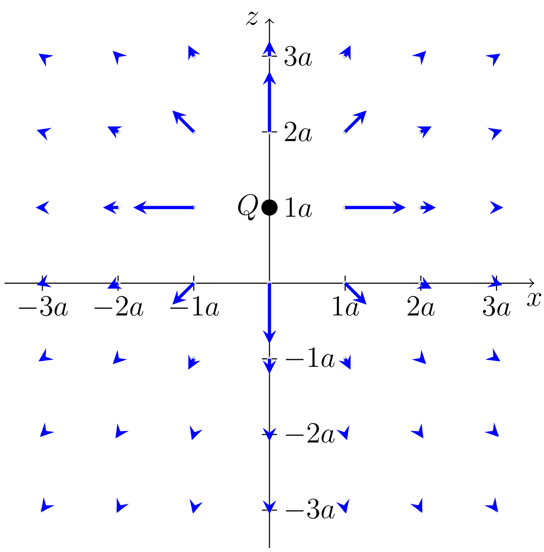


Fig. 1.5 Drawing of the electric field in the xz-plane from a point charge at $(0, 0, a)$.

1.2.2 Superposition principle for the electric field

The superposition principle can be used to find the net force on a charge q at $\mathbf{r}$ from several charges Q_i at $\mathbf{r}_i$:

$$\mathbf{F} = \sum_i \mathbf{F}_i = \sum_i \frac{qQ_i}{4\pi\epsilon_0} \frac{\mathbf{R_i}}{R_i^3} = q \sum_i \underbrace{\frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R_i}}{R_i^3}}_{\mathbf{E}_i} = q \sum_i \mathbf{E}_i = q\mathbf{E} \; . \tag{1.17}$$

The superposition is therefore also valid for electric fields:

$$\mathbf{E} = \sum_i \mathbf{E}_i = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R_i}}{R_i^3} \; . \tag{1.18}$$

Electric field from a set of point charges

For a set of charges Q_i in positions $\mathbf{r}_i$, the net electric field in a postion $\mathbf{r}$ is the sum of the electric fields from each of the charges:

$$\mathbf{E}(\mathbf{r}) = \sum_i \mathbf{E}_i(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}_i}{R_i^2} = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R}_i}{R_i^3} \; . \tag{1.19}$$

where $\mathbf{E}_i(\mathbf{r})$ is the field from charge Q_i, $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$, and $R_i = |\mathbf{R}_i|$.

Inserting $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$ gives the explicit expression:

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \; . \tag{1.20}$$

Note on the vector R

The vector $\mathbf{R}$ plays a special role in the expression for the electric field (and for Coulomb's law). We often write only $\mathbf{R}$ for simplicity — and then leave it to be implicily understood that the vector depends on both the position of the charge Q_i and on the position in space, $\mathbf{r}$. For example, in the expression for the field from a set of charges Q_i

$$\mathbf{E} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{R_i^2} \hat{\mathbf{R}}_i \; , \tag{1.21}$$

each of the vectors $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$. A common mistake is to insert $\mathbf{r}_i$ instead of $\mathbf{R}$.

Notice that the $\mathbf{R}_i$ vector points from the charge Q_i to the position $\mathbf{r}$. It is common to call the position $\mathbf{r}$ the *observation point*. The vector $\mathbf{R}_i$ therefore points from the charge (Q_i) to the observation point.

1.2.3 Example: Electric field from two charges

Let us start with a specific example to learn about the use of the $\mathbf{R}$-vector. We look at the electric field from two charges Q_1 and Q_2 at $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ respectively. To find the total field, we use the superposition principle, adding the fields from charge Q_1 and Q_2. We want to find the electric field at some position $\mathbf{r}$ (the observation point, P) as illustrated in Fig. 1.6.

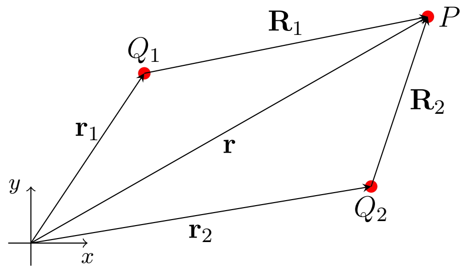


Fig. 1.6 Illustration of two charges Q_1 and Q_2. The electric field at a position $\mathbf{r}$ (the observation point P) is the sum of the contributions from each of the two charges.

First we find an expression for the electric field $\mathbf{E}_1(\mathbf{r})$ from the charge Q_1:

$$\mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1^2}\hat{\mathbf{R}}_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1^2}\frac{\mathbf{R}_1}{R_1} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1^3}\mathbf{R}_1 \; . \tag{1.22}$$

We can get an explicit result by inserting $\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1$:

$$\mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1^3}\mathbf{R}_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1\,(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} \; . \tag{1.23}$$

We can even find the solution in Cartesian coordinates by inserting $\mathbf{r} = (x, y, z)$:

$$\mathbf{E}_1(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{Q_1\,(x-x_1, y-y_1, z-z_1)}{\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right)^{3/2}} \; . \tag{1.24}$$

Now, this is a rather complicated expression. We have developed it here in its full to illustrate that the $\mathbf{R}$-vector tends to simplify the mathematics, but when we do the full calculation we have to include all the details.

We can use this to find the electric field from two charges, Q_1 and Q_2. The general expression becomes complicated, but follows directly from the superposition principle:

$$\begin{aligned}\mathbf{E}(x,y,z) &= \mathbf{E}_1(x,y,z)+\mathbf{E}_2(x,y,z)\\ &= \frac{1}{4\pi\epsilon_0}\frac{Q_1\,(x-x_1,y-y_1,z-z_1)}{\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right)^{3/2}}+\\ &\quad \frac{1}{4\pi\epsilon_0}\frac{Q_2\,(x-x_2,y-y_2,z-z_2)}{\left((x-x_2)^2+(y-y_2)^2+(z-z_2)^2\right)^{3/2}}\,. \end{aligned} \tag{1.25}$$

1.2.4 Example: Electric field from a dipole

We can use the same approach to find the electric field from an *electric dipole*. An electric dipole consists of two charges of the same magnitude, but of opposite sign placed close together. Seen from far away the net charge is zero for this two-particle system, but the electric field is still not zero[2]. An example of a dipole consists of a charge $Q_1 = Q$ at $\mathbf{r}_1 = (d/2,0,0)$ and $Q_2 = -Q$ at $\mathbf{r}_2 = (-d/2,0,0)$.

The electric field at $\mathbf{r} = (x,y,z)$ from these two charges is then (from the example above):

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1+\mathbf{E}_2\\ &= \frac{1}{4\pi\epsilon_0}\frac{Q_1\,(x-x_1,y-y_1,z-z_1)}{\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right)^{3/2}}+\\ &\quad \frac{1}{4\pi\epsilon_0}\frac{Q_2\,(x-x_2,y-y_2,z-z_2)}{\left((x-x_2)^2+(y-y_2)^2+(z-z_2)^2\right)^{3/2}}\\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{(x,y-d/2,z)}{\left(x^2+(y-d/2)^2+z^2\right)^{3/2}}-\right.\\ &\quad \left.\frac{(x,y+d/2,z)}{\left(x^2+(y+d/2)^2+z^2\right)^{3/2}}\right)\,. \end{aligned} \tag{1.26}$$

Again, this is a complicated, but not difficult result. Later we will find an approximation for this result when $r = |\mathbf{r}| \gg d$. Here, we will use this as a basis for visualizing the electric field from a dipole.

[2] The dipole is the simplest perturbation to a trivial system of a single charge. The dipole is similar to the harmonic oscillator and represents the simplest non-trivial model of an attractive interaction. Indeed, this correspondence is even deeper as we can often write the field from a complex charge distribution in terms of multipole expansion, starting with a dipole term followed by more complicated poles of higher order.

1.2.5 Visualizing electric fields

How can we visualize the field from a dipole or more generally from any set of charges Q_i at positions $\mathbf{r}_i$? In order to visualize the field, we need to calculate the electric field $\mathbf{E}(\mathbf{r})$ at various positions $\mathbf{r}$ in space, and then illustrate the field by, for example, drawing a vector from each of these positions as illustrated in Fig. 1.7a.

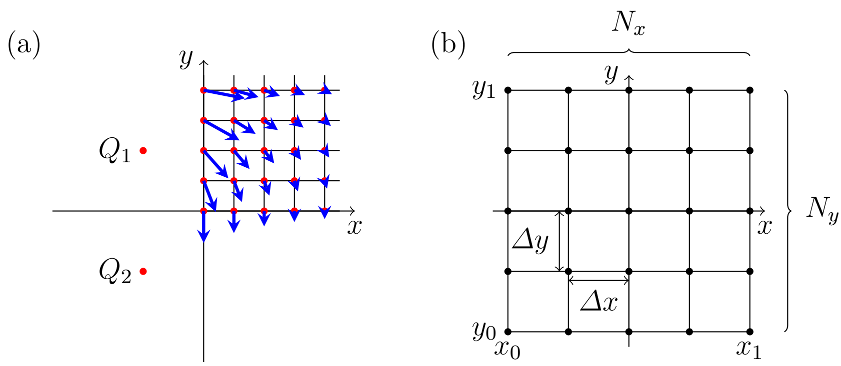


Fig. 1.7 We visualize the field by obtaining the field on a grid of positions and then draw small arrows to illustrate the field.

Calculating the field on a grid. To do this we need to evaluate the field at various positions (x, y, z). Here, we will illustrate how to do this in two dimensions (x, y). We will do this in four steps: (1) We will write a function to calculate the contribution to the electric field from a single point charge. (2) We will generate a grid of regularly spaced (x, y)-points on which to calculate the electric field. (3) We will calculate the field at these position using superposition. (4) We visualize the vectors values as arrows or as stream lines.

Function for single point charge. We want to write a function `efield(r,q0,r0)` that calculates the electric field at a position $\mathbf{r}$ from a charge q_0 at position $\mathbf{r}_0$. For simplicity, we calculate the field $\mathbf{E}'$ without the prefactor: $\mathbf{E}' = 4\pi\epsilon_0\mathbf{E}$:

$$\mathbf{E}' = \mathbf{E}\,4\pi\epsilon_0 = Q\frac{\mathbf{R}}{R^3}\ , \tag{1.27}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. The code to do this directly follows the mathematical expression for the field: We first find $\mathbf{R}$, which we call `R`, then the length $R = |\mathbf{R}|$, which we call `Rnorm`, and then we calculate the electric field $\mathbf{E}' = Q\mathbf{R}/R^3$:

```
import numpy as np
import matplotlib.pyplot as plt
def efield(r,q0,r0):
    # Find E*4*pi*epsilon0 at r from a charge q at position r0
    R = r-r0
    Rnorm = np.linalg.norm(R)
    return q0*R/Rnorm**3
```

We test the code by checking the value for $q_0 = 1$, $r = (2,0)$ and $r_0 = (0,0)$, which should give $\mathbf{E}' = q_0(2,0)/2^3 = (1/4,0)$:

```
q0 = 1.0
r0 = np.array([0,0])
r = np.array([2,0])
print(efield(r,q0,r0))
```

```
[0.25 0.  ]
```

This is indeed correct. It is always smart to check the validity of a function with a few simple tests — and it is good programming practice to write these test into your program as unit tests.

Generating a square grid of points. Now, we need to calculate the values of the electric field for **r**-values on a square grid of points as illustrated in Fig. 1.7b. In this case, the grid spans x-values from x_0 to x_1 and y-values from y_0 to y_1. There are N_x boxes along the x-axis, which means that the distance between two points is $\Delta x = (x_1 - x_0)/N_x$. Similarly, there are N_y points along the y-axis with $\Delta y = (y_1 - y_0)/N_y$. We want to generate a grid with these spacings, that is, we want to generate a set of points $\mathbf{r}_{i,j} = (x_{i,j}, y_{i,j})$ so that $x_{i,j} = x_0 + i\Delta x$ and $y_{i,j} = y_0 + j\Delta y$, where $i = 0,1,\ldots,N_x$ and $j = 0,1,\ldots,N_y$. Notice that there are N_x boxes but $N_x + 1$ points along the x-axis and similarly along the y-axis.

There is a function in Python that automatically makes such as grid. We first define the positions of the points along the x- and y-axes using `linspace` and then create the grid using `meshgrid`. Let us start by making a grid with $N_x = N_y = 5$ points and where $x_0 = -L_x$, $x_1 = L_x$, $y_0 = -L_y$, and $y_1 = L_y$:

```
Lx = 2
Ly = 4
x = np.linspace(-Lx,Lx,5)
y = np.linspace(-Ly,Ly,5)
rx,ry = np.meshgrid(x,y)
```

Let us examine the resulting points generated. What are the values of `rx` and `ry`?

```
print(rx)
```

```
[[-2. -1.  0.  1.  2.]
 [-2. -1.  0.  1.  2.]
 [-2. -1.  0.  1.  2.]
 [-2. -1.  0.  1.  2.]
 [-2. -1.  0.  1.  2.]]
```

```
print(ry)
```

```
[[-4. -4. -4. -4. -4.]
 [-2. -2. -2. -2. -2.]
 [ 0.  0.  0.  0.  0.]
 [ 2.  2.  2.  2.  2.]
 [ 4.  4.  4.  4.  4.]]
```

Notice the numbering used by Python may be confusing: The element `rx[1,0]` is -2 and not -1, because the first index in the array is the row and the second is the column. Therefore the element `rx[j,i]` corresponds to $x_{i,j}$. We need to remember this when we generate, visualize and make calculations in Python.

Visualization of $\mathbf{r}$. First, let us visualize the vector field we have generated, that is, visualize the identiy function $\mathbf{f}(\mathbf{r}) = \mathbf{r}$. We use the function `quiver` to draw vectors at each of the grid points. The first two arguments to `quiver` provide the grid coordinates, and the second two arguments provide the components of the vector field to be visualized. Notice that both the grid points and the vector field must be matrices of the same dimensions. Fig. 1.8 shows the result of the following commands. Notice the use of `set_aspect` on the axis to generate a plot with the correct aspect ratio.

```
import numpy as np
import matplotlib.pyplot as plt
fig = plt.figure(figsize=(12,6))
ax1 = plt.subplot(1,2,1)
plt.quiver(rx,ry,rx,ry)
ax2 = plt.subplot(1,2,2)
plt.quiver(rx,ry,rx,ry)
ax2.set_aspect('equal', 'box')
```

Creating the field with a loop. Now, we use this method to visualize the electric field from a single point charge at $\mathbf{r}_0 = 0$. First, we generate the lattice in the form of the `rx` and `ry` matrices. For each point $\mathbf{r}_{i,j} = (x_{i,j}, y_{i,j})$ we then calculate the components of the electric field, $\mathbf{E}(\mathbf{r}_{i,j})$, and store the results in a matrix. This is done in the following program,

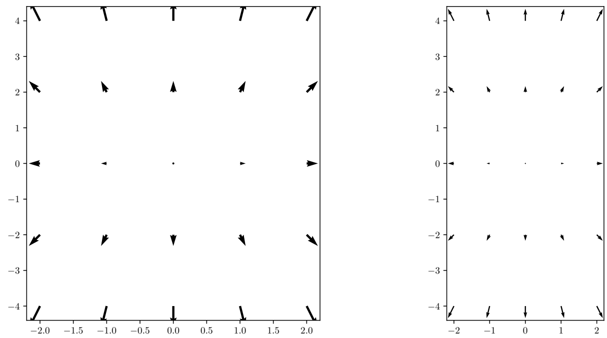


Fig. 1.8 Visualization of the field $\mathbf{r}$. (a) Direct visualization. (b) With equal axes.

which uses the function `efield` defined above. Notice that `rx`, `ry`, `Ex` and `Ey` are two-dimensional arrays. To loop through all their elements using only a single index, we can flatten the array to a one-dimensional array using the built-in method `flat`. We visualize the field on a 20×20 grid from -10 to 10 in both x- and y-directions. The resulting plot is shown in Fig. 1.9.

```
import numpy as np
import matplotlib.pyplot as plt
r0 = np.array([0,0]) # Position of charge
q0 = 1.0 # Charge of charge
Lx = 10
Ly = 10
N = 20
x = np.linspace(-Lx,Lx,N)
y = np.linspace(-Ly,Ly,N)
rx,ry = np.meshgrid(x,y)
# Set up empty matrix for electric field
Ex = np.zeros((N,N))
Ey = np.zeros((N,N))
# Calculate the field
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i]])
    Ex.flat[i],Ey.flat[i] = efield(r,q0,r0)
# Visualization
plt.quiver(rx,ry,Ex,Ey)
plt.axis('equal')
```

Vector visualization using colors. For electric fields from point charges, the magnitude of the field diverges close to the charges. This means that the arrows used to visualize the field become very large close to the charges and small far away from the charges. It is therefore sometime useful to separate the visualization of the direction and the magnitude

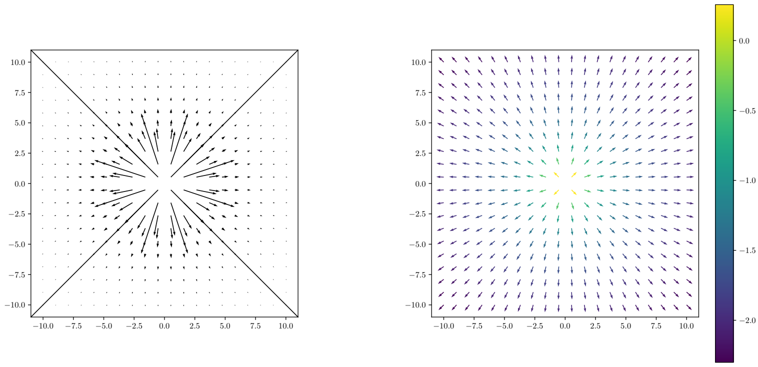


Fig. 1.9 Visualization of the field from a single charge at the origin using arrows, where the length of the arrow indicate the magnitude of the field (left), and where the color of the arrow indicate the magnitude of the field (right).

of the field. This can be done by drawing unit vectors to indicate the direction of the field and then illustrate the magnitude with a color-scheme. This is done by the following short program. The resulting visualization is shown in Fig. 1.9.

```
# Calculate the magnitude of the field
Emag = np.sqrt(Ex**2 + Ey**2)
# Calculate unit vectors in the fields direction
nEx = Ex / Emag
nEy = Ey / Emag
# Visualize
plt.quiver(rx,ry,nEx,nEy,np.log10(Emag))
plt.colorbar()
```

Here, we have also used the logarithm of the magnitude in order to better visualize the range of values because the range of magnitudes for the field varies by orders of magnitude from far away to close to the charges. Experiment with these methods to find a visualization scheme you like.

1.2.6 Example: Visualizing the field from point charges

We now have the tools needed to visualize the electric field set up by various charge distributions. For a set of charges Q_i at positions $\mathbf{r}_i$ we use the superposition principle and the methods introduced above to add together the contributions from each charge.

Electric field from a dipole. We start by visualizing the field from a dipole with a charge $q_1 = Q$ at $\mathbf{r}_1 = (0, a)$ and a charge $q_2 = -Q$ at $\mathbf{r}_2 = (0, -a)$, where a is a length. The program follows exacly the same structure as above, but we have modified it slightly to instead draw the arrows with logarithmic length and not the actual length. The resulting visualization is shown in Fig. 1.10. (You also need to include the function `efield` from above).

```
import numpy as np
import matplotlib.pyplot as plt
a = 1.0
q1 = 1.0
r1 = np.array([0,a])
q2 = -1.0
r2 = np.array([0,-a])
Lx = 5
Ly = 5
N = 21
x = np.linspace(-Lx,Lx,N)
y = np.linspace(-Ly,Ly,N)
rx,ry = np.meshgrid(x,y)
# Set up electric field
Ex = np.zeros((N,N),float)
Ey = np.zeros((N,N),float)
# Calculate the field
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i]])
    Ex.flat[i],Ey.flat[i] = efield(r,q1,r1) + efield(r,q2,r2)
# Calculate field magnitude and unit vectors
Emag = np.sqrt(Ex**2 + Ey**2)
minlogEmag = np.nanmin(np.log10(Emag.flat))
scaleE = np.log10(Emag) - minlogEmag
uEx = Ex / Emag
uEy = Ey / Emag
# Visualize using both arrows and colors
ax1 = plt.subplot(1,2,1)
#scaleE = 1.0
plt.quiver(rx,ry,uEx*scaleE,uEy*scaleE)
ax1.set_aspect('equal', 'box')
ax2 = plt.subplot(1,2,2)
plt.quiver(rx,ry,uEx,uEy,np.log10(Emag))
ax2.set_aspect('equal', 'box')
```

Notice the use of `nanmin` to find the minimum of an array while ignoring any `nan`-values. The `nan`-values may appear if the magnitude of the electric field is zero and the logarithm therefore is undefined. The `quiver`-function does not draw any arrow in a point where the vector is a `nan`, so these points are simply ignored in the visualization.

Electric field from many charges. We can update the computational method to include a list of charges, Q_i, and a corresponding list of positions, $\mathbf{r}_i$. First, we generate the lists `Q` and `R`, here for a system of three charges:

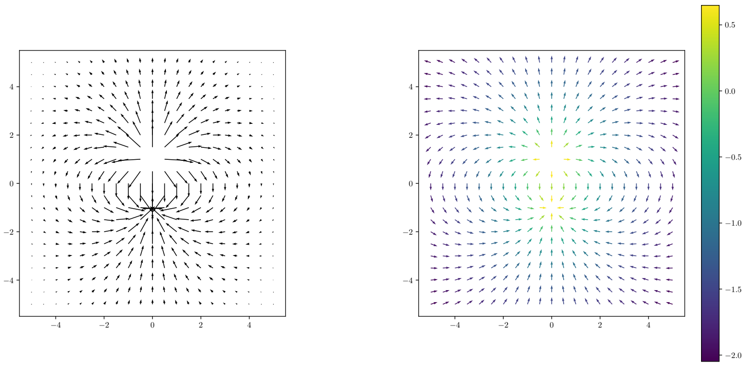


Fig. 1.10 Visualization of the field from a dipole.

```
R = []
Q = []
r0 = np.array([1,0])
q0 = 1
R.append(r0)
Q.append(q0)
r0 = np.array([0,0])
q0 = -2
R.append(r0)
Q.append(q0)
r0 = np.array([-1,0])
q0 = 1
R.append(r0)
Q.append(q0)
```

We write a function `efieldlist` to calculate the field from a list of charges instead of a single charge:

```
def efieldlist(r,Q,R):
    # Find E*4*pi*epsilon0 at r from a charge q at position r0
    E = np.zeros(np.shape(r))
    for i in range(len(R)):
        Ri = r - R[i]
        qi = Q[i]
        Rinorm = np.linalg.norm(Ri)
        E = E + qi*Ri/Rinorm**3
    return E
```

And finally, we write a function to find the field:

```
def findfield(R,Q,x0,x1,y0,y1,Nx,Ny):
    x = np.linspace(x0,x1,Nx)
    y = np.linspace(y0,y1,Ny)
    rx,ry = np.meshgrid(x,y)
    # Set up electric field
    Ex = np.zeros((Nx,Ny),float)
```

```
    Ey = np.zeros((Nx,Ny),float)
    # Calculate the field
    for i in range(len(rx.flat)):
        r = np.array([rx.flat[i],ry.flat[i]])
        Ex.flat[i],Ey.flat[i] = efieldlist(r,Q,R)
    return x,y,rx,ry,Ex,Ey
```

And a function to visualize a field

```
def visfield(x,y,rx,ry,Ex,Ey):
    Emag = np.sqrt(Ex**2 + Ey**2)
    uEx = Ex / Emag
    uEy = Ey / Emag
    ax = plt.subplot(1,1,1)
    plt.quiver(rx,ry,uEx,uEy,np.log10(Emag))
    ax.set_aspect('equal','box')
    return
```

We use these functions to calculate and visualize the electric field from the system of three charges defined in the lists `Q` and `R`:

```
L = 3 # Grid goes from -L to L in x and y directions
N = 21 # Number of grid points in x and y directions
x,y,rx,ry,Ex,Ey = findfield(R,Q,-L,L,-L,L,N,N)
visfield(x,y,rx,ry,Ex,Ey)
```

Lines of charge. Or you can visualize the field from two lines of charge: One line at $y = 1.5a$ from $x = -L/2$ to $L/2$ with a total charge Q, and one line at $y = -1.5a$ from $x = -L/2$ to $L/2$ with a total charge $-Q$. We approximate a line charge by placing out n point charges with charge $Q_i = Q/n$ uniformly spaced along the line. The position of charge i is then $x_i = -L/2 + i/(n-1)L$ for $i = 0, 1, \ldots, n-1$.

```
# Generating two lines of charges
L = 6
Q0 = []
R0 = []
n = 50
for i in range(n):
    xi = -L/2 + i/(n-1)*L
    yi = 1.5
    R0.append(np.array([xi,yi]))
    Q0.append(1.0)
for i in range(n):
    xi = -L/2 + i/(n-1)*L
    yi = -1.5
    R0.append(np.array([xi,yi]))
    Q0.append(-1.0)
x,y,rx,ry,Ex,Ey = findfield(R0,Q0,-5,5,-5,5,31,31)
visfield(x,y,rx,ry,Ex,Ey)
```

The resulting visualization is shown in Fig. 1.11. This approach motivates the next section, where we go from a set of charges to a continuous distribution of charges in space.

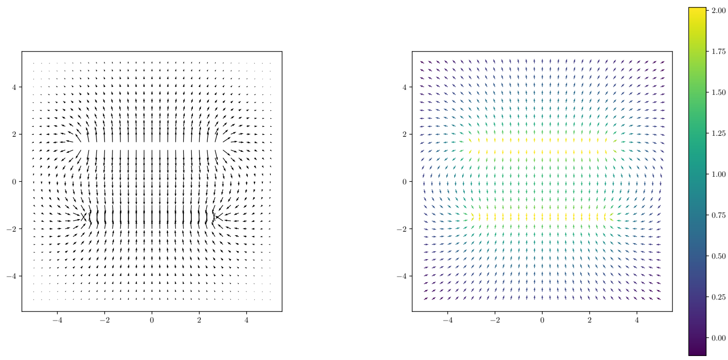


Fig. 1.11 Visualization of the field from two lines with opposite charges.

Dynamic visualization. This method also opens for dynamic visualization in Python using the `interact` function. For example, we can visualize a dipole, where the distance between the two charges can be changed by a slider:

```
# Generating an interactive dipole
from ipywidgets import interact, interactive, fixed, interact_manual
import ipywidgets as widgets
def f(d):
    Q0 = []
    R0 = []
    R0.append(np.array([0,d/2]))
    Q0.append(1.0)
    R0.append(np.array([0,-d/2]))
    Q0.append(-1.0)
    x,y,rx,ry,Ex,Ey = findfield(R0,Q0,-10,10,-10,10,21,21)
    visfield(x,y,rx,ry,Ex,Ey)
interact(f, d=widgets.IntSlider(min=1, max=18, step=1, value=2));
```

Visualization in 3d. The methods developed here are easily extended to model system in three dimensions. Indeed, the function `efieldlist` accepts both two-dimensional and three-dimensional inputs. We demonstrate this by visualizing the electric field from two planes of dimensions $L \times L$ located a $z = a$ and $z = -a$, with charges Q and $-Q$ respectively. We assume that the charge is uniformly distributed across the planes. Here, we also demonstrate the use of real physical length-scales, where lengths are measured in meters and the electric field is measured in V/m. We must then rewrite `efieldlist` to return the field in units of V/m:

```
import scipy.constants as sc
def efieldlist_units(r,Q,R):
    E = np.zeros(np.shape(r))
    for i in range(len(R)):
        Ri = r - R[i]
        qi = Q[i]
        Rinorm = np.linalg.norm(Ri)
        E = E + qi*Ri/(Rinorm**3*4*np.pi*sc.epsilon_0)
    return E
```

We generate the two planes using the following program:

```
# Make a plane in real coordinates
L = 1e-3 # in m (meter)
Q = 1e-6 # in C (Coulomb)
n = 50 # number of elements
a = 0.5e-3 # in m (meter)
Q0 = []
R0 = []
n = 50
for ix in range(n):
    for iy in range(n):
        xi = -L/2 + ix/(n-1)*L
        yi = -L/2 + iy/(n-1)*L
        zi = a
        R0.append(np.array([xi,yi,zi]))
        Q0.append(Q/(n*n))
        zi = -a
        R0.append(np.array([xi,yi,zi]))
        Q0.append(-Q/(n*n))
```

Now, we could calculate the field in three dimensions, that is, on a cubic lattice. Instead, we will only study the field in the xz-plane, that is, for $y = 0$. We call this function `findfield3d`. The function is very similar to our previous `findfield`-function. Notice the small differences: The function `efieldlist` returns a vector in three dimensions, but we do not need to use the `Ey` component. Also, the input to `efieldlist` is a three dimensional vector, but where $y = 0$. The program takes as input the scale L used for the visualization and the number of points N where the field is calculated along each dimension. We find the field at N points uniformly spaced from $-L$ to L in the x and z directions:

```
def findfield3d(R,Q,x0,x1,y0,y1,Nx,Ny):
    x = np.linspace(x0,x1,Nx)
    z = np.linspace(y0,y1,Ny)
    rx,rz = np.meshgrid(x,z)
    # Initialize matrix for electric field
    Ex = np.zeros((Nx,Ny),float)
```

```
    Ez = np.zeros((Nx,Ny),float)
    # Calculate the field
    for i in range(len(rx.flat)):
            r = np.array([rx.flat[i],0,rz.flat[i]])
            Ex.flat[i],Ey,Ez.flat[i] = efieldlist_units(r,Q,R)
    return x,z,rx,rz,Ex,Ez
```

We call this function and visualize the field in the xz-plane with

```
N = 20 # Number of points in each direction
dl = 0.001 # Visualized on the interval -L<x<+L, -L<z<+L
x,z,rx,rz,Ex,Ez = findfield3d(R0,Q0,-dl,dl,-dl,dl,N,N)
plt.figure(figsize=(6,6))
plt.quiver(rx/L,rz/L,Ex,Ez)
plt.plot([-1/2,1/2],[a/L,a/L],'-r')
plt.plot([-1/2,1/2],[-a/L,-a/L],'-b')
plt.xlabel('$x/L$')
plt.ylabel('$z/L$')
```

We can also visualize the field along a line at $z = 0$ and $x = 0$. Here we find the electric field in V/m by reintroducing the factor $1/(4\pi\epsilon_0)$ that was not included in the `efieldlist` function. Notice that the physical constant ϵ_0 is part of the `scipy.constants` library.

```
import scipy.constants as sc
plt.figure(figsize=(10,3))
plt.subplot(1,2,1)
plt.plot(x/L,Ez[int(N/2),:]/(4*np.pi*sc.epsilon_0))
plt.xlabel('$x/L$')
plt.ylabel('$E_x (V/m)$')
plt.subplot(1,2,2)
plt.plot(Ez[:,int(N/2)]/(4*np.pi*sc.epsilon_0),z/L)
plt.ylabel('$z/L$')
plt.xlabel('$E_z (V/m)$')
```

The resulting plots are shown in Fig. 1.12 and Fig. 1.13.

1.2.7 Visualization using field lines

A practical and insightful way to visualize the electric field, and other fields, is through *field lines*. A field line is a continuous curve constructed in such a way that the tangent in any point along the line points in the direction of the electric field in that point. The local density of field lines is proportional to the local magnitude of the electric field: When the field lines are closely spaced, the field is strong, and when they are sparsely spaced, the field is weaker. Field lines radiate from positive charges and terminate on negative charges. However, we may have to consider charges at infinity to make a system neutral. Field lines cannot intersect.

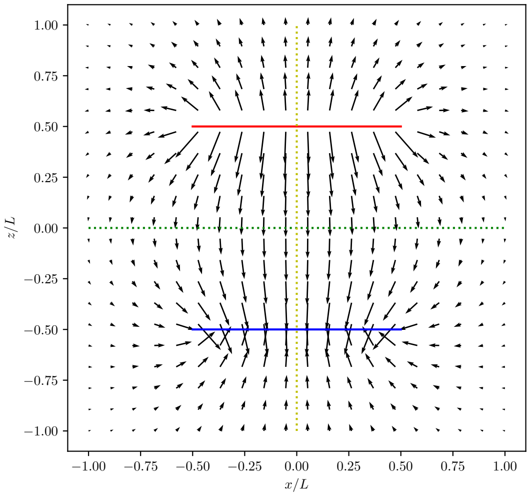


Fig. 1.12 Visualization of the field from two planes with opposite charges.

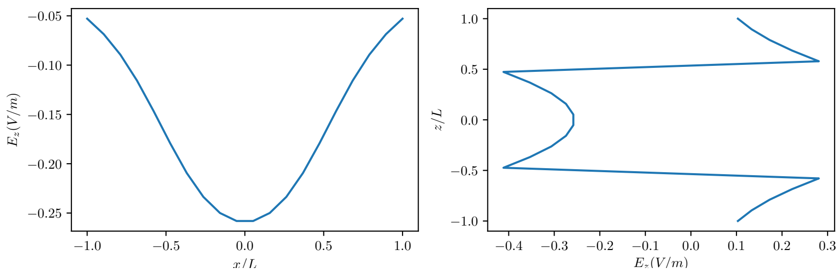


Fig. 1.13 Visualization of the field from two planes with opposite charges. (a) The field $E_z(x, 0, 0)$ along the x-axis corresponding to a cross-section with $z = 0$ (green line in Fig. 1.12). (b) The field $E_z(0, 0, z)$ along the z-axis corresponding to a cross-section with $x = 0$ (yellow line in Fig. 1.12).

You will often find visualizations of field lines in textbooks or illustrations of electromagnetic phenomena. But how can we construct and visualize field lines?

Vector visualization using streamlines. It may be tempting to use built-in functions in Python that visualize streamlines, such as `streamplot`. The following program illustrates the use of `streamplot` to visualize the field from a dipole using the functions `efieldlist` and `findfield` developed above:

```
import numpy as np
import matplotlib.pyplot as plt
# Set up the dipole field
Q0 = []
R0 = []
```

```
R0.append(np.array([-1,0]))
Q0.append(1.0)
R0.append(np.array([1,0]))
Q0.append(-1.0)
# Calculate the field
x,y,rx,ry,Ex,Ey = findfield(R0,Q0,-5,5,-5,5,51,51)
# Visualize the field
plt.subplot(1,2,1)
plt.quiver(rx,ry,Ex,Ey)
plt.subplot(1,2,2)
plt.streamplot(rx,ry,Ex,Ey)
```

The resulting plots are shown in Fig. 1.14. From this plot it is clear that not all the lines start at the positive charge and end at the negative charge. This plot includes curved lines that have tangents that are directed along the electric field, just as field lines, but the stream lines do not all start at positive charges and terminate at negative charges. The plot does therefore not show the intensity of field lines, which gives information about the strength of the field. The `streamplot` function does therefore not visualize the field lines. It does, however, still provide an interesting and useful visualization of the field.

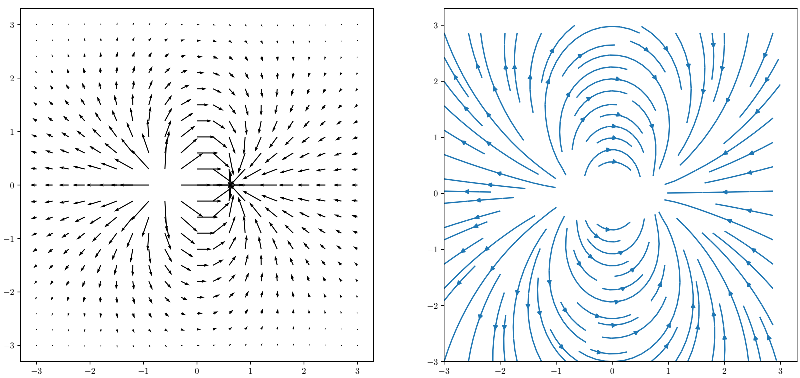


Fig. 1.14 Visualization of the field from a dipole using arrows (of logaritmic length) and stream lines using `streamplot`.

Understanding field lines. What are field lines? We can think of them as the trajectories of positive charges with no inertia moving in the field. (This is like the motion of a particle in a highly viscous fluid or the motion of a submicron particle in water). Newton's second law for this particle would be $q\mathbf{E} - D\mathbf{v} = m\mathbf{a} = 0$, where the D-term is due to viscosity and we assume that the mass is very small (no inertia).

Therefore, $\mathbf{v} = (q/D)\mathbf{E}$, that is, the velocity points in the direction of the electric field!

This provides us with a nice intuitive notion about the field lines. We also realize that field lines cannot cross each other. Why? Because the direction of a field line out of a point depends on the direction of $\mathbf{E}$. If two field lines were to cross, there would have to be two field lines out of a point, which would mean that the field would have to point in two different directions in the same point.

To visualize the field lines, we start with a given density of field lines (number) of field lines starting from each charge q. For a small sphere with radius a around this charge, the density of field lines would be $N/(4\pi a^2)$, where N is the number of field lines. If we are close to the charge, we assume that the electric field is dominated by the field from the charge, which is $q/(4\pi\epsilon_0 a^2)$. We therefore see that if we choose N so that it is proportional to the charge q, then the density of field lines is proportional to the electric field.

We can therefore visualize the field by starting a number of field lines proportional to the charge at positive charges. (Similarly, we will terminate field lines at negative charges.) Close to the charges, the density of field lines will correspond to the magnitude of the electric field, and this is also true everywhere in space, as we will see later.

Notice that this argument is only correct in three dimensions. If we make this construction in two dimensions instead, the density of lines passing through a circle of radius r would be $N/(2\pi r)$, which does not go like $1/r^2$.

Finding field lines using `streamplot`. The function `streamplot` *can* be used to find field lines, but then we need to specify the starting ponits of the field lines. We can start field lines from a small circle of radius r around the charges, where the number of points on the circle is proportional to the charge. (Note the comment on two and three dimensional systems above). We need to create a list of these starting points and submit them to `streamplot`. First, we generate lists of the points in the array `start_xy` (here for the dipole created above):

```
numlines = 32
start_x = []
start_y = []
Ri = np.array([-1,0])
rad = 0.25
for ni in range(numlines):
    ang = 2*np.pi*ni/numlines
    start_x.append(Ri[0]+rad*np.cos(ang))
```

```
    start_y.append(Ri[1]+rad*np.sin(ang))
Ri = np.array([1,0])
rad = 0.25
for ni in range(numlines):
    ang = 2*np.pi*ni/numlines
    start_x.append(Ri[0]+rad*np.cos(ang))
    start_y.append(Ri[1]+rad*np.sin(ang))
start_xy = np.column_stack([start_x, start_y])
```

We then call `streamplot` with this list as input:

```
plt.streamplot(rx,ry,Ex,Ey,start_points=start_xy,density=100)
plt.plot(start_x, start_y, '.k')
```

The resulting plot is shown in Fig. 1.15. Notice that we need to specify a high number for `density` when calling `streamplot`. You should explore this parameter, but notice that if you choose a too small number, you will not get a correct visualization. You should also explore what happens if you only include points around the positive charges — can you explain what happens?

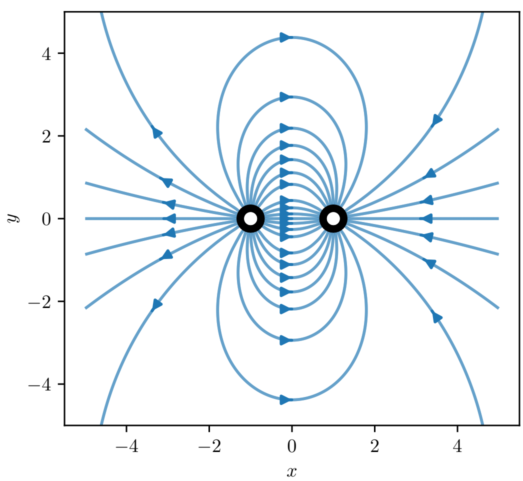


Fig. 1.15 Visualization of the field from a dipole with field lines using `streamplot`. The black dots show where the field lines were started.

Finding field lines numerically. (This section is technical and describes how e.g. `streamplot` calculates the field lines. You can skip to the next section without loss of continuity). We visualize the field lines by integrating the equations of motion for test charges moving along the field lines. In this section, we will implement the drawing explicitly for clarity and completeness.

First we build a function to draw a field line starting from a given point. Then we distribute field lines in space. We start a test charge in a point $\mathbf{r}_0$ and integrate the equations of motion:

$$\frac{dr}{dt} = Cq\frac{\mathbf{E}}{E}dt, \ . \tag{1.28}$$

where C is constant of choice. There is, of course, no real physical time involved, we only want to trace out a line. We continue integrating in both positive and negative direction until (i) the line comes within a distance d from one of the charges, (ii) the line extends beyond a maximum range, or (iii) we have generated a given maximum number of steps. Situation (iii) should not occur, but we include it to catch bugs or unexpected behaviors due to e.g. integration errors. We introduce a maximum range L so that the lines have to stay within $-L < x < L$ and $-L < y < L$. Notice that we integrate in both positive and negative direction from the starting point, that is, for both positive and negative time.

```
def findfieldline(R,Q,L,r0,radius):
    # R: list of positions of charges
    # Q: list of charges
    # L: maximum range for calculation
    # r0 = starting point
    # radius = radius around each charge
    dt = 0.01
    ri = [r0]
    for dir in [-1,1]:
        r = r0
        stop = 0
        nstep = 0
        while stop==0 and nstep<10000:
            nstep = nstep + 1
            E = efieldlist(r,Q,R)
            Enorm = E/np.linalg.norm(E)
            r = r + dir*Enorm*dt
            if (dir>0):
                ri.append(r) # Add to end
            else:
                ri.insert(0,r) # Add before beginning
            # Check if outside range
            if (r[0]<-L or r[0]>L or r[1]<-L or r[1]>L):
                stop = 1
                break
            # Check if it hits a charge
            for ii in range(len(Q)):
                dr = r - R[ii]
                if (np.linalg.norm(dr)<radius):
                    stop = 1
                    break
    return ri
```

Notice the trick we use to make one continuous line by integrating both forward and backward in time: We add each new point at the end when we move forward and at the beginning at the list of points when we move backward.

We set up the charge distribution as previously — here with a dipole:

```
Q2 = []
R2 = []
R2.append(np.array([-1,0]))
Q2.append(1.0)
R2.append(np.array([1,0]))
Q2.append(-1.0)
```

Now, we need to select where to start the field lines. In general, we would like the density of field lines to be proportional to the magnitude of the electric field. For the dipole, we assume that close to one of the charges, the field is dominated by one of the charges. We therefore select the points to be equally spaced on circle with a small radius around the positive charge. (For other problems, you will need to consider where to start field lines. For example, if the system is not neutral, it is useful also to start lines far away from the charges. We will return to this later.) We put all the starting points into a list `pointlist`:

```
numlines = 16
Ri = np.array([-1,0])
rad = 0.5
pointlist = []
for ni in range(numlines):
    ang = 2*np.pi*ni/numlines
    ri = np.array([Ri[0]+rad*np.cos(ang),Ri[1]+rad*np.sin(ang)])
    pointlist.append(ri)
```

Then we are ready to find and draw the lines. We select the radius around each charge to be `rad=0.1` and the maximum size of the box to be $L = 30$:

```
rad = 0.1
L = 30
for ni in range(len(pointlist)):
    ri = pointlist[ni]
    fieldl = findfieldline(R2,Q2,L,ri,rad)
    x,y = list(zip(*fieldl))
    plt.plot(x,y,'-k')
    ni = int(len(x)/2)
    plt.arrow(x[ni], y[ni], (x[ni+1]-x[ni]), (y[ni+1]-y[ni]), \
        fc="k", ec="k", head_width=0.2, head_length=0.2)
plt.axis('equal')
plt.xlim(-5,5)
plt.ylim(-5,5)
```

The resulting plot is shown in Fig. 1.16. Notice that a field line is missing. We expected a field line from the right charge in the positive x-direction. See if you can understand why this line not was drawn with this algorithm. You can add this line, by adding another initialization point for field lines directly to the right of the negative charge, for example, at $(1.1, 0)$.

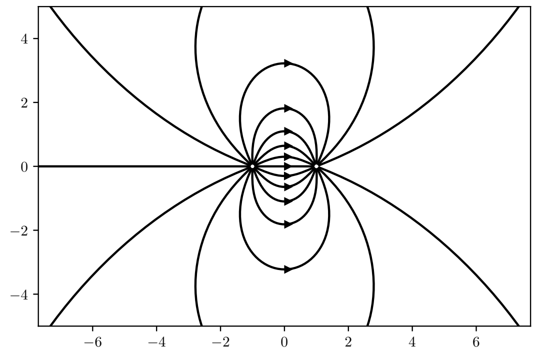


Fig. 1.16 Visualization of the field from a dipole using field lines.

1.2.8 Example: Electric field from a dipole

What is the electric field from a dipole? A *dipole* is a set of two opposite charges, q and $-q$ placed a distance d apart. Let us assume the system consists of a charge $-q$ at $(-d/2, 0, 0)$ and a charge q at $(d/2, 0, 0)$. We will study the dipole in three ways. First, we find the exact result in one dimension, that is, at a position x. Then we explore how this field behaves numerically and measure the behavior when x is large. Finally, we will compare with an approximate analytical solution when x is large. In the next chapter, we will find an approximate expression as a function of $\mathbf{r}$ at large distances.

One-dimensional solution. We find the electric field in a point $(x, 0, 0)$ along the x-axis using the superposition principle:

$$
\begin{aligned}
\mathbf{E} &= \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} && (1.29) \\
&= \frac{1}{4\pi\epsilon_0} \left(q \frac{(x,0,0) - (d/2,0,0)}{|x - d/2|^3} - q \frac{(x,0,0) - (-d/2,0,0)}{|x + d/2|^3} \right) && (1.30) \\
&= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(x - d/2)^2} - \frac{1}{(x + d/2)^2} \right) \hat{\mathbf{x}} \,. && (1.31)
\end{aligned}
$$

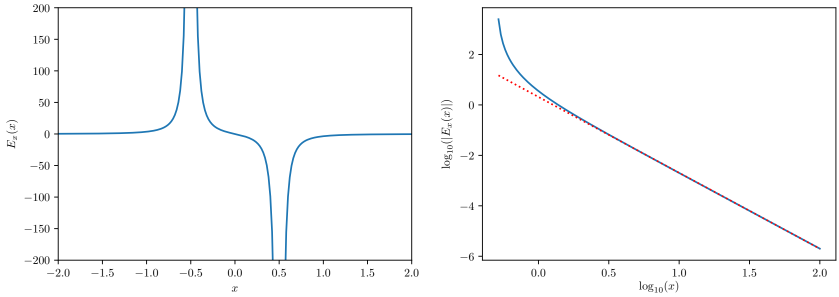


Fig. 1.17 (a) Plot of $E_x(x)$ for a one-dimensional dipole. (b) Plot of $E_x(x)$ when $x \gg 1$ on a log-log-scale.

Numerical analysis of behavior for large x. We plot the behavior of $E_x(x)$ as a function of x in Fig. 1.17 using the script:

```
import numpy as np
import matplotlib.pyplot as plt
x = np.linspace(-100,100,10001)
d = 1.0
E = -1/(x-d/2)**2+1/(x+d/2)**2
plt.plot(x,E)
plt.xlim(-5,5)
```

We see that $E_x(x)$ decays rapidly for $x > d/2$ (and for $x < -d/2$). But how rapidly? Let us look at the functional form of this decay. We do this by plotting $\log(E_x)$ as a function of x? Using a log-log plot is a commonly used trick to discover a power-law behavior. The electric field, $E_1(x)$ from a single point charge is a power law that decays as $1/x^2$. If we plotted $\log(E_1)$ as a function of x, we would get $\log(E_1) = \log(Cx^{-2}) = \log C - 2\log x$. If we plot the data this way, we can read the power law exponent, here -2, directly as the slope of the plot of $\log(E_1)$ versus $\log(x)$. We have done the same in Fig. 1.17. We see that when $x >> d/2$ the curve is approximately a straight line, but for smaller values of x, the curve deviates from a line. We can find the behavior of this line by fitting a linear function when $\log(x)$ is larger than 0.25. We do this using `polyfit` to fit a first order polynomial and plot the resulting fitted values:

```
j = np.where(np.log10(x)>0.25)
logx = np.log10(x[j])
logy = np.log10(abs(E[j]))
plt.plot(logx,logy,'oy') # Plot data
pol = np.polyfit(logx, logy, 1)
logfit = np.poly1d(pol)
plt.plot(logx,logfit(logx),':r') # Plot fitted line
```

```python
print(pol)
```

```
[-3.00893297  0.31654262]
```

The output from `polyfit` shows as its first output that the slope of the curve is approximately -3, which means that the dipole field approximately behaves as x^{-3} when $x \gg 1$. This is illustrated by the dotted red curve in Fig. 1.17.

Theoretical analysis of behavior for large x. We can derive the behavior when $x \gg 1$ theoretically. We rewrite (1.31) as:

$$
\begin{aligned}
E_x = & \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x-d/2)^2} - \frac{1}{(x+d/2)^2}\right) \\
= & \frac{q}{4\pi\epsilon_0}\frac{(x+d/2)^2-(x-d/2)^2}{(x-d/2)^2(x+d/2)^2} \\
= & \frac{q}{4\pi\epsilon_0}\frac{2xd}{(x-d/2)^2(x+d/2)^2} \, .
\end{aligned}
\tag{1.32}
$$

When $x \gg 1$, we can approximate $x - d/2 \simeq x$ and $x + d/2 \simeq x$, getting:

$$
E_x = \frac{q}{4\pi\epsilon_0}\left(\frac{2xd}{(x-d/2)^2(x+d/2)^2}\right) \simeq \frac{1}{4\pi\epsilon_0}\frac{2dq}{x^3} \, . \tag{1.33}
$$

This is the same as we saw numerically. We see that the *strength* of the dipole interaction — the field set up by the dipole — depends on dq: We can increase the field by increasing either d or q. We call this quantity the *dipole moment* and write $p = dq$. Generally, we say that a dipole field decays as $1/r^3$, whereas the field from a single charge decays as $1/r^2$. (Notice, that if we insert $x - d/2 \simeq x$ and $x + d/2 \simeq x$ directly into (1.31) we would get that the field is zero everywhere. This is typically an indication that we did not include sufficient terms in the series expansion.)

1.2.9 Example: Induced dipole interactions in one dimension

We started this chapter with a demonstration of a balloon being attracted to (and sticking to) a wall, but we did not describe the details of this demonstration. Another popular demonstration is to show that if we rub a balloon against our hair, it will attract an empty aluminum can as illustrated in Fig. 1.18. How can we understand this process? The aluminum does not have a net charge, so why is it attracted to the charge of balloon? We call this effect an *induced dipole interaction*. This effect is very important and common on the atomic scale. Let us address this in a simplified situation, and at the same time demonstrate how we make a simplified model of a complex physical situation.

Fig. 1.18 Illustration of how a charged balloon attracts an empty aluminum (soda) can. (Photos by Aurora Malthe-Sørenssen).

Induced dipole. Fig. 1.19 illustrates a situation where a charge q is placed near a neutral atom, e.g. Argon. We assume that the core of the atom does not move, but that the distribution of charge (gray) around the atom may deform. What do we expect to happen? If the charge q is positive, we expect it to pull the electron cloud slightly toward q. We assume that q is placed in the origin and that the positive core of the atom is at x. The effect of q is to move a charge $-\Delta q$ a small distance d. Since the atom is neutral, this means that the atom will be negative on the left side and positive on the right side, which we model as a charge $-\Delta q$ at the position $x - d$ and a charge Δq at x. We assume that the displacement d is much smaller than x.

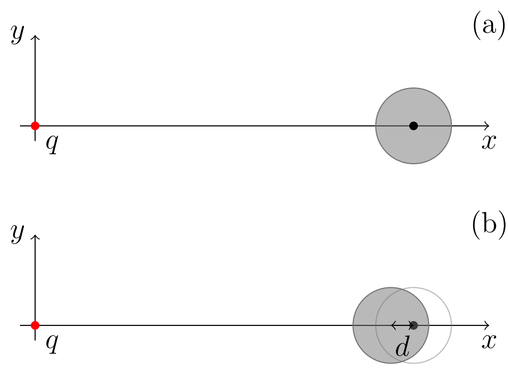


Fig. 1.19 Illustration of an induced dipole. We place a charge q near an atom with a center and a charge cloud. (a) Initial situation. (b) Sitation after the charge distribution around the atom has equilibrated to a new configuration. (This usually happens very fast).

Size of the induced dipole. The charge q therefore *induces* a dipole of strength $2d\Delta q$. How large is $2d\Delta q$? Again, we have to make a simplified model for this: We will assume that the response is linear, that is, that $d\Delta q$ is proportional to the electric field from the charge q at the position x:

$$2d\Delta q = C' \frac{1}{4\pi\epsilon_0} \frac{1}{x^2} = C/x^2 , \tag{1.34}$$

where $C = C'/(4\pi\epsilon_0)$ is a constant that we do not know.

Field from the induced dipole. What is the force from the induced dipole on the charge q? The electric field from the dipole at a distance x is:

$$E_x = \frac{2d\Delta q}{4\pi\epsilon_0 x^3} , \tag{1.35}$$

in the *positive* x-direction, with $2d\Delta q = C/x^2$. (Check the sign with yourself. Can you convince yourself that the force on the charge q is attractive independently of the sign of q?) The electric field at the position of the charge q, due to the induced dipole at x, is therefore:

$$E_x = \frac{C}{4\pi\epsilon_0 x^5} . \tag{1.36}$$

The attractive force from the dipole on charge q is therefore

$$F_x = qE_x = \frac{qC}{4\pi\epsilon_0 x^5} . \tag{1.37}$$

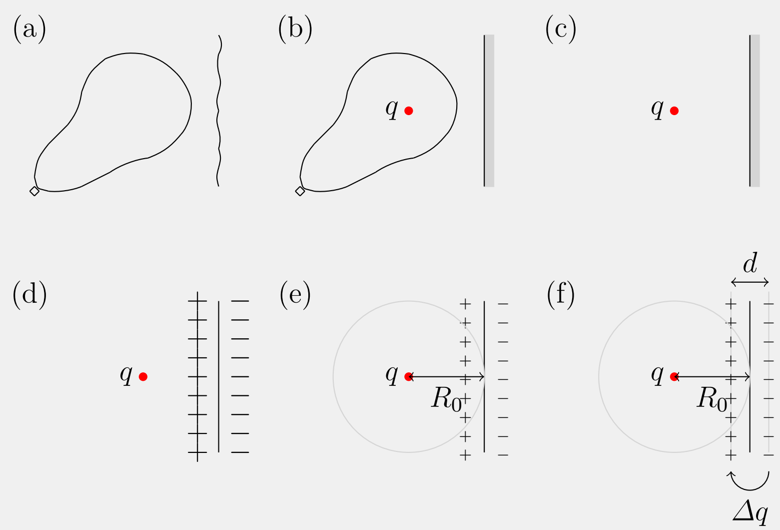
The force from the charge on the dipole is the reaction force to this force, and therefore has the same magnitude, but points in the negative x-direction.

Modeling a physics system

An important aspect of our analysis of the induced dipole system is what we call *modeling*. Modeling is to take a physical situation and convert it into a simplified, cartoon system that represents the essential physical properties we need to address the system.

The difficult part is to figure out what physical aspects need to be included in our model, and how to model these aspects. This takes experience and a bit of courage. In physics, we want to create a model that is as simple as possible, but not simpler. And we want to model the system using components that we can understand and use in computations.

For example, let us analyze a balloon close to a wall as illustrated in the figure part (a). The balloon has a charge q (b). We simplify it as a point charge q (c). We model the wall as displaced charges (d). The distance from the balloon to the wall is R_0 (e). We model the displacement of the charges in the wall as a charge Δq displaced by a distance d, where $d\Delta q$ is proportional to the electric field from q at the surface of the wall: $d\Delta q \simeq Cq/R_0^2$. This is similar to a charge Δq attached to a small spring with a deflection proportional to the force. We could continue to refine the model, but stop here. This allows us to explain the phenomenon, and maybe even measure C and R_0?



Implications for the soda can. We can use a similar type of argument for the force on the soda can. The soda can is a conductor and the electrons are therefore rather free to move around. Introducing a charge q near the conductor will therefore displace the electrons, forming a structure similar to a dipole, which will lead to an attractive force.

1.3 Continuous distributions of charge: charge density

We have learned that we can use the superposition principle to find the net electric field from many charges. For example, we saw earlier that we could make something that looked like a line of charges by placing many charges close to each other. Fig. 1.20 illustrates a set of charges Q_i. The net electric field from these charges is

$$\mathbf{E} = \sum_i \frac{Q_i}{4\pi\epsilon_0}\frac{\mathbf{R}_i}{R_i^3} \ . \tag{1.38}$$

When there are many, many small charges it becomes simpler not to think of individual point charges, but instead introduce a volume density of charges — a charge density. If we divide the system in Fig. 1.20 into small boxes of volume Δv_j containing a net charge ΔQ_j, then the total electric field is:

$$\mathbf{E} = \sum_j \frac{\Delta Q_j}{4\pi\epsilon_0}\frac{\mathbf{R}_j}{R_j^3} \ , \tag{1.39}$$

where the sum is over all the boxes j and $\mathbf{R}_j = \mathbf{r} - \mathbf{r}_j$ refers to the center $\mathbf{r}_j$ of box j. As the boxes become smaller and smaller, this approaches an integral over the volume elements Δv_i:

$$\mathbf{E} = \sum_j \frac{\Delta Q_j}{\Delta v_j}\frac{\Delta v_j}{4\pi\epsilon_0}\frac{\mathbf{R}_j}{R_j^3} = \sum_j \rho_j \frac{\Delta v_j}{4\pi\epsilon_0}\frac{\mathbf{R}_j}{R_j^3} \rightarrow \int_v \frac{\rho_v(\mathbf{r}')\mathrm{d}v}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} \ , \tag{1.40}$$

where $\rho_j = \Delta Q_j/\Delta v_j$. In the limit when Δv_j becomes small, ρ_j approaches the charge density, $\rho(\mathbf{r}_j)$. The sum is over all $\mathbf{r}_j$ in some volume region. When we rewrite this as an integral, the integral is over the positions $\mathbf{r}'$, which corresponds to the center of the box of volume $\mathrm{d}v$, thus $\mathbf{r}'$ corresponds to $\mathbf{r}_j$. In the integral, we therefore write the charge density as $\rho(\mathbf{r}')$ and we interpret $\mathbf{R}$ as $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. We should therefore write the integral as

$$\mathbf{E}(\mathbf{r}) = \int_v \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}v' \ , \tag{1.41}$$

Note that the integral is *not* over $\mathbf{r}$, but over $\mathbf{r}'$! We will therefore often use the notation $\mathrm{d}v'$ to make this distinction clear. We discuss this distinction in more detail below.

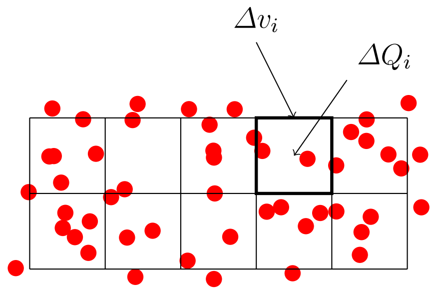


Fig. 1.20 Illustration of a set of charges divided into small volumes of size ΔV_i. Volume i has a net charge Q_i , which is the sum of the charges inside voluem ΔV_i.

Charge density

The volume charge density is:

$$\rho = \rho_v = \frac{\mathrm{d}Q}{\mathrm{d}v} \ , \tag{1.42}$$

with unit $\mathrm{C/m^3}$.

The net charge in a volume. The charge density is generally a function of position, $\rho(\mathbf{r})$. The net charge in a volume v is the sum of all the charges inside the volume, which is given by the integral over the volume v. We write a v below the integral sign to indicate that the integral is over a specific volume v in space:

$$Q_v = \int_v \rho(\mathbf{r})\mathrm{d}v \ . \tag{1.43}$$

Tricks of notation. Sometimes we use a compressed notation when we describe distributions of charge. Instead of specifying the volume of a small element as a differential, $\mathrm{d}v$, we instead specify the (net) charge of the small volume by writing $\mathrm{d}q$. We still think of this as the charge of a small element in a particular point $\mathbf{r}$ in space. Sometimes we will write

$$Q = \int_Q \rho \mathrm{d}q \ . \tag{1.44}$$

What we really mean by this is that the integral is over a volume element $\mathrm{d}v$ and that the charge in this volume element is $\mathrm{d}q = \rho(\mathbf{r})\mathrm{d}v$ as illustrated in Fig. 1.21 However, this notation also makes it easier to think of other charge elements, such as charges distributed over a surface (an area charge) or charged distributed along a curve (a line charge).

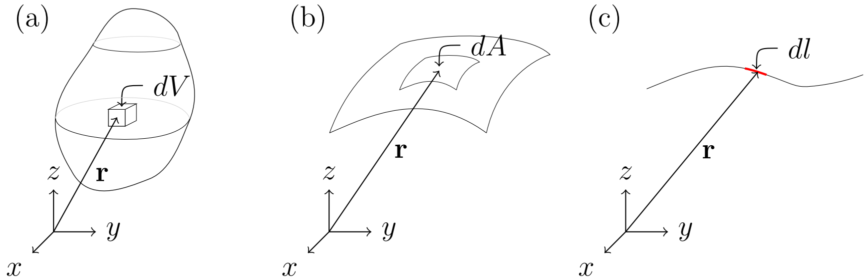


Fig. 1.21 Illustration of (a) volume charge distribution, (b) surface charge distribution, and (c) line charge distribution.

Surface charge density. Following the same reasoning as for a volume charge density, we can describe charges that are distributed on an area with an *area charge density*, $\rho_s(\mathbf{r})$ as illustrated in Fig. 1.21. In the literature, you may sometimes find the notation σ used for a surface charge density instead of ρ_s, but we will use ρ_s here. This is the charge per unit surface area

$$\rho_s = \frac{\mathrm{d}Q}{\mathrm{d}A} \tag{1.45}$$

with unit $\mathrm{C/m^2}$. The charge in a surface with area S is then given by an integral over the area S:

$$Q_S = \int_S \rho_s(\mathbf{r})\mathrm{d}S \ . \tag{1.46}$$

Line charge density. Finally, we also describe charges that are distributed on a line with a line charge density ρ_l (or often λ) as illustrated in Fig. 1.21. This the charge per unit length:

$$\rho_l = \lambda = \frac{\mathrm{d}Q}{\mathrm{d}l} \ , \tag{1.47}$$

with unit C/m. The charge in a length L is then given by an integral over the curve:

$$Q_L = \int_C \rho_l(\mathbf{r})\mathrm{d}l \ . \tag{1.48}$$

1.3.1 Finding the electric field from the charge density

How can we use the volume charge density to find the electric field? We will do this by summing up the contributions from all the small charge elements $\mathrm{d}q$ making up the system. We need to be very precise with our notation. We want to find the electric field $\mathbf{E}(\mathbf{r})$ in a point $\mathbf{r}$. We call this point the *observation point*. We want to include the contribution to this field from all the charges. For discrete charges, the charges were placed in positions $\mathbf{r}_i$ and we summed over all i. For a continuous distribution of charges, we sum the contributions of all charge elements $\mathrm{d}q = \rho(\mathbf{r}')\mathrm{d}v'$ at $\mathbf{r}'$ to the electric field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$. The contribution from one such element is:

$$\mathrm{d}\mathbf{E} = \frac{\rho(\mathbf{r}')\mathrm{d}v'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \ . \tag{1.49}$$

The contributions from all charges in a volume v is then found by the integral:

$$\mathbf{E} = \int_v \frac{\rho(\mathbf{r}')\mathrm{d}v'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \ . \tag{1.50}$$

Notice that the integral is over $\mathrm{d}v'$. What does this mean? It may become more obvious in Cartesian coordinates: In this case $\mathbf{r}' = (x', y', z')$ and the integral is over $\mathrm{d}v' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$. Notice that the *integration variable* is $\mathbf{r}'$, and that $\mathbf{r}$ is a *constant* in this integration! We use the triple integral notation if we want to write the integral explicitly out in Cartesian coordinates:

$$\mathbf{E} = \iiint_v \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \ . \tag{1.51}$$

Using the $\mathbf{R}$ notation. It is common and often practical to introduce the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Notice that the $\mathbf{R}$ goes *from* the element $\mathrm{d}v'$ at $\mathbf{r}'$ *to* the observation point at $\mathbf{r}$. We can then rewrite the integral for the electric field as:

$$\mathbf{E} = \int_v \frac{1}{4\pi\epsilon_0}\frac{\rho(\mathbf{r}')\mathrm{d}v'}{R^2}\hat{\mathbf{R}} \ . \tag{1.52}$$

Here, it is important to remember that $\mathbf{R}$ depends on the integration variable $\mathbf{r}'$. You can therefore generally not put $\mathbf{R}$, $\hat{\mathbf{R}}$ or R outside the integral! They are generally not constants. However, there may be special cases where for example the magnitude R is a constant in the integral, while the direction $\hat{\mathbf{R}}$ is not. We will use the $\mathbf{R}$ notation because it is compact, but we will be careful to unpack its meaning in each case. A common novice mistake is to assume that $\mathbf{R}$ is a constant in the integral or to assume that $\mathbf{R}$ (or $\mathbf{r}$) is the integration variable instead of over $\mathbf{r}'$.

Electric fields from surface and line charge densities. We can make exactly the same arguments for surface and line charge densities.

For a *surface density* ρ_a, the electric field from a surface S is

$$\mathbf{E} = \int_S \frac{1}{4\pi\epsilon_0} \frac{\rho(\mathbf{r}')\mathrm{d}S'}{R^2} \hat{\mathbf{R}} \, . \tag{1.53}$$

For a *line density* ρ_l, the electric field from a line L is

$$\mathbf{E} = \int_L \frac{1}{4\pi\epsilon_0} \frac{\rho(l')\mathrm{d}l'}{R^2} \hat{\mathbf{R}} \, . \tag{1.54}$$

Method for finding the electric field

There is a robust method to find the electric field $\mathbf{E}(\mathbf{r})$ from a discrete or continuous charge distribution. Given the charges and their positions, we find the field using the following procedure:

- Model the problem: Choose a coordinate system to describe the system that reflects the symmetries of the system.
- Make a drawing: Draw a sketch of the system where you indicate the charges and the observation point $\mathbf{r}$.

Discrete charges:

- Find the distance R_i from the observation point $\mathbf{r}$ to charge Q_i at $\mathbf{r}_i$: $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$ for each charge.
- Sum the contributions from all charges Q_i:

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}_i|^3} \, . \tag{1.55}$$

Continuous charges:

- Describe a small element $\mathrm{d}q$ at a position $\mathbf{r}'$. Relate $\mathrm{d}q$ to a volume; surface; line element $\mathrm{d}q = \rho \mathrm{d}v'$; $\rho_s \mathrm{d}S'$; $\rho_l \mathrm{d}l'$.
- Find the distance R from the observation point $\mathbf{r}$ to the small element at $\mathbf{r}'$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$.
- Integrate over all $\mathbf{r}'$ that make up the body of interest analytically or numerically. Remember that the integration variable is $\mathbf{r}'$ and not $\mathbf{r}$! For a volume charge density the integral is:

$$\mathbf{E}(\mathbf{r}) = \int_v \frac{\rho \mathrm{d}v'}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} = \int_v \frac{\rho \mathrm{d}v'}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, . \tag{1.56}$$

1.3.2 Example: Electric field from a ring charge

A ring of radius a in the xy-plane with center in the origin has a total charge Q which is uniformly distributed along the ring. Find the electric field from the ring.

First, we develop a model of the system: The length of the ring is $L = 2\pi a$. Because the charge is uniformly distributed along the ring, the line charge density is $\rho_l = Q/L = Q/(2\pi a)$. We plan to find the electric field in the point $\mathbf{r}$ by summing the contributions from all parts of the ring. A small part of length $\mathrm{d}l$ at a position θ has a charge $\mathrm{d}q = \rho_l \mathrm{d}l$. We will sum up all such contributions. The system is illustrated in Fig. 1.22 where $\mathbf{r}$ is chosen along the z-axis.

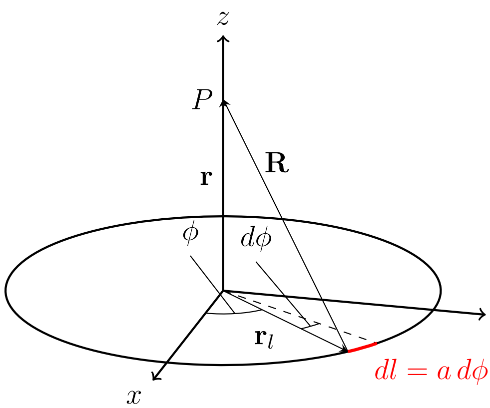


Fig. 1.22 An illustration of a ring with a charge Q.

Analytical solution along the z-axis. We start by finding the electric field at a point $\mathbf{r} = z\hat{\mathbf{z}}$ along the z-axis. We choose this observation point because the system has rotational symmetry around the z-axis, and we therefore expect the solution to be simpler along this axis. We describe the position of an element along the ring using the angle ϕ. A small length $\mathrm{d}l$ along the ring is then equal to $\mathrm{d}l = a\mathrm{d}\phi$, and the charge of this element is $\mathrm{d}q = \rho_l a \mathrm{d}\phi$.

The position of the element $\mathrm{d}l$ is $\mathbf{r}' = (a\cos\phi, a\sin\phi, 0)$. We find the field by summing up all contributions through the integral

$$\mathbf{E} = \int_Q \frac{\mathrm{d}q}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} = \int_l \frac{\rho_l(\mathbf{r}')\,\mathrm{d}l'}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} \,, \tag{1.57}$$

where the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}' = (0,0,z) - (a\cos\phi, a\sin\phi, 0) = (-a\cos\phi, -a\sin\phi, z)$. We see that

$$R^2 = a^2\left(\cos^2\phi + \sin^2\phi\right) + z^2 = a^2 + z^2 , \tag{1.58}$$

where we have used the identity $\cos^2\phi + \sin^2\phi = 1$. This means that R^2 does not depend on ϕ. We rewrite the integral to be an integral over ϕ:

$$\mathbf{E} = \int_l \frac{\rho_l \mathrm{d}l'}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} = \int_0^{2\pi} \frac{\rho_l a \mathrm{d}\phi}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^2} = \frac{\rho_l a}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\mathbf{R}}{R^3}\mathrm{d}\phi . \tag{1.59}$$

We insert the expression for $\mathbf{R}$ and R, getting:

$$\mathbf{E} = \frac{\rho_l a}{4\pi\epsilon_0}\int_0^{2\pi}\frac{(-a\cos\phi, -a\sin\phi, z)}{(a^2+z^2)^{3/2}}\mathrm{d}\phi . \tag{1.60}$$

We see that the integrals over $\cos\phi$ and $\sin\phi$ are zero. The resulting field $\mathbf{E}$ along the z-axis therefore only has a component in the z-direction:

$$\mathbf{E}(0,0,z) = \frac{\rho_l a}{4\pi\epsilon_0}2\pi\frac{z}{(a^2+z^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{z}{(a^2+z^2)^{3/2}} . \tag{1.61}$$

It is good practice always to check this result in the limits. Limits typically means for very large or very small values of z. Here, the limit of large z corresponds to the case when $z \gg a$ meaning that we are far away from the ring compared to the size of the ring. In this case, $a^2 + z^2 \simeq z^2$ and the electric field approaches:

$$\mathbf{E} \to \frac{Q}{4\pi\epsilon_0}\frac{z}{z^3} \quad , \quad z \gg a . \tag{1.62}$$

Is this result reasonable? Yes, we recognize this as the electric field from a point charge with charge Q placed at the origin. When we are far away from the ring, we expect that we can approximate it as a point charge, and this is indeed what we have found.

Symmetry argument. Let us see how we can solve this problem using symmetry instead of brute force. The symmetry argument may be more elegant, but it does require that you are good at finding and using symmetries to solve the problem. This takes some practice, but is a useful skill in physics.

In this case, we notice that for each element $\mathrm{d}\phi$ at a position ϕ, there is also an element on the opposite side of the ring (at $\phi+\pi$), and that the xy contributions from these two components cancel, thus leaving only the

z-component. Thus, we can limit ourselves to finding the z-component. This simplifies the integral. We can instead formulate an integral in terms of dl and not go to the explicit parameterization of the curve through the angle ϕ. In this case, the integral for the z-component of $\mathbf{E}$, E_z, becomes:

$$\mathbf{E} = \int_l \frac{\rho_l \mathrm{d}l}{4\pi\epsilon_0} \frac{R_z}{R^3} , \tag{1.63}$$

where R_z is the z-component of $\mathbf{R}$. We see from the geometry that R_z is simply z and that $R^2 = a^2 + z^2$ from Pytagoras. Hence the integral becomes

$$\mathbf{E} = \int_l \frac{\rho_l \mathrm{d}l}{4\pi\epsilon_0} \frac{z}{R^3} = \int_l \frac{\rho_l \mathrm{d}l}{4\pi\epsilon_0} \frac{z}{\left(a^2+z^2\right)^{3/2}} , \tag{1.64}$$

where there are no parts inside the integral that depend on the position along the ring. Hence, the integral is:

$$\begin{aligned}
\mathbf{E} &= \int_l \frac{\rho_l \mathrm{d}l}{4\pi\epsilon_0} \frac{z}{\left(a^2+z^2\right)^{3/2}} \\
&= \frac{\rho_l}{4\pi\epsilon_0} \frac{z}{\left(a^2+z^2\right)^{3/2}} \int_l \mathrm{d}l \\
&= \frac{\rho_l 2\pi a}{4\pi\epsilon_0} \frac{z}{\left(a^2+z^2\right)^{3/2}} ,
\end{aligned} \tag{1.65}$$

where we have used that $\int_l \mathrm{d}l = 2\pi a$. This is indeed the same result as we found above.

Solution outside the z-axis. If we want to find the solution outside the z-axis, we can either follow the same type of argument, but we need to evaluate the integral numerically, or we can start from a discretization of the ring into finite linear segments, and then sum of the contributions from each of these segments. We will address both methods, which essentially are identical, although the discretization is different.

Numerical integration of a charged ring with an angular discretization. We can use the same approach as introduced above, but instead introduce a general point $\mathbf{r} = (x, y, z)$ for $\mathbf{R}$ so that

$$\begin{aligned}
\mathbf{R} = \mathbf{r} - \mathbf{r}_l &= (x, 0, z) - (a\cos\phi, a\sin\phi, 0) \\
&= (x - a\cos\phi, y - a\sin\phi, z) .
\end{aligned} \tag{1.66}$$

The integral for the electric field is

$$\mathbf{E} = \int_0^{2\pi} \frac{\rho_l a \mathrm{d}\phi}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} \ , \tag{1.67}$$

when we insert $\rho_l = Q/(2\pi a)$ and $\mathbf{R} = (x - a\cos\phi, y - a\sin\phi, z)$, we get

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{2\pi} \int_0^{2\pi} \frac{(x - a\cos\phi, y - a\sin\phi, z)}{\left((x - a\cos\phi)^2 + (y - a\sin\phi)^2 + z^2\right)^{3/2}} \mathrm{d}\phi \ . \tag{1.68}$$

This integral we can solve by numerical integration. Notice that x, y, and z are constants in this integral, it is ϕ that is the integration variable and varies in the integral. We can discretize the integral by introducing N elements $\Delta\phi = 2\pi/N$ and then summing over $\phi_i = i\Delta\phi$ for $i = 0, 1, \ldots, N$:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 \, 2\pi} \sum_i \frac{(x - a\cos\phi_i, y - a\sin\phi_i, z)\, \Delta\phi}{\left((x - a\cos\phi_i)^2 + (y - a\sin\phi_i)^2 + z^2\right)^{3/2}} \ . \tag{1.69}$$

We leave the numerical implementation of this sum as a task for you.

Numerical integration of charged ring with piecewise linear discretization. We can take a different and more physically motivated approach to the discretization of the ring. The idea is to divide the ring into small pieces of length Δl placed along the circle, and then sum the contibutions from each of these elements to the electric field at a position $\mathbf{r}$.

If we divide the ring into N elements, then the charge of an element is Q/N. The (line) element starts at $\mathbf{r}_i = a(\cos\phi_i, \sin\phi_i, 0)$ and ends at $\mathbf{r}_{i+1} = a(\cos\phi_{i+1}, \sin\phi_{i+1}, 0)$ where $\phi_i = i2\pi/N$ for $i = 0, 1, \ldots, N-1$. We may consider the position of the element to be the midpoint of this line, which would be at $(\mathbf{r}_i + \mathbf{r}_{i+1})/2$. However, we do not make a large mistake if we instead use $\mathbf{r}_i$ as the position of element i. (Indeed, the error becomes smaller as the number of elements N increases.)

We can then find the total electric field $\mathbf{E}$ as a sum of the contributions from each of the i elements:

$$\mathbf{E}(\mathbf{r}) = \sum_i \mathbf{E}_i = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\mathbf{R}_i}{R_i^3} \ , \tag{1.70}$$

where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$.

This method can be directly implemented in Python. We use the `efieldlist_units` method we developed previously, adding all the elements $Q_i = Q/N$ at $\mathbf{r}_i$ to a list Q and R of charges. First, we set up the list:

```
# Requires functions efieldlist_units, findfield3d, and visfield
# Create lists Q and R with ring elements
a = 1.0 # radius of ring
N = 20 # number of elements in discretization of ring
q = 1.0 # charge of ring
rcenter = np.array([0,0,0]) # center of circle
Q = [] # Empty list for charges
R = [] # Empty list for positions
deltaphi = 2*np.pi/N
Qi = q/N
for i in range(N):
    phi_i = i*deltaphi
    r_i = np.array([a*np.cos(phi_i),a*np.sin(phi_i),0])+rcenter
    Q.append(Qi)
    R.append(r_i)
```

Then we calculate the field on a grid and visualize the results. Because the system is rotationally symmetric around the z-axis — the ring is identical if we rotate it around the z-axis — we expect the field to have the same symmetriy. It is therefore sufficient to calculate and visualize the field in the xz-plane. Any other plane, such as the yz-plane, will be identicial. We use the findfield3d function to calculate the field:

```
# Calculate field in xz-plane
L = 2 # Size of region to visualize
Nx = 31 # Number of grid points
x,z,rx,rz,Ex,Ez = findfield3d(R,Q,-L,L,-L,L,Nx,Nx)
# Visualize the field
visfield(x,z,rx,rz,Ex,Ez)
```

The resulting plot can be seen in Fig. 1.23.

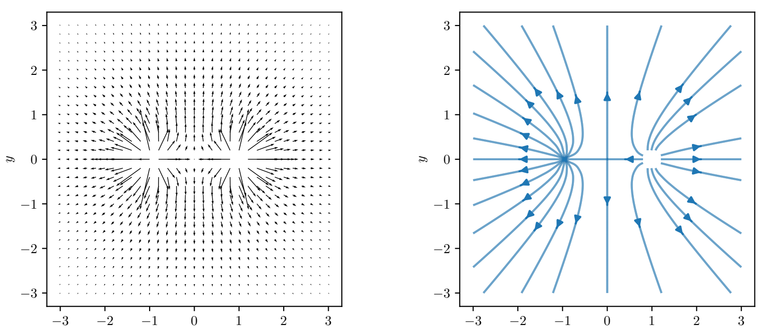


Fig. 1.23 Plot of the calculated field in the xz-plane.

1.4 Summary

Coulomb's law. The force on a point charge q at $\mathbf{r}$ from a point charge Q at $\mathbf{r}'$ is

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,. \tag{1.71}$$

Electric field. The electric field $\mathbf{E}$ at a point $\mathbf{r}$ due to a set of charges is defined as

$$\mathbf{E} = \frac{\mathbf{F}}{q} \,, \tag{1.72}$$

where $\mathbf{F}$ is the net force on a test charge q due to the set of charges.

Properties of the electric field.

- The electric field from a point charge Q at $\mathbf{r}'$ is

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,. \tag{1.73}$$

- The vector $\mathbf{R}$ is from the charge Q at $\mathbf{r}'$ to the observation point at $\mathbf{r}$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.
- The electric field obeys the **superposition principle** : $\mathbf{E} = \sum_i \mathbf{E}_i$.
- The electric field can be visualized using a vector plot, `quiver(rx,ry,Ex,Ey)` or with a streamline plot `streamplot(rx,ry,Ex,Ey)`

Charge distributions. We describe a continuous distribution of charges using a volume; surface; or line charge density with charge elements $\mathrm{d}q = \rho_l \mathrm{d}v$; $\rho_s \mathrm{d}S$; $\rho_l \mathrm{d}l$.

The electric field from a **volume charge density** in the volume v is given by an integral over $\mathbf{r}'$:

$$\mathbf{E} = \int_v \frac{\rho(\mathbf{r}')\mathrm{d}v'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,, \tag{1.74}$$

by a **surface charge density**, ρ_s, on a surface S is

$$\mathbf{E} = \int_S \frac{\rho_s(\mathbf{r}')\mathrm{d}S'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,, \tag{1.75}$$

and by a **line charge density** along a curve C is:

$$\mathbf{E} = \int_C \frac{\rho_l(\mathbf{r}')\mathrm{d}l'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \,. \tag{1.76}$$

1.5 Exercises

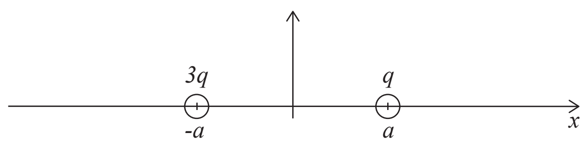
1.5.1 Test yourself

Exercise 1.1: Two charges

Two charges q and $3q$ are placed as shown in the figure. Draw the forces acting on the charges.

Exercise 1.2: Two charges

Two charges $3q$ and q are placed on the x-axis at $(-a, 0)$ and $(a, 0)$ as shown in the figure.

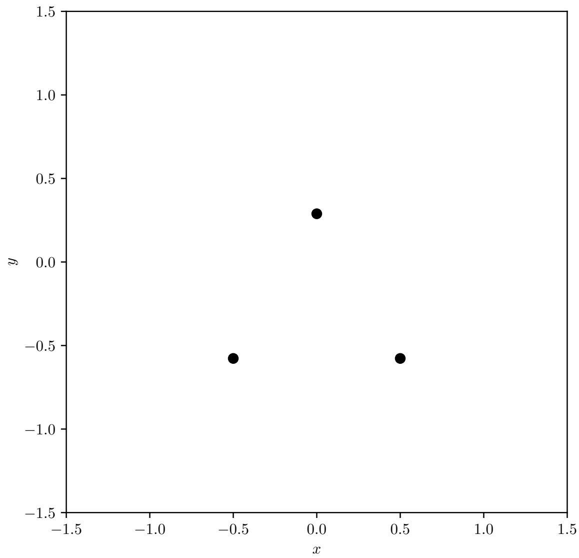


a) Draw a vector diagram of the electric field along the x-axis.

b) Sketch the electric field $E_x(x)$ as a function of x along the x-axis.

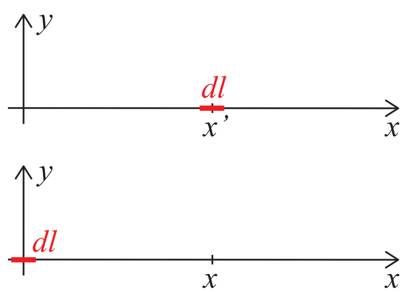
Exercise 1.3: Three charges

Three identical charges are placed at the vertices of an equilateral triangle of side a as illustrated in the figure. Draw the electric field in the plane using a vector diagram.



Exercise 1.4: Line element

The figure below shows a line element of length dl of a material with a uniform line charge density ρ_l.



a) What is the contribution d$\mathbf{E}$ from the line element dl at x' to the electric field at the origin?

b) What is the contribution d$\mathbf{E}$ from the line element dl at the origin to the electric field at x?

1.5.2 Discussion exercises

Exercise 1.5: Zero field

Draw a diagram with two point charges so that the electric field is zero somewhere and show where that position is. For a system of two charges, is there always a point where the electric field field is zero?

Exercise 1.6: Field and force

What is the relationship between the terms "field" and "force"? What are their units?

Exercise 1.7: Constant field

How would you illustrate an electric field that is a constant (in space and time)? How does this field affect a positive and a negative charge at $(0, 0)$ and $(1, 1)$? Make drawings to illustrate your answers.

Exercise 1.8: Peeling tape

If you peel two strips of transparent tape off the same roll and immediately let them hang near each other, they will repel each other. If you then stick the sticky side of one to the shiny side of the other and rip them apart, they will attract each other. Give a plausible explanation.

Exercise 1.9: Mechanisms for charge transfer

If a glass rod is rubbed on a silk cloth, the rod becomes positively charged, but if you rub the rod on fur, it becomes negatively charged. How can you explain the mechanisms of charge transfer?

1.5.3 Tutorials

Exercise 1.10: Electric dipole

An electric dipole consists of two particles: particle 1 with a charge Q at $\mathbf{r}_1 = (a, 0, 0)$ and particle 2 with a charge $-Q$ at $\mathbf{r}_2 = (-a, 0, 0)$.

a) Make a sketch of the system. What is the direction of the electric field $\mathbf{E}$ for a point on the y-axis?

b) What are the two $\mathbf{R}$-vectors you used in this argument? (The $\mathbf{R}$-vector appears in the expression for the electric field from a single charge: $\mathbf{E} = Q/(4\pi\epsilon_0)\mathbf{R}/R^3$.)

c) For a point in a plane through $x = a$ parallel to the yz-plane, that is for points $\mathbf{r} = (a, y, z)$, what are the two $\mathbf{R}$-vectors needed to find the electric field?

d) Find an expression for the electric field $\mathbf{E}(a, y, z)$.

e) What type of coordinate system would you use to take advantage of the symmetry of the problem? Provide an argument for your choice.

f) *(Challenging)* What is the electric field $\mathbf{E}$ expressed in this coordinate system for a general point $\mathbf{r} = (x, y, z)$?

Exercise 1.11: Line charge

A rod of length L has a charge Q. We place the rod along the x-axis with its center at the origin.

a) Make a drawing of the system. What assumptions would you need to make to approximate this as a line charge with a line charge density ρ_l? Find ρ_l.

b) A small piece of the rod from x to $x + \mathrm{d}x$ has a length $\mathrm{d}x$. What is the charge of this piece?

c) We want to calculate the electric field $\mathbf{E}(0, y, 0)$ along the y-axis. What is the contribution to the field from the piece of length $\mathrm{d}x'$ at x' expressed using the $\mathbf{R}$-vector? Write down an explicit expression for the $\mathbf{R}$-vector.

d) Write down an expression for the electric field $\mathbf{E}(0, y, 0)$ in terms of an integral. Explain what variable you integrate over. (Ensure that your result is a vector!)

e) Explain how you can use the symmetry of the problem to simplify the calculation. What type of symmetry do we have in this problem. Use this symmetry to explain where you know the field if you calculate it in the point $(0, y, 0)$.

Exercise 1.12: Plane charge

An infinite plane with a homogeneous charge density ρ_s is placed at $z = 0$.

a) What does the word homogeneous mean in this context?

b) What is the total charge of the plane?

c) We want to find the field in a point $\mathbf{r} = (x, y, z)$. Use a symmetry argument to find the direction of the electric field in this point.

d) What is the contribution $\mathrm{d}\mathbf{E}$ to the electric field at $\mathbf{r} = (x, y, z)$ from a small piece $(x', x' + \mathrm{d}x')$, $(y', y' + \mathrm{d}y')$ at $(x', y', 0)$? First, make a drawing of the system. Explain why we use x' and not x. What is the $\mathbf{R}$-vector? Find an expression for $\mathrm{d}\mathbf{E}$.

e) Find an integral-expression for the electric field $\mathbf{E}$ in a point $\mathbf{r} = (x, y, z)$ from charges in the region $0 < x' < a$, $0 < y' < b$, $z' = 0$. (You do not need to solve the integral).

Exercise 1.13: Angle between suspended charges

Two charges of identical mass m with charges $q_1 = q$ and $q_2 = 2q$ as suspended from the same point in strings of length L. You can assume that the gravitational force is much larger than the electrostatic force on each charge.

a) Draw a force diagram for each charge.

b) Find an approximate expression for the angles θ_1 and θ_2 that each charge makes with respect to the vertical.

c) Why did you have to assume that the electrostatic force was small compared with gravity to solve this problem. How can you solve this problem is you do not make this assumption?

Exercise 1.14: Adding and subtracting using superposition

We use superposition to find the net electric field on a charge from a distribution of individual charges or from a continous distribution of charges. However, superposition can also be used in other, creative ways as a "trick" for solving problems. In this exercise, look for new ways to apply superposition and reflect on how you used it in your reasoning.

a) Six identical charges Q are placed at the vertices of a hexagon edge length L. What is the net force on a test charge q placed at the center of the hexagon?

b) We now remove one of the charges. There are now 5 equal charges present at five of the vertices of the hexagon. What is the net force on the test charge now?

c) How can you generalize your result to the case where you place n equidistant charges Q on a circle of radius r? What is the electric field in the center of the circle? What is the electric field in the center of the circle if you remove one charge at $(r, 0, 0)$?

d) The electric field at a distance r from the center of a uniformly charged sphere of radius a and volume charge density ρ is

$$\mathbf{E} = \begin{cases} \frac{\rho}{3\epsilon_0} r \hat{\mathbf{r}} & r < a \\ \frac{\rho}{3\epsilon_0} \frac{a^3}{r^2} \hat{\mathbf{r}} & r \geq a \end{cases} .$$

Explain how you can use superposition to find the electric field inside a spherical shell of uniform charge density ρ with inner radius b and outer radius a.

Hint 1. A spherical shell with outer radius a and inner radius b plus a sphere of radius b is a sphere of radius a.

Hint 2. $\mathbf{E}_a = \mathbf{E}_{\text{shell}} + \mathbf{E}_b$.

Exercise 1.15: Visualization of the electric field

a) Draw a figure illustrating the electric field from a single, positive point charge in the origin. Use arrows to indicate the direction and magnitude of the field.

The following minimally working Python program to find the electric field from a single point charge at the origin.

```
import numpy as np
import matplotlib.pyplot as plt

def efieldq(q0,r,r0):
    # Input: charge q in Coulomb
    #        r: position to find field (in 1,2 or 3 dimensions) in meters
    #        r0: position of charge q0 in meters
    # Output: electric field E at position r in N/C
    dr = r-r0
```

```
    drnorm = np.sqrt(dr.dot(dr))
    epsilon0 = 8.854187817e-12
    return q0/(4.0*np.pi*epsilon0)*dr/drnorm**3

r0 = np.array([0.0,0.0])
q0 = 1.0
L = 5
N = 21
x = np.linspace(-L,L,N)
y = np.linspace(-L,L,N)
rx,ry = np.meshgrid(x,y)
Ex = np.zeros((N,N),float)
Ey = np.zeros((N,N),float)
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i]])
    Ex.flat[i],Ey.flat[i] = efieldq(q0,r,r0)
plt.quiver(rx,ry,Ex,Ey)
plt.axis('equal')
```

b) Explain and comment the code - write comments directly into the code. What is the `flat` command used? How can you change the range and resolution of the visualization?

c) Use the program to show the visual difference between the fields from a positive and from a negative charge. Are the results reasonable?

d) Rewrite the program to find the field from a charge $q_0 = 0.1\mu\text{C}$ at $\mathbf{r}_0 = (0.01, 0)\text{m}$.

e) Use the rewritten program to find the force acting on (i) a charge $Q = 0.1\text{mC}$ at $\mathbf{r} = (1, 0)\text{mm}$; (ii) a charge $Q = -0.1\text{mC}$ at $\mathbf{r} = (1, 0)\text{mm}$; (iii) a charge $Q = 0.2\text{mC}$ at $\mathbf{r}_0 = (1, 2)\text{mm}$.

f) Use the program to find the field from two positive and equal charges $q = 0.1\mu\text{C}$ at $(0, 0)\text{mm}$ and $(0.5, 0)\text{mm}$.

g) Use the program to find the net field from two charges: $q_1 = -0.1\mu\text{C}$ at $\mathbf{r}_0 = (0, 0)\text{mm}$ and $q_1 = 0.1\mu\text{C}$ at $\mathbf{r}_0 = (0.5, 0)\text{mm}$.

1.5.4 Homework

Exercise 1.16: Point charges

A point charge $q_1 = 4.0$ nC is on located the x-axis at $x = 2.0$ m. Another point charge, $q_2 = -6.0$ nC, is located on the y-axis at $y = 1.0$ m.

a) Draw a figure of the charges in a coordinate system and sketch the forces acting on the two charges.

b) Calculate the force $\mathbf{F}_2$ on the charge q_2 and the magnitude $|\mathbf{F}_2|$ of the force on the charge q_2.

c) What is the force $\mathbf{F}_1$ on the charge q_1?

Exercise 1.17: The electric field from groups of point charges

We address a system with a single charge q in $\mathbf{r} = 0$.

a) Use Python to make an arrow plot that shows the electric field $\mathbf{E}(x, y)$ in the xy-plane and a plot that show the stream lines in the xy-plane.

b) Plot $E(r) = |\mathbf{E}(r)|$, where $r = |\mathbf{r}|$. Explain how to use a plot of $E(r)$ to obtain the functional form of $E(r)$ and show that it is $E(r) \propto r^{-2}$.

We will then address a system of two charges: a charge q in $\mathbf{r} = (a, 0)$ and a charge $-q$ in $\mathbf{r} = (-a, 0)$, where a is a characteristic length.

c) Make an arrow plot of the electric field in the xy-plane and a plot that shows the stream lines in the xy-plane.

d) Plot $E(r)$. From the plot, find the functional form, $E(r)$, in the limit of large r ($r \gg a$).

e) Will these results change significantly if both charges have the same sign? How?

We will then address a system of four charges — a quadrapole — with q in $\mathbf{r} = (\pm a, 0)$ and $-q$ in $\mathbf{r} = (0, \pm a)$.

f) Make an arrow plot of the electric field in the xy-plane and a plot of the stream lines in the xy-plane.

g) Using the method you have developed to find the functional form of $E(r)$ in the limit of large r ($r \gg a$).

Exercise 1.18: The electric field above a finite disk

In this exercise we study the electric field along the z-axis (the symmetry axis) from a disk of radius a and constant surface charge density ρ_A.

a) What is the contribution to the electric field in a position z from an element from r to $r + \mathrm{d}r$ and from θ to $\theta + \mathrm{d}\theta$. (You can assume $z > 0$).

b) What is the electric field in a position z (for $z > 0$)?

c) Find the electric field $\mathbf{E}$ in the limit where z is small ($z \to 0$ from above). Interpret the result.

d) Sketch $|\mathbf{E}|$ as a function of z for $z > 0$.

Exercise 1.19: Work on point charges

Three point charges with the same charge Q are placed at each corner A, B, C in an equilateral triangle with side lengths a. One charge is displaced along the dashed line from A to the midpoint of A' on the side BC. The charges in B and C are kept in place throughout the displacement. Calculate the work done to carry out this displacement.

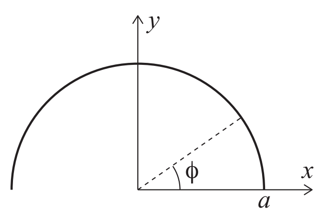
Exercise 1.20: Plate with a hole

Find $|\mathbf{E}|$ at a height z above an infinitely large, plane surface with a hole with radius a. The surface has a constant surface charge density ρ_s.

Hint. Superposition.

Exercise 1.21: Half circle

We will here study a half-circle shaped line charge with radius a as illustrated in the figure.



a) Assume that the line has a uniform charge density with a total charge Q. Find the electric field $\mathbf{E}$ in the center of the half circle, that is, in the origin of the figure.

b) Let as assume that the half circle is *not* uniformly charged, but that the line charge density is given as

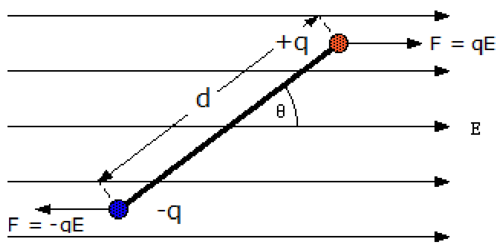
$$\rho_l(\phi) = \frac{Q}{2a} \sin \phi \ . \tag{1.77}$$

Sketch the line charge density, and show that the total charge for the half circle is still Q.

c) Find the magnitude and direction of the electric field in the origin for the charge distribution in **b** . Which of the charge distributions (from **a** or **b**) gives the largest magnitude for the electric field?

Exercise 1.22: Dipole in a uniform electric field

In this exercise we will take a look at an electric dipole in a uniform electric field. The distance between the two charges in the dipole is d as shown in the figure.



a) What is the net force on the dipole?

b) Find the torque around the center of mass of the dipole in terms of the dipole moment $\mathbf{p} = Q\mathbf{d}$ and the electric field $\mathbf{E}$.

Hint. The torque is $\tau = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the the vector going from the center of mass to where the force $\mathbf{F}$ is applied.

c) For what angles θ do we have stable and unstable equilibrium? And when do we have maximum torque?

d) Show that for small values of θ you get a simple harmonic motion. Here you need to use the small angle approximation ($sin(\theta) \approx \theta$) and $\tau = I\alpha$, where I is the moment of inertia and α is the angular acceleration.

Hint. A simple harmonic oscillator has the differential equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \ , \tag{1.78}$$

where m and k are constants and x describes the harmonic motion. The solution is

$$x(t) = Acos(\omega t + \phi) \quad , \quad \omega = \sqrt{\frac{k}{m}} \ , \tag{1.79}$$

where A and ϕ are constants.

e) What is the angular velocity?

f) Find a formula for θ.

g) Plot the angle over two periods.

Hint. The formula for moment of inertia for point mass

$$I = \sum_i m_i {r_i}^2 \tag{1.80}$$

m_i is the mass and r_i the distance from the point of rotation to the point mass.

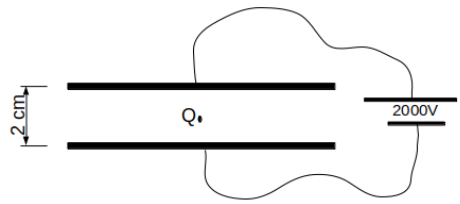
h) Find the numerical solution (do not use small angle approximation) and plot both results over four periods. Use `SciPy`'s `odeint` package to solve the differential equation. The initial conditions are $q = e$, $d = 3 \times 10^{-9}$m, $m = 3 \times 10^{-5}$kg and $E = 1000$N/C.

i) Expand the code and plot the difference between the numerical and approximate solution over time. Find where the difference between them is above 0.05 rad. Plot red dots over the numeric solution where that is the case.

Exercise 1.23: Milikan oil drop experiment

(By Sigurd Sørlie Rustad)
In 1909 Robert A. Milikan and Harvey Fletcher tried to measure the elementary electric charge. They did this by measuring the electric field needed to levetate a ionized oil drop. They found the elementary electric charge to be $1.5924(17) \times 10^{-19}$*, about 0.6% difference from the current value.*
(https://en.wikipedia.org/wiki/Oil_drop_experiment)

In this exercise we are going to do something that resembles the Milkan oil drop experiment. Consider the setup shown on the figure. The oil drop is in equilibrium and the mass is 4.9×10^{-15}kg.



a) Draw a sketch to show the forces acting on the oil drop.

b) What is the sign of the charge on the oil drop?

c) Find the total charge on the oil drop.

d) How many electrons does that amount to?

Exercise 1.24: Field from a hemisphere shell

(By Sigurd Sørlie Rustad)
In this exercise we are going to calculate the field from a hemisphere shell (see figure 1.24). The shell has constant charge density ρ and radius R.

a) Calculate the electric field, $\mathbf{E}$, in the origin.

Exercise 1.25: Visualize electric field

(By Sigurd Sørlie Rustad)
Here we are going to visualize the electric field from several particles in two dimensions. We are going to do this in parts.

a) Consider a particle in (x_1, y_1) with charge Q, write down the general electric field from this in position (x, y).

b) Write a Python program that takes in position and charge from a particle, a point in space, and returns the electric field vector in that point. Make a vector arrow plot of the electric field.

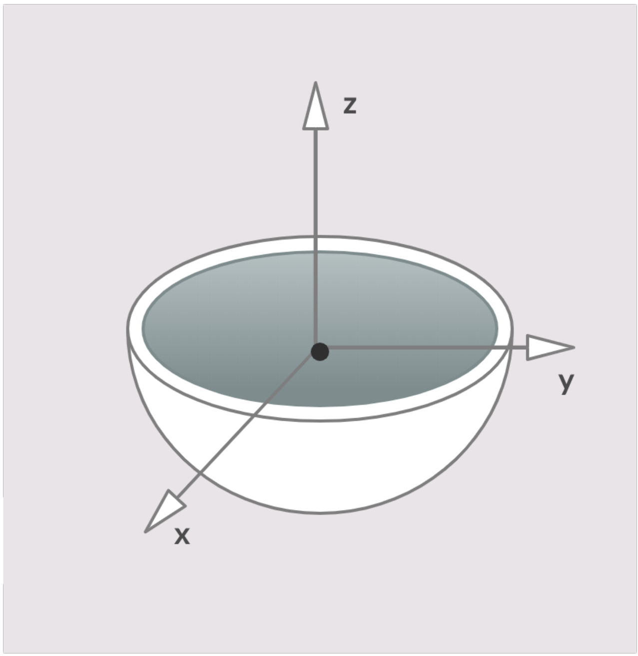


Fig. 1.24 Illustration of a hemisphere shell. The origin is equidistant to the hemisphere and the hemisphere is oriented such that it has rotational symmetry along the z-axis.

Hint 1. It can be a good idea to scale ϵ_0 so you get reasonable sizes.

Hint 2. Use `quiver` from `matplotlib.pyplot` to visualize the field.

c) Expand the function you made to take in an arbitrary number of particles and then returns the resulting electric field. Make a vector arrow plot.

Exercise 1.26: Parallel lines

a) A charge q is in vacuum in the point $\mathbf{r}_0 = x_0\hat{\mathbf{x}} = (x_0, 0, 0)$ on the x-axis. What is the electric field in the point $\mathbf{r} = (x, y, z)$? Make a drawing to illustrate the vectors you have used in your calculation and the direction of the field at $\mathbf{r}$.

* * *

b) An infinitely long line charge with uniform line charge density ρ_l is placed along the x-axis. There are no other charges present and the system is in vacuum. Find the electric field $\mathbf{E}$ at $\mathbf{r} = (x, y, z)$.

c) An infinitely long line charge with uniform line charge density ρ_l is instead placed on a line parallel to the x-axis through $y = d$. There are no other charges present and the system is in vacuum. Find the electric field $\mathbf{E}$ in the point $\mathbf{r} = (x, y, z)$.

d) Let us assume that the system consists of two infinitely long line charges that are parallel with the x-axis. One line passes through $(0, 0, 0)$ and the other passes through $(0, d, 0)$. Both lines have uniform line charge densities ρ_l. Show that the electric field $\mathbf{E}(\mathbf{r})$ in the xy-plane is:

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon_0}\left(\frac{1}{y} + \frac{1}{y-d}\right)\hat{y} \; . \tag{1.81}$$

We will now study a system that consists of a line along the x-axis and through the origin with a line charge density ρ_l, which is not uniform, but varies with x:

$$\rho_l(x) = \begin{cases} 0 & x < 0 \\ \rho_l(x) & 0 \le x \le L \\ 0 & x > L \end{cases} \; . \tag{1.82}$$

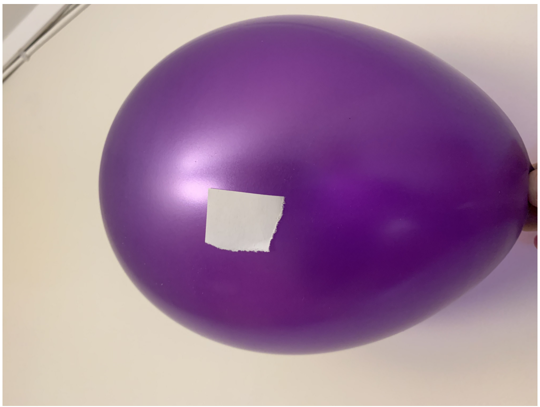
e) Find an expression on integral form for the electric field $\mathbf{E}(\mathbf{r})$ in $\mathbf{r} = (x, y, z)$ from the line charge density $\rho_l(x)$ on the x-axis. (You do not need to solve the integral.)

f) Assume that you may approximate the integral with a sum over N elements of width $dx_i = L/N$ at $x_i = (i + 1/2)dx_i$ for $i = 0, 1, \ldots, N-1$. Every element contributes with a point charge $dq_i = \rho_l(x_i)dx_i$. Write a short program to find an approximate value for the electric field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$ by summing the fields from the point charges dq_i.

1.5.5 Modeling projects

Exercise 1.27: Paper on balloon

A small piece of paper attaches to a balloon so strongly that it does not fall down as shown in the figure.



Explain how you would model this system.

Exercise 1.28: Wavy surface

If you cut into a material such as a mineral, the fresh surfaced may have effective surface charges.

a) How will you model the electric field from a freshly cut mineral surface?

b) If the surface is flat, what is the electric field outside the surface?

c) Assume that the surface is a fracture. We model the fracture as a sine-wave with amplitude A and wavelength λ. Write a Python program to find the electric field outside the surface. How can you use the result from the previous exercise to test your results?

d) A fracture creates two sine-wave surfaces that are displaced a small distance d from each other. One surface has positive charge and the other negative charge. What is the electric field between the surfaces?

e) Assume that the surface is carefully constructed to be have a rectangular wave shape with an amplitude A and a wavelength λ. Modify your Python program to find the electric field outside the surface. What are the main differences between the two systems?

f) How would the charged surfaces affect a water molecule? You can assume that you can model the water molecule as a dipole.

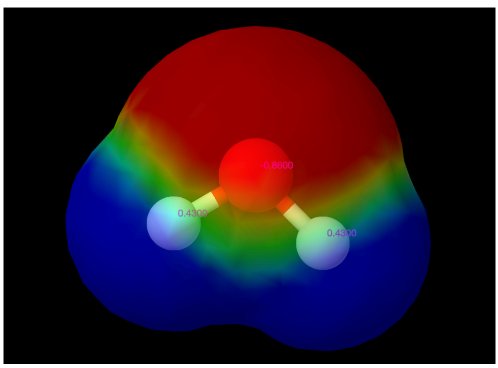
g) (*Challenging*) A real fracture surface has a more complicated shape called a self-affine fractal. An example of a self-affine fractal is a random walk. (Although the scaling behavior of the random walk is different from most fracture surfaces, they both are examples of self-affine fractals). You can generate a random walk of length L with the following script

```
from pylab import *
L = 100
y = cumsum(randn(L))
x = arange(L)
y = y - (y[-1]-y[0])/L*x
plot(x,y)
axis('equal')
```

Use this model of a fracture to study the electric field inside a fracture.

Exercise 1.29: Water models

In this project we will develop and study an increasingly complex model for a water molecule. The figure illustrates the charge distribution around a water molecule as calculated by jmol[3] in the molview.org[4] application, where red are negative charges and blue are positive charges. The dipole moment of water is 1.85D, where D is a unit called Debye: $1\text{D} \simeq 3.33564 \cdot 10^{-30}\text{Cm}$. We will now build a model for a water molecule.



a) How would you model a water molecule as a set of point charges based on the illustration in the figure. (You need to look up realistic numbers and make your own assumptions here).

[3] http://jmol.org
[4] http://molview.org

b) Write a Python script based on your model to find the electric field around the water molecule.

c) For a dipole, the field far ($r \gg d$) from the dipole along the dipole axis is $2p/(4\pi\epsilon_0 r^3)$, where $p = dq$ is the dipole moment. How can you use this to estimate the dipole moment from the electric field?

d) Estimate the dipole moment of your water model and compare with the $p = 1.85$D. Comment on this result.

e) There are many different types of water models used for molecular-scale modeling where the water molecule is modelled as various set of point charges. Look up and describe the models called SPC and TIP4P. Calculate the electric field for from these two models and compare them. Why do you think the TIP4P model was introduced? Do you think the SPC model an exact description of the system?

Exercise 1.30: Ionic bonds

(From Hornyak and Marion.)

An ionic bond occurs between two charged atoms separated by a distance $2a$. For singly charged ions, ions with the charge $\pm e$, the dipole moment would be $2ae$. We often measure molecular dipole moments in units of eÅ, where e is the charge of an electron, $e = 1.602 \times 10^{-19}$C and one Angstrom is 10^{-10}m. Let us now assume that we know the interatom separation, which can be establish by independent methods such as light scattering. This means that we know $2a$. In this case, if the dipole moment is different from $2ae$ it means that only a part of the charge is displaced or that the charges are not fully displaced the interatomic distance. It depends on the degree the bond is not completely ionic. It is common to characterize the bond by how large a portion it is ionic, that is, by the ratio $p/(2de)$, where p is the measured dipole moment of the molecule. Most bonds are partly ionic and partly covalent.

For example, the lithium fluiride (LiF) diatom has an interatomic separation of $2d = 1.52$Å and a dipole moment $p = 1.39e$Å, while hydrogen iodide (HI) has an interatomic separation $2d = 1.62$Å and a dipole moment $p = 0.080e$Å.

a) What is the degree of ionic bonding in these two molecules?

b) A water molecule has an angle $\theta \simeq 105°$ between the two hydrogen atoms. The O-H distance is $2s = 0.97$Å. The observed dipole moment of H_2O is $p = 0.387e$Å. What fraction of the O-H bond is ionic?

2 Electric potential

In this chapter we will introduce the concept of electric (scalar) potential. This is a powerful and important concept in electromagnetism. We will develop the concept in analogy with your intuition from gravitational fields and potential energy. We will define the electric potential and demonstrate how we can find the potential from the electric field and the electric field from the potential. And we will introduce the potential from a single charge, from a set of charges and from continuous charge distributions. In many situations, it is easier to calculate the electric potential than the electric field. We will therefore use the potential as a method to find the electric field. We will demonstrate how to find the electric potential numerically, visualize the potential and calculate the corresponding electric field. Hopefully, when you have finished this chapter you will have built an intuition for the electric potential and skills on how to calculate it. And you should be able to use the potential to analyze more complex electrostatic situations.

2.1 Motivation for the electric potential

You most probably have heard the term potential directly or indirectly. You may know that a battery has a voltage of 1.5 Volts. The term voltage refers to the electric potential between the two poles of the battery. Similarly, you may have heard about high voltage as found in power lines or in lightning strikes (Fig. 2.1). How does this concept of voltage correspond to the concept of the electric field? We will try to build this intuition here.

Fig. 2.1 Electric potentials in (a) A 1.5 V battery, (b) a 10 kV power line, (c) a 10 GV lightning strike.

You have developed an intuition for the relation between (conservative) forces and potential energy in mechanics. The relation between the electric field and the electric potential (the voltage) is similar. Gravity is a conservative force field. It is a force field in the sense that it is a force acting everywhere in space on masses due to particles with masses. It is conservative in the sense that the work done by the force is independent on the path, and it is therefore possible to introduce a potential energy. You may recall that for a conservative force, $\mathbf{F}$, the potential energy in a point A was defined as the line integral along an arbitrary path from an (arbitrary) reference point 0 to a point A:

$$U(A) - U(0) = \int_A^0 \mathbf{F} \cdot \mathrm{d}\mathbf{l} \tag{2.1}$$

We can therefore determine the potential energy from the force field. Similarly, we recall that $\mathbf{F} = -\nabla U$, so we can find the force field from the potential energy. The potential energy provided us with a different way to reason about the motion of gravitational forces — a supplement to reasoning with forces which proved useful in many situations.

We will build on this intuition when we introduce the electrical potential. We will first demonstrate that electrostatic forces are convervative and use this to introduce a potential for the electric field.

2.1.1 Electric forces are conservative

First, let us demonstrate that the force from an electric field is conservative, that is, that the work done by the force does not depend on the path taken. Coloumb's law describes the interactions between two charges: a charge Q_1 setting up the field and a charge q moving in the field. Let us place the origin of the coordinate system at Q_1. The force on q from the field from Q_1 is then:

$$\mathbf{F} = q\mathbf{E} = q\frac{Q_1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}}{R^2} \ . \tag{2.2}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_1 = \mathbf{r}$, since $\mathbf{r}_1 = \mathbf{0}$. The work on a charge q as it moves along a path C from A to B is illustrated in Fig. 2.2 and is defined as:

$$W_{AB} = \int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{l} = q\frac{Q_1}{4\pi\epsilon_0}\int_A^B \frac{\hat{\mathbf{R}}\cdot \mathrm{d}\mathbf{l}}{R^2} \tag{2.3}$$

Let us show that the path does not depend on the path C: We approximate the path by dividing it into small pieces that follow a circular arc with Q_1 as its center, or a radial path directly towards Q_1. We have illustrated a few such segments in Fig. 2.2. The total integral is (approximately) the sum of the integrals along these smaller paths.

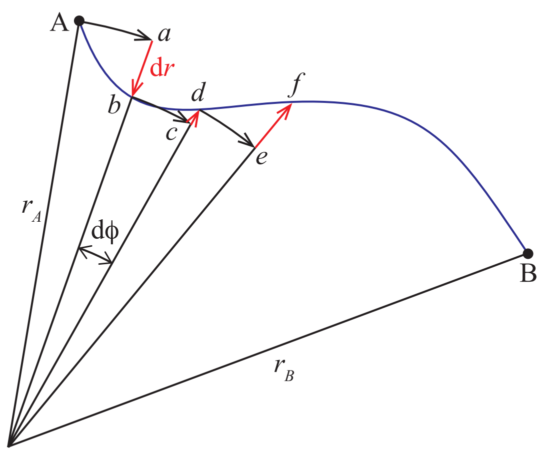


Fig. 2.2 Illustration of the work done on a charge q as it moves along the path C from A to B.

Along a **circular segment**, such as the segment from b to c, $\mathbf{R}$ points radially from Q_1, whereas the direction of the path is along the circle segment of radius r, $\mathrm{d}\mathbf{l} = r\mathrm{d}\phi\hat{\boldsymbol{\phi}}$, where $\hat{\boldsymbol{\phi}}$ is a unit vector along the circular arc. But the circular arc is normal to $\mathbf{R}$. Therefore, $\hat{\mathbf{R}}\cdot\mathrm{d}\mathbf{l}$ is zero for this segment. The same is true for all the segments that are along circular paths centered on Q_1. These segments will not contribute to the integral.

Along a **radial segment**, such as the segment from a to b, the displacement $\mathrm{d}\mathbf{l}$ is along the vector $\mathbf{R}$: $\mathrm{d}\mathbf{l} = \mathrm{d}l\hat{\mathbf{R}} = \mathrm{d}r\hat{\mathbf{R}}$, where $\mathrm{d}r$ is the change in the distance r from the origin where Q_1 is placed. This integral is therefore simply in the scalar distance r to Q_1. The same is true for all the segments that are along radial paths centered on Q_1. The total integral is therefore

$$W_{AB} = q\frac{Q_1}{4\pi\epsilon_0}\int_{r_A}^{r_B}\frac{\mathrm{d}r}{r^2} = -q\frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) . \tag{2.4}$$

This means that work does not depend on the path C, but only on the end points. We recall from mechanics that we call such a force field a *conservative* force field, and that we can introduce a potential energy for the force field from a single charge as the work needed to move the charge from a point A to a reference point:

$$U_A = W_{A,\mathrm{ref}} = U(r) . \tag{2.5}$$

We can use the *superposition principle* to argue that if this is true for a single charge, it must also be true for any charge distribution through the superposition principle: The total work is the sum of the work done by the individual forces $\mathbf{F}_i$ from each charge, Q_i:

$$U_A = W_{A,\mathrm{ref}} = \int_A^{\mathrm{ref}} (\mathbf{F}_1 + \mathbf{F}_2 + \ldots)\,\mathrm{d}\mathbf{l} \tag{2.6}$$

$$= \int_A^{\mathrm{ref}} \mathbf{F}_1 \cdot \mathrm{d}\mathbf{l} + \int_A^{\mathrm{ref}} \mathbf{F}_2 \cdot \mathrm{d}\mathbf{l} + \ldots \tag{2.7}$$

$$= W_1 + W_2 + \ldots = U_{A,1} + U_{A,2} + \ldots \tag{2.8}$$

This is the potential energy for a test particle with charge q. We can rewrite this in terms of the electric field instead, since $\mathbf{F} = q\mathbf{E}$:

$$U_A = \int_A^{\mathrm{ref}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = q\int_A^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = qV_A , \tag{2.9}$$

where we have introduced V_A, which we call the electric potential. The electric potential is therefore related to the potential energy, just as the electric field is related to the force: We find the potential energy of a charge by multiplying the potential with the charge.

2.2 Properties of the electric potential

We define the *electric potential* $V(\mathbf{r})$ for an electic field $\mathbf{E}(\mathbf{r})$ as

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \,, \tag{2.10}$$

where the integral is a line integral along any curve from $\mathbf{r}$ to a reference point for which $V = 0$. It is common to choose the reference point infinitely far away.

- *The electric potential integral does not depend on the path, only on the end points.*

- *The electric potential obeys the superposition principle.*

The electric potential in a point A from a set of charges Q_i is the sum of the electric potential for each of the charges:

$$V_A = V_{A,1} + V_{A,2} \ldots = \sum_i V_{A,i} \tag{2.11}$$

- *The scalar potential is measured in joule/coulomb which is called Volt, V.*

The unit of the electric field is then V/m, volt per meter.

- *The integral around a closed path is zero.*

Because the integral $W_{AB}/q = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{l}$ is the same for all paths C from A to B, the integral for any closed loop is zero. Why? As illustrated in Fig. 2.3 we can always divide a closed path into two paths, one from A to B and one from B to A. The total integral along the path is the sum of these two integrals:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_{C_1} \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_{C_2} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0 \,, \tag{2.12}$$

where the second integral is the negative of the first because it goes in the opposite direction. Thus the total integral around a closed loop is zero.

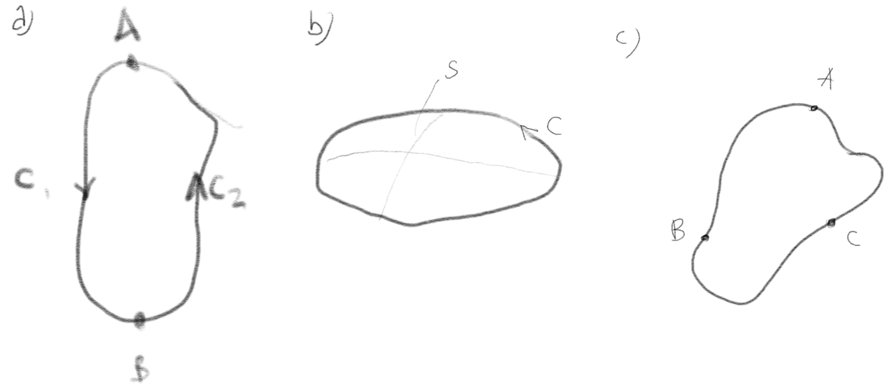


Fig. 2.3 (a) Illustration of how a closed loop can be divided into two curves C_1 from A to B and C_2 from B to A. (b) Illustration of a closed curve C and a corresponding surface S enclosed by C. (c) A circuit with three points A, B and C.

- *The curl of* $\mathbf{E}$ *is zero.*

Because the integral around any closed path is zero, we can apply Stoke's theorem:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 0 \; , \tag{2.13}$$

where S is a surface enclosed by the closed curve C as illustrated in Fig. 2.3. Since this is true for any surface (and any corresponding closed curve), we find that

$$\nabla \times \mathbf{E} = 0 \; . \tag{2.14}$$

This is true for **static** fields. We will later find modifications to this for time-varying fields.

- *The electric potential is only defined relative to a reference point.*

Notice that for finite charge distributions it is common to choose the reference point to be at infinity. A consequence is that the difference in potentials, V_{AB}, between two points A and B does not depend on the reference point:

$$V_{AB} = V_A - V_B = \int_A^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} - \int_B^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{2.15}$$

$$= \int_A^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_{\mathrm{ref}}^{B} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{2.16}$$

$$= \int_A^{B} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \; . \tag{2.17}$$

- *The electric potential obeys Kirchoff's law of voltages: The sum of potential differences along a closed loop (such as a circuit) is zero.*

This is a consequence of the path integral of the electric field around a closed path being zero. For the case in Fig. 2.3 we find:

$$V_{AB}+V_{BC}+V_{CA} = \oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \int_A^B \mathbf{E}\cdot\mathrm{d}\mathbf{l}+\int_B^C \mathbf{E}\cdot\mathrm{d}\mathbf{l}+\int_C^A \mathbf{E}\cdot\mathrm{d}\mathbf{l} = 0 \quad (2.18)$$

2.3 Electric potentials from charge distributions

Let us now see how to find the electric potential $V(\mathbf{r})$ at a position $\mathbf{r}$ due to various charge distributions: a single charge; a set of charges; and charge densities for lines, surfaces and volumes.

2.3.1 Example: Single charge Q at the origin

We find the electric potential by applying the definition. We choose the reference point to be at infinity:

$$V_A = \int_A^\infty \mathbf{E}\cdot\mathrm{d}\mathbf{r} \, . \quad (2.19)$$

For a single charge Q at the origin, the electric field is:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} \, , \quad (2.20)$$

which gives

$$V(\mathbf{r}) = \int_{\mathbf{r}}^\infty \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot\mathrm{d}\mathbf{r}}{r^3} = \int_r^\infty \frac{Q}{4\pi\epsilon_0}\frac{\mathrm{d}r}{r^2} = \frac{Q}{4\pi\epsilon_0 r} \, . \quad (2.21)$$

Here, we have used that $\mathbf{r}\cdot d\mathbf{r} = r\,dr$. This result demonstrates that the electric potential decays as $1/r$, while the electric field decays as $1/r^2$ — as we would guess because the potential is the integral of the field. This is useful rule of thumb that can be extended also to e.g. dipoles where the field decays at $1/r^3$, while the potential decays as $1/r^2$.

2.3.2 Example: Single charge Q at $\mathbf{r}_1$

How does this result change if the charge Q_1 is in the position $\mathbf{r}_1$ instead? We introduce a new coordinate system $\mathbf{r}' = \mathbf{r} - \mathbf{r}_1$. In this coordinate system, the charge is in the origin, hence the potential is

$$V(r') = \frac{Q_1}{4\pi\epsilon_0 r'} \,, \tag{2.22}$$

and we insert $\mathbf{r}' = \mathbf{r} - \mathbf{r}_1$, getting

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_1|} \,. \tag{2.23}$$

2.3.3 Example: A set of charges Q_i at $\mathbf{r}_i$

For a set of charges we can use the superposition principle for the electric potential. The total potential is therefore

$$V = \sum_i V_i = \sum_i \frac{Q_i}{4\pi\epsilon_0 R_i} \,, \tag{2.24}$$

where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$. As a function of $\mathbf{r}$ we get:

$$V(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_i|} \,. \tag{2.25}$$

2.3.4 Example: Continuous charge distibutions

We can extend this also to a continuous distribution of charges:
For a *volume charge density* ρ, we get:

$$V(\mathbf{r}) = \int_v \frac{\rho \, \mathrm{d}v}{4\pi\epsilon_0 R} \,, \tag{2.26}$$

where R is the distance from the point $\mathbf{r}$ to the volume element dv.
For a *surface charge density* ρ_A, we get:

$$V(\mathbf{r}) = \int_S \frac{\rho_A \, \mathrm{d}S}{4\pi\epsilon_0 R} \,, \tag{2.27}$$

where R is the distance from the point $\mathbf{r}$ to the surface element dS.

For a *line charge density* ρ_l, we get:

$$V(\mathbf{r}) = \int_C \frac{\rho_l \, \mathrm{d}l}{4\pi\epsilon_0 R} \,, \tag{2.28}$$

where R is the distance from the point $\mathbf{r}$ to the line element dl.

2.4 Relation between electric potential and electric field

You may recall from mechanics that there is a relationship between the force $\mathbf{F}$ and the potential energy: $\mathbf{F} = -\nabla U$. This relation transfers directly to the electric potential:

$$\mathbf{E} = \frac{\mathbf{F}}{q} = -\frac{1}{q}\nabla U = -\nabla\left(\frac{U}{q}\right) \tag{2.29}$$

$$= -\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right) \,. \tag{2.30}$$

Relation between the electric field and the electric potential

The electric potential $V(\mathbf{r})$ can be found from the electric field $\mathbf{E}$ by:

$$V(\mathbf{r}) = \int_r^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \,, \tag{2.31}$$

and the electric field $\mathbf{E}(\mathbf{r})$ can be found from the electric potential $V(\mathbf{r})$ by:

$$\mathbf{E} = -\nabla V \,. \tag{2.32}$$

Notice that in order to find the electric field from the potential, we need to know not only the electric potential in a single point. We must know how the potential varies around that point because the field depends on the (spatial) derivatives of the electric potential. Similarly, in order to find the electric potential from the field, we need to know not only the electric field in a single point or in a few points. We must know the how the electric field varies in space in order to calculate the path integral to determine the potential.

The compactness of the electric potential. It may be surprising that the electric field, which is a vector field with three components E_x, E_y, E_z, can be represented by the scalar electric potential, $V(x, y, z)$. This is because there are additional conditions for the electric field: It must also satisfy the condition that $\nabla \times \mathbf{E} = 0$, which essentially brings three additional differential equations for the electric field. Notice also that because the electric field can be written as the gradient of the potential, the curl of the electric field is necessarily zero, because the curl of a gradient is identically zero: $\nabla \times (\nabla V) = 0$.

Proof of the relation between field and potential. If we look at the potential difference between A and a point B which is an infinitesimal displacement (dx, dy, dz) away, we get:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = V_B - V_A \tag{2.33}$$

$$= -\int_A^B \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\left(E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z\right) \ , \tag{2.34}$$

Since this is valid for any choice of dx, dy, and dz, we get that

$$E_x = -\frac{\partial V}{\partial x} \ , \ E_y = -\frac{\partial V}{\partial y} \ , \ E_z = -\frac{\partial V}{\partial z} \ . \tag{2.35}$$

Method for finding the electric potential

There is a robust method to find the electric potential $V(\mathbf{r})$ from a discrete or continuous charge distribution. Given the charges and their positions, we find the potential using the following procedure:

- Model the problem: Choose a coordinate system to describe the system that reflects the symmetries of the system.

Discrete charges:

- Find the distance R_i from the observation point $\mathbf{r}$ to charge Q_i at $\mathbf{r}_i$: $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$ for each charge.
- Sum the contributions from all charges Q_i:

$$V(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_i|} \ . \tag{2.36}$$

Continous charges:

- Describe a small element dq at a position $\mathbf{r}'$. Relate dq to a volume; surface; line element $\mathrm{d}q = \rho \mathrm{d}V'$; $\rho_s \mathrm{d}S'$; $\rho_l \mathrm{d}l'$.

- Find the distance R from the observation point $\mathbf{r}$ to the small element at $\mathbf{r}'$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.
- Integrate over all $\mathbf{r}'$ that make up the body of interest analytically or numerically. Remember that the integration variable is $\mathbf{r}'$ and not $\mathbf{r}$! For a volume charge density this is:

$$V(\mathbf{r}) = \int_v \frac{\rho \mathrm{d}V'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \ . \tag{2.37}$$

2.5 Applications of the electric potential

This relationship between the electric potential and the eletric field opens for a new way to calculate the electric field: First we find the electric potential, and then we find the field from the potential. In many cases, it is mathematically simpler to find the potential than to find the field. This approach may also be used numerically.

2.5.1 Interpreting the electric potential

A linear potential. Fig. 2.4 illustrates the electric potential V for three different cases. In Fig. 2.4a the electric potential decreases linearly from a high potential at $x = 0$ to a lower potential at $x = L$. How can we use the potential to gain insight into the electric field? We know that the electric field is given at $E_x = -\partial V/\partial x$ in the one-dimensional case. The electric field therefore points from a high potential toward lower potentials. In Fig. 2.4a the electric field points to the right as illustrated by the arrow. The field is proportional to the slope of the potential. Here, the slope is constant, which means that the electric field is uniform. The electric field tells us what direction a force on a positive charge acts. Positive charges will therefore move from high to low potential. Negative charges will move from low to high potential.

One-dimensional potential. Fig. 2.4b illustrates a more complicated one-dimensional electric potential. The electric field points in the direction the field is decreasing and its magnitude is illustrated by the length of the arrows.

Two-dimensional potential. Fig. 2.4c illustrates a two-dimensional electric potential illustrated by its *equipotential surfaces*. An *equipotential surface* is a surface where the potential is a constant: $V =$ const.. The

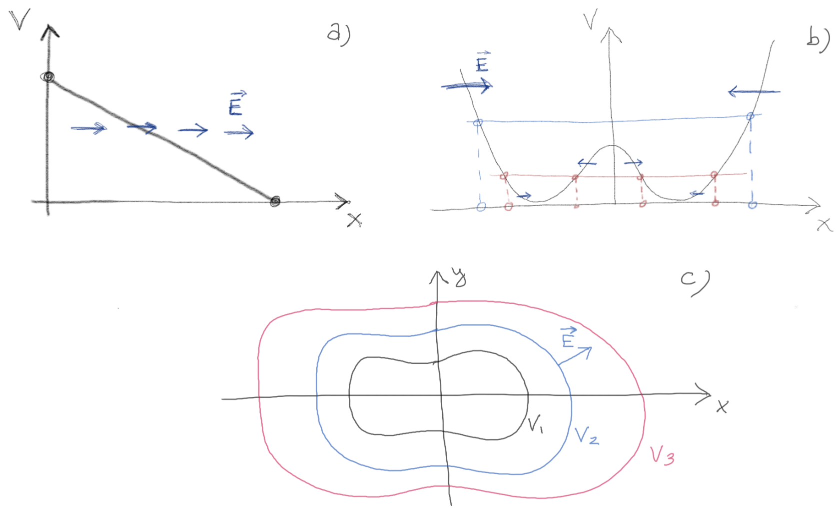


Fig. 2.4 Illustration of equipotential surfaces and the eletric field for a linear potential (a), a one-dimensional potential (b), and a two-dimensional potential (c).

gradient is normal to the equipotential surface and points in the direction that V increases the fastest. Since $\mathbf{E} = -\nabla V$, we see that the electric field is normal to the equipotential surface and points in the direction that V decreases the fastest. The electric field points from high potential values towards lower potential values. Fig. 2.4c illustrates the equipotential curves for a two-dimensional electric field. How can you see from the equipotential curves where the gradient is steep? This is seen by regions where the equipotential surfaces are close: The gradient and hence the electric field has a larger magnitude along the y-axis than along the x-axis because the distance between the equipotential lines (surfaces) are longer along the x-axis than along the y-axis. In the following examples, we will calculate the electric potential and use equipotential curves and surfaces to visualize the potential.

2.5.2 Example: Dipole

The electric potential from a dipole. *Find the electric potential from a dipole consisting of a charge Q at $y = a$ and a charge $-Q$ at $y = -a$.*

The system is illustrated in Fig. 2.5. We find the potential using the superposition principle, adding the contributions from the two charges. We want to find the potential in the point $\mathbf{r}$. The two charges are at

$\mathbf{r}_1 = a\hat{\mathbf{y}}$ and $\mathbf{r}_2 = -a\hat{\mathbf{y}}$. The total potential is

$$V = V_1 + V_2 = \frac{Q}{4\pi\epsilon_0 R_1} + \frac{-Q}{4\pi\epsilon_0 R_2} \;, \tag{2.38}$$

where $\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1$ and $\mathbf{R}_2 = \mathbf{r} - \mathbf{r}_2$.

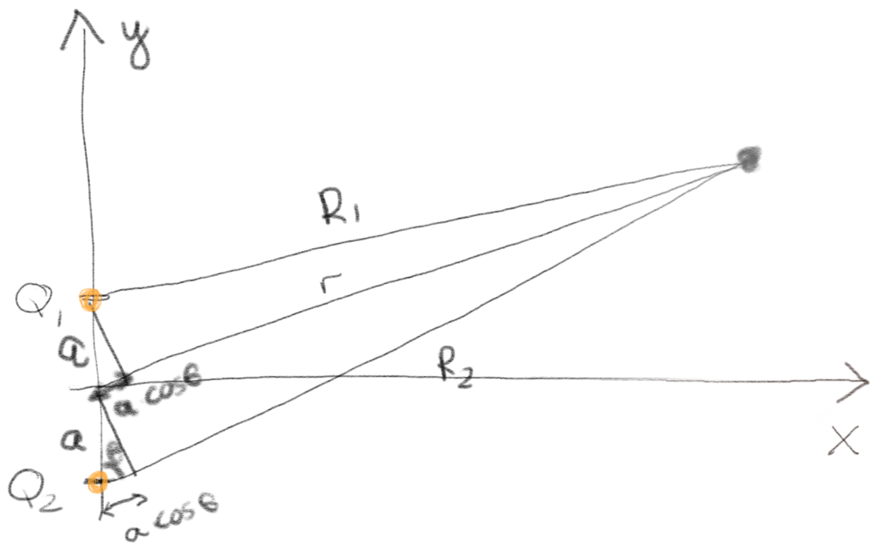


Fig. 2.5 Illustration of dipole.

Electric potential for $r \gg a$**.** *Find an approximation for the electric field at* $\mathbf{r}$ *when* $r \gg a$*.*

It is not immediately obvious how we should approximate the potential for large r. We need to know a *trick*. I do not expect you to discover such a trick by yourself, but when you have seen it here, you may be able to use such a trick in another situation.

The trick is to rewrite the two length R_1 and R_2 using the angle θ from the axis of the dipole to the point $\mathbf{r}$. When $r \gg a$ we can assume that the triangles in the figure are approximately normal. We see from Fig. 2.5 that $R_1 = r - a\cos\theta$ and $R_2 = r + a\cos\theta$.

We can rewrite the potential as

$$V = \frac{Q}{4\pi\epsilon_0 R_1} + \frac{-Q}{4\pi\epsilon_0 R_2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \;. \tag{2.39}$$

We insert the expressions for R_1 and R_2 and also approximate $R_1 R_2 = r^2 - a^2\cos^2\theta \simeq r^2$ (when $r \gg a$), getting

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \simeq \frac{Q}{4\pi\epsilon_0}\left(\frac{2a\cos\theta}{r^2}\right) . \tag{2.40}$$

(Notice that the dipole decays as r^{-2}, whereas a single charge decays as r^{-1}.)

Again, we introduce a *trick* to rewrite the $\cos\theta$ expression as $2Qa\cos\theta = 2Qa\hat{\mathbf{y}}\cdot\hat{\mathbf{r}}$. Therefore, we can introduce the quantity $\mathbf{p} = Q2a\hat{\mathbf{y}}$, which we call the *dipole moment*, so that the electric potential becomes:

$$V(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} \tag{2.41}$$

Visualization of the dipole potential. We visualize the potential using the same approach as for the electric field. First, we introduce a function to find the potential in a point $\mathbf{r}$ from a set of charges Q_i at positions $\mathbf{r}_i$. We know that the potential from this charge is

$$V(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0 R_i} , \tag{2.42}$$

where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$. We calculate this directly in Python:

```
import numpy as np
import matplotlib.pyplot as plt

def epotlist(r,Q,R):
    # Find V*4*pi*epsilon0 at r from a charges Q at positions R
    V = 0
    for i in range(len(R)):
        Ri = r - R[i]
        qi = Q[i]
        Rinorm = np.linalg.norm(Ri)
        V = V + qi/Rinorm
    return V
```

We set up the charges for the dipole:

```
# Setup charges
R = []
Q = []
r0 = np.array([1,0])
q0 = 1
R.append(r0)
Q.append(q0)
r0 = np.array([-1,0])
q0 = -1
R.append(r0)
Q.append(q0)
```

Then we construct a lattice of $\mathbf{r}_j$-values and calculate the value $V_j = V(\mathbf{r}_j)$ for each position on the lattice:

```
Lx = 3
Ly = 3
N = 21
x = np.linspace(-Lx,Lx,N)
y = np.linspace(-Ly,Ly,N)
rx,ry = np.meshgrid(x,y)
# Set up electric potential
V = np.zeros((N,N),float)
# Calculate the potential
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i]])
    V.flat[i] = epotlist(r,Q,R)
```

Finally, we visualize the resulting potential as a two-dimensional image:

```
plt.figure(figsize=(8,8))
plt.contour(rx,ry,V,200)
plt.colorbar()
plt.axis('equal')
```

The resulting plot is shown in Fig. 2.6.

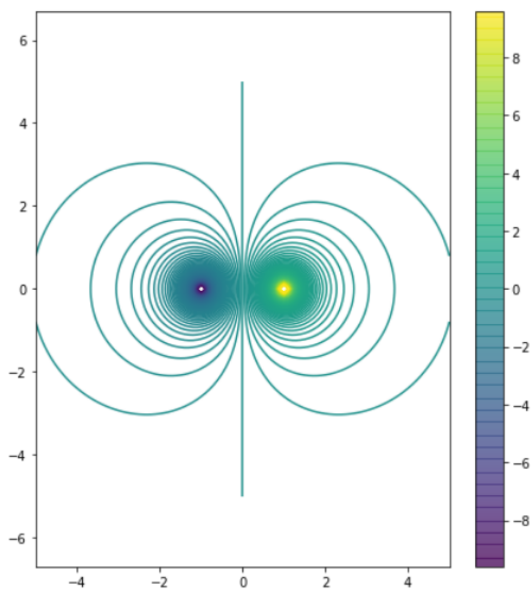


Fig. 2.6 Plot of the equipotential surfaces for a dipole.

Numerical calculation of the field from the potential for a dipole. Based on these results, we want to estimate the electric field using $\mathbf{E} = -\nabla V$. The gradient operator, ∇, is provided in Python. We can therefore find the gradient directly, using `gradient`, but we must now remember that in Python the first coordinate in a matrix refers to the y-position, and the second coordinate refers to the x-position, whereas we usually use the opposite order in Cartesian coordinates (x first and then y, (x, y)). We must therefore remember to collect `Ex,Ey` in reverse order from the gradient function:

```
Ey,Ex = np.gradient(-V) # Notice reverse order Ey,Ex
# Calculate field magnitude and unit vectors
Emag = np.sqrt(Ex**2 + Ey**2)
minlogEmag = min(np.log10(Emag.flat))
scaleE = np.log10(Emag) - minlogEmag
uEx = Ex / Emag
uEy = Ey / Emag
# Visualize using both arrows and colors
levels = np.arange(-3.5, 1.5+0.2, 0.2)
cmap = plt.cm.get_cmap('plasma')
plt.figure(figsize=(16,8))
ax1 = plt.subplot(1,2,1)
plt.contourf(rx,ry,V,10, cmap=cmap, levels=levels, extend='both')
plt.quiver(rx,ry,uEx*scaleE,uEy*scaleE)
ax1.set_aspect('equal', 'box')
ax2 = plt.subplot(1,2,2)
plt.contourf(rx,ry,V,10, cmap=cmap, levels=levels, extend='both')
plt.streamplot(rx,ry, Ex, Ey)
ax2.set_aspect('equal', 'box')
```

Here we have used logarithmic lengths of the vectors to visualize the magnitude of the field, we have scaled the color scale for the potential and used a different colormap. The resulting plot is shown in Fig. 2.7.

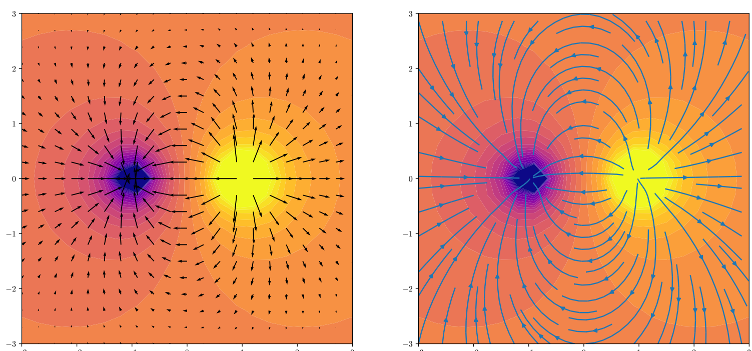


Fig. 2.7 Plot of the equipotential surfaces for a dipole.

Sometimes you would like to calculate the field with a high resolution, but then only draw arrows in some of the points you have calculated. This is done using the `slice` function in Python. The following only plots every 4-th arrow:

```
skip = (slice(None, None, 4), slice(None, None, 4))
plt.quiver(rx[skip],ry[skip],uEx[skip]*scaleE[skip],uEy[skip]*scaleE[skip])
```

Symbolic calculation of the field from a dipole. We can also find the electric field directly from the potential as $\mathbf{E} = -\nabla V$. We demonstrate how we can do this using Sympy. We use the package `CoordSys3D` to define a three-dimensional Cartesian coordinate system. The various unit vectors are then denoted `R.i`, `R.j` and `R.k` and the coordinates x, y, z are `R.x`, `R.y`, `R.z` respectively. The $\mathbf{r}$-vector is then `R.x*R.i + R.y*R.j + R.z*R.k`. We introduce the dipole moment $\mathbf{p} = (a, b, c)$ which is `a*R.i + b*R.j + c*R.k` and define the potential as (where we have removed the $1/(4\pi\epsilon_0)$ prefactor for simplicity):

```
import sympy as sy
from sympy.vector import CoordSys3D,gradient
R = CoordSys3D('R')
r = R.x*R.i + R.y*R.j + R.z*R.k
a = sy.symbols('a')
b = sy.symbols('b')
c = sy.symbols('c')
p = a*R.i + b*R.j + c*R.k
V = r.dot(p)/r.dot(r)**1.5
E = -sy.gradient(V)
```

The result is:

$$\mathbf{E}' = \left(\frac{3.0 x_R (x_R a + y_R b + z_R c)}{\left(x_R^2 + y_R^2 + z_R^2\right)^{2.5}} - \frac{a}{\left(x_R^2 + y_R^2 + z_R^2\right)^{1.5}} \right) \hat{\mathbf{x}} \quad (2.43)$$

$$+ \left(\frac{3.0 y_R (x_R a + y_R b + z_R c)}{\left(x_R^2 + y_R^2 + z_R^2\right)^{2.5}} - \frac{b}{\left(x_R^2 + y_R^2 + z_R^2\right)^{1.5}} \right) \hat{\mathbf{y}} \quad (2.44)$$

$$+ \left(\frac{3.0 z_R (x_R a + y_R b + z_R c)}{\left(x_R^2 + y_R^2 + z_R^2\right)^{2.5}} - \frac{c}{\left(x_R^2 + y_R^2 + z_R^2\right)^{1.5}} \right) \hat{\mathbf{z}} \quad (2.45)$$

which we recognize can be simplified to:

$$\mathbf{E}' = \frac{3\mathbf{r}\,(\mathbf{r} \cdot \mathbf{p})}{r^5} - \frac{\mathbf{p}}{r^3} \; . \quad (2.46)$$

We reintroduce the prefactor $1/(4\pi\epsilon_0)$, getting,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3\mathbf{r}\,(\mathbf{r}\cdot\mathbf{p})}{r^5} - \frac{\mathbf{p}}{r^3}\right) . \tag{2.47}$$

2.5.3 Example: Charged ring

Find the electric potential from a uniformly charged ring with charge Q and radius a.

Analytical solution along the axis of the ring. The system is illustrated in Fig. 2.8. The ring can be described as a line charge with a line charge density $\rho_l = Q/L = Q/(2\pi a)$. The contribution to the potential from an infinitesimal element $d\phi$ of charge $dq = \rho_l(ad\phi)$ is

$$dV = \frac{dq}{4\pi\epsilon_0 R} , \tag{2.48}$$

where the length R is the distance from the point $\mathbf{r} = (0,0,z)$ on the axis to a point at the ring. We see from Fig. 2.8 that $R^2 = a^2 + z^2$ for all angles ϕ. The potential is therefore

$$V = \int_0^{2\pi} \frac{\rho_l a \mathrm{d}\phi}{4\pi\epsilon_0\left(a^2+z^2\right)^{1/2}} \tag{2.49}$$

$$= \frac{\rho_l a}{4\pi\epsilon_0\left(a^2+z^2\right)^{1/2}}\int_0^{2\pi}\mathrm{d}\phi \tag{2.50}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{\left(a^2+z^2\right)^{1/2}} . \tag{2.51}$$

We see that when $z \gg a$, the potential is the same as for a point charge Q located at the origin.

Numerical calculation of the potential at any point in space. In order to find the electric field $V(\mathbf{r})$ in an arbitrary point $\mathbf{r}$, we need to sum up the contributions from all elements $d\theta$ of the ring. We will first find the contribution from one element, and then integrate numerically to find the contributions from all elements.

A small element $\mathrm{d}\,\theta$ of the ring has a length $\mathrm{d}\,l = a\mathrm{d}\,\phi$, a charge $\mathrm{d}\,q = \rho_l \mathrm{d}\,l = \rho_l a \mathrm{d}\,\phi$, and is at a position $\mathbf{r}' = (a\cos\phi, a\sin\phi, 0)$, where ϕ is the angle with the x-axis. The contribution from this element to the electric potential is:

$$\mathrm{d}\,V = \frac{\mathrm{d}\,q}{4\pi\epsilon_0 R} = \frac{\rho_l a\ d\,\phi}{4\pi\epsilon_0 R} , \tag{2.52}$$

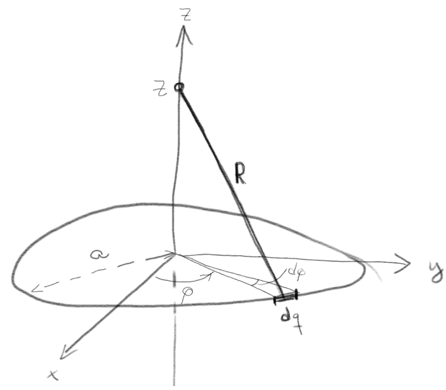


Fig. 2.8 Illustration of a charged ring.

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

To find the total electric field, we sum the contributions from $\phi = 0$ to $\phi = 2\pi$. We divide the interval into N pieces of size $\mathrm{d}\,\phi = 2\pi/N$, and sum the contributions. This is done in the following Python function:

```
import numpy as np
import matplotlib.pyplot as plt
def potentialfromring(r,a,rhol,N):
    # dV = \frac{\rho a d\phi}{4 \pi \epsilon_0 R}
    V = 0
    dphi = 2*np.pi/N
    for i in range(N):
        phi = i*dphi
        dl = a*dphi
        r0 = np.array([a*np.cos(phi),a*np.sin(phi),0])
        R = np.linalg.norm(r-r0)
        dV = rhol*dl/R
        V = V+dV
    return V
```

We use this function to calculate the potential in the xz-plane with 90 points from -3 to 3 for both x and z:

```
# Calculate potential in the xz plane
NL = 90
N = 100
L = 3
a = 1.0
rhol = 1.0
x = np.linspace(-L,L,NL)
y = np.linspace(-L,L,NL)
rx,ry = np.meshgrid(x,y)
# Set up electric potential
V = np.zeros((NL,NL),float)
```

```
# Calculate the potential
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],0,ry.flat[i]])
    V.flat[i] = potentialfromring(r,a,rhol,N)
```

We can then visualize the result using the same methods as demonstrated above. The resulting visualization of the vector field and the stream lines are shown in Fig. 2.9.

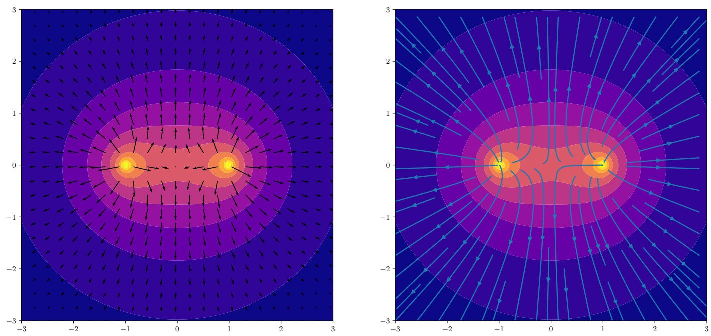


Fig. 2.9 Illustration of the electric potential and the electric field from a charged ring.

2.5.4 Example: Charged line segment

Find the electric potential from a finite line charge

First, let us specify the system. We assume the line has a length L and a line charge density ρ_l. We place the line along the y-axis from $y = -L/2$ to $y = L/2$ as illustrated in Fig. 2.10. Then, let us find the contribution $\mathrm{d}\,V$ to the potential in a point $\mathbf{r}$ from a small element of length $\mathrm{d}\,y'$ at the position $\mathbf{r}' = (0, y', 0)$:

$$\mathrm{d}\,V = \frac{\mathrm{d}\,q}{4\pi\epsilon_0 R} \;, \tag{2.53}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. We will now use this to find the electric potential at a point on the x-axis analytically, and then find the electric potential for any point numerically.

Analytical solution on an axis normal to the line. For a point $\mathbf{r} = (x, 0, 0)$, the contribution to the electric potential from an element at y' is:

$$\mathrm{d}\,V = \frac{dq}{4\pi\epsilon_0 R} \;, \tag{2.54}$$

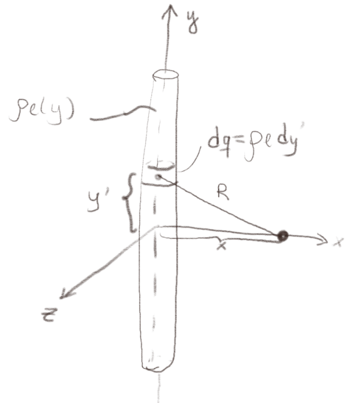


Fig. 2.10 Illustration of a charged line along the y-axis.

where $\mathbf{R} = (x, 0, 0) - (0, y', 0) = (x, -y', 0)$ and $R = (x^2 + (y')^2)^{1/2}$. This gives

$$\mathrm{d}\,V = \frac{\rho_l(y')\mathrm{d}\,y'}{4\pi\epsilon_0 \left(x^2 + (y')^2\right)^{1/2}} \,. \tag{2.55}$$

We find the potential by summing up all these contributions from $y' = -L/2$ to $y' = L/2$. Notice that the integration variable is y' and not x!

$$V = \int_{-L/2}^{L/2} \frac{\rho_l(y')\mathrm{d}\,y'}{4\pi\epsilon_0 \left(x^2 + (y')^2\right)^{1/2}} \,. \tag{2.56}$$

How can we solve this integral? You may look it up in a table of integrals or simply use Sympy to solve it:

```
import sympy as sy
x = sy.Symbol('x')
y = sy.Symbol('y')
f = 1/sy.sqrt(x**2 + y**2)
V = sy.integrate(f,y)
print(V)
```

```
asinh(y/x)
```

We insert the end point $y' = -L/2$ and $y' = L/2$, getting

$$V = \frac{\rho_l}{4\pi\epsilon_0}\left(\operatorname{asinh}\left(\frac{L}{2x}\right) - \operatorname{asinh}\left(\frac{-L}{2x}\right)\right) \,. \tag{2.57}$$

This function is plotted in Fig. 2.11.

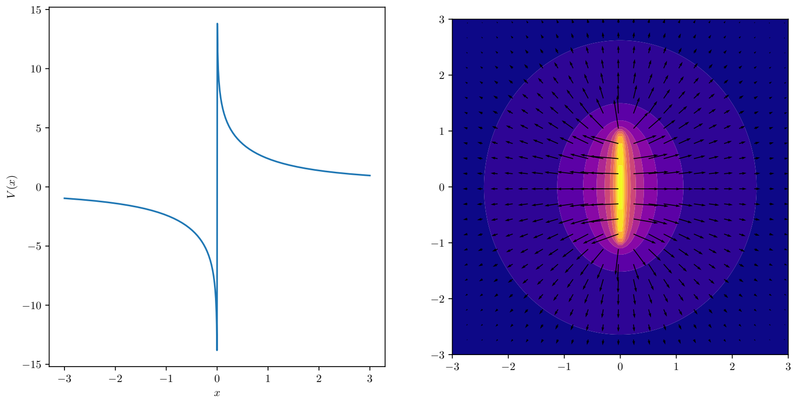


Fig. 2.11 Electric potential from a line charge from $y = -1$ to $y = 1$. (a) Plot of $V(x, 0, 0)$. (b) Plot of the electric potential and the electric field in the xy-plane.

Numerical solution for an arbitrary point. To find the solution at any point $\mathbf{r} = (x, y, z)$, we calculate the integral numerically. The contribution from a small element dy' at y' is:

$$\mathrm{d}\,V = \frac{\mathrm{d}\,q}{4\pi\epsilon_0 R}\;, \tag{2.58}$$

where $\mathbf{R} = (x, y, z) - (0, y', 0) = (x, y - y', z)$ and $R = (x^2 + (y - y')^2 + z^2)^{1/2}$. This gives

$$\mathrm{d}\,V = \frac{\rho_l(y')\,\mathrm{d}\,y'}{4\pi\epsilon_0\,(x^2 + (y - y')^2 + z^2)^{1/2}}\;. \tag{2.59}$$

We find the potential by summing up all these contributions from $y' = -L/2$ to $y' = L/2$. We divide the interval into N pieces of length $\mathrm{d}\,y' = L/N$ each and sum the contributions numerically:

```python
import numpy as np
import matplotlib.pyplot as plt
def potentialfromline(r,L,rhol,N):
    # dV = \frac{\rho dy}{4 \pi \epsilon_0 R}
    V = 0
    dl = L/N
    for i in range(N):
        y0 = -L/2+i*dl
        r0 = np.array([0,y0,0])
        R = np.linalg.norm(r-r0)
        dV = rhol*dl/R
        V = V+dV
    return V
```

We visualize the result by calculating the field in the xy-plane. The resulting field is shown in Fig. 2.11 using the same visualization methods as developed above.

```
NL = 90
N = 100
L = 3
l = 2.0
rhol = 1.0
x = np.linspace(-L,L,NL)
y = np.linspace(-L,L,NL)
rx,ry = np.meshgrid(x,y)
V = np.zeros((NL,NL),float)
# Calculate the potential
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i],0])
    V.flat[i] = potentialfromline(r,l,rhol,N)
```

2.5.5 Example: Three-dimensional visualization

So far, we have addressed how to visualize the potential and the electric field in two dimensions. It is, however, simple to visualize both the electric potential and the electric field in three dimensions using the module `plotly`.

Installing `plotly`. To install `plotly` using anaconda you write the following command in a terminal:

Terminal

```
conda install plotly
```

Generating a 3d potential field. The methods we have developed to compute the electric potential are easily extended to three dimensions. The `epotlist` function indeed already supports calculations in one, two, and three dimensions. We can therefore set up a potential from a single charge Q in the origin with the following program:

```
import numpy as np
def epotlist(r,Q,R):
    # Find V*4*pi*epsilon0 at r from a charges Q at positions R
    V = 0
    for i in range(len(R)):
        Ri = r - R[i]
        qi = Q[i]
        Rinorm = np.linalg.norm(Ri)
        V = V + qi/Rinorm
```

```
    return V
Q = []
R = []
Q1 = 1.0
R1 = np.array([0,0,0])
Q.append(Q1)
R.append(R1)
```

To visualize the potential, we need to calculate it on a three-dimensional grid of $\mathbf{r}$-values. We generate this grid using `np.meshgrid` as we did in two dimensions:

```
# Calculate the potential
Lx = 3
Ly = 3
Lz = 3
N = 21
x = np.linspace(-Lx,Lx,N)
y = np.linspace(-Ly,Ly,N)
z = np.linspace(-Lz,Lz,N)
rx,ry,rz = np.meshgrid(x,y,z)
```

We then need to loop through all the $\mathbf{r}$-values, and find the electric potential in each such point. This is done by using the `rx.flat[i]` method, where we only use a single variable `i` to loop through the entire three-dimensional matrix:

```
# Set up electric potential
V = np.zeros((N,N,N),float) # Initialize V
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i],rz.flat[i]])
    V.flat[i] = epotlist(r,Q,R)
```

Visualization of the potential with isosurfaces. In two dimensions, we visualized the electric potential $V(\mathbf{r})$ using isopotential contours. That is, we found the points $\mathbf{r}$ where $V(\mathbf{r}) = V_i$ for a set of values V_i of the potential. We know that the gradient ∇V is normal to the contour, so the contour provides a useful insight also to the electric field: The electric field is normal to the contour and since $\mathbf{E} = -\nabla V$, and ∇V points the direction the electric potential increases the most, then the electric field points in the direction the electric potential decreases the most.

This concept can be extend to three dimensions by plotting the isosurface, that is, the surfaces that consists of points $\mathbf{r}$ so that $V(\mathbf{r}) = V_i$ for a sequence of values V_i. Again, we can interpret these surfaces the same way as the contour lines in the two-dimensional plot. We visualize these using `plotly` with the following program:

```
import plotly.graph_objects as go
```

```
fig = go.Figure(data=go.Isosurface(
    x=rx.flatten(),
    y=ry.flatten(),
    z=rz.flatten(),
    value=V.flatten(),
    isomin=0.0,
    isomax=0.5,
    surface_count=5, # nr of isosurfaces
    caps=dict(x_show=False, y_show=False)
    ))
fig.show()
```

Notice how we use `flatten` inside the call to pass on the values for the x, y, and z positions of the grid values and the values of the potential at these points. The values `isomin` and `isomax` sets the lower and upper limits for the isosurface, and `surface_count` sets the number of isosurfaces. The resulting plot is shown in Fig. 2.12a.

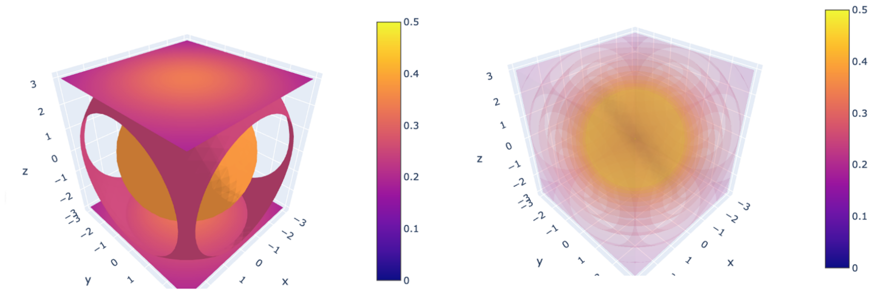


Fig. 2.12 (a) Plot of isosurfaces of $V(\mathbf{r})$ from a single charge $+Q$ in the origin. (b) Translucent plot of the electric potential $V(\mathbf{r})$ from a single charge $+Q$ in the origin.

Activity

Change the potential to that of a dipole with $Q_1 = 1$ at $\mathbf{r}_1 = (1, 0, 0)$ and $Q_1 = -1$ at $\mathbf{r}_2 = (-1, 0, 0)$ measured in dimensionless simulation units. Remember to change the range of isosurfaces, because the potential is not negative close to the negative charge and positive close to the positive charge.

Visualization of the potential using volume visualization. We can also visualize the isosurfaces by making them semi-transparent and color-coding them with the potential. This is done using the `volume` visualization tool in `plotly`. The following program visualizes the field from a single charge $Q = 1$ in the origin, and the resulting plot is show in Fig. 2.12b.

```
fig = go.Figure(data=go.Volume(
    x=rx.flatten(),
    y=ry.flatten(),
    z=rz.flatten(),
    value=V.flatten(),
    isomin=0.0,
    isomax=0.5,
    opacity=0.1,
    surface_count=20, # nr of isosurfaces
    ))
fig.show()
```

Visualization of the electric field. We can find the electric field from the electric potential using a numerical derivative approximation for $\mathbf{E} = \nabla(-V)$ using the `np.gradient` function in `numpy`:

```
Ey,Ex,Ez = np.gradient(-V)
```

Notice the ordering of the components to reflect the Python indexing scheme. We can visualize this field in three dimensions by drawing small arrows, in this case cones, the are directed in the direction of the field and colored by the magnitude of the field. This is done using the `cone` command in `plotly`:

```
fig = go.Figure(data=go.Cone(
    x=rx.flatten(),
    y=ry.flatten(),
    z=rz.flatten(),
    u=Ex.flatten(),
    v=Ey.flatten(),
    w=Ez.flatten(),
    sizemode="absolute",
    sizeref=2,
    anchor="tip",
    ))
fig.show()
```

Here, you see that we transfer the x, y, and z coordinate of the grid in the same way as before, and that we transfer the x, y, and z components of the electric field to the `u`, `v`, and `w` variables inside the function. The last three options are not necessary here, but are useful options that you may want to explore to make your visualizations more interesting. The resulting plot for a single charge $+Q$ in the origin and for a dipole charge $Q_1 = +Q$, $Q_2 = -Q$ in $\mathbf{r}_1 = (1, 0, 0)$ and $\mathbf{r}_2 = (-1, 0, 0)$ are shown in Fig. 2.13.

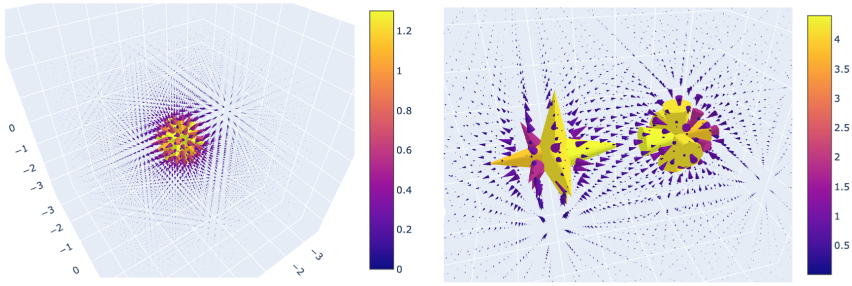


Fig. 2.13 Plot of the electric field **E** using cones to represent the direction and magnitude of the field for (a) a single charge $+Q$ in the origin, and (b) a dipole in the Translucent plot of the electric potential $V(\mathbf{r})$ from a single charge $+Q$ in the origin.

2.5.6 Example: Electric potential from a water molecule

An important aspect in physics is *modeling* — how we take a physical system and develop a simplified model that we can address theoretically. How can we address a problem like finding the electric potential from a water molecule? We do not really know what the charge distribution is for a water molecule. And the charge distribution may depend on the situation: Is the water molecule alone? Is it close to other water molecules? Is it close to a charged surface? When we model, we need to make reasonable assumptions, and we will train making such assumptions gradually. However, to some degree this requires courage and curiosity — you need to dare to make a simple model and you need to explore the consequences.

How would we model a water molecule? A simplified model would be to assume that we can model the molecule as a set of point charges at the positions of the atoms in the molecule: a charge $Q_{\mathrm{O}} = -0.7e$ at the oxygen atom and a charge $Q_{\mathrm{H}} = 0.35e$ at the hydrogen atoms [**?**] (See e.g. Fyta[1] for data). We will then assume that the angle between the hydrogen atoms is $\theta = 104.5°$ and that the distance between the oxygen and the hydrogen atom is $a = 0.95$Å [**?**]. These values are taken from scientific articles, but you could also have chosen reasonable values — at least reasonable enough for our purposes from a simple web search. However, you should always provide information about where you have found the values you use.

We place the oxygen atom at the origin and then place the hydrogen atoms symmetrically around the y-axis. The position of the two hydrogen

[1] `https://www2.icp.uni-stuttgart.de/ icp/mediawiki/images/3/35/SimmethodsII_ss13_lecture6_watermode`

atoms are then $\mathbf{r}_i = a(\pm \sin\theta/2, \cos\theta/2)$. We set up this system in Python

```
theta = 104.5/180.0*np.pi # degrees -> rad
a = 0.95 # AAngstrom
QO = -0.7 # e
QH = -QO/2
# Setup charges
R = []
Q = []
r0 = np.array([0,0])
R.append(r0)
Q.append(QO)
r0 = a*np.array([np.sin(theta/2),np.cos(theta/2)])
R.append(r0)
Q.append(QH)
r0 = a*np.array([-np.sin(theta/2),np.cos(theta/2)])
R.append(r0)
Q.append(QH)
```

And generate the electric potential with a reasonable resolution:

```
# Calculate the potential
Lx = 3
Ly = 3
N = 101
x = np.linspace(-Lx,Lx,N)
y = np.linspace(-Ly,Ly,N)
rx,ry = np.meshgrid(x,y)
# Set up electric potential
V = np.zeros((N,N),float)
# Calculate the potential
for i in range(len(rx.flat)):
    r = np.array([rx.flat[i],ry.flat[i]])
    V.flat[i] = epotlist(r,Q,R)
```

The resulting electric potential and the electric field is visualized using the same methods we developed above, giving the result shown in Fig. 2.14.

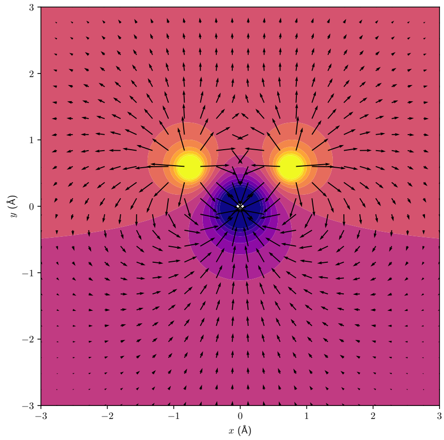


Fig. 2.14 Electric potential from a simplified model of a water molecule.

2.6 Summary

Electrical potential.

- The electrical potential $V(\mathbf{r})$ for an electrical field $\mathbf{V}(\mathbf{r})$ is defined as

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\mathrm{ref}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} \ , \tag{2.60}$$

where the integral is a path integral along any line from $\mathbf{r}$ to the reference point where $V = 0$.

- The electrical potential is defined relative to a reference point. It is only differences in electrical potential that have physical significance.

Electric field.

- The electric field is related to the electric potential through $\mathbf{E} = -\nabla V(\mathbf{r})$.
- The curl of a (static) electric field is 0: $\nabla \times \mathbf{E} = \mathbf{0}$.

Electric potential from a charge distribution.

- The electric potential in $\mathbf{r}$ from a single point charge Q in $\mathbf{r}'$ is

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 R} = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \ , \ \mathbf{R} = \mathbf{r} - \mathbf{r}' \ , \ R = |\mathbf{R}| \ . \tag{2.61}$$

- The electrical potential obeys the *superposition principle*: $V(\mathbf{r}) = \sum_i V_i(\mathbf{r})$.
- The electrical potential from a volume charge density ρ is:

$$V(\mathbf{r}) = \int_v \frac{\rho \mathrm{d}v'}{4\pi\epsilon_0 R} = \int_v \frac{\rho(\mathbf{r}')\mathrm{d}v'}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|} \tag{2.62}$$

- The electrical potential from a surface charge density ρ_s is:

$$V(\mathbf{r}) = \int_S \frac{\rho_s \mathrm{d}S'}{4\pi\epsilon_0 R} = \int_S \frac{\rho_s(\mathbf{r}')\mathrm{d}S'}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|} \tag{2.63}$$

- The electrical potential from a line charge density ρ_s is:

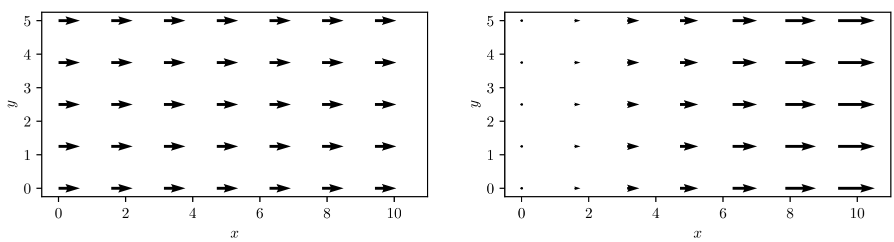
$$V(\mathbf{r}) = \int_C \frac{\rho_l \mathrm{d}l'}{4\pi\epsilon_0 R} = \int_C \frac{\rho_l(\mathbf{r}')\mathrm{d}l'}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|} \tag{2.64}$$

2.7 Exercises

2.7.1 Test yourself

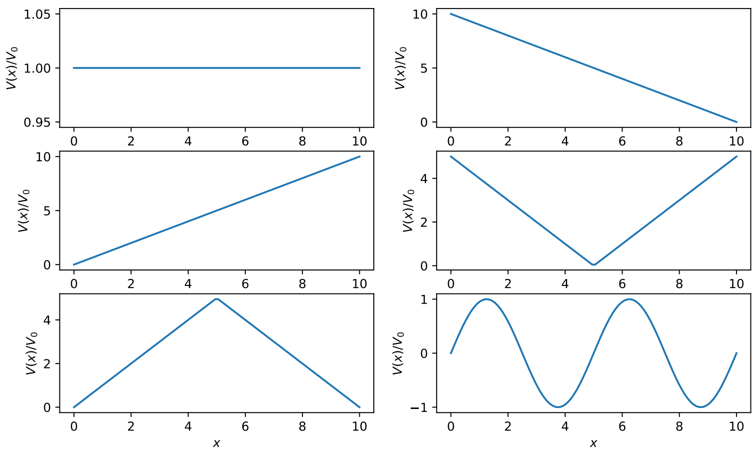
Exercise 2.1: From field to potential

The figure below illustrates two electric fields $\mathbf{E}$ in the xy-plane. Sketch the electric potential $V(x,0)$ along the x-axis and $V(0,y)$ along the y-axis for the two situations. (Assume $V(0,0)=0$).



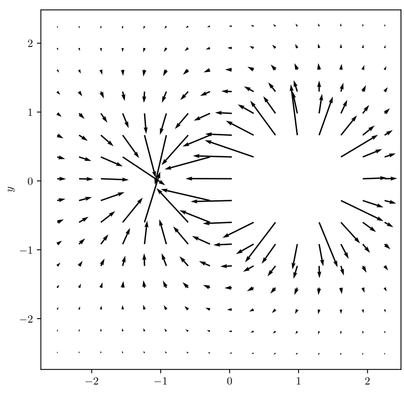
Exercise 2.2: From potential to field

The figure below illustrates electric potentials, $V(x)$, for different physical situations. Sketch the corresponding electrical fields with arrows on top of the figure. Can you also sketch the electric field $E_x(x)$?



Exercise 2.3: Maximum and minimum of the potential

The figure illustrates the electric field around a dipole. The electric potential is zero at infinity.



a) Where is the minimal value for the electric potential?

b) Where is the maximum value for the electric potential?

c) Where is the electric potential zero?

Exercise 2.4: Potentials and sparks

A spark in air occurs where the electric field exceeds a maximum value beyond which the air becomes a conducting plasma. You may have experienced sparks between your finger and another object if you are charged with static electricity. Let us model this process by assuming that your finger has an electrical potential V and that for example your table lamp has potential zero. Explain what happens to the electric field as your finger approaches the lamp. (You can assume that the potential of your finger and the lamp does not change before a spark occurs).

2.7.2 Discussion exercises

Exercise 2.5: Choosing your path

The potential (relative to a point at infinity) midway between two charges of equal magnitude and opposite sign is zero. Is it possible to bring a test charge from infinity to this midpoint in such a way that no work is done in any part of the displacement? If so, describe how it can be done. If it is not possible, explain why.

Exercise 2.6: Finding the field from the potential

If the electric potential at a single point is known, can $\mathbf{E}$ at that point be determined? If so, how? If not, why not?

Exercise 2.7: Zero field

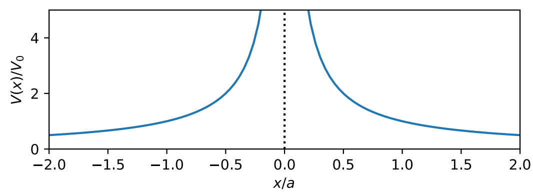
If $\mathbf{E}$ is zero throughout a certain region of space, is the potential necessary also zero in this region? Why or why not? What *can* be said about the potential?

Exercise 2.8: The zero-field path

If $\mathbf{E}$ is zero everywhere along a certain path the leads from point A to point B, what is the potential difference between those two points? Does this mean that $\mathbf{E}$ is zero everywhere along *any* path from A to B? Explain.

Exercise 2.9: Motion in a simple field

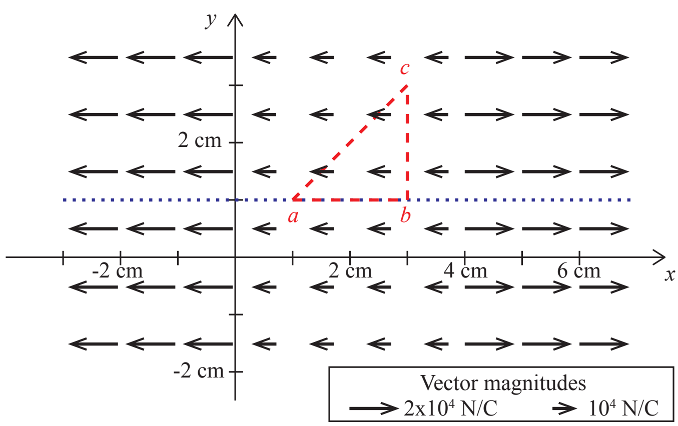
The figure illustrates the scalar potential, $V(x)$, resulting from some charge configuration. Discuss what possible types of motion a positively and a negatively charged particle can have by comparing with the potential energy of a cart rolling down a hill. Illustrate the motions in an energy diagrams ($U(x)$ for the particle and for the cart).



2.7.3 Tutorials

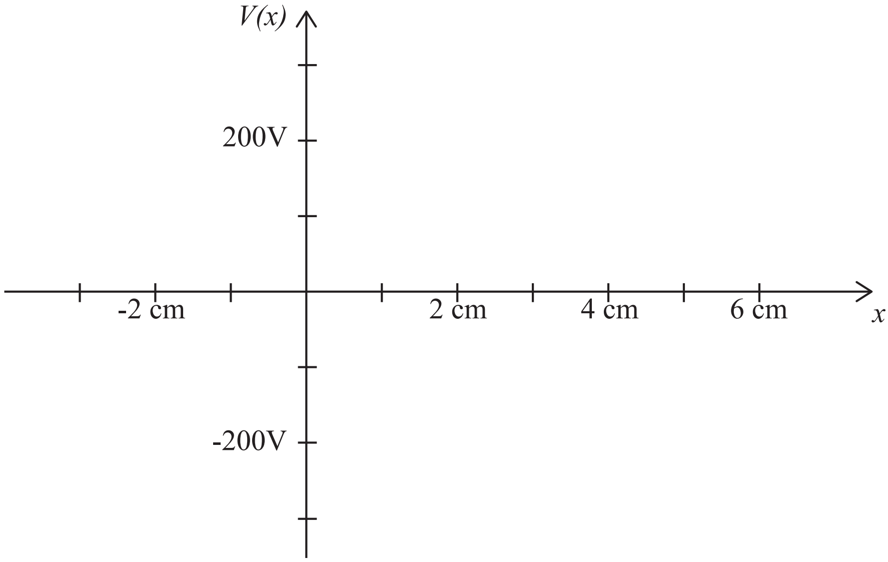
Exercise 2.10: Electric fields and electric potential

Below is a drawing of electric field vectors to illustrate the electric field. The *length* of the vector is proportional to the magnitude of the electric field. The *direction* of the vector is in the direction of the electric field. The electric field is defined in all points in space; we just can't draw all the vectors.



a) Suppose the reference point (zero) for the electric potential is at the origin (0cm,0cm). What are the electric potentials at points a, b, and c?
$V_a =$ $V_b =$ $V_c =$

b) With the same choice of reference point, plot the electric potential as a function of position along the dotted line in the figure.



You put a $+50\mu$C charge at point a (1cm,1cm), and move it to point b (3cm,1cm) at a constant, slow velocity.

c) What force vector (F_x, F_y) do you need to apply to keep the velocity constant?

d) How much work did you do to move the particle?

e) What is the potential energy difference $U_b - U_a$?

f) What is the electric potential difference $V_b - V_a$?

g) What is the relation between the potential energy and the electric potential?

Exercise 2.11: Gradients

In this exercise we will practice basic skills to find the scalar potential, interpret gradients in the potential, and find the field from the potential.

Two identical charges, q, are placed at $(a,0,0)$ and $(-a,0,0)$, where a is a given length.

a) Find the electrical potential $V(x)$ everywhere along the x-axis. Is the potential negative when $x < a$? Is the potential zero at $x = 0$? Explain.

b) Plot or sketch the potential along the x-axis and discuss the direction of the electric field.

c) Use the sketch of the potential to draw equipotential points (at a constant spacing in potential).

d) Find the potential $V(y)$ everywhere along the y-axis.

e) Sketch equipotential curves in the xy-plane for this system.

Exercise 2.12: Vector calculus

(From Johannes Skaar)

a) Several types of line integrals often appear in physics and mathematics. Some of these are:

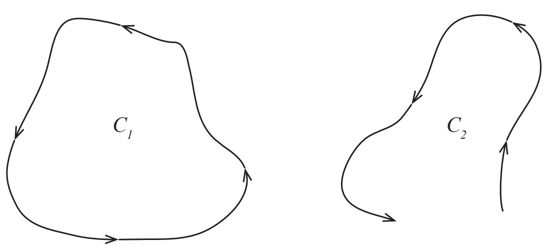
$$(1)\ \int_C F dl \qquad (2)\ \int_C F d\mathbf{l} \qquad (3)\ \int_C \mathbf{F} dl \qquad (4)\ \int_C \mathbf{F}\cdot d\mathbf{l} \tag{2.65}$$

Which of these integrals are appropriate for the folloing physical situations?

(A) If the mass density of a wire is F, what is the total mass of the wire?

(B) The work done by a force $\mathbf{F}$ when an object is moved along a curve C.

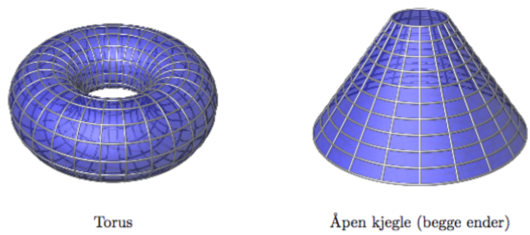
b) Let $\mathbf{F}$ and F be constants different from 0 and use the integration paths C_1 and C_2 from the figure below. In which of case(s) (1)-(4) is the integral along these curves 0? Draw $d\mathbf{l}$ for a few points on each curve.



c) For surface integrals the corresponding integrals are:

$$(1) \iint_S F dS \qquad (2) \iint_S F d\mathbf{S} \qquad (3) \iint_S \mathbf{F} dS \qquad (4) \iint_S \mathbf{F} \cdot d\mathbf{S} \quad (2.66)$$

Which integrals become zero? (You can still assume that $\mathbf{F}$ and F are constants.) Start by considering a torus and a cone as illustrated in the figure.



d) How is the direction of $d\mathbf{S}$ defined for the torus? Sketch $d\mathbf{S}$ in three points.

e) Consider the surface S_1 defined by the curve C_1, in which direction does $d\mathbf{S}$ point for S_1?

Exercise 2.13: Line charge

(We will now address the problem of a line charge, but using potential instead of electric field, as we did previously.)

A rod of length L has a charge Q. We place the rod along the x-axis with its center at the origin.

a) We want to calculate the electric potential $V(0, y, 0)$ along the y-axis. What is the contribution to the potential from the piece of length dx at x?

b) Write down an expression for the electric potential $V(0, y, 0)$ in terms of an integral. Explain what variable you integrate over.

c) Anne argues that with this integral it is very simple to find the electric field $E_y = -\partial V/\partial y$: This is simply what is inside the integral, since derivation is the opposite of integration. Is Anne right? Explain.

Exercise 2.14: Continuous distribution of charges

(From Danny Caballero, MSU)

In this exercise we will see how we use the principle of superposition to find the potential and electric field from a continuous distribution of charges.

Part 1: A charged rod. We would like to find the electric field from a rod of length L with a uniformly distributed charge Q.

a) Make a drawing of the rod. How would you orient and place the rod in your coordinate system?

b) Add a small element of length dl to your drawing. What is the charge of this part of the rod?

We would like to find the electric field in the plane normal to the rod going through the center of the rod.

c) What is the contribution from the element dl to the electric field in a distance r from the center of the rod, and in the plane normal to the rod and going through the center of the rod. Use your drawing actively in your argument.

d) What integral do you need to perform to find the electric field at position r? Ensure that you also include the limits of the integral.

You may need to use the following formulas:

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^3\sqrt{1+(u/a)^2}} + C \tag{2.67}$$

$$\int \frac{du}{(a^2+u^2)^{1/2}} = \operatorname{asinh}\left(\frac{u}{a}\right) + C \tag{2.68}$$

(The interested student may want to solve these integrals using Sympy)

e) Find the solution to the definite integral to find the electric field.

Part 2: Numerical integration of the electric field. Suppose you have a vertically oriented rod of total charge $Q = +1\mu\mathrm{C}$, centered at the origin with a length of 1m.

f) Determine the electric field at the location $(0.1, 0, 0)$m. (*Is your answer a vector? Because it should be.*)

This result is only valid at the center of the rod. Let us develop a model to find the electric field anywhere in space. (You can later use the same approach also to find the scalar potential).

g) Let us represent the uniformly charged rod by a sequence of small pieces, where we represent each piece as a point charge located at the center of that piece. Calculate the electric field at $(0.1, 0, 0)$m when you divide the rod into 4 pieces. Compare with the exact results.

h) Let us instead divide the rod into N pieces at locations $\mathbf{r}_i$ and charges q_i. Find expressions for $\mathbf{r}_i$ and q_i.

i) Write a computer program to perform the summation — corresponding to a numerical integration. Compare the value at $(0.1, 0, 0)$m with the exact result for various values of N. (You can find a sketch of the computer program Electricfield-IntegrateRod-Student.ipynb[2] at JupyterHub).

j) (Advanced and optional) Use the same program to find the electric field on a grid in the xy-plane and visualize this grid.

2.7.4 Homework

Exercise 2.15: Dipole and gradient

Two identical charges, q, are placed in $(a, 0, 0)$ og $(-a, 0, 0)$, $a > 0$.

a) Implement the Python programs from the text to find the electric potential in the xy-plane from the two charges. Illustrate the potential with contour lines.

b) Find the electric field in the xy-plane numerically by taking the gradient of the electric potential and illustrate the field with a vector plot. Is the electric field normal to the contour lines?

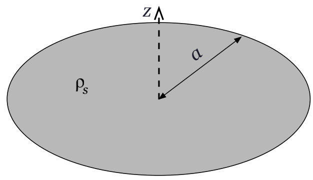
c) Let us make one of the charges negative, making the system a dipole. Repeat parts **a** og **b** for this new system.

d) The vector plots are maybe not so pretty. Plot the same data using streamlines instead. What are the advantages and disadvantages of a streamline plot?

Exercise 2.16: Potential and field above a disk

In this exercise we will address the electric field along the z-axis above the center of a disk in the xy-plane. The disk has radius a and a constant surface charge density ρ_s.

[2] `https://jupyterhub.uio.no/`



a) Find the contribution $\mathbf{dE}$ to the electric field at a height z above the disk from an area element $\mathbf{d}S$. You may assume $z > 0$.

Hint. It may be useful to work in cylinder coordinates. Remember that the area element in the $r\phi$-plane when we integrate in cylinder coordinates is $r\mathbf{d}r\mathbf{d}\phi$

b) Find the electric field $\mathbf{E}$ from the disk at the height z.

Hint. Integrate over the disk $\mathbf{d}q = \rho_s \mathbf{d}S$. It may help to start by introducing a symmetry argument so that you only need to find one of the components of the $\mathbf{E}$-field.

c) Use the result from the previous exercise to find the electric potential V in the same point.

d) Find the electric field $\mathbf{E}$ in the limit where z is small (that is $z \to 0$ from above). Interpret the result.

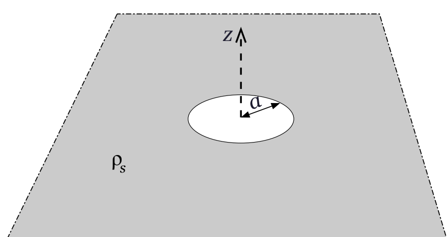
e) Sketch/plot $|\mathbf{E}|$ as a function of z for $z > 0$.

f) Show that $\mathbf{E}$ approaches the field from a point charge for large z.

Hint. Use the series expansion $(1 + x)^a \approx 1 + ax$ when $x \ll 1$.

Exercise 2.17: Plate with hole

Find the magnitude of the electric field $|\mathbf{E}|$ at a height z along the z-axis over an infinitely large, plane surface with a circular hole with radius a. The center of the whole is in the origin. The surface has a uniform charge density ρ_s.



Hint 1. Superposition

Hint 2. The field over an infinite plate without a hole is the same as the field from an infinite plate in the limit $z \to 0$.

Exercise 2.18: Dipole field

The electric potential at large distances from a dipole $\mathbf{p}$ at the origin is

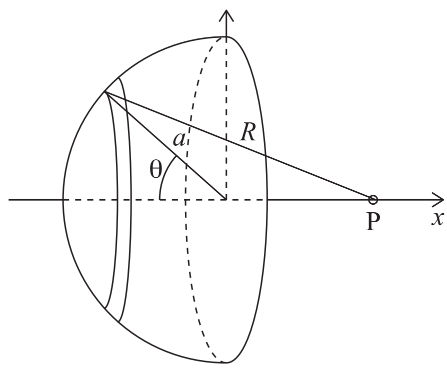
$$V(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} \ . \tag{2.69}$$

Show that electric field at large distances is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(-\frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} \right) \ . \tag{2.70}$$

Exercise 2.19: Half spherical shell

The figure shows half of a spherical shell of radius R and uniform surface charge density ρ_s.



a) Find the electrical potential along the x-axis.

Hint 1. Use that $dq = \rho_s 2\pi a\, a \sin\theta d\, d\theta$.

Hint 2. Integrate using Sympy.

b) Show that the electrical potential is approximately $V(x) \simeq Q/(4\pi\epsilon_0)$ when $x \gg a$, where $Q = 2\pi a^2 \rho_s$.

c) Find the electric field E_x along the x-axis for $x > 0$.

Hint. Use Sympy.

2.7.5 Modeling projects

Exercise 2.20: Studying a charge distribution

In this modeling project you will study an electrostatic system of your own choice using the tools we have developed so far. This will most probably feel uncomfortable, because we are used to answer questions, not pose the questions ourselves. However, this project aims to introduce you to how we think about and use physics in practice, and as you would be using physics as a tool in research or development.

The goal of this project is to (1) choose a charge distribution of interest, (2) make a simplified model for the electric field or potential from the charge distribution that you can find an expression for analytically, (3) implement a computational model of electric field or potential from the charge distribution, (4) compare the results from the computational model with results from your simplified model, (5) visualize the results from the computational model, and (6) write a brief report on your whole study. We suggest that you use a Jupyter notebook to develop your project.

This project is rather open, so let us provide you with a few examples of projects that you may pursue:

Example 1: Charge distribution around a CO_2 molecule. What is the electric field and potential around a CO_2 molecule? We start by finding a simplified model for the charge distribution of the CO_2 molecule: We place a charge $2q$ to represent the C in the origin and a charge $-q$ in positions $(-a, 0, 0)$ and $(a, 0, 0)$. This system we can solve exactly. We find the potential and field along the x-axis analytically and plot the results. Then we find the potential and field in the xy-plane, and visualize the results. We study how the electric field decays when $r \to \infty$ by plotting $\log_{10}(E)$ as a function of $\log_{10}(x)$ for $x \gg a$. We write a brief Jupyter notebook describing the results.

Example 2: Inside a charged box. What is the electric potential and electric field inside a box with charged surfaces? The full model is in two dimensions as a square box centered in the origin with sides $2L$ and a charge Q uniformly distributed on each of the four sides. First, we simplify the model to consist of four charges, Q, placed at the center of each of the four sides. We find the electric potential and field in the center of the box for this system. We also calculate the electric potential and field along the x-axis. Then we use 100 charges to model each of the 4 sides (400 charges in total) to calculate the potential and the electric field. We visualize the potential and field in space and compare the solution with the simplified model. Is the model ok along the x-axis. Where in the box are the deviations the greatest between the simplified model and the full model? We write a brief Jupyter notebook to describe the study. You can vary this study by e.g. choosing different charges on the various four sides, studying a three-dimensional system and comparing with a two-dimensional system and so on.

Grading rubric. To help you understand how this project will be assessed we provide you with a simple rubric that can help you see if you demonstrate the expected competences. You will need to satisfy all 7 points to get the project approved.

Element	Points (0/1)	Competence
Description	1	There is a description of the system both in terms of a physical system, and with all the needed variables defined.
Sketch	1	There is a sketch of the system, with an axis, length and charges shown.
Simplified model	1	There is a description of a simplified model. All the positions and charges are described. There are arguments for the approximations used.
Mathematical solution	1	Coulombs law for the field and/or the potential has been used to find the field/potential for the simplified model. The results are plotted.
Code	1	The full model is solved using a Python program to find the field and potential in space. The field and potential is visualized.
Comparison	1	The full model is compared with the simplified model in selected points, along selected lines, or in selected limits. Differences and similarities are commented.
Written report	1	There is a report that explains the simplified and advanced models with mathematics, programs, plots and visualizations of the fields/potentials. The report may be a Jupyter notebook.

3 Gauss' law

We have learned that we can find the electric field for any distribution of electric charges. In this chapter, we will introduce Gauss' law, which provides a different approach to finding the electric field. We will introduce Gauss' law on integral form and demonstrate how this law can be used to find the electric field in systems with a high degree of symmetry. We will also demonstrate that Coulomb's law is a consequence of Gauss' law and vice versa. Gradually, as you build intuition for electromagnetism, you will see that Gauss' law appears to be more fundamental than Coulomb's law and provides you with a very compact description of the laws of electromagnetism. Gauss' law will also be our first encounter with one of Maxwell's equation which provide a beautify and compact description of electromagnetism. Here, we will focus on building up the basic concepts needed to apply Gauss' law: flux, symmetries, and surface and volume integrals. We will learn to apply Gauss' law in a variety of cases, and understand both its limitations and its strength.

3.1 Gauss' law on integral form

Gauss' law

Gauss' law states that the flux of the electric field through a *closed* surface S is proportional to the net total charge in the volume enclosed by the surface:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_S}{\epsilon_0} \tag{3.1}$$

Here, S is any closed surface and Q_S is the net charge in the volume inside the surface S. (We provide a proof of Gauss' law further down.)

- The surface S must be a *closed* surface. The charge Q_S is the net total charge inside the surface S. The net charge inside the surface S is either the sum of all the charges in the volume v enclosed by the surface S: $Q_S = \sum_i Q_{i,\text{in}}$ or the integral of a volume charge density ρ_v over the volume v enclosed by the surface S: $Q_S = \int_v \rho_v \mathrm{d}v$.
- The integral $\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$ is called the flux of the electric field through the surface S. We introduce the electric flux and how to calculate it in the following.

3.1.1 Electric flux

The electric flux, $\mathrm{d}\phi$, through a small surface $\mathrm{d}\mathbf{S}$ is defined as

$$\mathrm{d}\phi = \mathbf{E} \cdot \mathrm{d}\mathbf{S} \ , \tag{3.2}$$

where $d\mathbf{S}$ is an *oriented* surface element that has both an area and a direction. The direction is given by the unit normal vector for the surface $\hat{\mathbf{n}}$ so that $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$. The directed surface element $\mathrm{d}\mathbf{S}$ points in the direction of the *positive* surface normal.

Flux and surface normal. The flux is given by the dot product of the field $\mathbf{E}$ and the surface element $d\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$. This is illustrated in Fig. 3.1. It is only the component of the field that is normal to the surface that contributes. Any component normal to the normal vector $\hat{\mathbf{n}}$ will not contribute to the flux.

There are two ways to think about this as illustrated in Fig. 3.2: (1) You can think of the flux as the projection of the electric field $\mathbf{E}$ onto the surface $\mathrm{d}\mathbf{S}$ with surface normal $\hat{\mathbf{n}}$. (2) Or you can think of the flux as the projection of the surface area $\mathrm{d}\mathbf{S}$ onto the electric field. In both cases $\mathrm{d}\Phi = E\mathrm{d}S\cos\theta$. In the first case, we interpret $E\cos\theta$ as the part of the electric field that is normal to the surface. It is only this part of the field that contributes to the flux. In the second case, we interpret $dS\cos\theta$ as the cross-sectional surface area — the projection of the area onto the

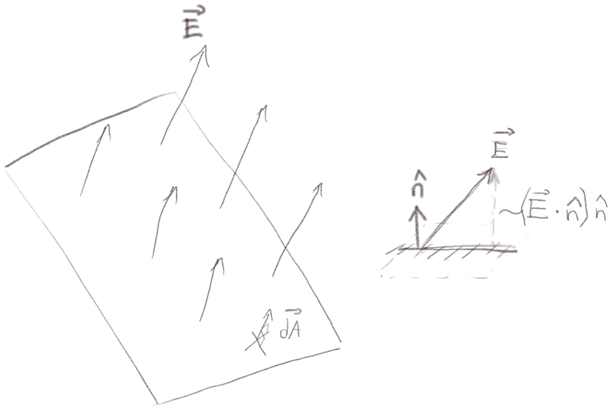


Fig. 3.1 Illustration of the electric field **E** and a small surface element with normal vector $\hat{\mathbf{n}}$.

direction of the electric field: It is only the part of the surface that has a surface normal parallel to the electric field that contributes to the flux.

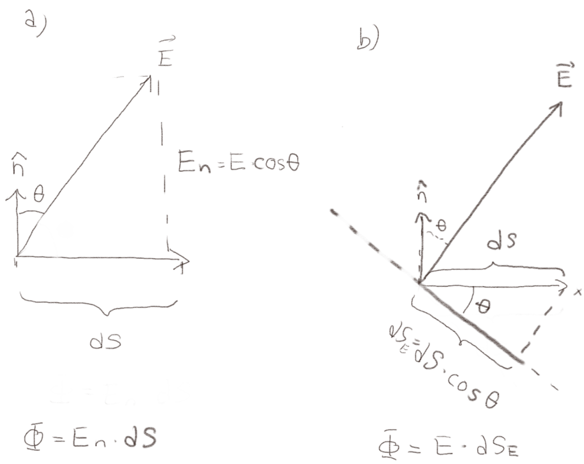


Fig. 3.2 Illustration of the electric field **E** and a small surface element with normal vector $\hat{\mathbf{n}}$.

Only the field component normal to a surface contributes to the flux. Fig. 3.3 illustrates an electric field and two possible surfaces. For the case when the normal vector $\hat{\mathbf{n}}$ for the surface is in the same direction as the electric field, $\mathbf{E}$, the flux is maximum, ϕ_{max}. For the case when the normal vector $\hat{\mathbf{n}}$ for the surface is in a direction normal to the electric field, $\mathbf{E}$, the flux is zero.

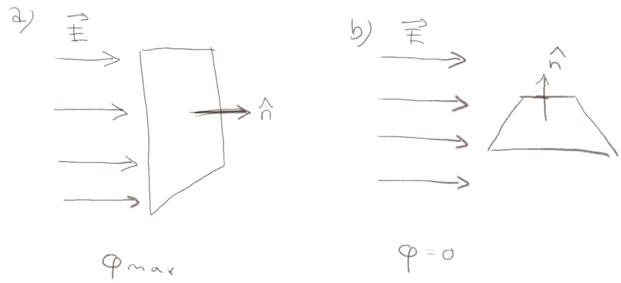


Fig. 3.3 Illustration of the electric field **E** and a small surface element with normal vector $\hat{\mathbf{n}}$ for two possible surfaces.

Direction of the surface normal. Notice that in the surface integral over the closed surface S:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} \mathrm{d}S = \frac{Q_{\text{in}}}{\epsilon_0} \;, \tag{3.3}$$

the surface normal $\hat{\mathbf{n}}$, and the directed surface element $\mathrm{d}\mathbf{S}$, points *outward*. This is the flux *out of the volume* enclosed by S. When we calculate this surface integral we need to sum up the contributions from all surfaces.

3.1.2 Example: Gauss' law for a single charge

Let us see how we can use Gauss' law to find the electric field for a single charge in the origin?

In this case we know that for any closed surface that encloses the origin, then the integral of the field over the surface is

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} \;. \tag{3.4}$$

The *trick* is to find a surface on which the electric field is constant, so that the surface integral becomes simple. We use *symmetry arguments* to find where the electric field is constant. For a single charge in the origin, we expect that the field must be the same for all possible rotations of the system — we expect the system to have spherical symmetry. We write the field in spherical coordinates as

$$\mathbf{E} = E_r \hat{\mathbf{r}} + E_\phi \hat{\mathbf{u}}_\phi + E_\theta \hat{\mathbf{u}}_\theta \;. \tag{3.5}$$

We realize that the field only can be directed in the radial direction. Otherwise the field would break the spherical symmetry. Thus we expect the field to have the form

$$\mathbf{E} = E_r(r, \phi, \theta)\hat{\mathbf{r}} \ . \tag{3.6}$$

In addition, the field cannot depend on ϕ or θ, because this would also break the symmetry. The field can not be larger in one direction, because the orientation of the coordinate system is arbitrary. Thus, we expect the field to have the form:

$$\mathbf{E} = E_r(r)\hat{\mathbf{r}} \ . \tag{3.7}$$

This describes the full symmetry of the field. The field points in the radial direction and only depends on the distance to the origin. The field is therefore constant on a surface that has a constant distance to the origin — on a sphere. And in this case the normal vector of the surface is $\hat{\mathbf{r}}$. The flux gives as the surface integral over a spherical surface at a distance r to the origin and therefore simplifies to:

$$\varPhi = E_r(r) \oint_S dS = E_r(r) 4\pi r^2 = \frac{Q}{\epsilon_0} \ . \tag{3.8}$$

We can now solve for the electric field $E_r(r)$:

$$E_r(r) = \frac{Q}{4\pi\epsilon_0 r^2} \ , \tag{3.9}$$

and

$$\mathbf{E} = E_r(r)\hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \ . \tag{3.10}$$

Notice two things in this argument.

- First we see that we used symmetry arguments to simplify the description of the electric field.
- Second, this method only worked because we chose a surface S where the electric field was constant. It would not have worked if we chose a cubic surface enclosing the charge.

It is thererfore essential to be able to recognize and use *symmetries* to use Gauss' law.

Method: Using Gauss' law to find the electric field

Based on this example, we can propose a general strategy for how to find the electric field for a charge distribution using Gauss' law.

- Find a set of surfaces that enclose a volume such that $\mathbf{E} \cdot \hat{\mathbf{n}}$ is constant on each such surface element. (It may be zero on some of the surfaces — zero is a constant!) We call these surfaces *Gauss surfaces* for the problem.
- This often requires that you find a simplified description of the field in a chosen coordinate system, such as $\mathbf{E} = E_r(r)\hat{\mathbf{r}}$.
- Find the flux integral.
- Use Gauss' law to find the electric field as a function of charge and position.
- Notice that the surface does not have to enclose all the charges — it is allowed and indeed often necessary to chose a surface that contains only some of the charges. However, the electric field must be a constant on the surfaces you have chosen.

3.1.3 Example: Electric field from an infinite line charge

Find the electric field from an infinite line

Specifying the problem. The line is along the z-axis and we assume that it has a uniform line charge density, ρ_l.

Drawing the system. We illustrate this system in the drawing in Fig. 3.4.

Symmetry. We look for symmetries in this system. First, we notice that the line charge is along the z-axis. We therefore expect the field to be symmetric around the z-axis. First, let us use a cylindrical coordinate system. We can write $\mathbf{E}$ in cylindrical coordinates (r, θ, z), where r is the distance to the z-axis, θ is the angle around the z-axis with $\theta = 0$ along the x-axis, and z is the position along the z-axis. Using this coordinate system we have that $\mathbf{E} = E_r\hat{\mathbf{u}}_r + E_\theta\hat{\mathbf{u}}_\theta + E_z\hat{\mathbf{z}}$, where $\hat{\mathbf{u}}_r$, $\hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{z}}$ are unit vectors.

We expect the field to depend on the distance from the z-axis, but not on the angle around the axis. The symmetry therefore implies that $E_r(r, \theta, z)$ only depends on r and z, $E_r = E_r(r, z)$.

Since the line is infinite along the z-axis, we expect the system to look the same no matter where we are along the z-axis. Thus we do not expect any z-dependence in the field. $E_r = E_r(r)$.

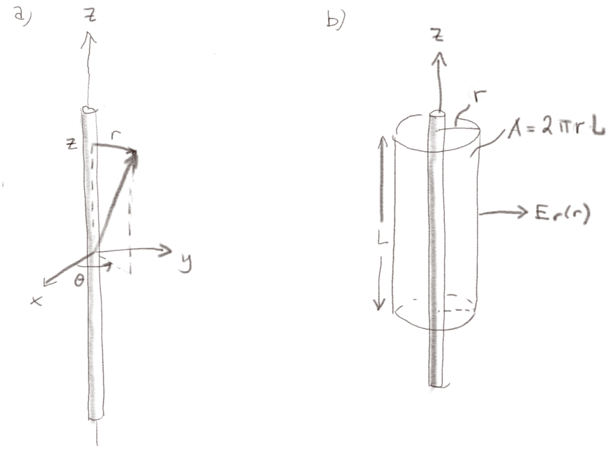


Fig. 3.4 Illustration of the electric field $\mathbf{E}$ from an infinite line charge along the z-axis.

This means that the field will look like $\mathbf{E} = E_r(r)\hat{\mathbf{u}}_r + E_\theta(r)\hat{\mathbf{u}}_\theta$. Finally, what about the field in the θ-direction? Here, there are two arguments: Either we could argue that since the charge distribution is the same in both the positive and negative z-direction, we do not expect the θ-direction to break that symmetry. Alternatively, we can argue that since the electric field around any closed loop, $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$, then we see that $\oint E_\theta \mathrm{d}l = 2\pi r E_\theta = 0 \Rightarrow E_\theta = 0$.

Finally, we have the symmetry of the problem written down in terms of how the electric field depends on the various variables. Usually, we do not need to be *this* systematic. I expect that you will be able to see such symmetries immediately and write down the corresponding form of the electric field.

Applying Gauss' law using the found symmetry. With a cylindrical symmetry, we use a cylinder around the z-axis as the surface for Gauss' law. On the surface of this cylinder, the electric field will be constant, because $E_r(r)$ only depends on r. We then apply Gauss' law on the cylinder surface in Fig. 3.4:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \, . \tag{3.11}$$

We divide the surface into three surfaces: The top and bottom edges of the cylinder and the curved cylinder surface with a radius r. On the *bottom* and *top* surfaces the electric field is radial and the surface normal is along the z-axis. Therefore $\mathbf{E} \cdot \hat{\mathbf{n}}$ is zero here. On the curved cylinder surface the electric field is constant $\mathbf{E} = E_r(r)\hat{\mathbf{u}}_r$ and directed along the surface normal $\hat{\mathbf{n}} = \hat{\mathbf{u}}_r$, which points outwards from the center of the cylinder. Therefore, $\mathbf{E} \cdot \hat{\mathbf{n}} = E_r(r)$, which is a constant on the whole surface. The integral is therefore simply $E_r(r)$ times the surface integral of the curved cylinder surface, $A = 2\pi r L$, where L is the length of the cylinder. We therefore find that the flux is

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 2\pi r L E_r \ . \tag{3.12}$$

Gauss's law states that this is equal to the net charge inside the volume:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} \ . \tag{3.13}$$

Here, the amount of charge inside is the charge line density times the length of the line: $Q = \rho_l L$. Gauss' law therefore gives:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 2\pi r L E_r = \frac{\rho_l L}{\epsilon_0} \ \Rightarrow \ E_r = \frac{\rho_l}{2\pi\epsilon_0 r} \ . \tag{3.14}$$

3.1.4 Example: Electric field from a spherical charge

A sphere with radius a has a uniformly distributed charge Q. Find the electric field everywhere in space.

Specifying the problem. The problems sketches a sphere of radius a. Inside the sphere there is a constant volume charge density ρ_v. We find the charge density from:

$$Q = \int_v \rho_v \, dv = \rho_v \int_v \mathrm{d}v = \rho_v \frac{4}{3}\pi a^3 = Q \ \Rightarrow \ \rho_v = \frac{3Q}{4\pi a^3} \ . \tag{3.15}$$

We can specify this further. The charge density is constant for $r < a$ and zero outside. This means that

$$\rho_v = \begin{cases} \rho_v & \text{when } r \leq a \\ 0 & \text{when } r > a \end{cases} \tag{3.16}$$

We want to find the electric field everywhere. This means that we want to find the electric field both for $r < a$ and for $r > a$!

Drawing the system. We illustrate this system in the drawing in Fig. 3.5.

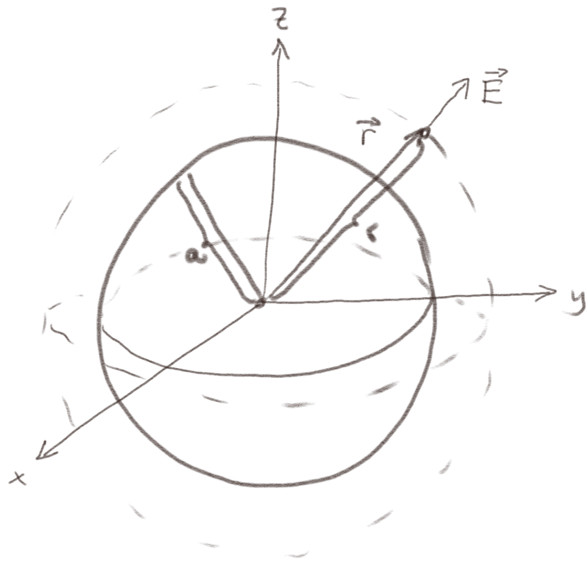


Fig. 3.5 Illustration of the electric field $\mathbf{E}$ from a spherical charge.

Symmetry. What symmetries does this system have? We realize the system has spherical symmetry. We realize that the symmetry indicates the electric field is radial $\mathbf{E} = E\hat{\mathbf{r}}$. If this was not the case, we could have rotated the sphere half a turn around an axis through the center of the sphere, and the electric field would have changed. But this does not change the distribution of charges, therefore the field cannot change. This contradiction means that the field cannot have a component that is not radial. Similarly, we argue that E only depends on r and not on θ or ϕ. The field is therefore $\mathbf{E} = E(r)\hat{\mathbf{r}}$.

Applying Gauss' law. We can then apply Gauss' law using this symmetry. We choose to apply Gauss' law on a spherical surface centered on the center of the spherical charge distribution. In this case, the electric field $\mathbf{E} = E(r)\hat{\mathbf{r}}$ is always pointing in the direction of the outward-pointing surface normal $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. The flux through a spherical surface is then

$$\Phi = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = E(r) \oint_S \mathrm{d}A = E(r)\, 4\pi r^2 = \frac{Q_{\mathrm{in}}}{\epsilon_0} \, . \tag{3.17}$$

Now, what is the charge Q_{in}? It is the charge *inside* the surface. If $r > a$ the surface encloses the whole charge Q and $Q_{\mathrm{in}} = Q$. In this case the field is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \;\; r > a \; . \tag{3.18}$$

However, when $r < a$, the charge Q_{in} is only the charge that is inside the sphere of radius r. Inside the sphere, the volume charge density if constant. We can therefore find the charge Q_{in} as the integral over the volume enclosed by a sphere of radius r, v_r:

$$Q_{\text{in}} = \int_{v_r} \rho \mathrm{d}v = \rho_v \int_{v_r} dv \tag{3.19}$$

$$= \rho_v \frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi a^3}\frac{4}{3}\pi r^3 = Q\left(\frac{r}{a}\right)^3 \; . \tag{3.20}$$

We can then find the electric field from

$$E(r)\, 4\pi r^2 = \frac{Q_{\text{in}}}{\epsilon_0} = \frac{Q(r/a)^3}{\epsilon_0} \;\Rightarrow\; E(r) = \frac{Qr}{4\pi a^3} \; . \tag{3.21}$$

The electric field is therefore:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{when } r > a \\ \frac{Qr}{4\pi\epsilon_0 a^3}\hat{\mathbf{r}} & \text{when } r \le a \end{cases} \; . \tag{3.22}$$

3.2 Gauss' law on differential form

Gauss' law as formulated so far is on *integral form*:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{in}}}{\epsilon_0} \; . \tag{3.23}$$

This law is true for any surface S. In general, the net charge inside the surface is given as an integral over a volume charge density ρ:

$$Q_{\text{in}} = \int_v \rho \, \mathrm{d}v \; , \tag{3.24}$$

where the volume v is the volume enclosed by the surface S.

We can rewrite the surface integral using the divergence theorem from vector calculus. This states that for a (continuous) vector field $\mathbf{E}$ we have that

$$\oint_S \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{f} \, \mathrm{d}v \; , \tag{3.25}$$

where the volume V is the volume enclosed by the surface S. We apply this to Gauss' law:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{E} \, \mathrm{d}v = \frac{1}{\epsilon_0} \int_v \rho \mathrm{d}v \; . \tag{3.26}$$

Because this is true for any volume V, the two expressions inside the integrals are also identical:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \; .} \tag{3.27}$$

This is called *Gauss' law on differential form.* This version of Gauss' law will be used frequently when we address electromagnetics waves and it is often the starting point for numerical computations of the electric field.

3.3 Proof of Gauss' law

Gauss law can be proved by proving it for a single point charge and then use the superposition principle to prove it for any charge distribution. We will follow the elegant approach from Johannes Skaar [**?**] where we use the divergence theorem on integral form and that the divergence of the electric field for a single point charge is zero everywhere except in the center of the charge. For a charge q inside a closed surface S, we know from the divergence theorem that the net flux of $\mathbf{E}$ through S is equal to the volume integral of the divergence of $\mathbf{E}$ over the volume v enclosed by S:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{E} \, \mathrm{d}v \; . \tag{3.28}$$

As long as the charge q is inside the closed surface S and therefore also the volume v, we can divide the volume into two separate parts: a spherical volume v_s of a sphere centered on the charge q and the remaining volume, $v_r = v - v_a$. The volume integral is therefore:

$$\int_v \nabla \cdot \mathbf{E} \, \mathrm{d}v = \int_{v_s} \nabla \cdot \mathbf{E} \, \mathrm{d}v + \int_{v_r} \nabla \cdot \mathbf{E} \, \mathrm{d}v \; . \tag{3.29}$$

We will now show that the second integral is zero, because the divergence of $\mathbf{E}$ is zero everywhere away from the point charge, and that the first integral can be solved explicitely.

What is the divergence of the field from a single point charge? We place the origin in the point charge, so that the field from the point charge is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = E_r \hat{\mathbf{r}} \; . \tag{3.30}$$

The divergence of a field in spherical coordinates, $\mathbf{E} = E_r\hat{\mathbf{r}} + E_\theta\hat{\mathbf{u}}_\theta + E_\phi\hat{\mathbf{u}}_\phi$, is

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial\left(r^2 E_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, E_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} \tag{3.31}$$

For a single point charge $E_\theta = E_\phi = 0$. We see that since E_r is proportional to $1/r^2$, the term $r^2 E_r$ is a constant, and its derivative is zero. All the terms are therefore zero and the divergence is zero. Notice that this is only true when $r > 0$. But we have carefully selected the volume v_r so that $r > 0$ everywhere in this volume. Therefore, the volume integral of the divergence of $\mathbf{E}$ over v_r is zero:

$$\int_{v_r} \nabla \cdot \mathbf{E}\, \mathrm{d}v = 0 \; . \tag{3.32}$$

Now, let us find the integral over the spherical volume v_s centered on the charge q. This is a sphere with radius a and center in the origin, where the charge q is. We apply the divergence theorem again and relate the volume integral to the surface integral:

$$\int_{v_s} \nabla \cdot \mathbf{E}\, \mathrm{d}v = \oint_{S_s} \mathbf{E} \cdot \mathrm{d}\mathbf{S} \; . \tag{3.33}$$

A surface element on this spherical surface of radius a is $\mathrm{d}\mathbf{S} = \mathrm{d}S\hat{\mathbf{r}}$ and at a distance $r = a$ we have that $\mathbf{E} = q/(4\pi\epsilon_0 a^2)\hat{\mathbf{r}}$ so that:

$$\oint_{S_s} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_{S_s} \frac{q}{4\pi\epsilon_0 a^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\mathrm{d}S = \frac{q}{4\pi\epsilon_0 a^2}\oint_{S_s} \mathrm{d}S = \frac{q}{4\pi\epsilon_0 a^2}4\pi a^2 = \frac{q}{\epsilon_0} \; . \tag{3.34}$$

We have therefore shown that when a charge q is inside a closed surface S:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_{v_s} \nabla \cdot \mathbf{E}\, \mathrm{d}v + \int_{v_r} \nabla \cdot \mathbf{E}\, \mathrm{d}v = 0 + \frac{q}{\epsilon_0} = \frac{q}{\epsilon_0} \; . \tag{3.35}$$

What if the charge q is not inside the surface, but outside the surface? In this case we do not need to subdivide the volume inside the surface into two parts, because the divergence will be zero everywhere in the volume inside the surface and the integral will be zero.

We have therefore demonstrated that Gauss' law on integral form is true for a single point charge q: the integral is q/ϵ_0 if the charge is inside the surface and zero if it is outside the surface. For a charge distrubution with charges Q_i in positions $\mathbf{r}_i$ where therefore find that the flux through a closed surface S is

$$\oint_S \sum_i \mathbf{E}_i \mathrm{d}\mathbf{S} = \sum_{i \text{ inside } S} \frac{Q_i}{\epsilon_0} = \frac{Q_{\text{inside}}}{\epsilon_0} \ . \tag{3.36}$$

3.4 Summary

Gauss' law on integral form. For a close surface S:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{in}}}{\epsilon_0} \ , \tag{3.37}$$

where Q_{in} is the net charge inside the surface S.

Application of Gauss' law. We use Gauss' law to determine the electric field for systems with a high degree of symmetry, so that $\mathbf{E} \cdot \mathrm{d}\mathbf{S}$ is constant E_0 on the surface. We can then find E_0 by applying Gauss' law and calculate the net charge inside the surface: $SE_0 = Q_{\text{in}}/\epsilon_0$, and therefore $E_0 = Q_{\text{in}}/(S\epsilon_0)$, where S is the area of the corresponding surface.

Gauss' law on differential form. Gauss' law on differential form is:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \ , \tag{3.38}$$

where ρ is the volume charge density.

3.5 Exercises

3.5.1 Discussion exercises

Exercise 3.1: Rubber balloon

(From Sears and Semanskys)

A rubber balloon has a single point charge in its interior. Does the electric flux through the balloon depend on whether or not it is fully inflated? Explain you reasoning?

Exercise 3.2: Two charges

(From Sears and Semanskys) For the system in the figure, what would be the fluxes through each of the surfaces A,B,C and D?

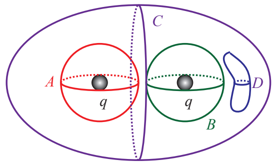


Fig. 3.6 Illustration of fluxes through surface A,B,C, and D.

Exercise 3.3: No charges, no field?

(From Sears and Semanskys)

A certain region of space bounded by an imaginary closed surface contains no charge. Is the electric field always zero everywhere on the surface? Is the flux always zero everywhere on the surface? Gauss' law relates the total flux to the charge contained by the surface—what will the total flux be here?

Exercise 3.4: Uniform charge density

(From Sears and Semanskys)

a) In a certain region of space, the volume charge density ρ has a uniform, positive value. Can $\mathbf{E}$ be uniform in this region?

b) Suppose that in this region of uniform positive ρ there is a "bubble" within which $\rho = 0$. What, if anything, can you say about the electric field within this bubble using Gauss' law?

Exercise 3.5: A different Coulumb's law

(From Sears and Semanskys) If the electric field of a point charge were proportional to $1/r^3$ instead of $1/r^2$, would Gauss' law still be valid? Explain your reasoning?

Hint. Consider a spherical surface centered around a single point charge.

3.5.2 Tutorials

Exercise 3.6: Symmetries

(From Steven Pollock)

In this exercise, we will use symmetry arguments and Gauss' law to find an expression for the electric field around an inifinitely long line of charge with constant charge density.

Usually, we begin by assuming that the electric field around the charged wire is entirely in the *radial* direction.

a) Give a brief symmetry argument for why the electric field should *not* have a *longitudinal* component (parallel with the wire).

b) Give a brief symmetry argument for why the electric field should *not* have a *tangential* component (circling around the wire).

c) Assuming that the electric field *is* purely radial, why would we choose an imaginary *cylinder* as our Gaussian surface? Why not a sphere or a cube? (Draw how you would orient the cylinder).

d) For an infinitely long wire with uniform charge density λ, what is the total charge on a small section of this wire of length L?

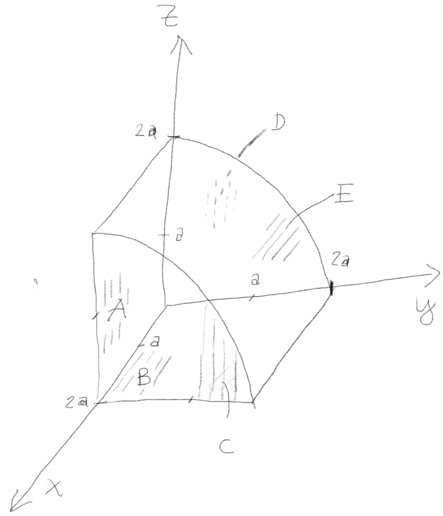
e) Use Gauss' law (on integral form) to solve for the electric field at any point around the wire. Define any symbols you use.

f) Challenge question (for fast teams): If the wire was not inifinite, but a segment of length L, would your formulation still hold, assuming you only want to know the electric field around the exact midpoint of the wire segment? Why/why not?

Exercise 3.7: Flux-trix

In this exercise you will learn to calculate fluxes for specific surfaces and to use Gauss' law to infer fluxes that may be difficult to calculate.

We will study an electric field $\mathbf{E} = E_0\hat{\mathbf{y}}$ generated by charges that are far away. We will now study the flux through several surfaces illustrated in the figure (assume all lengths are in units of a, which is a given length). Surface A and B are planar, surface E is a cylindrical shell, and surfaces C and D are the top and bottom caps that ensure that the whole volume is closed.



a) What are the surface normals for surfaces A, B, C, and D? How would you describe the surface normal for surface E?

b) Find the flux through surfaces A, B, C, and D.

c) Use Gauss' law to find the flux through surface E. How would you find this without using Gauss' law? Discuss how your methods can be generalized to other electric fields.

Let us now use a similar principle to find the flux in another situation. A charge q is placed at the origin. This is the only charge in the system. The charge sets up an electric field $\mathbf{E}(\mathbf{r})$.

d) What is the flux of $\mathbf{E}(\mathbf{r})$ through a triangular surface with corners at $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$? Does the result depend on the value of a?

Hint. Make a sketch. Is there another surface you can easily find the flux through? How can you use Gauss' law?

Exercise 3.8: SLAC lightning strike

(*From Steven Pollock, University of Colorado - Boulder*).

SLAC (Stanford Linear Accelerator Center) is where quarks (including the charm quark), and the tauon (like a heavier electron) were discovered. Charged particles are accelerated inside a long metal cylindrical pipe, which is 2 miles long and has a radius $a = 6$cm. All the air is pumped out of this pipe, known as the "beam line."

One afternoon, the beam line is struck by lightning, which gives it a uniform surface charge density ρ_s. After the lightning strike, Stanford physicists want to start accelerating particles in the beam line, but they are concerned that the charge density might affect the beam particles, causing them to crash into the wall of the pipe and burn a hole through it. Air and dirt would rush into the empty pipe causing months of expensive delay. You will investigate whether the surface charge of the beam line could affect the beam particles.

a) First, what is the infinitesimal area, dS, of a small patch on a cylindrical shell centered on the z-axis? Assuming you use this dS in a surface integral over a closed surface, give the vector direction of dS.

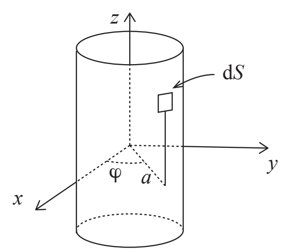


Fig. 3.7 Illustration of the system.

b) What direction does the **E**-field point at all points in space? Explain in detail how you know.

c) Use Gauss's Law to find the **E**-field at all points in space.

d) Does the charge ρ_s on the beam line affect the particles being accelerated inside it? Could it affect the electronic equipment outside the tunnel?

Exercise 3.9: Overlapping clouds of charge

(*From Danny Caballero, Michigan State University*)

a) For a cloud of charge of radius R with uniform charge density ρ_0, find the electric field inside and outside the cloud using Gauss' law.

b) Consider to oppositely charged clouds, both of radii R and uniform charge densities. They overlap with their centers separated by d. Find the electric field in the overlapping region.

c) Sketch the electric field lines in the overlapping region.

d) Discuss what happens as d becomes small compared with R. Discuss what the resulting (total, physical) charge distribution in space looks like.

e) (*Advanced, can be skipped*) Write a Python program to visualize the electric field in space. Place the center of the line connecting the two charges in the origin.

3.5.3 Homework

Exercise 3.10: Fluxes from point charges

A point charge $q_1 = 4.0$ nC lies on the x-axis in $x = 2.0$ m. Another point charge $q_2 = -6.0$ nC lies on the y-axis in $y = 1.0$ m.

a) A spherical surface S is centered in the origin with radius 0.5 m. What is the electric flux out of this surface, that is, $\Phi_E = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$

b) What is the flux out of the spherical surface if the radius is 1.5 m or 2.5 m?

c) Would it be useful to use Gauss' law to find the electric field $\mathbf{E}(x, y, z)$ in this system?

Exercise 3.11: Field from charge distributions

We have a total charge Q. Find the electric field $\mathbf{E}$ everywhere in space when:

a) Q is a point charge.

b) Q is uniformly distributed over the volume of a sphere with radius a so that the charge density ρ_v is:

$$\rho_v = \frac{Q}{4\pi a^3/3} \ . \tag{3.39}$$

c) Q is uniformly distributed on a spherical shell with radius a so that the surface charge density ρ_s is

$$\rho_s = \frac{Q}{4\pi a^2} \ . \tag{3.40}$$

d) Q is uniformly distributed over the volume of a sphere with radius a so that the charge density is proportional to the distance r from the center of the sphere, that is, $\rho_v = kr$ where k is a constant.

Hint. Determine k by calculating $Q = \int_v \rho \mathrm{d}v$. The geometry in this subexercise indicates that spherical coordinates is a good choice. Remember to use the correct volume element in the integral.

Exercise 3.12: Two wires (A)

Two very long wires lie parallell to each other in vacuum. The magnitude of their linear charge density ρ, is equal, although one is positive and the other is negative. The distance between the wires is x.

a) Find the direction of the $\mathbf{E}$-field half-way between the two wires without calculating it.

Hint. Compare the situation to that of finding the electric field strength in the half-way point between two equal but opposite charges. (Draw field lines)

b) Calculate the field strength at this point.

Hint. Use cylindrical symmetry and Gauss' law

c) Without calculating, compare the $\mathbf{E}$-field at a distance $x/3$ from the positive wire (and $2/3x$ from the negative) to the $\mathbf{E}$-field at a distance $x/3$ from the negative wire (and $2/3x$ from the positive).

Hint. See hint a)

d) Calculate the field strength at the locations in c).

Hint. See hint b)

Exercise 3.13: Two wires (B)

Repeat the previous exercise, but with to positively charged wires.

a) Find the direction of the $\mathbf{E}$-field half-way between the two wires without calculating it.

b) Calculate the field strength at this point.

c) Without calculating, compare the $\mathbf{E}$-field at a distance $x/3$ from one wire (and $2/3x$ from the other) to the $\mathbf{E}$-field at a distance $x/3$ from the first wire (and $2/3x$ from the second).

d) Calculate the field strength at the locations in c).

Exercise 3.14: Concepts in vector calculus

Define the following concepts and explain them in your own words:

a) flux

b) divergence

c) curl

d) conservative field

Write down and explain in your own words the contents of the following theorems:

e) the divergence theorem

f) Stokes' theorem

Exercise 3.15: Electrons in a box

a) A cubic box of sidelength $a = 1$ meter contains only electrons in a vacuum. You may assume that the **E**-field is normal on the faces of the cube, and that the electric field strength is constant on all faces with a magnitude of $E = 6.03\ \mu\text{N/C}$. How many electrons are in the box?

Hint. Use Gauss' law on a Gaussian surface equal to the surface of the box.

Exercise 3.16: Integrals in vector calculus

a) Calculate the integral

$$I = \int_v (\nabla \cdot \mathbf{F}) \mathrm{d}v \ , \tag{3.41}$$

where $\mathbf{F} = r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and the volume v is a sphere with radius R placed in the origin:

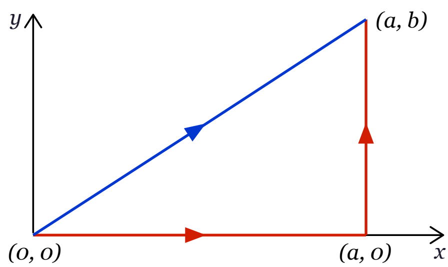
- By direct calculation
- By using the divergence theorem.

b) Calculate the curve integral

$$I = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \,, \tag{3.42}$$

where $\mathbf{F} = (xy^2 + 2y)\hat{\mathbf{x}} + (x^2y + 2x)\hat{\mathbf{y}}$,

- along the curve C_1 consisting of the two straight lines that connects the points $(0,0)$, $(a,0)$ and (a,b) in the figure.
- along the curve C_2 that consists of the straight line connecting the points $(0,0)$ and (a,b).
- These answers are the same, why? Explain using Stokes' theorem.



Exercise 3.17: Two line charges

a) Find the electrical field from an infinitely long line charge directed along the z-axis through a point $(a,0,0)$ with uniform line charge density ρ_l.

b) Find the electric field from two infinitely long lines charges directed along the z-axis: A line charge with line charge density ρ_l through $(a,0,0)$ and a line charge with line charge density $-\rho_l$ through $(-a,0,0)$.

c) Find an approximate value for the electric field when $x^2 + y^2 \gg a^2$ and $y = 0$. Express the result in terms of the dipole moment per unit length, $\mathbf{p} = 2a\rho_l\hat{\mathbf{x}}$.

4 Polarization and dielectrics

So far we have only addressed the electric field in a vacuum. What happens with the electric field inside a material such as water, a ceramic, or a metal? We will start by addressing how the electric field behaves inside an insulator and then later discuss the electric field inside a conductor. Inside an insulator, the charges are bound. An imposed electric field will displace the bound charges small distances, causing many small dipoles to be formed. These dipoles can be described as a distribution of bound charges, which will induce an electric field, which in turn will change the total electric field in the material. We will introduce the polarization vector, $\mathbf{P}$, to describe the average effect of the dipoles, the spatial distribution of bound volume charges, $\rho_{b,v}(\mathbf{r})$, surface charges $\rho_{b,s}(\mathbf{r})$, and the displacement field $\mathbf{D}$. We will demonstrate that we can introduce a modified version of Gauss' law for $\mathbf{D}$ that only depends on the free and not the bound charges. This provides us with a set of tools to extend our electrostatic theory to dielectric materials.

4.1 Dielectrics

A dielectric or an insulator is a material with very little free charges such as (pure) water, a ceramic, plastics, or glass. We discern between two types of dielectric: *polar* and *non-polar* as illustrated in Fig. 4.1. Polar systems consist of many small, permanent dipoles such as water molecules, that align with an external electric field, whereas a non-polar system consists of many small atoms with charge distributions that are displaced by an imposed electric field. What are the effects of these alignments or displacements?

4.1.1 Non-polar dielectrics

In a non-polar dielectric without any electric field, the electrons are distributed symmetrically around the positive charges, so that there are no net dipoles. However, if we apply an external electric field, the electron clouds around atoms or molecules tend to be displaced: The negative electron cloud will be displaced in a direction opposite the electric field, whereas the positive nucleus is displaced in the opposite direction. The net result is illustrated in Fig. 4.1. An equilibrium will form between the electric field pulling the electrons and the nucleus apart and the attractive forces between the negative electrons and the positive nucleus drawing them back together. As a result, each atom becomes a small dipole because of a small displacement of the electron cloud around the atom. The stronger the field, the more the electron cloud will be displaced, and the stronger is each dipole. We expect the induced dipole moment $\mathbf{p}$ of an individual atom to be proportional to the electric field:

$$\mathbf{p} = \alpha \mathbf{E} \ , \tag{4.1}$$

where α is called the *atomic polarizabiltiy*. The value of α depends on details in the structure of the atoms or the configuration they are in (in a molecule, a crystal, etc). However, if the electric field is too strong, the atom will be pulled completely apart, ionizing it, such as in a lightning. The free ions and electrons then become a conducting material. (For systems consisting of molecules or crystals, the situation is a bit more complicated as they may polarize more easily in some directions than in other. This may lead to a more complicated relationship between the effective polarization vector $\mathbf{p}$ and the electric field described by a polarization tensor.)

4.1.2 Example: Simple model for atomic polarizability

As part of our modeling approach, let us see how we can make a simplified model for atomic polarizability. We model an atom as a nucleus with charge q and the surrounding electronic cloud as a uniform, spherical

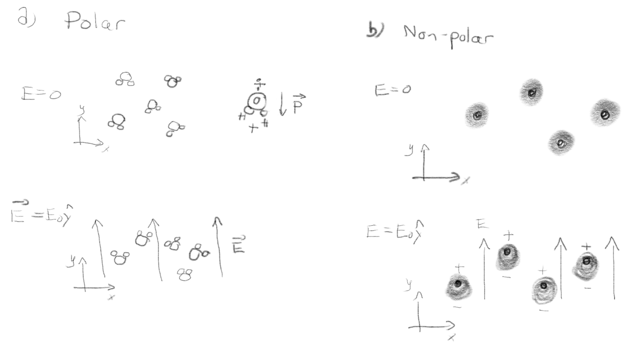


Fig. 4.1 Illustration of the polarization of a non-polar and a polar material.

distribution of charge with radius a and total charge $-q$. When the atom is put in an external electric field E_0, the nucleus and the electron cloud are displaced a distance d, which we assume is (much) smaller than a. In equilibrium, the force on the electron cloud from the external field must be equal to the force from the nucleus on the electron cloud, which again is equal to the force from the electron cloud on the nucleus. What is the electric field inside the electron cloud? We recall that the electric field at a distance r from the center inside a uniform, spherical charge distribution is $E_r(r) = qr/(4\pi\epsilon_0 a^3)$. In equilibrium, the distance between the nucleus and the center of the cloud is d, so that the force on the nucleus is $F = qqd/(4\pi\epsilon_0 a^3)$. This must be equal to the force from the field, qE_0, so that $qE_0 = qqd/(4\pi\epsilon_0 a^3)$. We recall that the dipole moment for two charges q and $-q$ at a distance d is $p = qd$. We therefore find that $p = qd = E_0 4\pi\epsilon_0 a^3 = \alpha E_0$. For this model, we see that the polarization (i.e. the dipole moment) is proportional to the applied field. It turns out that this model is within a factor four of the correct result for many simple atoms.

4.1.3 Polar dielectric

In a polar dielectric the electrons are distributed relative to the positive charges so that molecules behave as individual dipoles. An example is water. Each water molecule is a small dipole with a dipole moment $\mathbf{p}$ that points from the oxygen atom towards the midpoint between the hydrogen atoms. If a polar dielectric is subject to an external electric field, there will be no net force on the dipoles, but the dipole molecules will tend to orient in the electric field with the positive part of the dipole pointing in the direction of the local field. Without an applied electric field, the dipoles will point in random directions, with no net effect, but with an applied electric field, the dipoles will tend to align with the field. The stronger the field, the stronger will be the alignment and the stronger the net dipoles.

Let us address in detail what happens when a molecule such as a water molecule is placed in an electric field $\mathbf{E}$. We model the water as a two-charge dipole with a charge q in a position $\mathbf{d}/2$ relative to the center of the atom, in the middle between the hydrogen atoms, and a charge $-q$ at the oxygen atom at a position $-\mathbf{d}/2$. If the field is uniform, the net force from the field is $q\mathbf{E} - q\mathbf{E} = 0$. However, the torque around the center of the molecule will be

$$\tau = \mathbf{d}/2 \times q\mathbf{E} + (-\mathbf{d}/2) \times -q\mathbf{E} = q\mathbf{d} \times \mathbf{E} \ . \tag{4.2}$$

We replace $\mathbf{p} = q\mathbf{d}$ so that

$$\tau = \mathbf{p} \times \mathbf{E} \ . \tag{4.3}$$

We see that this will make the molecule rotate so that the dipole moment will align with the electric field.

4.1.4 Polarization

The reponse of both a polar and a non-polar dielectric due to an external field is the formation of a net set of dipoles that point more or less in the direction of the electric field. We say that the material becomes *polarized*. We describe the effect by the *polarization* $\mathbf{P}$, which is the dipole moment per unit volume. The sum of the dipole moments of all the dipoles (with index i) in a small volume element $\mathrm{d}v$ is then:

$$\mathbf{P}\mathrm{d}v = \sum_{i \text{ in } \mathrm{d}v} \mathbf{p}_i = N_v \frac{1}{N_v} \sum_{i \text{ in } \mathrm{d}v} \mathbf{p}_i = N_v \langle \mathbf{p} \rangle \ , \tag{4.4}$$

where N_v is the number of dipoles in the element $\mathrm{d}v$ and $\langle \mathbf{p} \rangle$ is the average dipole moment in the volume $\mathrm{d}v$.

The polarization vector depends on total field. The mechanisms we have described for polarization implies that the polarization vector in a point $\mathbf{r}$ will depend on the electric field in this point. We write this the following way: $\mathbf{P}(\mathbf{E})$. The polarization vector is a function of the (local) electric field. Notice that the polarization is a function of the *total* electric field. Notice that it is the total field – including the field set up by all the dipoles in the material. This may initially seem a bit counterintuitive. The polarization is a function of the total electric field, but the polarization will also affect the electric field, which again will affect the polarization. We often call such relationships, or equations, self-consistent solutions. Mathematically, it is not difficult to set up such as relation, and we also understand the physics of this process: The polarization is due to the electric field that affects the atoms/molecules, and this electric field must be the total electric field, including the field set up by the displacement of the molecules themselves.

Linear dielectrics. We saw that for a single atom the polarization was aligned with the electric field $\mathbf{p} = \alpha \mathbf{E}$ and for polar dielectrics we also expect the effective polarization to be aligned with the electric field. Materials that have this property also on the macroscopic scale are called *linear dielectrics*. We characterize a linear dielectric by the electric susceptibility χ_e:

$$\boxed{\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}} \tag{4.5}$$

The electric susceptibility is unitless. In vacuum $\chi_e = 0$ and in air $\chi_e \simeq 0$. Many materials are linear dielectrics, but not all. We previously mentioned that some molecules may have an anisotropic displacement of the charges when in an electric field, and similar effects may occur in a crystal, where the electron cloud may be more easily displaced in some directions. These effects may imply that the dielectric is anisotropic or non-linear. Most materials we will discuss in this text are linear dielectrics.

4.2 Bound charges

We have now argued that the effect of an applied field on a dielectric material is a reorientation of permanent dipoles (for a polar dielectric), or a local displacement of the electron cloud to create dipoles (non-polar dielectric). While there are no free charges in the material, the small induced or aligned dipoles will lead to a distribution of charges in the dielectric material. We call these charges *bound charges*, because they are bound to the atoms or molecules and are not free to move around. Our plan is to relate the polarization $\mathbf{P}$ to a density of bound volume and surface charges. We can then use this distribution of charges to calculate the electric field due to the bound charges, that is, the electric field due to the polarization.

Fig. 4.2 illustrates the alignment of dipoles that make up the polarization $\mathbf{P}$ inside a dielectric material. There are no net charges inside the volume, only the bound charges. Each small dipole in itself does not contribute with any net charge — it has a positive and a negative end, with equal and opposite charges. However, if we look at a part of the system, such as the part inside the volume in Fig. 4.2a, then the bounding surface intersects some of the dipoles so that one side of the dipole is inside the volume and one side is outside the volume. This will lead to sets of net charges on the inside (and outside) of the surface.

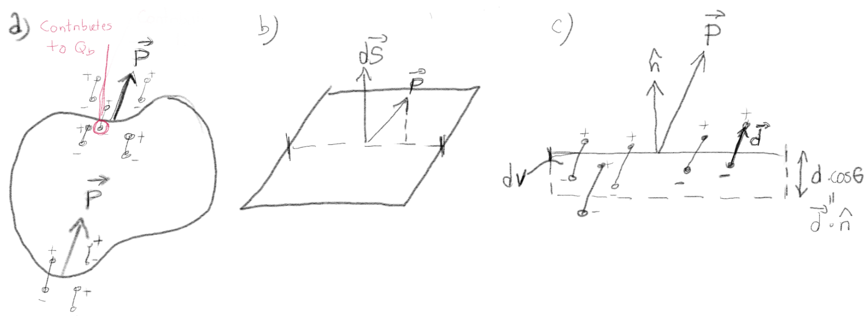


Fig. 4.2 Illustration of the bound charges inside a volume v (a) and a small volume element dv (b,c).

Bound charge within a volume. How much net bound charge is within the volume near the surface? We look at a small surface element $d\mathbf{S}$ with surface normal $\hat{\mathbf{n}}$ as illustrated in Fig. 4.2c. In this point, the polarization vector is $\mathbf{P}$. We assume that the polarization is due to small dipoles with dipole moment $\mathbf{p} = Q\mathbf{d}$. We see that only dipoles that cross the surface contribute to the bound charge within the volume. Dipoles that are completely within the volume do not contribute to the net charge inside the volume since both the negative and the positive charges in the dipole are inside the volume and the *net* contribution is therefore zero. However, dipoles that cross the surface contribute to the net charge because only one part of the dipole is inside the volume. We see that all dipoles that are within a range $d\cos\theta = \mathbf{d} \cdot \hat{\mathbf{n}}$ cross the surface and hence contribute to the net bound charge inside the volume. In the figure, each dipole contributes with a charge $-Q$ to the bound charge. If N_v is the number of dipoles per unit volume, then the number of dipoles in a volume $\mathrm{d}v = d\cos\theta \mathrm{d}S$ is $N_v \mathrm{d}v$. We can rewrite this as

$$N_v \mathrm{d}v = N_v d \cos\theta \mathrm{d}S = N_v \mathbf{d} \cdot \hat{\mathbf{n}} \mathrm{d}S = N_v \mathbf{d} \cdot \mathrm{d}\mathbf{S} \,, \tag{4.6}$$

The contributions from these dipoles to the bound charge $\mathrm{d}Q_b$ in the volume $\mathrm{d}v$ is then

$$\mathrm{d}Q_b = (-Q) N_v \mathrm{d}v = -Q N_v \mathbf{d} \cdot \mathrm{d}\mathbf{S} = -N_v (Q\mathbf{d}) \cdot \mathrm{d}\mathbf{S} \tag{4.7}$$

$$= -N_v \mathbf{p} \cdot \mathrm{d}\mathbf{S} = -\mathbf{P} \cdot \mathrm{d}\mathbf{S} \,, \tag{4.8}$$

where we have used that $p = Q\mathbf{d}$ and that $N_v \mathrm{d}v\mathbf{p} = \mathbf{P}\mathrm{d}v$ and therefore that $N\mathbf{p} = \mathbf{P}$. In order to find the net bound charge from the whole surface, we integral over the surface:

$$\boxed{Q_b = -\oint_S \mathbf{P} \cdot \mathrm{d}\mathbf{S}} \,. \tag{4.9}$$

We call this the *bound charge* because this charge is bound to the material. It is not free to move. And it is a charge that appears due to the application of an external electric field. Without an electric field, there will be no bound charge.

4.2.1 Bound volume charge density

In (4.9) we found the total bound charge Q_b in a volume v enclosed by the surface S. We can rewrite this in terms of the volume charge density of the bound charge $\rho_{v,b}$:

$$Q_b = -\int_S \mathbf{P} \cdot \mathrm{d}\mathbf{S} = \int_v \rho_{v,b} \mathrm{d}v \,. \tag{4.10}$$

We can apply the divergence theorem to this surface integral, getting

$$Q_b = -\int_v \nabla \cdot \mathbf{P} \rho_{v,b} \mathrm{d}v = \int_v \rho_{v,b} \mathrm{d}v \,. \tag{4.11}$$

This is valid for any surface S enclosing a volume v, and therefore the arguments of the integrals must also be equal:

$$-\nabla \cdot \mathbf{P} = \rho_{v,b} \,. \tag{4.12}$$

This implies that in a region where the polarization $\mathbf{P}$ is uniform, the divergence is zero, and hence the volume density of bound charges is zero, $\rho_{v,b} = 0$.

4.2.2 Bound surface charge density

What is the surface charge density on the interface of a dielectric, that is, on the boundary between a dielectric and vacuum? We look at a small volume Δv of height Δh and area ΔS as illustrated in Fig. 4.3. We assume that there is a surface charge density of bound charge, $\rho_{s,b}$, on the interface surface so that the bound charge inside the volume Δv is $\Delta Q_b = \rho_{s,b}\Delta S$. From (4.9) we know that the bound charge inside the volume is also given as the integral of $\mathbf{P}$ over the enclosing surface S:

$$\Delta Q_b = \rho_{s,b}\Delta S = -\oint_S \mathbf{P} \cdot \mathrm{d}\,\mathbf{S} \tag{4.13}$$

As $\Delta h \to 0$ the contributions to the integral from the side walls of the cylinder goes to zero. In vacuum, there is no dielectric, and the polarization vector $\mathbf{P}$ is zero. We are therefore left with the contribution to the integral from the surface inside the dielectic. This surface points into the dielectric away from the interface, thus the surface element is $\Delta\mathbf{S} = -\Delta S\hat{\mathbf{n}}$. The integral is therefore approximately:

$$\rho_{s,b}\Delta S = -\oint_S \mathbf{P} \cdot \mathrm{d}\,\mathbf{S} = -\mathbf{P} \cdot \Delta\mathbf{S} = \mathbf{P} \cdot \hat{\mathbf{n}}\Delta S \tag{4.14}$$

and therefore we have found that the surface charge density of bound charge is

$$\rho_{s,b} = \mathbf{P} \cdot \hat{\mathbf{n}} \tag{4.15}$$

Bound charge density

The **bound volume charge density** $\rho_{v,b}$ is related to the polarization $\mathbf{P}$ through

$$\rho_{v,b} = -\nabla \cdot \mathbf{P}\ . \tag{4.16}$$

The **bound surface charge density** $\rho_{s,b}$ at the surface of a dielectric material with surface normal $\hat{\mathbf{n}}$ is:

$$\rho_{s,b} = \mathbf{P} \cdot \hat{\mathbf{n}}\ . \tag{4.17}$$

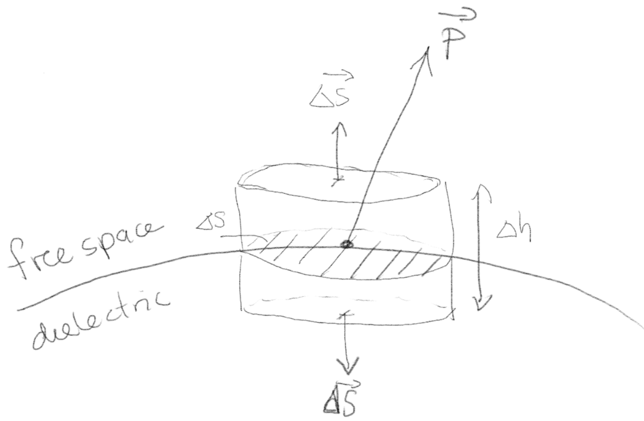


Fig. 4.3 Illustration of the bound charges inside a volume v (a) and a small volume element dv (b,c).

4.2.3 Example: Infinite polarized slab

An infinte dielectric slab of thickness d in has a uniform polarization $\mathbf{P}$ *in the direction normal to the slab surface. Find the bound surface charge densities and the electric field in the slab.*

Fig. 4.4a illustrates the geometry of the system. We choose the x-direction along $\mathbf{P}$, so that $\mathbf{P} = P\hat{\mathbf{x}}$. Because the polarization is uniform inside the slab, we know that $\rho_{v,b} = \nabla \cdot \mathbf{P} = 0$ inside the slab. However, there may still be a surface charge density. For surface S_1, the surface normal is $\hat{\mathbf{n}}_1 = \hat{\mathbf{x}}$. We find the surface charge density from $\rho_{s,b} = \mathbf{P} \cdot \hat{\mathbf{n}}_1 = P$. Similarly, for surface S_2 we have $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$ and $\mathbf{P} \cdot \hat{\mathbf{n}}_2 = P\hat{\mathbf{x}} \cdot -\hat{\mathbf{x}} = -P = -\rho_{s,b}$.

We can then use the surface charge densities to find the electric field. For a surface charge density ρ_s on an infinite surface, we recall that the electric field only has a component normal to the surface, $\mathbf{E} = E_x(x)\hat{\mathbf{x}}$, and that the field is mirror symmetric around the surface: $E_x(x) = -E_x(-x)$. We find the electric field from Gauss' law on a cylindrical Gauss ==surface as illustrated in Fig. 4.4b. Only the cylinder end surfaces ΔS contribute to the flux $\Phi = 2E_x\Delta S = \rho_s\Delta_S/\epsilon_0$ and therefore $E_x = \rho_s/(2\epsilon_0)$.

We find the electric field from the superposition of the field from the two bound surface charge densities $\rho_{s,b}$ and $-\rho_{s,b}$. We find that for $x > d/2$, the net field is $E_x = \rho_s/(2\epsilon_0) - \rho_s/(2\epsilon_0) = 0$. For $-d/2 < x < d/2$, the net field is $E_x = -\rho_s/(2\epsilon_0) - \rho_s/(2\epsilon_0) = -\rho_s/\epsilon_0 = -P/\epsilon_0$. For $x < -d/2$, the two contributions cancel, $E_x = -\rho_s/(2\epsilon_0) + \rho_s/(2\epsilon_0) = 0$. The electric field (from the polarization $\mathbf{P}$) is therefore $\mathbf{E} = -P\hat{\mathbf{x}}$ inside the slab.

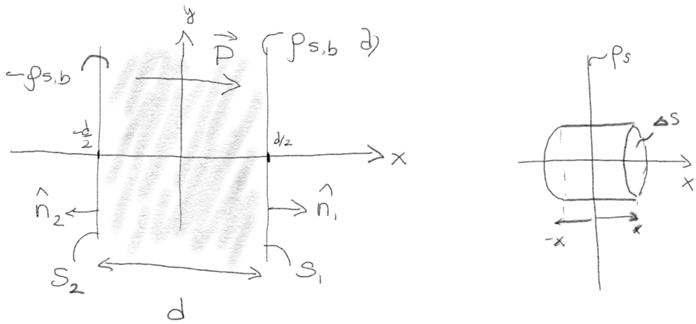


Fig. 4.4 Illustration of an infinite dielectric slab of thickness d with a uniform polarization $\mathbf{P}$.

Method: Finding the electric field from polarization

We have now seen than when dielectric materials are subject to electric fields, they will become polarized with a polarization $\mathbf{P}$. If we know the polarization, we can use this to find the density of bound charges. The volume density of bound charge, $\rho_{v,b} = -\nabla \cdot \mathbf{P}$, and the surface density of bound charge at an interface between the dielectric and vacuum (or air) is $\rho_{s,b} = \mathbf{P} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the surface normal pointing from the dielectric into to vacuum/air. We can then calculate the electric field $\mathbf{E}$ and the electric potential V from the charge densities by integrating Coulomb's law on the appropriate form as we have learned to do previously.

4.3 Generalized Gauss' law

We have now seen that the polarization $\mathbf{P}$ depends on the electric field, where for linear dielectric media $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$, and that we can find the bound charges from the polarization. How can we put all of this in a common framework? We can do this be rewriting Gauss law to include the bound charges due to polarization in addition to the free charges, and then use the same methods we developed previously to find the electric field.

Gauss law states that

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{in}}}{\epsilon_0} = \frac{Q_f + Q_b}{\epsilon_0} \ . \tag{4.18}$$

where Q_{in} is the total charge inside the volume, which is the sum of the free charges and the bound charges. What are the free charges? We use the word *free charge* to specify the charges we discussed in previous chapter and contrast these charges from the *bound* charges that are due to polarization of a dielectric material. We would like to have a version of Gauss law that only includes the free charges, which are the ones that we know where are without solving an equation for the polarization. We rewrite Gauss' law by replacing the bound charges Q_b with the integral over the polarization:

$$Q_b = -\oint_S \mathbf{P} \cdot \mathrm{d}\mathbf{S} \ . \tag{4.19}$$

getting

$$\epsilon_0 \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = Q_f - \oint_S \mathbf{P} \cdot d\mathbf{S} \ . \tag{4.20}$$

We rewrite so that only the free charge is on the right-hand side:

$$\oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} = Q_f \ . \tag{4.21}$$

Displacement field

We therefore introduce a new quantity, the **displacement field** or simply the **D**-field:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \ . \tag{4.22}$$

Gauss' law in a dielectric material then becomes:

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q_{\text{free in } S} \ , \tag{4.23}$$

where $Q_{\text{free in } S}$ is the free (non-bound) charge inside the surface S.

Linearly polarizable media. For linearly polarizable media $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. For such a medium we get

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 \left(1 + \chi_e\right) \mathbf{E} = \epsilon_r \epsilon_0 \mathbf{E} = \epsilon \mathbf{E} \ , \tag{4.24}$$

where $\epsilon_r = (1 + \chi_e)$ is called the *relative permittivity* and $\epsilon = \epsilon_r \epsilon_0$ is called the *absolute permittivity* or simply the permittivity.

In general, a material is *linearly polarizable* if the relation between $\mathbf{P}$ and $\mathbf{E}$ is linear. If the material is also *isotropic*, the relation does not depend on the direction $\mathbf{E}$ and χ_e and ϵ_r are scalars. If χ_e does not depend on the position in space we say that the medium is *homogeneous*.

Effect of dielectric on the electric field. A distribution of free charges set up an electric field $\mathbf{E}$ in vacuum. If we replace the vacuum with a dieletric material with relative permittivity ϵ_r, what happens to the electric field inside the dielectric? We notice that $\mathbf{D}$ only depends on the free charges. The $\mathbf{D}$-field is therefore the same in both situations. However, the electric field will be $\mathbf{E} = \mathbf{D}/(\epsilon_r \epsilon_0)$ in the dielectric. This is a factor $1/\epsilon_r$ smaller than in vacuum.

Typical values of ϵ_r. Typical examples of linear, isotropic media are water, air, gases, and glass. (Crystals are often anisotropic). The relative permittivity of water is 81. Electric fields are therefore reduced by a factor 81 in water compared to in vacuum. You can find the relative permittivity of many materials in physical tables.

Material	ϵ_r	E_c(MV/m)
Vaccum	1	
Air	1.0005	3
Paper	3-5	16
Wood	2-5	10
Polystyrene	2.5	24
Oil	4	12
Glass (pyrex)	5	14
Water	81	
Porcelain	6	12
Rubber (neoprene)	6.7	12
Mica	7	150
Strontium titanate	300	8
Barium titanate	1200	100

4.3.1 Gauss' law on differential form

Gauss' law can be rewritten on differential form using the divergence theorem:

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{D} \mathrm{d}v = Q_{in} = \int_v \rho_{\mathrm{free}} \mathrm{d}v \tag{4.25}$$

and therefore we get Gauss' law on differential form:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_{\text{free}}} \tag{4.26}$$

It is common just to use ρ for the free charges.

There is no Coulomb's law for $\mathbf{D}$. Notice that even if Gauss' law for the displacement field $\mathbf{D}$ looks very similar to Gauss law for the electric field — you simply have to replace the total charge ρ with the free charges ρ_f — it is in general not possible to find the displacement field by integrating the free charge density in space, as we did for the electric field. This is because the divergence alone is not sufficient to determine the field: you also need to know the curl of the field. For the electric field, the curl is zero, but the curl of the displacement field is not always zero. The curl of the displacement field is

$$\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P} \ . \tag{4.27}$$

and the curl of $\mathbf{P}$ is not in general zero. This means that there is in general also no potential for $\mathbf{D}$ because $\nabla \times \mathbf{D}$ is not in general zero.

Method: Using Gauss' law to find the electric field for linear dielectrics

We should now update our method of applying Gauss' law to the general case with linear dielectric materials:

- Find a set of surfaces that enclose a volume such that $\mathbf{D} \cdot \hat{\mathbf{n}}$ is constant on each such surface element. (It may be zero on some of the surfaces — zero is a constant!) For linear dielectrics you may use your physics knowledge of the symmetries of $\mathbf{E}$, because $\mathbf{D} = \epsilon \mathbf{E}$.
- This often requires that you find a simplfied description of the field in a chosen coordinate system, such as $\mathbf{D} = D_r(r)\hat{\mathbf{r}}$.
- Find the flux integral.
- Use Gauss' law to find the displacement field as a function of charge and position.
- Find the electric field from $\mathbf{E} = \mathbf{D}/\epsilon$.
- Notice that the surface does not have to enclose all the charges — it is allowed and indeed often necessary to chose a surface that contains only some of the charges. However, the electric field must be a constant on the surfaces you have chosen.

4.3.2 Dielectric breakdown

Materials are dielectric only up to a given field strength. From the models we proposed above, it is clear that if the electric field becomes too large, the electron cloud will be pulled completely away from the positive charges in the nucleus, ionizing the material. This leads to the formation of free charges, and the material becomes an conductor. In many cases this gives a channel of ionized electrons which is often seen as a lightning. For air this happens for fields that are larger than $E \geq 3 \cdot 10^6 \text{V/m}$. This limit is called the *dielectric strength*, E_c. We will look at dielectric breakdown in detail later when we address lightning strikes.

Material	E_c (MV/m)	Material	E_c (MV/m)
Air	3	Glass	30
Wood	10	Silicon (Si)	30
Porcelain	11	Mica	200
Paper	15	Fused quartz	1000
Rubber	25	Si_3N_4	1000

4.3.3 Example: Point charge in a dielectric medium

Find the electric field from a point charge Q in the origin in a linear, isotropic dielectric material with permittivity ϵ.

We apply the same approach as we did for Gauss' law for a single charge. Because $\mathbf{D} = \epsilon \mathbf{E}$, we can assume that the $\mathbf{D}$-field has the same symmetries as the $\mathbf{E}$-field. We use Gauss' law on a sphere with radius r, we get that

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = D 4\pi r^2 = Q \,, \tag{4.28}$$

which gives that

$$\mathbf{E} = \frac{D}{\epsilon} = \frac{Q}{4\pi\epsilon r^2} \hat{\mathbf{r}} \,. \tag{4.29}$$

What if the dielectric medium only stretches out a distance a? In that case, the electric field is given with the expression above for $r < a$. When $r > a$ the field is the same as for a point charge in vacuum, that is,

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2} \hat{\mathbf{r}} & r < a \\ \frac{Q}{4\pi\epsilon_o r^2} \hat{\mathbf{r}} & r > a \end{cases} \tag{4.30}$$

What happens at the boundary between the two regions? We expect that there will be surface charges on the outside of the spherical surface at $r = a$. These charges will reduce the field inside the sphere.

4.3.4 Example: Bound surface charge on a spherical dielectric

What is the bound surface charge density on the outer interface of a spherical dielectric of radius a and dielectric constant ϵ with a single charge q at the center?

For $r < a$ the electric field is $\mathbf{E} = Q/(4\pi\epsilon r^2)\hat{\mathbf{r}}$. We know that that polarization is related to $\mathbf{E}$ through $\epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\epsilon_r\mathbf{E}$, which gives $\mathbf{P} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$. We can therefore find the surface charge density when $r \to a$ as

$$\rho_S = \mathbf{P}\cdot\hat{\mathbf{n}} = \epsilon_0(\epsilon_r - 1)E = \epsilon_0(\epsilon_r - 1)\frac{Q}{4\pi\epsilon_0\epsilon_r a^2} = \frac{(\epsilon_r - 1)}{\epsilon_r}\frac{Q}{4\pi a^2}\ . \quad (4.31)$$

Method: Find the bound charges in a linear dielectric

If we have found $\mathbf{D}$, we can also find both the electric field $\mathbf{E} = \mathbf{D}/\epsilon$ and the polarization $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$. We can then use this to find the bound volume charge density from $\rho_{v,b} = -\nabla\cdot\mathbf{P}$ and the bound surface charge densities from $\rho_{s,b} = \mathbf{P}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal vector on the interfacial surfaces.

4.3.5 Example: Coaxial cable filled with a dielectric medium

Find the electric field inside and outside a coaxial cable with inner radius a with charge density ρ, and outer radii b and c with charge density $-\rho$, filled with a dielectric with permittivity ϵ in the region between the inner and the outer radius.

The system is illustrated in Fig. 4.5.

We plan to use Gauss' law on integral form. We notice that the systsem has cylindrical symmetry. We use a cylinder surface with radius r and length L. We assume that material from $0 < r < a$ has permittivity ϵ_0, that the material from $a < r < b$ has permittivity ϵ, and that the material from $b < r$ has permittivity ϵ_0.

We assume that the $\mathbf{E}$- and $\mathbf{D}$-fields only have radial components. The flux through the top and bottom surfaces in a cylinder of length L will

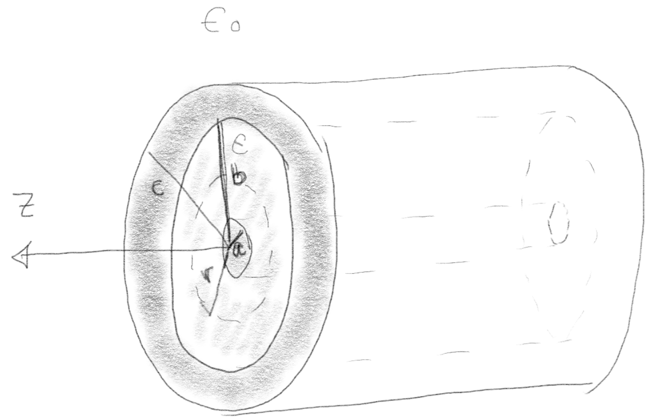


Fig. 4.5 Illustration of a coaxial cable.

therefore be zero, since the field will be normal to the surface normal here. Gauss' law therefore gives

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = 2\pi r L D = Q \tag{4.32}$$

We then study the regions separately:

When $r < a$**.** When $r < a$, the charge is the charge inside the Gauss surface of radius r. In this range, the charge density is ρ so that the charge inside a cylinder of radius r is $Q = \pi r^2 L \rho$. We therefore find

$$2\pi r L D = \pi r^2 L \rho \Rightarrow D = \frac{1}{2}\rho r \tag{4.33}$$

We use that $\mathbf{E} = \mathbf{D}/\epsilon_0$, giving

$$E = \frac{1}{2\epsilon_0}\rho r \tag{4.34}$$

When $a < r < b$**.** When $a < r < b$, the charge inside is the charge in the inner cylinder, $Q = \pi a^2 L \rho$. We therefore find

$$2\pi r L D = \pi a^2 L \rho \Rightarrow D = \frac{\rho r}{2a^2} \; E = \frac{\rho r}{2\epsilon a^2} \; . \tag{4.35}$$

When $b < r < c$**.** When $b < r < c$, the charge inside is the charge in the inner cylinder plus the part of the outer cylinder that is inside the Gauss surface at r:

$$Q = \pi a^2 L \rho + \pi (r^2 - b^2)(-\rho) L \; . \tag{4.36}$$

This gives that for the field:

$$2\pi r L D = \pi a^2 L\rho - \pi r^2 \rho L + \pi b^2 \rho L \Rightarrow D = \frac{\pi\rho}{2r}\left(a^2 + b^2 - r^2\right) . \quad (4.37)$$

and similarly for the electric field:

$$E = \frac{\pi\rho}{2\epsilon_0 r}\left(a^2 + b^2 - r^2\right) \quad (4.38)$$

When $c < r$**.** When $c < r$ the charge inside is the charge in the inner and the outer cylinder: $Q = \pi a^2 L\rho - \pi(c^2 - b^2)L\rho$. If this is non-zero, there will be a field outside the cylinder. If the charge densities and thickness is such that the net charge is zero, there will not be any field outside the outer cylinder. The field outside the cylinder will be

$$D = \frac{\rho}{2r}\left(a^2 + b^2 - c^2\right) \ , \ E = \frac{\rho}{2\epsilon_0 r}\left(a^2 + b^2 - c^2\right) . \quad (4.39)$$

What value of ϵ to use. Notice that the value of ϵ that is used for the conversion between **D** and **E** is the local value. For a Gauss surface, this means that it is the value *at* the surface, not the value inside the surface.

4.4 Boundary conditions for E and D

What happens on the interface between two (dielectric) media, such as the interface between a plastic and air, or between water and your skin?

We know the electric field **E** and the displacement **D** on one side of the interface, what is it on the other side? The situation is illustrated in Fig. 4.6.

4.4.1 Tangential boundary condition

Strategy: Use that the line integral of **E** over a closed loop is zero to related the field on one side to the other.

We construct an integration loop C with a very small width Δw as illustrated in the figure. Even if there is a surface charge density σ_s on the surface, the integral around a closed loop is zero. This is also valid across an interface.

$$\mathbf{E}_1 \cdot d\mathbf{l} + \mathbf{E}_2 \cdot (-d\mathbf{l}) = 0 \quad (4.40)$$

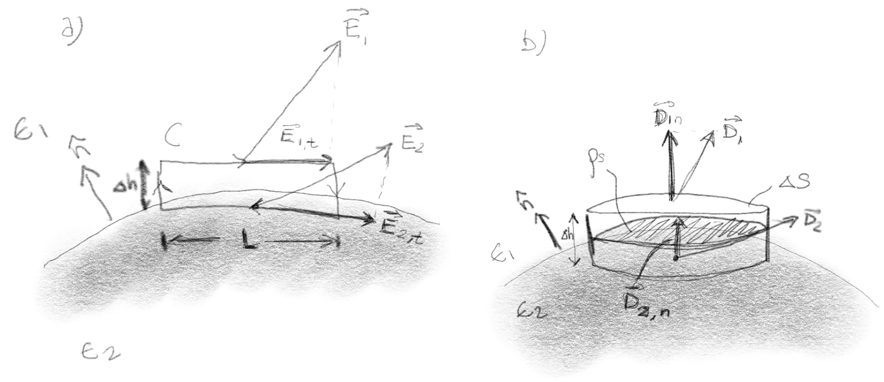


Fig. 4.6 Illustration of the boundary conditions for the **E** and **D** fields.

Since $d\mathbf{l}$ is tangential to the surface, this means that the tangential components are equal:

$$\boxed{E_{1t} = E_{2t}} \tag{4.41}$$

4.4.2 Normal boundary condition

Strategy: Use Gauss' law to relate the normal component of the field on each side of the interface to the charge on the interface.

We construct an integration cylinder S of width Δw and area ΔS. We apply Gauss' law for dielectrics, getting

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \mathbf{D}_1 \cdot \hat{n} \Delta S + \mathbf{D}_2 \cdot (-\hat{n}) \Delta S = q = \sigma_s \Delta S \;, \tag{4.42}$$

where we have assumed that as Δw becomes small, this contribution is negligible, and that $\hat{n}$ is a normal vector to the surface. The result is that the normal components of **D** is related to the surface charge density:

$$\boxed{D_{1n} - D_{2n} = \sigma_s} \;. \tag{4.43}$$

Two dielectric media. If both media are dielectric, there are no free surface charges (it is only the free charges that are included in Gauss' law for **D**), and the normal components of **D** are equal across the surface: $D_{1n} = D_{2n}$. This is the case for a point charge in a dielectric medium. There is no change in **D** across the boundary, but there is a change in **E** as we found in the discussion on boundary conditions for **P** above.

Boundary conditions for the electric field

At an interface between a dielectric material 1 and adielectric material 2 with a surface normal $\hat{\mathbf{n}}$ pointing from material 2 to material 1, we have the following boundary conditions:

$$E_{1,t} = E_{2,t} \ , \tag{4.44}$$

and

$$\mathbf{D}_1 \cdot \hat{\mathbf{n}} - \mathbf{D}_2 \cdot \hat{\mathbf{n}} = \rho_s \ . \tag{4.45}$$

4.4.3 Example: Cavities in a dielectric

A linear dielectric has an electric field $\mathbf{E} = E_0 \hat{z}$ and a displacement field $\mathbf{D} = \epsilon \mathbf{E}$. The dielectric has (a) a thin cylindrical hole or (b) a wide, short cylinderical hole directed along the field. What are the fields inside the holes (cavities)?

For a long, thin cylinder directed along $\mathbf{E}$, boundary conditions on the side of the cylinder states that the tangential fields must be the same on the inside and outside of the cylinder. Because the electric field only has a tangenetial component here, the field inside the cylinder is the same as the field in the dielectric: $\mathbf{E}_1 = \mathbf{E}$.

For a wide, short cylinder directed along $\mathbf{E}$, the boundary conditions states that $D_{1n} = D_{2n}$, and therefore that $\epsilon_0 E_1 = \epsilon E_0$, and $E_1 = E_0(\epsilon/\epsilon_0)$. If the dielectric is water and the hole is a dilute gas, then the field inside the hole is about 80 times that of the field in the water.

4.5 Summary

Polarization. The polarization vector $\mathbf{P}$ is the dipole moment per unit volume

$$\mathbf{P} = \frac{1}{v} \sum_{i \text{ in } v} \mathbf{p}_i \ . \tag{4.46}$$

The polarization depends on the **total electric field**. In a **linear dielectric**:

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \ , \tag{4.47}$$

where $\mathbf{E}$ is the local electric field. The constant χ_e is called the electrical susceptibility.

Bound charges. The **bound charge** Q_b inside a closed surface S is

$$Q_b = -\oint_S \mathbf{P} \cdot d\mathbf{S} \; . \tag{4.48}$$

The **volume bound charge density** $\rho_{v,b}$ is

$$\rho_{v,b} = -\nabla \cdot \mathbf{P} \; , \tag{4.49}$$

and the **surface bound charge density** $\rho_{s,b}$ is

$$\rho_{s,b} = \mathbf{P} \cdot \hat{\mathbf{n}} \; , \tag{4.50}$$

where $\hat{\mathbf{n}}$ is the surface normal.

Displacement field. The **displacement field D** is defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.51}$$

Gauss' law in a dielectric material is

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{free in } S} \; , \tag{4.52}$$

where the charge $Q_{\text{free in } S}$ is the free (not bound) charge inside the surface S.

In **linearly polarizable media** $\mathbf{D} = \epsilon \mathbf{E}$, where $\epsilon = \epsilon_r \epsilon_0$ is called the absolute permittivity.

Gauss' law on differential form becomes $\nabla \cdot \mathbf{D} = \rho_{v,\text{free}}$.

Boundary conditions for the electric field. The tangential components of the electric field is continuous across a boundary between two different dielectric materials, $E_{1,t} = E_{2,t}$. The normal component of the displacement field is related to the free surface charge density ρ_s across a boundary between two different dielectric materials, $D_{1,n} - D_{2,n} = \rho_s$.

4.6 Exercises

4.6.1 Test yourself

Exercise 4.1: Torque on dipole

A dipole with two charges $-Q$ and Q a distance d apart is in a field $\mathbf{E} = E_0 \hat{\mathbf{y}}$.

a) What orientations of the dipole gives zero torque around its center?

b) What orientations give the maximum torque?

c) Does it matter where the center of the dipole is located?

Exercise 4.2: Bound charges

A dielectric sphere with radius a has a uniform polarization $\mathbf{P} = P\hat{\mathbf{z}}$.

a) What is the bound volume charge density inside the sphere?

b) Sketch the bound surface charge density $\rho_{s,b}(\theta)$ on the surface of the sphere as a function of the angle θ in the xz-plane.

c) What is the net bound charge on the whole sphere?

Exercise 4.3: Charge in dielectric

A charge q is placed in the origin in (1) vacuum and (2) an infinite, isotropic, linear dielectric.

a) Is the displacement field at a distance a from the charge larger, equal, or smaller in case (1) than in case (2)?

b) Is the electric field at a distance a from the charge larger, equal, or smaller in case (1) than in case (2)?

c) Is the polarization at a distance a from the charge larger, equal, or smaller in case (1) than in case (2)?

Exercise 4.4: Charge in hole

A charge q is placed in the origin in the center of a spherical empty hole (vacuum) or radius a surrounded by an infinite, isotropic, linear dielectric.

a) What is the electric field as a function of r, the distance to the charge, in the hole?

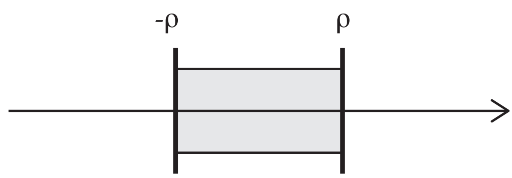
b) What is the displacement field as a function of r, the distance to the charge, in the hole?

c) What is the displacement field as a function of r outside the hole? Sketch the displacement field from $r < a$ to $r > a$.

d) What is the electric field as a function of r outside the hole? Sketch the electric field from $r < a$ to $r > a$.

Exercise 4.5: Continuity

The figure illustrates a system consisting of an infinite slab of a dielectric with dielectric constant ϵ in vacuum. There are surface charge densities ρ_s and $-\rho_s$ on the two surfaces.



a) Sketch the free and bound surface charge densities in the system.

b) Sketch the $\mathbf{D}$-field in the system.

c) Sketch the $\mathbf{E}$-field in the system.

4.6.2 Discussion exercises

Exercise 4.6: Temperature dependence

Water is polarized when subject to an electric field. Do you think the polarization increases or decreases if you increase the temperature. Explain your reasoning.

Exercise 4.7: Continuous change

Does the scalar potential and the electric field always change continuous across boundaries? Argue that they change continuously or find examples where they do not.

Exercise 4.8: Continuity of fields

A charge q lies in the origin and is enclosed by a spherically shaped dielectrical material of radius a, permittivity ϵ and its center in the origin. Are the electrical fields ($\mathbf{E}$ and $\mathbf{D}$) continuous across the interface from the sphere to the vacuum outside? Explain your reasoning.

4.6.3 Tutorials

Exercise 4.9: Dielectric plate in uniform field

We will now build intuition and skills by addressing a particular situation in detail and develop a model for the distribution, the polarization, the electric field and the displacement field in the system.

We start with a system consisting of empty space with a uniform electric field $\mathbf{E}_0 = E_0\hat{\mathbf{y}}$.

a) What kind of charge distribution may be the cause of such a field? What assumptions would you then need to make about the system?

We then place an infinitely long linearly dielectric plate of thickness d and dielectric constant ϵ in the xz-plane. We will now try to find the electric field, the distribution of bound charges, and the polarization everywhere.

b) Draw the plane into a coordinate system and specify where the top and the bottom of the plane are in the coordinate system.

c) Make a sketch of the behavior of bound dipoles when the dielectric is placed into the uniform electric field. What are the consequences for the net bound charge in the system? Explain the sign of the bound charges on the interfaces between the plate and the vacuum.

d) You co-student John states that this problem is simple to solve. The polarization inside the plate is simply $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}_0$. Do you agree with him? Explain your reasoning.

We will now solve the problem in two different ways. In method 1, we ill assume a given surface density of bound charges, use this to find the electric field due to the charges, then find the polarization, and finally find a self-consistent solution so that the polarization due to the total field is consistent with the charge density related to the polarization. In method 2, we will use the displacement field approach to solve the same problem.

Method 1: Starting from the bound charges.

e) Assume that there is a bound surface charge $\rho_{s,b}$ on the top surface and a bound charge $-\rho_{s,b}$ on the bottom surface. What is the electric field due to these charge densities? What is the total electric field?

f) What is the polarization, $\mathbf{P}$, for this total electric field?

g) Find the charge density on the top surface from the polarization and use this to find an equation for $\rho_{s,b}$. Solve this equation and find expressions for $\mathbf{P}$ and $\mathbf{E}_T$.

Method 2: Using the displacement field. While method 1 provides us with insight into the physics of the problem, the methods is rather cumbersome. Let us now instead use the displacement field to solve the same problem.

h) What is the displacement field, $\mathbf{D}$, outside the dielectric?

i) What are the boundary conditions at the interface between the dielectric and the vacuum outside?

j) Show that $E_T = (\epsilon_0/\epsilon)E_0$.

Exercise 4.10: A simple model for polarization

(Adapted from Steven Pollock)

We will now use a similar approach to calculate the polarization and the effective field in a medium due to polarization.

a) For an infinite xy-plane at $z = 0$ with a surface charge density σ, use Gauss' law to find the electric field at a position z both for $z > 0$ and for $z < 0$.

Let us now address a slab of plastic which is large in the xy-plane and of a thickness h in the z-direction. We assume that the inside of the plastic slab consists of a positive (ρ_0) and a negative ($-\rho_0$) charge density. The charge densities overlap and hence cancel when there is no electric field in the interior. If a uniform electric field $\mathbf{E} = E\hat{z}$ is acting inside the material, the two charge densities are shifted a distance Δz.

b) What is the charge density on the top (large z) and bottom (small z) of the slab when it is in an electric field $\mathbf{E} = E\hat{z}$? Make a sketch to illustrate the physical process.

c) What is the induced electric field, E_i, inside the slab due to these induced surface charge densities? (E_i is the field from the induced charges, not the total electric field).

d) Assume that the displacement Δz of the charge densities is proportional to the total local electric field, $\Delta z = cE_{tot}$, where c is a constant. Find an expression for the induced field inside the slab, E_i, as a function of the total electric field, E_{tot}, inside the slab.

e) What is the total electric field inside the slab if the slab is placed in an external field $\mathbf{E}_0 = E_0\hat{z}$?

f) You can assume that the external field $\mathbf{E}_0$ is the field set up by what we would call external/free charges. Can you relate your result to the general relation $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$? (For a linear dielectric $\mathbf{P} = \epsilon_0\xi_e\mathbf{E}$).

4.6.4 Homework

Exercise 4.11: Polarized cube

A dielectric cube with sides of length a and dielectric constant ϵ is placed in the first octant with a corner in the origin. There is vacuum outside the cube.

a) Assume that the cube has a uniform polarization $\mathbf{P} = P_0\hat{\mathbf{x}}$. What is the density of bound charges inside the cube? And on the sides?

b) Assume that the cube has a polarization $\mathbf{P}(x, y, z) = P_0 xy/a^2\hat{\mathbf{x}}$. What is the distribution of bound charges in and on the cube now?

Exercise 4.12: Long cylinder

An infinitely long cylinder of radius a and dielectric constant ϵ has a uniform volume charge density ρ and is placed in air.

a) Find the electric field inside and outside the cylinder.

b) Find the bound charge on the cylinder.

Exercise 4.13: Hollow sphere

A hollow spherical shell of inner radius a, outer radius b and dielectric constant ρ surrounds a uniformly charged sphere of radius a.

a) Find the displacement field $\mathbf{D}$.

b) Find the electric field $\mathbf{E}$.

c) Find the bound volume charge density.

5 Laplace equation

So far we have found the electric field $\mathbf{E}$ from the spatial distribution of charges by either summing or integrating up the contributions to the electric field; by integrating the contributions to the electric potential and then taking the gradient of the potential to find the electric field; or in highly symmetric situations we can use Gauss' law on integral form to relate the electric field to the charge inside a Gauss surface. The integration methods are robust — they work as long as we know the charge distribution — but it can be difficult to find the integrals in practice.

In these cases is can be useful to reformulate the problem in a different form — as a differential equation. We have in general found two sets of differential equations for the electric field: Gauss' law on differential form, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$; and that the curl of the electric field is zero, $\nabla \times \mathbf{E} = 0$. This gives us a set of partial differential equations that can be solved, but it may again be practically challenging. However, if we introduce the potential V so that $\mathbf{E} = -\nabla V$, these equations can be reformulated in a simpler form because $\nabla \cdot \mathbf{E} = -\nabla^2 V = \rho/\epsilon_0$ and $\nabla \times \nabla V = 0$ always because the curl or a gradient always is zero. This equation is called *Poisson's equation*. Together with appropriate boundary conditions, solving Poisson's equations gives us the potential which again gives us the electric field. Solving Poisson's equation is therefore equivalent to finding the potential or electric field from an integral over the charges that provides us with new, powerful tools to study electrostatic systems, but it requires new mathematical and numerical methods.

In this chapter we will demonstrate how we can solve problems in electrostatics by solving Poisson's equation, we will demonstrate fundamental mathematical properties of the equations such as the uniqueness of solutions given a set of boundary conditions, and we will show how Poisson equation can be used to model complex electrostatic systems.

5.1 Motivational example

Fig. 5.1 illustrates a lightning strike. How would we model such a system using the physics we have learned so far? We would need to simplify the system: We could simplify it to a cloud and the ground as illustrated in Fig. 5.2. We assume that the cloud is charged and therefore has a potential V_1 relative to the potential V_0 at the ground. What then happens during the lightning?

Fig. 5.1 Image of lightning from `https://www.youtube.com/watch`?v=nBYZpsbu9ds.

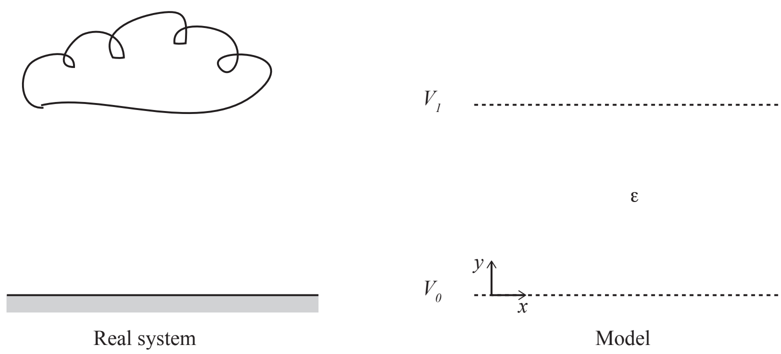


Fig. 5.2 Real system and model for a lightning strike.

We would need to know some additional physics for this: The effect of dielectric breakdown. When the electric field (that is, the potential difference) across a small (dielectric) region exceeds a maximum value, the charges (electrons) are no longer attached (bound), but are ripped from their bound positions, and become free. We call this *dielectric breakdown*. The result is the motion of charges, which we will discuss further down, but also that the potential effectively changes. However, in order to find out if and where there is dielectric breakdown we would need to find the electric field in space, for example by finding the electric potential. How can we find the electric potential everywhere in space?

5.1.1 Finding the electric potential

This is where Poisson's equation provides a solution. In general, to find the potential from a given set of charges, we need to solve the integral:

$$V(\mathbf{r}) = \int_v \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|r - r'|} \mathrm{d}v' \tag{5.1}$$

Alternatively, we would need to find the electric field, which is given from Gauss' law on differential form:

$$\nabla \cdot \mathbf{D} = \rho \ , \tag{5.2}$$

For a linear dielectric $\mathbf{D} = \epsilon \mathbf{E}$ so Gauss' law on differential form becomes:

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \rho \ . \tag{5.3}$$

If we assume that ϵ is uniform, we can put it outside the differential (∇) operator, getting

$$\epsilon \nabla \cdot \mathbf{E} = \rho \ \Rightarrow \ \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \ . \tag{5.4}$$

We then insert that $\mathbf{E} = -\nabla V$, getting:

$$\nabla \cdot \mathbf{E} = -\nabla^2 V = \frac{\rho}{\epsilon} \ . \tag{5.5}$$

We call this equation *Poisson's equation*:

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon} \; . \tag{5.6}$$

In the special case when $\rho = 0$, we get *Laplace's equation*:

Laplace's equation

$$\nabla^2 V = 0 \; . \tag{5.7}$$

Notice that is ρ is zero everywhere, there will be no electric fields. The idea of Laplace's equation is that there may be plenty of charges elsewhere or at the boundaries of the system — and these charges set up electric fields — but there are no free charges inside the system where we want to solve Laplace's equation.

In addition to these equations, in order to find V, we need to specify the *boundary conditions* for the problem. In order to find the potential in a region v, we may for example need the values for the potential on a surface S enclosing v. For example, in the one-dimensional case introduced above, we may need to know the values of the potential V at the boundaries. This is indeed why it is called *boundary conditions*. We will later come back to what types of boundary conditions are necessary for the solution to be unique, and what different types of boundary conditions we may have for Poisson's or Laplace's equation.

5.1.2 Example: Laplace's equation in one dimension

Problem. *For a system of length L with $V(x=0) = V_0$ and $V(x=L) = V_1$. What is the potential in between when there are no charges in this range and the material is a homogeneous dielectric with a constant ϵ.*

Fig. 5.3 Illustration of 1d problem.

Solution. The potential is given by Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = 0 \tag{5.8}$$

The solution to this equation is $V(x) = A + Bx$.

The boundary conditions then provides the constants A and B. We notice that we need two boundary conditions to specify the problem. (This means that if we only knew that $V(0) = V_0$, we would not be able to find both constants).

The boundary conditions gives that $V(0) = A = V_0$ and $V(L) = V_0 + BL = V_1$, and therefore $B = (V_1 - V_0)/L$.

Problem. What is the potential if the charge distribution is $\rho(x) = \rho_0 \sin(x/L)$?

Solution. In this case, we need to find the potential from Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\epsilon} = -\frac{\rho_0}{\epsilon} \sin(x/L) \ . \tag{5.9}$$

This equation has the solutions $V(x) = B\cos(x/L) + C\sin(x/L) + Dx + E$, which inserted gives

$$\frac{\partial^2 V}{\partial x^2} = -B/L^2 \cos(x/L) - C/L^2 \sin(x/L) = -\frac{\rho_0}{\epsilon} \sin(x/L) \ , \tag{5.10}$$

where we see that $B = 0$ and $-C/L^2 = -\rho_0/\epsilon$, that is, $C = \rho_0 L^2/\epsilon$. The two constants D and E must be determined from the boundary conditions.

5.2 Laplace's equation and other differential equations

5.2.1 Laplace's equation vs the equations of motion

You already know a differential equation that may look similar to Laplace's equation. When you solve Newton's law for motion for a time-varying external force $F(t) = mf(t)$:

$$\frac{d^2 x}{dt^2} = f(t) \ , \ x(t_0) = x_0 \ , \ \frac{dx}{dt}(t_0) = v_0 \ . \tag{5.11}$$

First, we notice that this equation looks similar to Laplace's equation in one dimension:

$$\frac{d^2V}{dx^2} = \rho(x)\ ,\ V(a) = V_a\ ,\ V(b) = V_b\ . \tag{5.12}$$

You may therefore be tempted to use some of the techniques you already know for one-dimensional equations of motion. However, there are important differences. One difference may be in the boundary conditions. For Newton's equation we usually have an *initial value* problem. We know x and its derivative at a given time $x(t_0) = x_0$ and $x'(t_0) = v_0$. For Laplace's equation we often have a boundary value problem, where we know the value of $V(x)$ for two different values of x, e.g. $V(a) = V_a$ and $V(b) = V_b$. This becomes particularly important for numerical solution methods. For an initial value problem we can simply integrate forward in time using a numerical, iterative scheme. However, for a boundary value problem, we cannot do this, since we need to ensure that we end up at a particular value $V(b) = V_b$. Thus we must use different numerical methods. Finally, Laplace's equation is generalized to a partial differential equation in higher dimensions, and Newton's equations do not have a corresponding generalization. We can therefore use some intuition from our previous experience, but we also need to build new intuition about partial differential equations and their solution methods.

5.2.2 Why is Laplace's equation so common?

You may not yet know this, but Laplace's equation is one of these equations that you will meet again and again in physics and other disciplines. Indeed, it is such a beautifully simple equation, $\nabla^2 V = 0$, which makes it mathematically interesting in itself. But the equation is also of fundamental importance in physics. One situation where Laplace's equation appear are in variations of the diffusion equation. The time development of a concentration field, $c(\mathbf{r};t)$, or a temperature field, $T(\mathbf{r};t)$ is given by the diffusion equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c \ \text{ or } \ \frac{\partial T}{\partial t} = \alpha \nabla^2 T\ . \tag{5.13}$$

where D is called the diffusivity and α the thermal diffusivity. These equations are called the *diffusion equation* and the *heat equation*. In the *stationary state*, that is when the time derivative is zero, this equation becomes Laplace's equation. Thus many types of transport equations result in Laplace's equation in the stationary state. (We will later see that this is also the case in electrostatics).

5.3 Boundary conditions

The solution of Laplace's or Poisson's equations depends on the boundary conditions, but what kind of boundary conditions do we need to find a solution? This requires answers to several questions: Is there a unique solution to the equations given a set of boundary conditions? And what kinds of boundary conditions are sufficient to ensure a unique solution? First, we will provide a proof of the uniqueness of the solution for a particular type of boundary conditions, and then we will discuss other types of boundary conditions.

5.3.1 Existence and uniqueness

We can find the electric field by solving Laplace's (or Poisson's) equation to find the potential and then find the electric field from the potential. But how do we know if a solution to Laplace's or Poisson's equation even *exist*? And if it exists, how do we know that it is *unique*?

Existence

First, we will simply borrow a result from mathematics: A solution to Poisson's and Laplace's equation exists.

If we find a solution, how do we know that it is the right or the only solution? For this we need to demonstrate uniqueness of the solution. We will do this in two steps. First, we will show that a function V that satisfies Laplace's equation in a region v, then V does not have any extrema on the inside of v: all the extreme are on the boundary, S. Second, we will use this to demonstrate the the solution to Laplace's equation in a volume v is uniquely determined if the potential V is specified on the boundary surface S.

The extremes of V are on the boundaries. Let us assume that V has a maximum (minimum) in a point inside v. This implies that V will decrease (increase) away from this point, which again implies that the gradient ∇V must point in towards (out from) the point. The electric field $\mathbf{E} = -\nabla V$ must therefore point out from (in towards) the point. Gauss' law then implies that there must be a positive (negative) free charge in this point. But this cannot be true for Laplace's equation, since

$\rho = 0$ in the region v. Consequently, V does not have any local maxima (or minima) inside v.

The value of V is the average of the surrounding values. This result is a special case: The value of V at a point $\mathbf{r}$ is the average value of V over a spherical surface or radius R around $\mathbf{r}$. We can prove this by first proving that this is the case for a single point charge outside the sphere, and therefore that it is the case for any charge distribution which is the sum of single point charges. For a single point charge q at a distance z from the center of a sphere of radius R, to potential at a point $\mathbf{r}'$ on the surface of the sphere is

$$V = \frac{q}{4\pi\epsilon_0 |z\hat{\mathbf{z}} - \mathbf{r}|} , \tag{5.14}$$

where $|z\hat{\mathbf{z}} - \mathbf{r}| = z^2 + R^2 - 2zR\cos\theta$, where θ is the angle between the z-axis and the point $\mathbf{r}'$. A surface element $\mathrm{d}S$ of an element of thickness $\mathrm{d}\theta$ is $\mathrm{d}S = 2\pi R \sin\theta \, R\theta \mathrm{d}\theta$ and the total area of the sphere is $4\pi R^2$. The average of V over the sphere is therefore:

$$
\begin{aligned}
V_{\text{avg}} &= \frac{1}{4\pi R^2} \int_S \frac{q}{4\pi\epsilon_0 (z^2 + R^2 - 2zR\cos\theta)^{1/2}} \mathrm{d}S && (5.15)\\
&= \frac{q}{4\pi R^2 4\pi\epsilon_0} \int \frac{R^2 \sin\theta \mathrm{d}\theta}{(z^2 + R^2 - 2zR\cos\theta)^{1/2}} && (5.16)\\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left((z+R) - (z-R)\right) && (5.17)\\
&= \frac{q}{4\pi\epsilon_0 z} && (5.18)
\end{aligned}
$$

We recognize this as the potential from a single point charge q at the center of the sphere. This is also not surprising at all, because the integral is the same as we would find for the potential of a uniform charge q on the surface in a point at a distance z from the center of the sphere, and in this case we know that the electric field from a uniformly charged sphere is the same as the potential from a single point charge at the center of the sphere, when the point is outside the sphere (if it is inside the sphere, the potential is zero).

This must therefore also be true for the superposition of any set of charges outside the sphere. We have therefore proved that the potential in a point $\mathbf{r}$ is the average of the potentials over a sphere centered in $\mathbf{r}$. This implies that V cannot have any local maxima or minima inside a region, they must occur at the boundaries. Because if V had a maximum (or a minimum) in a point $\mathbf{r}$, it would be possible to find a sphere around $\mathbf{r}$ so that all the points on the sphere would be smaller than (or larger) than the value at $\mathbf{r}$ — this is what is means to be a local maximum/minimum.

Averaging property of V. We also notice this *averaging property* of V: V in a point is the average of the values on a sphere around that point. We will see that this is also the case for Laplace's equation on discrete form and forms the basis for a method to solve Laplace's equation — the method of relaxation.

Laplace's equation has a unique solution. We can use this to prove that Laplace's equation has a unique solution in v given that the values of V on the boundaries are specified, $V(S)$. Let us assume that there are *two* solutions to Laplace's equation, V_1 and V_2, that both satisfy Laplace's equation in v. We then define $V = V_1 - V_2$. If the two solutions

V_1 and V_2 are identical at the boundaries, we notice that V must be zero at the boundaries. Because V is a solution to Laplace's equation it cannot have a maximum or a minimum at the boundary, hence $V = 0$ in the whole volume v, and we conclude that $V_1 = V_2$. The solution must therefore be unique. This implies that is does not matter how we find a solution: If we find a solution that satisfies the boundary conditions, then the solution is *the* solution, the unique solution to the problem.

5.3.2 Different types of boundary conditions

Dirichlet boundary conditions. The typical boundary condition for Poisson's equation is to specify the value of the potential at the boundary. This type of boundary condition is called a *Dirichlet boundary condition*. Boundaries can be either the external boundaries or internal boundaries. Fig. 5.4 illustrates boundary conditions for a one dimensional cases and a two-dimensional case. Dirichlet boundary condition here correspond to specifying the value of the potential at specific positions $V(x_i) = V_i$

von Neumann boundary conditions. Another type of boundary condition is to specify not the value of V, but instead the derivative of V, $\partial V/\partial n$, in a given direction n, typically normal to the boundary. This type of boundary condition is called *von Neumann boundary conditions*. In the one-dimensional case, von Neumann boundary conditions corresponds to specifying $\partial V/\partial x$ at the boundaries, which corresponds to specifying the electric field at the boundaries, because $E_x = -\partial V/\partial x$.

The same interpretation is valid in higher dimensions, where we interpret $\partial V/\partial n$ as the gradient in the direction along n, which again is related to the electric field. A common type of von Neumann boundary conditions, is that the derivative is zero at the boundaries. In the case of the diffusion equation, we interpret von Neumann boundary conditions as the flux into the system at the boundaries. In the case of the electric potential, we can also interpret the von Neumann condition as the flux — the flux of the electric field in the direction of n: $\mathbf{E} \cdot \hat{\mathbf{n}}$, across a small surface area of size unity.

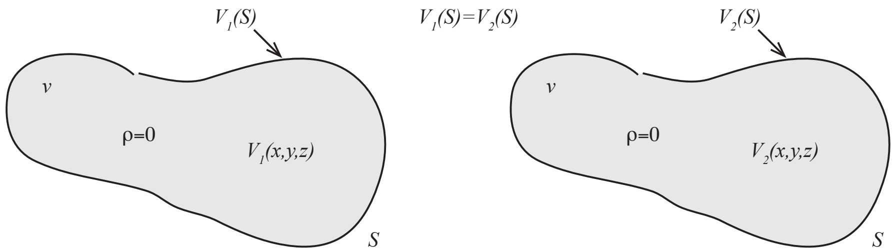


Fig. 5.4 Illustration of the boundary conditions. FIX THIS FIGURE

5.4 Poisson's equation in various coordinate systems

For your convenience, we provide Poisson's equation, $\nabla^2 V = -\rho/\epsilon$, in various coordinate systems:

5.4.1 Cartesian coordinates

In Cartesian coordinates Poisson's equation is:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\rho(x,y,z)/\epsilon \; . \tag{5.19}$$

5.4.2 Cylindrical coordinates

In cylindrical coordinates we describe a position using the cylindrical radius r, the azimuthal angle ϕ around the z-axis (measured with $\phi = 0$ at the x-axis), and the z-coordinate, z: $V = V(r,\phi,z)$. The ∇^2 operator applied to V in cylindrical coordinates is:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\rho(r,\phi,z)/\epsilon \; . \tag{5.20}$$

5.4.3 Spherical coordinates

In spherical coordinates we describe a position using the spherical radius r, the azimuthal angle θ from the x-axis, and the angle ϕ with the z-axis: $V = V(r,\theta,\phi)$. The ∇^2 operator applied to V in spherical coordinates is:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \; . \tag{5.21}$$

5.4.4 Choosing the right version of Poisson's equation

How do we know which version of Poisson's equation to use? This follows from the symmetry of the boundary conditions. If the boundaries are specified with cylindrical symmetry, use cylindrical coordinates. If boundary conditions have spherical symmetry use spherical coordinates.

5.4.5 Example: Cylindrical system

Problem. A cylindrical system consists of two concentric cylindrical surfaces with radius a and b. Assume that the potential at $r = a$ is V_a and at $r = b$ is V_b. There are no free charges for $a < r < b$. Find the potential for $a < r < b$.

Solution. We recognize that the system has cylindrical symmetry, and we expect $V = V(r)$, so that there is no ϕ or z dependence. Because there are no free charges between the cylinder surfaces, the system must obey Laplace's equation in this region. We use Laplace's equation in cylindrical coordinates, but only include the r-derivative:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \tag{5.22}$$

For $r > a > 0$, we multiply by r, getting

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \tag{5.23}$$

This means that what is inside the paranthesis must be a constant, that is,

$$r\frac{\partial V}{\partial r} = A \;\Rightarrow\; \frac{\partial V}{\partial r} = \frac{A}{r} \tag{5.24}$$

This means that $V = A\ln r + B$. We determine the constants A and B from the boundary conditions $V_a = A\ln a + B$ and $V_b = A\ln b + B$. We subtract the equations, getting $V_a - V_b = A(\ln a - \ln b) = V\ln(a/b)$ and therefore $A = (V_a - V_b)/\ln(a/b)$. We find B from $V(a) = V_a = A\ln a + B$, which gives us $B = V_a - A\ln a = V_a - \ln a\,(V_a - V_b)/\ln(a/b)$.

5.5 Numerical solutions: Finite difference methods

Generally, it is difficult to find analytical solutions to Poisson's equations in two or three dimension unless the system has a very clear symmetry. When we cannot find an analytical solution, we may instead use a numerical method. There are numerous numerical methods to solve Poisson's equation spanning from simple methods, such as Finite Difference method, to advanced methods such as finite element methods. Here, we will focus on a very simple, but inefficient, solution method that also provides insight into the physics of the problem.

5.5.1 Discrete lattice

We introduce a method to solve Laplace's equation, but the same approach may be used to solve Poisson's equation. We would like to solve the equation

$$\nabla^2 V = 0 \tag{5.25}$$

with a given set of boundary conditions. We will solve this on a *discrete lattice*. That is, we will find the solution at lattice positions $(x_i, y_i) = (i\Delta x, j\Delta y)$ as illustrated in Fig. 5.5.

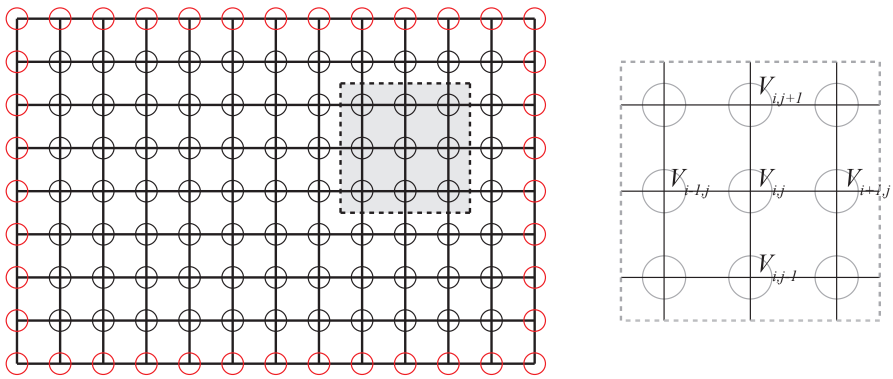


Fig. 5.5 Illustration of the discretized grid.

5.5.2 Discretizing the equation on a square/cubic lattice

We can discretize the equation on this lattice by first finding the discrete derivative and then taking the deriative once more. This is usually done on what is called a staggered grid: We find the derivative at in-between positions $x_i + \Delta x/2$ and $x_i - \Delta x/2$, and then use this to find the second derivative.

The derivative at $x_i + \Delta x/2$ is approximately:

$$\frac{\partial V}{\partial x}(x_i + \Delta x/2) \simeq \frac{V(x_i + \Delta x, y_i) - V(x_i, y_i)}{\Delta x} \;, \tag{5.26}$$

and similarly

$$\frac{\partial V}{\partial x}(x_i - \Delta x/2, y_i) \simeq \frac{V(x_i, y_i) - V(x_i - \Delta x/2, y_i)}{\Delta x} \;, \tag{5.27}$$

The second derivative in (x_i, y_i) is then:

$$\frac{\partial^2 V}{\partial x^2} \simeq \frac{\frac{\partial V}{\partial x}(x_i + \Delta x/2, y_i) - \frac{\partial V}{\partial x}(x_i - \Delta x/2, y_i)}{\Delta x} \;, \tag{5.28}$$

and similarly for the y-direction:

$$\frac{\partial^2 V}{\partial y^2} \simeq \frac{\frac{\partial V}{\partial y}(x_i, y_i + \Delta y/2) - \frac{\partial V}{\partial y}(x_i, y_i - \Delta y/2)}{\Delta y} \;, \tag{5.29}$$

We can simplify the notation by introducing $V(i\Delta x, j\Delta x) = V_{i,j}$ when $\Delta x = \Delta y$. We then get that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0 \;. \tag{5.30}$$

We can solve for $V_{i,j}$, getting

$$V_{i,j} = \frac{1}{4}\left(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right) \;. \tag{5.31}$$

Notice that this looks like a local average. The solution is such that the value in each point is the average of the values at the nearest neighbor around it. We therefore expect the solution to be very smooth.

This is a linear system of equations, which must be supplemented by the boundary conditions. Here, we will specify the boundary conditions by Dirichlet boundary conditions through a function $B(x, y)$ on a boundary S. The boundary conditions are that $V(x, y) = B(x, y)$ for all points (x, y) on the boundary S.

5.5.3 Solving the discretized equations using Jacobi iterations

There are various numerical schemes to solve these sets of equations. First, we will solve this system of equations using a simple, iterative scheme (which is an inefficient method to solve the system of equations, but the method is easy to understand) called Jacobi iterations. We see that in the *final solution* for each point (x_i, y_i), the potential is the average of the surrounding potentials. Our strategy is therefore the following: If we iterate this average many times, we will slowly smooth things out and converge towards a solution.

This suggests the following method:

1. Initialize $V_{i,j}$.
2. On the boundary we set $V_{i,j} = B_{i,j}$, that is, equal to the boundary condition $B(x, y)$.
3. For each point in the interior of the volume v, calculate new potentials as the average of the surrounding potentials. Repeat until it converges (sufficiently well).

5.5.4 Numerical implementation

First, we make a function to solve the equation using Jacobi iterations. We start by importing necessary libraries:

```
import numpy as np
import matplotlib.pyplot as plt
from numba import jit
```

The last part, `numba`, is a library that is used to accelerate loops in functions. Here, it is simply used as a method to speed the calculation up by orders of magnitude.

Then we write a function using the algorithm above:

```
@jit
def solvepoisson(b,nrep):
    # b = boundary conditions, =NaN where we will calculate the values
    # nrep = number of iterations
    # returns potential on the same grid as b
    V = np.copy(b)
    for i in range(len(V.flat)):
        if (np.isnan(b.flat[i])):
            V.flat[i] = 0.0
    Vnew = np.copy(V) # See comment in text below
```

```
    Lx = b.shape[0] # x-size of b matrix
    Ly = b.shape[1] # y-size of b matrix
    for n in range(nrep):
        for ix in range(1,Lx-1):
            for iy in range(1,Ly-1):
                if (np.isnan(b[ix,iy])):
                    Vnew[ix,iy] = (V[ix-1,iy]+V[ix+1,iy]+V[ix,iy-1]+V[ix,iy+1])/4
                else:
                    Vnew[ix,iy] = V[ix,iy]
        V,Vnew = Vnew,V # Swap points to arrays V and Vnew
    return V
```

Why do we introduce the new array `znew` and not only update `z` in each step? If we did this, we would gradually change `z` as we were moving through the loops, and we would no longer use the values in the previous step to calculate the next step, instead we would use some of the values from the old step and some values from the new step. This would become a mess and would not correspond to the algorithm presented above. Also notice the use of `nan` in the array `b`, which represents the boundary conditions. In the positions where `b` contains a value, this is the boundary condition in this point, whereas in the positions where `b` does not contains any numerical value, where it is set to `nan`, there are no boundary conditions. (This method is improved below to also work for internal boundary conditions. Here, it only works for boundary conditions at the external boundaries, and all the values at the external boundaries must be set).

5.5.5 Defining boundary conditions

We are now ready to use this function to find the potential for a given set of boundary conditions. Let us start with a square system that is 40×40 units, where the values at the external boundaries are given. For example, let us assume that $B(x, 0) = 1$ and that $B(x, L) = B(0, y) = B(L, y) = 0$. We set up this problem

```
L = 40
b = np.zeros((L,L),float)
b[:] = np.float('nan')

b[0,:] = 0.0
b[L-1,:] = 0.0
b[:,0] = 1.0
b[:,L-1] = 0.0

plt.imshow(b)
```

```
plt.colorbar()
```

This shows the boundary conditions as shown in Fig. 5.6a.

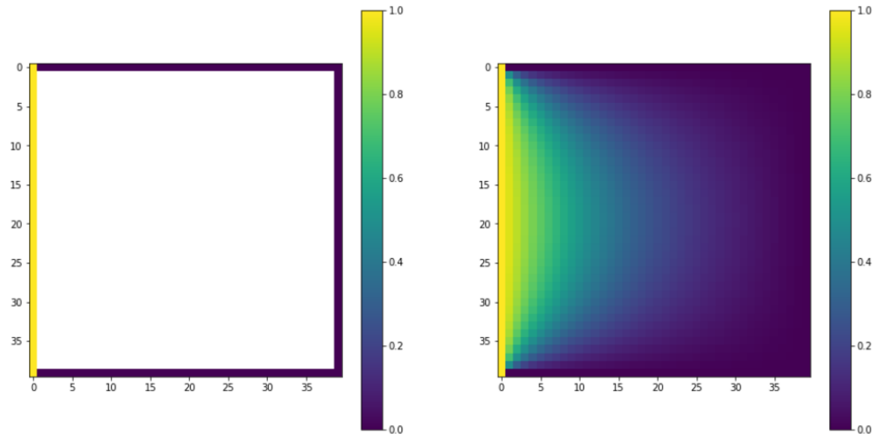


Fig. 5.6 (a) Illustration of the boundary conditions $B(x, y)$ shown with a color-scale. The colors indicate where $B(x, y)$ is defined. White indicates values where `B = nan`. These are the values where we will find the potential. (b) Plot of the resulting potential $V(x, y)$.

5.5.6 Finding the potential

We can then apply Jacobi's method to find the potential $V(x, y)$ with the given boundary conditions $B(x, y)$.

```
nrep = 2000
V = solvepoisson(b,nrep)
plt.imshow(V)
plt.colorbar()
```

The resulting visualization of $V(x, y)$ is shown in Fig. 5.6b.

5.5.7 Finding the electric field

When we have found the potential, we are ready to also calculate the electric field using $\mathbf{E} = -\nabla V$. This is implemented using the `gradient`-function. Notice the reversal of the coordinates because the first index in an array is the y-direction in Python, as we saw previously.

```
plt.contourf(V)
Ey,Ex = np.gradient(-V)
x = np.linspace(0,L-1,L)
y = np.linspace(0,L-1,L)
plt.streamplot(x,y,Ex,Ey,linewidth=1,density=1,arrowstyle='->',arrowsize=1.5)
plt.axis('equal')
```

The resulting plot is shown in Fig. 5.7.

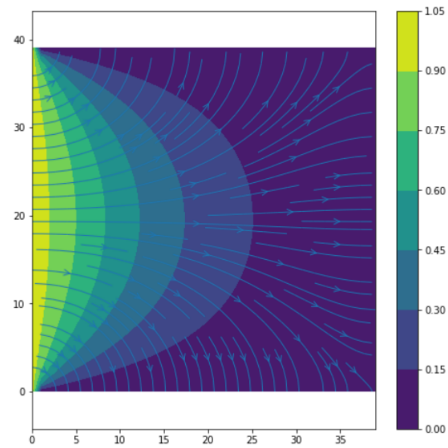


Fig. 5.7 Plot of $V(x, y)$ and an illustration of the electric field $\mathbf{E}$.

5.5.8 Electric field in physical units units

The results from the numerical solution of Laplace's equation are on the form $V_{i,j} = V(x_i, y_j)$. If we use the simple `plt.contourf(s)` to visualize this, the axes are simply the i and j values. However, we often want to visualize the results in real physical units. Let us assume that $\Delta x = 1\,\text{mm}$ and that the system is centered in the origin in the xy-plane. The values for x_i then spans from $-(N/2)\Delta x$ to $(N/2)\Delta x$, where $N = 40$ is the number of lattice units in both x and y directions. We can visualize the potential $V(x, y)$ in physics coordinate units using:

```
deltax = 1e-3 # m
N = L
xu = np.linspace(-(N/2)*deltax,(N/2)*deltax,N)
yu = np.linspace(-(N/2)*deltax,(N/2)*deltax,N)
plt.figure(figsize=(8,8))
plt.contourf(xu,yu,V)
plt.xlabel('$x$ (m)')
plt.ylabel('$y$ (m)')
```

The resulting plot is shown in Fig. 5.8. Now, if the voltages $V_{i,j}$ are calculated in real physical values, for example in units of Volt, we can also calculate the electric field in real physical units, Volts/meter (V/m). The electric field in the x-direction is found from

$$E_x = -\frac{\partial V}{\partial x} \simeq -\frac{V(x_i + \Delta x, y_j) - V(x_i, y_j)}{\Delta x} \; . \tag{5.32}$$

To find the electric field in physical units, we therefore need to divide the `gradient` by Δx, that is:

```
Eyu,Exu = np.gradient(-V/deltax)
```

We can then plot the electric field in the x direction for the line along $y = 0$ using:

```
plt.plot(xu,Exu[int(N/2),:])
plt.xlabel('$x$ (m)')
plt.ylabel('$E_x$ (V/m)')
```

where we recall that the enumaration of Python array are `Ex[iy,ix]` so that x is the second index. This is why we choose `int(L/2)` as the first index, this is the line in the middle of the `N` elements along the y-axis. And we use the index `:` to denote all indicies along the x-axis. The resulting plot is show in Fig. 5.8.

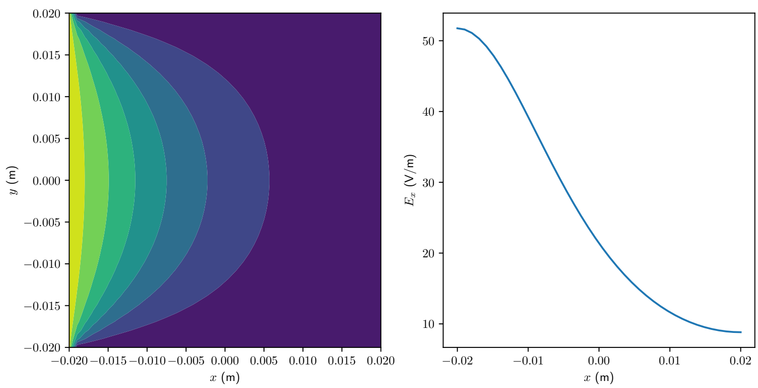


Fig. 5.8 (a) Plot of $V(x,y)$ using physical units for the axes. (b) Plot of $E_x(x)$ using physical units for the field and for the axis.

5.5.9 Adding complexity

We can then add more complex boundaries, for example with oscillating values of the potential at the boundaries as in the follow script:

```
# Let us try some more exciting boundary conditions
L = 100
z = np.zeros((L,L),float)
b = np.copy(z)
b[:] = np.float('nan')

b[0,:] = 0.0
b[L-1,:] = 0.0
x = np.arange(0,L)
b[:,0] = np.sin(2*2*np.pi*x/L)
b[:,L-1] = np.cos(2*2*np.pi*x/L)

plt.figure(figsize = (16,8))
plt.subplot(1,2,1)
plt.imshow(b)

nrep = 100000
s = solvepoisson(b,nrep)
plt.subplot(1,2,2)
plt.imshow(s)
plt.colorbar()
```

The resulting plot is shown in Fig. 5.9.

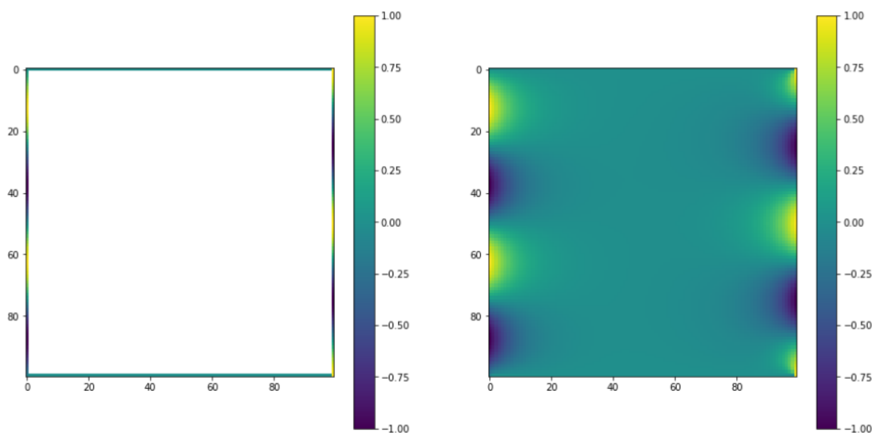


Fig. 5.9 Plot of $B(x,y)$ (a) and $V(x,y)$ (b).

5.5.10 More robust solution method

The solution method provided above only works when we specify the boundary condition on all the external boundaries. What is we would like to only specify the potential at certain points inside the lattice, and not at the boundaries. Strickly, we would then need to have an infinite system, but this is not realistic in a numerical calculation. Instead, we can introduce a different type of boundary condition at the external boundaries - a Neumann boundary condition specifying the derivative $\partial V/\partial x$ or $\partial V/\partial y$ at the boundary. If we specify that the derivative is zero, this means that the potential is flattening out, but we do not specify at which value. We take this to represent that the potential becomes constant far away, but we need to be careful and check our interpretation afterward.

We rewrite `solvepoisson` to be more robust in the function `solvepoissonvonneumann2d`:

```
@jit
def solvepoissonvonneumann2d(b,nrep):
    # b = boundary conditions
    # nrep = number of iterations
    # returns potentials
    V = np.copy(b)
    for i in range(len(V.flat)):
        if (np.isnan(b.flat[i])):
            V.flat[i] = 0.0
    Vnew = np.copy(V)
    Lx = b.shape[0]
    Ly = b.shape[1]
    for n in range(nrep):
        for ix in range(Lx):
            for iy in range(Ly):
                ncount = 0.0
                pot = 0.0
                if (np.isnan(b[ix,iy])):
                    if (ix>0):
                        ncount = ncount + 1.0
                        pot = pot + V[ix-1,iy]
                    if (ix<Lx-1):
                        ncount = ncount + 1.0
                        pot = pot + V[ix+1,iy]
                    if (iy>0):
                        ncount = ncount + 1.0
                        pot = pot + V[ix,iy-1]
                    if (iy<Ly-1):
                        ncount = ncount + 1.0
                        pot = pot + V[ix,iy+1]
                    Vnew[ix,iy] = pot/ncount
                else:
```

```
                    Vnew[ix,iy]=V[ix,iy]
        V, Vnew = Vnew, V # Swap pointers to arrays
    return V
```

Notice here the use of the `if (np.isnan(b[ix,iy])):` command, which ensures that it is only where `b` is equal to `nan` that new values are calculated in each step. In positions where `b` is not equal to `nan`, the boundary conditions values are used instead.

Another fine detail of this implementation is what happens at an external boundary point if a Dirichlet boundary condition (a value for V) is not specified. In this case, a von Neumann boundary condition with zero derivative is used instead. (We leave this as an exercise).

This allows us to address different types of geometries. Such as this case with two finite planes with given potentials.

```
L = 100
b = np.zeros((L,L),float)
b[:] = np.float('nan')

# Finite length capacitor
L14 = np.int(L/4)
L34 = 3*L14
b[L14:L34,L14]=1.0
b[L14:L34,L-L14]=-1.0

plt.subplot(1,2,1)
plt.imshow(b)
plt.colorbar()
plt.subplot(1,2,2)
nrep = 100000
s = solvepoissonvonneumann2d(b,nrep)
plt.imshow(s)
plt.colorbar()
```

The resulting plots are shown in Fig. 5.10. This method is robust and can be used to model many types of systems in detail.

5.5.11 Dielectrics

How would we need to modify the methods to model systems where the dielectric properties vary in space, that is, where $\epsilon = \epsilon(x, y)$ for a two-dimensional system?

In this case, we need to re-evaluate how we derived Poisson's equation. We start from Gauss' law on differential form $\nabla \cdot \mathbf{D} = \rho$ and then use that $\mathbf{D} = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ and that $\mathbf{E} = -\nabla V$, getting:

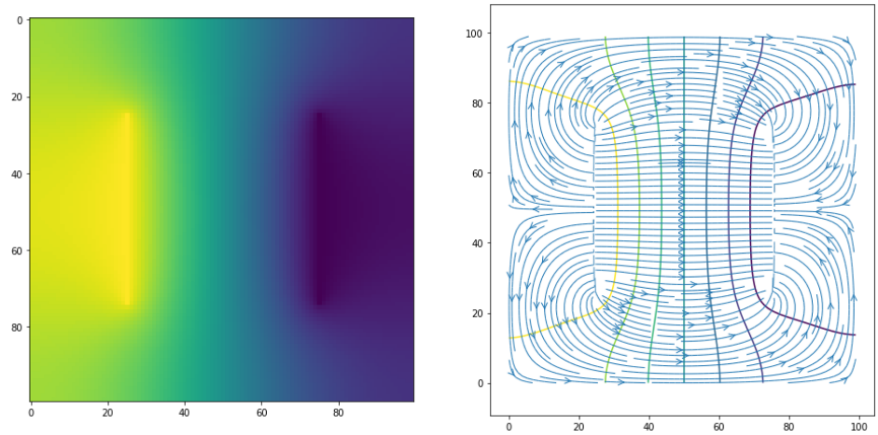


Fig. 5.10 Plot of $V(x, y)$ (a) and $\mathbf{E}(x, y)$ (b).

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \nabla \cdot (\epsilon(-\nabla V)) = -\nabla \cdot \epsilon \nabla V \ . \tag{5.33}$$

We can discretize this equation independently in the x and y directions. For the x-components, we get:

$$\frac{\partial}{\partial x} \epsilon \frac{\partial V}{\partial x} \ . \tag{5.34}$$

We use the *staggered* approach introduced above, and calculate $\partial V/\partial x$ at $x_i + \Delta x/2$:

$$\left(\epsilon \frac{\partial V}{\partial x} \right) (x_i + \Delta x/2) \simeq \epsilon(x_i + \Delta x/2, y_i) \frac{V(x_i + \Delta x, y_i) - V(x_i, y_i)}{\Delta x} \ , \tag{5.35}$$

and similarly

$$\left(\epsilon \frac{\partial V}{\partial x} \right) (x_i - \Delta x/2) \simeq \epsilon(x_i - \Delta x/2, y_i) \frac{V(x_i, y_i) - V(x_i - \Delta x, y_i)}{\Delta x} \ . \tag{5.36}$$

We introduce the notation $V(i\Delta x, j\Delta x) = V_{i,j}$ and $\epsilon(x_i + \Delta x/2, y_i) = \epsilon_{i+\frac{1}{2},j}$ and approximate the x-component of $\nabla \cdot \epsilon \nabla V$ as

$$\frac{\partial}{\partial x} \epsilon \frac{\partial V}{\partial x} \simeq \frac{\epsilon_{i+\frac{1}{2},j} \frac{V_{i+1,j} - V_{i,j}}{\Delta x} - \epsilon_{i-\frac{1}{2},j} \frac{V_{i,j} - V_{i-1,j}}{\Delta x}}{\Delta x} \ . \tag{5.37}$$

This means that Laplace's equation, $\nabla \cdot \epsilon \nabla V = 0$, becomes:

$$\epsilon_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\epsilon_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)+ \tag{5.38}$$

$$\epsilon_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\epsilon_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)=0 \tag{5.39}$$

We solve for $V_{i,j}$, getting:

$$V_{i,j}=\frac{1}{4\bar{\epsilon}_{i,j}}\left(\epsilon_{i+\frac{1}{2},j}V_{i+1,j}+\epsilon_{i-\frac{1}{2},j}V_{i-1,j}+\epsilon_{i,j+\frac{1}{2}}V_{i,j+1}+\epsilon_{i,j-\frac{1}{2}}V_{i,j-1}\right) \tag{5.40}$$

where

$$\bar{\epsilon}_{i,j}=\frac{1}{4}\left(\epsilon_{i+\frac{1}{2},j}+\epsilon_{i-\frac{1}{2},j}+\epsilon_{i,j+\frac{1}{2}}+\epsilon_{i,j-\frac{1}{2}}\right) \tag{5.41}$$

We can now use exactly the same procedure to solve these equations as we developed previously — for example Jacobi iterations as introduced above, we only need to include the values of the dielectric constant ϵ at the staggered grid positions.

5.6 Modeling: Lightning

We now have the tools to start addressing lightning using our simplified model. We start by assuming that the top of our system represents a cloud with a potential V_1 and that the bottom represents the ground with a potential V_0. We now have the tools to calculate the potential $V(x,y)$ everywhere in the air using the numerical model. For simplicity we will scale the system so that the potential is $V_1 = 0$ at $B(x,L)$, because this corresponds to the top of the screen when we visualize the system in Python. (Python by default prints (0,0) in the top left corner of an image) and then we will assume that $V_0 = 1$ at $B(x,0)$.

How can we introduce a lightning in this case? We need a model for what happens during a lightning. We will assume that lightning propagates due to dielectric breakdown: Lightning propagates into regions of air and that cells where the potential difference is the largest, that is where the electric field is the largest, are most likely to have a dielectric breakdown. In addition, we will assume that the dielectric breakdown only can occur one cell at a time, starting from the cell with zero potential. Whenever a cell experiences a dielectric breakdown we will set the potential to be zero at this cell as well, because the cell becomes a good conductor due to the breakdown, so that the potential effectively is the same. (We will see in the next chapter that the potential is the same in a conductor, otherwise charges will flow until the potential is the same).

How do we implement this in practice? We introduce a matrix `zeroneighbor`, which is `NaN` only in the points that are neighbors to a place where the potential is zero. This means that the lightning can only grow into cells where `zeroneighbor` is `NaN`. We then introduce a probability for the breakdown. We assume that sites with higher potential (difference, but since it is relative to a potential of zero, the difference is equal to the value in the cell) has a higher probability to experience dielectric breakdown. We assume that the probability is $P_i = c_i V_i^\eta$, for a cell I to break down, where V_i is the potential (difference compared to zero). η is a parameter, which here is set to $\eta = 1$, that is, the probability is proportional to the potential difference. We introduce randomness by selecting the value c_i to be a number uniformly distributed between 0 and 1. The cell with the highest value of P_i breaks down. When the cell breaks down, we change the boundary conditions for Laplace's equation, since the boundaries are changed: The cell gets a potential $V_i = 0$. And we must also update `zeroneighbor` to ensure that the neighbor of the new cells also become possible new cells for the lightning to propagate into.

This is all implemented in the following code.

```
# First, we set up the boundary conditions
Lx = 100
Ly = 100
z = np.zeros((Lx,Ly),float)
b = np.copy(z)
c = np.copy(z)
b[:] = np.float('nan')

# Where is the potential 1.0?
b[:,0] = 1.0

# Where is the potential 0.0?
b[:,Ly-1]=0.0

# For the lightning simulations, we will only allow the lightning
# to propagate into positions that are adjacent to where the
# potential is 0.0. We therefore create a matrix that is 0.0 everywhere
# except at the sites that are neighbors to where b (the boundary value potential)
# is zero
zeroneighbor = np.copy(z)
zeroneighbor[:] = 0.0
zeroneighbor[:,Ly-2] = np.float('nan')

nrep = 10000 # Number of jacobi steps
```

```
eta = 0.2
ymin = Ly-1
ns = 0
while (ymin>0):
    # First find potential
    s = solvepoissonvonneumann2d(b,nrep)
    # Probabilities depend on potential to power eta
    sprob = s**eta
    # We multiply with random number, uniform between 0 and 1
    sprob = sprob*np.random.uniform(0,1,(Lx,Ly))*np.isnan(zeroneighbor)
    # Multiply with isnan(zeroneighbor) to ensure only 'nan' points can be chosen
    [ix,iy] = np.unravel_index(np.argmax(sprob,axis=None),sprob.shape)
    # Update boundary conditions
    b[ix,iy] = 0.0
    # Update neighbor positions
    if (ix>0):
        zeroneighbor[ix-1,iy]=np.float('nan')
    if (ix<Lx-1):
        zeroneighbor[ix+1,iy]=np.float('nan')
    if (iy>0):
        zeroneighbor[ix,iy-1]=np.float('nan')
    if (iy<Ly-1):
        zeroneighbor[ix,iy+1]=np.float('nan')
    ns = ns + 1
    c[ix,iy] = ns
    if (iy<ymin):
        ymin = iy
    print("ymin = ",ymin)

print("Finished")

plt.figure(figsize=(16, 4))
plt.figure(figsize=(8,8))
plt.imshow(c.T)
plt.gca().invert_yaxis()
plt.colorbar()
```

Notice the use of the variable `ns` to color the lightning in the sequence in which it grew. The first cells that broken down have low values and the last have high values. The resulting simulation is shown in Fig. 5.11. Since the process has randomness, a new simulation will generate a new lightning.

I encourage you to play with this model to try to understand the conditions for various phenomena in lightnings.

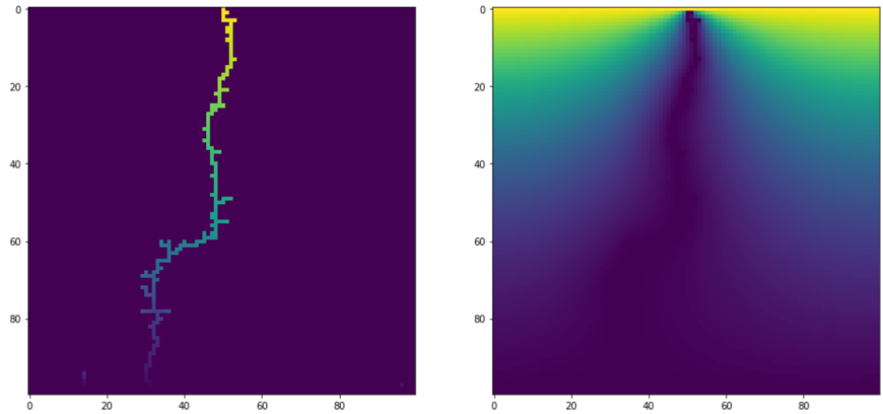


Fig. 5.11 Plot of (a) `ns` (b) $V(x, y)$.

5.7 Modeling: Water droplet in electric field

We will use our theory to explore what happens to a water droplet in an electric field. When we model a system such as this, we have to be precise and we have to determine what types of approximations we want to make.

First, what do we mean by a water droplet? We assume that the droplet is round — spherical — with radius a. We will also assume that it does not change its shape. (We may want to modify this assumption, if we are adventurous). What is the difference between water and vacuum? You may think that water is a conductor, but clean water is generally not a conductor but instead a dielectric with dielectric constant $\epsilon_r \simeq 80$. If the water contains ions, it may indeed be conducting, but here we assume that it is a dielectric.

Our model of is therefore a sphere of radius a with dielectric constant ϵ_r. The droplet is placed in a uniform field. For simplicity, we place the uniform field along the x-axis. Our plan is to solve Laplace's equation numerically and analytically to find the potential V, and then find the electric field from the potential, $\mathbf{E} = -\nabla V$.

We will find the electric field everywhere in this system in two ways: First by solving Laplace's equation numerically, and then by solving Laplace's equation analytically.

5.7.1 Numerical solution

We model the system on a $L \times L$ lattice and want to find $V(x, y) = V(i\Delta x, j\Delta x) = V_{i,j}$ for $i, j = 0, 1, \ldots, L-1$ by solving Laplace's equation numerically. But how do we set up an external, uniform electric field? We know there is a uniform field between two plates. Therefore, we put one plate at $i = 0$ with potential V_0 and one plate at $i = L - 1$ with potential V_1. The field will point from $i = 0$ to $i = L - 1$ is $V_0 > V_1$. We introduce a unit V_0 for the potential and set $V_{0,j} = V_0$ and $V_{L-1,j} = -V_0$. For simplicity, we measure all potentials in units of V_0 so that $V_{0,j} = 1$ and $V_{L-1,j} = -1$.

How can we model the sphere? First, we simplify the system and only study a two-dimensional slice through the center of the sphere — a thin disk. (We expect the electric field to be entirely in this plane for all points in the plane, also in the full three-dimensional case). This disk has radius a and is placed in the origin of our coordinate system at $i, j = L/2, L/2$.

We will now need to implement Laplace's equation with non-uniform dielectric constant ϵ: $\nabla \cdot \epsilon \nabla = 0$, where we introduce $\epsilon = \epsilon_r \epsilon_0$. However, ϵ_0 is uniform, so the equation becomes $\nabla \cdot \epsilon_r \nabla V = 0$, where V is measured in units of V_0. In addition, we measure lengths in units of Δx.

For each point i, j we define $\epsilon_{r,i,j}(r)$, which depends on the distance r from the origin. When $r < a$, $\epsilon_{r,i,j} = 82$, whereas for $r \leq a$, $\epsilon_{r,i,j} = 1$.

Modified Python code. We modify the python code to include that ϵ_r varies in space. We find $\epsilon_{r,i+1/2,j}$ as the average of the values at i, j and $i + 1, j$: $\epsilon_{r,i+1/2,j} = (\epsilon_{r,i,j} + \epsilon_{r,i+1,j})/2$. We also include periodic boundary conditions in both directions to avoid edge effects. Periodic boundary conditions corresponds to solving the problem on top of a cylinder, so that $V_{L,j} = V_{0,j}$ and $V_{-1,j} = V_{L-1,j}$ (and similarly in the y-direction).

The new function to solve Laplace's equation is now:

```python
import numpy as np
import matplotlib.pyplot as plt
import numba
@numba.jit(cache=True)
def solvepoissonperiodic(b,epsi,nrep):
    """ b = boundary conditions
    epsi = epsilon (same size as b)
    nrep = number of iterations
    returns potentials """
    V = np.copy(b)
    for i in range(len(V.flat)):
        if (np.isnan(b.flat[i])):
            V.flat[i] = 0.0
    Vnew = np.copy(V)
```

```
    Lx = b.shape[0]
    Ly = b.shape[1]
    for n in range(nrep):
        for ix in range(Lx):
            for iy in range(Ly):
                etot = 0.0
                pot = 0.0
                if (np.isnan(b[ix,iy])):
                    ix1 = ix-1
                    if (ix1<0):
                        ix1 = Lx-1
                    epsilon = (epsi[ix1,iy]+epsi[ix,iy])/2
                    etot = etot + epsilon
                    pot = pot + epsilon*V[ix1,iy]
                    ix1 = ix+1
                    if (ix1>Lx-1):
                        ix1 = 0
                    epsilon = (epsi[ix1,iy]+epsi[ix,iy])/2
                    etot = etot + epsilon
                    pot = pot + epsilon*V[ix1,iy]
                    iy1 = iy-1
                    if (iy1<0):
                        iy1 = Ly-1
                    epsilon = (epsi[ix,iy1]+epsi[ix,iy])/2
                    etot = etot + epsilon
                    pot = pot + epsilon*V[ix,iy1]
                    iy1 = iy+1
                    if (iy1>Ly-1):
                        iy1 = 0
                    epsilon = (epsi[ix,iy1]+epsi[ix,iy])/2
                    etot = etot + epsilon
                    pot = pot + epsilon*V[ix,iy1]
                    Vnew[ix,iy] = pot/etot
                else:
                    Vnew[ix,iy]=V[ix,iy]
        V, Vnew = Vnew, V # Swap pointers to arrays
    return V
```

And we set up and run the program using:

```
L = 200
z = np.zeros((L,L),float)
b = np.copy(z)
e = np.copy(z)
b[:] = np.float('nan')
e[:] = 1.0
# Potential at the edges for uniform field
b[:,0]=1.0
b[:,L-1]=-1.0
# Setup circular water droplet
R = 10
for ix in range(L):
    for iy in range(L):
```

```
        dx = ix-L/2
        dy = iy-L/2
        dd = np.sqrt(dx*dx+dy*dy)
        if (dd<R):
            e[ix,iy] = 80
# Visualize system before simulation
plt.subplot(1,2,1)
plt.imshow(b)
plt.subplot(1,2,2)
plt.imshow(e)
```

```
nrep = 100000
s = solvepoisson(b,e,nrep)
plt.subplot(1,2,1)
plt.imshow(s)
plt.colorbar()
plt.subplot(1,2,2)
Ey,Ex = np.gradient(-s)
x = np.linspace(0,L-1,L)
y = np.linspace(0,L-1,L)
Emag = Ey*Ey + Ex*Ex
plt.contourf(Emag)
plt.streamplot(x,y, Ex, Ey,density=1,arrowstyle='->',arrowsize=1.5)
plt.colorbar()
```

The resulting figures are shown in Fig. 5.12

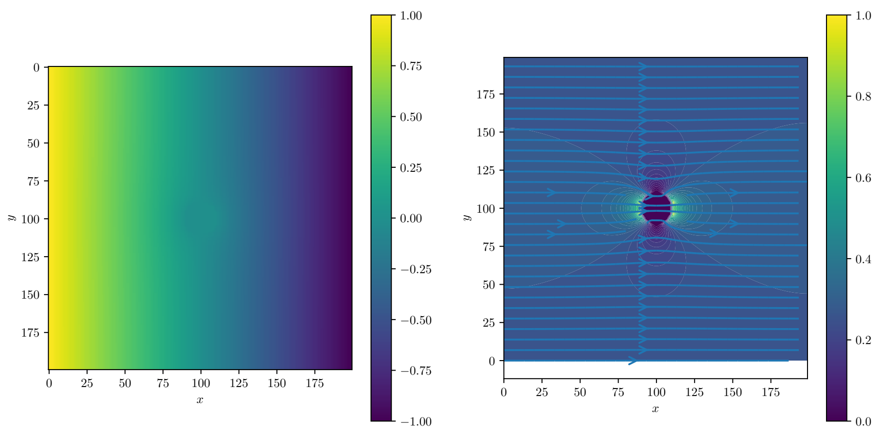


Fig. 5.12 Plot of (a) $V(x, y)$ and (b) the magnitude of the electric field and stream lines.

Discussion of numerical results. The resulting electric field looks like it is almost constant within the disk and that the numerical value of the electric field is much smaller. We indeed expect the magnitude of the field to be much smaller. We know that for a dielectric plate in an external field, the field inside the disk is reduced to $1/\epsilon_r$ of the uniform field. We expect a similar effect here, although we need to study in detail how much the field is reduced. Fig. 5.13 shows plots of $E_x(x, L/2)/\min(E_x(x, L/2))$, which clearly shows that the field is approximately uniform far from the droplet and inside the droplet, and that the field falls significantly inside the droplet. Let us now see if we also can develop an analytical theory for this particular problem.

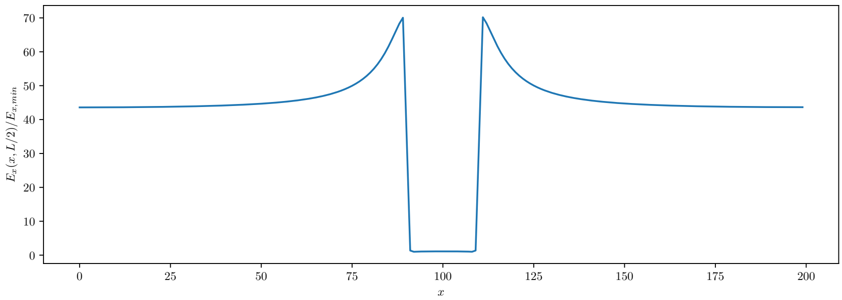


Fig. 5.13 Plot of $E_x(x, L/2)$ through the center of the sphere.

5.7.2 Analytical solution: Spherical coordinates

We want to solve Laplace's equation for this system. We choose spherical coordinates with the z-axis along the external field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$. We measure position using r, θ and see that $x = r\cos\theta$ as illustrated in Fig. 5.14. The boundary conditions for Laplace's equation are that that potential is continuous at $r = a$: $V_1(a) = V_2(a)$, where 1 is on the inside and 2 is on the outside. Because there are no free charges in the system, the normal component of the displacement field is also continuous across the boundary at $r = a$. The electric fields are therefore discontinuous and $D_1 = \epsilon_1 E_1 = \epsilon_2 E_2 = D_2$. This implies that $\epsilon_r \partial V_1/\partial r = \partial V_2/\partial r$. And far away from the sphere, we expect the system to have an electric field $E_0\hat{\mathbf{z}}$ so that $V = -E_0 z = -E_0 r\cos\theta$.

We must therefore find solutions to Laplace's equation that are proportional to $\cos\theta$. There are only two possible functions that are solutions to Laplace's equation in spherical coordinates that are proportional to

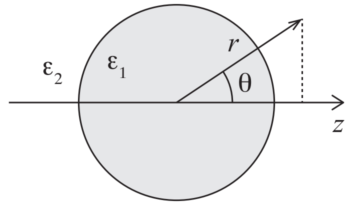


Fig. 5.14 Illustration of the coordinate system used for the analytical calculation.

$\cos\theta$: $V = Cr\cos\theta$ and $V = (C'/r^2)\cos\theta$ (check for yourself that these are solutions to Laplace's equation in spherical coordinates) [1] .

We use this to find an analytical solution to Laplace's equation in spherical coordinates which is consistent with the boundary conditions. To fit to the far-away field $\mathbf{E}_0$, the potential must have the form

$$V_2 = -E_0 r\cos\theta + \frac{B}{r^2}\cos\theta \ . \tag{5.42}$$

We assume that the field is uniform inside the droplet, as observed in the numerical simultion. (We can show that this is true by using the full solution to Laplace's equation in spherical coordinates). This means that $V_1 = Cr\cos\theta$. We use boundary conditions to find the unknowns, but drop the $\cos\theta$ term, since this is present in all the expressions: We see that $V_1(a) = Ca = V_2(a) = -E_0 a + (B/a^2)$. Also, we use that $\partial V_2/\partial r = (-E_0 - 2B/a^3) = \epsilon_r \partial V_1/\partial r = \epsilon_r C$. We solve for B and C and find that:

$$C = -\frac{3E_0}{\epsilon_r + 2} \ , \tag{5.43}$$

and therefore $V_1 = -\frac{3E_0}{\epsilon_r+2} r\cos\theta$ and

$$\mathbf{E}_1 = \frac{3}{\epsilon_r + 2}\mathbf{E}_0 \ . \tag{5.44}$$

inside the water droplet.

Outside the water droplet, we find that $B = (\epsilon_r - 1)/(\epsilon_r + 2)a^3 E_0$ and

$$V_2(r, \cos\theta) = -E_0 r\cos\theta + \frac{\epsilon_r - 1}{\epsilon_r + 2}\frac{a^3}{r^2}E_0\cos\theta \ . \tag{5.45}$$

[1] In the case of azimuthal symmetry, the general solutions to Laplace equation' in spherical coordinates are given as $V(r,\theta) = \left(A_l r^l + B_l/r^{l+1}\right) P_l(\cos\theta)$, where $P_l(x)$ are Legendre polynomials, which can be defined by Rodrigues formula $P_l(x) = 1/(2^l l!)\,(d/dx)^l\,(x^2 - 1)^l$. The first few are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$. The general solution is a sum over all possible values of l: $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l/r^{l+1}\right) P_l(\cos\theta)$.

The electric field outside the droplet is then $\mathbf{E}_1 = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$:

$$\mathbf{E}_2 = \mathbf{E_0} + 2E_0 \frac{\epsilon_r - 1}{\epsilon_r + 2}\frac{a^3}{r^3}\cos\theta\hat{\mathbf{r}} + E_0 \frac{\epsilon_r - 1}{\epsilon_r + 2}\frac{a^3}{r^3}\sin\theta\,\hat{\boldsymbol{\theta}} \ . \tag{5.46}$$

We can compare these exact results with the numerical results we found above. In Fig. 5.15 we plot both the numerical and the analytical results. We plot the electric field in terms of E_0, which in the numerical case is $\Delta V/\Delta L = 1 - (-1)/200$. Notice in the magnification that there is a significant mismatch between the numerical and the analytical results. You may be tempted to believe this is due to a problem with the numerical solution. And it is. But in a fundamental way. Numerically we have solved the two-dimensional problem, which is for a cylindrical system, but analytically we have solved for a spherical system. These systems are very different! We would see this if we plotted the functional shape of $E(r)$ as well.

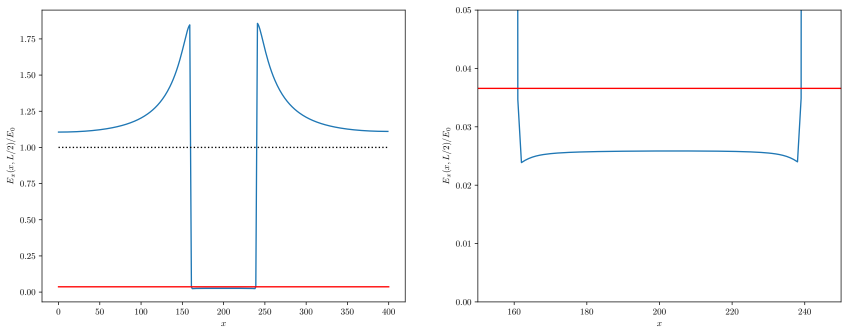


Fig. 5.15 Plot of the electric field $E_x(x)$ for the numerical simulations and the analytical solution in spherical coordinates.

If we instead solve the system in cylindrical coordinates with no variation in the z-direction. In this case, the solutions are

$$V_1 = -\frac{2E_0}{\epsilon_r + 1} r\cos\phi \tag{5.47}$$

$$V_2 = \frac{\epsilon_r - 1}{\epsilon_r + 1}\frac{a^2 E_0}{r}\cos\phi - E_0 r\cos\phi \ . \tag{5.48}$$

and the electric field inside the disk is:

$$\mathbf{E}_1 = \frac{2}{\epsilon_r + 1}\mathbf{E}_0 \ . \tag{5.49}$$

In Fig. 5.16 we compare this with the numerical results, and we see that the match now is very good. Let this serve as a warning — be careful with how you interpret your two-dimensional simulation in three dimension. In this case, we were really interested in a water droplet, which is round, that is, spherical. We should therefore really have redone the simulations, but in three dimensions.

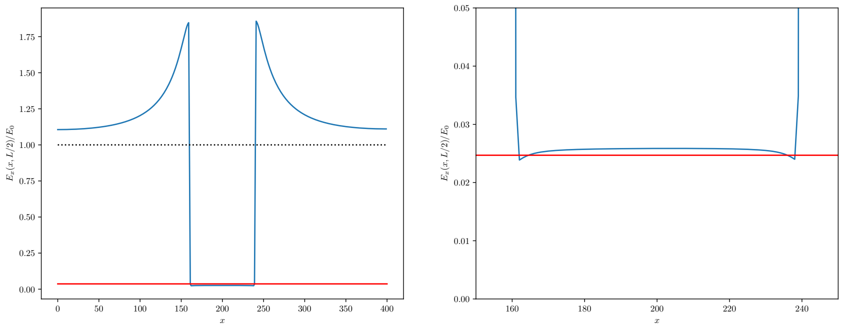


Fig. 5.16 Plot of the electric field $E_x(x)$ for the numerical simulations and the analytical solution in clindrical coordinates.

5.7.3 Electric field from water droplet

The electrical potential from the spherical droplet outside the droplet has the shape:

$$V_2(r, \cos\theta) = -E_0 r \cos\theta + \frac{\epsilon_r - 1}{\epsilon_r + 2} \frac{a^3}{r^2} E_0 \cos\theta \ . \tag{5.50}$$

We recognize the far-field component as due to the constant potential. We also recognize the $(1/r^2)\cos\theta$-potential term as the potential from a dipole:

$$V_{DP} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{r} \cdot \mathbf{p}}{r^3} = \frac{p \cos\theta}{4\pi\epsilon_0 r^2} \ . \tag{5.51}$$

This means that water droplet behaves as a dipole with dipole moment:

$$p = 4\pi\epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 2} a^3 E_0 \ . \tag{5.52}$$

The strength of the dipole increases linearly with E_0. In addition, we see that increases with $a^3 = V/(4\pi/3)$. Therefore, we also notice that the strength of the dipole is proportional to the volume of the water droplet.

5.7.4 Bubble inside water?

What do you think would happen in the reverse situation, where there is a bubble inside a water body with an external uniform electric field? You can use exactly the same method to address this, but we need to realize that $\epsilon_r = \epsilon_1/\epsilon_2$. In the case of a water droplet, we see that $\epsilon_r = 80$, but in the case of a bubble in water, we get $\epsilon_r = 1/80$.

What are the consequences of this? For the water droplet, the electric field inside the droplet is decreased, but for the bubble in water, the electric field inside the bubble is increased compared to the external field.

Finally, we also observe that when $\epsilon_1/\epsilon_2 \to \infty$, we get $p = 4\pi\epsilon_0 a^3 E_0$, which we will see later also is the effective dipole moment of a metal sphere in vacuum.

5.8 Numerical solutions: Implicit methods

Above we introduced the method of Jacobi iterations to find an approximative solution to Laplace's equation for the two-dimensional boundary value problem. This is just one of many methods to solve Laplace's equation in general and the discretized version of Laplace's equation in particular. There are also other strategies such as finite element methods and machine learning methods. Here, we will generalize the solution methods for the discretized equations and discuss their properties.

We found that in two dimensions Laplace's equation is discretized to:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0 \,, \tag{5.53}$$

where boundary conditions can be in the form of specified values of $V(x, y)$ on a boundary S. Finding a *solution* to this discretized version of Laplace's equation means to find values $V_{i,j}$ that satisfies both (5.53) and the boundary conditions. The method of Jacobi iterations provides an *approximative* solution. We did not find the values $V_{i,j}$ that exactly satisfied (5.53), but values that were sufficiently near the exact solution. Let us be more precise by following the approach introduced in [**?**], where we look at a small lattice in order to enumerate the problem exactly.

5.8.1 Description of the system

We address the electric potential $V(x, y)$ on a quadratic system of length $L \times L$ with boundary conditions $V = 0$ on the lines $x = 0$, $y = 0$ and $x = L$ and $V = V_0$ on the line $y = L$. The system is discretized using a $N \times N$ elements square lattice with spacing $\Delta x = L/N$ so that $V_{i,j} = V(i\Delta x, j\Delta x)$, where i and j run from 0 to $N - 1$. This system is illustrated in Fig. 5.17 with the detailed enumeration for $N = 6$. For this system, there are 16 internal points, where we need to find the values for $V_{i,j}$ from (5.53), and there are 20 boundary values that are given.

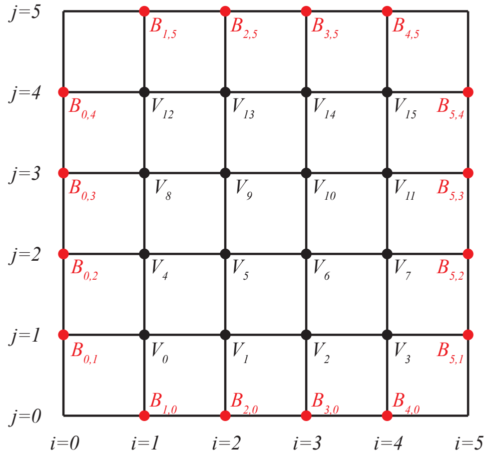


Fig. 5.17 Illustration of an 6×6 system of points for the solution of $V(x, y)$ on the lattice $V_{i,j}$. Boundary nodes are illustrated in red and internal nodes are illustrated in black.

5.8.2 Equations for the unknown values of $V_{i,j}$

Each value of $V_{i,j}$ are determined from (5.53). For point that do not have any boundary values as its nearest neighbors, such as for $(2, 3)$, the equation is

$$V_{3,3} + V_{1,3} + V_{2,4} + V_{2,2} - 4V_{2,3} = 0 \;, \tag{5.54}$$

where all the values $V_{i,j}$ are unknowns. However, for points that are near the boundary, such as for $(1, 3)$, the equation still has the same form:

$$V_{2,3} + V_{0,3} + V_{1,4} + V_{1,2} - 4V_{1,3} = 0 \;, \tag{5.55}$$

but we replace $V_{0,3}$ with its boundary value $B_{0,3}$, which is a given constant:

$$V_{2,3} + B_{0,3} + V_{1,4} + V_{1,2} - 4V_{1,3} = 0 \;, \tag{5.56}$$

We get one such linear equation for each of the 16 internal points: 16 equations in total and 16 unknown values of $V_{i,j}$. We therefore have a set of linear equations that we need to solve to find the solution. These equations can be solved using an approximative method, such as Jacobi iterations, or with an exact method, such as Gaussian elimination.

5.8.3 Enumeration

However, if we are to write all 16 linear equations, it is better to introduce a new enumeration scheme for the points, where we use a single index n for each point (i, j). Here, we use the scheme: $n = (j-1)(N-2) + (i-1)$ so that $n = 0, 1, \ldots, 15$. This scheme is illustrated in Fig. 5.17. We also change the sign of the equation to simplify the notation. For point $(2, 3)$ we get:

$$V_{3,3} + V_{1,3} + V_{2,4} + V_{2,2} - 4V_{2,3} = 0 \;\rightarrow\; -V_{10} - V_8 - V_{13} - V_5 + 4V_9 = 0 \;. \tag{5.57}$$

and for point $(1, 3)$ the equation becomes:

$$V_{2,3} + B_{0,3} + V_{1,4} + V_{1,2} - 4V_{1,3} = 0 \;\rightarrow\; -V_9 - B_{0,3} - V_{12} - V_4 + 4V_8 = 0 \;, \tag{5.58}$$

where we notice that we need to keep track of the boundary value $B_{0,3}$. We can rewrite this on standard form for linear equations, where the constants are on the right hand side:

$$-V_9 - V_{12} - V_4 + 4V_8 = B_{0,3} \;, \tag{5.59}$$

5.8.4 System of equations

We write down all the 16 equations for the internal points. On matrix form this gives us the following system of equations:

$$\begin{bmatrix}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \\ V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \\ V_{15} \end{bmatrix}
=
\begin{bmatrix} B_{0,1} + B_{1,0} \\ B_{2,0} \\ B_{3,0} \\ B_{4,0} + B_{5,1} \\ B_{0,2} \\ 0 \\ 0 \\ B_{5,2} \\ B_{0,3} \\ 0 \\ 0 \\ B_{5,3} \\ B_{0,4} + B_{1,5} \\ B_{2,5} \\ B_{3,5} \\ B_{5,4} + B_{4,5} \end{bmatrix}$$

This is a matrix equation on the form $Ax = B$, where the unknown x here is the potentials, V. Notice how you can read out the form of the equation from this matrix. You see that there is a 4 on the diagonal (this would be 2 in one dimension and 6 in three dimensions), which corresponds to the $4V_i$ in the equation. You also see that there is a -1 for each of the nearest neighbors: These are the -1 that are next to the 4 on the diagonal: $-V_{i+1,j}$ etc. These are replaced by zeros for some of the rows — for the nodes that are next to a boundary. Then there are -1's that are offset by two zeros. These correspond to the nodes that are on the row above or below the current row. The number of positions they are offset from the diagonal (with value 4) correspond to the number of nodes in a row. See for yourself that the values in the matrix makes sense when you compare with Fig. 5.17.

5.8.5 Implicit numerical solution

We now have a system of linear equations that we can solve with any solution method that is efficient and precise. There are many tools in Python to help you solve such systems of linear equations. Here, we will set up the equations and solve them with a simple tool.

Setting up the matrices. We now generate the matrix $A_{n,m}$ and the vector B_n by inserting the relevant values for each (i, j). However, we have to remember that Python uses the inverse numbering of the indexes so that the first index is the y-coordinate (the row) and the second index is the x-coordinate (the column). We look through each point (i, j) and insert a 4 at the corresponding $A_{n,n}$. We then insert a -1 for each neighbor $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$ at the corresponding position in A, unless they are part of the boundary, in which case we instead add the boundary value to the B-vector. We write a function to set up the system of equations, solve the system, and put the values back into an array that includes both the internal and the boundary values for V:

```
def poissonimplicit(b):
    N = b.shape[0]
    # Setting up system of equations
    LM = (N-2)*(N-2)
    A = np.zeros((LM,LM),float)
    B = np.zeros((LM),float)
    for j in range(1,N-1):
        for i in range(1,N-1):
            # 4Vi,j-Vi+1,j-Vi-1,j-Vi,j+1-Vi,j-1
            n = (j-1)*(N-2) + (i-1)
            #print("ix,iy = ",i,j,", n = ",n)
            A[n,n] = 4 # Diagonal
            if (i>1):
                A[n,n-1] = -1
            else:
                B[n] = B[n] + b[j,i-1]
            if (i<N-2):
                A[n,n+1] = -1
            else:
                B[n] = B[n] + b[j,i+1]
            if (j>1):
                A[n,n-(N-2)] = -1
            else:
                B[n] = B[n] + b[j-1,i]
            if (j<N-2):
                A[n,n+(N-2)] = -1
            else:
                B[n] = B[n] + b[j+1,i]
    x = np.linalg.solve(A,B)
    # Generate full V matrix
    V = np.zeros((N,N),float)
    for j in range(0,N):
        for i in range(0,N):
            if (i<1)or(i>=N-1)or(j<1)or(j>=N-1):
                V[j,i] = b[j,i]
            else:
                n = (j-1)*(N-2) + (i-1)
                V[j,i] = x[n]
    return V
```

We then set up the boundary condition in the array `b` and solve the system:

```
# Setting up the boundaries
N = 6
b = np.zeros((N,N),float)
b[:,0] = 0.0
```

```
b[:,N-1] = 0.0
b[0,:] = 0.0
b[N-1,:] = 1.0
# Solve the boundary value problem
V = poissonimplicit(b)
# Plot the result
plt.contourf(V)
```

The resulting plot is shown in Fig. 5.18. Notice that this is an exact solution of the linear system of equations, which is an approximation to Laplace's equation. It is not necessarily an exact solution to Laplace's equation! (For this particular problem we can find the exact solution to Laplace's equation and compare with the numerical solution. We will leave this as an exercise.)

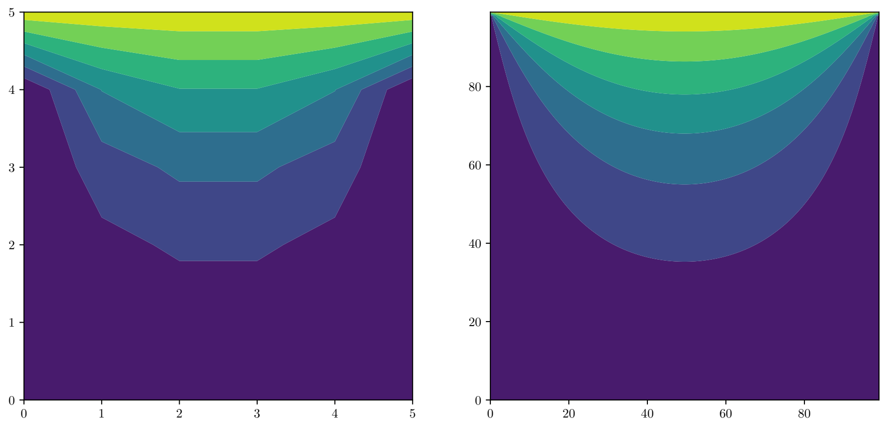


Fig. 5.18 Plot of $V(x,y)$ for the implicit solution of Laplace's equation for $N=6$ (left) and $N=100$ (right).

Improved implementation. The implementation we have provided here is simple and naive and is meant to be pedagogical and not efficient, however, it does work well up to system of N of a few hundred. If you try a high resolution (a large value of N) you will notice that the matrix A mostly will contain zeros. We call such matrices *sparse*. There are more efficient methods to set up and solve sparce matrices in Python. In order to make the solution method efficient you will both need to implement this as a sparse matrix and you may need to be more careful in how you choose solution methods for solving the system of linear equations. We will discuss this in an exercise.

5.9 Numerical solutions: Machine learning methods

So far we have attempted to solve Laplace's equation by finding values of $V(x,y)$ on a lattice in such a way that the discretized version of Laplace's equation is satisfied on the lattice. Let us try a completely different approach: Let us describe $V(x,y)$ by a large set of parameters w_j: $V(x,y;w_j)$, and then we will try to choose values of w_j so that $V(x,y;w_j)$ is a solution to Laplace's equation and satisfies all the boundary conditions.

This is a very different approach to the finite difference scheme. In general, we do not expect to choose the parameters so that the function fits exactly, we will instead try to minimize the deviation from a solution that satisfies both the differential equation and the boundary conditions. To do this, we will use a neural network to represent the function $V(x, y; w_j)$, where the parameters are the weights and biases in the neural network. And we will use methods from machine learning to find a good set of parameters. This provides us with a continuous solution — a solution for any point in the volume we are solving Laplace's equation. This example was inspired by an example by Diogo Ferreira[2].

5.9.1 Model formulation

We want to develop a method to solve Laplace equation to find the potential $V(x, y)$:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{5.60}$$

in a region $0 \leq x \leq a$, $0 \leq y \leq b$ with a given set of boundary conditions.

The idea is that we will represent the potential $V(x, y)$ with a neural network. The neural network has a set of parameters consisting of weights w_{ij} and biases b_{ij} that describe the potential. With sufficiently many parameters, it can be shown that a neural network can represent any function.

We will then determine the parameters so that this function is an (approximate) solution to the differential equation and the boundary conditions. We can improve the approximation by increasing the size of the network and by improving the choice of parameters.

One-dimensional potential. First, let us see how we can describe a one dimensional potential, $V(x)$, by a neural network. The neural network is a function that takes an input parameter, x, transforms this into a set of internal variables, h_i, the hidden nodes, and then finally, transform the h_i into the output, $V(x)$.

Each hidden value h_i is determined by the weight $w_{x,i}$ of the connection between x and the hidden parameter i and the bias of the connection b_{0i} through the following relation

$$h_i = g(w_{x,i}x + b_{0i}) \ . \tag{5.61}$$

where the function $g(u)$ is called the activation function. Here, we will use a hyperbolic tangent for the activation function, $g(u) = \tanh(u)$.

The output, $V(x)$, is found by summing all the hidden values, again adding weights and biases:

$$V(x) = \sum_i w_{1i}h_i + b_{1i} \tag{5.62}$$

where we have used the notation w_{1i} for the weights that are added to the hidden values and b_{1i} to the biases added to the hidden values. In some cases we may also apply the transfer function, but here we simply use the weighted sum of the hidden parameters in what is called the *hidden layer*.

Networks may consist of one of several such hidden layers. A network with several hidden layers is called a *deep neural network*. Here, we will only use a single hidden layer.

The function $V(\{w, b\}; x)$ is a function of all the parameters $\{w_{x,i}, w_{1i}, b_{ij}\}$ needed to determine the function. The goal is to determine the parameters so that this function satisfies the differential equation and the boundary conditions.

[2] `https://towardsdatascience.com/how-to-solve-an-ode-with-a-neural-network-917d11918932`

Two-dimensional potential. How can we extend this to two dimension? In this case, we have two inputs to each hidden parameter h_i, one from x and one from y with weights $w_{x,ij}$ and $w_{y,ij}$ respectively. Otherwise, the description is the same. We only include one bias per hidden parameter, h_i. We sum the inputs and apply the transfer function to get the hidden parameter, h_i:

$$h_i = g(w_{x,0i}x + w_{y,0i}y + b_{0i}) \ , \tag{5.63}$$

and we then sum all the hidden parameters to find the output value:

$$V(x,y) = \sum_i w_{1i}h_i + b_{1i} \ . \tag{5.64}$$

Again, we need to find the parameters that make this function satisfy the differential equation and the boundary conditions. (And again we do not apply the transfer function $g(u)$, but only include the sum).

Satisfying the differential equation. How can we ensure that this function satisfies the differential equation? We can insert it into the differential equation $\nabla^2 V = 0$. That is, we can calculate $\nabla^2 V - 0$ and choose parameters that make this as small as possible (as close to zero as possible) .

We do this by looking at the square of the value of $\nabla^2 V - 0$, and we call this the loss, $L(\{w,b\})$:

$$L(\{w,b\}) = \left(\nabla^2 V - 0\right)^2 \ . \tag{5.65}$$

Notice that the loss is a function of the parameters $\{w,b\}$ of the neural network.

In addition, the function should satisfy the boundary conditions. We do this by also including the boundary conditions in the loss function. For example, if we have the boundary condition $V(0,y) = V_0$, we need to include this at a set of points y_i at the boundary, and include the average of the square deviations, averaged over the points y_i:

$$L(\{w,b\}) = \left(\nabla^2 V - 0\right)^2 + \langle (V(0,y_i) - V_0)^2 \rangle \ . \tag{5.66}$$

Similarly, we must include the values on the other boundaries. In this particular case we would like to use the boundary conditions $V(x,0) = \sin(x)$, $V(0,y) = 0$, $V(a,0) = 0$, $V(0,b) = 0$.

This gives us the loss function:

$$L(\{w,b\}) = \left(\nabla^2 V - 0\right)^2 + \langle (V(x_i,0) - \sin(x_i))^2 \rangle \tag{5.67}$$

$$+ \ \langle (V(0,y_i) - 0)^2 \rangle + \langle (V(a,y_i) - 0)^2 \rangle \tag{5.68}$$

$$+ \ \langle (V(x_i,b) - 0)^2 \rangle \ . \tag{5.69}$$

Now, the task is to find the set of parameters $\{w,b\} = \{w_{x,i}, w_{y,i}, w_{ij}, b_{ij}\}$ that makes $L(\{w,b\})$ as small as possible. We have therefore changed the problem of solving the differential equation to a minimization problem.

5.9.2 Initiating the neural network

We initiate the neural network by defining the function `V`, which will correspond to the potential V. We introduce the neural network through this function. All the weights are placed in one array, `params`.

```
# Import necessary libraries
import jax.numpy as np
import matplotlib.pyplot as plt
from jax.experimental import optimizers
from jax import jit
# We set up the neural network with n nodes in the hidden layer
n = 100
def V(params, x, y):
    w0 = params[:n]
    b0 = params[n:2*n]
    w1 = params[2*n:3*n]
    w2 = params[3*n:4*n]
    b1 = params[4*n]
    hidden = np.tanh(x*w0 + y*w1 + b0)
    output = np.sum(hidden*w2) + b1
    return output
```

We apply `grad` to find the derivatives of `V` with respect to the second and third parameter, which corresponds to x and y. (Notice the use of Python numbering, so that `params` is number 0). This function is provided by `jax` and is very fast and accelerated by a GPU or a TPU if you have this on your machine.

```
from jax import grad
# Then we define the derivative of f
dVdx = grad(V, 1)
dVdy = grad(V, 2)
ddVddx = grad(dVdx, 1)
ddVddy = grad(dVdy, 2)
```

In order to use these later, we need them to be vectorized so that they can work on a whole vector in a fast way. This is done using the `vmap` function which is part of `jax`.

```
from jax import vmap
# And the vectorized versions
V_vect = vmap(V, (None, 0, 0))
dVdx_vect = vmap(dVdx, (None, 0, 0))
dVdy_vect = vmap(dVdy, (None, 0, 0))
ddVddx_vect = vmap(ddVddx, (None, 0, 0))
ddVddy_vect = vmap(ddVddy, (None, 0, 0))
```

We select all the initial weights of the neural network to be random numbers. Here, we choose them from a normal distribution (but it is not important exactly how we choose them, although they should not all be zero initially). Notice that we use the built-in `random` function from `jax` and not from `numpy`.

```
# We define initial weights of the neural net
# Random weights with normal distribution
from jax import random
key = random.PRNGKey(0)
params = random.normal(key, shape=(4*n+1,))
```

5.9.3 Loss function

When we calculate the loss function L, we cannot calculate the differential equation everywhere. Instead, we choose a set of points (x_i, y_i) where we calculate the differential equation, and then we find

$$\left(\nabla^2 V - 0\right)^2 = \langle \left(\nabla^2 V(x_i, y_i) - 0\right)^2 \rangle \, , \tag{5.70}$$

where the average is over all the points (x_i, y_i). We can choose these points in any way we like. We may choose them to be on a lattice, or we may choose them to be random points on the interval of x and y that we are addressing. Here we choose them randomly in the region $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$.

```
import jax.ops
inputs = np.pi*random.uniform(key, shape=(1000, 2))
inputs = jax.ops.index_update(inputs, jax.ops.index[:,0], \
           np.pi*random.uniform(key, shape=(1000,)))
```

5.9.4 Boundary conditions

Similarly, we need to define the boundary conditions. We do this on a set of points on the boundaries. We introduce arrays for the points x_i and y_i along the x and the y axis:

```
# We define the grid used for the boundary conditions
nx = 31
ny = 31
x = np.linspace(0, np.pi, num=nx)
y = np.linspace(0, np.pi, num=ny)
```

And then we introduce the boundary conditions. Here, we need to find the values for $V(x, 0)$, where is should be set to $V(x, 0) = \sin(x)$:

```
# Define boundary conditions
def boundary_values_1(x, y):
    return np.sin(x)

boundary_condition_1 = np.zeros(nx)
boundary_condition_1 = jax.ops.index_add(boundary_condition_1, \
   jax.ops.index[:], boundary_values_1(x, 0.0))

# We check that this looks right by plotting the values
plt.plot(boundary_condition_1)
```

The resulting plot is show in Fig. 5.19.

5.9.5 Loss function

Then, we introduce the loss function:

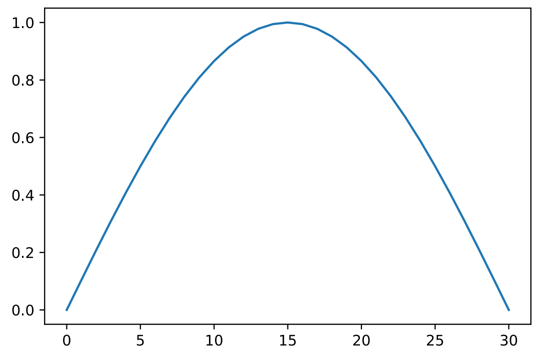


Fig. 5.19 Plot of the boundary conditions along one of the axis. The values along the other axis are zero.

$$L = \langle \left(\nabla^2 V(x_i, y_i) - 0\right)^2 \rangle + \langle (V(x_i, 0) - \sin(x_i))^2 \rangle \tag{5.71}$$

$$+ \langle (V(0, y_i) - 0)^2 \rangle + \langle (V(a, y_i) - 0)^2 \rangle \tag{5.72}$$

$$+ \langle (V(x_i, b) - 0)^2 \rangle \, . \tag{5.73}$$

```
# Define loss function
@jit
def loss(params, inputs):
    ix = inputs[:,0]
    iy = inputs[:,1]
    eq = ddVddx_vect(params, ix, iy)+ ddVddy_vect(params, ix, iy)

    bc1 = V_vect(params, x, y[0]*np.ones(nx)) - boundary_condition_1
    bc3 = V_vect(params, x[0]*np.ones(ny), y)
    bc2 = V_vect(params, x[-1]*np.ones(ny), y)
    bc4 = V_vect(params, x, y[-1]*np.ones(nx))
    return np.mean(eq**2) + np.mean(bc1**2) + np.mean(bc2**2) + \
        np.mean(bc3**2) + np.mean(bc4**2)
```

5.9.6 Minimizing the loss function

The final part of the method is then to introduce a method to minimize the loss function. Here we will use a simple method called gradient decent. This is a very simple method, but we include it here to demonstrate that rather simple methods work. You may want to replace this with a more advanced method from one of the many machine learning libraries in Python. (We use `jit` to ensure functions are compiled *just in time* during the minimization. This speeds the calculation up significantly).

```
# Learning
epochs = 10000
learning_rate = 0.001
momentum = 0.99
velocity = 0.

grad_loss = jit(grad(loss, 0))
```

```
for epoch in range(epochs):
    if epoch % 1000 == 0:
        print('epoch: %3d loss: %.6f' % (epoch, loss(params, inputs)))
    gradient = grad_loss(params + momentum*velocity, inputs)
    velocity = momentum*velocity - learning_rate*gradient
    params += velocity
```

If the loss L is converging towards zero, we have found a good solution and we are ready to visualize. (If you try to fiddle with the parameters, I advice you to only change `learning_rate` and the number of minimization steps, `epochs`. If you get a `nan` for the loss, you typically need to reduce the `learning_rate`.) If you have a background in physics, you may see how the choice of names for the variables corresponds to what happens if you relax (move) a system along with the direction of its gradients.

```
# We visualize V(x,y) on a grid spanned by the points x,y from boundary conditions
xv, yv = np.meshgrid(x, y)
import numpy as np
solution = np.zeros((nx, ny))
for i, ix in enumerate(x):
    for j, iy in enumerate(y):
        solution[j,i] = V(params, ix, iy)

# We plot in 3d. The figure can be rotated if plotted outside of the notebook
# This is why we use %matplotlib above
from mpl_toolkits.mplot3d import axes3d
import matplotlib.pyplot as plt

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

X, Y = np.meshgrid(x, y)
# Plot a basic wireframe.
ax.plot_wireframe(X, Y, solution, color="black")
```

The resulting plot is show in Fig. 5.20.

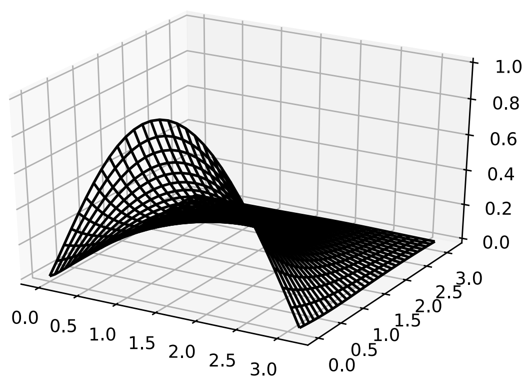


Fig. 5.20 Plot of the solution to Laplace equation in two dimensions using machine learning.

5.10 Summary

Poisson's equation can be used to find the electric potential in a volume with a given charge density ρ: $\nabla^2 V = -\rho/\epsilon_0$.

Laplace's equation can be used to find the electric potential in a volume with zero charge density: $\nabla^2 V = 0$.

Laplace's and Poisson's equations have a unique solution on a volume v if the boundary conditions on the surface enclosing v are given.

Dirichlet boundary conditions provide a given value for the potential at the boundaries. **von Neumann boundary condition** provide a given value for the gradient of the potential at the boundaries.

We have now established several methods to relate the charge density ρ, the electric field $\mathbf{E}$ and the electric potential V:

$$
\begin{aligned}
\rho_v \to \mathbf{E} &: \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v \mathbf{R}}{R^2}\mathrm{d}v' \\
\mathbf{E} \to \rho_v &: \quad \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad (\nabla \times \mathbf{E} = 0) \\
\rho_v \to V &: \quad V = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v}{R}\mathrm{d}v' \\
V \to \rho_v &: \quad \nabla^2 V = -\rho_v/\epsilon_0 \\
V \to \mathbf{E} &: \quad \mathbf{E} = -\nabla V \\
\mathbf{E} \to V &: \quad V = -\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{l}
\end{aligned}
$$

5.11 Exercises

5.11.1 Test yourself

Exercise 5.1: Laplace operator

Which of the following scalar potentials are or can be solutions to Laplace's equation and Poisson's equation?

a) $V(x) = ax + b$

b) $V(x) = A \sin \omega x$

c) $V(x) = c$

d) $V(x, y) = Ax + By$

e) $V(r, \theta, \phi) = a/r + b$

f) $V(x, y) = ax^2 y$

5.11.2 Discussion exercises

Exercise 5.2: Laplace operator

In one dimension, a linear curve is a solution to Laplace equation. Are there any other solutions that are not linear? In two dimensions, a plane may also be a solution to the Laplace equation. Are there any other solutions and what would that depend on?

Exercise 5.3: Discontinous potential

Is it possible to have a discontinuous potential? Provide examples or arguments for why it is not possible. (Hint: $\mathbf{E} = -\nabla V$).

5.11.3 Tutorials

Exercise 5.4: Spatial variation of potentials (Laplace equation)

We will study a system in the xy-plane. At $x = 0$ and $x = L$ the scalar potential is $V_0 = 0$. The volume charge density between $x = 0$ and $x = L$ is zero, and the dielectric constant is ϵ.

a) Make a sketch of the system. What quantities is it useful to include in the sketch?

b) What is the scalar potential in the region between $x = 0$ and $x = L$? Be very precise in how you determine this and what assumptions you make. What is the electric field in this region?

We now introduce a new feature in the system: The potential at $x = L/2$ is $V_1 > V_0$. The volume charge density is still zero, and the dielectric constant is ϵ.

c) Update your drawing to reflect this.

d) Find the scalar potential $V(x, y)$ everywhere between $x = 0$ and $x = L$. What is the electric field?

e) What kind of system could this situation represent?

f) What is the surface charge density on the surfaces at $x = 0$, and $x = L$. Discuss what happens at $x = L/2$.

Exercise 5.5: Surface charges

Consider a sphere with radius a with a charge Q uniformly distributed on its surface.

a) What is the electric field as a function of r? (Find the behavior for both $r < a$ and $r > a$)

b) What is the scalar potential, $V(r)$?

c) Does $V(r)$ satisfy Laplace equation? (Where does it/does it not satisfy the equation).

d) Use your result for E to find the surface charge at $r = a$.

Exercise 5.6: Laplace's equation in two dimensions

The Python programs provided in the text provides skeleton programs to solve Laplace's equation on a rectangular area with a given set of Dirichlet boundary conditions on the boudary S, $V(S)$.

a) Apply the program to the test case where $V(x = 0, y) = 0$ and $V(x = L, y) = V_1$. How can you check that the results are correct? Find the exact solution and compare with the numerical results.

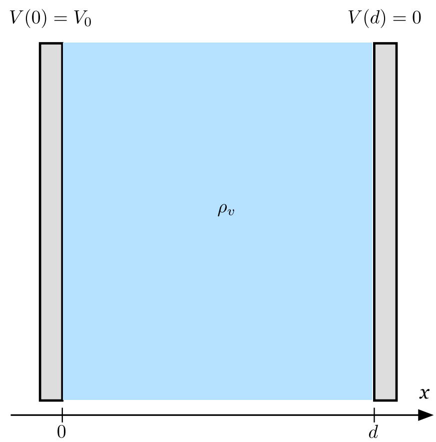
b) Use the program to find V and $\mathbf{E}$ for the case when $V = 0$ for $x = 0$ and $x = L$, and $V = V_1$ for $y = 0$ and $y = L$. Discuss the potential and field you get. Does it correspond to your intuition?

c) Use the program to explore a situation of choice. Predict what you will get first, and then run the program to test your intuition.

5.11.4 Homework

Exercise 5.7: Charge distribution between metal plates

Two plane metal plates with a spacing of $d = 1$ cm have a uniformly distributed volume charge density $\rho_v = -10^{-5}$ C/m^3 in the area between them. The medium between the plates have relative permittivity $\epsilon_r = 1$. One plate is grounded ($V = 0$) while to other plate is at a potential $V_0 = 10$ V.



a) Find the electric potential between the plates as a function of x when we assume that the plates have infinite extent.

Hint. Poisson's equation gives a second order differential equation in one variable. Solve this equation with boundary conditions $V(0) = V_0$ and $V(d) = 0$.

b) Find the electric field as a function of x.

c) For which value of x does the potential have a minimum? Find V_{min}.

d) Sketch the potential $V(x)$, and the x-component of the electric field as a function of x.

Exercise 5.8: Poisson's equation in one dimension

Poisson's equation is

$$\nabla^2 V = -\frac{\rho}{\epsilon}\,, \tag{5.74}$$

and is valid in this form when the permittivity does not vary over the area we solve the equation.

We can solve Poisson's equation for example using finite differences by approximating the second derivative of the potential as:

$$\frac{\mathrm{d}^2 V_i}{\mathrm{d}x^2} \approx \frac{V_{i+1} - 2V_i + V_{i-1}}{\varDelta x^2} \tag{5.75}$$

where $V_i = V(x_i)$, and we evaluate V in discrete points x_i.

We see from the form of the finite difference approximation that we cannot solve the equation by forward interation, as we do when we integrate the equations of motion using Forward Euler or Euler-Cromer's method. If we had started on the left side and tried to find V_{i+1} from V_i and V_{i-1}, we would not necessarily reached the correct boundary condition on the right hand side. Similarly, if we started on the right hand side, we would not have reached the correct boundary condition on the left hand side.

The solution is to use an *implicit* solver. We can do this by posing the complete set of equations as a matrix equation and solving this matrix equation by inverting the matrix,

$$A\overrightarrow{V} = \frac{\overrightarrow{\rho}}{\epsilon} \tag{5.76}$$

$$\overrightarrow{V} = A^{-1}\overrightarrow{\rho} \ . \tag{5.77}$$

Where A is a matrix and $\overrightarrow{V}$ and $\overrightarrow{\rho}$ are column vectors.

a) Construct the matrix A so that $A\overrightarrow{V} = \frac{1}{\epsilon}\overrightarrow{\rho}$, where $\overrightarrow{V}$ and $\overrightarrow{\rho}$ now are column vectors that contain the values of respectively V and ρ in discrete points. For now ignore what happens at the boundaries of the region of solution.

We need to make an additional consideration when it comes to the boundary values. When we multiply the matrix with V, the resulting boundary values we get for ρ will not reflect the elements outside the diagonal, as is the case for the elements inside ρ.

b) Modify the first and the last element in ρ so that you can choose a specific potential value on the boundary of the region (Dirichlet boundary conditions).

c) Write a program that takes the boundary conditions and the charge distribiution as inputs, and return the potential. Test the program for the charge distribution from exercise 5.7.

Exercise 5.9: Poisson's equation

(By Sigurd Sørlie Rustad)

In this exercise we will look at Poisson's equation.

a) Using Gauss' Law, derive Poisson's equation.

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \ . \tag{5.78}$$

b) Consider a cylinder with height $h = 1\text{m}$ and radius $r = 1\text{m}$. The inside of the cylinder is hollow, and the walls are very thin. Use Poisson's equation to find the electric potential on the walls of the cylinder. The bottom of the cylinder has electric potential $V = 10\text{V}$, and the top of the cylinder has zero potential. The cylinder is electrically neutral.

Hint. If you fold the cylinder out you get a square.

c) Find a difference equation that approximates the solution Poisson's equation, in one dimension.

Hint. Use the approximation

$$\frac{\mathrm{d}V_n}{\mathrm{d}x} \approx \frac{V_{n+1} - V_n}{\Delta x} \tag{5.79}$$

d) Imagine that we do not know the theoretical solution. We then need to find it numerically. Use the difference equation over to solve Poisson's equation numerically for the cylinder.

Hint 1. Notice that you only know the top and bottom conditions. You need to test for different initial conditions until you find a solution that fits.

Hint 2. To test the initial conditions, you need to compare your result (with the tested initial conditions) with the potential at the bottom of the cylinder (that you know).

Hint 3. A good testing range is $V_{n+1} \in [10\text{V}, 8\text{V}]$.

Exercise 5.10: Potential

In this exercise we will address a one-dimensional system with charge density $\rho(x)$. Assume that the electric potential is $V(0) = 0$ in the point $x = 0$ and $V(L) = V_0$ in the point $x = L$.

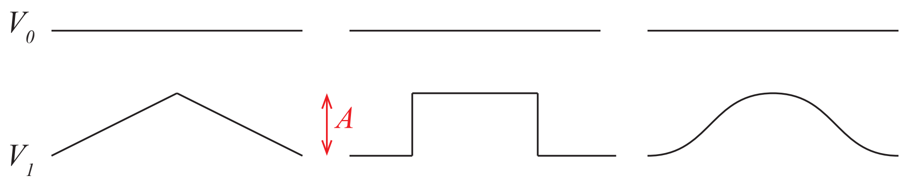
a) Find the electric potential on the interval $0 < x < L$ when $\rho(x) = 0$.

b) Find the electric potential on the interval $0 < x < L$ when $\rho(x) = \rho_0$, which is a constant.

5.11.5 Modeling projects

Exercise 5.11: Dielectric breakdown in periodic geometries

You are building a small electrical component that has an irregular surface structure. Three such structures are shown in the figure. You will apply a huge voltage across the gap – between surfaces A and B – and you worry that you may get dielectric breakdown.



Your plan is to use the program that we have developed to find the potential:

```
import numpy as np
import numba
@numba.jit(cache=True)
def solvepoissonperiodic(b,nrep):
    """ b = boundary conditions
    nrep = number of iterations
    returns potentials """
    V = np.copy(b)
    for i in range(len(V.flat)):
        if (np.isnan(b.flat[i])):
            V.flat[i] = 0.0
    Vnew = np.copy(V)
    Lx = b.shape[0]
    Ly = b.shape[1]
    for n in range(nrep):
        for ix in range(Lx):
            for iy in range(Ly):
                etot = 0.0
                pot = 0.0
                if (np.isnan(b[ix,iy])):
                    if (ix>0):
                        etot = etot + 1
                        pot = pot + V[ix-1,iy]
                    if (ix<Lx-1):
                        etot = etot + 1
                        pot = pot + V[ix+1,iy]
                    iy1 = iy-1
                    if (iy1<0):
                        iy1 = Ly-1
                    etot = etot + 1
                    pot = pot + V[ix,iy1]
                    iy1 = iy+1
                    if (iy1>Ly-1):
                        iy1 = 0
                    etot = etot + 1
                    pot = pot + V[ix,iy1]
```

```
                    Vnew[ix,iy] = pot/etot
                else:
                    Vnew[ix,iy]=V[ix,iy]
        V, Vnew = Vnew, V # Swap pointers to arrays
    return V
```

to explore the electric field (the difference in potential between two neighboring points). We expect the dielectric material to break down first where the electric field is the largest.

You can specify the structure in the `b`-array and then calculate the electric potential and field. For example, you can specify a cosine-shaped surface in the following way:

```
# Make more complicated system
L = 200
b = np.zeros((L,L),float)
b[:] = np.float('nan')
b[L-1,:]=-1.0
A = 30
P = 1.0
for iy in range(L):
    h = int(5+A - A*np.cos(iy/L*2*np.pi*P))
    b[0:h,iy] = 1.0
plt.imshow(b)
```

Notice that the x and y axes are interchanged when the system is visualized, so that what is along the horizontal axis is the second index in the arrays.

a) Study the three suggested structures to see which one would be best for this purpose.

b) Explore how the resolution (`L`) and discreteness of the simulation affects the results. (Remember to include the lattice spacing Δx when you calculate the electric field correctly: $E_{x,i,j} = (V_{i+1,j} - V_{i,j})/\Delta x$). Can we use this type of modelling to answer this type of question?

c) Extra challenge: Explore how your results would change if you tilted the lattice by 45 degrees for the sawtooth pattern. Select the angle of the sawtooth pattern to be 45 degrees, so that it aligns with the modeling lattice, and the size of the simulation box appropriately.

6 Ideal conductors

So far we have studied static charges. Now we will start to let them free. We will start to address the behavior of conductors, materials that have charges that are free to move around. In this chapter, we will study the properties of ideal conductors — where charges can move freely around — in equilibrium, that is, after the charges has settled down. Later, we will address non-ideal conductors and currents. We introduce the basic properties of ideal conductors, for example, that they have zero electric field inside and hence are equipotential surfaces, and both analytical and numerical tools to study electric field and charge distributions in and around conductors. Ideal conductors will be frequenly used as a model for real systems. For example, whenever we draw an electric circuit we will assume that all the wires we draw are ideal conductors, or we may assume that a part of a system is an ideal conductor to calculate the electric potential or field without knowing the detailed charge distribution.

6.1 Conductors

Conductors are materials that contain a large number of charges — usually electrons — that can move around in the material. Metals are a class of materials that behave as conductors. Metals are found to the left in the period table (see Fig. 6.1) and the electrons in metals behave as a sea of electrons that can move around with very little resistance. This is essentially a quantum mechanical effect, but you can gain some insight into this also from a classical model. Metals have weakly bound electrons in the outer shells, which in metallic crystals are essentially free to move around as illustrated in Fig. 6.2. The metal may be neutral and still allow the electrons to move: the positive atoms remain, while the electrons move, allowing for a redistribution charges within the system. However, not only metals are conductors. Water with dissolved salt is also a conductor, because the positive and negative salt ions will move if an electric field is applied.

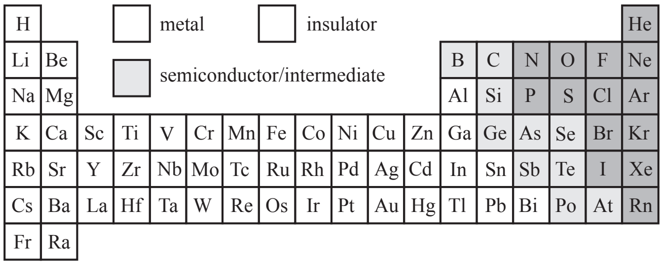


Fig. 6.1 Illustration of periodic table with metals and insulators.

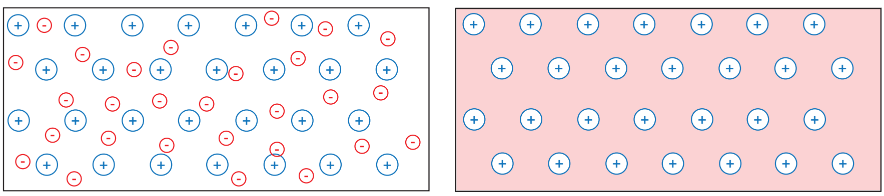


Fig. 6.2 Metallic atoms have electrons in the outer shell. In a crystal of metallic ions these electrons form a sea of conducting electrons that can move around with low resistivity illustrated by the red, smeared-out electrons on the right.

The charges in a conductor experience a form of resistance from the material. We call this effect electric resistance and will discuss it in detail later. In an *ideal conductor* we assume that the motion of the charges in response to an electric field is very rapid, so that it quickly reaches a state of equilibrium. What happens if we place an (ideal) conductor in an external electric field? This is illustrated in the sequence of figures in Fig. 6.3. In the top figure, we illustrate the uniform electric field. Inside the circular conductor, there are charges that are free to move. Positive charges will move in the direction of the field (if they are free to move) and negative charges will move in the direction opposite the electric field. (If only the electrons can move, the remaining atoms will be positively charged, and the effect will be the same as if both positive and negative charges can move). The negative charges will move in the direction of $-\mathbf{E}$ until it reaches the end of the conductor — the surface. However, as the charges move, they will set up an electric field and change the total electric field in the system. The middle figures illustrate the electric field set up by the charges on the surface of the conductor. The charges will build up at the surface until the electric field inside the conductor is zero: If the electric field is not zero, the charges will continue to move. The bottom figure shows the sum of the external field and the field set up by the surface charges in equilibrium. The electric field will then be zero inside the conductor. What are the consequences of this for an ideal conductor?

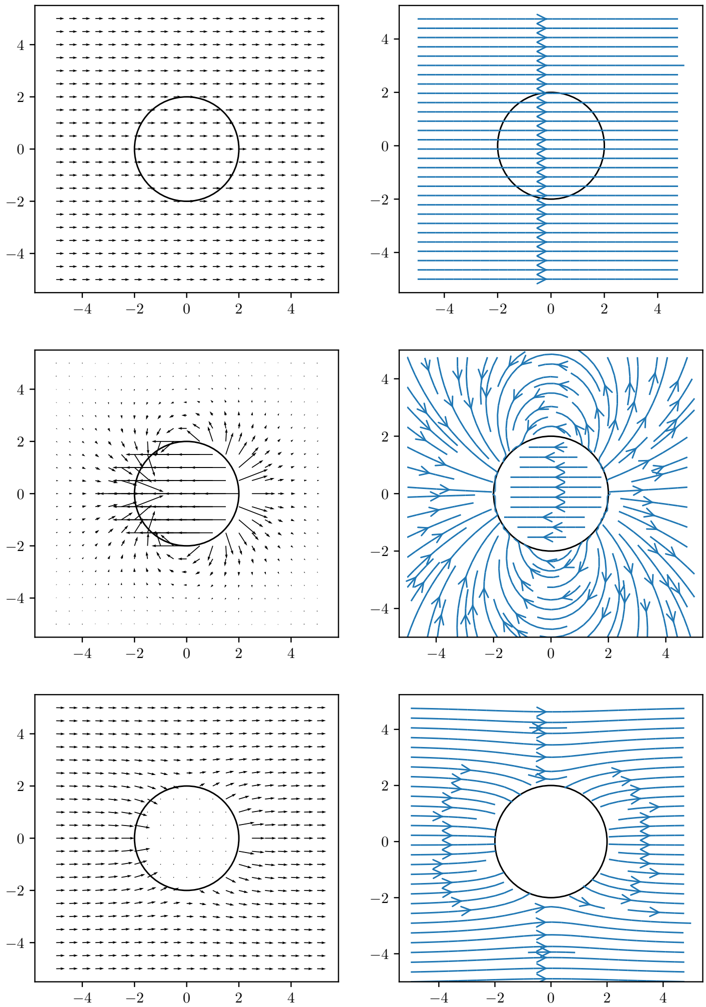


Fig. 6.3 Top: Uniform electric field and conducting sphere. Middle: Field inside sphere due to surface charges. Bottom: Total field is zero inside conducting sphere.

6.2 Properties of an ideal conductor in equilibrium

When the ideal conductor is in equilibrium, that is, when all the charges have finished moving, the ideal conductor will therefore have the following properties which also are illustrated in Fig. 6.4:

Zero field inside a conductor. The electric field is zero inside an ideal conductor, $\mathbf{E} = 0$. If this was not the case, the electric field would move charges until they fall to rest on the surface of the conductor. The field from these charges will then compensate the external field so that the net field inside is zero.

No free charges inside a conductor. There are no free charges inside and ideal conductor. We can see this from Gauss' law: Because $\mathbf{E} = 0$ inside the conductor, Gauss' law on differential form gives $\nabla \cdot \mathbf{E} = 0 = \rho/\epsilon_0$. All the charges must therefore be on the surface. The surface charge density is often called an *induced surface charge density*, since it is often the consequence of an external field.

Conductors are equipotential surfaces. The conductor must be an equipotential surface because the electric field is zero inside the conductor. We can see this by finding the potential difference between two points A and B inside the conductor: $V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{l} = \int_A^B 0 d\mathbf{l} = 0$, because $\mathbf{E} = 0$ inside the conductor.

There is no tangential field immediately outside a conductor. The tangential electric field is zero immediate outside the conductor. This is a consequence of the boundary conditions for the electric field and that the electric field is zero inside the conductor. The boundary condition across the boundary from inside to outside the conductor are $E_{2t} = E_{1t}$. Because E_t is zero inside the conductor, E_t must also be zero immediate outside the conductor.

The electric field field immediately outside a conductor is $E_n = \rho_s/\epsilon$. The electric field immediately outside the conductor is normal to the surface of the conductor and is given as $E_n = \rho_s/\epsilon_0$. Because the tangential component is zero, there can only be a normal component of the field immediately outside the conductor. We can find the magnitude of this field from the boundary conditions, which states that $D_{2n} - D_{1n} = \rho_s$. On the inside the electric field and therefore also the displacement field is zero, $D_{1n} = 0$. We therefore get that $D_{2n} = \epsilon_0 E_n = \rho_s$ and $E_n = \rho_s/\epsilon_0$ when the outside is vacuum. If the conductor is a dielectric, we need to divide by ϵ instead. In general, we therefore have that immediately outside the conductor $\mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s/\epsilon$. We can use this to find the electric field given that we know the surface charge density, and also the opposite: We can find the surface charge density from the electric field.

Properties of an ideal conductor

- $\mathbf{E} = 0$ inside the ideal conductor.
- $\rho = 0$ inside an ideal conductory.

- The conductor is an equipotential surface.
- $E_t = 0$ immediately outside the conductor.
- $E_n = \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s/\epsilon_0$

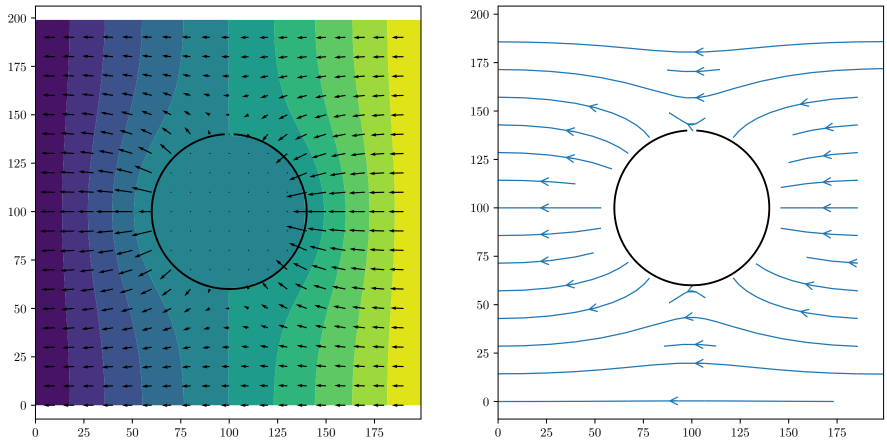


Fig. 6.4 Illustration of a conducting disk in a constant electric field showing that $\mathbf{E} = 0$ inside, the potential is constant inside the conductory, and $\mathbf{E}$ is normal to the surface.

6.2.1 Example: A spherical conductor

Problem. A spherical ideal conductor with radius a has a total charge q. Find the electrical potential and field everywhere.

Approach. To solve this problem, we will use key properties of ideal conductors. We know that the electric field inside the conductor is zero and that all the charges are on the surface. There are several ways to approach this problem.

1. We could use the spherical symmetry to assume that the charge is uniformly distributed on the surface. We have then mapped the problem onto a known problem — the case of a charged, spherical shell with radius a. For this system we know that the electric field outside the shell is the same as for a point charge in the origin (the center of the sphere), that is, $\mathbf{E} = E_r\hat{\mathbf{r}}$, $E_r = q/(4\pi\epsilon_0 r^2)$, and that the potential also will be the potential from a single point charge in the origin, $V(r) = q/(4\pi\epsilon_0 r)$.

2. Alternatively, we could use that the surface is an equipotential surface and then solve Laplace's equation. We leave this as an exercise.

Alternative problem. We could have formulated the problem differently, by instead saying that the spherical ideal conductor has a potential V_0. We can then still use the same approaches as before, but we would need to solve to find the charge q from the potential V_0: $V_0 = q/(4\pi\epsilon_0 a)$. However, for a problem formulated with the potential of the conductor, it may be tempting to instead use Laplace's equation.

More complex problem. If the conductor is not a sphere, but a more complicated shape, such as a super-ellipse, then we generally cannot solve the problem analytically. We may then instead solve the problem numerically, by first solving Laplace's equation for the given set of boundary conditions, and then use the potential to find the electric field. We demonstrate this in the following example.

6.3 Method: Solving Laplace's equation with conductors

The previous example suggests a general method to study systems with conductors. Such systems are important, because conductors and ideal conductors are common parts of electromagnetic models, systems and circuits. In such systems, we often place a conductor in contact with an electric potential at a given value. What does this mean? In practice, it may imply that we place it in contact with a battery with a given potential difference between its two poles. In our models or theoretical considerations, it means that the conductor has a given value of the potential, V. Because the conductor is an equipotential surface, it means that the whole conductor will be at the same potential. This means that systems with conductors can be addressed using Laplace's or Poisson's equation with the conductors are Dirichlet boundary conditions with a given value of the potential. In this case, we do not need to know the amount of charge present at the conductor, instead we know the potential. The distribution of charges can be calculated afterwards, when we have found the potential and the electric field. Thus we can propose a new method to find the electric potential and fields in electrostatic problems:

Method for problems with conductors

1. Determine what parts of the system are conductors and what their potentials are. These are the boundary conditions.
2. Solve Laplace's or Poisson's equation with the conductors as boundary conditions.
3. Calculate the electric field from the electric potential, $\mathbf{E} = -\nabla V$.
4. Calculate the surface charge density from the electric field, $\rho_s = \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}}$.

6.3.1 Example: Numerical calculation of field from conductor

We apply this method to find the electric field from a conductor with a non-trivial shape. Let us assume that a conductor shaped as a super-ellipse has a potential V_0 and that the potential is zero at inifinity. The shape of the super-ellipse is given as $(x/a)^n + (y/b)^n = 1$ where $n = 4$. What is the electric field and surface charge density for this system?

We plan to solve this problem numerically in the xy-plane. We generate a system of size $L \times L$ and place the conductor with its center in $(L/2, L/2)$. The boundary conditions for the problem is that the potential is V_0 everywhere on the conductor and 0 at infinity. Because we cannot model an infinite system, we may implement the outer boundary condition either by setting the potential to be 0 at the outer boundary or by setting the potential to have zero derivative at the out boundary. (We can also study the difference between the two assumptions).

Thus, we want to solve Laplace's equation (because there are no free charges): $\nabla^2 V = 0$ on the square from $(0,0)$ to (L,L). We set up the system to use our previously developed function `solvepoissonvonneumann2d`. First, we set up the boundary conditions:

```
L = 200
b = np.zeros((L,L),float)
b[:] = np.float('nan')
alpha = 4.0
ax = 30.0
ay = 40.0
rx0 = 100
ry0 = 100
for ix in range(L):
    for iy in range(L):
        d = ((ix-rx0)/ax)**alpha+((iy-ry0)/ay)**alpha
```

```
        if (d<1.001):
            b[ix,iy] = 1.0
```

Here, we have implicitely introduced the boundary conditions that the derivative is zero across the outer boundary. We could instead have specificed a given value of the potential by e.g. setting `b[:,0]=0.0` and similarly for the other boundaries.

We solve the Laplace's equation and plot the results using:

```
s = solvepoissonvonneumann2d(b,100000)
plt.contourf(s,10)
Ey,Ex = np.gradient(-s)
x = np.linspace(0,L-1,L)
y = np.linspace(0,L-1,L)
X,Y = np.meshgrid(x,y)
plt.quiver(X,Y,Ex,Ey)
```

The resulting plot is shown in Fig. 6.5.

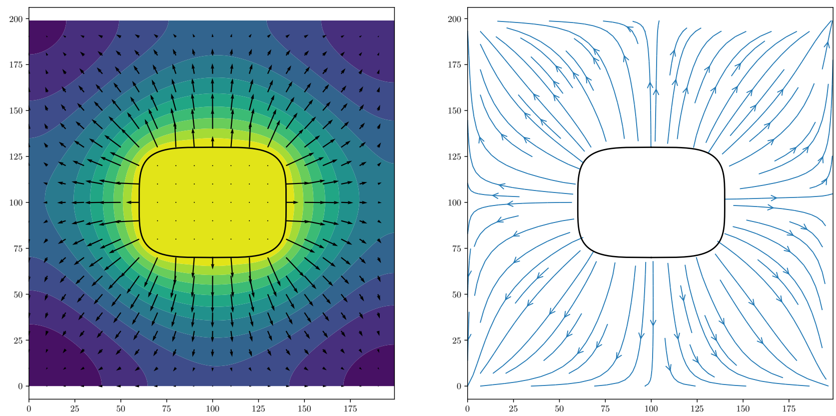


Fig. 6.5 Plot of the eletric potential and the electric field around a super-ellipse-shaped conductor calculated from Laplace's equation.

6.3.2 Example: Modeling the shape of a lightning rod

Does a lightning preferrable strike a thin, long pole or a rounded hill?

In order to answer a question like this, we need to convert the question into a physics problem we can address with the physics we know. We need to *make a model* of the problem and answer the *model problem*.

Model problem. We already know that lightnings occur due to dielectric breakdown of air, which is due to a high electric field. Our plan is to make a model for a lightning rod that can address both a thin pole and a round object, calculate the electric field around this rod, and find how the field depends on the geometry of the rod.

We model the rod as a sphere, which can represent either the tip of the rod or the roundness of a hill. The sphere has a radius a and we assume that it is an ideal conductor. Because it is an ideal conductor, the whole sphere will have the same potential, V_0 as illustrated in Fig. 6.6.

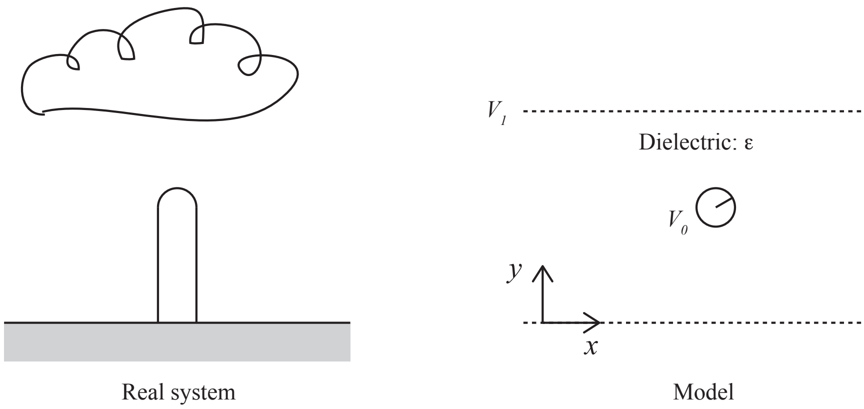


Fig. 6.6 Sketch of a real lightning rod (left) and our simplified model of the lightning rod system (right).

Electric field inside the sphere. The electric field inside the sphere is zero.

The potential outside the sphere. We can find the potential outside the sphere by solving Laplace's equation. According to Fig. 6.6 we have sketched that the potential is V_1 in the cloud, which is far above the ground and the sphere. We do not know how to solve this problem analytically now, but we can solve it numerically. Let us therefore first solve a simpler problem analytically, and compare with the numerical solution. We simplify the problem to a sphere with potential V_0 and a potential $V_1 = 0$ infinitely far away.

We need to solve Laplace's equation $\nabla V^2 = 0$ in spherical coordinates. The spherical symmetry implies that the potential only will depend on r, the distance to the center of the sphere, and not on any of the angles θ

or ϕ, because the system is symmetric — it does not change — under any rotation around the center of the sphere [1].

The ∇^2 operator in spherical coordinates is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \, , \tag{6.1}$$

which gives

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \, . \tag{6.2}$$

This means that

$$r^2\frac{dV}{dr} = \text{const.} = C_1 \Rightarrow \frac{dV}{dr} = \frac{C_1}{r^2} \, . \tag{6.3}$$

We integrate from r to ∞:

$$\int_r^\infty \frac{dV}{dr}dr = V(r) - \underbrace{V(\infty)}_{=0} = \int_r^\infty \frac{C_1}{r^2}dr = -\frac{C_1}{r} \, . \tag{6.4}$$

We also know that $V(a) = V_0$, which gives us $C_1/a = -V_0$:

$$V(r) = \frac{V_0 a}{r} \, . \tag{6.5}$$

This is the electric potential in our simplified model of the lightning rod. Now, we find the electric field to determine where it is largest and where dielectric breakdown will occur.

Electric field outside the sphere. We find the electric field from the potential using $\mathbf{E} = -\nabla V$. We use polar coordinates where

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \, . \tag{6.6}$$

Which gives us

$$\mathbf{E} = \frac{V_0 a}{r^2}\hat{r} \, . \tag{6.7}$$

The electric field is therefore largest immediately outside the sphere, where r is the smallest. The electric field here is $E_n = V_0/a$ (and $E_t = 0$).

Consequences of the model. The result in our model is that the electric field is largest immediately outside the sphere. A dielectric breakdown

[1] This symmetry depends on our assumption that the potential is zero infinitely far away. If we instead has assumed that the potential was zero on a plane representing the cloud, we could not have assumed spherical symmetry.

that depends on the electric field will therefore initiate here. The electric field becomes larger for smaller radius a of the sphere. Indeed, we can induce a breakdown by making a sufficiently small. (But this will probably vialate other aspects of our model). This indicates that lightning rods should be sharp to ensure that the lightning strikes the rod.

Surface charge density. We could immediately use this to find the surface charge density since $E_n = \sigma/\epsilon$ and therefore $\sigma = \epsilon V_0/a$. This means that the way we have formulated the problem, with a given potential on the sphere and zero potential at infinity, the sphere has a given net charge which is $Q = 4\pi a^2\sigma = 4\pi a\epsilon V_0$. This is a feature of using Laplace's equation and the properties of ideal conductors. We do not have to know the charge, but find the charge on the conductor as a result of the calculation when we assume a particular value for the potential. This is the charge that sets up the electric field outside the conductor.

Checking with Gauss' law. Let us check this by using Gauss' law. We assume that the charge Q is uniformly distributed on the spherical surface because the system has perfect spherical symmetry. We can therefore apply Gauss' law on a spherical Gauss surface of radius r around the center of the sphere: $\oint_S \mathbf{D} \cdot d\mathbf{S} = D4\pi r^2 = \epsilon E4\pi r^2 = Q$, which gives us $E = Q/(4\pi\epsilon r^2)$. We insert the value for $Q = 4\pi\epsilon a V_0$, getting, $E = V_0 a/r^2$. When $r \to a$ this approaches $E = V_0/a$ on the surface. The results we found from Laplace's equation are indeed the same as for Gauss' law.

Asymmetric system. What if we instead model a system which is more similar to the system sketched in Fig. 6.6. We introduce a model with a sphere at a height h above the ground. The sphere and the ground has potential $V_0 = 1$ and the clouds above are modeled as a flat plane with potential $V_1 = 0$. This is implemented in the following program, which uses the function `solvepoissonvonneumann2d`

```
L = 200
b = np.zeros((L,L),float)
b[:] = np.float('nan')
b[0,:] = 1.0 # Ground
b[L-1,:] = 0.0 # Cloud
# Make sphere
h = 50.0
a = 10.0
for ix in range(L):
    for iy in range(L):
        d = np.sqrt((iy-L/2)**2 + (ix-h)**2)
        if (d<a):
            b[ix,iy] = 1.0
nrep = 100000
```

```
s = solvepoissonvonneumann2d(b,nrep)
Ey,Ex = np.gradient(-s)
Emag = np.sqrt(Ex*Ex+Ey*Ey)
x = np.linspace(0,L-1,L)
y = np.linspace(0,L-1,L)
X,Y = np.meshgrid(x,y)
plt.contourf(Emag,10)
plt.quiver(X,Y,Ex,Ey)
```

The resulting plot of the magnitude of the electric field is shown in Fig. 6.7. We see that the maximum of the magnitude of the electric field also in this case is close to the top of the sphere. The small deviations we observe close to the top we attribute to effects of the discrete lattice used for the numerical solution. We can perform several such simulations for various values of a and study how $E_{\max}$ depends on a. This is also plotted in Fig. 6.7. We see that the magnitude of the field decays with increasing a, but not as fast as predicted from the simplified model. In the simplified model, we found that $E = V_0/a$. In the numerical simulation, we see that when we increase a by a factor of 2, that is, from $a = 5$ to $a = 10$ in numerical units, the value of $E_{\max}$ does not reduce by a factor of 2, but with a smaller factor. Here, we recall earlier issues we had with the difference between two-dimensional simulations and three-dimensional results. The numerical results here are really for a cylindrical geometry, and not with a sphere, and we therefore expect significant differences. We leave it as an exercise to study this problem in a three-dimensional system. (Notice that a three-dimensional simulation will require more computational resources).

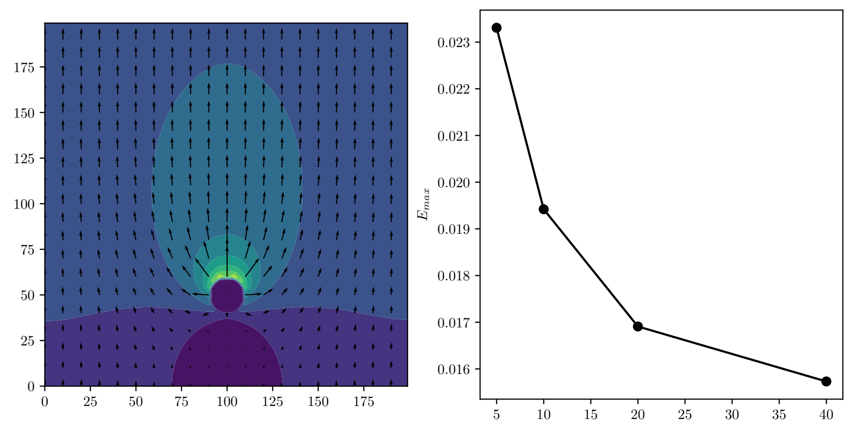


Fig. 6.7 Plot of the magnitude fo the electric field and the maximum magnitude of the electric field as a function of the radius a of the sphere.

(Advanced material) Cylindrical coordinates. Laplace's equation in cylindrical coordinates when we assume no ϕ or z dependence is

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \, , \tag{6.8}$$

which means that

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \tag{6.9}$$

$$r\frac{\partial V}{\partial r} = A \, , \tag{6.10}$$

where A is a constant, which gives

$$\frac{\partial V}{\partial r} = \frac{A}{r} \, , \tag{6.11}$$

and

$$V(r) = A \ln r + B \tag{6.12}$$

We notice that this solution cannot be adapted to the boundary conditions that $V(a) = V_0$ and $V(\infty) = V_1 = 0$. Instead, we must solve this on a finite disk of radius R, setting $V(R) = 0$ for a given, large value of R:

$$V(a) = V_0 = A \ln a + B \quad , \quad V(R) = 0 = A \ln R + B \tag{6.13}$$

which gives $B = -A \ln R$ so that $V(r) = A \ln r - A \ln R = A \ln(r/R)$ and $V(a) = A \ln(a/R) = V_0$ and $A = V_0/\ln(a/R)$. Thus, we get

$$V(r) = V_0 \ln(r/R)/\ln(a/R) \tag{6.14}$$

The electric field is then found from $\mathbf{E} = \nabla V$, which in cylindrical coordinates is:

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} = \frac{V_0}{r \ln(a/R)}\hat{\mathbf{r}} \, . \tag{6.15}$$

The electrical field immediately outside the sphere is found at $r \to a$, $E_r(a) = -V_0/(a \ln(a/R))$. The behavior of the electric field is therefore somewhat modified from the result for the spherical system.

6.3.3 Example: Faraday cage

If you are inside an elevator, you may have noticed that your cell phone signal may be weak. Why is this? We may understand this from a simplified model of the elevator. We assume that the elevator consists of a box of metal — a conductor with a hole inside it. What is the electrical field inside a hole inside a conductor? This is illustrated in Fig. 6.8.

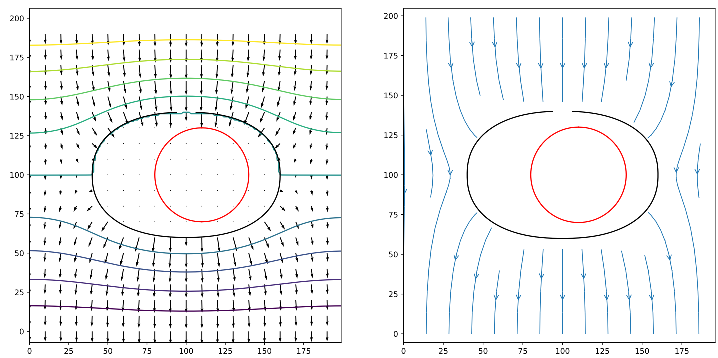


Fig. 6.8 We use this model to address the electric field immediately outside a conducting sphere with a potential V_0

We know that the conductor has constant potential. The surface of the hole therefore also has a constant potential. We can therefore solve Laplace's equation with a constant potential on the boundary and no free charges inside. (We have assumed there are no free charges inside the elevator). One solution to this problem is that the potential is constant inside the whole. Because the solutions to Laplace's equation are unique, this is also the only solution: The potential is constant both in the conductor and in any hole inside a conductor. The electric field is $\mathbf{E} = -\nabla V$, which therefore also must be zero.

A hole inside a conductor is therefore electrostatically shielded from the outside. Independent of the complexity of the external electric field, the electric field inside the conductor and inside hole in the conductor is zero. (There may be a response time to changes in the electric field, because the surface charges on the conductor need to reorganize, but this time scale is typically very short).

6.4 Method: Imaging methods and mirror charge

Another classical method used to solve complex problems with conductors is the use of mirror charges. We demonstrate this method through an example: A point charge Q is located a distance h above a grounded, conducting plane with potential $V_0 = 0$ as illustrated in Fig. 6.9. How can we find the surface charge distribution, $\rho_s(x, y)$, in the conductor plane?

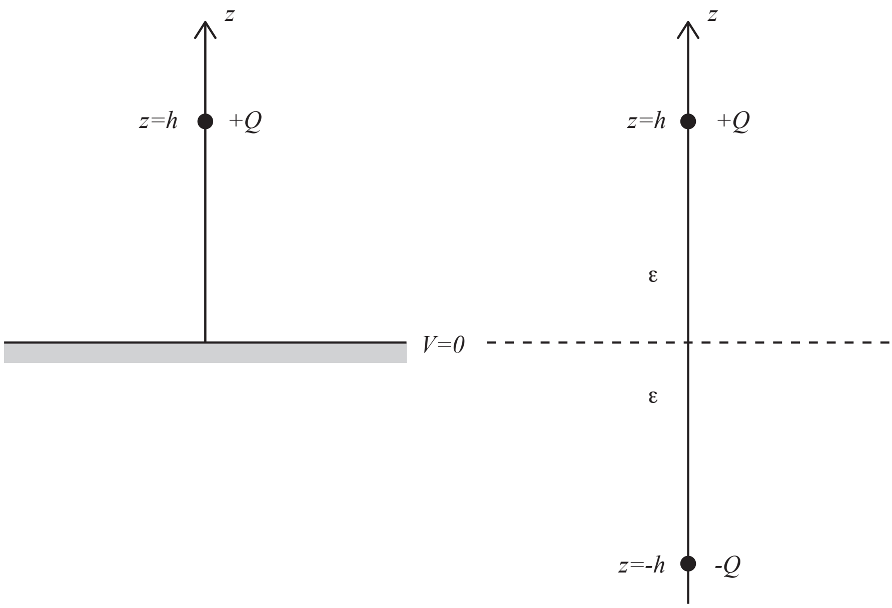


Fig. 6.9 (a) A point charge $+Q$ at $z = h$ above a conducting plane. (b) We remove the plane and replace it with a mirror charge $-Q$ at $z = -h$.

The trick of mirror charges. We know that immediately outside the conductor, the electric field in the normal direction is $E_n(z = 0) = \rho_s/\epsilon$, and $E_t(z = 0) = 0$. The electric field in in the plane at $z = 0$ is a superposition of the field from the point charge Q and the charges in the plane. Because the plane is a conductor, it is an equipotential surface. The electric potential is therefore the same everywhere on the plane.

Now, we will introduce a *trick*. We can use the uniqueness property of the potential field. If we can find an electric potential that satisfies the boundary condition $V = 0$ on the plane, we know that this must be a unique solution to the problem. We therefore compare the two cases (a) and (b) illustrated in Fig. 6.9. Case (a) corresponds to the charge Q and the conducting plane, whereas case (b) corresponds to the charge Q and a charge $-Q$ in a position mirrored across the conducting plane. For both these systems the potential is zero at $z = 0$. In the part $z > 0$ we can solve Laplace's equation to find the electric potential with the given boundary condition at $z = 0$ and the charge Q. This would give us a solution for V for $z > 0$. However, we also see that the system with the charge Q and the mirror charge $-Q$ satisfies this boundary condition at $z = 0$. The electric potential from the sum of the charge Q and the charge $-Q$ must therefore be a solution to Laplace's equation for $z > 0$. However, because the solution is unique, this is *the* solution to Laplace's equation for $z > 0$. We have therefore found the potential using the method of mirror charges. We can then find the electric field from the electric potential, or we can simply find the electric field from the charge and the mirror charge, which very much simplifies the problem.

Simplified solution at $x = 0, y = 0$. We can apply the mirror charge method to find the electric field in any position $z \geq 0$. At $x = 0, y = 0, z = 0$ we find that

$$\mathbf{E} = -\frac{Q}{4\pi\epsilon h^2}\hat{z} - \frac{Q}{4\pi\epsilon h^2}\hat{z} = -2\frac{Q}{4\pi\epsilon h^2}\hat{z} \tag{6.16}$$

General solution in the plane. We can find the field in a position $(x, y, 0)$ by realizing that the field only has a component along the z-axis due to symmetry. Thus, the field from $+Q$ is

$$\mathbf{E}_+ = -\frac{Q}{4\pi\epsilon}\frac{h\hat{z}}{(x^2 + y^2 + h^2)^{3/2}} , \tag{6.17}$$

and similarly for $-Q$, giving:

$$\mathbf{E} = -\frac{2Q}{4\pi\epsilon}\frac{h\hat{z}}{(x^2 + y^2 + h^2)^{3/2}} , \tag{6.18}$$

and the charge distribution is given by $\rho_s = \epsilon\mathbf{E} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \hat{\mathbf{z}}$:

$$\rho_s = -\frac{2Q\epsilon}{4\pi\epsilon}\frac{h}{(x^2 + y^2 + h^2)^{3/2}} , \tag{6.19}$$

6.5 Summary

Ideal conductors have a sea of charges that immediately move in response to an electric field, until the total electric field inside the conductor is zero.

There are no free charges inside an ideal conductor.

An ideal conductor is an *equipotential surface.*

There is no tangential field immediately outside an ideal conductor.

The electric field immediately outside an ideal conductor is $E_n = \rho_s/\epsilon_0$.

We can use the property that the ideal conductor is an equipotential surface as a boundary condition to Laplace's equation. This allows us to find the electrical potential and therefore the electrical field around conductors.

6.6 Exercises

6.6.1 Test yourself

6.6.2 Discussion exercises

Exercise 6.1: Copper between charges

We place two charges Q_A and Q_B a distance $2L$ from each other. Then we place a neutral copper sphere at the midpoint between Q_A and Q_B. Sketch the distribution of charges in the copper sphere if (i) both charges are positive, (ii) both charges are negative, (iii) the charges have opposite signs. What is the electric field and the potential inside the sphere in these situations?

Exercise 6.2: Where are the charges

"The free charges in a conductor are always on the external surface of the conductor". Is this statement always true? Sketch and explain.

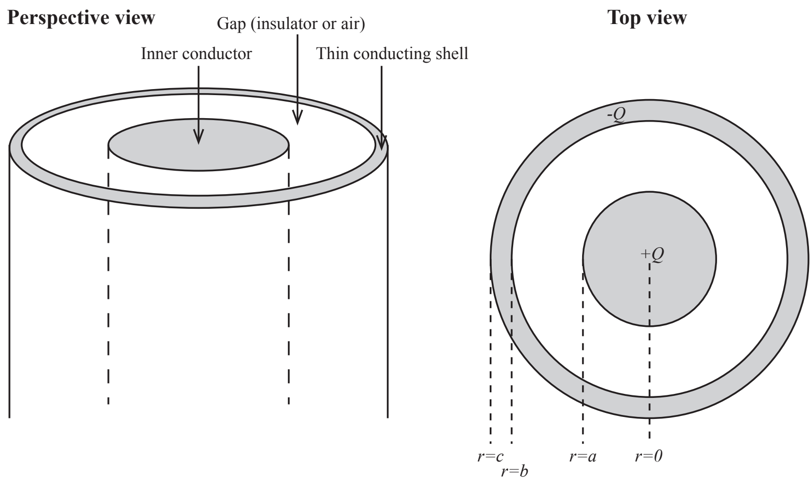
6.6.3 Tutorials

Exercise 6.3: Understanding conductors

(*From Steven Pollack, University of Colorado — Boulder*).

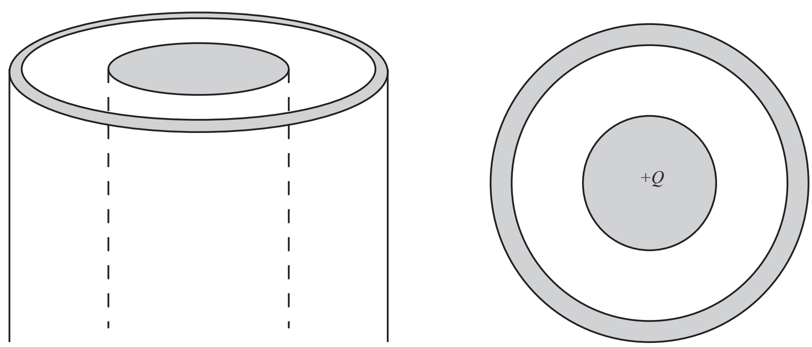
A coax cable is essentially one long conducting cylinder surrounded by a conducting cylindrical shell (the shell has some thickness). The two conductors are separated by a small distance. (Neglect all fringing fields near the cable's ends).

a) Draw the charge distribution (little + and - signs) if the inner conductor has a total charge of $+Q$ on it, and the outer conductor has a total charge $-Q$. Be precise about exactly where the charge will be on these conductors, and how you know.

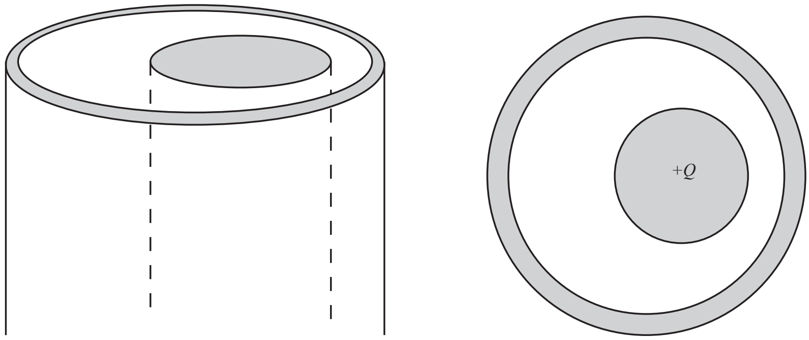


b) If you were calculating the potential difference, ΔV, (for the configuration above), between the center of the inner conductor ($r = 0$) and infinitely far away ($r = \infty$), what regions of space would have a (non-zero) contribution to your calculation?

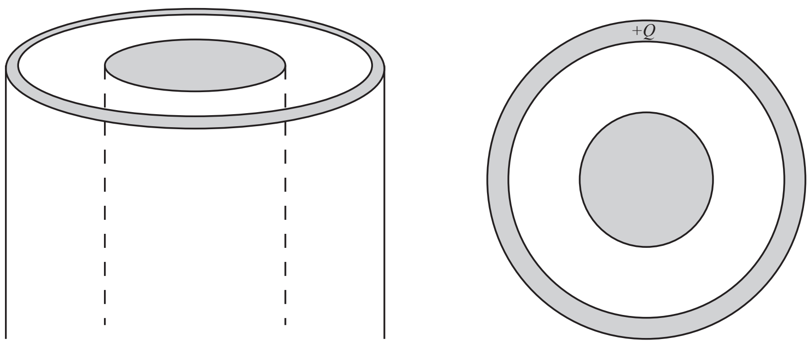
c) Now, draw the charge distribution (litte - and + signs) if the inner conductor has a total charge $+Q$ on it, and the outer conductor is electrically neutral. Be precise about exactly where the charge will be on these conductors, and how you know.



d) Consider how the charge distribution would change if the inner conductor is shifted off-center, but still has $+Q$ on it, and the outer conductor remains electrically neutral. Draw the new charge distribution (little + and - signs) and be precise about how you know.



e) Now, instead of the total charge $+Q$ being on the inner conductor, sketch the charge distribution (little + and - signs) if the *outer* conductor has a total charge $+Q$ on it, and the inner conductor is electrically neutral. Be precise about exactly where the charge will be on these conductors, and how you know.

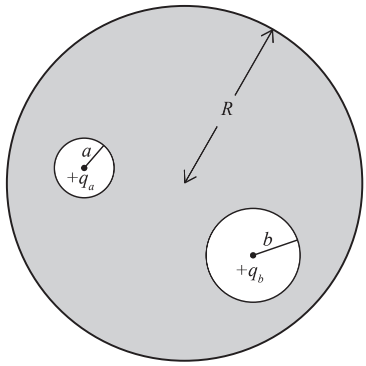


f) For this case ($+Q$ on outer conductor, inner conductor neutral) what is the potential difference, ΔV, betwen the center of the inner conductor ($r = 0$) and the outer conductor ($r = c$)?

Exercise 6.4: Superposition in conductors (shielding)

(*From Danny Caballero, Michigan State University*)

We carve out two spherical cavities form a metal sphere of radius R (as shown below). The first cavity (radius, a) has a charge $+q_a$ placed at the center of the cavity. Similarly, the second cavity (radius, b) has a charge $+q_b$ placed at the center of that cavity.



a) Sketch the surface charge everywhere. Explain how you know the surface charge looks this way.

b) Determine the surface charges (σ_a, σ_b, and σ_R)

c) Sketch the electric field everywhere. If the field is zero anywhere, indicate this explicitly. Explain how you know the electric field looks this way.

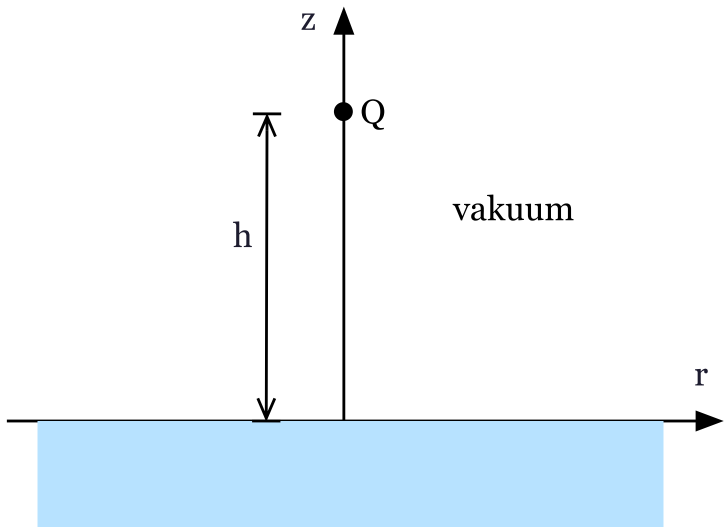
d) Determine the magnitude of electric field outside the conductor and inside each cavity.

e) If we brought an external charge, q_c, near the conducting sphere, how do your answers to the questions above change? You may answer in words, pictures, or both.

6.6.4 Homework

Exercise 6.5: Punktladning over ledende plan

En punktladning Q befinner seg i en avstand h over et uendelig stort, ledende plan:



a) Finn potensialet V langs z-aksen for $z \in [0, h)$. Finn også det elektriske feltet her.

Hint. Speilladningsmetoden

b) Finn den induserte ladningstettheten $\rho_s(r)$ på lederens overflate.

Hint. Bruk $\rho_s = \epsilon_0 E$, der E evalueres rett ovenfor lederen.

c) Vis at den totale induserte ladningen på overflaten er lik $-Q$.

d) Skisser de elektriske feltlinjene overalt.

e) Finn kraften på punktladningen.

7 Capacitance

In this chapter we will introduce the concept of capacitance. Capacitance is a property of a component that tells us about its capacity to store electric charge. It depends on the geometry of the component: how the conductors are placed relative to each other. We call such components a capacitor, and we will frequently use such components as models of electrical properties of real systems or as components in electric circuits. In this chapter, we will focus on how the capacitance depends on the geometry of the system, how to calculate the capacitance of a system, and how to construct complex capacitors from several simple capacitors. We will also introduce the energy stored in a capacitor and use this to introduce the energy in an electric field.

7.1 Motivation

Capacitance is a concept that we will introduce to describe the ability for set of conductors to store electrical charge, and a *capacitor* is the physical realization of this system. A capacitor is a thing and capacitance is the property of that thing. (You may already have such an intuition about resistance and a resistor when it comes to electrical resistivity — this analogy is, as we will see, quite useful).

Fig. 7.1 illustrates two (ideal) parallel plate conductors that are separated from each other by a distance d. We call such a configuration a parallel-plate capacitor. If the system starts as neutral and we move a charge Q from the bottom plate to the top plate, the bottom plate will have a change $-Q$ and the top plate a charge Q. The electric field between the plates will point from the $+Q$ plate to the $-Q$ plate and the magnitude of the field will be $E = Q/(A\epsilon)$ [1], where A is the area of the plate. The difference in electrical potential for this constant electric field is then $V = Ed = Qd/(A\epsilon)$. We define the *capacitance*, C, of this system as

$$C = \frac{Q}{V} = \frac{QA\epsilon}{Qd} = \frac{A\epsilon}{d} \ . \tag{7.1}$$

We see that the capacitance only depends on the geometry of the system: the distance d between the plates, the area A of the plates, and the dielectric constant of the material they are embedded in.

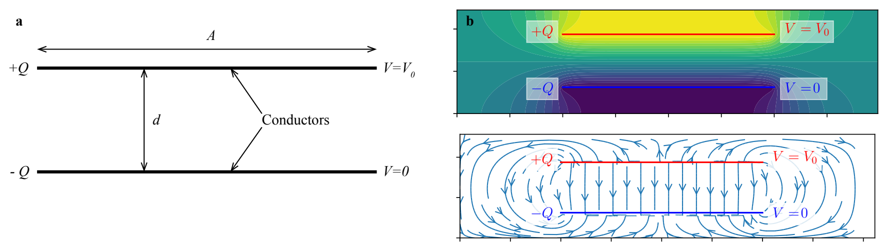


Fig. 7.1 (a) Illustration of a parallel plate capacitor consisting of two plates of area A separated by a distance d. There is a potential difference V between the plates generated by moving a charge Q from the bottom plate to the top plate so that the top plate has a charge Q and the bottom plate has a charge $-Q$. (b) The electrical potential V around a finite-sized parallel-plate capacitor.

The capacitance C defined as $C = Q/V$ is a property of the geometry of a system of conductors. This not only valid for two parallel plates, but for any configuration, because Coulomb's law for the electric field as well as for the electric potential is proportional to the charge: $V = \sum_i Q_i/(4\pi\epsilon_0 R_i)$: If all the charges are doubled, the field and the potential are doubled as well. This implies that the ratio of the charge and the potential is a constant, independent of the charge.

What does the capacitance tell about this system? As is hinted in the name, the capacitance describes a systems capacity to carry charge at

[1] We have calculated this several times before. We apply Gauss' law to each of the plates. For the top plate we use a cylindrical Gauss surface with an axis normal to the plate and area S. The field will be symmetric around the plate and only depend on the distance to the plate. Gauss' law therefore gives that $2SE = \rho_s S/\epsilon$ and $E = \rho/(2\epsilon)$, where $\rho = Q/A$ is uniformly distributed on the plate. (The charges are only on the surfaces of the plate, be we look at a volume that goes through the plate). The total electric field is therefore $E = 2\rho/(2\epsilon) = Q/(A\epsilon)$.

a given potential difference. We see this from $C = Q/V$ which gives us $Q = CV$. The amount of charge the system can store at a given potential difference V between two conductors is proportional to C. Thus if you were to carry us much charge as possible in a briefcase with a 9V battery to hold it, you would want to have a system with as large a capacitance as possible.

We call a system with a capacitance C a *capacitor*. A capacitor is a common component in electric circuits with a useful function as we will address later. Fig. 7.2 shows examples of capacitors. Capacitance is also a useful aspect of a system that we may want to include in a model of that system. For example, a cell membrane as illustrated in Fig. 7.2 may be considered to act as a capacitor and have a given capacitance. Here, we will develop methods to calculate the capacitance for simple and complex geometries and for systems that consists of combinations of several capacitors. Later, we will see how we use capacitance and capacitors as elements in our models of real electromagnetic systems and in our analysis of electric circuits.

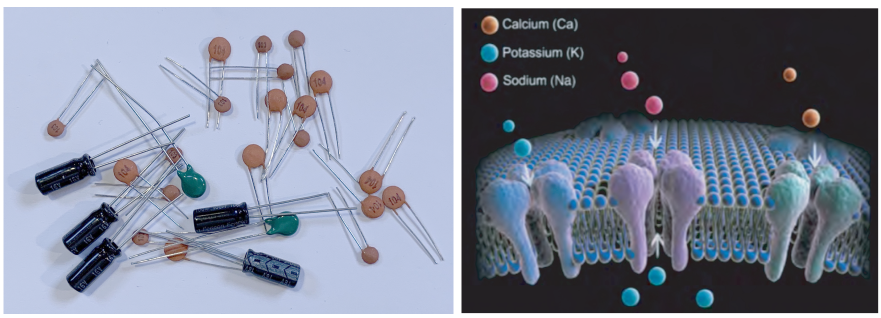


Fig. 7.2 (a) Pictures of capacitors used in electric circuits. (b) A cell membrane consists of a set of lipid molecules (called a lipid bilayer) that separate the inside and outside of the cell. There is litte charge transport across the lipid bilayer, and therefore we may consider the cell membrane a capacitor. There are special tunnels across the cell membrane to transport ions called ion channels and ion pumps. The typical thickness of a nerve cell membrane is 8-10 nm and the voltage difference, called the membrane potential, is typically 30-90 mV. (The inside is negatively charged). The typical electric field is therefore $E = V/d = 80\text{mV}/10\text{nm} = 8\text{MV}$.

Capacitance

The **capacitance** of a system of two conductors with a potential difference V is

$$C = Q/V \;, \tag{7.2}$$

where Q is the charge moved from one conductor to another to get the potential difference V. Thus one conductor has the charge $+Q$ and the other conductor has the charge $-Q$ and the resulting potential difference between the two conducutors is V.

The unit of capacitance is F (farad), which is C/V.

(This may open for some confusion, because we use C for capacitance and C for Coulomb, V for volt and V for electric potential. However, notice the difference in fonts between the unit V, which uses the normal font, and V, which uses the mathematical font.)

7.1.1 Applications of capacitors

Capacitors have many practical applications and are important parts of realistic models of electromagnetic systems. For example, capacitors are often used to measure distances. A parallel-plate capacitor can be made very sensitive to small changes in the distance d between the two plates, $C = A\epsilon/d$, and this can be used to precisely measure nano-scale distances. A parallel-plate capacitor can also be very sensitive to small particles entering the area between the plates, effectively changing the dielectric properties of this area and thereby changing the capacitance. Many different types of sensors are based on these types of principles.

7.1.2 Example: Parallel plate capacitor

Let us now use the definition of capacitance to calculate the capacitance of a parallel plate capacitor as illustrated in Fig. 7.3. We have two general plans: (1) First assume a charge distribution of $+Q$ on one conductor and $-Q$ on the other, use this to find the electric field $\mathbf{E}$, use the electric field to find the potential difference $V = \int_0^1 \mathbf{E} \cdot \mathrm{d}\mathbf{l}$, and then find the capacitance from $C = Q/V$. (2) Solve Laplace's equation to find the electric potential, $V(\mathbf{r})$, use the potential to find the electric field, $\mathbf{E} = -\nabla V$, and then relate the electric field to the charge, for example, using that $E_n = \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s/\epsilon_0$ on the surface of the conductors.

Method 1: From charge to field to potential.

Find the field: We find the electric field using Gauss' law. We assume that the plates are large, so that we can ignore edge effects and assume that

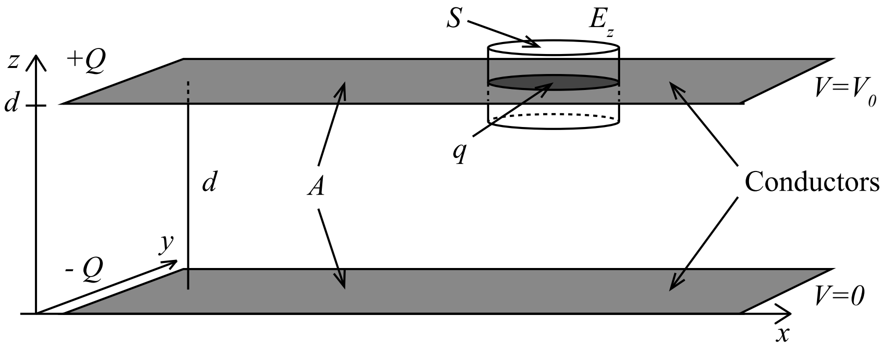


Fig. 7.3 Illustration of a parallel plate capacitor consisting of two plates of area A separated by a distance d. There is a potential difference V between the plates generated by moving a charge Q from the bottom plate to the top plate so that the top plate has a charge Q and the bottom plate has a charge $-Q$.

the electric field only has a z-component, $\mathbf{E} = E_z\hat{\mathbf{z}}$, in the region between the plates. We address the fields from each conductor independently, and then use the superposition principle to find the total field. We choose a cylindrical Gauss surface with top and bottom surface areas A as illustrated in the figure. Because the electric field is parallel to the z-axis, there is no flux through the sides surfaces of the cylinder. We also assume that the electric field from the plate is symmetric around the plate. If we choose a Gauss cylinder which has a height $h/2$ on each side of the plate, the field on each side will have the same magnitude and opposite direction. The flux through the top and bottom plates are then $\Phi = E_z\hat{\mathbf{z}} \cdot S\hat{\mathbf{z}} + (-E_z)\hat{\mathbf{z}} \cdot S(-\hat{\mathbf{z}}) = 2ES$. We assume that the total charge Q is uniformly distributed along the plate. (We do not need to assume how the charge is distributed in the z-direction). The total charge inside the cylinder is $q = (Q/A)S$. Gauss' law then gives

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \Phi = 2E_zS = \frac{q}{\epsilon_0} = \frac{2\rho_s S}{\epsilon_0} = \frac{2QS}{2A\epsilon_0} = \frac{QS}{A\epsilon_0} \ , \tag{7.3}$$

and therefore $E_z = Q/(2A\epsilon_0)$. We get a similar result for the bottom plate. The total field between the plates is therefore $E_z = Q/(2A\epsilon_0) + Q/(2A\epsilon_0) = Q/(A\epsilon_0)$ in the negative z-direction.

Find the potential difference: The potential difference between the top and the bottom plate is then

$$V_1 - V_0 = \int_0^1 \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_d^0 -E_z dz = Qd/(A\epsilon_0) \ . \tag{7.4}$$

Find the capacitance: Finally, we find the capacitance from:

$$C = \frac{Q}{V} = \frac{Q}{Qd/(A\epsilon_0)} = \frac{A\epsilon_0}{d} \ . \tag{7.5}$$

Method 2: From potential to field to charge.

Find the potential using Laplace's equation: We assume that the distance d is small, so that the potential only depends on z and not on x and y. We want to solve $\nabla^2 V = \mathrm{d}^2 V/\mathrm{d}z^2 = 0$ for $0 < z < d$, where $V(0) = 0$ and $V(d) = V$.

We recognize that the solution must be on the form $V(z) = c_1 z + c_2$. The boundary conditions determines the constants c_1 and c_2: $V(0) = 0$ gives $c_2 = 0$ and $V(d) = c_1 d = V$ gives $c_1 = V/d$. Therefore $V(z) = Vz/d$. (We notice that in method 1 we only found $V(z = d)$ and not the general form $V(z)$.)

Find the electric field: We find the electric field from $\mathbf{E} = -\nabla V = -(V/d)\hat{z}$.

Find the surface charge density and the charge Q: The surface charge density on the bottom side of top plate must be given by $\mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s/\epsilon_0$, where $\mathbf{E} = -(V/d)\hat{\mathbf{z}}$ and $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, so that $\rho_s/\epsilon_0 = (V/d)$. The total charge on the bottom side of the top plate is then $Q = \rho_s A = V\epsilon_0 A/d$. Similarly, the total charge on the top side of the bottom plate is $-V\epsilon_0 A/d$. Notice that there is no charge on the top side of the top plate or the bottom side of the bottom plate, because the electric field is zero outside the plates on these sides. (See the comments under interpretation below).

Find the capacitance: We find the capacitance as:

$$C = \frac{Q}{V} = \frac{(V/d)A\epsilon_0}{V} = \frac{A\epsilon_0}{d} \ . \tag{7.6}$$

Interpretation. First, we notice that we can increase the capacitance by decreasing the spacing d, or by increasing the area A or the dielectric constant. What may be the disadvantage of decreasing the distance d between the plates? We see that for a given V, the electric field is $E_z = (V/d)$. Decreasing d while keeping V constant will increase the charge stored in the capacitor, but also increase the electric field. If d becomes too small, we may have dielectric breakdown, leading to the failure of the capacitor and charges leaking from one conductor to another.

How would the results be modified if we instead had a *dielectric material* with dielectric constant ϵ between the plates? The only change in the calculations we did above would be to change ϵ_0 with ϵ, increasing the capacitance.

The electric field inside the conductor must be zero. How is this achieved in this system? When we calculate the electric field, it may be tempting to assume that the charge Q is distributed uniformly on both surfaces (top and bottom) of the conductors. However, this would not be correct in this case. In the case of a single conductor with a net charge Q alone in space, the charges would have to be distributed uniformly on both surfaces to ensure that the electric field is zero inside the conductor. However, in the case of two parallel conducting plates, it is the *total field* that must be zero inside the conductors: For the top conductor the total field is the sum of the electric field from the bottom conductor and the electric field from the surface charges on the top conductor. For the parallel plate system, there will be a surface charge density $\rho_s = Q/A$ on the bottom side of the top plate and $-Q/A$ on the top side of the bottom plate. The electric fields from both of these sheets of charge will be non-zero only between the plates, everywhere else the superposition of the electric fields will be zero: both inside the conductors, above the top plate and below the bottom plate. This is clearly visible from the numerical solution in Fig. 7.1, where we see that the potential is approximately uniform on the top side of the top plate and the bottom side of the bottom plate, indicating that that electric field is zero.

7.1.3 Example: Effect of dielectric materials

What happens if we charge the capacitor illustrated in Fig. 7.3 to that it holds a total charge Q, and then introduce a dielectric material with dielectric constant ϵ into the space between the two plates?

We charge the capacitor by applying a voltage V_0. In vacuum, the capacitance is $C_0 = A\epsilon_0/d$. The charge on the capacitor is then $Q_0 = C_0V_0$. The electric field inside the capacitor with vacuum is then $E_0 = V_0/d$. We then decouple the capacitor, so that it retains its charge Q, but we no longer keep the potential difference by applying an external voltage. The charge on the capacitor then remains Q_0.

Then, we insert the dielectric into the gap. The charge remains the same, but the capacitance of the new system is $C_1 = \epsilon A/d$, where

$C_0 = (A\epsilon_0/d)$ so that $C_1 = (\epsilon/\epsilon_0)C_0 = \epsilon_r C_0$. This means that the potential also changes to:

$$V_1 = \frac{Q_1}{C_1} = \frac{Q_0}{C_1} = \frac{C_0 V_0}{\epsilon_r C_0} = \frac{V_0}{\epsilon_r} \ . \tag{7.7}$$

The electric field is then reduced to $E = E_0/\epsilon_r$ as expected. This is due to the polarization of the medium, that sets up a field that acts in the opposite direction, hence reducing the effective field.

Notice that if we performed the experiment *in a different way*, by applying a voltage $V_2 = V_0$ to the system with the dielectric, then the charge, Q_2, in the system would be different: $Q_2 = C_1 V_0$, but the electric field would be the same as in vacuum, $E_2 = V_2/d = V_0/d = E_0$.

7.2 Method: Calculating the capacitance

From these examples we can extract a *general method* that we can use to calculate the capacitance of a system:

Method: Finding the capacitance of a system

Method 1: From charge to field to potential:

1. We assume that one conductor has a charge $+Q$ and that the other conductor has a charge $-Q$.
2. We calculate the electric fields due to these charges using e.g. Gauss' law or Coulomb's law and utilizing possible symmetries in the system, and find the total electric field $\mathbf{E}$ using the superposition principle.
3. We calculate the potential difference between the two conductors, $V = \int_0^1 \mathbf{E} \cdot \mathrm{d}\mathbf{S}$, using the total electric field we found.
4. We find the capacitance from $C = Q/V$, where we insert our value for V and the value for Q we started with.

Method 2: From potential to field to charge:

1. We solve Laplace's equation to find the potential $V(\mathbf{r})$ for the system. Boundary conditions are given by assuming that one

conductor has a potential V_1 and the other a potential V_0. Usually, we choose $V_0 = 0$.
2. We find the electric field $\mathbf{E} = -\nabla V$.
3. We find the surface charge density on the conductors from $\rho_s = \epsilon \mathbf{E} \cdot \hat{\mathbf{n}}$. We sum/integrate the surface charge density to find the total charge Q on one of the conductors. (The other conductor should then have a charge $-Q$).
4. We find the capacitance from $C = Q/V$, where we insert our value for Q and the potential difference $V = V_1 - V_0$ we started with.

7.3 Combining several capacitors

The methods we have introduced above allow us to calculate the capacitance of a single capacitor consisting of two conductors. Capacitors are common elements in electric circuits and common features in models we build of electromagnetic systems. However, in many cases we want to combine several capacitors with individual capacitances C_i into one combined component with capacitance C. For example, we may connect several capacitor components together, in parallel, in series or in a combination. In this case, we can derive expressions for the combined capacitance if we can assume that the capacitors are independent components that do not interact. What do we mean by not interacting? This means that the electric field from one component do not interact with another component, for example, leading to a displacement of the charge distribution. For a set of parallel plate capacitors this is usually the case, because the electric field outside of the space between the capacitors is generally very small. (We assumed it to be zero in the case of a thin and large capacitor).

First, let us introduce a compact way to describe a system of several (non-interacting) capacitors. We do this in the form of a circuit diagram. A circuit diagram consists of a set of thin lines that represent ideal conductors connecting various parts of the components. In general, we do not care about the shape of these lines: We draw them as simple as possible often using straight lines, because the way we draw them has no impact on our system. (This will change when we discuss magnetic induction later). Because the lines represent ideal conductors, the electric potential is the same everywhere along the line: The lines connect components at the same potential. Fig. 7.4 illustrates an example diagram. We draw a capacitor as two parallel plates. Notice that the potential on each side of the capacitor will not be the same, but will depend on the charge Q on the capacitor. The voltage drop, the difference in potential across the component, is $V = C/Q$, where Q is the charge on the capacitor. Usually, in circuits, we will have voltage sources that provide a given potential difference and not charge sources, so that we will know the potential difference V and calculate the charge $Q = CV$.

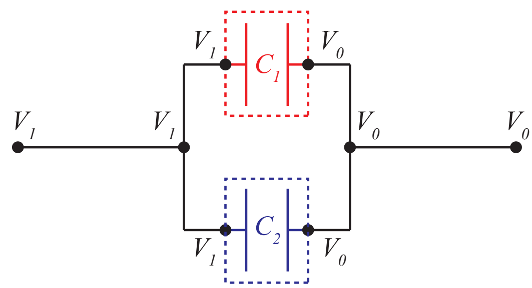


Fig. 7.4 Illustration of circuit diagram of two capacitors in parallel. The electric potential at various points (filled circles) along the circuit are shown. The top capacitor is shown in red and the bottom in blue. However, usually components are drawn in the same color as the connecting wires.

When we draw individual component in a circuit diagram we implicitly mean that they do not interact: The electric (or later magnetic) field inside a compent do not interact with the fields inside other components.

Let us now look at two examples in detail: The total capacitance of a system of several capacitors in series and in parallel. Based on these two rules, we can find the combined capacitance of any system of capacitors.

7.3.1 Example: Parallel coupling of capacitors

First, we address the case of n capacitors, $C_1, C_2, \ldots, C_n$, connected in parallel as illustrated in Fig. 7.5. The goal is to find the value C for a single capacitor to replace all the capactors so that the effective capacitance is the same.

We notice that all the n capacitors have the same potential on the top, V_1, and on the bottom, V_0, side. This is indeed how we interpret the solid line drawn between the components. The charge on each capacitor is Q_i as indicated in the figure, where $C_i = Q_i/V$, where $V = V_1 - V_0$ is the same for all the capacitors.

All the conductors on the top side are connected so that they are at the same potential. It is like they are one large conductor. Similarly for the bottom side. The total charge on the top side is $Q = \sum_i Q_i$ and the total charge on the bottom side is $-Q = \sum_i -Q_i$. The total capacitance of the whole system is then

$$C = \frac{Q}{V} = \frac{\sum_i Q_i}{V} = \sum_i \frac{Q_i}{V} = \sum_i C_i \ . \tag{7.8}$$

This proves that there is a simple additive law for (independent) capacitances in parallel, that is, for capacitors whoose *field do not interact.*

$$C = C_1 + C_2 + C_3 + \ldots + C_n = \sum_i C_i \ . \tag{7.9}$$

We can therefore replace these n capacitors with a single capacitor with the capacitance C.

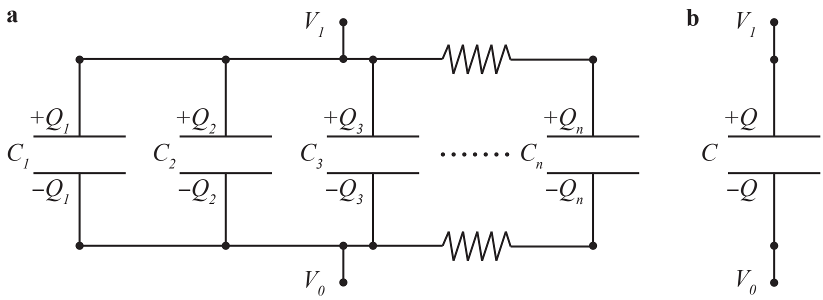


Fig. 7.5 (a) Illustration of circuit diagram of n capacitors in parallel. (b) The n capacitors are replaced by a single capacitor with capacitance C.

7.3.2 Example: Series coupling of capacitors

Fig. 7.6 illustrates a series coupling of n capacitors with capacitances $C_1, C_2, \ldots, C_n$. We want to replace them by a single capacitor with capacitance C. What should the value of C be? (We assume that the electric field from one capacitor does not affect another capacitor).

We notice from the figure that the right part of capacitor 1 is connected to the left side of capacitor 2. This means that the total charge on these two capacitors must be conserved: What is taken from the right side of capacitor 1 is added to the left side of capacitor 2, and so on. Therefore, all the capacitors will have the same charge Q.

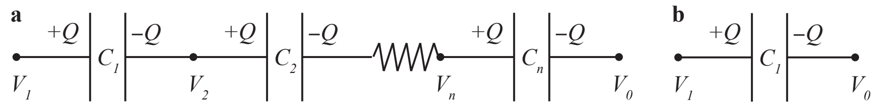


Fig. 7.6 (a) Illustration of circuit diagram of n capacitors in series. (b) The n capacitors are replaced by a single capacitor with capacitance C.

What about the voltage differences across the capacitors? We have denoted the voltage on the left V_1 and on the right V_0, (which we without loss of generality can set to zero, $V_0 = 0$). We denote the voltage to the left of capacitor i, V_i, as shown in the figure. The voltage difference across the first capacitor with capacitance C_1 is then $\Delta V_1 = V_1 - V_2$. In addition, we know that $C_1 = Q/(\Delta V_1)$ so that $\Delta V_1 = Q/C_1$. Similarly, for the second capactor, C_2, we have that $\Delta V_2 = V_2 - V_3 = Q/C_2$.

We sum the voltage differences for all the capacitors, noticing that subsequent pairs of potentials (e.g. $-V_2 + V_2$) sum to zero:

$$\sum_i \Delta V_i = V_1 - V_2 + V_2 - V_3 + \ldots + V_n - V_0 = V_1 - V_0 \tag{7.10}$$

$$= \frac{Q}{C_1} + \frac{Q}{C_2} + \ldots + \frac{Q}{C_n} \tag{7.11}$$

$$= Q \sum_i \frac{1}{C_i} = Q\frac{1}{C} \,. \tag{7.12}$$

The total effective capacitance is therefore:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_n} = \sum_i \frac{1}{C_i} \,. \tag{7.13}$$

7.4 Energy stored in an electric field

What is the energy stored in the electric field? What does this question even mean? For a capacitor, we can calculate how much energy is needed to build up a given charge difference, Q, between the two capacitors. There will be an electric field and a voltage associtated with this charge difference. We expect there also to be an energy associated with this charge difference and hence also with the electric field: The energy needed to set up the field. We expect to get this energy back if we discharge the capacitors. Our plan is to start to study a simple capacitor, introduce an energy associated with the electric field in this capacitor, and generalize this to build a general description of the energy associated with an electric field.

7.4.1 Energy of a parallel plate capacitor

Fig. 7.7 illustrates a part of the charging process of a parallel-plate capacitor. We start from a parallel-plate capacitor where both plates have the same potential and no charge. We charge the capacitor in increments by moving a small amount of charge dq from one conductor to another. The figure illustrates a step in this process when the charge on the top plate is q, and the potential difference is $V(q) = q/C$. This means that the work $\mathrm{d}W$ needed to move a charge $\mathrm{d}q$ from one conductor to another is $\mathrm{d}W = V(q)\mathrm{d}q = (q/C)\mathrm{d}q$.

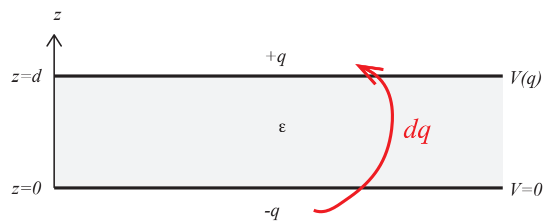


Fig. 7.7 Illustration of a parallel-plate capacitor when the charge is $+q$ on the top surface, $-q$ on the bottom surface, and the potential is $V(q)$. A small charge dq is transferred from the bottom surface to the top surface.

The total work is the integral of the work during each of these steps from $q = 0$ to $q = Q$.

$$W = \int \mathrm{d}W = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{1}{2}\frac{Q^2}{C} \,. \tag{7.14}$$

We interpret this work as the potential energy, U, stored in the capacitor. We insert $Q = CV$ and get:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2 \,. \tag{7.15}$$

If we connect a motor which is driven by the discharge of the charge, the total amount of work done by the motor will be U. Although we here argued for the behavior of a parallel plate capacitor, our argument is general and applies to any capacitor with capacitance C. We can therefore generalize our result:

Energy of a capacitor

The potential energy stored in a capacitor C with a charge Q and/or a potential V is:

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} \ . \tag{7.16}$$

7.4.2 Energy of a charge distribution

The energy stored in a capacitor corresponds to the work done to create the charge distribution. Just like for the capacitor, let us see what happens if we create a charge distribution of N charges Q_i at positions $\mathbf{r}_i$ by adding one charge at a time to the system. We place the first charge, Q_1, in position $\mathbf{r}_1$. Then we bring the second charge in from infinity to its position $\mathbf{r}_2$. The work to do this is $W_2 = Q_2 V_1(\mathbf{r}_2)$, where $V_1(\mathbf{r}_2)$ is the scalar potential from charge Q_1. We notice that this work only depends on the distance between the two charges.

$$W_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}_2 - \mathbf{r}_1|} \ . \tag{7.17}$$

Then we bring in the third charge. The work needed to bring this new charge to a position $\mathbf{r}_3$ is the sum of the potential energies from the interactions with each of the previous charges *plus* the energy we needed to make the system of two charges:

$$W_3 = Q_3 \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}_3 - \mathbf{r}_1|} + Q_3 \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}_3 - \mathbf{r}_2|} + Q_2 = Q_3 \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}_3 - \mathbf{r}_1|} + Q_3 \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}_3 - \mathbf{r}_2|} + Q_2 \frac{}{4} \tag{7.18}$$

We see that this energy is the sum of potential energies for all the pairs of charges, where we call the potential energy of a pair of charges the potential energy of one charge in the field set up by the other charge or vice versa. (Because $|\mathbf{r}_i - \mathbf{r}_j| = |\mathbf{r}_j - \mathbf{r}_i|$). This repeats itself each time we add a new charge. The total energy therefore becomes the sum of the potential energy for each pair of charges. We might be tempted to write this as:

$$W_N = \sum_i \sum_j \frac{Q_i Q_j}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|} \; . \tag{7.19}$$

However, there are two problems with this expression. First, we should not include the potential energy of a charge i with itself. Therefore i should not be equal to j in this sum. Second, if we do this sum, we will add both a term for the potential energy of a charge i in the field set up by charge j *and* the potential energy of a charge j in the field set up by charge i. The two potential energies are identical and we should only add one potential energy for each such interaction. In the sum we have counted each such interaction twice. We must divide by two to only include each interaction once. The energy is therefore:

$$W_N = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{Q_i Q_j}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|} \; . \tag{7.20}$$

We rewrite this as

$$W_N = \frac{1}{2} \sum_i Q_i \sum_{j \neq i} \frac{Q_j}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|} \; , \tag{7.21}$$

and notice that the sum over j is simply the potential at the point $\mathbf{r}_i$ due to the other $N-1$ charges at positions $\mathbf{r}_j$. We can therefore write the energy of the system as:

$$W = \frac{1}{2} \sum_i Q_i V(\mathbf{r}_i) \; . \tag{7.22}$$

Where $V(\mathbf{r}_i)$ is interpreted as the potential at a point $\mathbf{r}_i$ due to all the other charges in the system. This is the amount of energy needed to create the charge distribution.

This result can be generalized to a continous charge distribution $\rho(\mathbf{r})$ to be:

$$W = \frac{1}{2} \int_v \rho(\mathbf{r}') V(\mathbf{r}') \mathrm{d}v' \; . \tag{7.23}$$

7.4.3 Energy density

We have now found an expression for the total energy need to create a charge distribution, which we interpret as the energy of the charge distribution. However, the charges generate electric fields. We can therefore also interpret this energy as the energy needed to create the corresponding electric fields, and we may think of the energy as being in the electric field, so that there is an energy density in space which is related to the electric field.

For the parallel place capacitor, we know that the electric field is only between the plates (for an infinite system, that is). In this case the energy U is

$$U = \frac{1}{2}CV^2 \ . \tag{7.24}$$

In this case, the electric field is uniform $E = V/d$ in the region between the two plates and the capacitance C is $C = \epsilon A/d$, where A is the area of the plate and d is the spacing. We insert $V = Ed$ and $C = \epsilon A/d$ into (7.24) and find that

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon A}{d}\left(Ed\right)^2 = \frac{1}{2}\epsilon E^2 Ad \ , \tag{7.25}$$

where we recognize Ad as the volume that the electric field occupies inside the capacitor. The *energy density*, which is the energy per unit volume:

$$u_e = \frac{U}{v} = \frac{U}{Ad} = \frac{1}{2}\epsilon E^2 \ . \tag{7.26}$$

For the parallel plate capacitor we then get that the total energy is $U = u_e \cdot Ad$. While we derived this result for a parallel-plate capacitor, the result is general for any linear, isotropic medium.

Energy density of the electric field

The energy density $u_u(\mathbf{r})$ of the electric field for a linear, isotropic medium is defined as:

$$u_e = \frac{1}{2}\epsilon E^2 \ . \tag{7.27}$$

Notice that $\mathbf{E}(\mathbf{r})$ does not have to be uniform and that the energy density in general varies in space. In general, the energy U in a volume v is the integral of the energy density over the volume:

$$U = \int_v u_e(\mathbf{r})\mathrm{d}v \ . \tag{7.28}$$

This result can be further generalized to be valid for any linear medium, including anisotropic media, by introducing $\mathbf{D} = \epsilon\mathbf{E}$:

$$u_e = \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \, . \tag{7.29}$$

7.4.4 Example: Spherical capacitor

Fig. 7.8 illustrates a spherical capacitor consisting of an inner conducting sphere of radius R_1 and an outer conducting spherical shell of inner radius R_2 and outer radius R_3. Find the capacitance of and the energy stored of this system.

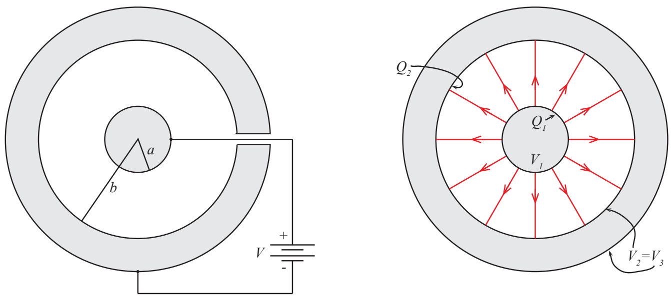


Fig. 7.8 Illustration of a spherical capacitor. (The gap in the drawing is to illustrate how we can connect the inner sphere to a voltage. Assume the outer spherical shell to be without any holes.)

Capacitance of spherical capacitor using method 1. We can use both method 1 and method 2 to solve this problem. Let us start by using method 1 where we assume a given charge Q, the find the electric field, from the electric field find the potential, and then find the capactice as $C = Q/V$.

We start by distributing the charges $+Q$ on the inner shell and $-Q$ on the outer shell. How is the charge distributed? On the inner shell, we know that there cannot be any free charges inside the conductor. Therefore, all the charges must be on the outer surface of the sphere. What about the outer shell? Also here, there cannot be any charges inside the conductor, so all the charges must be on the surfaces of the conductor. We also know that the electric field must be zero inside the outer conductor shell. How must the charges be organized to ensure this? We expect the charge distribution and the electric field to have spherical symmetry. We can use Gauss' law on a spherical Gauss surface inside the outer conductor. The electric field on this surface must be zero, and therefore the net charge inside the Gauss surface must be zero. The net charge on the surface at a is $+Q$. The net charge on the surface at b must therefore be $-Q$ to ensure the net charge to be zero. All the $-Q$ charge on the outer shell is therefore on the inner surface at b.

We can then find the electric field in the region $a < r < b$ by using Gauss' law with a spherical Gauss surface of radius r. Gauss' law gives that $4\pi r^2 E_r = Q/\epsilon$, and therefore $E_r = Q/(4\pi\epsilon r^2)$.

Then we find the potential difference between the inner and outer conductors. We first find the potential for all $a < r < b$:

$$V(r) - V(a) = \int_r^a \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \frac{Q}{4\pi\epsilon} \int_r^a \frac{\mathrm{d}r}{r^2} = -\frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{r}\right) . \tag{7.30}$$

The potential is highest in $r = a$ and decays towards $r = b$ — as expected when there is a positive charge at $r = a$. We find the potential difference ΔV as $\Delta V = V(a) - V(b)$, which is

$$\Delta V = -(V(b) - V(a)) = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right) . \tag{7.31}$$

The capacitance is therefore

$$C = \frac{Q}{\Delta V} = 4\pi\epsilon \frac{1}{\frac{1}{a} - \frac{1}{b}} = 4\pi\epsilon \frac{ab}{b-a} . \tag{7.32}$$

This result can also be simplified to the case when there is only a single sphere of radius a in a large grounded cage. In this case, $b \to \infty$ and $C = 4\pi\epsilon a$.

Energy stored in spherical capacitor. We now have an expression for both the electric field, $\mathbf{E}(\mathbf{r})$, the electric potential $V(\mathbf{r})$ and the capacitance. How can we use this to find the energy stored in the capacitor?

Our plan is to use the energy density, $u_e = \frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E}$, and integrate this over the whole volume of the capacitor. In the region $a < r < b$ the electric field is

$$E(r) = \frac{Q}{4\pi\epsilon}\frac{1}{r^2} , \tag{7.33}$$

so that the energy density is

$$u_e = \frac{1}{2}\epsilon E^2 = \frac{Q^2}{32\pi^2\epsilon r^4} . \tag{7.34}$$

What is the energy, $\mathrm{d}U$, in a shell between r and $r + \mathrm{d}r$?

$$dU = u dv = u(4\pi r^2 dr) = \frac{1}{2}\epsilon \frac{Q^2}{(4\pi\epsilon)^2} \frac{1}{r^2} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon} \frac{dr}{r^2} . \tag{7.35}$$

The energy for the whole region is the integral from $r = a$ to $r = b$:

$$U = \frac{Q^2}{8\pi\epsilon} \int_a^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon} \frac{b-a}{ab} . \tag{7.36}$$

We can now use that $C = 4\pi\epsilon ab/(b-a)$, getting:

$$U = \frac{1}{2} \frac{Q^2}{C} , \tag{7.37}$$

as it should be!

7.5 Summary

The **capacitance** C for a set of two conductors is defined as $C = Q/V$ when the two capacitors have the charges $+Q$ and $-Q$ and their potential difference is V.

The **capacitance** C **of a parallel plate capacitor** with area A, spacing d and dielectric constant ϵ is $C = A\epsilon/d$.

We find the capacitance of a system using two main methods. In **method 1**, we assume that the conductors have charges $+Q$ and $-Q$, calculate the electric field, from the field we find the potential difference V, and then we find $C = Q/V$. In **method 2**, we assume that the two conductors are equipotential surfaces with a potential difference V and solve Laplace's equation for the potential between the conductors, we find the electric field from the potential, and the surface charge density from the electric field, sum the surface charge density to find the charge Q and $-Q$ and calculate $C = Q/V$.

We can **combine capacitors** $C_1, C_2, \ldots, C_n$, to one effective capacitor C by combining capacitors *in parallel* getting $C = \sum_i C_i$ or *in series* getting $1/C = \sum_i 1/C_i$.

The **energy stored** in a capacitor is $U = Q^2/(2C)$ or $U = CV^2/2$.

The **energy density** of an electric field is $u_e = (1/2)\epsilon E^2$ or $u_e = (1/2)\mathbf{D} \cdot \mathbf{E}$. The total energy of a volume is $U = \int_v u_e dv$.

7.6 Exercises

7.6.1 Test yourself

Exercise 7.1: Capacitance

What is the definition of capacitance? Explain the symbols you use.

Exercise 7.2: Parallel plates

Two thin metal plates are placed in parallel in isolating brackets at a distance d. You can assume that the plates are so large and so close to each other that you can ignore edge effects and that all the charge are on the internal surfaces of the plates (on the side facing the other plate).

The surface charge density on one plate is $+\rho_s$ and on the other it is $-\rho_s$. How does the following quantities change (if they do change) when the two plates are moved closer to each other?

a) How does the surface charge density on each plate change?

b) How does the magnitude of the electric field between the plates change?

c) How does the absolute value of the potential difference between the plates change?

d) How does the capacitance to the two plates change?

Exercise 7.3: Ladningsfordeling på leder

(Fra Steven Pollock, UC-Boulder) Gitt et par veldig store, flate, ledende kondensator-plater med totale ladninger $+Q$ på den øverste platen og $-Q$ på den nederste platen. Hvis vi ser bort fra kant-effekter, hvordan er ladningen fordelt i likevekt? (A) Uniformt gjennom hver plate, (B) Uniformt på hver side av hver plate, (C) Uniformt på toppen av $+Q$ platen og på bunnen av $-Q$ platen, (D) uniformt på bunnen av $+Q$-platen og på toppen av $-Q$-platen, (E) Noe annet.

Exercise 7.4: Ladningsfordeling på ledende plater

Du har to veldig store, parallell-plate-kondensatorer, begge med samme areal A og med samme ladning Q. Kondensator 1 har dobbelt så stort gap som kondensator 2. Hvilken har lagret mest energi? (A) 1 har lagret dobbet så mye som 2, (B) 1 har lagret mer enn dobbelt så mye som 2, (C) De har det samme, (D) 1 har dobbelt så mye som 1, (E) 2 har mer enn dobbelt så mye som 1.

Exercise 7.5: Ladningsfordeling på ledende plater 2

Du har to parallell-plate-kondensatorer, begge med samme areal A og samme gap. Kondensator 1 har dobbelt så mye ladning som 2. Hvilken har størst kapasitans, C? Og mest lagret energy, U? (A) $C_1 > C_2$, $U_1 > U_2$; (B) $C_1 > C_2$, $U_1 = U_2$; (C) $C_1 = C_2$, $U_1 = U_2$; (D) $C_1 = C_2$, $U_1 > U_2$; (E) En annen kombinasjon.

Exercise 7.6: Capacitance

a) In your own words, what is:

- A capacitor
- Capacitance
- dielectric medium

b) Does the capacitance of a device depend on the charge residing within it?

c) Does the capacitance of a device depend on the applied voltage?

Hint. What does the V in $C = Q/V$ represent?

d) Is the capacitance necessarily 0 if the plates are not charged?

Hint. What is the capacitance in a parallel-plate capacitor?

e) You have two capacitors of different capacitance C_1 and C_2. You wish to construct a circuit with the highest possible total capacitance using these two capacitors. Should you wire them in parallel or in series?

Hint. Express the capacitance over the parallel connection, C_p and the serial conneciton C_s in a suitable way, then find the relation C_p/C_s

f) What is the largest and the smallest capacitance you can get from connecting four 2.0 μF capacitors?

Hint. The largest is when they are all in parallel. The lowest when they are all in series.

g) A capacitor has a capacitance of 5.0 μF when connected to a 9.0-V battery. How much energy is stored in the capacitor?

7.6.2 Discussion exercises

Exercise 7.7: Dielectric materials

How can you explain with words why the capacitance of a parallel plate capacitor increases if you place a dielectric material with higher dielectric constant in the spacing between the plate.

Exercise 7.8: Capacitance in pieces

You have a parallel plate capacitor with sides L and a spacing of d. You cut it into four identical pieces. What is the capacitance of one of the four pieces compared to the original capacitor?

Exercise 7.9: The shape of a capacitor

"A capacitor always consists of two identical condutors that have been translated relative to each other". Is this true? Explain.

7.6.3 Tutorials

Exercise 7.10: Capacitance of the axon

A nerve cell is illustrated in Fig. 7.9. The cell has a long axon, which is approximately cylindrical in shape. A cross section of the (cylindrical) axon shows that the cell consists of a center, a thin outer membrane and a myelin sheath outside the membrane. We will assume that the interior and the exterior of the cell can be considered as good conductors. We will model this system as a cylindrical capacitor: We model the inner cylindrical part of the cell as an ideal conductor ($r < a$); we model the cell membrane and the myelin sheath as a cylindrical shell with dielectric constant ϵ ($a < r < b$); and we model the exterior as an ideal conductor ($b < r$).

We will find the capacitance of this system by (i) making a drawing of the system, (ii) drawing the expected charge distributions, (iii) finding the electric field for this charge distribution, (iv) finding the potential by integrating the electric field, and (v) finding the capacitance by relating the charges to k the potential difference.

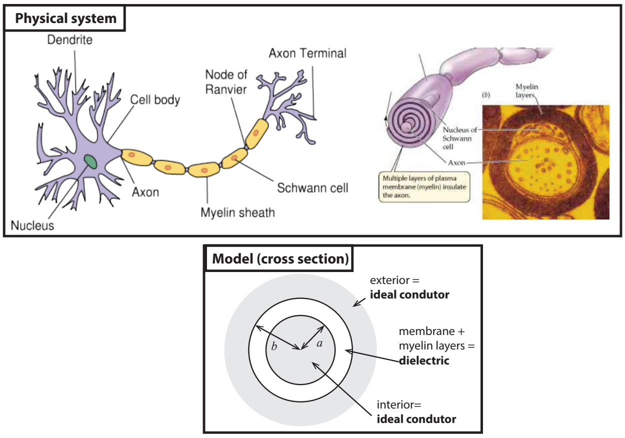


Fig. 7.9 Illustration of a nerve cell.

We start from an initially neutral system. We then apply a potential difference V across the dielectric shell. In this process, a charge Q is transferred from the outer shell to the inner shell. Afterwards, the exterior of the dielectric shell is at $V = 0$ and the inner part is at V.

a) Make a sketch of the distribution of charges on a cross section of the cylinder. What principles do you use when you make this drawing? Is the surface charge density the same on the inner and outer surfaces?

b) Find the electric field everywhere in terms of Q.

Hint. Choose a Gaussian surface and use Gauss' law.

c) Find the potential $V(r = a)$ at the inner cylinder, when the potential $V(r = b) = 0$ at the outer cylinder.

d) Find the capacitance of the cylindrical shell.

Exercise 7.11: Capacitance in a system with less symmetry

We will here study a system of two parallel cylindrical conductors of length L and radius a. They are directed along the z-direction and placed with their centers at $(0, 0, 0)$ and $(d, 0, 0)$. They are surrounded by a dielectric ϵ.

7.6.4 Homework

Exercise 7.12: Coaxial cable

A coaxial cable consists of an inner cylindrical conductor with radius a and outer cylindrical shell conductor with inner radius b. A dielectric with permittivity ϵ is placed between the conductors. The inner conductor has a potential V_0 while the outer conductor is grounded (has potential 0).

a) Find the $\mathbf{E}$-field between the conductors and the capacitance per unit length C'.

b) Calculate the numerical value for C' when $\epsilon = 3\epsilon_0$ and $\frac{b}{a} = 7$.

c) Find how much electric energy is stored per unit length of the cable.

d) What is the net force acting on the outer conductor from the inner conductor?

Exercise 7.13: Shocking bed

(Made by Bror Hjemgaard)

A physicist is in prison for not labeling his axes, and while laying in his metal bunk-bed he worries that the static electricity from the mattress might charge the bunk bed into a dangerously charged capacitor.

a) The bunk beds are separated vertically by 1.5 meters, and both have a surface area of 1.80 m^2. What is the capacitance of the bunk-bed?

b) If the inmates manage to place opposite charges of magnitude 1.00 nC on the bunk-beds, what is the voltage across the capacitor?

c) The voltage calculated in b) may be surprising to many, but to reassure yourself, calculate the energy stored in the capacitor. Should the physicist be worried for his health?

8 Current and resistance

We have seen that under equilibrium conditions — after a very long time — all charges in conductors have moved to positions so that the system is in equilibrium and there are no more charges moving. In this case, the conductor is an equipotential surface and all field lines are perpendicular to the surface.

In this chapter, we will address what happens when charges are moving due to an electric field: The electric field is not zero and charges move. We can induce this situation by applying a potential difference across a resistor.

The existence of an electric field means that charges will move inside the conductor, leading to an electric *current*, but in most materials there will also be an internal resistance to flow, corresponding to a *resistance*. In this chapter, we will introduce and quantify the concept of current density and how it is related to the local electric field through Ohm's law through a constant of proportionality called the resistivity. We will introduce the term *resistor* for a component that has a resistivity, and we will introduce a systematic method to calculate the resistance of resistors. We will also introduce conservation laws for currents, which follow from the conservation of charge. Finally, we will introduce numerical methods to calculate the current distribution in non-ideal conductors, so that we can model the flow of charge in complex situations.

8.1 Current

Fig. 8.1 illustrates a cross-section through a resistor conductor. Let us assume that charges are moving along the conductor so that a charge Δq passes through the cross-sectional area A in a short time interval Δt. We define the *current*, I, through the cross-sectional area A as the instantaneous rate at which positive charges are passing through the surface: $I = \Delta q/\Delta t = dq/dt$.

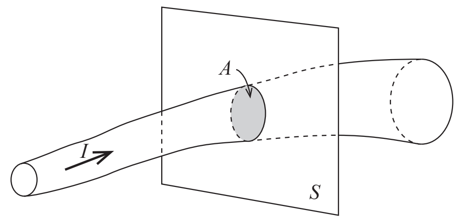


Fig. 8.1 Illustration of a conductor of cross-section A passing through a surface S. An amount of charge Δq is passing through the surface along the conductor in a short time Δt.

Definition of current

The **current** I through a surface S is defined as the instantanenous rate at which positive charges are passing through the surface:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{8.1}$$

where $\mathrm{d}q$ is the amount of charge passing through the surface S in an infinitesimal time $\mathrm{d}t$. Current is measured in coulombs per second (C/s), which is called **Ampere** in the SI system. Ampere is considered a basic unit in the SI system, such as kg, meters or seconds.

8.1.1 Microscopic picture

How can we relate the concept of current to the microscopic picture of charges and electric fields we have introduced so far? What happens to a particle with charge q, such as an electron, inside a real conductor? We often call such a particle a charge carrier. Fig. 8.2 illustrates a real conductor such as a metal. Such a system consists of a sea of electrons that can move and a stationary lattice of positive charges, ions, that are vibrating, but otherwise not moving far from their positions in the lattice. If we apply an electric field $\mathbf{E}$ to this system, a small charge q, such as an electron, will be subject to a force $\mathbf{F} = q\mathbf{E}$.

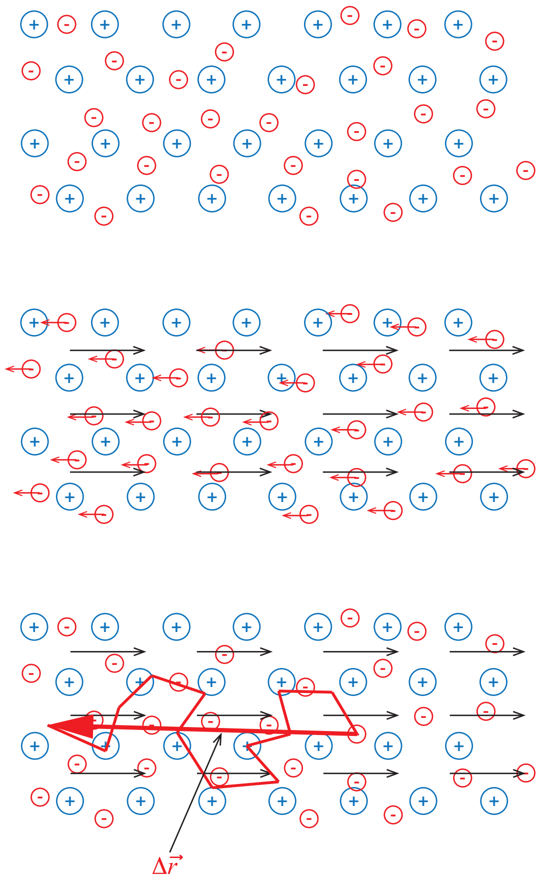


Fig. 8.2 (Top) Illustration of a metal with a sea of freely moving electrons and a vibrating lattice of positive ions. (Middle) An electric field is applied, indicated by black arrows, leading to force on all the electron illustrated by red arrow. (Bottom) The motion of an electron is due to both the force from the external electric field and forces from collisions (interactions) with the lattice ions, resulting in a directed random walk.

In Fig. 8.2 we have illustrated the forces on one electron when we apply an external electric field and the subsequent motion. It bounces across the system under the influence of both the force from the external field and due to forces from the ions making up the lattice, creating a motion of many small random steps, but with an overall drift in the direction of the force from the electric field. After a time Δt the charge has moved a distance $\Delta\mathbf{r}$ as illustrated in the figure. This gives rise to an *average* or *drift* velocity, where all the small fluctuations have been averaged out, $\mathbf{v}_d = \Delta\mathbf{r}/\Delta t$.

Let us now connect this picture with our definition of current by summing up the contributions from all the charge carriers in a small volume and finding how much charge moves through a surface $\mathrm{d}\mathbf{S}$ per unit time. We describe the charge carriers by N, the number of charge carriers per unit volume. The number of charge carriers in a small volume $\mathrm{d}v$ is then $N\mathrm{d}v$. We assume that each of these charge carriers have a charge q and move with a drift velocity $\mathbf{v}_d$. In a small time interval $\mathrm{d}t$, each charge moves a distance $\mathbf{v}_d\mathrm{d}t$. Fig. 8.3 illustrates the situation. All the charges that are in a volume $\mathrm{d}v = v_d\mathrm{d}t\cos\alpha\,\mathrm{d}S$ will have moved thorugh the surface $\mathrm{d}S$ in the time $\mathrm{d}t$, where α is the angle between the surface normal to $\mathrm{d}\mathbf{S}$ and the velocity $\mathbf{v}_d$ so that $\mathrm{d}\mathbf{S}\cdot\mathbf{v}_d = \mathrm{d}Sv_d\cos\alpha$ and $\mathrm{d}v = \mathbf{v}_d\mathrm{d}t\cdot\mathrm{d}\mathbf{S}$.

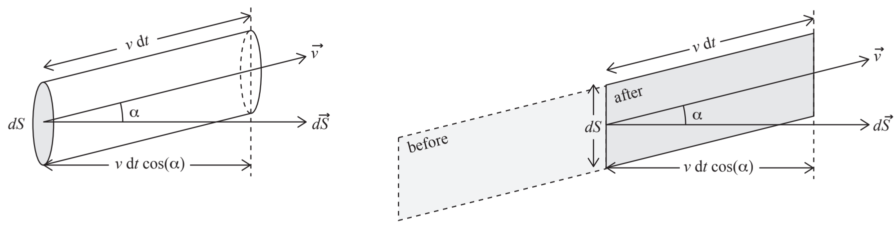


Fig. 8.3 Illustration of the volume $\mathrm{d}v = v\mathrm{d}t\mathrm{d}S\cos\alpha$ moving through a surface $\mathrm{d}\mathbf{S}$ in a time interval $\mathrm{d}t$.

The amount of charge in the volume $\mathrm{d}v$ is then $\mathrm{d}q = Nq\mathrm{d}v$. The amount of charge flowing through the surface $\mathrm{d}\mathbf{S}$ in the time interval $\mathrm{d}t$ is $Nq\mathbf{v}_d\mathrm{d}t\cdot\mathrm{d}\mathbf{S}$. This is the $\mathrm{d}q$ in the definition of the current $\mathrm{d}I$ through the surface $\mathrm{d}\mathbf{S}$:

$$\mathrm{d}I = \frac{\mathrm{d}q}{\mathrm{d}t} = \underbrace{Nq\mathbf{v}_d}_{\mathbf{J}}\cdot\mathrm{d}\mathbf{S} = \mathbf{J}\cdot\mathrm{d}\mathbf{S}\ , \tag{8.2}$$

where we introduce the term current density, $\mathbf{J}$, so that

$$\mathrm{d}I = \mathbf{J}\cdot\mathrm{d}\mathbf{S}\ . \tag{8.3}$$

The total current I through (an open) surface S is therefore the sum of all these elements $\mathrm{d}I$ integrated (summed) over the whole surface:

$$I = \int_S \mathbf{J}\cdot\mathrm{d}\mathbf{S}\ . \tag{8.4}$$

Current density

We define the current density $\mathbf{J}$ so that the current $\mathrm{d}I$ through a surface element $\mathrm{d}\mathbf{S}$ is

$$\mathrm{d}I = \mathbf{J} \cdot \mathrm{d}\mathbf{S} \; . \tag{8.5}$$

We can relate the current density to the volume density of charge carriers N, their charge q and their drift velocity $\mathbf{v}_d$ through

$$\mathbf{J} = Nq\mathbf{v}_d \; . \tag{8.6}$$

If there is more than one type of charge carrier, the current density is the sum of the contributions from all the charge carriers, $\mathbf{J} = \sum_i N_i q_i \mathbf{v}_{d,j}$.

8.2 Ohm's law and a microscopic theory for the drift velocity

Our intuition is that the drift velocity $\mathbf{v}_d$ must depend on the electric field, $\mathbf{E}$. How can we use this to find a relation between the current density $\mathbf{J}$ and the electric field? Here, we will introduce a microscopic theory for the motion of charge carriers and use this to derive Ohm's law: $\mathbf{J} = \sigma\mathbf{E}$, where σ is called the conductivity of the material.

Our goal is to relate the drift velocity $\mathbf{v}_d$ to a classical microscopic picture of the processes in a metal by making a *microscopic model* of the process. (Notice that the movement of electrons in a conductor may be better described using quantum mechanics). We will assume that the electrons move about like in a gas, but collide with the vibrating ions in the crystal. Between each collision, we will assume that the electrons are only affected by the force from the electric field. This means that the velocity $\mathbf{v}$ a time t after a collision, and before a new collision, is:

$$\mathbf{v} = \mathbf{v}_0 + (q/m)\mathbf{E}t \; , \tag{8.7}$$

where m is the mass of the electron, $q = -e$ is the charge of the electron, and $\mathbf{v}_0$ is the velocity after a collision. Let us assume that time between each collision is τ. We call this the *mean free time*. How far does it come between each collision? The displacement between two collisions is $\Delta\mathbf{r}$ (which is the integral of $\mathbf{v}$ over a time τ):

$$\Delta \mathbf{r} = \int_0^\tau (\mathbf{v}_0 + (q/m)\mathbf{E}t)\,\mathrm{d}t = \left(\mathbf{v}_0 + \frac{1}{2}\frac{q}{m}\mathbf{E}\tau\right)\tau \; . \tag{8.8}$$

Now, we take the average over many collision. (We use the symbols $\langle a \rangle$ to indicate a time average). The initial velocities, $\mathbf{v}_0$ are in random directions, thus their average is zero. Averaged over many collisions, we find:

$$\langle \Delta \mathbf{r} \rangle = \frac{1}{2}\frac{q}{m}\mathbf{E}\tau^2 \; , \tag{8.9}$$

and the average velocity, the *drift velocity*, is:

$$\mathbf{v}_d = \langle \mathbf{v} \rangle = \frac{\langle \Delta \mathbf{r} \rangle}{\tau} = \frac{q\tau}{2m}\mathbf{E} \; . \tag{8.10}$$

The current density is therefore:

$$\mathbf{J} = Nq\mathbf{v}_d = \frac{Nq^2\tau}{2m}\mathbf{E} = \sigma\mathbf{E} \; . \tag{8.11}$$

This relation is general for many types of materials, and is called *Ohm's law*:

$$\mathbf{J} = \sigma\mathbf{E} \tag{8.12}$$

Ohm's law (on local form)

Many real conductурs obey Ohm's law on local form:

$$\mathbf{J} = \sigma\mathbf{E} \; , \tag{8.13}$$

where σ is called the *conductivity*, $\mathbf{J}$ is the current density and $\mathbf{E}$ is the electric field. It tells us how well a material conducts (electrical) current. For an insulator $\sigma = 0$, while for a superconductor / ideal conductor $\sigma = \infty$.

Notice, that the *resistivity* ρ is $\rho = 1/\sigma$, which is what is often found in tables. The conductivity σ depends strongly on the temperature in a material. (In statistical physics you will learn about theories for the temperature dependence of σ based on quantum mechanics.)

8.2.1 Conductors and resistors

You may now be confused because we earlier said that the electric field had to be zero inside an ideal conductor. Now we have to modify this picture. For good conductors such as metals, the electric field needed to drive an electric current is so small, that we will usually assume that the voltage drop along a real conductor is negligible: We will assume that the potential is constant along the conductor. However, non-conductors, such as semi-conductors and isolators, are made of poorly conducting materials, where there is a non-negligible field needed to drive a current and where we cannot ignore the potential drop along the insulator. We will usually model systems as combinations of *conductors*, which are the ideal conductors with no potential drop, and *resistors*, which is our model of the parts of a system that has a non-negligible resistivity.

We usually group materials into *conductors*, *semi-conductors*, and *insulators*. The following table provides the resistivities for various common conductors (C), semi-conductors (S) and insultars (I) as 20° (Data from Handbook of Chemistry and Physics, 78th ed.)

Material	ρ (Ohm m)	Material	ρ (Ohm m)
Silver (C)	1.59×10^{-8}	Salt water (S)	4.4×10^{-2}
Copper (C)	1.68×10^{-8}	Diamond (S)	2.7
Gold (C)	2.21×10^{-8}	Silicon (S)	2.5×10^{3}
Aluminum (C)	2.65×10^{-8}	Water (I)	2.5×10^{5}
Iron (C)	9.61×10^{-8}	Wood (I)	$10^{8} - 10^{11}$
Graphene (C)	1×10^{-8}	Glass (I)	$10^{10} - 10^{14}$

8.3 Resistance

Let us apply this theory to find the current through a wire with a given conductivity σ, constant cross-sectional area S and length L as illustrated in Fig. 8.4. One side is at potential V and the other at $V = 0$. The wire has the same shape and cross-sectional area all across.

In order to find the current, we need to find the current density $\mathbf{J}$ and integrate across the cross-section to find the current. How can we find the current density and relate it to the potential difference V? Our plan is to use Ohm's law to relate the current density to the electric field through Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, and then use this to relate the current to the potential difference. We can do this by either first finding the electric field and then use this to find the current density, or by first finding the

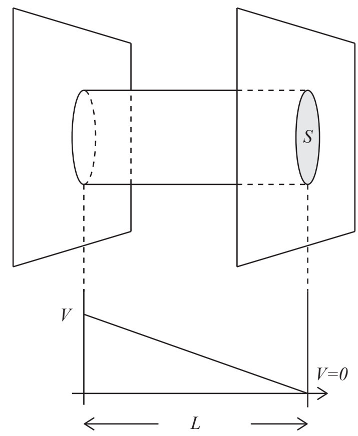


Fig. 8.4 A wire with conductivity σ and a constant cross sectional area S and length L is subject to a potential difference V.

current density using symmetry consideration and use this to find the electric field. We will here demonstrate both these methods.

8.3.1 Method: From electric field to current density

Our plan is to find the electric field, and then use the electric field to find both the current density and the potential difference. But how can we find the electric field? It may be tempting to assume that the component is similar to a parallel-plate capacitor. If the cross-sectional area S is large compared to the spacing L, we would know that the electric field would be uniform *when there are no moving charges*. However, we now have a different situation. The length L is not small compared to e.g. the diameter $a = 4\sqrt{S}/\pi$, and are we sure we can use the same types of arguments when charges are moving as when charges are stationary? We demonstrate in an example below that the electric field inside the conductor indeed is uniform also in this case, as for the parallel-plate capacitor, but for different reasons. (We can still use Laplace's equation, but we need to apply different boundary conditions).

We place the system so that L is along the x-axis. For a uniform field with a potential difference V, we see that the electric field is $E = V/L$ from $\mathbf{E} = -\nabla V$, and $E_x = -\partial V/\partial x = \Delta V/\Delta x = V/L$. Ohm's law gives the current density as $J_x = \sigma E_x = \sigma V/L$, which also is uniform in space. The current across the area S is then:

$$I = \int_S \mathbf{J} \cdot \mathrm{d}S\hat{\mathbf{x}} = \int_S J_x \mathrm{d}S = \sigma(V/L) \int_S \mathrm{d}S = \sigma(V/L)S \; . \tag{8.14}$$

We can now express the potential difference in terms of the current:

$$V = \underbrace{\frac{L}{\sigma S}}_{=R} I \tag{8.15}$$

The voltage (difference) V is therefore proportional to the current I. We call this constant of proportionality the *resistance* R. We have now recovered Ohm's law for a resistor:

Ohm's law for a resistor

The voltage drop V across a resistor is given as

$$V = RI \; , \tag{8.16}$$

where R is a property of the resistor called the **resistance**. The resistance for a specific resistor can be calculated or measured experimentally. Resistance is measured in Ohm which is defined as Volt/Ampere. This is the law you may recall as Ohm's law. To differentiate it from $\mathbf{J} = \sigma \mathbf{E}$, we will call $V = RI$ Ohm's law and $\mathbf{J} = \sigma \mathbf{E}$ Ohm's law on local or microscopic form.

8.3.2 Method: From current density to electric field

An alternative approach to find the resistance R for a resistor is to assume a particular symmetry for the current density and then use this to find an expression for the electric field and the potential difference.

For this particular problem we realize that the system is the same for any cross-section along the component. We therefore expect the current density $\mathbf{J}$ also to be uniform along the x-axis. Similarly, the conductor is the same at any point in the cross section, so expect the current density also to be uniform across the whole cross section. There are no currents normal to the boundaries, as this would mean that current is leaking from the conductor out into the vacuum. This combined with the requirement that the current density is uniform, means that the current density is $\mathbf{J} = J_x \hat{\mathbf{x}}$ everywhere inside the conductor and zero outside the conductor.

Ohm's law on local form then gives us the electric field as $\mathbf{E} = \mathbf{J}/\sigma = (J_x/\sigma)\hat{\mathbf{x}}$. We can then find the potential difference from the left (0) to the right (1) across the length L by

$$V_1 - V_0 = \int_0^1 \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_0^L \frac{J_x}{\sigma} dx = \frac{J_x}{\sigma} L \; . \tag{8.17}$$

which gives $J_x = V\sigma/L$, where the potential difference $V = V_1 - V_0$. We can also relate the current density to the current by

$$I = \int_S \mathbf{J} \cdot \mathrm{d}S\hat{\mathbf{x}} = \int_S J_x \mathrm{d}S = J_x \int_S \mathrm{d}S = J_x S \; . \tag{8.18}$$

So that $J_x = I/S$. We equate these two expressions for the current density, getting:

$$\frac{V\sigma}{L} = \frac{I}{S} \;\Rightarrow\; V = \frac{L}{\sigma S} I = RI \; . \tag{8.19}$$

We have therefore found the same result as above, but with a different starting point. The method we have used here is very similar to the method we used to find the capacitance for a capacitor.

8.4 Method: Calculating the resistance

Finding the resistance of a resistor from first principles in electromagnetism is a process you should understand and a skill that you should learn. From these examples we see the pattern of a general method for calculating the resistance of a resistor — following an approach which is very similar to what we developed for a capacitor. (We use the numbering from the capacitance method, which is the opposite of the order in which we solve the problem above).

Method: Finding the resistance of a system

Method 1: From current density to field to potential:

1. We assume that the current density, $\mathbf{J}$, has a particular symmetry or form inside the conductors based on considerations of the physics and geometry of the system.
2. We calculate the current, I, through a cross-section, S, of the conductor using $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$.
3. We calculate the electric field $\mathbf{E}$ using Ohm's law on local form: $\mathbf{E} = \mathbf{J}/\sigma$.

4. We calculate the potential difference between the two ends of the conductor, $V = \int_0^1 \mathbf{E} \cdot \mathrm{d}\mathbf{S}$, using the electric field we found.
5. We find the resistance from $R = V/I$, where we insert our value for V and the value for I we found from the current density.

Method 2: From potential to field to current:

1. We solve Laplace's equation to find the potential $V(\mathbf{r})$ for the system. Boundary conditions are given by assuming that there is no flow normal to the conductors.
2. We find the electric field $\mathbf{E} = -\nabla V$.
3. We find the current density from Ohm's law on local form $\mathbf{J} = \sigma\mathbf{E}$. We sum/integrate the current density to find the current through a cross-section S through the conductor: $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$.
4. We find the resistance from $R = V/I$, where we insert our value for I and the potential difference $V = V_1 - V_0$ we started with.

We will mostly use method 2 in this text. Method 1 is generally more difficult, as you need to solve Laplace's equation with boundary conditions that can be complex for complex geometries.

8.4.1 Example: Spherical resistor

Let us now use these methods to solve a more complicated problem: What is the resistance of a system consisting of a spherical shell resistor? The situation is illustrated in Fig. 8.5.

Our plan is to use method 2 to address this problem. We assume that a constant, stationary current I runs from the inner to the outer side of the spherical shell. Because the system has a spherical symmetry, we will assume that the current density also has spherical symmetry so that $\mathbf{J} = J_r \hat{\mathbf{r}}$. (We do not expect any currents in the Φ or θ directions). The amount of charge per unit time $I = dq/dt$, through a spherical shell of radius r, where $R_1 < r < R_2$, must be the same for all r, otherwise charge would disappear. We calculate the current from the current density:

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 J_r \ . \tag{8.20}$$

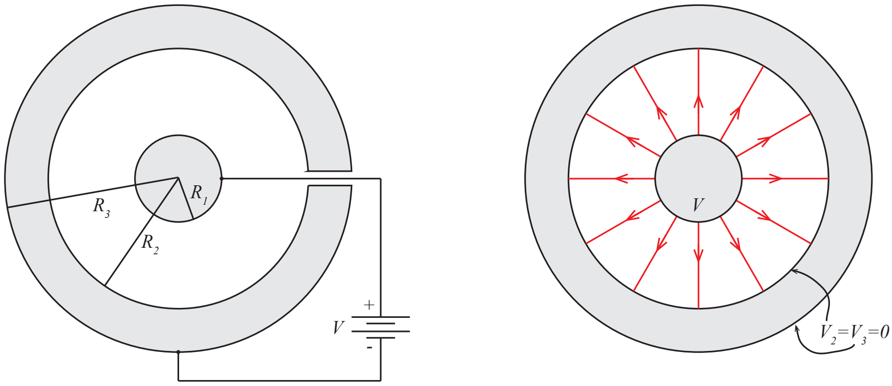


Fig. 8.5 Illustration of a cross-section through a resistor shaped as a spherical shell with inner radius R_1 and outer radius R_2. A potential difference V is applied across the shell.

and the current density is

$$\mathbf{J} = \frac{I}{4\pi r^2}\hat{\mathbf{r}} \;\text{ for } R_1 < r < R_2 \tag{8.21}$$

The electric field is found from Ohm's law on local form: $\mathbf{J} = \sigma\mathbf{E}$ and therefore $\mathbf{E} = \mathbf{J}/\sigma$

$$\mathbf{E} = \frac{I}{4\pi\sigma r^2}\hat{\mathbf{r}} \; . \tag{8.22}$$

We find the potential difference, V, from the electric field, which is the potential difference between the shell and the sphere, $V(R_2) = 0$ and $V(R_1) = V$. Hence we find

$$V = V(R_1) = \int_{R_1}^{\infty} \mathbf{E}\cdot d\mathbf{r} = \int_{R_1}^{R_2} \frac{I}{4\pi\sigma r^2} dr = -\frac{I}{4\pi\sigma}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \; . \tag{8.23}$$

Now, we are ready to relate V to I, finding that

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \; . \tag{8.24}$$

(Notice how this reminds us of what we did for the spherical capacitor. Indeed this is a general result, that there is a relation between capacitance and resistance of a system. We can see this by looking at the current through a spherical surface, S, $I = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_S \sigma\mathbf{E}\cdot d\mathbf{S} = (\sigma Q/\epsilon)$, thus $I/V = (\sigma/\epsilon)(Q/V)$, where $G = I/V$ is the conductance and C is the capacitance! This is a nice duality relationship.)

8.4.2 Example: Using Laplace's equation to find the field

Let us revisit the system in Fig. 8.4: A resistor with a constant cross-sectional area S of length L and let us assume that the cross-section is a circle of radius a, $S = \pi a^2$, so that the resistor is a cylinder of radius a and length L. How could we apply method 1 to this system? We would need to solve Laplace's equation within the cylinder.

What are the boundary conditions for Laplace's equation? The potential is constant V_0 and V_1 at boths ends of the cylinder, where we can set $V_0 = 0$. What about the outer surface of the cylinder? We know that there cannot be any charges flowing (leaking) from the resistor into the vacuum outside it. This means that the current density cannot have a component normal to the outer surface, that is, $\mathbf{J} \cdot \hat{\mathbf{n}} = 0$. Using Ohm's law on local form we therefore get that $\mathbf{E} \cdot \hat{\mathbf{n}} = 0$, which again means that $\partial V/\partial n = 0$ on the outer surface of the cylinder. This means that either V or its derivative is specified on all the surfaces. The potential V is therefore uniquely determined.

We therefore only need to find *a* solution and it will be *the* solution. For this problem, it is easy to guess the solution. If the cylinder is oriented along the x-axis, the potential $V = V_1 x/L$ satisfies the boundary conditions.

We can then find the electric field:

$$\mathbf{E} = -\nabla V = -\frac{V_1}{L}\hat{\mathbf{x}} \, , \tag{8.25}$$

which is uniform and has the form we assumed above. Notice that this problem can be solved because we assume that all the current is in the conducting material only (inside the resistor). It would be much more difficult to solve Laplace's equation without the boundary condition $\partial V/\partial n = 0$ on the surface, but instead use that the potential is zero at infinity.

8.5 Energy loss

As the electrons (or any charge carrier) moves along the conductor due to the electric field, they collide with vibrating lattice ions in the the underlying crystal as illustrated in Fig. 8.2. This will transfer energy from the electrons to the lattice vibrations, increasing the thermal energy of the metal (and thus the temperature of the metal). But how much energy is dissipated?

Let us see what happens in a small volume unit dv with Ndv charge carriers, where N is the charge carrier volume density. During a small time interval dt, the charges in this volume will move a length $\mathrm{d}\mathbf{r} = \mathbf{v}_d \mathrm{d}t$. The electric field acts on the electrons. The force on each electron is therefore $\mathbf{F} = q\mathbf{E}$ (where $q = -e$). The work, dW_1, done by the electric field on one charge q over the time interval dt is

$$\mathrm{d}W_1 = q\mathbf{E} \cdot \mathrm{d}\mathbf{r} = q\mathbf{E} \cdot \mathbf{v}_d \mathrm{d}t \tag{8.26}$$

and the work, dW, done an all Ndv charges, each with a charge q, is therefore:

$$\mathrm{d}W = N\mathrm{d}v\mathrm{d}W_1 = N\mathrm{d}v\, q\mathbf{E} \cdot \mathbf{v}_d \mathrm{d}t = \underbrace{Nq\mathbf{v}_d}_{=\mathbf{J}} \cdot \mathbf{E}\mathrm{d}v\mathrm{d}t \ , \tag{8.27}$$

The power loss, P, is $P = \mathrm{d}W/\mathrm{d}t = \mathbf{J} \cdot \mathbf{E}\mathrm{d}v$ and the power loss per unit volume, p_J, is therefore:

$$p_J = \frac{\mathrm{d}W}{\mathrm{d}v} = \mathbf{J} \cdot \mathbf{E} \tag{8.28}$$

We call this the power loss, because this is the work the electric field does to move the electrons. This work is therefore converted into thermal energy, and thus "lost" from the electric system. We often call this the dissipated energy, and you can feel its effect by an increase in temperature in a system.

Power loss

The power loss per unit volume, p_J, is

$$p_J = \mathbf{J} \cdot \mathbf{E} \ . \tag{8.29}$$

We find the total power loss, P, for a system (such as a resistor) by integrating the power loss per unit volume:

$$P = \int_v p_J \mathrm{d}v = \int_v \mathbf{J} \cdot \mathbf{E}\mathrm{d}v \ . \tag{8.30}$$

For a resistor, we can relate the work dW done on the charge $\mathrm{d}Q = qN\mathrm{d}v$ to the electric potential: The work done on a charge dQ as it is moved from one side of the resistor with potential V_1 to the other side with potential V_0 is $\mathrm{d}W = \mathrm{d}Q(V_1 - V_0) = \mathrm{d}QV$, where V is the potential drop across the resistor. For a constant V and I, the stored electric energy will be unchanged, so this energy is dissipated (converted to heat). The power loss is therefore:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}QV}{\mathrm{d}t} = V\frac{\mathrm{d}Q}{\mathrm{d}t} = VI \; . \tag{8.31}$$

Power loss for a component

The power loss for a resistor with a current I and potential drop V is

$$P = VI \; . \tag{8.32}$$

We often only call this the *power*, the *power loss* or the *power dissipated* in the component.

This is called the **Joule heating law** . For a resistor with resistance R, we know that $V = RI$ so that $P = RI^2$ or $P = V^2/R$.

8.5.1 Example: Power in a spherical resistor

What is the power loss in the spherical resistor illustrated in Fig. 8.5?

Let us compare the two methods: We find the power by either integrating the power loss per unit volume across the resistor or by using the expression for the power loss, P.

The power loss per unit volume is $p = \mathbf{E} \cdot \mathbf{J}$. We found that $\mathbf{J} = I/(4\pi r^2)\hat{\mathbf{r}}$ and that $\mathbf{E} = \mathbf{J}/\sigma$. Then

$$p_J = \mathbf{J} \cdot \mathbf{E} = \frac{1}{\sigma}\left(\frac{I}{4\pi r^2}\right)^2 \; . \tag{8.33}$$

We find the total power dissipated in the component by integrating over the volume from R_1 to R_2. A volume component $\mathrm{d}v = 4\pi r^2 \mathrm{d}r$. The integral is:

$$P = \int_v p_J \mathrm{d}v \tag{8.34}$$
$$= \int_{R_1}^{R_2} \frac{1}{\sigma} \left(\frac{I}{4\pi r^2} \right)^2 4\pi r^2 \mathrm{d}r \tag{8.35}$$
$$= \frac{I^2}{4\pi\sigma} \int_{R_1}^{R_2} \frac{\mathrm{d}r}{r^2} \tag{8.36}$$
$$= \frac{I^2}{4\pi\sigma} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tag{8.37}$$
$$= I^2 R \,, \tag{8.38}$$

where we have used the result we found above that $R = 1/(4\pi\sigma)(\,(1/R_1) - (1/R_2)\,)$. This means that we have demonstrated that the result we get is the same as using $P = I^2 R$ directly.

8.6 Conservation of charge

We know that the net charge is conserved. This is a fundamental principle. Charges can arise, as when an electron is emitted from a neutral atom, creating a negative electron ($-e$) and a positive ion ($+e$), but the net change in charge is zero. Similarly, a positron with charge $+e$ and an electron with charge $-e$ may combine form neutral photons (light), but the net charge remains zero. What consequences does the conservation of charge have for the flow of current? It means that charges can move, in the form of current, but they do not appear or disappear.

Let us formulate this conservation law — *the conservation of charge* — mathematically and apply it to discuss current. A volume v is enclosed by the surface S as illustrated in Fig. 8.6a. The charge Q inside the volume is the integral of the charge density, ρ:

$$Q = \int_V \rho \mathrm{d}v \,, \tag{8.39}$$

and the current out of the surface S is the integral of the current density over the surface S:

$$I_S = \oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \,. \tag{8.40}$$

In a small time interval $\mathrm{d}t$ the total change in charge inside the surface, $\mathrm{d}Q$ must correspond to the charge flowing through the surface in this time interval, $I_S \mathrm{d}t$. The charge is reduced (the change is negative), if the flow out of the surface is postive:

$$-\,\mathrm{d}Q = I_S \mathrm{d}t \;\Rightarrow\; -\frac{\mathrm{d}Q}{\mathrm{d}t} = I_S \; . \tag{8.41}$$

As always we check the sign to ensure it is correct: We notice that the integral over the surface tells us the amount of charge flowing *out of the surface*, which must correspond to a reduction in the charge inside, $-\mathrm{d}Q$. This is a formulation of the principle of conservation of charge.

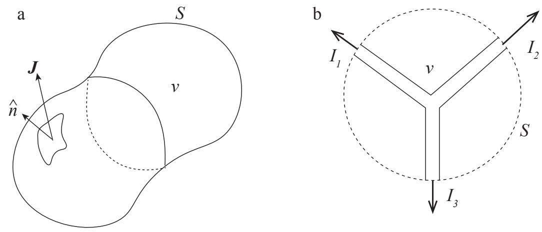


Fig. 8.6 (a) Illustration of a volume v enclosed by the surface S. (b) Illustration of a the currents I_1, I_2 and I_3 flowing in a conductor through the closed surface S.

We can rewrite the conservation of charge in integral form, by using the expression for Q from (8.39):

$$\oint_s \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{d}{\mathrm{d}t} \int_v \rho \mathrm{d}v \; , \tag{8.42}$$

If we study systems where all the currents are stationary, that is independent of time, then there cannot be a current into a volume or out of a volume forever, because this would lead to an infinite build-up (removal) of charge. Thus for a stationary system we have that:

$$\oint_s \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0 \; . \tag{8.43}$$

Fig. 8.6 illustrates a similar situation, but where all the currents are localized to a conductor such as a copper wire. In this case, the total current I_S out through the surface S is the sum of the currents $I_S = I_1 + I_2 + I_3$, where the currents I_1, I_2 and I_3 are positive in the direction out of the surface. For a stationary system, where $\mathrm{d}Q/\mathrm{d}t = 0$, conservation of charge then implies that

$$\sum_i I_i = 0 \; . \tag{8.44}$$

We call this law *Kirchoff's current law.* This is only valid for constant currents. When the currents are time dependent, it is possible to build up charge inside a volume, such as on a capacitor.

Conservation of charge

For a volume v enclosed by a surface S the net charge flowing out through the surface S must correspond to the reduction in charge inside the volume v:

$$\oint_s \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{d}{dt} \int_v \rho \mathrm{d}v \ . \tag{8.45}$$

For a stationary system, where all the currents are constant and there is no build-up or removal of charges, we get **Kirchoff's current law**:

$$\oint_s \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0 \ . \tag{8.46}$$

For an intersection of n conductors (wires), **Kirchoff's current law** becomes:

$$\sum_{i=1}^{n} I_i = 0 \ . \tag{8.47}$$

where each current I_i is the current *out of* the intersection (or *into* the intersection — the point is that they must all be either into or out of the intersection).

8.6.1 Example: Combining resistors

How can we find the effective resistance of a resistor that consists of several parts? Let us demonstrate this through two examples: Two resistors in series and two resistors in parallel.

Resistors in series. Fig. 8.7 illustrates two resistors A and B with resistances R_A and R_B placed after each other in *series*. They are connected to an ideal conductor on the left side and on the right side and they are joined by an ideal conductor as illustrated in the figure. What is the effective resistance of this combined system, that is, if we replaced this system with a single resistor, what would be the resistance R of this resistor?

The effective resistance R relates the voltage across the whole system, $V_2 - V_0 = V$, and the current, I, through the system: $R = V/I$. In a stationary state, when no charges are building up inside the system, the current through any cross-section must be the same. The current is therefore the same through both component A and B: $I_A = I_B = I$. The voltage drop across component A is $V_A = V_2 - V_1$ and the voltage drop across component B is $V_B = V_1 - V_0$. For each of the components we know that $V_A = R_A I_A = R_A I$ and $V_B = R_B I_B = R_B I$. We therefore see that the total voltage drop, V is:

$$V = V_2 - V_0 = V_A + V_B = R_A I + R_B I = \underbrace{(R_A + R_B)}_{=R} I = RI \ , \quad (8.48)$$

where $R = R_A + R_B$ is the effective resistance for the whole component. We can expand the same reasoning to a seqeuence of n resistors R_i with effective resistance $R = \sum_i R_i$.

Resistors in parallel. Fig. 8.7 illustrates two resistors A and B with resistances R_A and R_B placed in parallel. The are connected to an ideal conductor on the left and the right side. What is the effective resistance R of this combined system?

Because both resistors are connected to an ideal conductor on the left and on the right side, the potential on the left side is the same for both resistors and the potential on the right side is the same for both resistors. The potential drops across the resistors, V_A and V_B, are therefore the same, $V_A = V_B = V$. What about the current? Due to the conservation of current, we know that the current I flowing through the whole system must be the sum of the currents in each of the resistors. (You can also argue for this using Kirchoff's laws of current: The current flowing out of the left junction, that is the left side of the resistors, is I_A, I_B and $-I$, so that $-I + I_A + I_B = 0$ according to Kirchoff's law, and $I = I_A + I_B$). The voltage drop across resistor A is then $V_A = V = I_A R_A$ and the voltage drop across resistor B is $V_B = V = I_A R_B$. The currents are therefore $I_A = V/R_A$ and $I_B = V/R_B$. The total current I is:

$$I = I_A + I_B = \frac{V}{R_A} + \frac{V}{R_B} = \underbrace{\left(\frac{1}{R_A} + \frac{1}{R_B}\right)}_{=R} V = RV \ , \quad (8.49)$$

where $1/R = 1/R_A + 1/R_B$ is the effective resistance for the whole component. We can expand the same reasoning to a seqeuence of n resistors R_i with effective resistance $1/R = \sum_i 1/R_i$.

More advanced combinations. We can address more advanced combinations by grouping the system into parts that are connected in

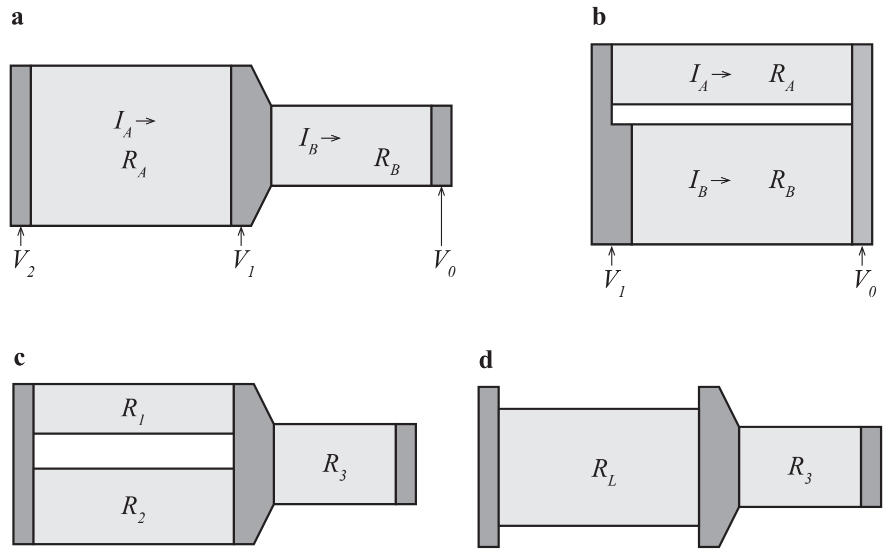


Fig. 8.7 Illustration of (a) a system of two resistors in series, (b) a system of two resistors in parallel, (c) a system of three resistors, (d) the same system but where the left two resistors in parallel have been combined to a new resistor with resistance R_L.

series or parallel. For example for the system illustrated in Fig. 8.7, we can first combine the two resistors in parallel on the left into one resitors $R_L = (1/R_1 + 1/R_2)^{-1}$ and then combine this new resistor R_L with the other resistor R_3 in series to find the total resistance $R = R_L + R_3 = (1/R_1 + 1/R_2)^{-1} + R_3$ as illustrated in the figure. (Notice that it would *not* be correct to group resistors R_1 and R_3 into one resistor in series and then combine this with resistor R_2 in parallel.)

Combinations of resistors

Series coupling. The effective resistance R of n resistors R_i connected in series is

$$R = \sum_i R_i \quad \text{(Series coupling)} \tag{8.50}$$

Parallel coupling. The effective resistance R of n resistors R_i connected in parallel is

$$\frac{1}{R} = \sum_i \frac{1}{R_i} \quad \text{(Parallel coupling)} \tag{8.51}$$

Combinations. You can find the effective resistance of any combination of resistors in both series and parallel by grouping the resistors hierarchically into groups that are connected in series or parallel.

8.6.2 Conservation of charge on differential form

We can rewrite the principle of conservation of charge on differential form in order to make it more generally applicable and so that you may recognize its similarity to other conservation laws. For a volume v enclosed by the surface S, the conservation of charge within the volume is formulated as:

$$\oint_s \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{d}{dt} \int_v \rho \mathrm{d}v \; . \tag{8.52}$$

We can reformulate the left side using the divergence theorem:

$$\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{J} \mathrm{d}v \; . \tag{8.53}$$

We combine this to get:

$$\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{d}{dt} \int_v \rho \mathrm{d}v = \int_v \nabla \cdot \mathbf{J} \mathrm{d}v \; . \tag{8.54}$$

This is valid for any volume v, which means that the equality also must hold for the arguments inside the integrals:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \; . \tag{8.55}$$

And when ρ is a constant (in time), we get

$$\nabla \cdot \mathbf{J} = 0 \; . \tag{8.56}$$

Continuity equation for charges

The **continuity equation** for charges, which formulates the principle of conservation of charge, is on differential form:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \; . \tag{8.57}$$

8.6.3 Laplace equation for conducting materials

Why can we use Laplace equation to find the potential inside a conductor? There are moving charges inside the conductor, so why can we assume that the charge density is zero?

For the case of non-ideal conductors, we can derive Laplace equation in a different way, by using the conservation of charges instead. If we address a stationary situation, then the conservation of charges state that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \; . \tag{8.58}$$

The term $\partial \rho / \partial t$ must be zero for a stationary state, otherwise charges will build up indefinitely inside the system. This means that

$$\nabla \cdot \mathbf{J} = 0 \; . \tag{8.59}$$

If we combine this with two other features: (i) that the current density is related to the electric field through Ohm's law: $\mathbf{J} = \sigma \mathbf{E}$, and (ii) that the electric field can be found from a potential $\mathbf{E} = -\nabla V$. We insert this in (8.59) and find:

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\sigma (-\nabla V)) = -\sigma \nabla^2 V = 0 \quad \Rightarrow \quad \nabla^2 V = 0 \; , \tag{8.60}$$

where we have assumed that the conductivity σ is uniform. We see that we recoved Laplace equation for the electric potential. We did not assume that the charge density was zero, but instead assumed that the conductity was uniform, that Ohm's law holds, and that the electric field could be found from a potential.

8.6.4 Comparison with continuum mechanics

(You may skip this part without any loss of continuity.)

If you have had continuum mechanics it may be instructional to observe the similarities between conservation of charge, the conservation of mass, and the conservation of energy (heat).

In the case of diffusion, we describe the diffusing material with a concentration $c(\mathbf{r}, t)$, which is the number of particles (atoms) of a particular material per unit volume or, alternatively, the mass of a type of particle per unit volume. The flux (current density) of the the particles is usually formulated using Fick's law, $\mathbf{j} = -D \nabla c$, where D is called the diffusion constant. The amount of particles (mass) in a volume v is then

$\int_v c \mathrm{d}v$, and the conservation of particles (mass) c can then be written as

$$\oint_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t} \int_v c \mathrm{d}v \tag{8.61}$$

on integral form. We can again rewrite the left side using the divergence theorem:

$$\oint_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{j} \mathrm{d}v \ . \tag{8.62}$$

On differential form we therefore get

$$\nabla \cdot \mathbf{j} = -\frac{\partial c}{\partial t} \ . \tag{8.63}$$

If we insert Fick's law, $\mathbf{j} = -D\nabla c$, we get the diffusion equation:

$$\frac{\partial c}{\partial t} = \nabla \cdot D \nabla c \ , \tag{8.64}$$

which simplifies to

$$\frac{\partial c}{\partial t} = D \nabla^2 c \ , \tag{8.65}$$

when D is uniform in space. In the stationary case, when $\partial c / \partial t = 0$, we get Laplace's equation, $\nabla^2 c = 0$.

8.7 Summary

The **current** I passing through a surface S is the amount of charge $\mathrm{d}q$ per unit time $\mathrm{d}t$ passing through the surface: $I = \mathrm{d}q/\mathrm{d}t$. Current is measured in **Ampere** which is a basic unit in the SI system.

We describe the motion of charges in space with the **current density** $\mathbf{J}(\mathbf{r})$. The current $\mathrm{d}I$ passing through a surface element $\mathrm{d}\mathbf{S}$ is

$$\mathrm{d}I0 = \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

and the current I passing through a surface S is

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \ .$$

Most materials obeys **Ohm's law** on microscopic form:

$$\mathbf{J} = \sigma \mathbf{E} \; , \tag{8.66}$$

where σ is a (temperature dependent) material property called the conductivity. The resistivity is the inverse quantity, $\rho = 1/\sigma$.

The **resistance** C of a resistor (an non-ideal condutor or an isolator) is defined as $R = V/I$, where V is the voltage drop across the resistor when a current I passing through it.

Ohm's law for a resistor is that the voltage drop across the resistor with resistance R is

$$V = RI \; ,$$

when a current I is passing through the resistor.

The **resistance** R **of a cylindrical resistor** with cross-sectional area A, length L and conductivity σ is $R = L/(A\sigma)$.

We find the **resistance** of a system using two main methods. In **method 1**, we assume that the current density $\mathbf{J}$ has a particular symmetry and use this to find the current I, the electric field $\mathbf{E}$, the potential difference V from the electric field, and then the resistance from $R = V/I$. In **method 2**, we find the potential $V(\mathbf{r})$ from e.g. Laplace's equation, the electric field from the potential, $\mathbf{E} = -\nabla V$, the current density from the electric field, $\mathbf{J} = \sigma\mathbf{E}$, the current from the current density, $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ and the resistance from $R = V/I$.

We can **combine resistors** $R_1, R_2, \ldots, R_n$, to one effective resistor R by combining resistors *in series* getting $R = \sum_i R_i$ or *in parallel* getting $1/R = \sum_i 1/R_i$.

The **power loss** due to resistance is $p_J = \mathbf{J}\cdot\mathbf{E}$. The power loss in a volume v is $P = \int_v p_J \mathrm{d}v$. The power loss for a resistor is $P = VI = RI^2 = V^2/R$.

The **conservation of charges** demand that $\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\int_v \rho \mathrm{d}v$ on integral form and $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$, where ρ is the volume charge density.

In a **stationary system** with no build-up of charges $\nabla \cdot \mathbf{J} = 0$ and **Kirchoff's current law** $\sum_i I_i = 0$ is valid for any junction.

8.8 Exercises

Learning outcomes. (1) Understand and use the current density and Ohm's law to find the resistance of a system. Here we will do this in a structured way for a simple system. (2) Use conservation of charge and Kirchoff's laws. Here we will use this to address flow through various cross sections of a conductor, and we will use this to address properties of current densities $\mathbf{J}(\mathbf{r})$.

8.8.1 Test yourself

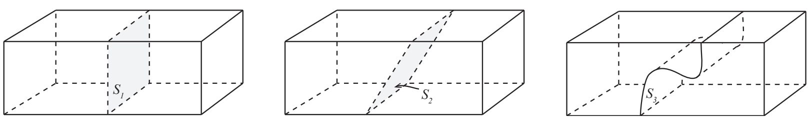
Exercise 8.1: Strøm av ioner

Positive ioner strømmer til høyre gjennom en væske, negative ioner strømmer til venstre. Den romlige tettheten og hastigheten til begge typer ioner er identiske. Er det en netto strøm gjennom væsken? (A) Ja, til høyre; (B) Ja, til venstre, (C) Nei, (D) Ikke tilstrekkelig informasjon.

8.8.2 Discussion exercises

Exercise 8.2: Strømtetthet og strøm

Du har en lang stav med kvadratisk tverrsnitt og det går en strøm med homogen strømtetthet rettet langsmed staven. Er strømmen gjennom flaten S_1 den samme som gjennom flaten S_2?



Exercise 8.3: Ladningsoppsamling

Er netto strøm av ladning ut av en lukket overflate alltid null? Hvis ikke, kan du komme med noen moteksempler?

8.8.3 Tutorials

Exercise 8.4: Current density

a) For a homogeneous current density $\mathbf{J} = J_0\hat{x}$, what is the current through all the surfaces of a cube with corners at $(-1,-1,-1)$,

$(-1,+1,-1)$, $(+1,-1,-1)$, $(+1,+1,-1)$, $(-1,-1,+1)$, $(-1,+1,+1)$, $(+1,-1,+1)$, and $(+1,+1,+1)$ all measured in units of a.

b) A block with uniform charge density ρ is moving with a constant velocity $\mathbf{v} = v_0\hat{z}$. What is the current density $\mathbf{J}$ at a specific point in space inside the block (the point does not move along with the block). What would the current density $\mathbf{J}$ be if the point moved along with the block?

Exercise 8.5: Cylindrical resistor

We will now look at charges that are leaking across the cylindrical membrane of a part of the axon of length L. We will assume that the myelin sheath is leaky (a conductor with conductivity σ) and that there is a radially-symmetric current density $\mathbf{J}$ leaking across it due to a potential difference. We will find the resistance R of a piece of the cylinder of length L. We assuming that the inner part of the cylidrical shell is connected to a potential V and that the outer part of the shell is connected to a potential $V = 0$.

We will find the resitance of this system by (i) making a drawing of the system, (ii) drawing the expected current density, (iii) finding the electric field for this current density, (iv) finding the potential by integrating the electric field, and (v) finding the resistance by relating the current to the potential difference. (Notice the similarity of this method and the method you used to find the capacitance!)

a) Make a drawing of the cylindrical system. Sketch the current density.

b) Given that there is a radially-symmetric current density, find the current density $\mathbf{J}$ expressed in terms of the total current I.

Hint. Integrate the current density along a cylindrical surface.

c) Find the electric field $\mathbf{E}$ by using the current-density version of Ohm's law.

d) Use the electric field to relate the current I to the voltage difference between the inner $V(r = a)$ and outer $V(r = b) = 0$ surfaces of the axon.

e) Find the resistance of the cylinderical shell. Is this the resistance for flow along the axon?

f) Interpret these expressions. Does the current you found actually depend on the length of the axon? Does the resistance?

8.8.4 Homework

Exercise 8.6: Lightning

Lightning strikes in one end of a lightning rod of steel and induces a current of 30000 A which lasts for 65 μs. The lightning rod is a 1 m long and 2 cm in diameter, and the other end is connected to the ground through a 40 m copper wire with a diameter of 5 mm. The conductivity of steel and copper is respectively $\sigma_{\text{steel}} = 5.0 \cdot 10^6 \Omega^{-1}\text{m}^{-1}$ og $\sigma_{\text{copper}} = 5.8 \cdot 10^7 \Omega^{-1}\text{m}^{-1}$.

a) Find the potential difference between the top of the lightning rod and the bottom of the copper wire as the current is passing through.

b) Find the total energy dissipated in the lightning rod and the copper wire from the lightning strike.

Exercise 8.7: Sfærisk symmetrisk strøm

Vi skal nå se på to konsentriske kuleskall, laget av metall, med radius a og b der $(a < b)$. I sjiktet mellom kuleskallene er det et svakt ledende materiale med konduktivitet σ. Husk at konduktivitet er definert som $\sigma \equiv 1/\rho$, der ρ er resistiviteten til materialet.

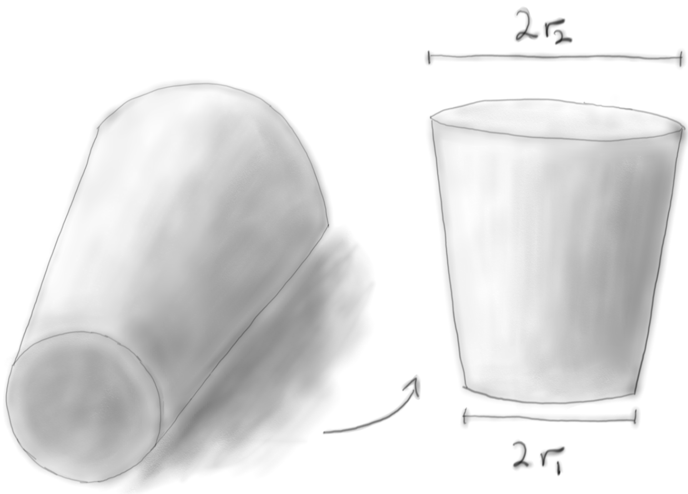
a) Anta at ved tiden $t = 0$ finnes det en ladning $+Q$ på det innerste kuleskallet, og en ladning $-Q$ på det ytterste skallet. Finn strømtettheten som funksjon av posisjon mellom kuleskallene, $\mathbf{J} = \mathbf{J}(r)$.

b) Finn strømmen $I(t = 0)$ fra det innerste kuleskallet til det ytterste.

c) Finn resistansen i materialet mellom kuleskallene.

Exercise 8.8: Resistivity in a conus-shaped conductor

We have a conus-shaped (a cut cone) conductors with resistivity ρ. The radii at the ends are r_1 and r_2 and the length is L. (*Notice that the method used here is elegant, but wrong. The results would imply that all equipotential surfaces are parallel to the end surfaces of the cone. This would mean that the electric field and hence the current density would have a component which is normal to the external surface, that is, current would flow out of the cone. Still, because this problem is often posed, we have included it, with the caveat that the solution is wrong.*)



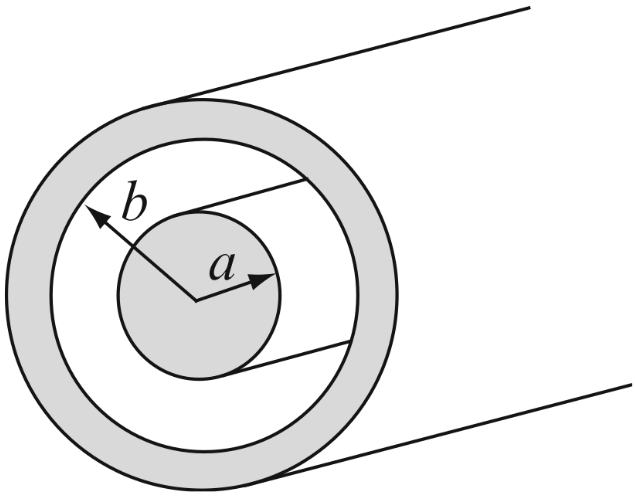
a) Find the resistance between two end disks in the conducutor.

b) Sjekk at resultatet ditt er konsistent med motstanden i en sylinderformet leder: $R = \rho L/\pi r^2$.

c) På en eller annen måte klarer vi å gi konusen en strømtetthet på $|\mathbf{J}| = ar$ rettet langs konusaksen. Finn forskjellen i strøm fra den ene enden til den andre. Vil denne lederen forbli nøytral over tid?

Exercise 8.9: Leaky coaxial cable

A 10 km long coaxial cable lies on the ocean floor. The internal radius is $a = 15$ mm, and the external is $b = 30$ mm. See the figure. The cable itself is made of a superconductor, and the voltage carried by the cable is $V_0 = 520$ V.



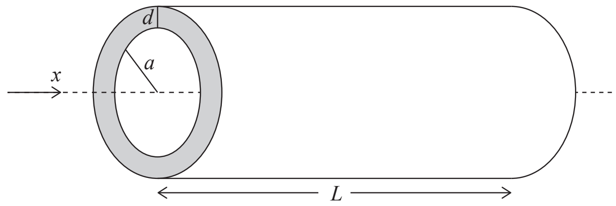
a) What is the current density **J** inside the cable?

b) At one point the cable gets a leak and the entire space between the conductors are filled with seawater ($\sigma = 4\Omega^{-1}\text{m}^{-1}$).
How much power is lost to the seawater inside the cable?

c) There is a generator connected to one of the ends of the cable. When the cable is filled with water, what is the resistance seen by this generator? Comment on this value in regards to the value of P_J.

Exercise 8.10: A cylindrical component

In this exercise, we will study the behavior of a cylindrical nerve cell as illustrated in the figure. We model a part of the cell as a cylindrical shell. The cylindrical shell has a length L. The inner radius is a. The thickness of the cylindrical shell is d. You can assume that L is much larger that a and d and that the system therefore has cylindrical symmetry.



First, we want to find the **capacitance** of the cylindrical shell. We assume that the regions inside and outside the cylindrical shell are ideal conductors. The cylindric shell is a dieletric material with dielectric constant ϵ.

a) Assume that there is a charge Q on the inner surface of the cylindrical shell and a charge $-Q$ on the outer surface of the shell. Find the electric field everywhere in space.

b) Find the scalar potential $V(r)$ as a function of the distance r to the axis of the cylindrical shell.

c) Show that the capacitance of the cylindrical shell is

$$C = \frac{2\pi\epsilon L}{\ln\left(1 + \frac{d}{a}\right)} \ . \tag{8.67}$$

Second, we want to find the **resistance** of the cylindrical shell. The cylindrical shell is a conductor with conductivity σ.

d) Find the current density $\mathbf{J}$ in the cylindrical shel when a current I passes the shell from the inner to the outer surface.

e) Find the electric field $\mathbf{E}$ in the cylindrical shell in this case.

f) Find the scalar potential $V(r)$ in the cylindrical shell and use this to show that the resistance of the cylindrical shell is

$$R = \frac{\ln\left(1 + \frac{d}{a}\right)}{2\pi\sigma L} \ . \tag{8.68}$$

Exercise 8.11: Two resistors

In this exercise we will study a cylindrical resistor of length L and radius a. The end surfaces of the cylinder are connected to conductors. The cylinder is divided into two cylindrical parts, each with a length $L/2$. One has a conductivity σ_1 and the other has a conductivity σ_2.

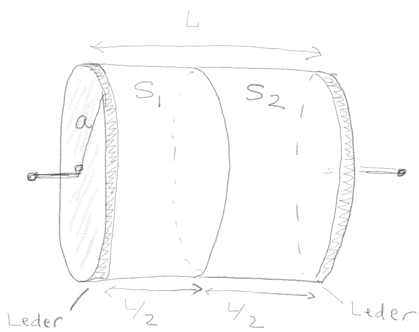
a) Find the electric field everywhere inside the resistor when it is carrying a current I.

b) Find the electric potential everywhere inside the resistor.

c) Find the resistance R to the resistor.

Exercise 8.12: Sylindrisk komponent

En sylindrisk komponent består av to sylindriske ledere med radius a og avstand L, hvor $L \ll a$. Området mellom sylindrene er fylt med to sylindre S_1 og S_2 med radius a som ligger etter hverandre. Hver av de to sylindrene har tykkelsen $L/2$. Systemet er illustrert i figuren.



a) Anta at S_1 har permittivitet ϵ_1 og S_2 har permittivitet ϵ_2. Finn kapasitansen til sylinderen.

b) Anta at S_1 har konduktiviteten σ_1 og S_2 har konduktiviteten σ_2. Finn resistansen til sylinderen.

9 Electric circuits — part 1

±± We now have all the components we need to start to describe and model systems as *circuits*. In this chapter we will introduce the concept of an electric circuit and develop the methods and tools needed to analyze, model and study electric circuits and the systems the are used to model. We will explain why charges are transported along a conductor, independently of its shape. We will introduce an efficient notation for various types of components such as (good) conductors (wires), resistors and capacitors. We will introduce and demonstrate Kirchoff's law of potentials and of currents and how they are used to describe circuits. Circuits can be used as simplified models for complex situation, and we will demonstrate how to simplify physical systems to simplified circuits. We will also introduce analytical and numerical methods to solve complex problems formulated as circuits — allowing you to understand and explore more complex physical phenomena such as signal propagation along nerve cell membranes, eddy currents and flow in disordered materials.

9.1 Electric circuits

Fig. 9.1 illustrates a system with two ideal conductors at potentials V_A and V_B. The left figure illustrates the resulting potential and electric field everywhere in space (found from Laplace's equation), when the system is in vacuum. We notice that the electric field points from V_A to V_B, but the field has a complicated shape. What happens if we place a conductor with a conductivity σ into this system as illustrated by the gray region in the right figure?

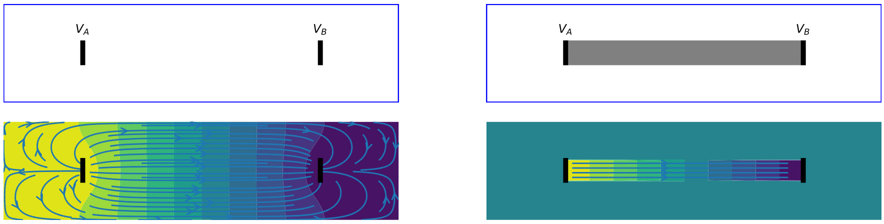


Fig. 9.1 (Left) Illustration of a system with two ideal conductors in vacuum at potentials $V_A > V_B$ and the resulting electric field. (Right) Illustration of a system with two ideal conductors at potentials $V_A > V_B$ and a real conductor with conductivity σ (in gray) and the resulting current density $\mathbf{J}$.

We solved this problem previously and found that the current density $\mathbf{J} = \sigma\mathbf{E}$ is parallel with the conductor as illustrated in the figure. The current density cannot have any component normal the to conductor, as this would indicate that charges are leaking from the conductor out into the vacuum outside. Therefore, the electric field also cannot have any component normal to the conductor. (In the figure on the right we have shown the current density for clarity. The electric field will generally not be zero outside the conductor, but it will only have a tangential component immediately outside the conductor).

But why does the electric field change when the conductor is added? Immediately after the conductor is added to the system, we will expect the electric field to be as illustrated on the left. As a consequence, charges will also move in directions normal to the conductor. However, these charges will not move outside the conductor, but stop at the outer surface of the conductor. This will change the electric field inside the conductor, and this process will continue until the surface charge ensure that there is no electric field normal to the surface of the conductor. Then we will have reached the situation in the figure on the right. This process is very fast, and we can effectively think of it as immediate.

A similar process occurs in a conductor that is curved. Fig. 9.2 illustrates a conductor with a curve. When an electric potential difference is applied across the ends of this conductor, electric charges will pile up on the surface of the conductor until the electric field, and therefore also the current density, does not have a component normal to the surface of the conductor. The end result is that the electric field will point along the conductor as illustrated in the figure. Again, this process is effectively immediate. This means that we do not need to think about the shape of the conductor. The electric field and the current will follow whatever shape the conductor has. Typically, when we make a drawing of a system with a conducting wire we will not care too much about the shape of the wire, but instead draw the wire in a way that is practical.

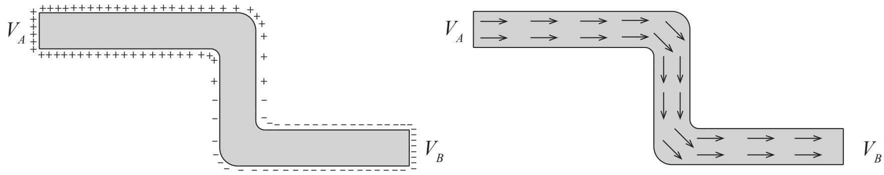


Fig. 9.2 Illustration of curved conductor. The surface charge density will organize in such a way that the electric field does not have a component normal to the surface of the conductor. The electric field will therefore point along the conductor.

9.1.1 Circuit diagrams

In Fig. 9.3 we have illustrated a system that consists of a good conductor, a wire made of a metal such as copper, and a bad conductor, an isolator used to form a resistor. The top figure shows the real system and the bottom shows the model drawing called a *circuit diagram*. A circuit diagram has several elements that are models of the real system, in this case, ideal conductors drawn as lines and resistors drawn as rectangle. We realize that along the good conductor, the potential is essentially constant. This means that the shape or length of the ideal conductor does not matter. (At least not yet, it will matter later when we introduce inductance). We therefore often draw good conductors as straight lines. We need to remember that all points along these lines have the same potential. In the figure we have illustrated that the potential is the same in several places along the conductor illustrated with small circles.

9.1.2 Drawing resistors

The conductor consists of two parts: A good conductor (a copper wire) and a bad conductor (forming a resistor). We have drawn the good conductor as a thin line and we draw the bad conductor as a rectangle,

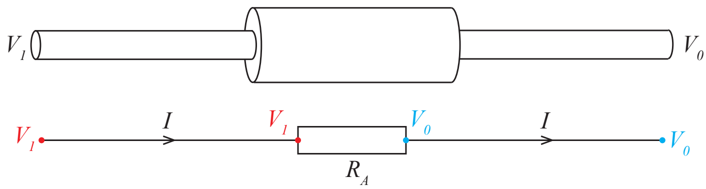


Fig. 9.3 (Top) Illustration of a system consisting of a good (ideal) conductor connected to a cylindrical insulator. (Bottom) Model in the form of a circuit diagram. Thin lines show ideal conductors. All points along a thin line has the same potential. Arrows indicate the direction of the currents.

which is still a conductor, but with much lower conductivity than the wire. Notice that the potential is not the same on each side of the resistor: There is a potential drop, $V = V_1 - V_0$, across the resistor. We characterize a resistor with a resistance R. (You know how to calculate the resistance if you know the shape of the resistor.) For a resistor we know that the voltage drop is related to the current through $V = RI$, where V is the voltage drop across the resistor, I the current through the resistor and R is the resistance.

We have now simplified the real system with an ideal system with two types of components: An ideal conductor drawn as a line and a resistor with resistances R respectively. We will often also simplify a real conductor, with a finite conductivity in this way, as a wire that is effectively an ideal conductor with zero potential drop connected in series with a resistor that represents the potential drop in the wire.

9.1.3 Charge conservation and the water analogy

Where do the charges go in the diagram in Fig. 9.3? In this case positive charges are transported from high potential V_1 to the lower potential V_0. If this was the complete circuit, we would expect charges to build up at V_0 and be removed at V_1, gradually resulting in an increase in the potential V_0 and a decrease in potential at V_1 until they are equal and no more current would flow.

To keep the potential difference and to keep the current flowing, we insert an element in the circuit called a voltage source or a *battery*. (See the symbol used in the figure). We will not go into the detailed workings of a battery here, but will simply assume that the potential of a charge is increased as it passes through the battery. This is illustrated in Fig. 9.4. Here, we have drawn the changes in the electric potential, which corresponds to the changes in the potential energy of a positive charge as it moves through the system. We may use a water analogy to guide our intuition about this system. When the charge moves along a wire, it does not change its potential. This is like a flat water channel. When the charge moves through a resistors, it decreased its potential. This is like a water-fall. When the charge moves through a battery, it increases its potential. This is like an elevator: It lifts the water up.

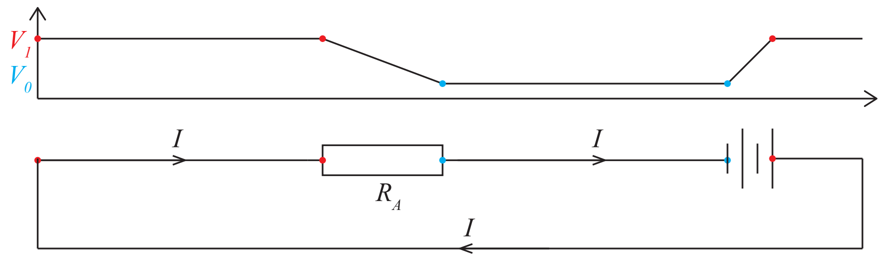


Fig. 9.4 (Top) Plot of the potential as a function of position along the top part of the circuit. (Bottom) Circuit diagram for the circuit. Notice the new symbol for the battery. Thin lines show ideal conductors. All points along a thin line has the same potential. Arrows indicate the direction of the currents.

Circuits are loops. Usually, we draw circuits as loops with batteries as voltage sources. A battery provides a particular voltage difference, and this drives the current in the circuit. We draw circuits as loops, inducating that the current does not disappear, but moves along the various components in the circuit. For the simple diagram illustrated in Fig. 9.4 the current is the same in any cross section along the circuit.

9.1.4 Loops and Kirchoff's voltage law

How does the electric potential vary along a loop such as the whole or a part of a circuit? We recall that we previously found what we called Kirchoff's voltage law:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0 \, , \tag{9.1}$$

for any closed loop C. (We will modify this law later in electrodynamics). But this integral is really the change electric potential around a loop. In Fig. 9.5 we illustrate a loop divided into segments and potential drops $\Delta V_i = V_{i+1} - V_i$ along the segments. Kirchoff's voltage law for this circuit is then

$$\sum_{i=1}^{n} \Delta V_i = 0 \; , \tag{9.2}$$

where $\Delta V_n = V_n - V_1$. Notice that for a resistor R we have a voltage *drop*, $V = RI$, whereas for a battery we have a voltage *gain*, ϵ, which is a property of the battery.

Notice also that Kirchoff's law is valid for *any closed circuit* or any *part of a closed circuit*. For example, it is valid along the circuits C_1, C_2 and C_3 illustrated in Fig. 9.5. Notice that the orientation of the loop can be in the positive direction (C_1) or in the negative direction (C_2). You are free to choose what suits you best. However, for loop C_3 you must ensure that there is a voltage drop over the top resistor but a voltage increase over the bottom resistor. We have voltage drops when we pass a resistor in the positive direction of the current and a voltage gain when we pass in the direction opposite the positive direction of the current. We therefore discuss the direction of the current in more detail below.

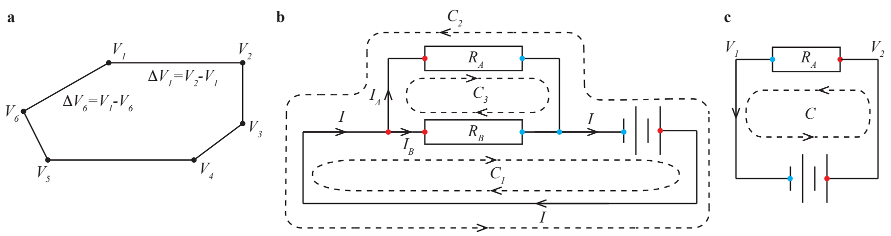


Fig. 9.5 (a) Illustration of a loop with vertices i, voltages V_i and voltage drops $\Delta V_i = V_{i+1} - V_i$. (b) Illustration of three loops C_1, C_2 and C_3 in a circuit. (c) Simple circuit with a single loop C.

Kirchoff's voltage law

Kirchoff's voltage law states that the sum of potential drops around any closed circuit (loop) is zero:

$$\sum_{i} \Delta V_i = 0 \; , \tag{9.3}$$

which follows from $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ for any loop C. Notice that across a component there is a voltage drop in the positive direction of the current and a voltage gain in the direction opposite the current.

9.1.5 Example: Applying Kirchoff's voltage law

Let us apply Kirchoff's voltage law to the circuit in Fig. 9.5c. We choose the loop C shown in the figure. Along the loop there are two changes in voltage: Across the resistor and across the battery. For the resistor the voltage change is $V_{2\to 1} = -RI$. Notice that this is a voltage drop. The potential V_1 is smaller than the potential at V_2. For the battery the voltage change is $V_{1\to 2} = e$, which is the voltage delivered by the battery. Kirchoff's voltage law gives:

$$\sum_i \Delta V_i = -RI + e = 0 \quad \Rightarrow \quad I = e/V \; . \tag{9.4}$$

We can therefore determine the current, given R and the voltage delivered by the battery.

9.1.6 Drawing directions for the currents

The way the conductors are connected provides additional information. In Fig. 9.5 the current I is flowing along the left conductor and through the battery. However, in the middle the conductors split. Intuitively, we understand that the current I is split into the currents I_A and I_B so that the current is conserved — just like current in a river or a garden hose.

We can analyze the junction using Kirchoff's current law: $\sum_i I_i = 0$, but then we need to be careful with the direction of the currents and the signs used: All the currents I_i must either be into the junction or out of the junction. The positive directions are drawn with arrows in the figure. Kirchoff's current law applied to the currents flowing out of the left junction is:

$$\sum_i I_i = (-I) + I_A + I_B = 0 \; , \tag{9.5}$$

where I is the current flowing into the junction from the left and therefore $(-I)$ is the current flowing out of the junction to the left.

Kirchoff's current law for circuits

For any junction in a circuit Kirchoff's current law applies:

$$\sum_i I_i = 0 \tag{9.6}$$

where all the currents I_i must flow either into the juction or out of the junction.

9.1.7 Example: Currents and voltage drops

We combine Kirchoff's voltage law and Kirchoff's current law to find the currents and voltage drops in the circuit illustrated in Fig. 9.5b. We recall that the lines indicate ideal conductors. All points along a line has the same potential. The potential is illustrated by the color of the dots at the junctions. There are only two potentials V_1 and V_0 in this system with a voltage difference $V = V_1 - V_0 = e$, which is the voltage delivered by the battery.

If we apply Kirchoff's voltage law to the loop C_1 we see that the voltage drop across resistor R_B, $-I_B R_B$, and the voltage increase e across the battery sums to zero: $\sum_i \Delta V_i = -I_B R_B + e = 0$, and we find $I_B = e/R_B$.

If we apply Kirchoff's voltage law to the loop C_2, we see that the direction of the loop is opposite the direction of the currents, so that the voltage change over that battery is $-e$ and the voltage change over resistor R_A is $R_A I_A$ so that $\sum_i \Delta V_i = -e + I_A R_A = 0$ and $I_B = e/R_A$.

We can then apply Kirchoff's current law to any of the junctions. We use the currents flowing out of the left junction. The two currents I_A and I_B are flowing out of this junction and the current $-I$ is also flowing out of this junction (since I is flowing into the junction). Kirchoff's current law therefore gives $\sum_i I_i = -I + I_A + I_B = 0$ and $I = I_A + I_B$, which indeed corresponds to our intuitive understanding: The current I flowing into the junction must equal the currents I_A and I_B flowing out of the junction to conserve currents.

Instead of applying Kirchoff's voltage law to loop C_2, we could also have applied Kirchoff's law to the loop C_3: $\sum_i \Delta V_i = -I_A R_A + I_B R_B = 0$. Notice that the voltage drop across R_B is positive, because the direction of the loop is in the opposite direction to the direction of the current I_B. However, this gives us a relation between I_A and I_B, which must be combined with two other equations to find the three unknowns I, I_A, I_B.

Circuits and systems of linear equations. The system described in Fig. 9.5b has three unknowns, I, I_A and I_B. In general, we can think of circuits as setting up systems of linear equations. (Sometimes systems of linear differential equations, as we will soon see). We then need a number of linearly independent equations to find the unknowns, and we find these equations from Kirchoff's laws and the definitions of voltage drops across resistors ($\Delta V = IR$) and other components. For the system in Fig. 9.5b we may get five equations: one for the application of Kirchoff's voltage law for each of the three loops, and two from the application of Kirchoff's current law for each of the two junctions. These equations will not be linearly independent: Kirchoff's current law for the two junctions will be the same, and there are only two independent equations from the three loops. However, for more complicated circuits, it may be difficult to recognize which laws give independent equations. You should then instead use standard procedures from linear algebra. For example, you can find all the relevant equations, and then reduce them to only the linearly independent ones using numerical or symbolic methods. (We demonstrate this in the exercises.)

Comparing with known results. For this particular problem we found that the total current was:

$$I = I_A + I_B = \frac{V}{R_A} + \frac{V}{R_B} = \left(\frac{1}{R_A} + \frac{1}{R_B}\right) V \; . \tag{9.7}$$

From this we also see that we can replace the two resistors in parallel with a single resistor with resistance $R = (1/R_A + 1/R_B)^{-1}$. This is the rule for combining resistors in parallel, which we found above. Sometimes we simplify problems be replacing complex resistors with simplified resistors, using the rules for combining resistors. This may provide you with better insight and oversight of the problem you are solving.

9.1.8 Example: Circuit analysis

We can now use these tools to study the circuits in Fig. 9.6. For these systems, the battery voltage and the resistances of the resistors are given. This is a situation that corresponds to a typical physical situation, where we have a circuit and want to figure out what happens. What happens here means to find out what currents are flowing, the potential drops across the various components, and the power consumption of each component and the total circuit.

Circuit a. In circuit a there are three unknown currents, I, I_A and I_B. Kirchoff's current law for the currents flowing out of junction 1 gives us

$$\sum_i I_i = -I + I_1 + I_2 \tag{9.8}$$

We apply Kirchoff's voltage law to loop C_1:

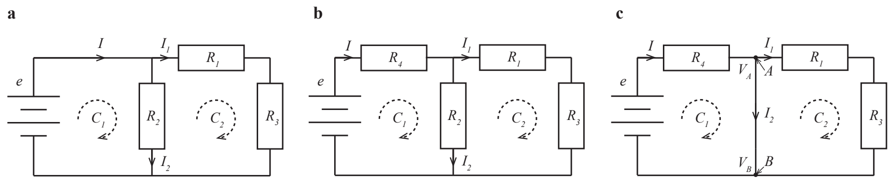


Fig. 9.6 Illustration of three circuits.

$$\sum_i \Delta V_i = e - R_2 I_2 = 0 \tag{9.9}$$

and similarly for loop C_2:

$$\sum_i \Delta V_i = -I_1 R_1 - I_1 R_3 + I_2 R_2 = 0 \ , \tag{9.10}$$

This gives us three equations with three unknowns, which can be solved by linear algebra. Here, we see that $I = I_1 + I_2$ from (9.8), $I_2 = e/R_2$ from (9.9), and that

$$I_1 = \frac{R_2}{R_1 + R_3} I_2 = \frac{R_2}{R_1 + R_3} \frac{e}{R_2} = \frac{e}{R_1 + R_3} \ , \tag{9.11}$$

which we also could have seen by using a different loop through R_1, R_3 and the battery.

Circuit b. Circuit b can be solved using the same approach as for circuit b. Try to redo the arguments and find the resulting currents yourself.

Circuit c. What happens if we replace resistor R_2 with a wire? This corresponds to *short-circuiting* the circuit. Because we have introduced a wire without any resitance, we know that points A and B will have the same potential. This means that there will not be any current through resistors R_1 and R_3, because the current will go through the resistanceless wire from A to B instead. Beware such short circuits in your analysis as they may lead to problems.

9.1.9 Power

We know that the current is not "used up" throughout a circuit. Indeed, the current is conserved because charges are conserved. In the circuit in Fig. 9.5c the current is the same everywhere, but the voltage drops across the resistor. This means that there is a work done on a charge to move it from one side of the resistor to the other, and the resistors heats up as a result. The energy for this work is delivered by the battery. We recall that the power consumption in a resistor is

$$P = VI = RI^2 = \frac{V^2}{R} \, . \tag{9.12}$$

We could therefore say that energy is used up, or converted from chemical energy in the battery to thermal energy, in the circuit. In physics we prefer not to use the word used up, in particular about energy, as energy conservation is one of the most fundamental laws of physics, and energy only passes from one form to another. We therefore say that energy is dissipated in the system. You will return to these questions when you study thermal and statistical physics.

9.2 Circuits with capacitors

So far we have only included resistors in our circuits, but we have also studied another component: the capacitor. How do we describe capacitors in circuits and how do circuits with capacitors behave?

A capacitor stores charge. We recall that a capacitor can function as a storage device for charge. A capacitor has a properties, capacitance, C, which relates the voltage across the capacitor and the charge stored in a capacitor

$$C = \frac{Q}{V} \, . \tag{9.13}$$

A capacitor typically consists of two conductors (for example two parallel plates) as illustrated in Fig. 9.7. When a capacitor is charged with a charge Q it has a charge $+Q$ on one of the plates and a charge $-Q$ on the other plate. The two plates are not in contact, so no charge is flowing from one plate to another across the gap between the plates. When the capacitor is charged with a charge Q, the side with a charge $+Q$ has a potential V_1 and the side with charge $-Q$ has a potential V_0 so that the potential difference, $V = V_1 - V_0$ is $V = Q/C$.

Circuit with capacitor. In Fig. 9.7 we have illustrated the real system consisting of two conductors with an electric field between them on the left and a simplified illustration of the system on the right. In the model we draw the capacitor as two parallel plates independently of how the real capacitor looks like, just like we draw a resistor as a rectangle

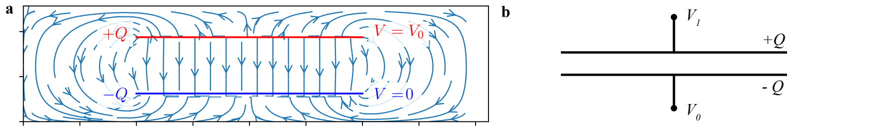


Fig. 9.7 (a) Illustration of a real capacitor with a streamplot of the electric field. (b) Model illustration of a capacitor used in a circuit diagram.

independently of the real shape of the resistor. This is the symbol we use in a circuit diagram to indicate a capacitor. The capacitor then only has the properties described by the model, that the voltage across the capacitor is $V = Q/C$ given the charge Q or that the charge $Q = CV$ given the voltage V. The voltage is higher on the $+Q$ side, implying that there is a *voltage drop* of $V = Q/C$ from the $+Q$ side to the $-Q$ side across the capacitor.

Switch is open. Fig. 9.8a illustrates a simple system consisting of a capacitor with capacitance C, a resistor with resistance R, a switch and a wire. Remember that all points along the wire has the same potential. When the switch is open, so that circuit is not connected, no current can flow in the wire. Let us assume that there is an initial charge Q on the capacitor. When the switch is open, there will not be any flow of current. Therefore, there will not be any potential drop across the resistor and $V_2 = V_0$. What happens when we close the switch and circuit becomes connected?

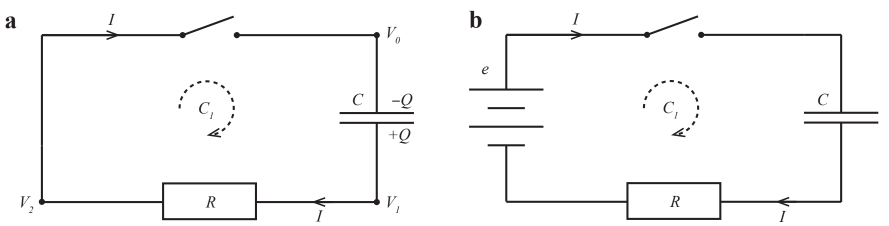


Fig. 9.8 (a) Illustration of circuit with capacitor C, resistor R and a switch. (b) Illustration of circuit with capacitor C, resistor R, battery e and a switch.

Switch is closed. When the switch is closed there is a wire from point 0 to 2 so these will have the same potential $V_2 = V_0$. There is a potential difference across the capacitor $V = V_1 - V_0 = (1/C)\,Q$, and there is a potential difference across the resistor $V_1 - V_0 = RI$. Kirchoff's voltage law along the loop C_1 therefore gives

$$-RI - \frac{1}{C}Q = 0 . \tag{9.14}$$

Notice that there is a *voltage drop* of $-IR$ over the resistor in the direction of the current, but that there is a *voltage gain* of $V = (1/C)\,Q$ over the capacitor because the voltage is higher on the $+Q$ side than on the $-Q$ side.

Current and charge in the circuit. But how are Q and I related? We recall that the current is the rate of change of the charge. There is a positive current I when the charge Q at the capacitor is flowing out of the capacitor into the conductor. Therefore, $I = -\mathrm{d}Q/\mathrm{d}t$. We therefore get:

$$R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = 0 , \tag{9.15}$$

which can be rewritten as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{1}{RC}Q . \tag{9.16}$$

Instead of a linear equation to find the current from the resistances, as we got for a circuit with only resistors, we now have a linear differential equation for the charge Q on the capacitor. When we have solved this equation, we can find the current by taking the time derivative $I = \mathrm{d}Q/\mathrm{d}t$. To solve the differential equation we also need the initial condition. We need to know what the charge Q of the capacitor is at a time $t = t_0$. Let us assume that the initial charge is $Q(t) = Q_0$ at $t = 0$. We recognize the differential equation, and that its solution is an exponential function of the form

$$Q(t) = Ae^{-t/(RC)} = Ae^{-t/\tau} , \tag{9.17}$$

where $\tau = RC$ is a characteristic time. We see that $Q(0) = Q_0$ gives that $A = 0$. You should convince yourself that this is a solution to the differential equation by inserting the solution in (9.16). The charge in the system is therefore $Q(t) = Q_0 e^{-t/\tau}$ and the current is $I = \mathrm{d}Q/\mathrm{d}t = -Q_0/\tau e^{-t/\tau}$.

Behavior a long time. What happens after a long time? As time increases, the charge and the current will decay, effectively reaching zero after a long time. The characteristic time τ only depends on the properties of the circuit, the resistance R and the capacitance C, and not on the initial charge Q_0 on the capacitor.

Adding a battery. What happens if we add a battery with voltage e to the system as illustrated in Fig. 9.8b? Assume that the initial charge is

$Q(t=0)=0$ and redo the calculation for a system with a battery. What is the charge and current in the system after an infinite time?

We apply Kirchoff's voltage law to the loop C_1

$$e + V - IR = 0 \tag{9.18}$$

where $V = (1/C)Q$ and $I = -\mathrm{d}Q/\mathrm{d}t$ as above. We can now decide to solve the problem either in terms of $Q(t)$ or in terms of $V(t) = (1/C)Q(t)$. (If we solve for $V(t)$ we need to use that $Q(t) = CV(t)$ and that $I = -\mathrm{d}Q/\mathrm{d}t = -C\mathrm{d}V/\mathrm{d}t$). The differential equation in Q is:

$$e + \frac{1}{C}Q + \frac{\mathrm{d}Q}{\mathrm{d}t}R = 0 \tag{9.19}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{e}{R} - \frac{1}{RC}Q \tag{9.20}$$

We solve this equation by first finding the homogeneous solution, when $e = 0$, which we found above, and then adding a constant to find the particular solution:

$$Q(t) = A + Be^{-t/\tau} \ , \tag{9.21}$$

where we know that $Q(0) = 0$, which gives $A = -B$ so that $Q(t) = A(1 - e^{-t/\tau})$. We also know that $\mathrm{d}Q/\mathrm{d}t$ at $t = 0$ is $-e/R$, giving that

$$\frac{A}{\tau}e^{-0/\tau} = -\frac{e}{R} \quad \Rightarrow \quad A = -\frac{e\tau}{R} \ . \tag{9.22}$$

and therefore

$$Q(t) = -\frac{e\tau}{R}\left(1 - e^{-t/\tau}\right) \ . \tag{9.23}$$

Hmmm. Why did we get a negative charge on the capacitor? We need to look back on the figure. We notice that after a long time, we expect that the potential on the top of the battery and on the top of the capacitor must be the same, e, and on the bottom of both the battery and the capacitor the potential is zero. However, we have assumed that the charge is positive on the bottom of the capacitor and negative on the top, but our calculate shows that the result is the opposite.

If you like, you could instead express the result in terms of the potential drop V across the capacitor, $V = -(1/C)Q$:

$$V(t) = -\frac{1}{C}Q = e\left(1 - e^{-t/\tau}\right) \ . \tag{9.24}$$

This shows that at $t = 0$, when the switch is closed, there is no charge and therefore no voltage drop across the capacitor. And after infinite time the potential drop across the capacitor is the same as the potential gain over the battery, since the current has then decayed to zero as illustrated in Fig. 9.9

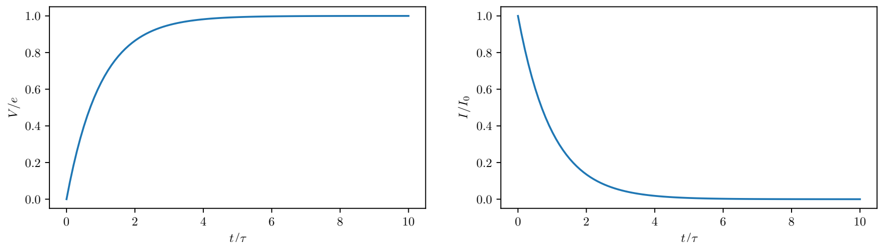


Fig. 9.9 Plot of the voltage $V(t)$ across the capacitor and the current I through the system as a function of time t for a system with a battery e, a capacitor C and a resistor R connected in series.

9.2.1 Components

We have now introduced several components to describe circuits as illustrated in Fig. 9.10.

- **Wire**: We draw wire or conductors as lines. We assume that the conductor is ideal and that all points along the line has the same potential.
- **Switch**: We introduce a switch that can break the circuit.
- **Resistors**: We draw resistors as small rectangles. Sometimes a zig-zag line is also used as symbols for resistors. The voltage drop across a resistor in the direction of the current is $-IR$.
- **Capacitors**: We draw capacitors as two parallel lines perpendicular to the wire. The voltage drop across a capacitor is $-(1/C)Q$.
- **Battery**: A battery provides a constant potential gain e.
- **Voltage source**: A voltage source may provide a time-varying voltage gain $V(t)$.

These basic components combined with Kirchoff's voltage and current law provides us with the tools to model and describe complex systems and solve their model behavior.

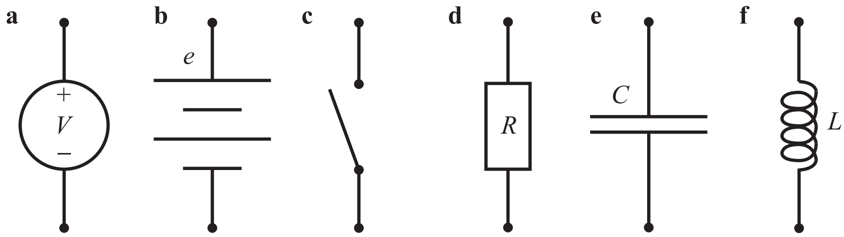


Fig. 9.10 Illustration of the various circuit symbols we have introduced. (a) Voltage source, (b) battery, (c) switch, (d) resistor, (e) capacitor, and (f) inductor (not introduced yet).

9.3 Circuits as models

Circuits can be real systems that you build yourself or they can be simplified models of real systems that allow us to address them using our knowledge of electromagnetics. For example, we know how to describe a real wire with a finite conductivity as illustrated in Fig. 9.11. However, if we want to include this wire in a description of a more complex system, we want to simplify its description. This can be done as illustrated in the figure as a system consisting of a wire of an ideal conductor, drawn as a line in the circuit diagram, connected in series with a resistor that represents the resistance of the whole wire. This is an example of a circuit model of a real system. The goal is that you should be able to simplify complex physical systems to simpler circuit-based systems and then analyze them using your knowledge of electromagnetics. Let us demonstrate this by a few examples.

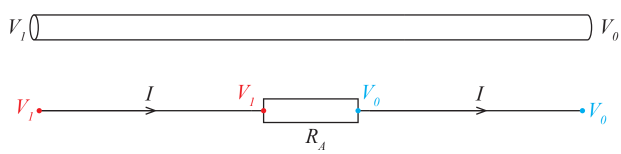


Fig. 9.11 Illustration of (a) a real wire with a finite conductivity and (b) the model of the system.

9.3.1 Example: Cell membrane potential

A cell membrane consists of a lipid bilayer with embedded ion channels and ion pumps. The lipid bilayer consists of a set of two layers of lipid molecules. Each lipid molecule is approximately linear, like a somewhat flexible rod. The lipid molecules are amphiphilic, which means that one end of the molecule is hydrophobic and one end is hydrophilic. In the outer lipid layer the hydrophilic end points outwards, while in the inner lipid layer the hydrophilic end point into the cell. The outside of the lipid bilayer is therefore polar, which means that the molecules form hydrogen bonds with the surrounding water molecules. The inner part of the lipid bilayer is non-polar. The lipid bilayer functions as a membrane, so that molecules that are on the outside of the layer cannot easily get through the layer. This allow cells to have different chemistry on the outside and inside of the cell membrane. See Fig. 9.12 for an illustration.

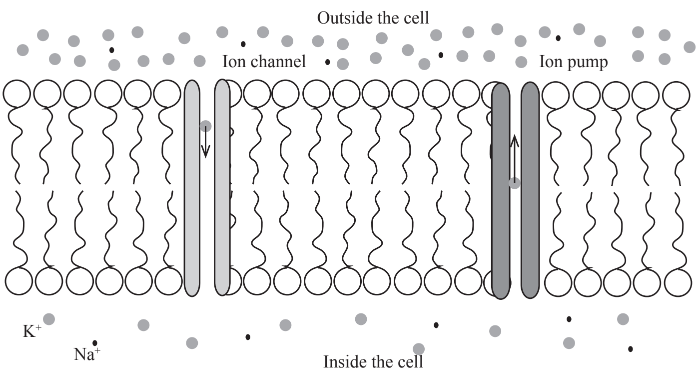


Fig. 9.12 Illustration of a cell membrane.

Cell membranes are not completely impermeable. They have various different types of channels in the form of complex molecules, proteins, that allow some molecules or ions to flow through the cell membrane. *Ion pumps* use energy to pump ions from one side of the membrane to the other. For example, the sodium-potassium pump, in each cycle transports three sodium ions out of the cell and two potassium ions into the cell. Since both ions have a charge of $+e$, this means that the pump builds up a charge difference between the outside and the inside of the cell. Due to this pump, the outside of the cell will have positive charge and the inside will have negative charge. There will therefore be an electric field from the outside of the cell to the inside of the cell. The inside of the cell will therefore have a lower electric potential than the outside. The ion pump uses chemical energy to perform work to transport charges across this potential. In addition, there are leaky ions channels that allows for example potassion ions to flow back into the cell. The net effect of both the ion pumps and ion channels for several types of ions is that there typically is a potential difference across the cell membran is about -80mV called the membrane potential for nerve cells.

How can we create a simplified model of this system? Fig. 9.13a contains a first iteration of a model. The cell membrane is illustrated as a dielectric medium of a given thickness and surface area. This represents a capacitor with a capacitance of $C = A\epsilon/d$. We assume that ion channels and ion pumps are small compared to the area and therefore do not affect the capacitance. But how should we represent the ion pump and ion channels in a model? The ion pump acts as a battery: It lifts charges up through a potential difference. We assume that all the ion pumps in total act as one battery with a voltage e. The ion channels acts as a leaky channel for ions: They will flow through the channel if there is a potential difference, but with some resistance. A potential difference is needed to drive the ions through the channel. We therefore model all the ion channels as one resistor with resistance R.

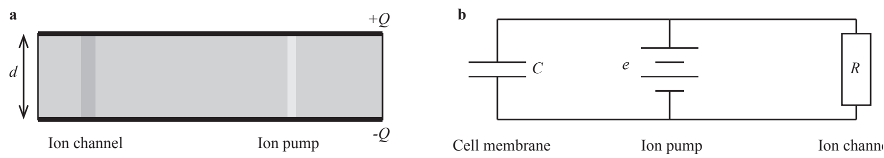


Fig. 9.13 Simplified models of a cell membrane. (a) Showing the cell membrane as a parallel-plate capacitor with ion pumps and ion channels. (b) Showing the corresponding circuit diagram.

We have now developed a model of the electric current and electric charges across the cell membrane as illustrated in Fig. 9.13b. The model consists of a battery, a capacitor and a resistor in parallel. This is a simplified model of one aspect of the physics of a cell membrane, and we can use this model to address the electrical properties of the cell membrane. We can ask questions like, what happens if the ion pump stops pumping? What happens if we change the number of ion channels and therefore change the resistivity of the leaky channel? We can answer these questions by studying the circuit and describing its behavior.

We can also improve this detail of this model. For example, we can assume that each ion pump has a voltage e_0 and that there is a given density N_P of ion pumps on the surface. The total number of ion pumps is then $N_P A$. The total effect of all of these pumps is like having $N_P A$ batteries in parallel. However, this does not change the potential difference of the battery, only the maximum power or current it can deliver, so that the total potential difference of the system still is $e = e_0$. Similarly, we can assume that there is density N_C of ion channels. Each ion channel contributes with a resistivity R_0 and there are $N_C A$ such resistors in parallel. The total resistivity is therefore $1/R = N_C A/R_0$.

9.3.2 Example: Cable equation

Let us extend the model for a cell to the axon, a long part of a nerve cell which transmits electric signals. The axon is approximately cylindrical in shape and consists of a cell interior, a cell membrane, and the extracellular system outside the cell as illustrated in Fig. 9.14. To make a model of this system, we divide the cylindrical axon into smaller cylindrical parts of length ΔL. The potential outside the cell is V_e. We model the behavior inside each such element by a single voltage V_i, in the middle of element i. The cell membrane acts as a capacitor with a capacitance C (which you know how to calculate for a cylindrical element), and there is a leaking current through the cell membrane through channels with an effective resistance R for each element. The capacitor C and the resistor R act in parallel for each such element.

In addition, an element is connected to the neighboring elements. Element i is connected to elements $i-1$ and $i+1$. There is a current going between the element with a resistance r. This makes up the model illustrated in Fig. 9.14. For simplicity we set $V_e = 0$.

Derivation of the equations of behavior. Let us find an equation for the behavior of element i in this system:

- We know that the current through the capacitor is related to the charge Q_i on the capacitor, $I_{C,i} = \mathrm{d}Q_i/\mathrm{d}t$. We can relate this to the voltage V_i across the capacitor, because $Q_i = CV_i$, getting $I_{C,i} = C\mathrm{d}V_i/\mathrm{d}t$. (The voltage drop across the capacitor is $V_i - V_e = V_i$ because $V_e = 0$.)
- We know that the current through the resistor R is related to the voltage drop, $I_{R,i} = (V_i - V_e)/R = V_i/R$.
- The current i_i through the resistor r is found through Ohm's law: $i_i = (V_i - V_{i+1})/r$.
- The current i_{i-1} through the resistor r is found through Ohm's law: $i_{i-1} = (V_{i-1} - V_i)/r$.

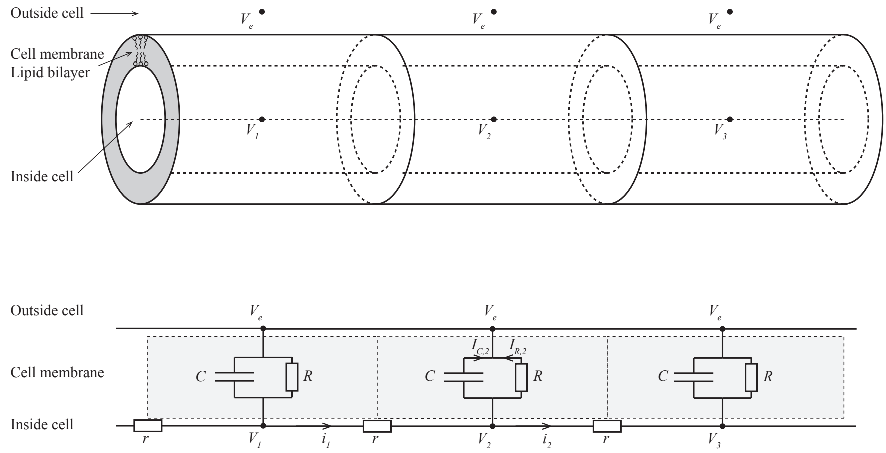


Fig. 9.14 Illustration of the physical system for the cable equation and the circuit model for the physical system.

We combine these results and apply Kirchoff's current law in the point V_i and get:

$$i_{i-1} = i_i + (I_{C,i} + I_{R,i}) \ , \tag{9.25}$$

which gives

$$\frac{V_{i-1} - V_i}{r} = \frac{V_i - V_{i+1}}{r} + \frac{V_i}{R} + C\frac{\mathrm{d}V_i}{\mathrm{d}t} \ . \tag{9.26}$$

We can rewrite this equation as:

$$C\frac{\mathrm{d}V_i}{\mathrm{d}t} + \frac{V_i}{R} = \frac{V_{i+1} - 2V_i + V_{i-1}}{r} \ . \tag{9.27}$$

Left boundary element. What about the left-most part of the system? We assume that $V_0(t)$ is a given voltage source, which we interpret as the signal sent out by the nerve cell. We therefore get:

$$C\frac{\mathrm{d}V_1}{\mathrm{d}t} + \frac{V_1}{R} = \frac{V_{1+1} - 2V_1 + V_0(t)}{r} \ , \tag{9.28}$$

where $V_0(t)$ is a given function of time.

Right boundary element. What about the right-most part of the system? For the last element, V_N, there is no element to the right. This means that i_N is zero, since there is no current going to the right of the last element. We therefore get the equation $i_{N-1} = I_{C,N} + I_{R,N}$ and

$$\frac{V_{N-1} - V_N}{r} = \frac{V_N}{R} + C\frac{\mathrm{d}V_N}{\mathrm{d}t} \ . \tag{9.29}$$

We can rewrite this equation as:

$$C \frac{\mathrm{d}V_N}{\mathrm{d}t} + \frac{V_N}{R} = \frac{V_{N-1} - V_N}{r} \ . \tag{9.30}$$

Interpretation of the main equation. How can we interpret these equations? If we look at (9.27) we see that it is a differential equation in time. But how can we interpret the right-hand side of the equation? This looks similar to the discrete version of Laplace's equation. The index i refers to the spatial position of the various elements. Indeed, we can describe the length of an element at Δx so that $V_i(t) = V(i\Delta x, t) = V(x,t)$ We can then interpret (9.27) as:

$$C \frac{\mathrm{d}V(x,t)}{\mathrm{d}t} + \frac{V(x,t)}{R} = \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{r} \ . \tag{9.31}$$

We recognize the right hand side as the second derivative of $V(x,t)$ with respect to x:

$$\frac{\partial^2 V}{\partial x^2} \simeq \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{\Delta x^2} \ . \tag{9.32}$$

We can therefore interpret (9.27) as:

$$C \frac{\mathrm{d}V(x,t)}{\mathrm{d}t} + \frac{V(x,t)}{R} = \frac{\Delta x^2}{r} \frac{\partial^2 V}{\partial x^2} \ . \tag{9.33}$$

This is similar to the diffusion equation, at least in the limit when $R \rightarrow \infty$. This equation is called the *Cable equation*.

Interpretation of the equation for element N. But how can we interpret the condition on the right hand side of the system, given as (9.30)? In a stationary state, the time derivative is zero, and we can interpret the right hand side of the equation as the spatial derivative of $V(x,t)$. In the stationary state, we therefore have that

$$\frac{\partial V(x,t)}{\partial x} = 0 \ , \tag{9.34}$$

which corresponds to a von Neumann boundary condition for $V(x)$ on the right hand side.

Comparison with Laplaces's equation. We also notice that in the stationary state, that is when the time derivative in the system is zero, we see from (9.33) that we get:

$$\frac{V(x,t)}{R} = \frac{\Delta x^2}{r} \frac{\partial^2 V}{\partial x^2} \; . \tag{9.35}$$

In the case when $R \to \infty$ this reduces to:

$$\frac{\partial^2 V}{\partial x^2} = 0 \; , \tag{9.36}$$

which is Laplace's equation.

Numerical scheme. The equations we found in (9.27) are discretized and it is simple to implement these equations in a simple Euler scheme to find the time development:

$$C \frac{\mathrm{d}V_i}{\mathrm{d}t} + \frac{V_i}{R} = \frac{V_{i+1} - 2V_i + V_{i-1}}{r} \; . \tag{9.37}$$

We start from a set of initial conditions, $V_i(t)$, and then find the solution at a time $t + \Delta t$ by using an explicit scheme where we approximate

$$\frac{\mathrm{d}V_i}{\mathrm{d}t} \simeq \frac{V_i(t+\Delta t) - V_i(t)}{\Delta t} \; . \tag{9.38}$$

This gives us the scheme:

$$V_i(t+\Delta t) = V_i(t) - \frac{\Delta t}{RC} V_i(t) + \frac{\Delta t}{rC} \left(V_{i+1}(t) - 2V_i(t) + V_{i-1}(t) \right) \; . \tag{9.39}$$

For the left hand side, we need to recall that $V_0(t)$ is the external signal provided on the left hand side. This is simply a given function in time, for example a square pulse, a sinusoidal function, or some other time-varying signal. For the right hand side, we use the same time discretization to find that:

$$C \frac{\mathrm{d}V_N}{\mathrm{d}t} \simeq C \frac{V_N(t+\Delta t) - V_N(t)}{\Delta t} = -\frac{V_N}{R} + \frac{V_{N-1} - V_N}{r} \; , \tag{9.40}$$

which gives

$$V_N(t+\Delta t) = V_N(t) - \frac{\Delta t}{RC} V_N(t) + \frac{\Delta t}{rC} \left(V_{N-1} - V_N \right) \; . \tag{9.41}$$

Python implementation. This numerical scheme can be almost directly implemented as it is into Python. We need two additional aspects: We need a function to specify the voltage source on the left side: `V0(t)`, and we need to specify the initial values for all the voltages, which corresponds to specifying the initial charges Q_i on the capacitors. Let us first assume that $V_i(0) = 0$. We store the whole time evolution in the two-dimensional array `V[j,i]`, where `j` represents the time and `i` represents the physical position along the cell.

```
import numpy as np
import matplotlib.pyplot as plt
# Setting realistic paramters
C = 1e-10 #Capacitance
R = 1e11
r = 1e6
V0 = 100e-3
# Define voltage pulse step
def Vs(t,V0,t0):
    return V0*(t<t0)
T = 0.1 # Simulation in seconds
t0 = T*0.001 # Pulse length
dt = 0.1*C*r
nsteps = int(T/dt) # Number of time steps
L = 100 # Number of capacitors
V = np.zeros((nsteps,L),float)
V[0,0] = V0 # Initial condition, t = 0
t = np.zeros((nsteps,1),float)
for j in range(0,nsteps-1):
    t[j+1] = t[j] + dt
    V[j+1,0] = Vs(t[j+1],V0,t0) # Left boundary condition (i=0)
    for i in range(1,L-1): # Internal elements
        V[j+1,i]=V[j,i]+(dt/C)*((V[j,i+1]-2*V[j,i]+V[j,i-1])/r-V[j,i]/R)
    # Right boundary condition (i=L-1)
    V[j+1,L-1] = V[j,L-1] + (dt/C)*((-V[j,L-1]+V[j,L-2])/r - V[j,L-1]/R)
# Plotting results
plt.subplot(2,1,1)
for i in range(0,L-1,10):
    plt.plot(t,V[:,i]/max(V[:,i]))
plt.subplot(2,1,2)
for j in range(0,nsteps-1,1000):
    plt.plot(V[j,:]/max(V[j,:]))
```

The resulting plots are shown in Fig. 9.15. In the top figure we show the voltage $V(x,t)$ as a function of time t at 8 different positions along the system. Notice that the value of V falls very rapidly as we study the system for larger values of x. This makes it difficult to interpret the results. We have therefore instead plotted $V(x,t)/V_{\max}$, where $V_{\max}$ is chosen so that when we plot $V(x,t)$ as a function of t for a given x it is the maximum of V over t for that given x, and similarly when we plot $V(x,t)$ as a function of x for a given t it is the maximum of V over x for that given t. This means that we can interpret the functional shapes, but we cannot compare the amplitudes of the curves. We clearly see how the initial signal, which is a square pulse for a short time t_0, is transmitted along the cable while being smeared out.

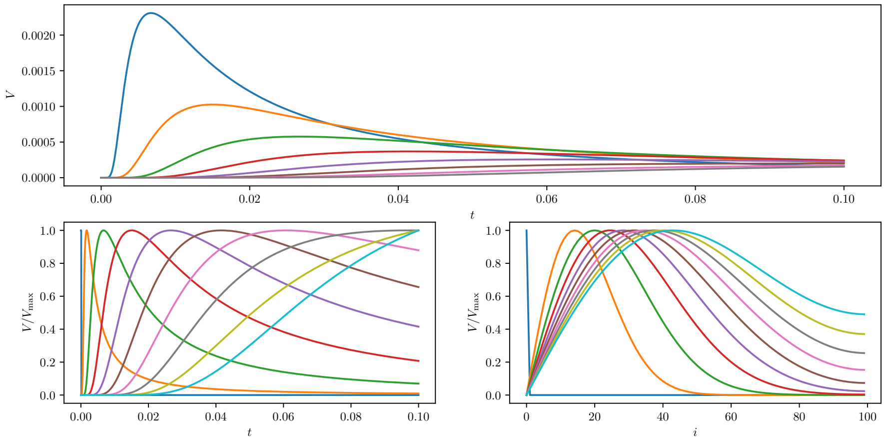


Fig. 9.15 Plots of $V(x,t)$ for the cable equation. Top figure illustrates the unscaled plots, whereas bottom figures illustrates normalized plots for clarity.

Exploring the model. You now have the basic tools needed to study the behavior of a signal traveling along a passive nerve cell. This provides you with a good basis to explore the behavior of this system and to extend the model to also include active components as you will find in real nerve cells and which ensure that the signal does not decay as rapidly as seen in the model presented here. This model can also be used to represent how signals decay in long electric signal cables, such as the telegraph cables laid across the Atlantic Ocean in the mid Nineteenth century. Indeed, these equations were used by Lord Kelvin for these purposes.

Stability of the numerical scheme. Notice that the numerical scheme we have used to solve this problem is an explicit scheme for solving the diffusion equation. In general, we know that this scheme, with $R \to \infty$, is numerically stable only if

$$\frac{\Delta t}{RC} \leq \frac{1}{2} \,. \tag{9.42}$$

Mapping onto real physical values.

9.4 Current distributions in disordered media

What would happen if we placed a porous material — a material with randomly placed holes — between two voltages? We would expect there to be variations in the currents in various places in the material, but how will the current vary? We now have the tools to model and study such systems by simplifying the problem to a circuit and then study the behavior of the circuit.

9.4.1 Model for porous medium

We want to model a two-dimensional simplified model of a random porous material such as a porous metal, a porous insulator, for example, such as in a porous rock which is common in the Earth's (upper) crust. How can we *model* this system as a circuit? Fig. 9.16 illustrates a simplified model of a porous material. In the model material, we have discretized the system into $N \times N$ voxels of size $d \times d$ each, which we call elements. Each element can be either present, with a conductivity σ, or absent, with zero conductivity as illustrated in the figure. We want to model the voltages and current through this system, when we apply an external voltage V_A on the left side and V_B on the right side, where we set $V_B = 0$ for simplicity.

Fig. 9.16 illustrates how we divide the system into a grid. At each grid point (i, j) there is a voltage $V_{i,j}$. We model the current from (i, j) to a neighboring grid point $(i + 1, j)$ using Ohm's law:

$$I_{(i,j);(i+1,j)} = \frac{1}{R_{(i,j);(i+1,j)}} \left(V_{i,j} - V_{i+1,j}\right) \ . \tag{9.43}$$

Where $R_{(i,j);(i+1,j)}$ is the resistance for the connection between points (i, j) and $(i + 1, j)$. For simplicity, we instead introduce the term *conductance*, G:

$$G_{(i,j);(i+1,j)} = \frac{1}{R_{(i,j);(i+1,j)}} \ . \tag{9.44}$$

When we say that we describe porous system or a random material, in this model this means that the conductances vary in space to reflect the porous system. Here, we will introduce a simplfied model for the conductance between the two elements at (i, j) and $(i+1, j)$: The conductance may be either zero, corresponding to a missing connection, or G, corresponding to a present connection. We introduce a probability p for the connection between (i, j) and e.g. $(i + 1, j)$ to be present.

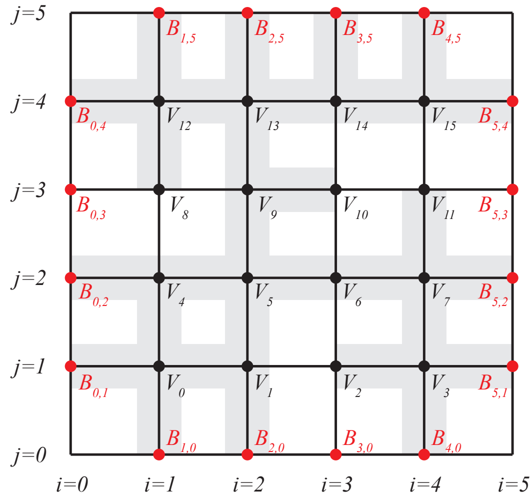


Fig. 9.16 Illustration of a 4×4 simulation system.

9.4.2 Applying Kirchoff's laws

The second step in our model is to realize that Kirchoff's current law for each element means that:

$$\sum_n I_n = I_{(i,j);(i+1,j)} + I_{(i,j);(i-1,j)} + I_{(i,j);(i,j+1)} + I_{(i,j);(i,j-1)} = 0 \quad (9.45)$$

We combine this with Ohm's law from (9.43) and get:

$$G_{(i,j);(i+1,j)} \left(V_{i,j} - V_{i+1,j}\right) + G_{(i,j);(i-1,j)} \left(V_{i,j} - V_{i-1,j}\right) + \quad (9.46)$$
$$G_{(i,j);(i,j+1)} \left(V_{i,j} - V_{i,j+1}\right) + G_{(i,j);(i,j-1)} \left(V_{i,j} - V_{i,j-1}\right) = 0\,. \quad (9.47)$$

This equation looks very much like the equations we found for voltage when we solved Laplace's equation if we set all the condutances to be the same. We will therefore try to solve this problem numerically using the same implicit scheme we developed for Laplace's equation.

9.4.3 Implicit numerical scheme

Let us first develop the numerical scheme for the small 4×4 system shown in Fig. 9.16. Each value of $V_{i,j}$ is determined from (9.47), where we must include both the values for the conductances and the values for the boundaries.

Let us first look at an internal element $V_{2,3}$. We also use the same enumeration scheme as we introduced previously, with a single index n for an element, where we form n as $n = (j-1)(N-2) + (i-1)$ so that $n = 0, 1, \ldots, 15$. Element $(2, 3)$ therefore corresponds to index $n = 9$. We can therefore use Fig. 9.16 directly to read out the neighboring elements. In the figure, we have illustrated elements with zero conductance with a white and elements with finite conductance (G) with gray. This means that the four conductances leading out of element 9 are: $G_{9,10} = G$, $G_{9,8} = 0$, $G_{9,5} = G$, and $G_{9,13} = G$. The equation for V_9 is therefore:

$$-V_8 - V_5 - V_13 + 3V_9 = 0 \; . \tag{9.48}$$

What happens to an element at the boundary? Let us look the element $n = 4$. The conductances leading out of element 4 are: $G_{4,5} = G$, $G_{4,(0,1)} = G$, $G_{4,8} = 0$, and $G_{4,0} = G$, where we have used the notation $(0, 1)$ for the conductance to a boundary element, $B_{0,2}$. What is the equation for V_4?

$$G(V_4 - V_5) + G(V_4 - B_{0,1}) + 0(V_4 - V_8) + G(V_4 - V_0) = 0 \; , \tag{9.49}$$

where we divide by G and reorder the equation to get:

$$-V_5 - V_8 - V_0 + 4V_4 = B_{0,2} \; . \tag{9.50}$$

At the top boundary we can either use Dirichlet boundary conditions and provide values for each boundary element $B_{i,j}$ needed, or we can use von Neumann boundary conditions, and instead assume that there is no current across the boundary, and that there is therefore no voltage difference across the boundary. Dirichlet boundary conditions for element $n = 13$ gives the equation:

$$-V_{12} - V_{14} - V_9 + 4V_{13} = B_{2,5} \; , \tag{9.51}$$

whereas von Neumann boundary conditions for element $n = 13$ gives the equation:

$$-V_{12} - V_{14} - V_9 + 3V_{13} = 0 \; , \tag{9.52}$$

which corresponds to assuming that $B_{2,5} = V_{13}$, which gives zero current into the boundary element $(2, 5)$. We will here use von Neumann's boundary condition in the y-direction and Dirichlet boundary conditions in the x-direction so that there will be a current flowing from the left to the right (or top to bottom in the Python scheme, where the first coordinate is along the vertical axis).

System of equations. We write down all the 16 equations for the internal points. On matrix form this gives us the following system of equations using von Neumann boundary conditions in the y-direction and Dirichlet boundary conditions in the x-direction:

$$
\begin{bmatrix}
3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \\ V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \\ V_{15}
\end{bmatrix}
=
\begin{bmatrix}
B_{0,1} \\ 0 \\ 0 \\ B_{5,1} \\ B_{0,2} \\ 0 \\ 0 \\ B_{5,2} \\ B_{0,3} \\ 0 \\ 0 \\ B_{5,3} \\ B_{0,4} \\ 0 \\ 0 \\ B_{5,4}
\end{bmatrix}
$$

This is a matrix equation on the form $Ax = B$ where the unknown x here contains the potentials, V_n.

Setting up the matrix. We now know how to set up the system of linear equations that we can solve with Python's linear equation solvers. However, first we need to set up the matrix $A_{n,m}$ and the vector B_n. We do this by looping through all elements (i, j), and then inserting all the relevant values for the equation used to determine $V_{i,j}$. For each neighbor (k, l) to (i, j) we insert the values from the equation $G_{(i,j),(k,l)}(V_{i,j} - V_{k,l}) = 0$, taking into consideration the value for the conductance and whether the element is a boundary element or an internal element. We write a function to set up the system equations, solve the system, and put the values back into an array of $V_{i,j}$ values as well as two arrays of the currents. This is done in the following program:

```
# Make them into subroutines
def findvoltagesp(N,p):
    # Setting up the boundaries
    b = np.zeros((N,N),float)
    b[:] = np.float('nan')
    b[0,:] = 0.0
    b[N-1,:] = 1.0
    # Setting up random conductances
    gx = 1.0*(np.random.rand(N,N)<p)
```

```
    gy = 1.0*(np.random.rand(N,N)<p)
    # Setting up system of equations
    LM = (N-2)*(N-2)
    A = np.zeros((LM,LM),float)
    B = np.zeros((LM),float)
    for j in range(1,N-1):
        for i in range(1,N-1):
            n = (j-1)*(N-2) + (i-1)
            # x- direction
            G = gx[j,i-1]
            if (i>1):
                A[n,n] = A[n,n] + G # Diagonal
                A[n,n-1] = -G
            # x+ direction
            G = gx[j,i]
            if (i<N-2):
                A[n,n] = A[n,n] + G # Diagonal
                A[n,n+1] = -G
            # y- direction
            G = gy[j-1,i]
            A[n,n] = A[n,n] + G # Diagonal
            if (j>1):
                A[n,n-(N-2)] = -G
            else:
                B[n] = B[n] + G*b[j-1,i]
            # y+ direction
            G = gy[j,i]
            A[n,n] = A[n,n] + G # Diagonal
            if (j<N-2):
                A[n,n+(N-2)] = -G
            else:
                B[n] = B[n] + G*b[j+1,i]
    Ainv = np.linalg.pinv(A)
    x = np.dot(Ainv,B)
    # Generate full V matrix
    V = np.zeros((N,N),float)
    for j in range(0,N):
        for i in range(0,N):
            if (i<1)or(i>=N-1)or(j<1)or(j>=N-1):
                V[j,i] = b[j,i]
            else:
                n = (j-1)*(N-2) + (i-1)
                V[j,i] = x[n]
    return V,gx,gy
```

The code is run with the following command

```
V,gx,gy = findvoltagesp(20,0.65)
```

To visualize the results we may visualize the potential as shown in Fig. 9.17. However, this does not provide as much insight into the physics as we would wish, because the potential varies slowly. Instead, we would like to study the current distribution. For each element, we calculate the absolute value of the current flowing into the element and then we visualize this quantity. This is implemented in the following program:

```
def findabscurrent(V,gx,gy):
    Nx = V.shape[0]
    Ny = V.shape[1]
    Ia = np.zeros((Nx,Ny),float)
    for i in range(1,Nx-1):
        for j in range(1,Ny-1):
            # Find contributions to current from each direction
            # x- direction
            G = gx[j,i-1]
            dV = abs(G*(V[j,i]-V[j,i-1]))
            # x+ direction
            G = gx[j,i]
            dV = dV + abs(G*(V[j,i]-V[j,i+1]))
            # y- direction
            G = gy[j-1,i]
            dV = dV + abs(G*(V[j,i]-V[j-1,i]))
            # y+ direction
            G = gy[j,i]
            dV = dV + abs(G*(V[j,i]-V[j+1,i]))
            Ia[j,i] = dV
    return Ia
```

and the results are plotted using:

```
plt.figure(figsize=(6,6))
ax1 = plt.subplot(1,1,1)
plt.imshow(Ia)
plt.gca().invert_yaxis()
ax1.set_aspect('equal', 'box')
```

The resulting plot is shown in Fig. 9.18 for four different values of p. Here, we observe interesting properties of the system. First, we realize that there is a value for p below which there is no flow from one side to another. This value is called the percolation threshold for the system. For the system we have chosen for the random connections, which is called a bond percolation lattice (because it is the bonds, the connections between the elements, that are chosen at random), the percolation threshold for a square lattice is known to be for $p = 1/2$. If we choose p below $1/2$, there is typically no connecting path from one side to another, whereas if $p > 1/2$, there is a connecting path. Just at $p = 1/2$ the system behaves as a fractal, with a complex geometrical structure which is clearly evident also from the way the current is distributed in the system in Fig. 9.18. You can read more about disordered systems and percolation theory in [**?**].

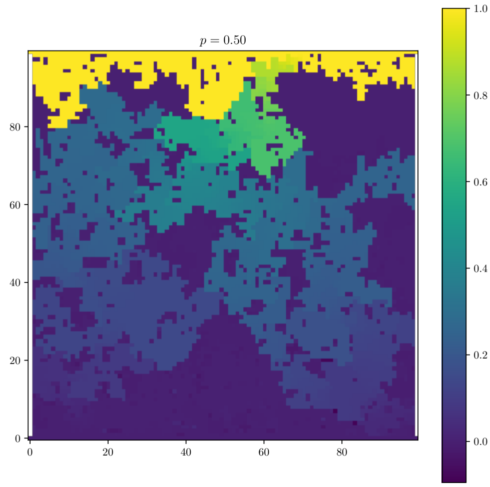


Fig. 9.17 Plot of $V(x, y)$ for a 100×100 simulation with $p = 0.50$.

Using sparse matrix representation. The numerical computations can be made much more efficient by introducing sparse matrices, which you may have learned in your linear algebra course. We will address a sparse matric implementation of the code in the exercises.

9.4.4 Relation to Laplace's equation

If we set all the G-values in (9.47) to be the same, we get the equation:

$$G\,(V_{i,j} - V_{i+1,j}) + G\,(V_{i,j} - V_{i-1,j}) + G\,(V_{i,j} - V_{i,j+1}) + G\,(V_{i,j} - V_{i,j-1}) = 0\;, \tag{9.53}$$

which also gives

$$(V_{i,j} - V_{i+1,j}) + (V_{i,j} - V_{i-1,j}) + (V_{i,j} - V_{i,j+1}) + (V_{i,j} - V_{i,j-1}) = 0\;. \tag{9.54}$$

This is identical to the discrete version of Laplace's equation. Can we understand this? We know that Ohm's law on microscopic form is $\mathbf{J} = \sigma\mathbf{E}$ where $\mathbf{E} = -\nabla V$. In addition, we know that when there is no local build-up in charges, we have that $\nabla \cdot \mathbf{J} = 0$. This gives us that

$$\nabla \cdot \mathbf{J} = \nabla \cdot (-\sigma \nabla V) = -\sigma \nabla^2 V = 0\;. \tag{9.55}$$

This is indeed Laplace's equation.

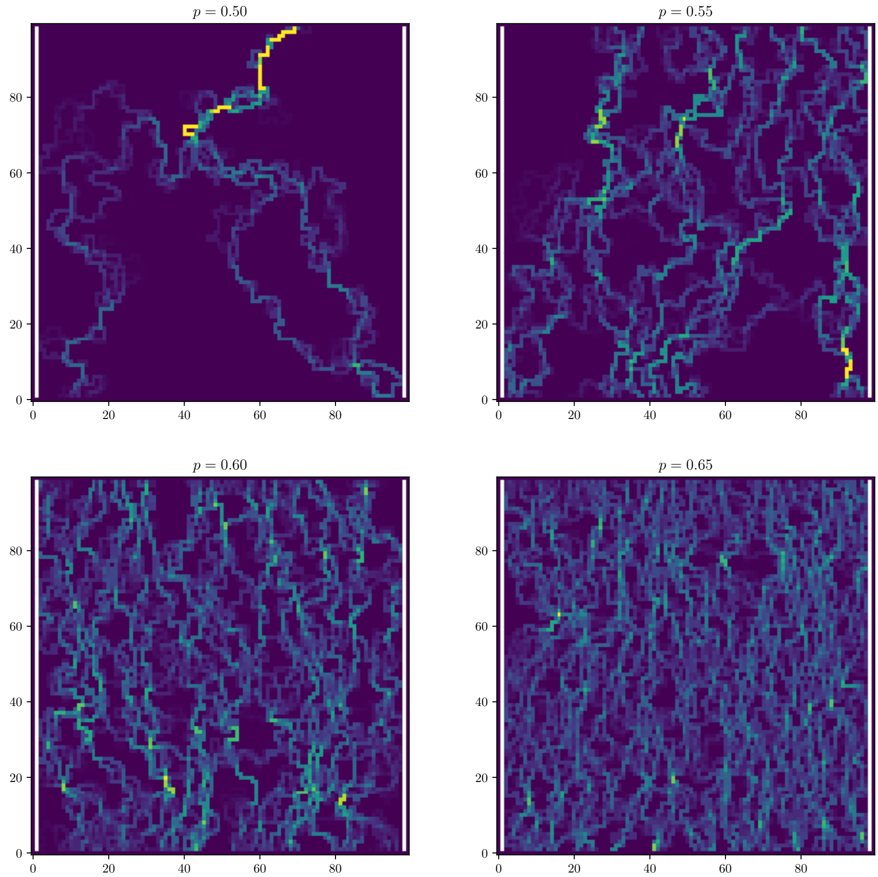


Fig. 9.18 Plot of $\sum_i |I_i|$ into each element for systems for various values of p. As p becomes larger, the current distribution becomes more homogeneous. For systems with $p < 1/2$ there are typically no connecting path from one side to another, and therefore no current.

9.5 Summary

The **electric field** point in the direction along a wire/conductor due to the immediate motion of charges to the surface of the conductor.

We analyze circuits using **Kirchoff's laws for circuits**:

- **Kirchoff's voltage law** states that the sum of voltage changes along a closed loop along a circuit is zero: $\sum_i \Delta V_i = 0$
- **Kirchoff's current law** states that the sum of currents flowing into (or out of) a junction is zero: $\sum_i I_i = 0$

The voltage drop of gain across various components are:

- Zero along a **wire** . The potential is the same along a wire
- $V = -IR$ for a **resistor** in the positive direction of the current
- $V = -(1/C)\,Q$ for a **capacitor** in the positive direction of the current
- $V = e$ for a battery

We use circuits as models for complex electromagnetic systems, where we simplify the physical situation by combining the components in a circuit diagram and then analyze the circuit.

9.6 Exercises

9.6.1 Test yourself

9.6.2 Discussion exercises

Exercise 9.1: Strøm i en kondensator

Er strømmen på hver side av en plate-kondensator alltid den samme? Forklar hvorfor eller hvorfor ikke.

Exercise 9.2: Lyset som gikk

Hvorfor går en (elektrisk) lyspære nesten alltid i stykker i det du skrur på lyset og ikke mens den er på?

9.6.3 Tutorials

Exercise 9.3: RC circuits

(*Based on a tutorial from University of Wisconsin - Madison*)

In this tutorial we will address resistors and capacitors in a circuit.

The capacitor C_1 has a charge of $1\,\mu$C with the sign as indicated in Fig. 9.19a.

a) What is the potential difference $V_C - V_D$? (The circuit is as drawn, with A and B the ends of the wires, not connected to anything else).

b) What is the potential difference $V_A - V_B$? (The circuit is as drawn, with A and B the ends of the wires, not connected to anything else).

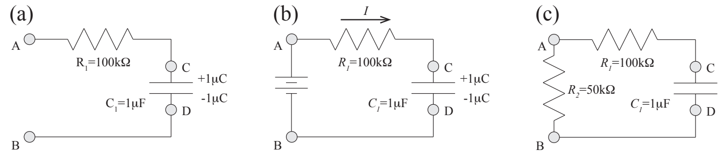


Fig. 9.19 Illustration of the RC circuit.

c) Now a 9V battery is connected to the circuit, the + terminal to A and the – terminal to B. What is the current through R_1 immediately after connecting the battery? (Current flowing toward the capacitor (from A to C) is defined to be positive).

d) After a very long time, with the battery still connected, what is the total amount of charge that has flowed from the battery through R_1?

e) After this very long time, the battery is disconnected, with terminals A and B open again. How much energy is stored in the capacitor?
A resistor $R_2 = 50\text{k}\Omega$ is then connected between A and B as shown in Fig. 9.19c.

f) Immediately after connecting R_2, what is the current flowing through R_1?

g) Will this current increase, decrease, or stay the same as time goes on?

9.6.4 Homework

Exercise 9.4: A simple circuit

Fig. 9.20 illustrates a simple circuit driven by a battery with emf V_0, and consisting of two resistors R_1 and R_2.

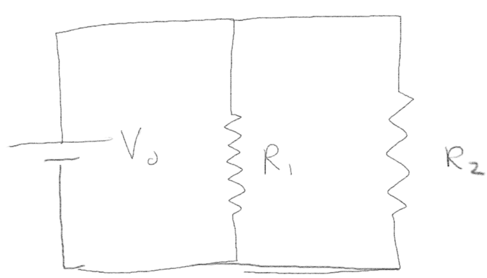


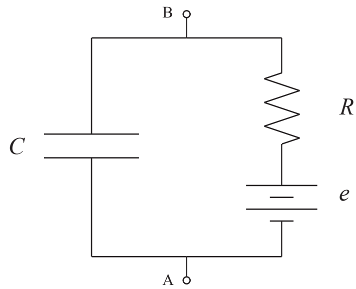
Fig. 9.20 Illustration of a simple circuit. .

a) Find the current I through the battery expressed in terms of V_0, R_1 and R_2.

b) What is the power develivered by the battery?

Exercise 9.5: A circuit-model for a cell

We study a circuit as a model for a cell membrane. The circuit consists of a capacitor C, a resistor R, and a battery with an emf e connected as illustrated in the figure. We measure the potential difference between points A and B, V_{AB}.



a) At the time $t = 0$ the capacitor is without charge. What is the current I_R through the resistor at $t = 0$?

b) What is the charge, Q_∞, at the capacitor when $t \to \infty$?

c) Show that an equation to describe the charge, Q, on the capacitor can be written as

$$\frac{dQ}{dt} = \frac{1}{\tau}\left(Q_\infty - Q\right) \ , \tag{9.56}$$

where $\tau = RC$.

d) Write a program to find the voltage $V_{AB}(t)$ as a function of time for this system.

Exercise 9.6: Wheatstone bridge

Fig. 9.21 illustrates a Wheatstone bridge, which is used to measure resistance. In the circuit, R_3 is variable resistance, R_x is the unknow resistance, and R_1 and R_2 are given, known resistances. The voltmeter has infinite resistance.

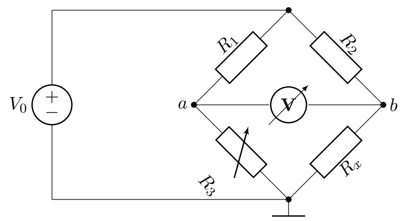


Fig. 9.21 Illustration of a Wheatstone bridge. .

a) Find the voltage difference $V_a - V_b$ which is read from the voltmeter expressed in terms of the four resistances.

b) The variable resistance R_3 is adjusted until the measured voltage is zero. Find the unknown resistance R_x from the other resistances.

Exercise 9.7: Lightning strike

In this problem we will develop a simple model for dielectric breakdown in a circuit. You may think of the system as a model for a lightning forming between clouds, or you may think of the model as an example of dielectric breakdown in a capacitor.

We start from the system illustrated in Fig. 9.22a consisting of a battery with emf V_0, a capacitor C and a resistor R_0.

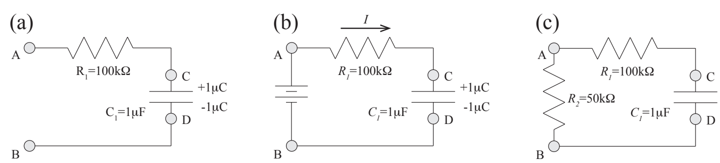


Fig. 9.22 Illustration of the RC circuit.

a) The circuit is open and has been open for a long time. Then at $t = 0$ the circuit becomes closed. What is the charge $Q(0)$ on the capacitor and the potential fall $V(0)$ over the capacitor immediately after the circuit is closed (at $t = 0$)?

b) What is the potential fall V over the capacitor and the charge Q on the capacitor when there is no current flowing in the circuit? (A very long time after the battery has been connected to the circuit).

c) Find the potential drop over the capacitor, $V(t)$.

We will model the dielectric breakdown of the capacitor by introducing a voltage-dependent resistor, R, in parallel to the capacitor as illustrated in Fig. 9.22b. The resistance depends on the voltage V across the resistor:

$$R(V) = \begin{cases} \infty & V < V_0/2 \\ xR_0 & V \geq V_0/2 \end{cases} . \tag{9.57}$$

where x is a small number, such as $1/1000$.

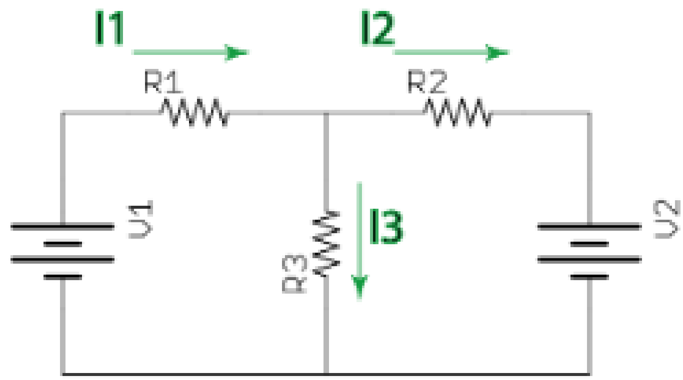
d) Write a python program to model the behavior of the system and plot the $V(t)$, the current $I_0(t)$ through resistor R_0 and the current $I(t)$ through the resistor R. Model the behavior from $t = 0$ when the circuit is closed.

Exercise 9.8: Kirchoff's circuit laws

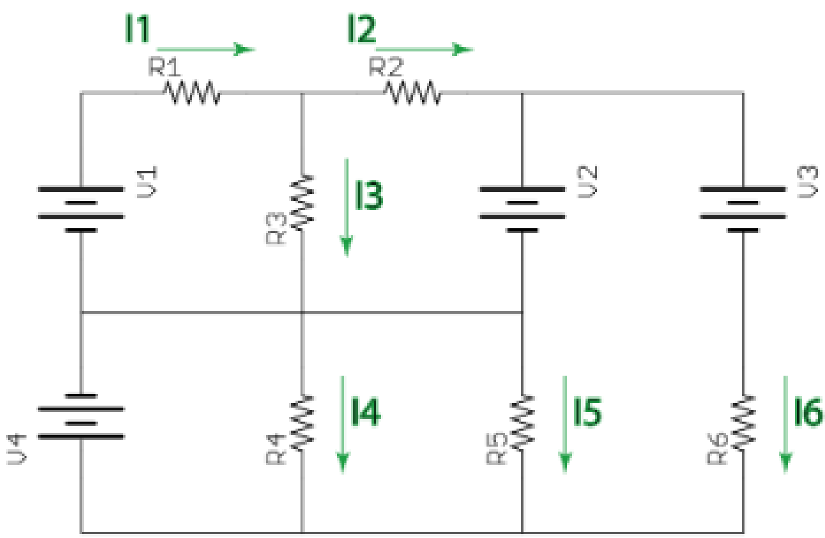
(By Sigurd Sørlie Rustad)
In this exercise we are going to look at Kirchoff's laws, and solve problems using linear algebra.

a) Write down Kirchoff's laws.

b) The voltages are $V_1 = 10\text{V}$ and $V_2 = 5\text{V}$. The resistances are $R_1 = 10\Omega$, $R_2 = 20\Omega$ and $R_3 = 30\Omega$. Using Kirchoff's laws, find the equations needed to solve for the currents. Write it as a matrix equation and use Python to inverse the matrix and find the solution.



c) Use the same approach for the circuit under as you did for the previous exersice. Use $R_4 = 40\Omega$, $R_5 = 50\Omega$, $R_6 = 60\Omega$, $V_3 = 12\text{V}$ and $V_4 = 24\text{V}$.



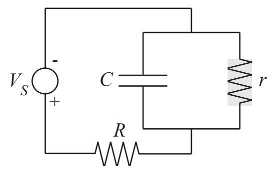
9.6.5 Modeling projects

Exercise 9.9: Voltage gated ion channels

The figure below shows a simple model for a small element in a cell membrane. The resistor r represents an ion channel, but this channel is special: When the potential drop over the resistor is less than v, then the resistance is infinite, but when the voltage drop is larger than v, the resistance is r:

$$r = \begin{cases} r & , V_r > v \\ \infty & , V_r \leq v \end{cases} , \tag{9.58}$$

where V_r is the voltage drop across the resistor.



a) Assume that the voltage V_s has been zero for a long time, so that hte system is stationary. What is the voltage V_C over the capacitor and the current through the resistor R immediately after the voltage V_s is set to $V_s > 0$?

b) First, assume that the voltage V_r over the resistor r is less than v. Show that the equation for the voltage V_C over the capacitor can be written as:

$$C \frac{dV_c}{dt} = \frac{V_s}{R} - \frac{V_C}{R} . \tag{9.59}$$

c) Assume that the voltage V_r over the resistor r is larger than v. What is now the equation for the voltage V_C?

d) Write a program to calculate the voltage V_C over the capacitor given the voltage source $V_s(t)$. You may assume that the voltage $V_C(0)$ has the value from part (a).

Magnetic field 10

In this chapter we will introduce Biot-Savart's law, which is the equivalent of Coloumb's law for magnetism. We will introduce the magnetic field in a way similar to how we introduced the electric field and use our intuition about electric fields to build a new intuition for magnetic fields. For while there are clear symmetries between electric and magnetic fields, there are also important differences. We will introduce methods to calculate the magnetic field from currents and current distributions, both analytically and numerically, and apply these to simple and complex situations. Finally, we will address how magntic fields induce forces and torques on current elements.

10.1 The magnetic field

While you may have some intuition about electric fields, our intuition about magnetic fields are usually connected to either the Earth's magnetic field, which we and many animals use to navigate, or to permanent magnets. However, to build an understanding of permanent magnets, we first need to build an intuition about magnetic fields and how they arise due to currents. Then we will be ready to see how permanent magnets are due to permanent microscopic currents just as polarized materials are due to small charge displacements.

While electrostatic is the theory of static charges and time-independent currents, we will now address the behavior of stationary currents. We will see that there is an experimentally based law describing the interactions between currents and a moving charge, just as there was an experimentally based law for the interaction between one charge and a static charge: Coloumb's law.

10.1.1 From Coulomb's law to the magnetic force law

We had great success with introducing the electric field in electrostatics. Let us recall how we proceeded to build a theory for electrostatics. We started from the experimental observation, Coulomb's law, which states that the force from a charge Q_1 on a charge Q_2 is given as

$$\mathbf{F}_{\text{on 2 from 1}} = Q_2 \frac{Q_1}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \ , \tag{10.1}$$

where the vector $\mathbf{R}$ is the vector from 1 to 2: $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$, where the position of charge Q_1 is $\mathbf{r}_1$ and the position of charge Q_2 is $\mathbf{r}_2$.

Electric field. We realized that it was useful to introduce the electric field, $\mathbf{E}$, from charge Q_1:

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \ , \tag{10.2}$$

and then find the force on a charge Q_2 from

$$\mathbf{F}_{\text{on 2 from 1}} = Q_2 \mathbf{E} \ , \tag{10.3}$$

where we understand that the electric field varies in space: $\mathbf{E} = \mathbf{E}(\mathbf{r})$. We also found that the superposition principle for forces gave us a superposition principle for the electric field: We found the electric field set up by many charges by summing or integrating the electric fields from each of the charges:

$$\mathbf{E} = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}_i}{R_i^2} = ß_v \frac{\rho}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}v \ , \ . \tag{10.4}$$

This allowed us to divide the problem of finding the forces between two charges into two parts:

- first find the electric field set up by a distribution of charges
- then find the force acting on a specific charge at a specific point

We then went on to develop effective methods to calculate and discuss electric fields.

Forces between moving charges and electric currents. We will follow a similar method when introducing the magnetic field. We will first describe an experimental law for the interaction between *moving* charges and electric currents and use this to introduce the concept of a magnetic field created by electric currents, and then address the force on a moving charge from a magnetic field.

Fig. 10.1a illustrates a system corresponding to the original experiments of Biot and Savart in 1820. They found that the force on a moving charge close to a long wire with a current I is proportional to the current I, the charge Q, the velocity v, and decays as $1/r$, where r is the distance to the wire. Hmmm. This reminds us of the behavior of a long line charge, where we know that the electric field decays as $1/r$. We also recall that the $1/r$ dependence was because we summed over all the charges along a long wire. Maybe something similar is at play for a force on a moving charge, but depending on the current instead of the static charge?

Indeed, detailed experimental studies have demonstrated that the force on a charge Q moving with a velocity $\mathbf{v}$ can be written as the sum of contributions from small *current elements* $I\Delta\mathbf{l}_i$:

$$\mathbf{F} = Q\mathbf{v} \times \left(\sum_i \frac{\mu_0}{4\pi} \frac{I\Delta\mathbf{l}_i \times \mathbf{R}_i}{R_i^3} \right) , \tag{10.5}$$

where the vector $\mathbf{R}_i$ is the vector from the current element $I\Delta\mathbf{l}_i$ to the moving charge Q as illustrated in Fig. 10.1b. If the particle with charge Q and velocity $\mathbf{v}$ is in position $\mathbf{r}$, and wire element $\Delta\mathbf{l}_i$ is at $\mathbf{r}_i$, then $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$ — just as for charge distributions.

In a *stationary situation*, currents must go in loops. The set of current elements must therefore form a *closed loop*. However, it is still often useful to decompose a closed loop into discrete or infinitesimally small current elements and sum up their contributions, but remember that this a mathematical construct. Individual current elements cannot exists in a stationary situation. (It is also possible for the current elements not to form a loop, but to go from infinity and to infinity, such as for an infinitely long wire.)

Introducing the magnetic field. Just like we saw for the electric field, we realize that it is conceptually useful to simplify this experimental expression by introducing a magnetic field, $\mathbf{B}(\mathbf{r})$ so that:

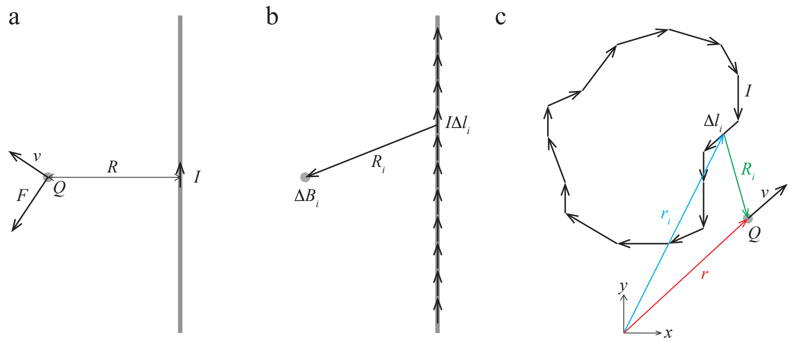


Fig. 10.1 (a) Illustration of the force on a moving charge from a very long wire carrying a stationary current I, (b) which is due to a sum of contributions from current elements $I\Delta\mathbf{l}_i$. (c) Illustration of a general current loop C consisting of current elements $I\Delta\mathbf{l}_i$ that combined forms a magnetic field $\mathbf{B}$ at the point of an observation charge Q.

$$\mathbf{F} = Q\mathbf{v} \times \underbrace{\frac{\mu_0}{4\pi} \sum_i \frac{I\Delta\mathbf{l}_i \times \mathbf{R}_i}{R_i^3}}_{=\mathbf{B}} = Q\mathbf{v} \times \mathbf{B} \; , \tag{10.6}$$

where

$$\mathbf{B} = \frac{\mu_0}{4\pi} \sum_i \frac{I\Delta\mathbf{l}_i \times \mathbf{R}_i}{R_i^3} \; . \tag{10.7}$$

The magnetic field $\mathbf{B}$ is a result of the current elements. Indeed, we consider the magnetic field to be generated by the current. This formulation of the magnetic field is called Biot-Savarts law, and is a fundamental law on the same level as Coloumbs law.

10.1.2 Biot-Savarts law and the force on a moving charge

The magnetic field $\mathbf{B}(\mathbf{r})$ is a vector field generated by loops of discrete ($I\Delta\mathbf{l}_i$) or infinitesimal ($I\mathrm{d}\mathbf{l}$) current elements and its value is given by Biot-Savart's law:

Biot-Savart's law on discrete and line-integral form

The magnetic field $\mathbf{B}(\mathbf{r})$ in a point $\mathbf{r}$ due to a current I in a closed loop C is

$$\mathbf{B} = \sum_i \frac{\mu_0}{4\pi} \frac{I\Delta\mathbf{l}_\times \mathbf{R}_i}{R_i^3} \quad \text{(discrete form)} \; . \tag{10.8}$$

for a set of discrete current elements $I\Delta\mathbf{l}_i$ at positions $\mathbf{r}_i$, where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$, or

$$\mathbf{B} = \oint_C \frac{\mu_0}{4\pi} \frac{I\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(integral form)} \ . \tag{10.9}$$

for a continuous curve of current elements $I\mathrm{d}\mathbf{l}$ at positions $\mathbf{r}'$ along the curve, where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\hat{\mathbf{R}} = \mathbf{R}/R$.

For clarity we can write this integral explicitly in terms of a parameterized curve $\mathbf{r}'(s)$ describing the closed loop C:

$$\mathbf{B} = \oint_C \frac{\mu_0}{4\pi} \frac{I\mathrm{d}\mathbf{l} \times \mathbf{R}}{R^3} = \oint_C \frac{\mu_0}{4\pi} \frac{I\frac{\mathrm{d}\mathbf{r}'(s)}{\mathrm{d}s} \times (\mathbf{r} - \mathbf{r}'(s))}{|\mathbf{r} - \mathbf{r}'(s)|^3} \mathrm{d}s \ . \tag{10.10}$$

A common mistake is not to realize that $\mathbf{R}$ varies along the curve. By now, you should be used to perform similar integrals over the charge density to find the electric field.

Physical constants in Biot-Savarts law. We see that Biot-Savart's law has similarities to Coulomb's law for the electric field. The constant μ_0 is called the *permeability of vacuum* and is given as

$$\boxed{\mu_0 = 4\pi \cdot 10^{-7}\mathrm{Ns}^2/\mathrm{C}^2 = \mathrm{H/m}} \tag{10.11}$$

where H, henry, is the unit for inductance. The magnetic field is measured in units of tesla, T = N/Am. A magnetic field of 1 Tesla is a huge field. For comparison, the magnitude of the Earth's magnetic field on the surface of the Earth as about 50μT.

The current element $I\mathrm{d}\mathbf{l}$**.** We will think of the element $I\mathrm{d}\mathbf{l}$ in the integral and the element $I\Delta\mathbf{l}_i$ in the sum as a *current element*. Mathematically, we see that the magnetic field is the sum of the contributions to the magnetic fields from each of the current elements:

$$\mathbf{B} = \oint_C \mathrm{d}\mathbf{B} = \sum_i \Delta\mathbf{B}_i \ , \tag{10.12}$$

where the magnetic field field from a current element is:

Biot-Savart's law for a current element

The contribution $d\mathbf{B}$ to the magnetic field at $\mathbf{r}$ from a current element $Id\mathbf{l}$ at $\mathbf{r}'$ is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{l} \times \mathbf{R}}{R^2}, \tag{10.13}$$

where $\mathbf{R}$ is the vector from the current element $Id\mathbf{l}$ at $\mathbf{r}'$ to $\mathbf{r}$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

We will often use this way to address problems, by integrating/-summing the contributions from current element. However, this is a mathematical way to think about the problem, because a stationary current must form always form closed loops.

10.1.3 Example: Magnetic field and magnetic force from a current element

Fig. 10.2 illustrates a moving charge Q moving with a velocity $\mathbf{v} = v\hat{\mathbf{y}}$ close to a current element $Id\mathbf{l} = Idl\hat{\mathbf{y}}$. What is the contribution to the magnetic field at charge Q from the current element and what is the contribution to the force on the charge from the current element?

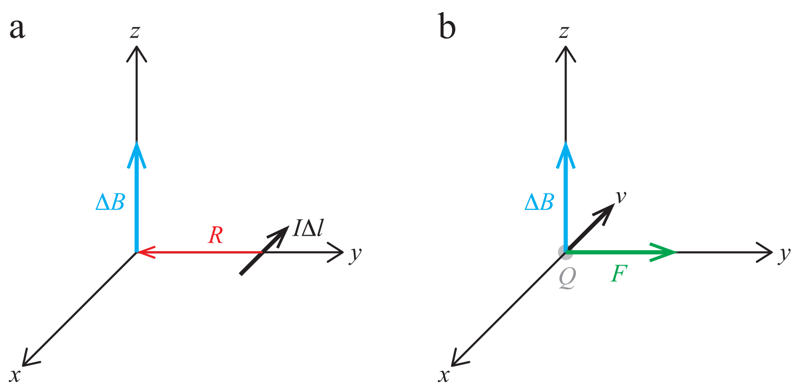


Fig. 10.2 (a) Illustration of the contribution to the magnetic field from a current element. (b) Illustration of the force on a moving charge due to the magnetic field along the z-axis.

First, we find the contribution to the magnetic field from the current element using Biot-Savart's law. We start by finding the $\mathbf{R}$-vector. We can do this directly from the figure or by vector artithmetic. From the figure, we see that $\mathbf{R}$ points from the current element to the position of charge Q, which is where we want to find the magnetic field. Therefore, $\mathbf{R} = -R\hat{\mathbf{x}}$. Alternatively, we can use that $\mathbf{r}' = R\hat{\mathbf{x}}$ and $\mathbf{r} = \mathbf{0}$ so that $\mathbf{R} = \mathbf{r} - \mathbf{r}' = -R\hat{\mathbf{x}}$.

Second, we want to use Biot-Savart's law to find the magnetic field

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\mathbf{l} \times \mathbf{R}}{R^3} . \tag{10.14}$$

We therefore need to find the cross product. We recall that in order to find the cross product we need to apply the *right-hand-rule*: We place the index finger along the first vector, that is along $\Delta\mathbf{l}$. Then we orient the hand so that the second vector, that is so that $\mathbf{R}$, points along the other three fingers. The thumb will then point in the direction of the cross-product. The cross product is *normal* (orthogonal) to the two vectors in the cross-product and the right-hand-rule gives the direction of the normal. We see from this that the vector points in the positive z-direction, that is, that $\Delta\mathbf{B} = \Delta B_z\hat{\mathbf{z}}$.

Now, we know the direction and magnitude of the magnetic field. We find the force on the charged particle from $\Delta\mathbf{F} = Q\mathbf{v} \times \Delta\mathbf{B}$. Again, we apply the cross product using the right-hand rule. We place the index finger in the y-direction and orient our right hand so that the other three fingers points in the direction of the second vector, that is $d\mathbf{B}$, which points in the positive z-direction. Out right thumb will then point toward the right, showing that the force is toward the current element.

10.1.4 Superposition principle for magnetic fields

The superposition principle for forces allow us to find the force on a charge $Q\mathbf{v}$ from a set of current loops simply by adding the contributions from each loop:

$$\mathbf{F} = \sum_i \mathbf{F}_i = \sum_i Q\mathbf{v} \times \mathbf{B}_i(\mathbf{r}) = Q\mathbf{v} \times \sum_i \mathbf{B}_i(\mathbf{r}) . \tag{10.15}$$

This shows that we can use the superposition principle for magnetic fields just as we did for electric fields: We can add the fields set up by different current elements and current loops to find the total magnetic field.

Superposition principle for magnetic fields

The total magnetic field in a point *vecr* is the sum of the magnetic fields $\mathbf{B}_i$:

$$\mathbf{B} = \sum_i \mathbf{B}_i \ . \tag{10.16}$$

10.1.5 Transfer of intuition from electric field to magnetic field

We can transfer much of the intuition and methods we have built up for electric fields to magnentic fields:

- The magnetic field is a spatial field: It varies in space because the vector $\mathbf{R}$ for each current element at a position $\mathbf{r}'$ is $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and this depends on $\mathbf{r}$.
- We can use the superposition principle to find the magnetic field from many current elements and current loops.
- We can calculate and study the magnetic field itself, just as we did for the electric field. We can build intuition for how the magnetic field varies in space.
- We must always keep track for the $\mathbf{R}$-vector in the expression for the magnetic field, just as we did with the $\mathbf{R}$-vector for electric fields.

10.1.6 Biot-Savart's law for line, surface and volume currents

We have so far considered currents that flow along a linear closed loop. However current distributions may be inhomogeneous in space yet stationary. For example, the current may be described by the volume current density $\mathbf{J}$ or a surface current density $\mathbf{J}_S$.

Biot-Savarts law for a volume current density. In the case of a volume current density $\mathbf{J}$, we can divide space into small current elements in the form of volumes $\mathrm{d}v$ with a surface $\mathrm{d}\mathbf{S}$ and a length $\mathrm{d}\mathbf{l}$. The current through this surface is $I = \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ and the current element is therefore

$$I\mathrm{d}\mathbf{l} = \mathbf{J} \cdot \mathrm{d}\mathbf{S}\mathrm{d}\mathbf{l} \ . \tag{10.17}$$

If we choose the surface $\mathrm{d}\mathbf{S}$ so that it is normal to $\mathbf{J}$ and $\mathrm{d}\mathbf{l}$ so that it points in the direction of $\mathbf{J}$, we can rewrite this as

$$I\mathrm{d}\mathbf{l} = J\mathrm{d}S\mathrm{d}\mathbf{l} = \mathbf{J}\mathrm{d}S\mathrm{d}l = \mathbf{J}\mathrm{d}v \ . \tag{10.18}$$

The current element is therefore $I\mathrm{d}\mathbf{l} = \mathbf{J}\mathrm{d}v$, and Biot-Savart's law for the current element is

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \hat{\mathbf{R}}}{R^2} \ . \tag{10.19}$$

Biot-Savart's law for a current density

The magnetic field from a current density $\mathbf{J}$ in a volume v is

$$\mathbf{B} = \int_v \mathrm{d}\mathbf{B} = \int_v \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v' \times \hat{\mathbf{R}}}{R^2} \ . \tag{10.20}$$

We can rewrite this with explicit coordinates as

$$\mathbf{B} = \int_v \frac{\mu_0}{4\pi}\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\mathrm{d}v' \ , \tag{10.21}$$

where the integral is over the coordinates $\mathbf{r}' = (x', y', z')$ which we write at $\mathrm{d}v' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$.

Biot-Savarts law for a surface current density. Similarly, for a surface current density $\mathbf{J}_A$ as illustrated in in Fig. 10.3 a current element $I\mathrm{d}\mathbf{l}$ becomes $J_s\mathrm{d}S\mathrm{d}\mathbf{l} = \mathbf{J}_S\mathrm{d}x\mathrm{d}l = \mathbf{J}_S\mathrm{d}S$. The total field is then a surface integral:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s\mathrm{d}S \times \hat{\mathbf{R}}}{R^2} \ . \tag{10.22}$$

This form of Biot-Savart's law is not used that often in practice, but we may use it for theoretical derivations later.

10.1.7 Example: Magnetic field from a circular current

We want to find the magnetic field from a circular cable with radius a in the xy-plane with the center at the origin. First, we will find the magnetic field on the z-axis and then more generally everywhere in space.

This looks very much like the case of a circle-shaped charge. However, we must now include the more complicated cross-products to find the magnetic field.

Magnetic field along the z-axis. First, we find the magnetic field along the z-axis, where this can be done analytically. We plan to divide the

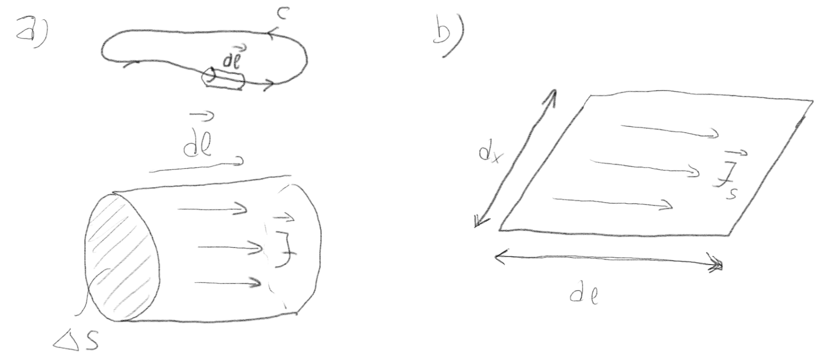


Fig. 10.3 (a) Illustration of a small line element dl from a cable along a curve C. (b) Illustration of a small surface element dS.

circle into small elements, find the contribution from an element, and then integrate over all the elements to find the total field.

We divide the circle into small elements d**l** as illlustrated in Fig. 10.4. The contribution from the element d**l** to the magnetic field is:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \mathrm{d}\mathbf{l} \times \hat{\mathbf{R}}}{R^2} , \tag{10.23}$$

where the **R**-vector is illustrated in the figure.

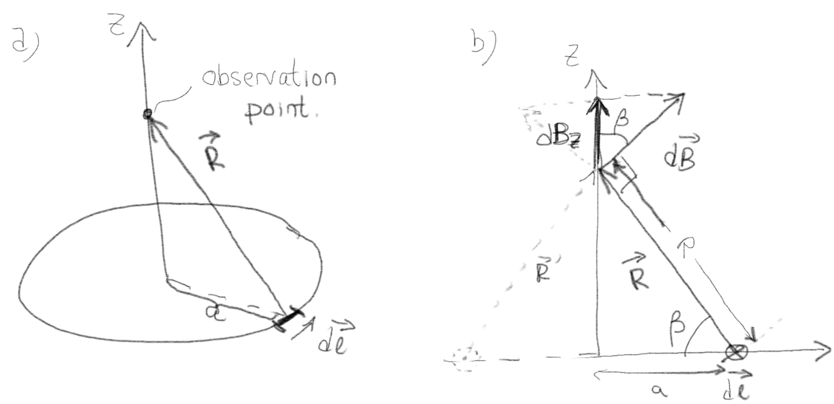


Fig. 10.4 (a) Illustration of a small line element dl from a cable along a curve C. (b) Illustration of a cross-section through the circle to show the geometry.

Using symmetry to simplify the integral. We can now use the symmetry of the system to simplify the integral. From the figure we notice that there are two contributions to the **B**-field. One contribution dB_z along the z-axis and one contribution $\mathrm{d}B_r$ normal to the z-axis. However, for each element $\mathrm{d}\mathbf{l}$, there will also be another element $\mathrm{d}\mathbf{l}'$ on the opposite side of the circle. Both these elements will contribute to with the same $\mathrm{d}B_z$, but their contributions $\mathrm{d}B_r$ will be of the same size and directed in the opposite directions. Thus, the $\mathrm{d}B_r$ components will cancel each other. Therefore, we only need to sum up the contributions $\mathrm{d}B_z$, and they will all be the same.

$$\mathrm{d}B_z = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}l\cos\beta}{R^2} \ , \tag{10.24}$$

where β is the angle between the z axis and $\mathrm{d}\mathbf{B}$ and $\cos\beta = a/R$. The integral is therefore simplified to:

$$B_z = \oint_C \frac{\mu_0}{4\pi}\frac{I\mathrm{d}l\cos\beta}{R^2} = \frac{\mu_0 I\cos\beta}{4\pi R^2}\oint_C \mathrm{d}l = \frac{\mu_0 I(a/R)}{4\pi R^2}2\pi a \ , \tag{10.25}$$

We now use that $R^2 = z^2 + a^2$, getting

$$B_z = \frac{\mu_0 I(a/R)}{4\pi R^2}2\pi a = \frac{\mu_0 I a^2}{2R^3} = \frac{\mu_0 I a^2}{2\left(z^2+a^2\right)^{3/2}} \ . \tag{10.26}$$

Using vector algebra to determine the integral. Instead of using a geometric argument, we can use vector algebra to find the magnetic field. To do this, we introduce a vector representation of the curve integral. We start by representing the two vector quantities, $\mathbf{R}$ and $\mathrm{d}\mathbf{l}$ in a Cartesian coordinate system. We look at an element $\mathrm{d}\mathbf{l}$ which is at an angle θ with the x-axis. This element is at a position $\mathbf{r}' = (a\cos\theta, a\sin\theta, 0)$, and the element points in a direction which is normal to this (which we can find by $\hat{\mathbf{z}} \times \mathbf{r}'$): $\mathrm{d}\mathbf{l} = \mathrm{d}l(-\sin\theta, \cos\theta, 0)$. We then find $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ for a point $\mathbf{r} = (0, 0, z)$, that is, $\mathbf{R} = (-a\cos\theta, -a\sin\theta, z)$. Finally, we find

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\mathbf{l}\times\mathbf{R}}{R^3} \ , \tag{10.27}$$

where $dl = a\mathrm{d}\theta$, giving

$$\mathrm{d}\mathbf{l}\times\mathbf{R} = \mathrm{d}l(-\sin\theta, \cos\theta, 0)\times(-a\cos\theta, -a\sin\theta, z) = (\mathrm{d}lz\cos\theta, \mathrm{d}lz\sin\theta, a\mathrm{d}l) \ . \tag{10.28}$$

Now, the integral over the circle is an integral for θ from 0 to 2π. This means that $\cos\theta$ and the $\sin\theta$ terms will be zero when we integrate, and we are left with the z-component only, giving the same result as above:

$$B_z = \int_0^{2\pi} \frac{\mu_0}{4\pi} \frac{I a^2 \mathrm{d}\theta}{R^3} = \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{3/2}} \,. \tag{10.29}$$

Finding the magnetic field numerically. To find the magnatic field at any point in space, we need to find the magnetic field numerically. We will do this by first parameterizing the curve carrying the current, in this case the circle, discretizing the curve and finally summing up the contributions to the magnetic field from each discrete element on the circle.

We already parameterized the curve above, describing the curve as $\mathbf{r}' = (a\cos\theta, a\sin\theta, 0)$, where θ is the angle with the x-axis. We divide this into N elements of angular extent $\Delta\theta = 2\pi/N$. Element i has a center at $\theta_i = i\Delta\theta$ and extends an angle $\Delta\theta/2$ in each direction. We assume that the element has a length $\mathrm{d}l$ which is given as the arc length, $\mathrm{d}l = a\Delta\theta$. The direction of $\mathrm{d}\mathbf{l}$ of the element is given as above, $\mathrm{d}\mathbf{l} = \mathrm{d}l(-\sin\theta_i, \cos\theta_i, 0)$.

With this discretization, we find the contribution to the magnetic field $\mathrm{d}\mathbf{B}(\mathbf{r})$ in a point $\mathbf{r}$ from element $\mathrm{d}\mathbf{l}$ at position $\mathbf{r}'$ from Biot-Savart's law:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \mathrm{d}\mathbf{l} \times \mathbf{R}}{R^3} \,, \tag{10.30}$$

where $\mathbf{R}$ is the vector from the element at $\mathbf{r}'$ to the observation point, $\mathbf{r}$, that is, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. We can then simply use the `cross`-function in Python to find the cross product.

First, we write a function `bfield` that find the magntic field in a point from the circle:

```
import numpy as np
import matplotlib.pyplot as plt
def bfield(r,a,N):
    # Find the magnetic field in the point r
    # N = number of points of resolution
    # a = radius of circle
    B = np.array([0,0,0])
    dtheta = 2*np.pi/N
    dl = a*dtheta
    for i in range(N):
        theta = i*dtheta
        rdv = np.array([a*np.cos(theta),a*np.sin(theta),0])
        R = r - rdv
        dlv = dl*np.array([-np.sin(theta),np.cos(theta),0])
        dB = np.cross(dlv,R)/np.linalg.norm(R)**3
        B = B + dB
    return B
```

Notice that we have here not included the prefactor $\mu_0/4\pi$, and that we define `B` as a vector and then sum up the contributions `dB` from the individual elements. Notice also how we can use a notation in the program which is very similar to the mathematical notation. This simplifies the translation from mathematics to computer program and makes it easier for us to check the mathematics in the program.

Using the `bfield`-function, we calculate the magntic field in the xz-plane. This provides a good visualization of the field because the field is rotationally symmetric around the z-axis. We find the field over a set of x and z values from -5 to $+5$.

```
# Define observer points
L = 20
N = 100
a = 1.6
x = np.linspace(-5,5,L+1)
z = np.linspace(-5,5,L+1)
x,z = np.meshgrid(x,z)
Bx = x.copy()
Bz = z.copy()
for ix in range(len(x)):
    for iz in range(len(z)):
        r = np.array([x[iz,ix],0,z[iz,ix]])
        Bx[iz,ix],By,Bz[iz,ix] = bfield(r,a,N)
```

Finally, we plot the field using both `quiver` and `streamplot`.

```
plt.figure(figsize=(16,8))
plt.subplot(1,2,1)
plt.quiver(x,z,Bx,Bz)
plt.subplot(1,2,2)
plt.streamplot(x,z,Bx,Bz)
```

The resulting plots are shown in Fig. 10.5. Notice that we here have used a color-coding to show the magnitudes of the field. The code for this visualization is:

```
nBx = Bx / np.sqrt(Bx**2 + Bz**2)
nBz = Bz / np.sqrt(Bx**2 + Bz**2)
BB = np.log10(np.sqrt(Bx**2+Bz**2))
plt.figure(figsize=(16,8))
plt.subplot(1,2,1)
plt.quiver(x,z,nBx,nBz,BB,cmap='jet')
plt.axis('equal')
plt.subplot(1,2,2)
plt.streamplot(x,z,Bx,Bz)
plt.axis('equal')
```

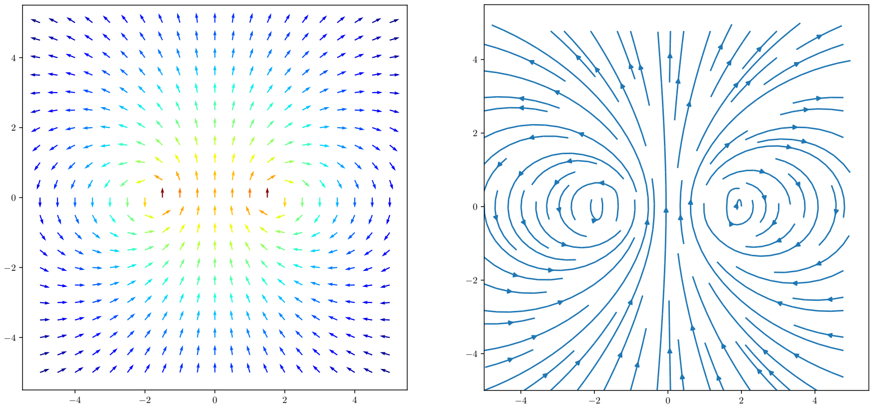


Fig. 10.5 Plot of the magnetic field in the xz-plane from a circular current in the xy-plane.

Comparing with an electric dipole. This looks similar to an electric dipole in the limit when the dipole distance becomes zero. We illustrate this similarity by also plotting the electric field in the xz-plane from a circular charge in the xy-plane. We calculate the field and plot the results.

```
# Comparing with an electric dipole
xx = x.copy()
zz = z.copy()
Ex = xx.copy()
Ez = zz.copy()
r1 = np.array([0,0,0.1])
q1 = 1.0
r2 = np.array([0,0,-0.1])
q2 = -1.0
for ix in range(len(xx)):
    for iz in range(len(zz)):
        r = np.array([xx[iz,ix],0,zz[iz,ix]])
        dr1 = r - r1
        dr2 = r - r2
        Ex[iz,ix],Ey,Ez[iz,ix] = q1*dr1/\
      np.linalg.norm(dr1)**3+q2*dr2/np.linalg.norm(dr2)**3
plt.figure(figsize=(16,8))
plt.subplot(1,2,1)
plt.quiver(x,z,Ex,Ez)
plt.subplot(1,2,2)
plt.streamplot(x,z,Ex,Ez)
```

The resulting plot (using a color scale for the vectors) is shown in Fig. 10.6. We see that there are similarities between the electric field and the magnetic field. The magnet indeed looks like a dipole. However, all the field lines are closed. In every region there are the same number of field lines coming in as therefore are going out. There are therefore no regions with a net divergence of the field.

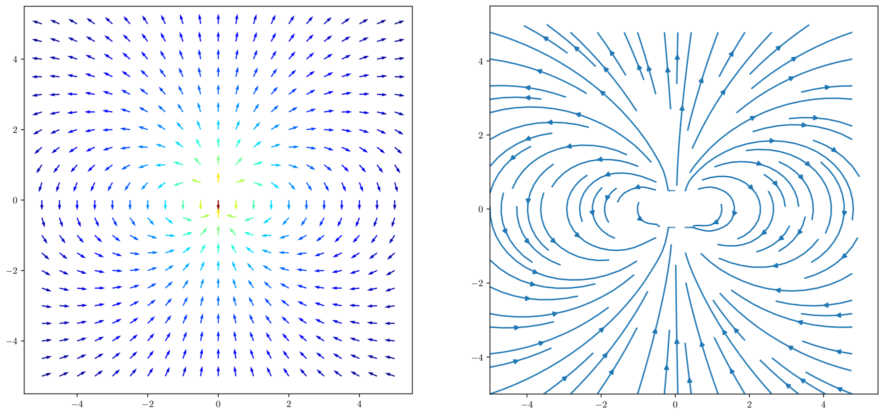


Fig. 10.6 Plot of the electric field in the xz-plane from a circular charge in the xy-plane.

10.1.8 Example: Magnetic field from arbitrary circuit

We have now found a way to find the magnetic field from a circular circuit. However, what if we instead specify a differently shaped circuit, indeed an arbitrarily shaped circuit, how can we find the magnetic field from such a circuit?

The idea is to find a way to describe the circuit in discrete form, in the form of small line segments, and then use this discretized form to find the magnetic field by the superposition principle. Fig. 10.7 show the discretization of a cirlce in 10 and 25 line segments. In each case, we can describe the circuit as a set of N points, $\mathbf{r}_i$, $i = 0, 1, \ldots, N-1$, describing the ends of the line segments. The first line segment is from point $i = 0$ to $i = 1$, the second from $i = 1$ to $i = 2$ and so on. Remember that the circuit is closed, so that the last line segment goes from $i = N - 1$ to $i = 0$.

The contribution to the magnetic field $\mathbf{B}$ at a point $\mathbf{r}$ from a line segement from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$ is given by Biot-Savart's law:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \mathrm{d}\mathbf{l} \times \mathbf{R}}{R^3} \ . \tag{10.31}$$

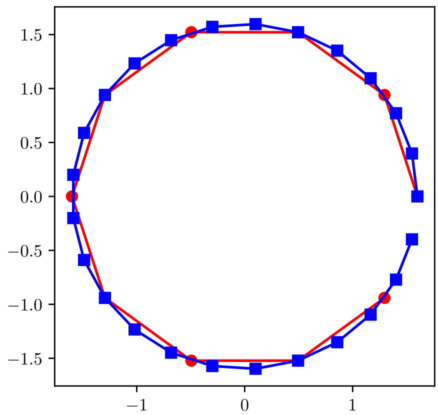


Fig. 10.7 Illustration of two discretizations of a circle into sets of 10 and 25 points.

What are the values for d$\mathbf{l}$ and $\mathbf{R}$ in this case? First, we see that d$\mathbf{l}$ points along the circuit, that is, it point from point i to point $i+1$, that is, d$\mathbf{l} \simeq \mathbf{r}_{i+1} - \mathbf{r}_i$. Second, we use the center of mass of the line segment as the position of the line segment. That is, the position $\mathbf{r}_{i,i+1}$ from the line segment from i to $i+1$ is $\mathbf{r}_{i,i+1} = (\mathbf{r}_i + \mathbf{r}_{i+1})/2$. This means that $\mathbf{R} = \mathbf{r} - \mathbf{r}_{i,i+1}$. We include the constants $I\mu_0/(4\pi)$ into a common constant that we use to scale the results.

Implementation of curve representation. We are then ready to implement the representation of the curve. We describe the curve with the list `list`, where each element represents $\mathbf{r}_i$. We set up such a list to describe a circle of radius a with N elements. The angular distance between two points is then $\Delta\theta = 2\pi/N$ and the position of point i is $\mathbf{r}_i = a(\cos\theta_i, \sin\theta_i, 0)$, where $\theta_i = i\Delta\theta$. We implement this with the following code segment:

```
# Setup list of point for a circle
N = 50
list = np.zeros((N,3),float)
dtheta = 2*np.pi/N
dl = a*dtheta
for i in range(N):
    theta = i*dtheta
    rdv = np.array([a*np.cos(theta),a*np.sin(theta),0])
    list[i,:] = rdv
```

Implementation of function to find the field. We then write a function to find the magnetic field in a point $\mathbf{r}$ due to a current I in the curve described by `list` in the function `bfieldlist`, where the results are scaled with the constant $\mu_0 I/(4\pi)$:

```
def bfieldlist(r,list):
    # Find the magnetic field in the point r
    # list = array(N,3) of points
    B = np.array([0,0,0])
    N = np.shape(list)[0]
    for i in range(N):
        i0 = i
        i1 = i + 1
        if (i1>N-1):
            i1 = 0
        rdv = 0.5*(list[i1,:]+list[i0,:])
        R = r - rdv
        dlv = list[i1,:]-list[i0,:]
        dB = np.cross(dlv,R)/np.linalg.norm(R)**3
        B = B + dB
    return B
```

Find the field in a region in space. We use this function to find the magnetic field $\mathbf{B}(\mathbf{r})$ on a grid in the xz-plane, following methods similar to that we introduced for the electric field.

```
# Find field in region in space
L = 20 # Number of points
b = 5 # Range of region
x = np.linspace(-b,b,L+1)
z = np.linspace(-b,b,L+1)
x,z = np.meshgrid(x,z)
Bx = x.copy()
Bz = z.copy()
for ix in range(len(x)):
    for iz in range(len(z)):
        r = np.array([x[iz,ix],0,z[iz,ix]])
        Bx[iz,ix],By,Bz[iz,ix] = bfieldlist(r,list)
```

Visualization of the magnetic field. Finally, we visualize the magnetic field using the methods we have previously developed.

```
nBx = Bx / np.sqrt(Bx**2 + Bz**2)
nBz = Bz / np.sqrt(Bx**2 + Bz**2)
BB = np.log10(np.sqrt(Bx**2+Bz**2))
plt.figure(figsize=(16,8))
ax1 = plt.subplot(1,2,1)
plt.quiver(x,z,nBx,nBz,BB,cmap='jet')
ax1.set_aspect('equal', 'box')
ax2 = plt.subplot(1,2,2)
plt.streamplot(x,z,Bx,Bz)
ax2.set_aspect('equal', 'box')
```

Test on more complex circuit. Let us now use this method to find the magnetic field from a system of M circular circuits, each with a radius a and spanning from $z = -a$ to $z = a$. First, we set up the `list` of points by extending the approach developed above:

```
M = 50
N = 50
list = np.zeros((N*M,3),float)
dtheta = 2*np.pi/N
dl = a*dtheta
icount = 0
for j in range(M):
    z = 2*a*j/M-a
    for i in range(N):
        theta = i*dtheta
        rdv = np.array([a*np.cos(theta),a*np.sin(theta),z])
        list[icount,:] = rdv
        icount = icount +1
```

Then we generate the magnetic field on the lattice of points in exactly the same way as above, and plot the results. The resulting plot is shown in Fig. 10.7. We notice that the magnetic field is approximately constant inside the set of circular circuits. We will explore such systems more in the following chapter while developing simplified methods to find the magnetic fields in systems with high degrees of symmetry.

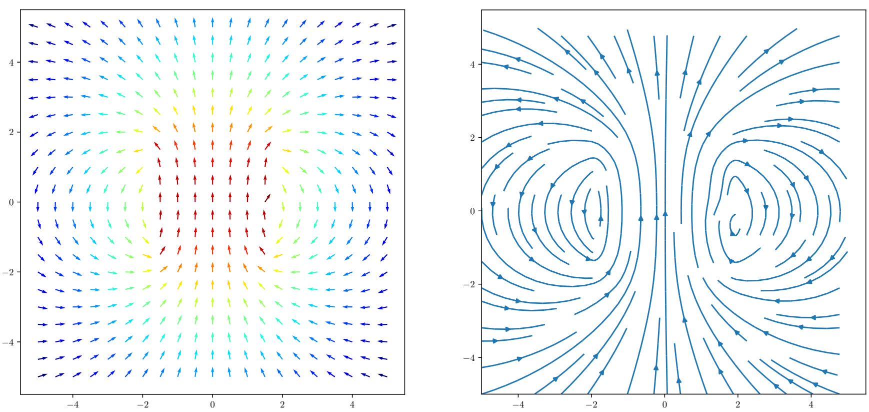


Fig. 10.8 Plot of the magnetic field in the xz-plane from a set of circular currents in the xy-plane.

Implement the methods as functions. You should now be ready to implement the various methods as functions to simplify the methods and make them simpler to use. For example, you may generate methods to calculate the magnetic field on a given regular grid and to visualize the field.

```
def gridbfield(line,L,b):
# Make grid of bfield values
# line: Circuit description
# b: range of grid
# L: number of grid points
    x = np.linspace(-b,b,L+1)
    z = np.linspace(-b,b,L+1)
    x,z = np.meshgrid(x,z)
    Bx = x.copy()
    Bz = z.copy()
    for ix in range(len(x)):
        for iz in range(len(z)):
            r = np.array([x[iz,ix],0,z[iz,ix]])
            Bx[iz,ix],By,Bz[iz,ix] = bfieldlist(r,list)
    return x,z,Bx,Bz
```

```
def visbfield(x,z,Bx,Bz):
    nBx = Bx / np.sqrt(Bx**2 + Bz**2)
    nBz = Bz / np.sqrt(Bx**2 + Bz**2)
    BB = np.log10(np.sqrt(Bx**2+Bz**2))
    ax1 = plt.subplot(1,2,1)
    plt.quiver(x,z,nBx,nBz,BB,cmap='jet')
    ax1.set_aspect('equal', 'box')
    ax1 = plt.subplot(1,2,2)
    plt.streamplot(x,z,Bx,Bz)
    ax1.set_aspect('equal', 'box')
    return
```

The code needed to make and visualize the field is now simplified to:

```
x,z,Bx,Bz = gridbfield(line,10,5)
visbfield(x,z,Bx,Bz)
```

10.2 Magnetic forces

We have now developed effective analytical and numerical methods to find the magnetic field from a given distribution of currents. However, we started from the force law for the force on a moving charge Q with velocity $\mathbf{v}$ from current distribution:

$$\mathbf{F} = Q\mathbf{v} \times \left(\sum_i \frac{\mu_0}{4\pi} \frac{I \Delta \mathbf{l}_i \times \hat{\mathbf{R}}}{R^2} \right) . \tag{10.32}$$

We now know how to find the right-hand part, which is the magnetic field. The force on a particle with charge Q and velocity $\mathbf{v}$ moving in a magnetic field $\mathbf{B}$ is therefore:

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B} \ . \tag{10.33}$$

This expression is called the **Lorentz force**.

10.2.1 Lorentz force

If the charge Q also is moving in an electric field $\mathbf{E}$, the total force is the sum of the forces from the electric and magnetic fields, which we call the Lorentz force:

Lorentz force law

The Lorentz force describes the force on a charge moving in an electric and a magnetic field:

$$\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \ . \tag{10.34}$$

10.2.2 Example: Charged particle in combined fields

Fig. 10.9a illustrates a charge Q moving with a velocity $\mathbf{v}$ that is orthogonal to the magnetic field $\mathbf{B}$. In this case the force, $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$, will be normal to both $\mathbf{B}$ and $\mathbf{v}$. Thus, the charge will move in a circular orbit.

Will the charge slow down due to the interaction with the magnetic field? No, because the force is always normal to the velocity. The magnetic force therefore does no work on the moving charge. It will continue in a circular orbit indefinitely.

What if the particle starts from rest? Unless there are other forces, the particle will remain at rest.

What if there is an electric field in the same direction as the magnetic field, as illustrated in Fig. 10.9b? If the particle starts at rest, the electric field will start to push it in the direction of the field (if Q is positive), In this case, the $\mathbf{E}$-field will do work, accelerating the particle in the direction of the field. The magnetic field will still only act in the direction normal to the velocity, which will make the particle go in a spiral. We notice that the velocity in the direction of the electric field, and in this

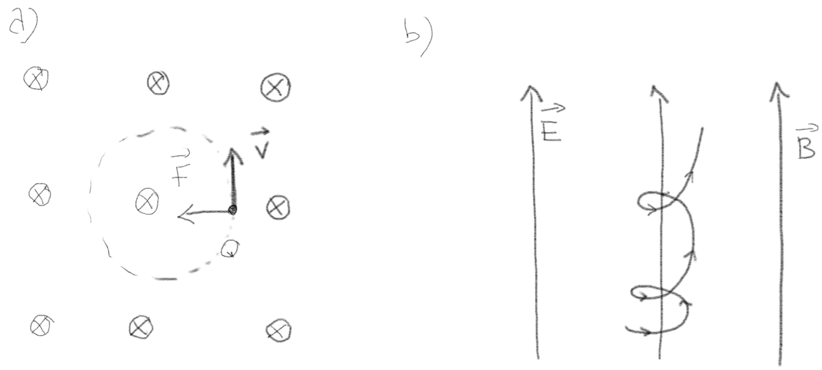


Fig. 10.9 (a) Illustration of a charged particle moving in a magnetic field. (b) Illustration of a charged particle in an electric and a magnetic field.

case also in the direction of the magnetic field, will not contribute to the magnetic force (because the component is parallel to $\mathbf{B}$ and hence will be zero in the cross product). The particle will therefore accelerate along the fields, but the circular motion will remain the same as initially.

10.2.3 Example: Hall effect

The Hall effect is an effect on charge carriers moving in a semi-conductor. In this case, we do not know if the charge carriers are positive or negative. How can we use the behavior of charges in a magnetic field to determine the sign of the charge carriers?

The trick is to apply a uniform magnetic field normal to the semi-conductor as illustrated in Fig. 10.10a. We apply an electric potential along the semi-conductor, resulting in a current density $\mathbf{J}$ along the semi-conductor. For practical purposes we can assume that the semi-conductor is a conductor where we do not know the sign of the charge carriers. Let us therefore analyze the two possibilities: Positive and negative charge carriers.

Positive charge carriers. If the charge carriers are positive, the positive charges are moving in the same direction as the current density $\mathbf{J}$. The magnetic force on a positive charge is therefore in the direction $Q\mathbf{v} \times \mathbf{B}$, which is to the right in Fig. 10.10b. This will mean that positive charges will accumulate on the right hand side, and the left hand side will be depleted of positive charges and hence negatively charged. This will result in an electric field, $\mathbf{E}_h$ directed towards the left, which is called

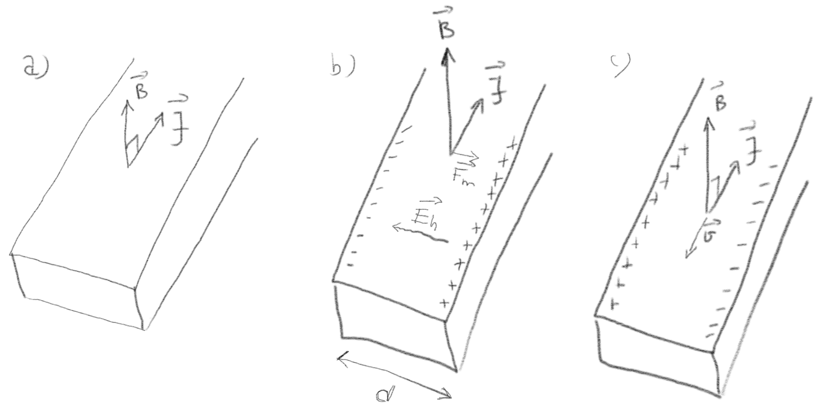


Fig. 10.10 (a) Illustration of the magnetic field and the current density in a semi-conductor. (b) Illustration of behavior of a positive charge. (c) Illustration of the behavior of a negative charge.

the *Hall field*. The charges will continue to build up until the force on a charge from the magnetic force is the same as the force from the electric field, E_h. In equilibrium the two forces will be the same, and charges will propagate along the semi-conductor: $QE_h = QvB$, which gives us that $E_h = vB$. We can measure E_h, this is the potential difference between the two sides of the semi-conductor. Thus, we can actually measure v for this system!

Negative charge carriers. What happens if the charge carriers are negative? First, we realize that a negative charge will move in the direction opposite of $\mathbf{J}$ in order to get the same net charge flowing through a cross-section of the semi-conductor. The magnetic force on the charge will therefore be in the same direction, to the right, as for positive charge carriers. But now negative charges will build up on the right-hand side, generating a potential difference $-E_h$. This means that we can determine if the charge carriers are positive or negative by measuring the Hall field or the Hall potential.

But there should not be any electric field inside a conductor. Does the argument here counter the assumption we made about conductors — that there should not be any electric field inside a conductor? No. This was under the assumption that the only force on a charge was $Q\mathbf{E}$. In this case, there could not be a net force on a charge in a conductor. However, when there is a magnetic field present, we must modify this. In the stationary case, the net force $\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B}$ must be zero inside the conductor. And this is indeed the case for the Hall effect.

10.2.4 Magnetic forces on a current element

What is the force on a small volume element $\mathrm{d}v$ with a current density $\mathbf{J}$? We assume that each charge Q in the element is moving with a velocity $\mathbf{v}$ and that there are $N\mathrm{d}v$ charges in $\mathrm{d}v$. The net force on the element is the sum of the forces on each charge, $\mathrm{d}\mathbf{F} = N\,\mathrm{d}v\,Q\mathbf{v}\times\mathbf{B}$, where we recognize $NQ\mathbf{v} = \mathbf{J}$, therefore:

Magnetic force on a current density element

The magnetic force from a current element $\mathbf{J}\mathrm{d}v$ from a magnetic field $\mathbf{B}$ is

$$\mathrm{d}\,\mathbf{F} = \mathbf{J}\mathrm{d}v\times\mathbf{B}\,. \tag{10.35}$$

Force on a current element. For a current element $I\mathrm{d}\mathbf{l} = \mathbf{J}\mathrm{d}v$ we get

$$\mathrm{d}\,\mathbf{F} = I\mathrm{d}\mathbf{l}\times\mathbf{B}\ . \tag{10.36}$$

and similarly for a surface current density $\mathbf{J}_S$, we get

$$\mathrm{d}\,\mathbf{F} = \mathbf{J}_S\mathrm{d}S\times\mathbf{B}\ . \tag{10.37}$$

For a piece of a wire of length L normal to the magnetic field, the force is therefore $F = ILB$.

Force on a closed circuit. We find the force on a closed circuit by integrating the contributions from each circuit element $\mathrm{d}\mathbf{l}$:

$$\mathbf{F} = \oint_C I\mathrm{d}\mathbf{l}\times\mathbf{B}\ . \tag{10.38}$$

For a **uniform magnetic field**, we can place $\mathbf{B}$ outside the integral, getting:

$$\mathbf{F} = \left(\oint_C I\mathrm{d}\mathbf{l}\right)\times\mathbf{B} = 0\times\mathbf{B} = 0\ . \tag{10.39}$$

10.2.5 Example: Force on a closed circuit

Fig. 10.11 shows two different situations for a closed circuit with a current I in a magnetic field. In Fig. 10.11a we see that the forces on all the elements are pointing into the center of the circuit, attempting to compress the circuit. The net force is zero. In Fig. 10.11b the forces along the two parts of the circuit (C_2 and C_4) that are parallel to the magnetic field are zero, whereas the forces on the two other sides (C_1 and C_3) act out of the plane of the circuit.

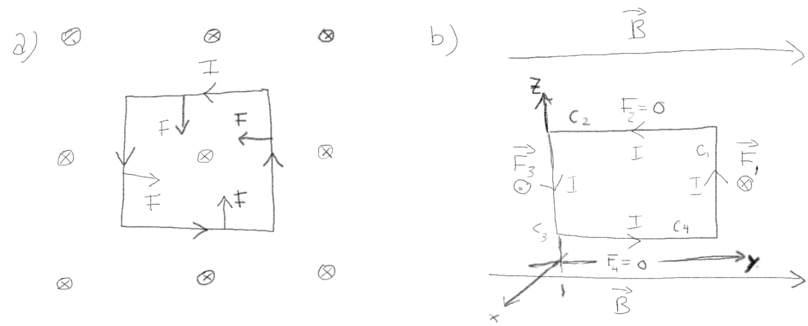


Fig. 10.11 Illustration of a closed circuit in a uniform magnetic field for two orientations of the magnetic field.

In this case the net force is zero, but the net torque around the z-axis is not zero. This is the principle of an electromotor. The torque around an axis z of a force $\mathbf{F}$ is $\tau = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is a vector from the axis to the point where the force acts and $\mathbf{F}$ is the force. The torque of $\mathbf{F}_2$ and $\mathbf{F}_4$ around the z-axis is the sum of the torques of each of the forces. The torque of $\mathbf{F}_4$ is zero since $\mathbf{r}$ is zero in this case. If the length of C_4 is b, the torque of $\mathbf{F}_2$ is

$$\tau = b\hat{\mathbf{y}} \times \mathbf{F}_2 \,, \tag{10.40}$$

where

$$\mathbf{F}_2 = \int_{C_2} I \mathrm{d}\mathbf{l} \times \mathbf{B} = I \int_{C_2} \mathrm{d}\mathbf{l} \times \mathbf{B} = -IaB\hat{\mathbf{x}} \,. \tag{10.41}$$

where a is the length of C_2. This gives

$$\tau = b\hat{\mathbf{y}} \times (-IaB\hat{\mathbf{x}}) = bIaB\hat{\mathbf{z}} = ISB\hat{\mathbf{z}} \,, \tag{10.42}$$

where $S = ab$ is the area of the circuit. The oriented area of the circuit is $\mathbf{S}$. The direction of this area is given by the right-hand rule as in the x-direction. The torque, τ, can therefore be written as

$$\tau = I\mathbf{S} \times \mathbf{B} \,. \tag{10.43}$$

We call the term $I\mathbf{S}$ the magnetic moment of the circuit. This is a term you will meet and use in many contexts in your future studies of physics.

idcmagnetic moment

Torque of magnetic moment

This is general result. We call the vector $\mathbf{m} = I\mathbf{S}$ the **magnetic moment** of the circuit. The torque of a magnetic moment $\mathbf{m}$ in the field $\mathbf{B}$ (around the center of the circuit) is:

$$\tau = \mathbf{m} \times \mathbf{B} \, . \tag{10.44}$$

What happens to a circuit in a magnetic field and electromotors. What happens to the circuit in Fig. 10.11b if it is hinged around the z-axis (or around an axis parallel to the z-axis through the center of the circuit)? The torque will make the circuit start to rotate. However, if the circuit has rotated an angle $\pi/2$, the magnetic moment of the circuit will have rotated to be parallel to the magnetic field, and the torque, $\tau = \mathbf{m} \times \mathbf{B}$ will be zero. If the circuit has some inertia, it will rotate past this angle. But the torque from the magnetic field would then stop the rotation. This is therefore only the start of an electromotor. To get the circuit to continue to rotate, we need to change the direction of the magnetic field so that the torque continues to have the same direction (sign) after a rotation of $\pi/2$. If the magnetic field is from a permanent magnetic, this means that we need to change the direction of the magnet. However, if the magnetic field is generated by the current flowing in a circuit, we can flip the direction of the field by changing the direction of the current in the circuit. This type of motor is called a brush-less motor. (In practice, we do not use a single circle to generate the magnetic field, but instead a series of circles as we will discuss in the next chapter).

Why is it called a brush-less motor? Where is the brush? This comes from the way an electromotor is constructed using a permament magnet. Instead of changing the direction of the magnetic field, we can also change the direction of the magnetic moment by changing the direction of the current in the circuit. In practice this is done by having a set of brushes that a part of the circuit slides along, which flips the direction of the current when the circuit has rotated $\pi/2$. This is illustrated in Fig. 10.12.

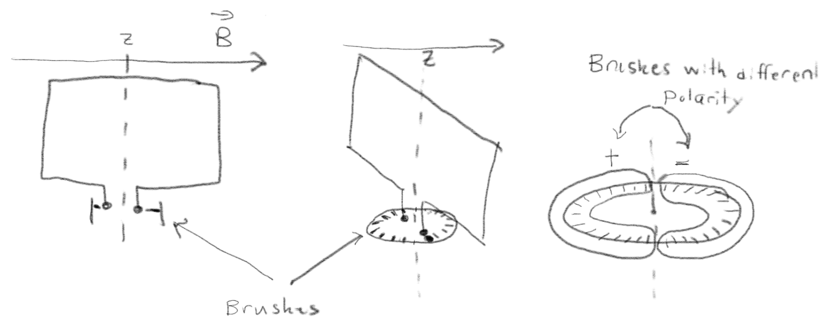


Fig. 10.12 Illustration of the brushes in an electromotor in a constant magnetic field $\mathbf{B}$. The direction of the current in the circuit changes direction by dividing the brush into two brushes with different potentials. When the motor rotates so that it is parallel to the magnetic field, the current changes direction.

10.3 Summary

The contribution to the **magnetic field** in a point $\mathbf{r}$ from a current element $I\mathrm{d}\mathbf{l}$ at $\mathbf{r}'$ is given by Biot-Savart's law for a current element:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \ ,$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

The **magnetic field B** from a current loop C is given by **Biot-Savart's** law for current loops:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\oint_C \mathbf{I}\mathrm{d}\mathbf{l} \times \mathbf{R}R^3 \ .$$

Magnetic fields obey the **superposition principle**:

$$\mathbf{B} = \sum_i \mathbf{B}_i$$

Biot-Savart's law on differential form states that the contribution $\mathrm{d}\mathbf{B}$ to the magnetic field at $\mathbf{r}$ from a current density $\mathbf{J}$ at $\mathbf{r}'$ is

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \hat{\mathbf{R}}}{R^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \mathbf{R}}{R^3} \ ,$$

where $\mathbf{R}$ is the vector from the volume element $\mathrm{d}v$ at $\mathbf{r}'$ to $\mathbf{r}$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

Biot-Savart's law on differential form for a current density $\mathbf{J}$ at $\mathbf{r}'$ is

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \hat{\mathbf{R}}}{R^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \mathbf{R}}{R^3} \ ,$$

where $\mathbf{R}$ is the vector from the volume element $\mathrm{d}v$ at $\mathbf{r}'$ to $\mathbf{r}$: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

The **Lorentz force** describes the force on a charge Q moving in an electric and a magnetic field:

$$\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \ .$$

The **magnetic force** on a current element $I\mathrm{d}\mathbf{l}$ from the interaction with the magnetic field $\mathbf{B}$ is

$$\mathrm{d}\,\mathbf{F} = I\mathrm{d}\mathbf{l} \times \mathbf{B} \ .$$

We call $\mathbf{m} = I\mathbf{S}$ the **magnetic moment** of a circuit. The torque of a magnetic moment $\mathbf{m}$ in the field $\mathbf{B}$ is:

$$\tau = \mathbf{m} \times \mathbf{B} \ .$$

10.4 Exercises

Learning outcomes. Denne uken arbeider vi med å bygge opp forståelse for hvordan $\mathbf{B}$-feltet oppstår, hvordan vi kan regne det ut, og hvordan det påvirker en ladning i bevegelse eller et strømelement. Vi kommer til å bruke mange av de samme teknikkene som vi brukte til å finne $\mathbf{E}$-feltet fra ladningsfordelinger. Det er spesielt viktig å lære seg hvordan vi bruker høyrehåndsregelen, hvilken retning $\mathbf{R}$ vektor har, og hvordan vi utfører linje, flate og volume-integraler over strømfordelinger.

10.4.1 Test yourself

Exercise 10.1: Biot Savart's law

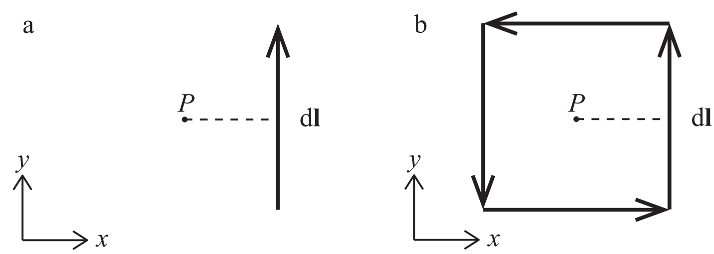
a) Write down Biot-Savart's law for a line current element.

b) Make a drawing of the line element where you show where you calculate the magnetic field, and where you draw the $\mathbf{R}$ vector. Also show the direction of the magnetic field.

c) What happens to the magnetic field if you double the current? If you double the distance from the element to the observation point?

Exercise 10.2: Superposition

The figure below illustrates a line of length $\mathbf{l}$ with a current I.



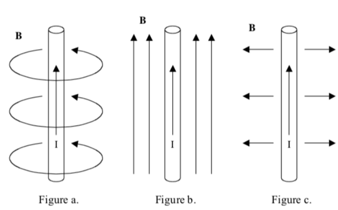
a) What is the direction of the magnetic field in the point P?

b) If the contribution to the magnetic field in part (a) is $\mathbf{B}_a$, what is the magnetic field due to four such line elements making up a complete circuit?

10.4.2 Discussion exercises

Exercise 10.3: Symmetrier for magnetfeltet

Figuren viser en tynn wire med en uniform strøm I. Dette setter opp et magnetfelt $\mathbf{B}$. Hvordan kan vi vite at dette magnetfeltet kun har en tangential (azimuthal) komponent? Hvorfor har det ikke en z-komponent eller en radielle komponent? Hvordan vil du argumenter for at det ikke er en z eller r-komponent i $\mathbf{B}$-feltet? Bruk symmetri eller andre egenskaper ved $\mathbf{B}$-feltet som du kjenner.



Hint. Det kan være lurt å tenke på divergensen til $\mathbf{B}$-feltet.

Exercise 10.4: Model for permanent magnet

La oss tenke oss at en modell for en permanent magnet er at magnetfeltet kommer fra ladninger (for eksempel elektroner) som beveger seg i atomene i form av spinn. Hvordan kan du bruke en slik modell til (i) å si noen om det magnetiske feltet rundt en permenent magnet, (ii) å si noe om hva som skjer om man deler en permanent magnet i to.

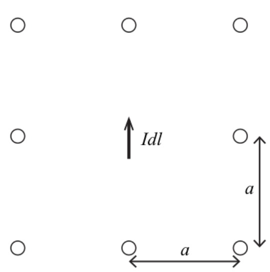
Exercise 10.5: Magnetisk nord

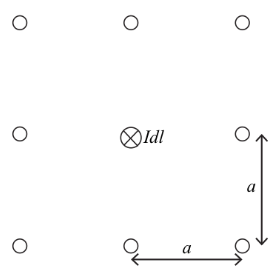
Hvordan kan du relatere jordens magnetfelt til modellene for elektromagneter du har lært om denne uken

Hint. Jorden har en flytende, metallisk kjerne.

Exercise 10.6: Oppvarming med $\mathbf{B}$-felt

Finn $\mathbf{B}$-feltet i punktene.





10.4.3 Tutorials

Exercise 10.7: Finding $\mathbf{B}$ from a line segment

In this tutorial we will focus on building skills in setting up the integrals needed to find the magnetic field from a current distribution using Biot-Savart's law.

We will see how we can find the magnetic field from a line current. There is a current I in a wire along the x-axis. The current runs in the positive x-direction. We want to calculate the resulting field at a position $\mathbf{r} = (x, y, z) = (0, 0, z)$.

a) Make a drawing of the system. Include a line element $d\mathbf{l}$ in your drawing, and the vector $\mathbf{R}$. Also express $\mathbf{R}$ on coordinate form $\mathbf{R} = (R_x, R_y, R_z)$.

b) Find an expression for the contribution $\mathrm{d}\mathbf{B}$ to the magnetic field at $\mathbf{r}$ from the line element $\mathrm{d}\mathbf{l}$. First do it geometrically (without writing vector on Cartesian form), then do it using the Cartesian coordinate form for $\mathbf{R}$. (Notice that vector algebra actually works).

c) Set up the integral to find the magnetic field $\mathbf{B}$ in the case when (i) the line is inifintely long, (ii) the line goes from minus infinity to $x = 0$, (iii) the line goes from $x = -L/2$ to $x = L/2$. (For case (ii) and (iii) we would need to connect the wire to other wires for the system to be physically reasonable with a constant current).

You may use that

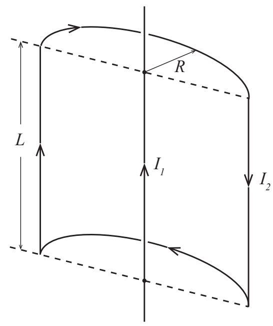
$$\int \frac{1}{(x^2 + z^2)^{3/2}} \mathrm{d}x = \frac{x}{z^2\sqrt{x^2 + z^2}} \,. \tag{10.45}$$

d) Find the $\mathbf{B}$-field in case (i), (ii) and (iii).

e) Check that when $z \gg L$ your result in (iii) gives back the result for the magnetic field from an infinite line current.

Exercise 10.8: Force on a current element

An inifinitely long, straight wire that carries a current I_1 is partialle surrounded by the loop shown in the figure. The loop has a length L and a radius R, and carries a current I_2. The axis of the loop coincides with the infinite wire. Calculate the force that acts on the loop.



The surface current density in a thin sheet at $z = 0$ is $\mathbf{J}_a = J_a\hat{x}$. Find $\mathbf{B}$ using Biot-Savart law. Carefully introduce a surface element, $dA = dx\,dy$, and the vector $\mathbf{R}$. It may simplify the calculation to use vector coordinates.

Hint. Divide the loop into four separate segments and find the force on each segment.

10.4.4 Homework

Exercise 10.9: Magnetic field above a circular circuit

We will study a circular circuit in the xy-plane, with radius a and with its center in the origin. A current I runs in the positive rotational direction (oppsite the clock).

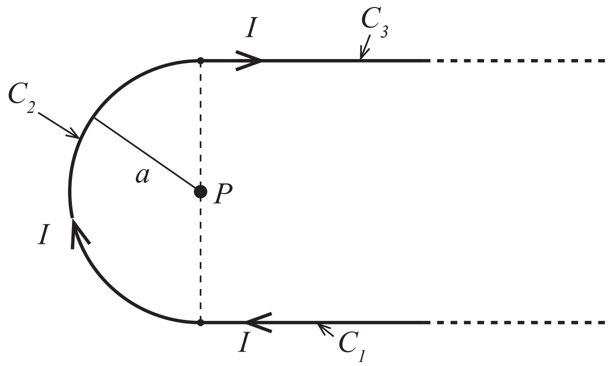
a) Find an expression for the vector $\mathbf{R}$ from a point $(x, y, 0)_{\text{Cartesian}} = (a, \phi, 0)_{\text{cylindric}}$ on the circuit and to a point $(0, 0, z)_{\text{Cartesian}} = (0, \phi, z)_{\text{cylindric}}$ on the z-axis. Express this vector using a and z. Also find an expression for $R^2 = |\mathbf{R}|^2$ og $\hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|}$

b) Find an expression from the contribution d$\mathbf{B}$ to the magnetic field in the point $(0, 0, z)$ from an infinitesimal current element d$\mathbf{I}$ $= I$ d$\mathbf{l}$ in the circular circuit.

c) Find the magnetic field in $(0, 0, z)$.

Exercise 10.10: Infinite bent wire

(By René Ask)



a) An infinite thin wire is bent about a point P in a half-circle of radius a with P as its center. Through the wire runs a constant current I. Find the magnetic field at point P.

b) Now that you have found the analytical expression for the magnetic field at P, write a code that computes the magnetic field at P numerically. You should use the answer you found analytically in the last problem to check that your code produces the correct result.

c) Extend your code so that it can compute the magnetic field at any point $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.

d) Extend the code you wrote so that it plots the streamlines of the magnetic field in the yz-plane at $x = 0$.

Exercise 10.11: Modelling the magnetic field of Earth

(By René Ask)

In this exercise, we want to model the processes that ultimately lead to the production of Earth's magnetic field. From Maxwell's equations (or Biot-Savart's law for that matter), we know that currents produce magnetic fields. Do model this process, we have to make some simplifying assumptions to make our computations tractable. In the core of the Earth, ion-flow (flow of charged iron atoms) of primarily iron create currents that produce magnetic fields both locally and globally. We're more interested in the global field.

a) To begin this modelling process, we can assume that Earth is simply an empty spherical shell with radius a that is rotating counter-clockwise with an angular velocity ω about the z-axis. For simplicity, let's assume that its surface is covered with a uniform charge density σ which represents the iron ions. Find the magnetic field at the sphere's center.

b) Now that we're a little more familiar with what we're dealing with, write a program that computes the magnetic field at any point $r = (x, y, z)$ and plot the streamlines of the magnetic field. Can you explain the physical reason behind the behaviour of the magnetic field inside the sphere? Do you think this is a reasonable approximation to Earth's magnetic field?

But before you work out a solution, wait! And read this short introduction to an integration method that will give a much faster code than what ordinary integration methods can. Also, you may not have learnt much probability theory yet so don't get too hung up on the details. Focus on understanding how you implement the algorithm instead. Since the computation of the magnetic field will involve multi-dimensional integrals, it's far more convenient to use an integration technique known as Monte-Carlo integration. Although this problem can just as easily be solved by simply extending the trapezoidal rule or midpoint rule or any other standard integration technique to two dimensions, such integration methods are computationally expensive in higher dimensions. The simplest form of Monte-Carlo integration boils down to the following: Suppose we have an integral $\int_a^b f(x)dx$ and suppose $p_x(x)$ is a uniform probability distribution on the interval (a, b). The probability distribution is then given by $p_x(x) = 1/(b-a)$ on this interval and is zero elsewhere. Let $p_u(u) = 1$ be the uniform distribution on the interval $(0, 1)$. It's more convenient to work with the distribution at the interval $(0, 1)$, so transforming from $p_x(x)$ to $p_u(u)$ is desirable. Such a transformation must conserve probability, which is to say

$$p_u(u)du = du = p_x(x)dx = \frac{dx}{b-a} \tag{10.46}$$

Integrating the equation gives

$$u = \int_a^x \frac{dx'}{b-a} = \frac{1}{b-a}(x-a), \tag{10.47}$$

which can be solved for x to give us the following formula

$$x(u) = (b-a)u + a. \tag{10.48}$$

Now what is this to be used for? Well, we can now sample $u \in (0,1)$ and then obtain the actual x-values on the interval (a,b). But that's not the whole story. I still haven't explained how this will all help us. Here it goes: we take our integral and rewrite it in the following way (recall that $du = p_u(u)du = p_x(x)dx$):

$$\int_a^b f(x)dx = \int_a^b \frac{f(x)}{p_x(x)} \underbrace{p_x(x)dx}_{=du} = \int_0^1 \frac{f(x(u))}{p_x(x(u))}du = (b-a)\int_0^1 f(x(u))du. \tag{10.49}$$

since $p_x(x) = 1/(b-a)$. What do we do with this expression? Well, the integral is now an expectation value of the function $f(x(u))$ with a uniform probability distribution on $(0,1)$. Therefore, we can approximate it in the following way:

$$\int_a^b f(x)dx = (b-a)\int_0^1 f(x(u))du \approx \frac{(b-a)}{N}\sum_{i=1}^{N} f(x(u_i)). \tag{10.50}$$

Now that's simple! And this can be extended to a d-dimensional integral in a straight forward manner:

$$\int f(\mathbf{x})d^d\mathbf{x} = \int_{a_1}^{b_1} \cdots \int_{a_d}^{b_d} f(x_1, ..., x_d)dx_1 \cdots dx_d \tag{10.51}$$

$$= \prod_{j=1}^{d}(b_j - a_j)\int_0^1 \cdots \int_0^1 f(x_1(u_1), ..., x_d(u_d))du_1 \cdots du_d \tag{10.52}$$

$$\approx \frac{\prod_{j=1}^{d}(b_j - a_j)}{N}\sum_{i=1}^{N} f(x_1(u_{1,i}), ..., x_d(u_{d,i})) \tag{10.53}$$

Let's pause for a minute and think about why this integration method is a better choice in higher dimensions. If we were to approximate a d-dimensional integral with, say, the midpoint rule, we would have d sums! With Monte-Carlo integration, there's always only a single sum regardless of how many dimensions we work in.

c) To make our model of Earth's currents a little more realistic, we can fill the inside of our Earth with a uniform (volume) charge density ρ. Modify your code from the previous problem to compute and visualize the magnetic field lines (using streamplot). You should compare the resulting streamlines to the ones you found using the spherical shell model. Which model do you think produce a magnetic field that most closely resemble Earth's?

Exercise 10.12: Derive field of magnetic dipole

(By: Sigurd Sørlie Rustad)
In this exercise we are going to derive the magnetic field for a magnetic dipole. First we are going to consider the field from an electric dipole. Lets say we have two oppositely charged particles a distance d from eachother. We place origin between the two particles and the particles are placed along the z-axis.

a) Find the electric field in $\mathbf{r} = (x, y, z)$, where $|\mathbf{r}| \gg d$.

b) Use the potential to find the electric field $\mathbf{E}(\mathbf{r})$.

c) It turns out that the magnetic field from a magnetic dipole has the same form as the field as an electric dipole, if you do the necessary substitutions. Do the substitutions and find the magnetic field $\mathbf{B}$.

Exercise 10.13: Visualize the field from a dipole

(By Sigurd Sørlie Rustad)
In this exercise we are going to visualize the field from a magnetic dipole with a streamplot. The field from a dipole is given by the equation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3}\right) \tag{10.54}$$

Make a function that takes magnetic moment $\mathbf{m}$, its location and the position $\mathbf{r}$ you want to evaluate the field. The output should be the resulting magnetic field. You only need to do it in 2D. Make a streamplot to visualize the field.
Extra challenge: Try to vectorize your code. If you do it properly it will be compatible with both 2D and 3D. NumPy has several good packages that you can use to vectorize your code. The ones used in the solution are `tensordot` to dotproduct along an axis and `linalg.norm` to find the length of several vectors along an axis.

(By Sigurd Sørlie Rustad)
Here we are going to visualize the magnetic force between two moving particles, in 3D. First we need to find the magnetic force between the two particles.

a) Write down the equation the for magnetic force between two moving particles.

b) Write a function (in Python or MatLab) that takes in the positions of the particles, their charges and their velocities, and outputs the magnetic force from one of the particles on the other.

c) Plot the velocity vectors for both particles and the force applied on one of the particles. Try for different initial conditions. Do you get what you expected?

Hint 1. To get reasonable sizes use a scaled version for the marmeability of vacuum.

Hint 2. Reasonable initial conditions are: $\mu_0 = 1$, $Q_1 = 1$, $Q_2 = 2$, $\mathbf{v}_1 = [0, -4, 0]$, $\mathbf{v}_2 = [0, 4, 0]$, $\mathbf{r}_1 = [0, 0, 0]$ and $\mathbf{r}_2 = [1, 0, 0]$.

Exercise 10.14: Modelling a 3d ion trap

(By René Ask)

At CERN, physicists trap antihydrogen, antiprotons and positrons in so called *Penning traps* to study their properties. In this exercise, we'll look at how we can go about and create such a trap with theory from electromagnetism. For our purpose we'll assume we're going to trap an ion with a charge $q > 0$ and a mass m. We'll assume that the trap is in a vacuum with no charge distibution within.

a) Imagine we want the particle to stay located in a small area in space. The simplest such area to represent mathematically would be a sphere, wouldn't it? To create a spherical trap, we can imagine that we set up a continuous distribution of charges to create a spherical harmonic oscillator (kinda like three decoupled springs with a mass attached to each) such that its potential is

$$V(x, y, z) = A(x^2 + y^2 + z^2), \tag{10.55}$$

for an appropriate constant A. This way the particle would oscillate back and forth about the origin with certainty. Unfortunately, nature is not that simple. Show that this potential cannot exist in a vacuum.

Hint. Use Laplace's equation.

b) To create a potential that can exist in a vacuum, we can modify the former as

$$V(x, y, z) = A(\alpha x^2 + \beta y^2 + \gamma z^2), \tag{10.56}$$

where $\alpha, \beta, \gamma \neq 0$ are real constants. Find constraints on these constants such that the potential can exist in a vacuum and show that from a suitable choice of constraints we can obtain the potential (there's more than one choice)

$$V(x, y, z) = A(x^2 + y^2 - 2z^2), \tag{10.57}$$

Hint. Again, use Laplace's equation.

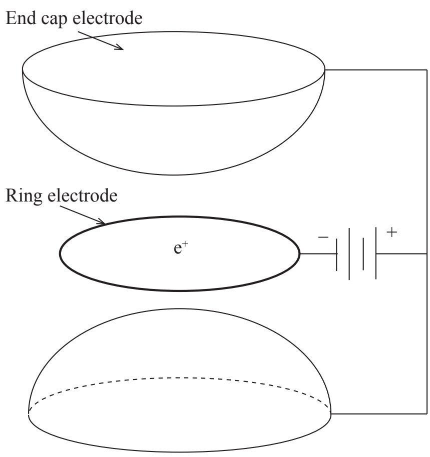


Fig. 10.13 A simple setup for the trap with potential $V(x, y, z)$.

c) Fig. 10.13 shows as simple setup to create the trap with the same form as the potential we're studying. To trap positive ions, we want the end caps to be held at a static positive potential V_0 and the ring to be held at at static negativ potential $-V_0$. Let the radius of the ring be r_0 and the end caps be placed a length z_0 from the center of the ring. Using cylindrical coordinates $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$, the potential can be written as

$$V(r, z) = A(r^2 - 2z^2). \tag{10.58}$$

Find the constant A and the relationship between r_0 and z_0 and use the results to show that the potential for the trap can be written as

$$V(r,z) = \frac{V_0}{r_0^2}(2z^2 - r^2). \tag{10.59}$$

d) Show that this potential cannot trap the ion.

Hint. Use the second derivative test (Hessian determinant). Or plot it and use it to explain why you can't trap the ion.

e) So using a static electric field doesn't seem to be enough (And indeed it isn't). Maybe we could add some more complicated feature to our trap. But we've already done alot of work with the field we already have, and even though it can't trap the atom in the radial direction, it could trap the ion in the z-direction if the ion has low enough energy. The *simplest* idea would be to add a uniform magnetic field. Why? Recall that the Lorentz' force $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$ will bend the particle in a direction that is normal to both the velocity and magnetic field. Now, figure out which direction the magnetic field should point in order to successfully trap the particle.

f) Combine the electric field and the uniform magnetic field to find the equations of motion of the particle. Solve these numerically to successfully trap a particle and make a 3D plot of its trajectory. You can play around with parameters, but to obtain a stable entrapment, $|\mathbf{B}| > \sqrt{8V_0 m/qr_0^2}$ is required (to derive this criterium is beyond the scope of this exercise).

Hint. Use Newton's 2.law. Use Euler-Cromer to solve it numerically (you need an algorithm that conserves energy).

Exercise 10.15: Magnetfeltet fra et uendelig strømførende plan

Vi har en uendelig stor, tynn ladet plate med uniform strømtetthet $\mathbf{J_s} = J_s\hat{\mathbf{x}}$ i xy-planet.

a) Hvilke symmetrier har vi i dette systemet? Hva har disse symmmetriene å si for det magnetiske feltet?

b) Finn det magnetiske feltet $\mathbf{B}(x,y,z)$ både over og under platen.

I figur 10.14 ser vi en strømførende ledning. Den delen av ledningen som har formen til en halvsirkel ligger inne i en region med et uniformt magnetfelt som peker rett ut fra oppgavearket. Retningen til ladningsbærerne indikeres av retningen til pilene. Finn kraften som virker på ledningen på grunn av magnetfeltet.

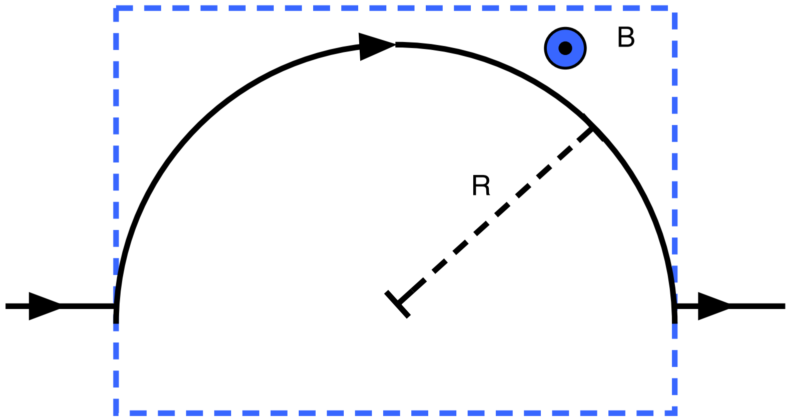


Fig. 10.14 En strømførende ledning som delvis ligger i et magnetfelt. Retningen til magnetfeltet er ut av arket.

Exercise 10.16: Magnetisk felt fra et linje-element

En strømkrets består av flere rette linjestykker som er koblet sammen. For å kunne finne det magnetiske feltet fra en slik strømkrets skal vi utvikle en metode til å finne det magnetiske feltet fra ett enkelt linjestykke.

a) Det går en strøm I gjennom et infinitesimalt linjestykke $d\mathbf{l}$ (et strømelement) langs x-aksen fra x_0 til $x_0 + dx$. Hva er det magnetiske feltet fra dette strømelementer i et punkt $\mathbf{r} = (x, y, z)$?

b) Anta at du kan finne det magnetiske feltet fra en strøm I gjennom et linjestykke langs x-aksen fra $x = a$ til $x = b$ ved å dele linjestykket opp i N like store deler (strømelementer) med lengde $(b - a)/N$ og summere de magnetiske feltene fra hvert slikt strømelement. Skriv et program som finner det elektriske feltet $\mathbf{B}$ i et oppgitt punkt $\mathbf{r}$.

c) Forklar hvordan du vil skrive et program for å finne det magnetiske feltet fra en strømløkke som går gjennom punktene $(a, 0, 0)$, $(b, 0, 0)$, $(b, d, 0)$, $(a, d, 0)$. (Du behøver ikke skrive programmet).

10.4.5 Modeling projects

Exercise 10.17: Simulating a particle accelerator

(By Bror Hjemgaard)

The aim of this exercise is to simulate a particle accelerator. But before we construct the accelerator, we need to take a look at the theory behind it. The accelerator can be broken down into two types of sections: linear sections and curved sections. The task of the linear section is to accelerate the particle, while the curved section curves the path of the particle. Our model will consist of 8 parts: 4 linear sections and 4 curved sections, making a loop. See the figure below. The curved sections of the accelerator are all quarts of a circle with radius R. The linear sections are all of length L.

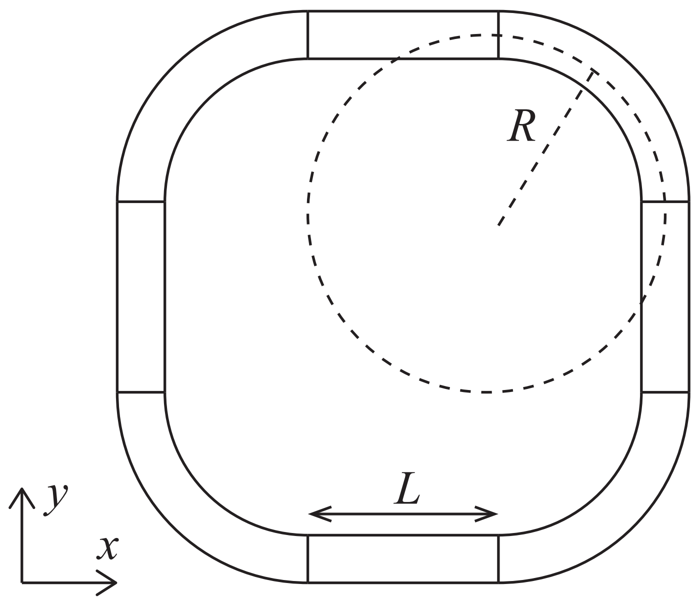


Fig. 10.15 The geometry of the accelerator.

a) Each of the sections only rely on one of either the electric force or the magnetic force. Using your knowledge of the electromagnetic fields, which section do you think uses which field?

Hint. Compare the vector-properties of Coulombs law and the Lorentz force (without **E**-field)

b) Write down the mathematical expressions of the force due to

- an electric field $\mathbf{E}$ on a particle with charge q.
- a magnetic field $\mathbf{B}$ on a particle with charge q and velocity $\mathbf{v}$

c) Using a sophisticated numerical integration method (not forward-Euler), write a (vectorized!) program that calculates the change in velocity and position due to the electric force over a small time interval dt

d) Between the linear sections the particle are in the curved sections where there is a magnetic field $\mathbf{B}$. In order for the field to curve the path of the particle (lets assume it is of positive charge) in the wished direction, what is the direction of the magnetic field?

Hint. Use the right-hand-rule

e) The curved sections of the accelerator are all quarts of a circle wtih radius R, and they are fitted with magnetic field generators that can vary its strength on command. How must we vary $\mathbf{B}$ in order to assure R is constant?

Hint. If we wish for the particles to follow a perfect circular path in the curved sections, we must make sure that the magnetic force is a *centripetal force*. Any centripetal force F on an object of mass m moving in a circle of radius R with constant rotational velocity v obeys $F = m\frac{v^2}{R}$.

f) Write a program that uses a numerical integration method (again, not forward-Euler) to find the change in position and velocity of a particle in a magnetic field, where the magnetic field is structured as discussed above.

g) We will now do the actual simulation of the particles. Provided is the python-script `CODE.PY` which contains the skeleton of a script that lets us simulate the accelerator. Missing from the code are the sections that actually calculate the particle displacement due to electromagnetic forces. Your job is to use the code you previously wrote to complete the script and run several simulations.

If you feel confident in your programming skills, it is highly recommended that you create your own simulator from scratch. The skeleton code is used as follows:

```
myAccelerator = Accelerator(L, E, R) # create the accelerator
myAccelerator.particle(m, q) # add particle
pos, vel = myAccelerator.run(dt, steps) # run the simulation
```

(You are not expected to study or rigorously understand the provided script. If you successfully completed the previous programming exercises, all that should be left is transferring the code to the skeleton.)

h) Run the simulation for a proton. Experiment with different field strengths and dimensions of the accelerator. Plot the path of the particle (animate it), as well as its speed against time. Explain the shape of the speed graph. Are the results as expected? (Please note that the time step dt must be very small.)

i) Is this simulation realistic? Are there factors we have forgot to take into account?

11 Ampere's law

In electrostatics we started from Coulomb's law, which we used to find the electric field, and used this to derive Gauss' law, $\nabla \cdot D = \rho$, which is useful for symmetric situations (and for forming Maxwell's equations which we can solve numerically). In addition we found that the electric field (in electrostatics) is curl-free: $\nabla \times \mathbf{E} = 0$. These laws for the foundation of Maxwell's equations and have proved important tools for the development of magnetostatics.

In magnetostatics, we have so far introduced Biot-Savart's law to find the magnetic field. In this chapter, we will introduce Ampere's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, which is what corresponds to Gauss' law for magnetic fields. In addition, we will argue that the magnetic field is divergence free, $\nabla \cdot \mathbf{B} = 0$, which is equivalent to stating that there are no magnetic monopoles. In combination, these two laws for a powerful theoretical tool that allows us to find expressions for the magnetic field in situations with a high degree of symmetry.

11.1 Magnetic flux

For electric fields, we found Gauss' law, which related the flux of the electric field to the net free charge: $\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = Q_{\mathrm{in}}/\epsilon_0$ on integral form and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ on differential form. We could derive this law from Coulomb's law. Electric fields are generated by charges, whereas magnetic fields are generated by current elements. Do we still have a similar law for magnetic fields?

We define the magnetic flux through a surface S as

$$\Phi_S = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \; . \tag{11.1}$$

For any *closed* surface, this flux is always zero! (We demonstrate this below). We can rewrite this on differential form using the divergence theorem, where we relate the flux of $\mathbf{B}$ through a closed surface S to the volume integral of the divergence of $\mathbf{B}$ throught the volume v enclosed by S:

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \mathrm{d}v = \; . \tag{11.2}$$

Because this is true for any closed surface S, the argument of the integral must be equal to zero, $\nabla \cdot \mathbf{B} = 0$.

Magnetic flux and the divergence of the magnetic field

The flux of the magnetic field $\mathbf{B}$ through any *closed* surface S is always zero:

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \; . \tag{11.3}$$

This also means that the divergence of the magnetic field is zero:

$$\nabla \cdot \mathbf{B} = 0 \; . \tag{11.4}$$

We do not know of any situations where these equations are not satisfied, that is, 'magnetic charge' or magnetic monopoles do not seem to exist.

11.1.1 Proof that the divergence of the magnetic field is zero

Our plan is to demonstrate that the divergence of the magnetic field contribution from any current element is zero. Therefore, the sum of many such current elements will also have zero divergence. We look at a current element $I\mathrm{d}\mathbf{l}$. The contribution to the magnetic field $\mathrm{d}\mathbf{B}(\mathbf{r})$ in the point $\mathbf{r}$ from this current element is given by Biot-Savart-s law:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \frac{\mathbf{J}\mathrm{d}v \times \hat{\mathbf{R}}}{R^2} \; , \tag{11.5}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, where $\mathbf{r}'$ is the position of the current element. For simplicity we place the element in the origin so that $\mathbf{r}' = 0$. This means that the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}' = \mathbf{r}$. The contribution $\mathrm{d}\mathbf{B}(\mathbf{r})$ from this current element to the magnetic field in $\mathbf{r}$ is then

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\mathrm{d}v \times \hat{\mathbf{r}}}{r^2} \,, \tag{11.6}$$

We want to find the divergence of $\mathrm{d}\mathbf{B}$ and apply the product rule:

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathbf{r}}}{r^2}\right) \,. \tag{11.7}$$

Notice that $\mathbf{J}$ here is the value for $\mathbf{J}$ in the origin. This value does not vary if we vary the reference point $\mathbf{r}$ and $\mathbf{J}$ is therefore a constant in the derivations. Therefore, $\nabla \times \mathbf{J}$ in this particular expression is zero. In addition, we see from the expression for the curl of a field in spherical coordinates that $\nabla \times \left(\hat{\mathbf{r}}/r^2\right)$ always is zero.

We therefore conclude that the divergence of $\mathrm{d}\mathbf{B}$ is zero, and, using the superposition principle, that the divergence of $\mathbf{B}$ is zero. This implies that the magnetic flux through any closed surface is zero.

Implications for a vector potential. We have previously found that $\nabla \times \mathbf{E} = 0$ allowed us to introduce a scalar potential V so that $\mathbf{E} = -\nabla V$. Similarly, the observation that $\nabla \cdot \mathbf{B} = 0$ for magnetic field allows us to introduce a vector potential $\mathbf{A}$ so that $\mathbf{B} = \nabla \times \mathbf{A}$. Such a definition will automatically ensure that the divergence of $\mathbf{B}$ is zero, because the divergence of a curl always is zero. You can find the introduction of the vector potensial $\mathbf{A}$ in the Appendix

11.2 Ampere's law

We found that electrostatics could be defined from the divergence and curl of the electric field, similarly, the laws of magnetostatics can be defined through the divergence and curl of the magnetic field. For the electric field, we have learned to apply Gauss' law to find the electric field in symmetric cases: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. The corresponding law for magnetic fields is Ampere's law, which describes the curl of the magnetic field. We must therefore replace integration over closed surfaces for Gauss' law with integration over closed loops for Ampere's law.

Ampere's law on differential form states that the curl of the magnetic field is $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. (We provide a proof of Ampere's law based on Biot-Savart's law below). We can transform the law to integral form using Stoke's theorem:

$$\int_S (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{B}\mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \ . \tag{11.8}$$

where we recognize that the curve C is the curve limited by the surface S. We recall that $\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ is the current I passing through the surface S. We can therefore rewrite Ampere's law as:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{net}} \ . \tag{11.9}$$

Ampere's law

Ampere's law on differential law states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \ . \tag{11.10}$$

Ampere's law on integral form states that:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \ . \tag{11.11}$$

where C is a curve along the boundary of S.

Ampere's law for a current loop states that:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{net}} \ . \tag{11.12}$$

where the current I_{net} is the net current through a (any) surface that has the curve C as its boundary.

Let us address the various terms in Ampere's law in detail.

1. The integral $\oint_C \mathbf{B} \cdot d\mathbf{l}$ is a line integral of the magnetic field along a closed curve C. You have seen such integrals before. However, just as with Gauss' law, we will usually not perform this integral, but instead take the integral along a path where $\mathbf{B}$ is constant or where $\mathbf{B} \cdot d\mathbf{l}$ is constant. Hence, we can calculate the value of B along the path.

2. The integral $\int_S \mathbf{J} \cdot d\,\mathbf{S}$ we recognize as the net current flowing through the surface S. The surface S in Ampere's law is not an arbitrary surface. It must be a surface which has the curve C as a boundary. In addition, the orientation of the surface S must correspond to the orientation of the path C according to the right-hand rule as illustrated in Fig. 11.1. Notice that there are many surfaces S that have C as a boundary. We will therefore choose a surface that simplifies the calculation.
   a. Instead of the integral $\int_S \mathbf{J} \cdot d\,\mathbf{S} = I$, we can use the current I. This current is the *net current* through the surface S. This means that we must add together all the currents going through the surface, taking into account the directions of the currents.

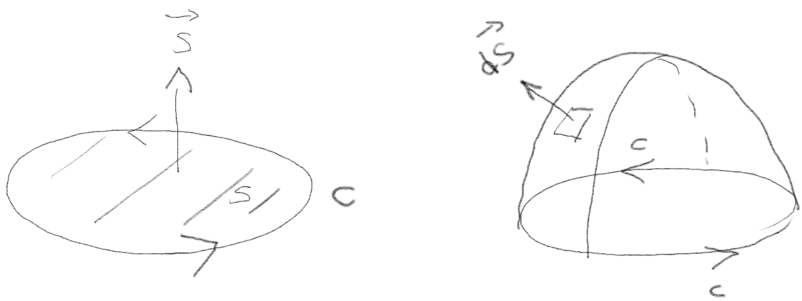


Fig. 11.1 Illustration of the relation between the curve C and the surface S. Notice that the direction of the surface corresponds to the direction of the curve according to the right-hand rule. If you place your right hand so that your thumb points up and let the other fingers curve along the curve in positive direction, your thumb is pointing in the positive direction of the surface S. We call this the orientation of the surface, which is described by the direction of the normal vector of the surface.

11.2.1 Example: Current through curves

Fig. 11.2 show three current I_1, I_2 and I_3 along the illustrated wires. What is the net current — and therefore also the Ampere integral $\oint \mathbf{B} \cdot \mathrm{d}\,\mathbf{l}$ — through each of the curves C_1, C_2, and C_3?

Curve C_1**.** We see that for curve C_1 there are no currents passing through a planar surface through C_1. Therefore $I = 0$ and:

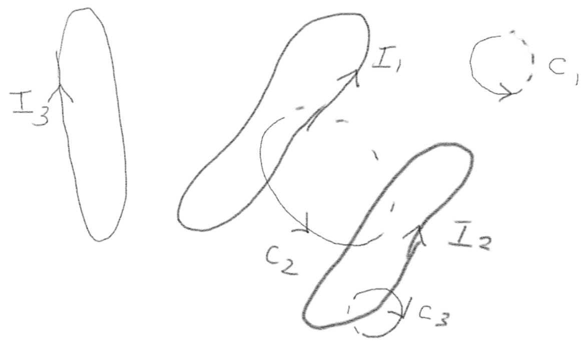


Fig. 11.2 Illustration of currents going along various wires and curves.

$$\oint_{C_1} \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = \mu_0 I = 0 \,. \tag{11.13}$$

Curve C_2**.** For curve C_2 there are two currents. First, we find the orientation of the planar surface S_2 with C_2 as the boundary. We see that current I_1 is going through a surface in the direction of the surface normal. Thus I_1 contributes positively to the total current through C_2. However, current I_2 has a direction that is opposite that of the surface normal to S_2. hence, I_2 contributes negatively to the total current. We find that:

$$\oint_{C_2} \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = \mu_0 \left(I_1 - I_2 \right) \tag{11.14}$$

Curve C_3**.** For curve C_3 there are only one current that passes through the curve, I_2. We find the orientation of the surface S_3 and see that I_2 is in the same direction as the surface normal, hence the current I_2 contributes positively to the total current through C_3. This gives that:

$$\oint_{C_3} \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = \mu_0 I_2 \tag{11.15}$$

11.2.2 Example: Infinitely long cylidrical conductor

We want to find the magnetic field $\mathbf{B}$ around a cylindrical conductor using Ampere's law. The idea is to find an integration loop C so that $\mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = B$ is a constant for this loop, $\oint_C \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = B \oint_C \mathrm{d}\,l = BL$, where L is the length of the loop. We can then find B from Ampere's law using $\oint_C \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = \mu_0 I$ by simply finding the current I through a surface with

C as its boundary. However, to realize this plan we first need to find a symmetry for the magnetic field. The situation is illustrated in Fig. 11.3.

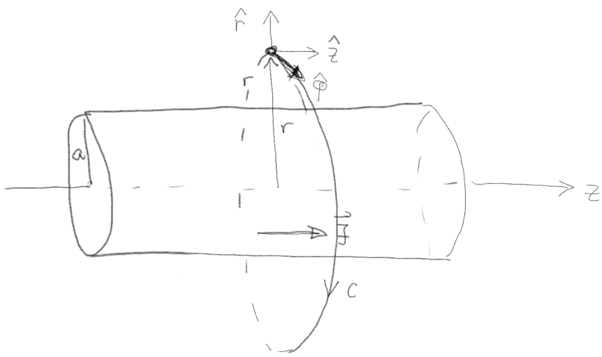


Fig. 11.3 Illustration of the current in and the magnetic field around an infinite conducting cylinder.

Symmetry of the current density. The magnetic field $\mathbf{B}$ is set up by the moving charges — the currents. We therefore first describe the symmetry of the current density in the cylinder. The current density must be the same everywhere along the cylinder, because all points along the cylinder have *translational symmetry.* The current can therefore not flow out towards the edges, because the current would then flow out toward the edges everywhere along the cylinder and charges would pile up at the surface or move outside the cylinder. Because the cylinder also has rotational symmetry around the z-axis, we also assume that $\mathbf{J}$ has rotational symmetry. Thus, $\mathbf{J} = J(r)\hat{\mathbf{z}}$, where r is the distance from the cylinder axis.

Symmetry of the magnetic field from the symmetry of the cylinder. The *translational* symmetry of the infinite cylinder implies that the magnetic field cannot depend on z. The rotational symmetry around the cylinder implies that the magnetic field cannot depend on the azimuthal angle ϕ. Thus, the field can only depend on the distance r from the axis, $B = B(r)$. However, the translational and rotational symmetry still opens for the $\mathbf{B}$-field to have components in the r-, ϕ- and z-direction: $\mathbf{B} = B_r(r)\hat{\mathbf{r}} + B_\phi(r)\hat{\boldsymbol{\phi}} + B_z(r)\hat{\mathbf{z}}$

Symmetry of the magnetic field from Biot-Savart's law. We must use different types of physics arguments to say more about the direction of the magnetic field. We know that the magnetic field is set up by the currents. From Biot-Savart's law we know that the contribution from a volume element dv is

$$d\mathbf{B} = \mu_0 \frac{\mathbf{J} dv \times \mathbf{R}}{4\pi R^2} \ , \tag{11.16}$$

where the vector $\mathbf{R}$ points from the element dv to the observation point. We argued above that $\mathbf{J} = J\hat{\mathbf{z}}$. This means that there will not be any component along z from $d\mathbf{B}$ due to the cross product (the cross product is normal to the z-axis). Therefore $B_z(r) = 0$.

Symmetry of the magnetic field from zero-divergence condition. Can the magnetic field have a component in the r-direction, $B_r(r)$? No, because this would imply that there is a net flux of the magnetic field out of a cylindrical surface. If we place a cylinderical (Gauss) surface around the z-axis, there would be no flux of the magnetic field out of the side surfaces (because $B_z = 0$). The flux out of the surface would then be $B_r(r)2\pi r L$. However, this must be zero because $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$. Therefore, $B_r(r) = 0$.

The simplified magnetic field. The argument presented here was long and systematic. When you get used to this you should be able to make these types of considerations quickly without all the details of the argument. We have now found that the magnetic field must have the form

$$\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} \ . \tag{11.17}$$

Application of Ampere's law to find the magnetic field. We are now ready to use Ampere's law to find the magnetic field. Because the field only depends on r we should choose a curve with a constant r so that the magnitude of the field is constant. We should also choose a curve so that the magntic field and the tangential vector $d\mathbf{l}$ along the curve always has the same relative angle. We do this by choosing a circular curve in the xy-plane with its center at the z-axis. If we choose the curve to be oriented in the positive rotational direction around the z-axis (according to the right-hand-rule), then $d\mathbf{l} = dl\hat{\boldsymbol{\phi}}$ and

$$\oint_c \mathbf{B} \cdot d\mathbf{l} = \oint_C B_\phi(r) d\,l = B 2\pi r = \mu_0 I(r) \ . \tag{11.18}$$

Where I is the current going through a cross section of the cylinder (inside the radius r). If $r > a$ then all the current I goes through the curve and $I(r) = I$ is the the total current in the cylinder. The magnetic field is therefore:

$$\mathbf{B} = B_\phi \hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\phi}} \ . \tag{11.19}$$

What if $r < a$ **?** There is nothing in our arguments that implies that r must be larger than a. However, if $r < a$ we must ensure that $I(r)$ only includes the part of the current that is inside the loop of radius r. If we assume that the current density $\mathbf{J}$ is uniform throughout the cylinder we can calculate the current $I(r)$:

$$I(r) = \oint_S \mathbf{J} \cdot d\,\mathbf{S} = J\pi r^2 \ . \tag{11.20}$$

But what is J? We can find this by setting $r = a$, for which $I(a) = I = J\pi a^2$, hence $J = I/(\pi a^2$, and

$$I(r) = \frac{I}{\pi a^2}\pi r^2 = I\frac{r^2}{a^2} \ . \tag{11.21}$$

The magnetic field is then for $r < a$:

$$\mathbf{B} = \frac{\mu_0 I(r)}{2\pi r}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I r}{2\pi a^2}\hat{\boldsymbol{\phi}} \ . \tag{11.22}$$

The complete field is then

$$\mathbf{B} = \begin{cases} \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} & , r \geq a \\ \frac{\mu_0 I r}{2\pi a^2}\phi & , r < a \end{cases} \tag{11.23}$$

Typical numbers for the magnetic field. If a current of $I = 10\text{A}$ is carried by the wire (the cylindrical conductor), what is the magnetic field at a distance of $r = 10\text{cm}$ from the wire? We find that

$$B = \frac{4\pi\, 10^{-7}\, 10}{2\pi\, 0.1}\text{T} = 2 \cdot 10^{-5}\text{T} = 20\mu\text{T} \ . \tag{11.24}$$

This is an typical magnitude of a magnetic field from the wires in your house.

Method: Applying Ampere's law to find the magnetic field

Based on this example, we propose a general strategy for how to find the magnetic field for a current distribution (from a current loop) using Ampere's law:

- Find a (integration) loop C the bound a surface so that $\mathbf{B} \cdot \mathrm{d}\mathbf{l} = B$ is a constant on this loop. (It may be zero on some parts of the loop — zero is a constant. And remember that it is only the component of $\mathbf{B}$ along d$\mathbf{l}$ that needs to be constant. Other elements are ignored in this approach. We call this loop the *Ampere loop* for the system.

- This usually requires that you find a simplified description of the magnetic field in a coordinate system that has the symmetry of the system, e.g. cylindrical coordinates for a cylindrical system we often find that $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}}$.
- You often also have to use that the divergence of $\mathbf{B}$ is zero to argue for the symmetry of the $\mathbf{B}$-field.
- Find the curve integral, which often can be simplified to only include the length of the curve: $\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B_l \oint_C \mathrm{d}l$
- Use Ampere's law $\mu_0 I = B \oint_C \mathrm{d}l$, to find the magnetic field as a function of current and position.
- Notice that the curve sometimes will enclose the current several times, for example when the wire is twisted several times around the magnetic field as in a solenoid. You must then include the current through all the loops, or multiply the current by the number of loops.

11.2.3 Example: Coaxial cable

A coaxial cable consists of a conducting (inifinitely thin) sheet of radius a and an (infinitely) thin outer shell at a radius b. Find the magnetic field when a current I is flowing in one direction in the center and in the opposite direction in the outer sheet. (Alternatively, we can assue that the conductors have a finite thickness, but that all the current is on the surface of the conductor).

We can use the same symmetry argument as for the inifinitely long cylinder, therefore $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}}$ and we are left with Ampere's law

$$\oint_c \mathbf{B} \cdot \mathrm{d}\,\mathbf{l} = B 2\pi r = \mu_0 I_{\mathrm{net}} \ . \tag{11.25}$$

When $r < a$ there is no current inside the loop, hence $B = 0$ in this region. When $r > b$ the *net* current is zero, hence $B = 0$ in this region as well. In the intermediate region we have that the net current is I. The magnetic field is therefore:

$$\mathbf{B} = \begin{cases} 0 & , r < a \\ \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} & , a \leq r \leq b \\ 0 & , b < r \end{cases} \tag{11.26}$$

11.2.4 Example: Toroid

There are several ways to increase the current through a surface: We can increase the current carried in a single wire or we can place many wires through the same surface. There are two main geometries where this is done systematically to generate a well-controlled magnetic field in a region of space: the *toroid* and the *solenoid*. First, let us address the behavior of the toroid.

Toroid geometry. The toroid consists of a donut shaped material with a wire wound around it as illustrated in Fig. 11.4. The wire connects to the outside on one side and is usually wound tightly. Typically we assume that it is wound so tightly that we can assume that the wire consists of circles that each are in the plane of the radial direction (radial from the center of the toroid) and the z-axis.

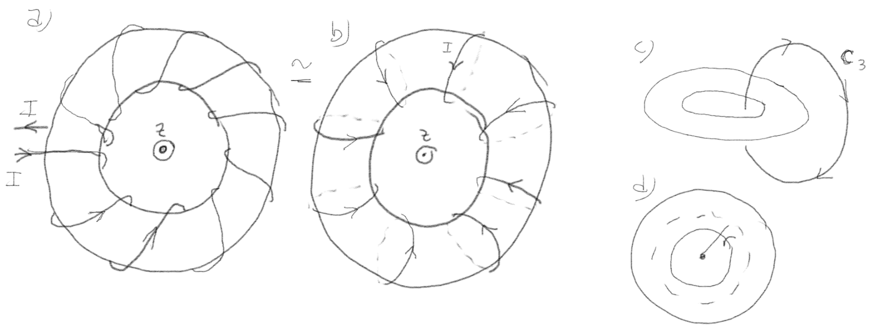


Fig. 11.4 Illustration of the toroid geometry.

Symmetry. What is the expected symmetry of the **B**-field in this case? We expect it to depend on the distance r from the center of the toroid. We do not expect it to depend on the angular position ϕ. We do not expect any field in the r-direction, since the current is in this direction. We also do not expect any z-component for the field since there is no current through the loop illustrated in Fig. 11.4d, and therefore $\oint_{C_3} \mathbf{B} \cdot d\mathbf{l} = 0$. We place the outside loop very far away and are then left with the B_z

component close to the toroid, which must be zero. The symmetry of $\mathbf{B}$ is therefore:

$$\mathbf{B} = B(r)\hat{\boldsymbol{\phi}} \, . \tag{11.27}$$

Applying Ampere's law. We can now apply Ampere's law along a path where $\mathbf{B} \cdot d\mathbf{l}$ is constant, that is, for a circle with radius r as illustrated in Fig. 11.4c. For this path we have:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B 2\pi r = I\mu_0 = -NI\mu_0 \, , \tag{11.28}$$

where we have used that the total current through the loop C is the sum of the currents in all the N windings of the wire around the toroid. This result is only valid inside the toroid. Outside the toroid there is not net current through the surface and the field is zero. This means that the field is:

$$\mathbf{B} = \begin{cases} -\frac{\mu_0 NI}{2\pi r}\hat{\boldsymbol{\phi}} & \text{inside the toroid} \\ 0 & \text{outside} \end{cases} \tag{11.29}$$

11.2.5 Example: Solenoid

A solenoid is what we get if we wind a wire around a long cylinder. The geometry is illustrated in Fig. 11.5. Notice how we use the standard symbols for vectors coming out of/going into the page to illustrate the direction of the currents. We assume that the wire has been wound N times along a length L of the cylinder, with N/L windings per unit length.

Solenoid and Toroid. We could assume that the solenoid is what we get for a toroid in the limit when the radius of the toroid becomes very huge. The result would therefore be a constant magnetic field inside the solenoid and zero field outside the solenoid. For the toroid the number N/L of windings per unit length is $N/L = N/2\pi r$, so that the field is $\mathbf{B} = \mu_0 I(N/L)\hat{\mathbf{z}}$, where z is the axis long the solenoid.

Direct calculation of the solenoid. Let us instead do a direct calculation for the solenoid starting from the symmetry of the magnetic field in the solenoid and then applying Ampere's law for a well-chosen curve.

Symmetry of the solenoid. First, we look at the ϕ-component of the field. We can use Amperes law for a circulat path around the cylinder. There is no net current through this curve C (see Fig. 11.5). Thus, Ampere's law gives:

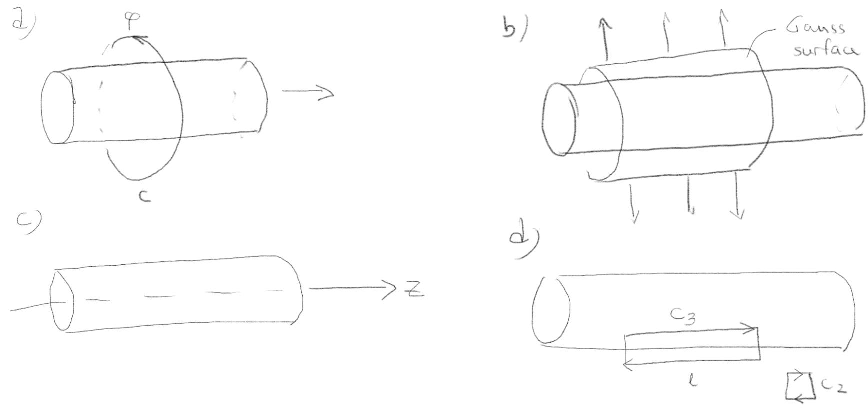


Fig. 11.5 Illustration of the solenoid geometry.

$$\oint_C \mathbf{B}\cdot \mathrm{d}\,\mathbf{l} = \oint_C B_\phi \mathrm{d}\,l = B_\phi 2\pi r = 0 \;\Rightarrow\; B_\phi = 0\,. \tag{11.30}$$

For the r-component in the direction radially out from the z-axis, we can apply the divergence theorem for the **B**-field. where

$$\int \mathbf{B}\cdot \mathrm{d}\,\mathbf{S} = B_r 2\pi r L = 0 \;\Rightarrow\; B_r = 0\,. \tag{11.31}$$

For the z-component we notive that $B_z(r)$ cannot depend on the the z or ϕ coordinates due to the symmetry: We assume that the cylinder is long and that edge effects are small. B_z therefore cannot depend on z cause the system has translational symmetry along the z-axis. Similarly, B_z cannot depend on ϕ because the system has rotational symmetry. We are therefore left with $\mathbf{B} = B_z(r)\hat{\mathbf{z}}$. Outside the cylinder the field must be zero. We place a circuit with one part of length l inside the cylinder (see Fig. 11.5d) and one part of length l outside the cylinder. The integral in Ampere's law is thefore:

$$\oint_C \mathbf{B}\cdot\ \mathrm{d}\mathbf{l} = Bl = \mu_0 I_{\mathrm{net}}\ . \tag{11.32}$$

In this case, $I_{\mathrm{net}} = (N/L)lI$, where N/L is the nuber windings per unit length so that $(N/L)l$ is the number of windings inside the curve of length l. The field is therefore

$$\mathbf{B} = B_z\hat{\mathbf{z}} = \mu_0 I\frac{N}{L}\ ,\hat{\mathbf{z}}\ . \tag{11.33}$$

The magnetic field is therefore uniform inside the solenoid. The solenoid is therefore the magnetic equivalent of the capacitor — a simple geometry where the field is uniform inside it.

11.3 Comparison of magnetostatics and electrostatics

We have now discovered the main laws of electro- and magneto-*statics*, and we have found that they have a nice kind of symmetry. For electrostatics we found that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \quad \text{(Gauss' law)} \tag{11.34}$$

$$\nabla \times \mathbf{E} = 0 \tag{11.35}$$

These two laws are called **Maxwell's equations** for electrostatics. We can use these to find the electric field for any charge distribution. It is only the second equation that will be modified (with Faraday's law) when we introduce a time-varying magnetic field. These equations can be found from Coulomb's law, and we can show that they reproduce Coulomb's law.

For magnetostatics we have that

$$\nabla \cdot \mathbf{B} = 0 \tag{11.36}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{(Ampere's law)} \tag{11.37}$$

These two laws are called **Maxwell's equations** for magnetostatics. Together with Maxwell's equations for electrostatics, they form the four Maxwell's equations. It is only the second equation that will be modified when we introduce time-varying electric fields. We can use these two equations to find the magnetic field for a given distribution of currents.

In addition to Maxwell's equations, we have Lorentz' law, which show how the fields interacts with charges:. The force on a charge Q is:

$$\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \; . \tag{11.38}$$

These laws essentially provides the entire description of (static) electromagnetism. We will modify them as we introduce dynamics, that is, time-varying fields. It is simple and beautiful!

11.4 Summary

The **magnetic flux** through a closed surface is zeroa charge Q_2 in $\mathbf{r}_2$ with velocity $\mathbf{v}_2$ is

$$\oint_A \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \; ,$$

and the **divergence of the magnetic field is zero**

$$\nabla \cdot \mathbf{B} = 0$$

Ampere's law on differential form states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \; .$$

Amperes's law on integral form states that

$$\oint_C \mathbf{B} \cdot \mathrm{dl} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \; ,$$

and **Ampere's law for a current loop** states that

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{net}} \; .$$

We find the magnetic field using Amperes law by:

- Finding an integration loop C where $\mathbf{B} \cdot \mathrm{d}\mathbf{l}$ is a constant due to symmetry
- Apply Ampere's law to find the magnetic field as a function of current and position.

There exists a **vector potential A** so that $\mathbf{B} = \nabla \times \mathbf{A}$. We can find the vector potential from

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_v \frac{\mathbf{R} \mathrm{d}v'}{R} \; ,$$

or from the **vector Poisson's equation**:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

The **divergence of the vector potential is zero**.

$$\nabla \cdot \mathbf{A} = 0 \; .$$

11.5 Exercises

Learning outcomes for tutorials. (1) Form symmetry arguments about the magnetic field, (2) Choose an appropriate loop so that the line integral over the **B**-field is easy to calculate, (3) Use line integrals over well-selected loops to find the **B**-field.

11.5.1 Test yourself

11.5.2 Discussion exercises

Exercise 11.1: Monopoler

La oss si at noen sier at de har funnet en magnetisk monopol. Hvordan ville vi kunne gå frem for å dokumentere eller motbevise dette? Hvilke egenskaper tror du en magnetisk monopol ville ha?

Exercise 11.2: Magnetisk storm

Elektrisk ladde partikler som strømmer ut fra solen kan i perioder med stor solaktivitet skape forstyrrelser i jordens magnetfelt. Hvordan kan dette skje?

Exercise 11.3: Field lines

a) The magnetic field on the inside of a toroid is non-zero, but outside it is zero. Argue why this could be the case using both symmetry and an Amperian loop.

b) What does this mean for the magnetic field of an infinite solenoid?

11.5.3 Tutorials

Exercise 11.4: Ampere's Law

(*Based on a tutorial by Steven Pollock*)

a) Write down Ampere's Law in integral form.

b) Imagine there is a constant magnetic field whose direction is given by the field lines shown in Fig. 11.6a. Find the integral $\int \mathbf{B} \cdot \mathrm{d}\mathbf{l}$ for each of the four segments of the loop. What is the integral around the whole loop?

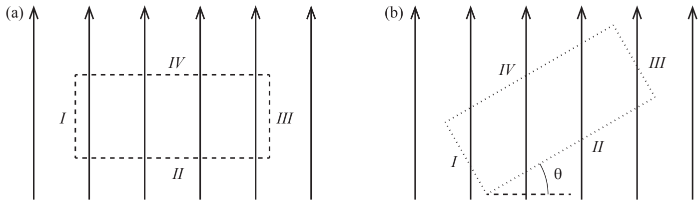


Fig. 11.6 Two integration loops in a magnetic field.

c) Discuss what would happen if you instead chose a loop that was tilted an angle θ, as illustrated with the dotted line in Fig. 11.6b.

d) Qualitatively explain how your results for questions b) and c) would change if your Amperian loop was a circle instead of a rectangle. Why is a rectangular Amperian loop better for this problem than a circular Amperian loop? Explain. What sort of situation might you want a circular Amperian loop for and why? Be explicit.

e) If $\oint_c \mathbf{B} \cdot d\mathbf{l} = 0$ for an Amperian loop (not necessarily the one in the questions above), can you conclude anything about the magnetic field $\mathbf{B}$? Explain.

f) What does it mean if $\oint_c \mathbf{B} \cdot d\mathbf{l}$ is not zero?

g) Consider the long fat cylindrical wire with a known, azimuthally symmetric current density $\mathbf{J}$ shown in Fig. 11.7. Look at the various loops shown in the figure, and decide what information, if any, Ampere's law applied to each loop might provide about $\mathbf{B}$. (i) Loop a (it is centered on the wire); (ii) loop b; (iii) loop c; (iv) loop d (also centered); (v) loop e (also centered); loop f.

Exercise 11.5: Application of Ampere's Law

While studying intensely for your physics final, you decide to take a break and listen to your stereo. As you unwind to a little music from

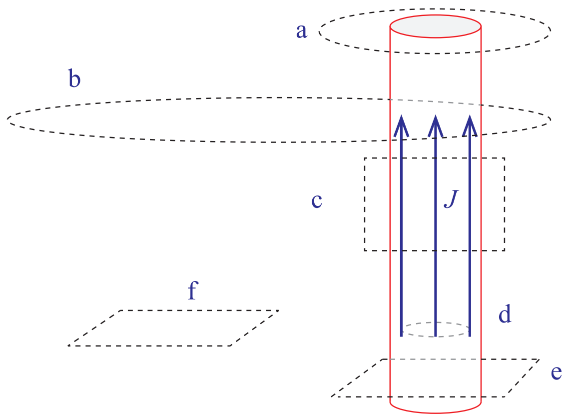


Fig. 11.7 Various integrations loops and a current density **J**.

your favourite Spotify list, your thoughts drift to newspaper stories about the dangers of household magnetic fields on the body. You examine your stereo wires and find that most of them are coaxial cables: essentially one conducting cylinder surrounded by a thin conducting cylindrical shell (the shell has some thickness). At some moment in time current is traveling up the inside conductor, and back down the conducting shell. As a way to practice for your physics final you decide to calculate the magnetic field at different radii.

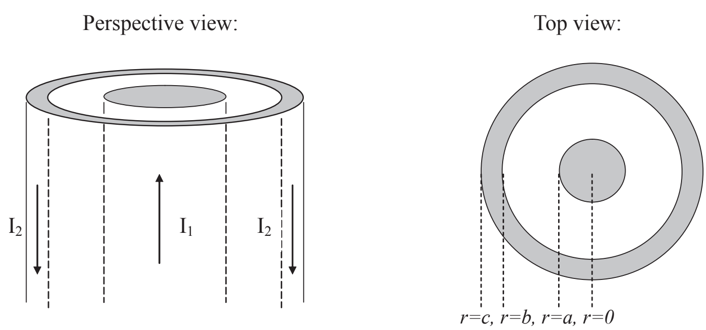


Fig. 11.8

a) How does the current density **J** vary throughout the body of a conducting wire? Is all the current concentrated right at the center of the wire, does it only flow on the outer edges, or, does it spread out uniformly across the cross-sectional area?

Now, using your model for $\mathbf{J}$, calculate $\mathbf{B}$ in the four regions (you may assume I_1 and I_2 have the same magnitude I):

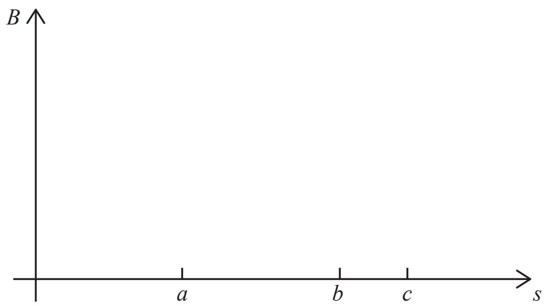
b) $s < a$

c) $a < s < b$

d) $b < s < c$

e) $s > c$

f) Based on your calculations, sketch a qualitative graph of B.



g) If I is 1 Amp (not an unreasonable size), what is the maximum value of B produced? Where is that maximum B produced? What is B at your location in the room? At either of these places, how does it compare with the earth's magnetic field (about 5×10^{-5} T)? What do you think about the newspaper's concerns? You may find the following number useful: $\mu_0 = 4\pi \times 10^{-7}\mathrm{N/A}^2$.

Exercise 11.6: Ampere and superposition

(*From Danny Caballero*)

A clever use of superposition should can make seemingly complicated situations easier to solve.

a) A long (infinite) wire (cylindrical conductor of radius R, whose axis coincides with the z-axis) carries a uniformly distributed current I_0 in the $+z$-direction. A cylindrical hole is drilled out of the conductor, parallel to the z-axis, (see Fig. 11.9 for geometry). The center of the hole is at $x = b$, and the radius is a. Determine the magnetic field in the hole region.

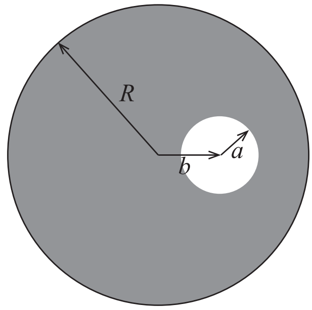


Fig. 11.9

b) If this is an ordinary wire carrying ordinary household currents, and the drilled hole has dimensions roughly shown to scale in the figure above, make an order of magnitude estimate for the strength of the B-field in that region. How does it compare to the earth's field? (*You should find that the B-field in the hole is uniform — that was just a little surprising to me!*)

Exercise 11.7: Magnetic field from a thin sheet

Use Ampere's Law to find the magnetic field around a thin sheet with a surface current density $\mathbf{J}_a = J_a \hat{x}$.

11.5.4 Homework

Exercise 11.8: Magnetfeltet fra en koaksialkabel

I denne oppgaven skal vi finne feltet fra en koaksialkabel. Dersom du har gjort tilsvarende oppgave fra ukens tutorial kan du gå rett til deloppgave d.

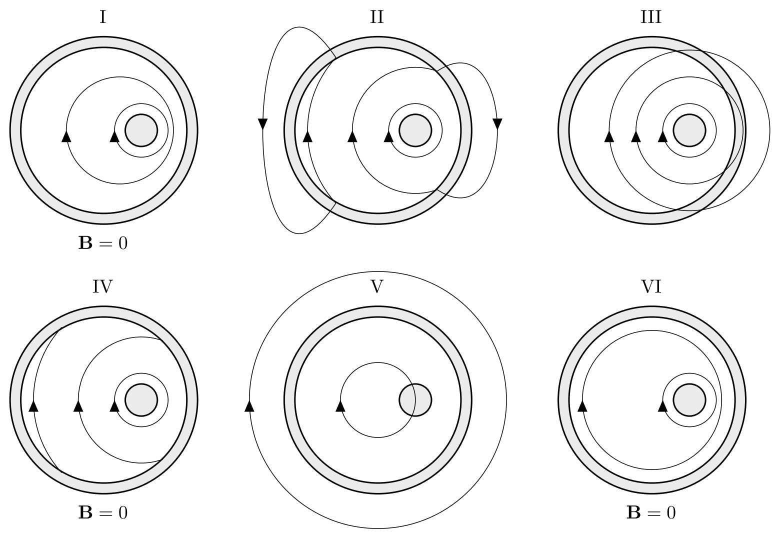
a) Finn B-feltet fra en uendelig lang, rett, sylindrisk, leder med radius a når du antar at en strøm I er jevnt fordelt over lederens tverrsnitt.

b) Finn B-feltet fra et uendelig langt sylinderskall med indre radius b og ytre radius c når det går en strøm I som er jevnt fordelt over lederens tverrsnitt.

c) Finn B-feltet fra en koaksialkabel der innerlederen har radius a og ytterlederen har indre radius b og ytre radius c. Det går en strøm I

i innerlederen, og en strøm I i ytterlederen. Begge strømmene er jevt fordelt over kabelens respektive tverrsnitt. (Finn feltet i alle avstander $r > 0$)

d) Anta nå at lederene forskyves slik at de blir liggende eksentrisk. Vi antar fortsatt at strømmen begge veier fordeler seg jevnt over henholdsvis inner- og ytterlederen. Figuren under viser seks forslag til grove skisser av det totale magnetiske feltet. Hvilken skisse er korrekt? Du kan gjerne begrunne svaret ved å vise at de resterende fem alternativene er umulige.



Vi ser på en tykk strømførende plate. Den er uendelig stor i x- og y-retning, og strekker seg fra $-h$ til h i z-retning. Platen har en strømtetthet $\mathbf{J} = J_0|z|\hat{\mathbf{x}}$ for $z \in (-h, h)$ og 0 ellers (utenfor platen).

Finn B-feltet (størrelse og retning) overalt i rommet (over, under og inni platen).

Exercise 11.9: Feltet fra en roterende ladd sylinder

Vi har et langt sylinderskall med radius a og flateladningstetthet ρ_s. Denne sylinderen roterer med vinkelhastighet ω

a) Hvilke symmetrier har vi i dette systemet? Hva kan vi si om det magnetiske feltet basert på disse symmetriene?

b) Dersom du skal bruke Amperes lov til å finne det magnetiske feltet fra sylinderen. Hva vil være et godt valg av Ampèreløkke?

Hint. Når sylinderen roterer med en ladningsfordeling på seg kan man se på det som en strøm.

c) Finn magnetfeltet (størrelse og retning) fra det roterende sylinderskallet.

Magnetfeltet i et område i rommet sentrert i origo har sylindersymmetri og er gitt ved $\mathbf{B} = B_0\hat{\boldsymbol{\phi}}$, der B_0 er en konstant og $\hat{\boldsymbol{\phi}}$ er asimut enhetsvektor i sylinderkoordinater.

d) Hva er strømtettheten i dette området i rommet?

e) Anta at ladningstettheten fra (a) bare gjelder ut til en radius R, og er 0 for $r > R$. Hva er magnetfeltet for $r > R$.

12 Magnetization

We have now introduced Ampere's law for magnetic fields, which plays a role similar to that of Gauss' law for electric fields. However, so far we have only addressed magnetic fields in vacuum. For electric fields, we found that polarization of dielectric materials introduce bound charges in addition to the free charges. We could either treat the bound charges explicitly in Gauss' law, or we could introduce a new concept, the displacement field $\mathbf{D}$ and reformulate Gauss' law in terms of the displacement field to simplify the treatment of polarized materials. In this chapter, we will introduce a similar type of approach to address magnetization. While polarization occured due to either the alignment of dipoles in a material or the induction of dipoles due to electron cloud distortions due to the electric field, magnetization occurs due to alignment of microscopic, permament currents, which in a classical picture can be interpreted as the motion of electrons spinning around their nuclei. Each of these microcurrents act as electromagnets, generating magnetic fields. These fields will usually cancel each other because they are aligned in random directions, but when a magnetic field is applied, they will align and the material becomes magnetized. Unlike for polarization, the magnetic field set up by magnetization may be parallel to the magnetic field (for paramagnetic materials) or opposite to the magnetic field (for diamagnetic materials). For some magnetic materials, the magnetization remains also after the applied magnetic field is turned off (for ferromagnetic materials). In this chapter we will therefore explain and address the type of magnetism you most probably are familiar with, *permanent magnets*, and we will demonstrate that all magnets, also permanent magnets, are due to moving electric charges. We will introduce the concepts of bound and free currents, the magnetization of a magnetic material, a modified version of Ampere's law in magnetic materials, and boundary conditions for the magnetic field. This will provide us will all the tools we need to find the magnetic field in any physical situation.

12.1 Magnetic fields in materials

12.1.1 Classical model

We will start from a simple classical model to build a model for the magnetic field in a material. In a classical model for atoms, electrons are orbiting the nucleus as illustrated in Fig. 12.1. In addition, the electrons have spin and there may be other contributions to the angular momentum of the atom, but let us first see the effect of a moving electron. It moves in a circular orbit with radius R with a period $T = 2\pi R/v$, where v is the velocity. This corresponds to a current of $I = e/T = ev/(2\pi R)$. The magnetic moment of this orbital is then

$$\mathbf{m} = -\frac{1}{2} ev R \hat{\mathbf{z}} \ , \tag{12.1}$$

where $\hat{\mathbf{z}}$ points in the direction of the axis of the motion.

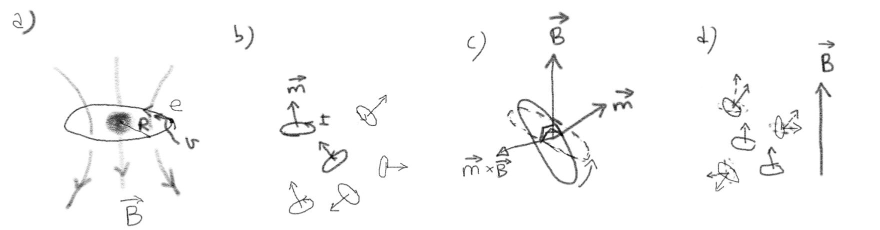


Fig. 12.1 (a) Illustration of an electro orbital in a classical interpretation and the resulting magnetic field. (b) A magnetic material consists of many, randomly oriented current loops. (c) When a magnetic moment is placed in a magnetic field, a torque will act on the current loop to align it with the magnetic field. (d) When a magnetic field is applied to a magnetic material with many current loops, they tend to align with the applied magnetic field, modifying the total magnetic field.

The orbiting electron comprising a small microcircuit will set up a magnetic field $\mathbf{B}_m$ as illustrated in Fig. 12.1. The magnetic field will point in the direction of the magnetic moment. A simple model of a magnetic material is that is constist of many such microscopic magnetic moments, where each moment corresponds to a small localized or bound current. The net magnetic field from all these microscopic magnetic moments will typically be zero if they are not aligned, but point in random directions.

If we apply a magnetic field to such a system, there will be a torque $\tau = \mathbf{m} \times \mathbf{B}$ acting on each such magnetic moment $\mathbf{m}$, as illustrated in Fig. 12.1. The torque will act to rotate the dipole so that the magnetic moment aligns with the applied magnetic field. As a consequence, the magnetic field $\mathbf{B}_m$ set up by the magnetic moments will act in the same direction as the applied magnetic field $\mathbf{B}$ [1]. This effect is similar to what we found when we addressed polarization for electric fields, and we will now introduce a magnetization $\mathbf{M}$ in analogy with the polarization $\mathbf{P}$. Just as for polarization, we note that the laws we have seen so far in magnetostatics, Biot-Savart's law and Ampere's law, are valid in a magnetic medium, but we need to include all currents, including the effects of the microscopic current loops. Just like for polarization, this is impractical, and we want to modify the theory by introducing a magnetization $\mathbf{M}$ to take care of the bound currents. Let us see how we can develop that theory.

12.1.2 Magnetization

A magnetic material consists of many microscopic current loops representing the bound currents with magnetic moments $\mathbf{m}$ as illustrated in Fig. 12.1. We interpret these magnetic moments as small circuits with current I so that their magnetic moment is

$$\mathbf{m} = I\mathbf{S}_m \; , \tag{12.2}$$

where $\mathbf{S}_m$ describes both the area and the direction of the magnetic moment (and therefore also the direction of the magnetic field inside the current loop). In a small volume element $\mathrm{d}v$ of the magnetic material, there will by many magnetic moments. We therefore describe the system by the *magnetization*, which is the sum of the magnetic moments in the volume element divided by the volume of the element (the volume density of magnetic moments):

[1] Notice that there are also mechanisms that act in the opposite direction, so that the magnetic moment may be turned in a direction opposite the applied magnetic field, an effect we call diamagnetism. We will briefly address mechanisms for diamagnetism below.

Magnetization

The magnetization $\mathbf{M}$ is defined as the net magnetic moment per unit volume:

$$\mathbf{M} = \frac{1}{\mathrm{d}v} \sum_{i \text{ in } \mathrm{d}v} \mathbf{m}_i \ . \tag{12.3}$$

which means that the total magnetic moment in $\mathrm{d}v$ is $\mathbf{M}\mathrm{d}v$. The magnetization $\mathbf{M}$ plays a role similar to the polarization $\mathbf{P}$. We interpret $\mathbf{M}$ as telling us about the *net* magnetic moment in a volume, which we expect to increase if the microscopic magnetic moments $\mathbf{m}_i$ are aligned.

First, we will simply assume that the material may be magnetized with a given magnetization $\mathbf{M}$, and see how this will impact Ampere's law. Our plan is to rewrite Ampere's law to discern between bound currents, that are related to the magnetization and the currents in the microscopic current loops, and the free currents, which are due to motion of charged particles over distance and which we can measure using an Ampere-meter.

12.1.3 Rewriting Ampere's law

Ampere's law states that

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I \ , \tag{12.4}$$

for any loop C and where the current I includes both the free current and the currents due to the microscopic magnetic moments. Our plan is to sum up the contributions from all the microscopic current loops along a curve C. Let us now assume that we can model the magnetization $\mathbf{M}$ as the effect of identical current loops each with magnetic moment $\mathbf{m}$ so that $\mathbf{M} = N\mathbf{m}$, where N is the number of current loops (magnetic moments) per unit volume. Fig. 12.2 illustrates the current loop and a small segment $\mathrm{d}\mathbf{l}$ along the loop. What are the contributions to the current through C from the microscopic current loops along $\mathrm{d}\mathbf{l}$?

First, we notice that it is only the current loops that enclose (run around) $\mathrm{d}\mathbf{l}$ that will contribute to the net current through C from the line segment $\mathrm{d}\mathbf{l}$. Loops that do not enclose $\mathrm{d}\mathbf{l}$ will either not intersect the surface S spanned by C at all, or they will intersect the surface in two

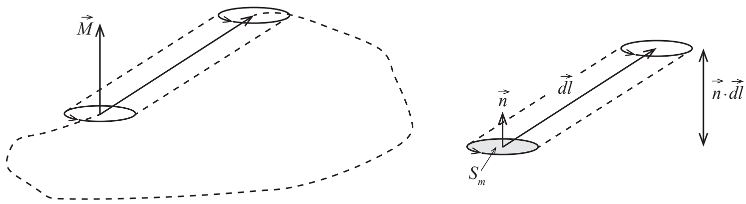


Fig. 12.2 Illustration of the current through a loop C and a smaller part of the loop.

points, so that the net contribution to the current through the surface will be zero.

Which current loops enclose d**l**? This depends on the orientation of the loops relative to the orientation of d**l**. The orientation of the loops are given by the direction of **M**. All the current loops with centers that are within the tilted cylinder with area S_m along d**l** will intersect the curve C only once, as illustrated in Fig. 12.2. The volume of this cylinder is $d\mathbf{v} = \mathbf{S}_m \cdot \mathrm{d}\mathbf{l}$. The number of magnetic moments in this volume is $N\mathrm{d}v$, and each magnetic moment will contribute with a current I. The contribution to the bound current $I_{b,C}$ through C from this element is therefore

$$\mathrm{d}I_{b,C} = I \cdot N\mathrm{d}v = I \cdot N\mathbf{S}_m \cdot \mathrm{d}\mathbf{l} = N\, I\mathbf{S}_m \cdot \mathrm{d}\mathbf{l}\ , \tag{12.5}$$

where we recognize $I\mathbf{S}_m$ as the magnetic moment of a individual current loop, $\mathbf{m} = I\mathbf{S}_m$:

$$\mathrm{d}I_{b,C} = N\,\mathbf{m} \cdot \mathrm{d}\mathbf{l}\ , \tag{12.6}$$

and we recognize $N\mathbf{m} = \mathbf{M}$. We therefore get:

$$dI_{b,C} = \mathbf{M} \cdot \mathrm{d}\mathbf{l}\ . \tag{12.7}$$

The total contribution from the bound currents along the whole curve C is then found by summing up the contributions from each d**l** and integrating along the curve C:

$$I_{b,C} = \oint_C \mathbf{M} \cdot \mathrm{d}\mathbf{l}\ . \tag{12.8}$$

The total current is therefore $I = I_{b,C} + I_C$, where I_C is the free (non-bound) current through C. We insert this back into Ampere's law

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I = \mu_0 \left(I_C + I_{b,C}\right) = \mu_0 I_C + \mu_0 \oint_C \mathbf{M} \cdot \mathrm{d}\mathbf{l}\ , \tag{12.9}$$

which can be rewritten as

$$\oint_C (\mathbf{B} - \mu_0 \mathbf{M})\, \mathrm{d}\mathbf{l} = \mu_0 I_C \; . \tag{12.10}$$

It is common to divide by μ_0 on both sides, giving:

$$\oint_C \left(\underbrace{\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}_{\mathbf{H}} \right) \mathrm{d}\mathbf{l} = I_C \; . \tag{12.11}$$

Here, we have defined the **H**-field:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \; . \tag{12.12}$$

We have therefore now reformulated Ampere's law for a magnetic material.

Ampere's law in a magnetic material

Ampere's law in a magnetic material is:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{C,\mathrm{free}} \; , \tag{12.13}$$

for any closed loop C. The current $I_{C,textfree}$ is the free current passing through the closed loop C. The **H**-field is defined as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \; , \tag{12.14}$$

where **B** is the magnetic field and **M** is the magnetization.

12.1.4 Interpretation of the bound current density

We know that the current I is related to the current density **J** of the free current through

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \; . \tag{12.15}$$

Can we find an expression for the bound current density? We found that the bound current through a closed loop C is

$$I_{b,C} = \oint_C \mathbf{M} \cdot \mathrm{d}\mathbf{l} \ . \tag{12.16}$$

We can rewrite this using Stoke's theorem as:

$$I_{b,C} = \oint_C \mathbf{M} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{M} \cdot \mathrm{d}\mathbf{S} \ , \tag{12.17}$$

where S is the surface enclosed by C. We can therefore interpret this as the bound current density:

$$I_{b,C} = \int_S \mathbf{J}_b \cdot \mathrm{d}\mathbf{S} = \int_S \nabla \times \mathbf{M} \cdot \mathrm{d}\mathbf{S} \ . \tag{12.18}$$

Because this is true for any surface S, it must also be true for the arguments of the integral, that is, $\mathbf{J}_b = \nabla \times \mathbf{M}$.

12.1.5 Ampere's law on differential form

We can also rewrite Ampere's law on differential form using Stoke's theorem. Ampere's law in a magntic material states that:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{C,\mathrm{free}} \ , \tag{12.19}$$

where the definition of the current is

$$I_{C,\mathrm{free}} = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \ , \tag{12.20}$$

and we apply Stoke's theorem to get:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \nabla \times \mathbf{H} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \ . \tag{12.21}$$

Because this is true for any surface S, the arguments of the integral must be identical. This gives us Ampere's law on differential form.

Ampere's law on differential form

Ampere's law on differential form states that

$$\nabla \times \mathbf{H} = \mathbf{J} \ , \tag{12.22}$$

where $\mathbf{J}$ is the free current density and

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \;, \tag{12.23}$$

and $\mathbf{M}$ is the magnetization.

12.1.6 Magnetic materials

Ampere's law for magnetic material is an analogous extension of Ampere's law to Gauss' law extension to dielectric materials. We can use Ampere's law to find the $\mathbf{H}$ field for a system, but in order to use that to find the magnetic field $\mathbf{B}$ we need to either have a theory to relate the magnetization to the magnetic field or a model for the magnetization.

Linear materials. For many magnetic materials, the magnetization is proportional to $\mathbf{H}$:

$$\mathbf{M} = \chi_m \mathbf{H} \;, \tag{12.24}$$

where χ_m is called the *magnetic suscpetibility* of the material. We call such material *linear (magnetic) materials*. For these materials, the magnetic field $\mathbf{B}$ is proportional to $\mathbf{H}$, because

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \quad \Rightarrow \quad \mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M}\right) \;, \tag{12.25}$$

which when $\mathbf{M} = \chi_m \mathbf{H}$ gives

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M}\right) = \mu_0 \left(1 + \chi_m\right) \mathbf{H} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{H} \;, \tag{12.26}$$

where we have introduced a notation similar to what we used for polarization. The relative permeability $\mu_r = 1 + \chi_m$ and the absolute permeability is $\mu = \mu_r \mu_0$. In vacuum, we have that $\mu = \mu_0$ and $\mathbf{B} = \mu_0 \mathbf{H}$.

Linear magnetic materials

For **linear magnetic materials** the magnetic field is proportional to $\mathbf{H}$:

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{H} \;, \tag{12.27}$$

where μ_r is called the *relative permeability* and μ is called the *absolute permeability*.

Material constants. The relative permeability is a material property. For magnetic materials the relative permeability can be smaller than one, which means that the material is diamagnetic.

Material	μ_r	Material	μ_r
Ferrite (manganese zinc)	350-20 000	Air	1.0000004
Nickel	100-600	Aluminum	1.00002
Copper	0.999994	Iron (99.8%)	5000
Water	0.999992	Pure iron	200 000
Vacuum	1	Metglas 2714A	10^6

Magnetic materials. We classify magnetic materials as *diamagnetic* when $\mu_r < 1$, *paramagnetic* when $\mu_r > 1$, but $\mu_r \simeq 1$, and *ferromagnetic* when $\mu_r \gg 1$.

Hysteresis. Notice that many ferromagnetic materials have a significant magnetic response, but that they often are non-linear and depend on the history of the material. A typical behavior of the magnetization of a ferromagnetic materials as a function of $\mathbf{H}$ is illustrated in Fig. 12.3. The magnetization does not only depend on the $\mathbf{H}$-field, but also on the history of the magnetization:

1. Let us assume that both the magnetization M and the applied field H starts as zero.
2. As we increase the applied field, H, the magnitization will also increase. First it increases linearly, but eventually it saturates and does not increase much more, because all domains (all microcurrents) are aligned, and the magnetization cannot increase further even if the H-field is increased. At this point the system has reached maximum magnetization M_{max}.
3. If we reduce H, the magnetic field will lag. When we have reduced the magnetic field H to zero, the magnetization is not zero, which means that there system is now a *permanent magnet* with a residual magnetization M_r
4. When H becomes negative, M will reduce further, eventually becoming zero and negative, until the magnetization again saturates at its maximum value in the opposite direction.
5. Notice that at some points along the curve, H and M might have opposite signs, showing that they point in different directions.

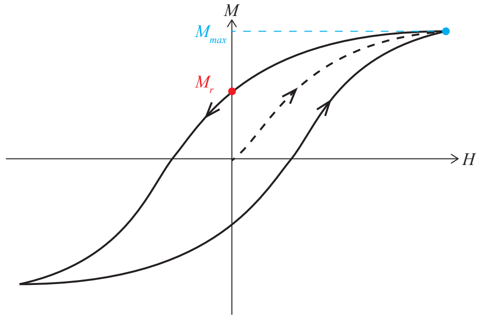


Fig. 12.3 Illustration of hysteresis loop for magnetization.

12.1.7 Maxwell's equations for static magnetic fields

We now have found maxwell equations for static magnetic fields:

$$\nabla \times \mathbf{H} = \mathbf{J} \ , \tag{12.28}$$

and

$$\nabla \cdot \mathbf{B} = 0 \ . \tag{12.29}$$

12.1.8 Example: Coaxial conductors in magnetic material

What is the magnetic field inside a coaxial cable with a magnetic material? We can use the same method as we used for Ampere's law in vacuum.

First, we realize that that symmetry of the **H**-field is the same as the symmetry of the **B**-field. This means that we expect the **H**-field to have cylindrical symmetry with only a component in the azimuthal direction that may depend on the distance r to the center of the cylinder:

$$\mathbf{H} = H_\phi(r)\hat{\boldsymbol{\phi}} \ , \tag{12.30}$$

We apply Ampere's law to a circular loop of radius r:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \oint_C H_\phi(r) \mathrm{d}l = H \oint_C \mathrm{d}l = H \, 2\pi r = I \ , \tag{12.31}$$

The current I is only non-zero in the range $a < r < b$. In this region, we find that

$$\mathbf{H} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}} \ , \tag{12.32}$$

The magnetic field is then

$$\mathbf{B} = \mu \mathbf{H} = \frac{\mu I}{2\pi r} \hat{\boldsymbol{\phi}} \,. \tag{12.33}$$

This result is identical to the result we found for the same system in vacuum, but we have replaced μ_0 with μ.

12.1.9 Surface currents

Let us see if we can understand what happens inside a permanent magnet? Fig. 12.4 describes a cylindrical permanent magnet. Let us assume that the cylinder has a uniform magnetization $\mathbf{M} = M_0 \hat{\mathbf{z}}$. This means that the bound current density inside the magnet is $\mathbf{J}_b = \nabla \times \mathbf{M} = 0$ inside the magnet. What happended to the bound currents? Fig. 12.4 illustrates the many parallel microscopic current loops in the material. We see that anywhere inside the material, there is always an adjacent loop. This means that any local current I along a loop is counteracted by an equal and opposite current I from a neighboring loop. Inside the material, the currents from these loops cancel, so that there is no net bound current density. However, at the external boundaries of the magnetic material, there are no counteracting currents. As a result, there is a surface current on the outer surface of the permament magnet, which points in the azimuthal direction. We therefore need to modify out theory of the bound currents to not only include a bound volume current density, but a bound surface current density on the outer surface of a magnetic material.

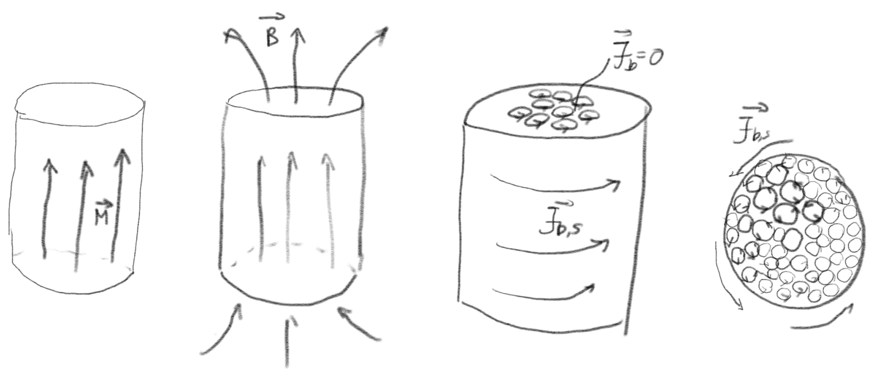


Fig. 12.4 Illustration of a permanent magnet with a uniform magnetization $\mathbf{M} = M_0 \hat{\mathbf{z}}$ inside.

Bound surface current density. We have sketched a small part of the surface of a magnetic material in Fig. 12.5. The surface normal is illustrated as $\hat{\mathbf{n}}$ and the magnetization is illustrated at $\mathbf{M}$. We form a current loop C in the plane formed by $\mathbf{M}$ and $\hat{\mathbf{n}}$. The loop has a length Δl and a thickness Δh, where we will let $\Delta h \to 0$. Inside the material, there is a magnetization $\mathbf{M}$, while outside the material, the magnetization is zero. The bound current flowing through the loop C is then given as:

$$I_b = \int_C \mathbf{M} \cdot \mathrm{d}\mathbf{l} = h \cdot \mathbf{M} + \Delta\mathbf{l} \cdot \mathbf{M} - h \cdot \mathbf{M} - \Delta\mathbf{l} \cdot \mathbf{M}_{\text{outside}} \ , \tag{12.34}$$

where we let $h \to 0$. We therefore get

$$I_b = \Delta\mathbf{l} \cdot \mathbf{M} = M\Delta l \cos\beta = M\Delta l \sin\alpha \ , \tag{12.35}$$

where β is the angle between $\mathbf{M}$ and $\Delta\mathbf{l}$ and α is the angle between $\hat{\mathbf{n}}$ and $\mathbf{M}$. The direction of this current is out of the plane, and we recognize it as a bound surface current density $J_{b,s}$ multiplied with the length Δl:

$$I_b = J_{b,s}\Delta l \ . \tag{12.36}$$

We therefore see that

$$J_{b,s}\Delta l = I_b = M\Delta l \sin\alpha \ , \tag{12.37}$$

and

$$J_{b,s} = I_b = M \sin\alpha \ . \tag{12.38}$$

The surface current density points out of the plane of the curve C, that is, in the direction of $\mathbf{M} \times \hat{\mathbf{n}}$. We also notice that $\mathbf{M} \times \hat{\mathbf{n}} = M \sin\alpha$, and therefore we write:

$$\mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}} \ . \tag{12.39}$$

We therefore have two terms for the bound current densities.

Bound current densities

The **bound volume current density** $\mathbf{J}_b$ inside a magnetic material with magnetization $\mathbf{M}$ is:

$$\mathbf{J}_b = \nabla \times \mathbf{M} \ . \tag{12.40}$$

and the **bound surface current density** $\mathbf{J}_{b,s}$ on the surface of a magnetic material (the interface between the magnetic material and vacuum/air) is

$$\mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}} \ , \tag{12.41}$$

where $\hat{\mathbf{n}}$ is the local surface normal.

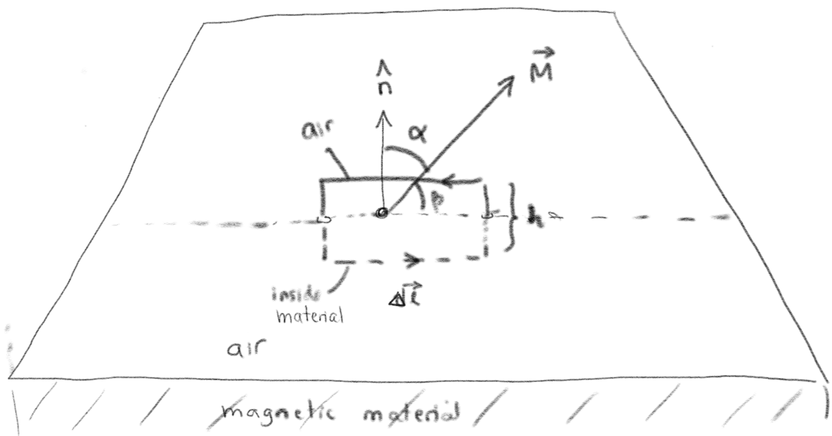


Fig. 12.5 The surface of a magnetic material with an illustration of the bound surface current.

12.1.10 Example: Permanent magnet

We can apply these new tools to address the magnetic field inside and outside of a cylindrical permanent magnet with a uniform magnetization $\mathbf{M}$.

We know that inside the magnet, the bound volume current density is $\mathbf{J}_b = \nabla \times \mathbf{M} = 0$. However, at the outer surface of the cylinder, the surface normal vector is $\hat{\mathbf{r}}$. The bound surface current density is therefore

$$\mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}} = M_z \hat{\mathbf{z}} \times \hat{\mathbf{r}} = M_z \hat{\boldsymbol{\phi}} \ . \tag{12.42}$$

Thus, there is a bound current similar to what we have in a finite solenoid. If the permanent magnet is long, we therefore expect the magnetic field to be the same as for a solenoid. That is, we expect the magnetic field to be uniform inside the permanent magnet and directed along the magnetization. We can also find the magnitude of the magnetic field by applying Amperes law to a current loop with a part Δl inside the magnet, a part $\varDelta l$ outside the magnet, and two small pieces $\varDelta h$ connecting them as illustrated in Fig. 12.6. We assume the magnetic field is zero outside the permanent magnet, similar to what we found for the long solenoid. Ampere's law along this loop gives:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\varDelta l = \mu_0 I = \mu_0 J_{s,b} \varDelta l = \mu_0 M_z \varDelta l \ . \tag{12.43}$$

That is, we found that $\mathbf{B} = \mu_0 \mathbf{M}$ inside the permanent magnet when we are far away from the edges. Because $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ this implies that $\mathbf{H}$ is zero if the cylinder is infintely long.

Notice that this approximation is only valid for an infinitely long cylinder. For a finite cylinder we need to perform a more careful calculation, even in the case when the magnetization is uniform. We will leave this for the exercises.

Fig. 12.6 Illustration of the magnetization, the bound surface current density and the integration loop C for a permanent magnet with a uniform magnetization $\mathbf{M}$

12.2 Boundary conditions for magnetic fields

For the electric field we derived boundary conditions at the interface between materials with different dielectric constants by applying Gauss' law on small volumes or that $\nabla \times \mathbf{E} = 0$ for small closed curves. Let us apply a similar method to find the boundary conditions for the magnetic fields across a boundary.

12.2.1 Normal boundary conditions

Fig. 12.7a illustrates a boundary between two materials 1 and 2 with differing magnetic properties. We relate the normal component of the magnetic field in material 1 and material 2 by introducing a cylindrical Gauss surface with a small height Δh and a surface area ΔS. The axis of the cylinder is oriented along the local surface normal $\hat{\mathbf{n}}$ of the interface. We apply the flux integral:

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \; , \tag{12.44}$$

where S is the surface of the cylinder. We assume that $\Delta h \to 0$ so that the flux contribution through the cylinder side is zero. The integral is then

$$\int_S \mathbf{B} \cdot d\mathbf{S} \simeq \mathbf{B}_1 \cdot \hat{\mathbf{n}} \Delta S - \mathbf{B}_2 \cdot \hat{\mathbf{n}} \Delta S = 0 \; , \tag{12.45}$$

which gives that

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}} \; . \tag{12.46}$$

The normal component of the magnetic field is therefore continuous across an internal interface, or equivalently, the normal components of the magnetic field is the same on both sides of an internal interface. (Notice that for electric fields, it was the tangnetial component of the electrical field that was the same on both sides. $E_{1,t} = E_{2,t}$.)

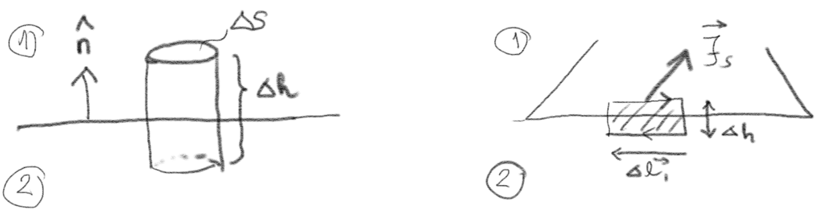


Fig. 12.7 Illustration of a boundary between two magnetic materials used to determine the normal and tangential boundary conditions for the magnetic fields.

12.2.2 Tangential boundary conditions

Fig. 12.7b illustrates a boundary between two materials 1 and 2 with differing magnetic properties. To compare the tangential components on each side of the interface, we introduce a small current loop C where the thickness $\varDelta h$ of the loop goes to zero, $\varDelta h \rightarrow 0$. We apply Ampere's law on this loop:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \underbrace{\oint_h \mathbf{H} \cdot \varDelta \mathbf{l}}_{=0} + \mathbf{H}_1 \cdot \varDelta \mathbf{l} + \underbrace{\oint_h \mathbf{H} \cdot \varDelta \mathbf{l}}_{=0} + \mathbf{H}_2 \cdot (-\varDelta \mathbf{l}) = J_s \varDelta l \ , \quad (12.47)$$

where J_s is the free surface current normal to the tangential direction. In the limit when $h \rightarrow 0$ we find

$$H_{1,t} - H_{2,t} = J_s \ , \quad (12.48)$$

which we can rewrite in vector notation as:

$$\hat{\mathbf{n}} \times \mathbf{H}_1 - \hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_s \ . \quad (12.49)$$

Boundary conditions for magnetic fields

At an interface between a magnetic material 1 and a magnetic material 2 we have the following boundary conditions:

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}} \ , \quad (12.50)$$

and

$$\hat{\mathbf{n}} \times \mathbf{H}_1 - \hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_s \ . \quad (12.51)$$

12.2.3 Example: Air-ferromagnet transition

Fig. 12.8 illustrates a curved interface between air (material a) and a ferromagnet with $\mu_b \gg 1$ (material b). What is the implications for the tangential and normal magnetic fields? There are no free surface currents.

The tangential fields are related by

$$H_{a,t} = H_{b,t} \quad \Rightarrow \quad \frac{B_{a,t}}{\mu_a} = \frac{B_{b,t}}{\mu_b} \ , \quad (12.52)$$

where we have used that $\mathbf{B} = \mu \mathbf{H}$. In this case, $\mu_a \simeq 1$, so that:

$$B_{b,t} \simeq \mu_b B_{a,t} \ . \quad (12.53)$$

For the case of a ferromagnetic material, where $\mu_b \gg \mu_a$, this means the magnetic field outside the ferromagnet will be negligible compared with

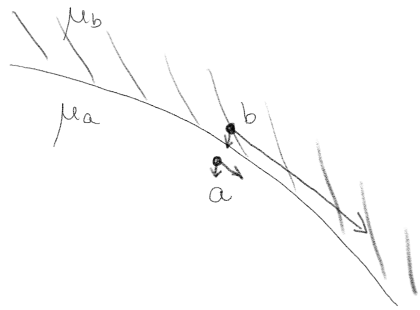


Fig. 12.8 Illustration of a boundary between a ferromagnetic material (b) and air (a).

the field inside the material. However, the normal component is the same in both materials.

This means that the magnetic field is much smaller on the outside of a material with a large μ. The magnetic field therefore tends to follow a ferromagnetic material.

12.3 Magnetic circuits

This last example demonstrates that the magnetic field tends to follow ferromagnetic materials. A *magnetic circuit* is a circuit made from materials that "conduct" magnetic flux so that the field lines close upon themselves. Fig. 12.9 illustrates an example of a magnetic circuit in the form of a toroidal magnet with a small gap. Magnetic circuits typically consists of ferromagnetic materials with current windings around parts of the cores, and may have small gaps. Here, we will argue that we can use our experience and intuition from electric circuits to discuss magnetic circuits. Magnetic circuits are important for many practical applications of electromagnetism. They can, and maybe should, be addressed by a full detailed study of the propagation of the magnetic field, but in practice we often use simplified rules to discuss their properties.

The similarity between magnetic and electric circuits are not perfect. In an electric circuit, the current density $\mathbf{J}$ is restricted to follow the conducting materials, with essentially zero current outside the conductors, because the conductivity of air or vacuum is practically zero. However, in a magnetic circuit, the corresponding "conductivity" (μ) is not zero outside the ferromagnetic conductors, so there is some leakage of the field out from the conductors into the surrounding air or vacuum. Here, we will assume that this leakage is small or negligible. In addition, a magnetic circuit will typically contain air gaps, where we will assume that the gap is so narrow that the fringe effects near the gap edges will not impact the way the flux is flowing in the circuit.

12.3.1 Analogy between magnetic flux and electric current density

Fig. 12.9 illustrates a magnetic circuit and an electric circuit. What are the similarities between these circuits?

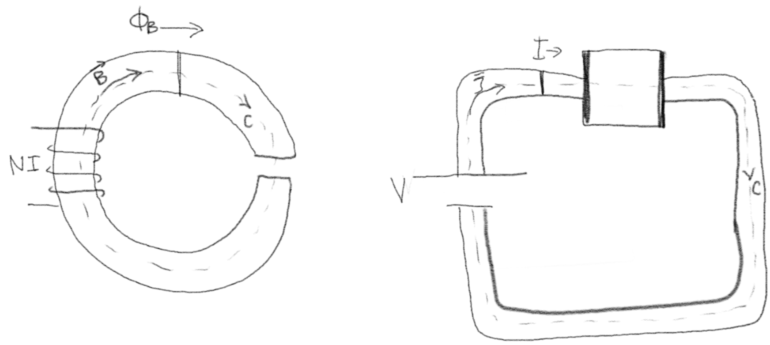


Fig. 12.9 Illustration of a magnetic and an electric circuit.

Current and flux. In the electric circuit, there is a current $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ through any cross-section of the circuit, and the current is conserved because the conductivity of the region outside the circuit is zero. (The current density $\mathbf{J}$ does not have a normal component on the boundary). In the magnetic circuit, there is a magnetic flux, $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ through any cross section of the circuit, and we will assume that the flux is conserved because the magnetic field lines tend to follow the magnetic material. (The magnetic field $\mathbf{B}$ is very small outside the magnetic material compared to inside the material as argued for above.)

Conservation of current and flux. Both the current out of a closed surface S and the flux out of a closed surface S is zero: $\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0$ and $\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$.

Relation to conductivity and permeability. The current density is related to the electric field through Ohm's law on microscopic level:

$\mathbf{J} = \sigma\mathbf{E}$, while the magnetic field is related to the $\mathbf{H}$-field through the law for magnetic materials: $\mathbf{B} = \mu\mathbf{H}$.

Drivers for E and H. The voltage source is driving the electric field in the electric circuit:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = V(t) \ , \tag{12.54}$$

while windings of a current-carrying wire is driving the magnetic field:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = NI \ . \tag{12.55}$$

Resistance and reluctance. We know that the resistance R relates the driving voltage to the current in an electric circuit:

$$I = \frac{V}{\sum R} \ , \tag{12.56}$$

and we will demonstrate that there is a similar relationship for the *reluctance* of a magnetic circuit:

$$\Phi = \frac{NI}{\sum R_m} \tag{12.57}$$

Summary of mathematical similarities. These mathematical similarties allow us to use aspects of our intuition from electrical circuits to address magnetic circuits:

Electrical circuit	Magnetic circuit
$\mathbf{J}$	$\mathbf{B}$
$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$	$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$
$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$
$\mathbf{J} = \sigma\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
σ	μ
$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = V$	$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = NI$

12.3.2 Example: Toroidal circuit

Find the magnetic flux through a thin toroid with an air gap of width d as illustrated in Fig. 12.10, driven by N turns of a wire with a current I. You can assume that the permeability of the toroid is so large that all the flux follows the toroid and the air gap is so small that you can assume there is no flux leakage across the gap (the fringe effects are negligible).

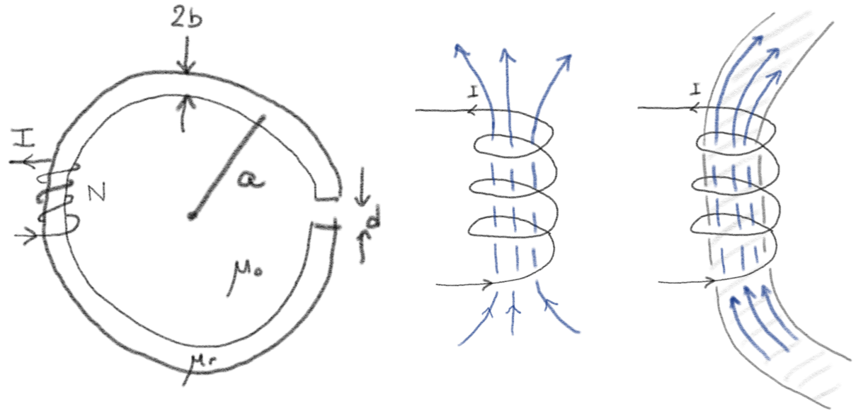


Fig. 12.10 (a) Illustration of magnetic circuit, (b) Illustration of the magnetic field from N winded wires with and without a magnetic material. The field tends to follow the magnetic material when $\mu_r \gg 1$.

First, let us address what sets up the magnetic field in the circuit. The N turns of wire are only wrapped around a small part of the toroid. Inside this area we expect the magnetic field to be approximately uniform because the thickness of the toroid (b) is small compared with the radius of the toroid (a). If there was no magnetic material, we would expect the magnetic field to spread out outside the region where the wires are winded as illustrated in the figure. However, due to the magnetic material and the large ratio in magnetic pemeability between the air and the magnetic material, we expect the magnetic field to follow the magnetic material instead. This is indeed why we call the system a magnetic circuit. The magnetic field follows the magnetic material also outside the region where the wired are winded.

We therefore expect the magnetic flux, Φ_B, to be conserved across any cross section along the wire, and since we have assumed that the magnetic field is uniform across the cross-section and equal to B, the flux is:

$$\Phi_B = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \pi b^2 B \ . \tag{12.58}$$

We also expect this relation to hold in the air gap, where we also assume that the magnetic field is uniform. We can therefore assume that the magnetic field B is the same all along the circuit. However, the H field will vary between the magnetic material and the air gap.

How can we find B? We can use Amperes law along an integration path C following the circuit as illustrated in the figure:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = NI \; . \tag{12.59}$$

Notice that the current through S is still NI, even if this current only passes through one part of the circuit. What is $\mathbf{H}$ along the circuit? We will assume that $\mathbf{H}$ is aligned with $\mathrm{d}\mathbf{l}$ everywhere: The field follows the circuit and does not leak out into the surrounding air. (This is only an approximation, due to the high ratio between μ of the material and μ_0 of the air, which typically is around 10^5 to 10^6). Inside the magnetic material we have that $B = \mu H = \mu_0 \mu_r H$, so that $H = B/(\mu_0 \mu_r)$ and inside the air gap we have that $B = \mu_0 H$ so that $H = B/\mu_0$. We recall that B is the same along the whole path, because the flux is the same, as we argued above. The length of the path inside the material is $2\pi a - d$ and the length inside the air gap is d, so that:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = NI = \frac{B}{\mu_0} d + \frac{B}{\mu_0 \mu_r} \left(2\pi a - d \right) \; . \tag{12.60}$$

We can therefore find B from this equation:

$$B = \frac{\mu_0 NI}{d + \frac{1}{\mu_r} \left(2\pi a - d \right)} \; . \tag{12.61}$$

And the flux is

$$\Phi_B = BS = \frac{NI}{\frac{d}{\mu_0 S} + \frac{2\pi a - d}{\mu_0 \mu_r S}} \; . \tag{12.62}$$

We can rewrite this in a way that demonstrates the similarity with the electrical circuit by introducing the reluctances $R_{m,t}$ for the toroid and $R_{m,g}$ for the air gap:

$$R_{m,t} = \frac{2\pi a - d}{\mu_0 S} \; , \tag{12.63}$$

$$R_{m,a} = \frac{d}{\mu_0 S} \; . \tag{12.64}$$

We can then rewrite the expression for the flux as:

$$\Phi_B = \frac{NI}{R_{m,t} + R_{m,g}} \; , \tag{12.65}$$

This is very similar to an expression we would get for an electrical circuit with two resistances $R_t + R_g$ in series, for which the current would be:

$$I = \frac{N}{R_t + R_g} . \tag{12.66}$$

12.3.3 Example: Magnetic circuit with varying cross section

Fig. 12.11 illustrates a magnetic circuit consisting of two magnetic materials with permeabilities μ_1 and μ_2 and with two different cross-sections, S_1 and S_2. The length of the part of the circuit with cross section S_1 is l_1 and the length of the part of the circuit with cross section S_2 is l_2. We assume that $\mu_1 \gg \mu_0$ and $\mu_2 \gg \mu_0$ so that the magnetic flux is constrain to the circuit (leakage is negligible). Find the magnetic flux Φ_B in the circuit.

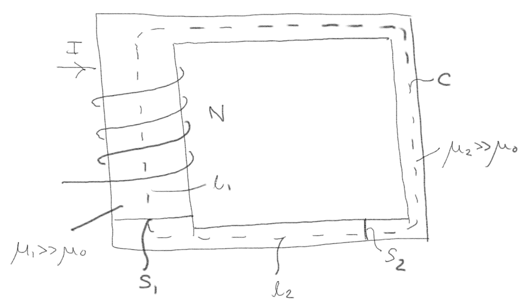


Fig. 12.11 Illustration of magnetic circuit of two materials 1 and 2 with different cross sections S_1 and S_2, different permeability μ_1 and μ_2 and different lengths $l1_1$ and l_2.

We assume that the magnetic field $\mathbf{B}$ everywhere is aligned along the magnetic circuit. (We realize there may be losses at the sharp corners in the drawing, but assume that these losses are negligible). The basic law we will use is that the magnetic flux is conserved along the circuit, so that Φ_B is the same across any cross section. This means that the flux in region 1 is the same as in region 2. We will also assume that the magnetic field is uniform across any cross section, that is, that the variations across a cross section is small. The magnetic field will, however, vary along the magnetic circuit. (This is the same as assuming that the current is the same along a circuit, that the current density may vary along the circuit, but that the current density is approximately uniform across a cross section). The conservation of flux gives us:

$$\Phi_1 = B_1 S_1 = \Phi_2 = B_2 S_2 = \Phi . \tag{12.67}$$

In addition, we will use Ampere's law along the path C around the circuit. The net current through this path is NI:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = NI \ . \tag{12.68}$$

We will also assume that $\mathbf{H} = (1/\mu)\mathbf{B}$ is aligned with the circuit and therefore parallel with $\mathrm{d}\mathbf{l}$ along the whole circuit. We use that $B_1 = \Phi/S_1$ and $B_2 = \Phi/S_2$ in the H-field:

$$H_1 = \frac{B_1}{\mu_1} = \frac{\Phi}{\mu_1 S_1} \quad , \quad H_2 = \frac{B_2}{\mu_2} = \frac{\Phi}{\mu_2 S_2} \tag{12.69}$$

Inserted in Ampere's law this gives:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = H_1 l_1 + H_2 l_2 \tag{12.70}$$

$$= \frac{\Phi l_1}{\mu_1 S_1} + \frac{\Phi l_2}{\mu_2 S_2} = IN \ , \tag{12.71}$$

which gives

$$\Phi = \frac{NI}{\frac{l_1}{\mu_1 S_1} + \frac{l_2}{\mu_2 S_2}} = \frac{NI}{R_{m,1} + R_{m,2}} \ , \tag{12.72}$$

where we have introduced the reluctances

$$R_{m,1} = \frac{l_1}{\mu_1 S_1} \quad , \quad R_{m,2} = \frac{l_2}{\mu_2 S_2} \ . \tag{12.73}$$

Again, like in the previous example, we recognize this as the same law we have for resistances. For reluctances connected in series we therefore expect:

$$R_m = \sum_i R_{m,i} \ , \tag{12.74}$$

and that the flux is related to the current NI through the circuit through

$$NI = \Phi R_m \quad , \quad \Phi = \frac{NI}{R_m} \tag{12.75}$$

Which means that we have a law similar to Ohm's law for the reluctance: $R_m = l/(\mu S)$, which is similar to what we found for a resistance with constant cross-sectional area S and length l: $R = l/(\sigma S)$, and we have found that we can add reluctances in a way similar to how we add resistances. This again illustrated the close analogy between magnetic and electrical circuits.

12.3.4 Kirchoff's laws for magnetic circuits

Based on the theoretical insights and examples from above we are now ready to formulate a set of Kirchoff's laws for magnetic circuits. These laws are not exact, as for electric circuits, but are approximate laws in the sense that they assume that the leakage of electric flux along the circuit is negligible.

Conservation of flux. The conservation of flux in magnetic circuits corresponds to conservation of current in electric circuit, and is valid also for junctions in the magnetic circuit. It follow from the lack of divergence of the magnetic field:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \Rightarrow \quad \sum_i B_i S_i = 0 \ , \tag{12.76}$$

where S is a surface surrounding the magnetic circuit, as illustrated in Fig. 12.12, and S_i are the cross-sectional surfaces. This law reduces to the conservation of flux along the circuit, as illustrated by the surface S^* in the figure.

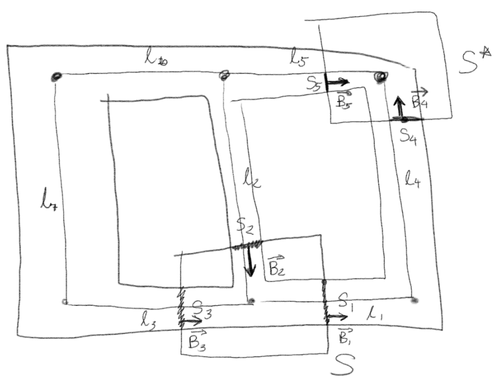


Fig. 12.12 Illustration of Kirchoff's laws for a magnetic circuit.

Generalized Ampere's law. A generalized Ampere's law for a magnetic circuit provides us with a law corresponding to Kirchoff's voltage law for magnetic circuits:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I \quad \Rightarrow \quad \sum_j H_j l_j = \sum_k N_k I_k \ , \tag{12.77}$$

where the sum over j is a sum over all the line elements of length l_j making up the circuit and the sum over k is the sum over the wire loops winding around the magnetic circuit.

12.4 Summary

The **magnetization M** is the net magnetic moment per unit volume:

$$\mathbf{M} = \frac{\sum_i \mathbf{m_i}}{\mathrm{d}v} ,$$

where the sum is over all the microloops i in the volume element $\mathrm{d}v$.

Ampere's law in a magnetic material is

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{C,\text{free}} ,$$

where the **H**-field is defined as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

The **bound current density** $\mathbf{J}_b$ is related to the magnetization through

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

and the **bound surface current density** $\mathbf{J}_{b,s}$ on the interface between a magnetic material and vacuum (air) with surface normal $\hat{\mathbf{n}}$ pointing from the magnetic material to the air is:

$$\mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}} .$$

Ampere's law on differential form is:

$$\nabla \times \mathbf{H} = \mathbf{J} .$$

For **linear magnetic materials** the magnetic field is proportional to **H**:

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{H}$$

where μ_r is called the relative permeability and μ is called the absolute permeability.

The **boundary conditions** for the magnetic fields are:

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}} \ ,$$

and

$$\hat{\mathbf{n}} \times \mathbf{H}_1 - \hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_s \ .$$

For **magnetic circuits** the magentic flux, $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ is conserved along the circuit as long as $\mu \gg \mu_0$ for the magnetic material. We can use concepts from electrical circuits as an approximation to understand and describe magnetic circuits.

12.5 Exercises

Learning outcomes. Explain basic concepts from magnetization; use Ampere's law for the **H**-field; and find the **B** for magnetized systems.

12.5.1 Discussion exercises

Exercise 12.1: Varm paramagnet

Hvorfor forventer vi at permeabiliteten til et paramagnetisk materiale blir mindre ved høyere temperatur?

Exercise 12.2: Evig magnet

Else har akkurat fått seg en flott, ny sylindrisk permanent-magnet. Hun har hørt at magneten har et magnetisk felt fordi det går en netto strøm I langs overflaten av magneten. Men hun har lært i elektromagnetisme at en strøm I i et materiale med motand R gir et effekttap RI^2. Hun måler R mellom to punkter på overflaten og finner at denne ikke er null. Hvorfor blir ikke magneten da varm?

Exercise 12.3: To halve magneter

(*Repetisjon av tidligere oppgave*) Hans har fått en permanent magnet formet som et rektangulært, flatt prisme. Den ene halvdelen er faget rød og har bokstaven N på seg, den andre halvdelen er farget sort og har bokstaven S på seg. Hans deler magneten i to deler, en rød og en sort bit, men blir forundret da hver av bitene igjen oppfører seg som en magnet med en nordpol og en sydpol. Hvordan kan du forklare Hans dette?

12.5.2 Tutorials

Exercise 12.4: Bound currents in a magnet

The focus on this tutorial is to develop an understanding of the concept of bound currents.

You know that currents create magnetic fields, but how much current does it take? You now have all the tools to estimate the bound current of an everyday toy magnet.

a) A compass needle is a magnet, and the arrow represents its north pole. If the compass needle (on a compass on the surface of the earth) is attracted towards the Earth as shown in Fig. 12.13, label the magnetic poles of the Earth ("N" or "S").

b) Which direction is the current flowing inside the Earth? Label it with arrows.

c) If the compass needle deflects as shown in Fig. 12.13, which pole of the toy magnet is closer to the compass? Label it ("N" or "S").

d) Which direction is current flowing on/in the toy magnet? Label it with arrows.

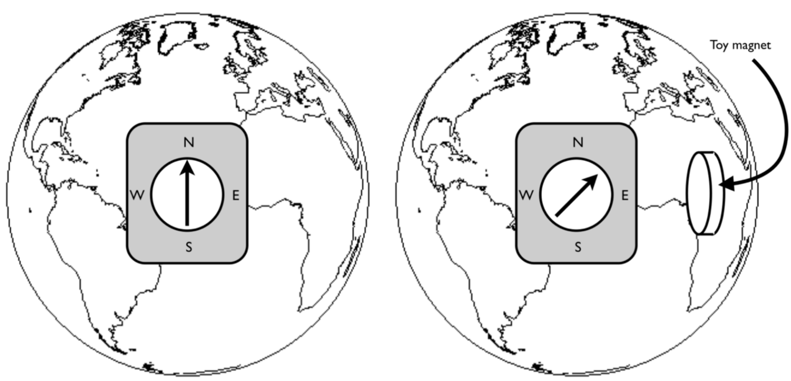
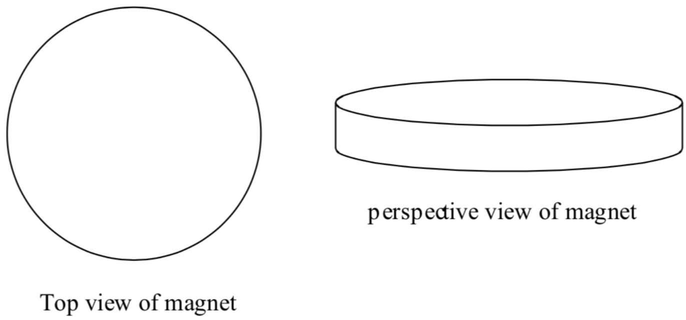


Fig. 12.13 Toy compass and the Earth.

e) Before you start estimating the bound currents of the toy magnet, do you expect there to be a bound surface current, bound volume current, or both? Sketch the atomic currents in the magnet below to illustrate your claim.



f) With your compass and toy magnet (and the Earth) oriented as shown in Figure 2, estimate the magnetic moment, m. You should use a meter stick. The following information may be useful:

$$\mathbf{B}_{dip}(r) = \frac{\mu_0|\mathbf{m}|}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \qquad \mu_0 = 4\pi \times 10^{-7}\mathrm{Tm/A} \tag{12.78}$$

The Earth's magnetic field is about 5×10^{-5}T.

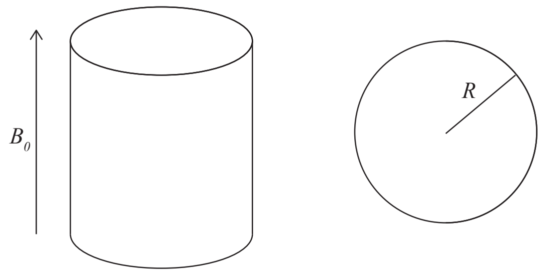
g) Now you can find M, the magnetic dipole moment per unit volume.

h) Estimate the bound current, I_b.

Exercise 12.5: Cylindrical magnet

In this tutorial we will focus on the relation between the external field $\mathbf{B}_0$*, the magnetization,* $\mathbf{M}$*, and the* $\mathbf{H}$*-field.*

A cylindrically shaped linearly magnetic material with magnetic susceptibility ξ_m is placed in an external, homogeneous magnetic field $\mathbf{B}_0$. The axis of the cylinder is directed along the magnetic field as illustrated in the figure. You can assume that the magnetic field is homogeneous and parallel to $\mathbf{B}_0$ inside the cylinder.



a) What is **H** outside the cylinder?

b) Write down Ampere's law for a magnetic material.

c) What type of Amperian loop would you choose to find **H** inside the cylinder.

d) What is **H** inside the cylinder?

e) Check that your result is consistent with the boundary conditions at the boundary between the cylinder and the vacuum outside.

f) What is **M** inside the cylinder for a diamagnetic and a paramagnetic material?

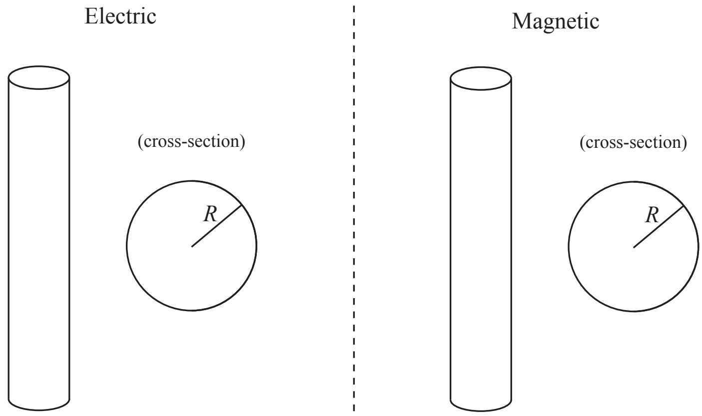
g) What is the magnetic field **B** inside the cylinder for a diamagnetic and a paramagnetic material?

h) How would you argue that the magnetic fields are homogeneous inside the cylinder?

Exercise 12.6: Similarities and differences between Gauss and Ampere

In this tutorial we will focus on the commonalities in the approaches to find the electric field using Gauss' law for a dielectric and to find the magnetic field using Ampere's law for a magnetizable material.

We will study an infinitely long cylinder of radius a in two scenarios. In scenario A the cylinder has a constant charge density ρ and consists of a dielectric material with dielectric constant ϵ, while the space outside the cylinder is vacuum. In scenario B the cylinder has a constant current density **J** directed along the axis of the cylinder with a magnetic permeability μ, while the space outside the cylinder is vacuum.



a) Write down Gauss' and Ampere's laws.
First, we will address the fields $\mathbf{D}$ (scenario A) and $\mathbf{H}$ (scenario B) outside the cylinder.

b) What symmetries would you use to apply Gauss' law to scenario A? What is the free charge? Find the field $\mathbf{D}$.

c) What symmetries would you use to apply Ampere's law to scenario B? What is the free current? Find the field $\mathbf{H}$.
Second, we will address the fields $\mathbf{D}$ (scenario A) and $\mathbf{H}$ (scenario B) inside the cylinder.

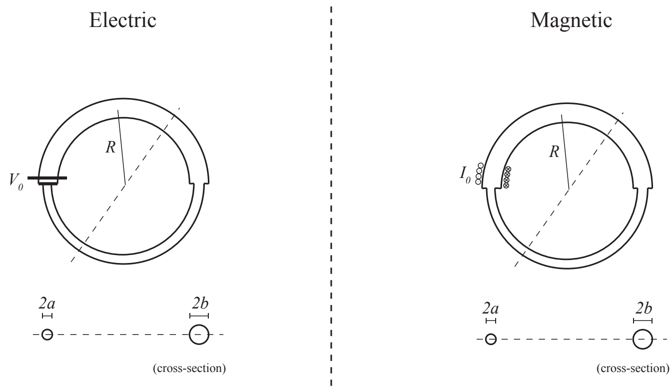
d) What symmetries would you use to apply Gauss' law to scenario A? What is the free charge? Find the field $\mathbf{D}$.

e) What symmetries would you use to apply Ampere's law to scenario B? What is the free current? Find the field $\mathbf{H}$.

Exercise 12.7: Magnetic circuits

The focus on this tutorial is to develop an understanding of the concept of magnetic circuits and how we can apply our intuition from electric circuits to address magnetic circuits.
In this tutorial we will address the correspondence between current flow in an electrical circuit (scenario A) with flux in a magnetic circuit (scenario B) as illustrated in the figure. The circuits consists of cylindrical half-donuts. The electric circuit has a conductivity σ and the magnetic circuit has a magnetic permeability $\mu \gg \mu_0$. You may assume there to

be air or vacuum outside the circuit. The electric circuit is driven by a constant emf V_0 from a battery. The magnetic circuit is driven by a constant current I_0 which is wrapped N times around the circuit cylinder along a very short length along the cylinder as illustrated in the figure.



a) Sketch the electric field **E** along the circuit in scenario A and the **H**-field in scenario B.

b) What line integral along the circuit would you use to find the electric field in scenario A and the H-field in scenario B?

c) Use your results from above to show that $E_1 \pi R + E_2 \pi R = V_0$ and that $H_1 \pi R + H_2 \pi R = N I_0$ where index 1 refers to the thick part of the cylinder and index 2 refers to the thin part of the cylinder.

d) How can you related **J** to **E** and similarly **B** to **H**?

e) What are the currents I_1 and I_2 expressed in terms of σ, E_i and the cross-sectional areas S_i? What are the fluxes Φ_1 and Φ_2 expressed in terms of μ, H_i and the cross-sectional areas S_i? (You can assume that **J** and **H** are uniform across a cross-section of the circuit).

f) What conservation law relates the currents along the circuit in scenario A and the fluxes along the circuits in scenario B?

g) Find the electric fields E_i expressed in terms of I, σ and S_i. Similarly, for scenario B find the magnetic fields H_i expressed in terms of Φ, μ and S_i.

h) Finally, put your results back into the integral around the circuit to find Kirchoff's law of voltage drops (scenario A):

$$\left(\frac{\pi R}{\sigma S_1}\right) I + \left(\frac{\pi R}{\sigma S_2}\right) I = V_0 \tag{12.79}$$

and similarly for scenario B:

$$\left(\frac{\pi R}{\mu S_1}\right) \Phi + \left(\frac{\pi R}{\mu S_2}\right) \Phi = N I_0 \tag{12.80}$$

12.5.3 Homework

Exercise 12.8: Sylinder i magnetfelt

(Johannes Skaar)

I et ellers uniformt magnetfelt bringes det inn et sylindrisk legeme. Sylinderen antas å være uendelig lang. Figur 12.14 viser magnetiske flukslinjer i et snitt vinkelrett på sylinderaksen. Oppgaven består i, for hver figur, å angi hva sylinderen består av, av følgende alternativer:

a) Diamagnetisk materiale, $0 < \mu_r < 1$.

b) Ideelt diamagnetisk materiale, $\mu_r = 0$

c) Paramagnetisk eller ferromagnetisk materiale, $\mu_r > 1$

d) Ideelt ferromagnetisk materiale, $\mu_r = \infty$.

e) Elektrisk leder ($\mu_r = 1$), som fører en elektrisk strøm som er jevnt fordelt over lederens *tverrsnitt*.

f) Elektrisk leder ($\mu_r = 1$), som fører en elektrisk strøm som er jevnt fordelt over lederens *overflate*.

Gitt en lang, tettviklet solenoide med totalt N viklinger på lengden l. Viklingene fører strømmen I. Finn den magnetiske flukstettheten $\mathbf{B}$ inne i solenoiden dersom

g) Solenoiden er en papirrull med relativ permeabilitet $\mu_r \approx 1$.

h) Solenoiden består av en jernkjerne med $\mu_r = 5000$.

Hint. Bruk Ampères lov. Symmetriargumentet for en solenoide kan du ta for gitt, dvs. du kan anta at feltet er i $\hat{\mathbf{z}}$-retning (langs aksen), og at det er neglisjerbart utenfor solenoiden.

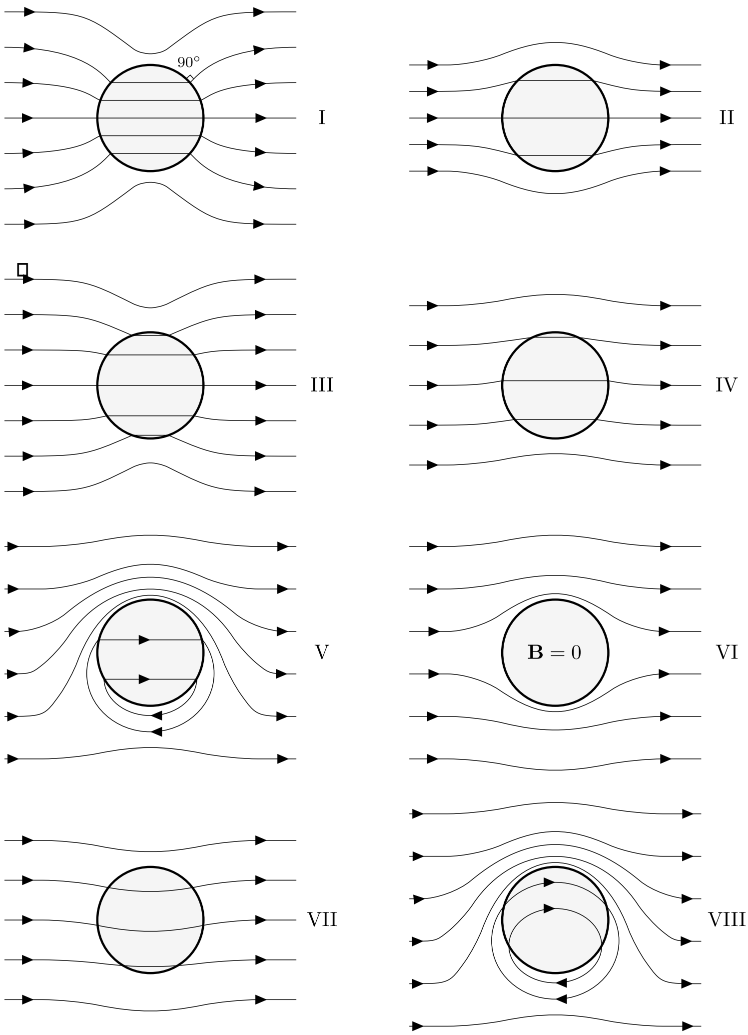
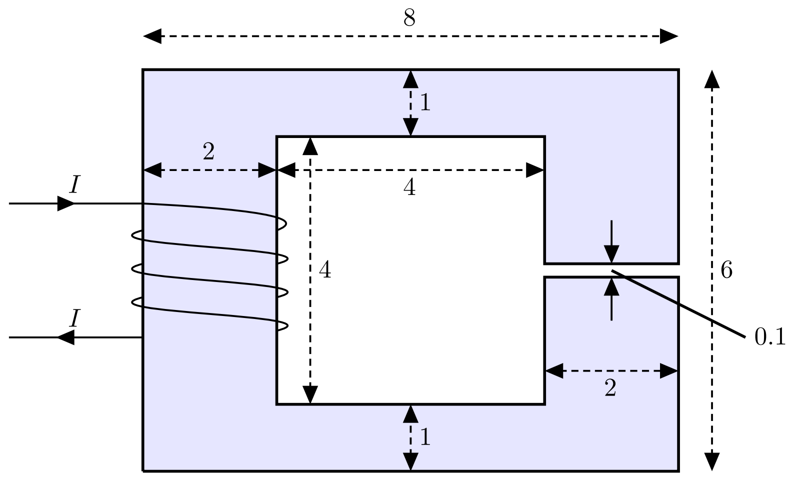


Fig. 12.14 Skisser av magnetiske flukslinjer for en sylinder i et magnetfelt.

Exercise 12.9: Magnetkrets

(Fra Johannes Skaar)



Figuren over viser en jernkjerne med alle mål gitt i cm. Kjernens tykkelse (vinkelrett på papirplanet) er 1 cm. Anta at den magnetiske flukstettheten $\mathbf{B}$ er jevnt fordelt over alle tverrsnittsareal S av kjernen, slik at $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = BS$. Kjernen har en relativ permeabilitet på 5000. Det er $N = 100$ viklinger, og strømmen I er 1 mA.

a) Anta at fluksen er konstant gjennom kretsen, dvs. se bort fra spredning av flukslinjer. Bruk Ampères lov til å vise at for denne magnetiske kretsen får vi fluksen

$$\Phi_m = \frac{NI}{\sum_i R_{mi}}, \tag{12.81}$$

der R_{mi} er reluktansen til del i i kretsen, definert som

$$R_{mi} = \frac{l_i}{\mu_i S_i}. \tag{12.82}$$

Her er l_i, μ_i, og S_i henholdsvis lengden, permeabiliteten og tverrsnittsarealet til den $i'te$ delen av kretsen. La integrasjonsveien være i midten av jernkjernen, slik at for eksempel den øverste delen av kretsen får lengden 6 cm og ikk 8 cm.

b) Finn tallsvar for den magnetiske fluksen Φ_m gjennom jernkjernen.

c) Hvor stor feil gjør vi om vi regner med uendelig relativ permeabilitet i jernkjernen?

Exercise 12.10: Surface current

Assume that there is a uniform surface current density in the x-direction, $\mathbf{J}_S = J_S\hat{x}$, everywhere in the xy-planet. This means that a current $I = LJ_S$ passes through a line section from $y = 0$ til $y = L$ in the positive x-direction.

a) Find the magnetic field $\mathbf{B}$ everywhere in space.

I sylinderkoordinater beskriver vi en posisjon med en radius r, en azimutal vinkel ϕ og en lengde z langs sylinderaksen. Anta en uniform overflatestrømtetthet $\mathbf{J} = J_0\hat{\phi}$ på en uendelig lang sylinderflate med radius a og akse langs z-aksen.

b) Find the $\mathbf{H}$-field and the magnetic field $\mathbf{B}$ everywhere in space when there is vacuum inside the cylinder.

c) Assume that there instead is a permanent magnet with magnetization $\mathbf{M} = -J_0\hat{\mathbf{z}}$ inside the cylinder. What is now $\mathbf{B}$ and $\mathbf{H}$ inside the cylinder? Provide a physical explanation for the result.

If the length L of the cylinder is much smaller than the radius a, we can describe the cylinder as a circular circuit with radius a and current $I = J_0L$. Assume that the cylinder is in vacuum.

d) What is contribution $d\mathbf{B}$ to the magnetic field $\mathbf{B}$ in in the point $\mathbf{r} = (x, y, z)$ from a small element $\mathrm{d}\mathbf{l}$ of the current loop in the position $a(\cos\phi, \sin\phi, 0)$ where ϕ is the angle from the x-axis?

e) Write a program to find the magnetic field $\mathbf{B}$ in a point $\mathbf{r} = (x, y, z)$ and visualize the field in the xz-plane.

Exercise 12.11: A simple magnetic trap

In this exercise we will study the magnetic field $\mathbf{B} = \frac{B_0}{b}\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\right)$, where B_0 is a constant and b is a length.

a) Sketch the $\mathbf{B}$-field in the xy-plane.

b) What is the divergence, $\nabla \cdot \mathbf{B}$, of the magnetic field?

We place a quadratic circuit with side L in the xy-plane centered in the origin and with surface normal in the positive z-direction.

c) What is the magnetic flux, Φ, through the circuit from the magnetic field?

d) The circuit is moved along the z-axis with the velocity. Find the emf, e, induced in the circuit.

e) Assume the circuit carries a current I. What is the force on the circuit from the magnetic field when $z = 0$?

f) What is the energy density in this magnetic field?

Faraday's law

13

So far we have only studied systems without any time variation, in electro-*statics* and magneto-*statics*. Now, we will start studying dynamic systems, in electro-*dynamics*. So far we have learned that charges generate electric fields (Gauss' law) and that currents generate magnetic fields (Ampere's law). We will now see that we can generate electric and magentic fields in a different way: A magnetic field that charnges (in time) generates an electric field and an electric field that changes (in time) generates a magnetic field. This ensures a nice symmetry, which is essential for the laws of electromagnetics to be the same across inertial systems. We will show that an emf, an electromotoric force, is generated in a circuit moving through a magnetic field. But, in the reference frame of the circuit, this may look like a time-varying magnetic field through the circuit. Faraday's law has this duality built-in: it is the rate of change of the flux through the circuit that induces an electromotoric force, independently of whether the change is due to a moving circuit or a time-varying magnetic field. However, in order to introduce Faraday's law, we need a new concept, the electromotoric force, which means that we need to start from revisting the electric field in circuits.

13.1 Emf: Electromotoric force

What is driving the current flowing in a circuit? Fig. 13.1 illustrates a circuit with a parallel-plate capacitor C, a wire with a finite conductivity and a resistor R. If the capacitor is charged up so that there are positive charges on the top side of the capacitor, there will be an electric field along the wire. As we discussed before, after a brief initial reorganization of charges on the surface of the wire, the electric field in the wire will point along the wire. In the resistor, the electric field is pointing from the high to the low potential as illustrated. The current is driven by the potential difference of the capacitor, but will eventually decay as charge is transported from the positive to the negative side of the capacitor.

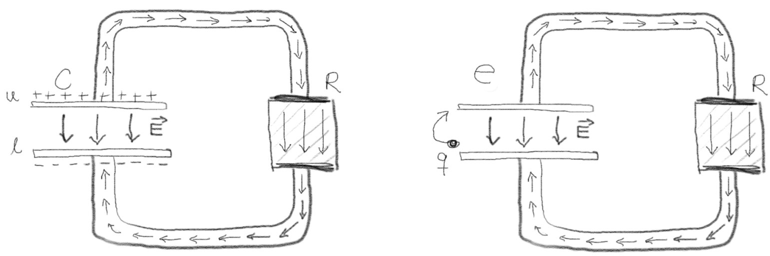


Fig. 13.1 Illustration of a circuit driven by a charged capacitor (left) and driven by a battery (right).

Instead of a capacitor, we could insert a mechanical battery. We could insert a device that would move a positive charge from the lower side to the upper side of the capacitor. This would require an external force. Inside the capacitor, there is an electric field $\mathbf{E}$. In order to move a charge from the lower side to the upper side of the capacitor, we would need a force per unit charge, $\mathbf{f}$, which is at least $\mathbf{f} = -\mathbf{E}$. This force would a property of the internal mechanics of the battery, or more generally, of any another mechanism that has the same effect as the battery of providing an external force per unit charge.

In this case, there are two forces driving charges around the circuit: one force from the electric field $\mathbf{E}$ and a force due to external forces per unit charge $\mathbf{f}$. These external forces may be due to various mechanicsms and there may be several such forces acting around the circuit, for example, a chemical force in a battery, a temperature gradient in a thermocouple, or light in a photoelectric cell. The *net effect* of these forces is the line integral of the force around the circuit:

$$e = \oint_C (\mathbf{f} + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} \, . \tag{13.1}$$

We call e the *electromotive force* or *emf* of the circuit. It is unfortunate that a potential difference is called a force, but this term is so ingrained in electromagnetics that we will use it, even if it is a bit of a misnomer.

Notice that since $\oint_c \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$ for any circuit, we could instead have written $e = \oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{l}$.

In the case of an *ideal battery*, as illustrated in Fig. 13.1, the net force on the charge in the battery is zero, so that $\mathbf{f} = -\mathbf{E}$. The potential difference between the upper and lower terminals of the battery (the upper, u, and lower, l, plates in the figure) is then

$$V = -\int_l^u \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int_l^u -\mathbf{f} \cdot \mathrm{d}\mathbf{l} = \int_l^u \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{l} = e \ , \tag{13.2}$$

where we have extended the integral to the entire loop because $\mathbf{f}$ is zero outside of the battery in the figure. We can therefore interpret the battery as a voltage source that keeps the potential difference at a given voltage corresponding to the emf of the battery, and we can interpret the emf as the work done per unit charge by the source around the circuit. We include the electric field $\mathbf{E}$ in our definition of the emf to include sources that are due to $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l}$ not being zero, which we will see is the case for Faraday's law.

Electromotoric force

The electromotoric force, emf, is defined as

$$e = \oint_C (\mathbf{f} + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} \tag{13.3}$$

where $\mathbf{f}$ is the external force per unit charge acting on charges along some part of the circuit. There may be several external forces acting along the circuit, and we interpret $\mathbf{f}$ as the superposition of these forces.

13.1.1 Example: Emf sources

What determines the voltage difference V of a battery? It is determined by an internal property of the battery, which is the force $\mathbf{f}$ per unit charge that the battery provides. Thus, it is usually the battery (or the voltage sources) that determines the voltage difference in a circuit.

Mechanical battery. In Fig. 13.2 we illustrate a *mechanical battery* illustrated as a conveyor belt that pushes a charge with a force $\mathbf{f}_{NC}$, where we have used the subindex $_{NC}$ to indicate that the force is Non-Coulombic, that is, an external force. Let us see what happens if we start this system from a neutral state, and then suddenly turn on a force that moves electrons to the right. The force on the first electron will only be from the external force. After a while, electrons will have moved to the left to the right, so that the right side will be negatively charged and the left side positively charged. Thus, there will be an additional force acting on the charges from the electric field. Eventually, the system will reach an equilibrium, so that the external force is equal to the electric field, that is, $\mathbf{E} = -\mathbf{f}_{NC}$. In this case, the potential difference will be $V = F_{NC}s/e = f_{NC}s$, where s is the distance between the plates. The potential difference and the emf of the battery therefore depends on the force per unit charge provided by the external mechanisms and by the geometry of the battery, here represented by the distance s between the plates.

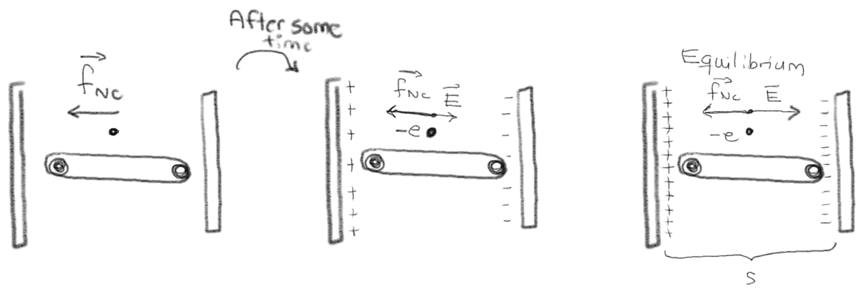


Fig. 13.2 Illustration of a mechanical battery.

Chemical battery. Fig. 13.3 illustrates a Daniel cell, which is a chemical battery named after the English physicist and chemist John Frederick Daniell (1790-1845). The battery consists of two electrodes, one with Zinc (Zn) and one with Copper (Cu) in a solution of SO_4^{2-} in water. In the solution, there are dissolved ions of Zn^{2+}, Cu^{2+}, and SO_4^{2-}. There is a membrane separating the two sides of the solution, as illustrated in the figure, and only SO_4^{2-} can pass through the membrane.

What happens in this system? When Zinc dissolves from the Zinc electrode, two electrons are released

$$Zn \rightarrow Zn^{2+} + 2e^- \; . \tag{13.4}$$

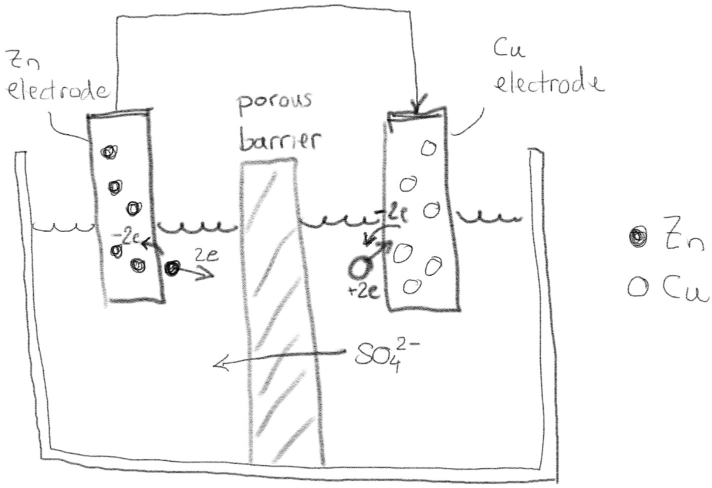


Fig. 13.3 Illustration of a chemical Daniell-cell battery.

The electrode reduction potential for this reaction is −0.76V. The charge of the two electrons travel to the Cu-electrode, where Copper ions are absorbed from the solution onto the electrode:

$$Cu^{2+} + 2e^- \rightarrow Cu \ . \tag{13.5}$$

The electrode reduction potential of this reaction is +0.34V. At the same time SO_4^{2-} diffuses through the membrane from the Copper to the Zinc side. The total voltage difference between the two electrodes due to this reaction is 1.1V per such cell. You will learn more about the physics underlying such processes when you learn about thermal and statistical physics.

13.2 Faraday's law

Now, we will address other sources of an emf in a circuit and one such source is either that circuit moves or changes shape or that the magnetic field through a circuit changes. We will demonstrate that these two forms of change are equivalently included in Faraday's law.

Fig. 13.4 illustrates a circuit moving through space with a velocity **v**. We will assume that the magnetic field is stationary and constant, and that we therefore may use the theory we already have developed in electrostatics and magnetostatics. There are two possible situations: Either the circuit is stationary and the magnetic field is moved with a velocity $\mathbf{v}$ or the magnetic field is stationary and the circuit is moved with a velocity $-\mathbf{v}$ (as illustrated on the left). The resulting behavior in the circuit must be identical in the two cases. Let us see how we can determine the behavior of the circuit from the theory we have developed.

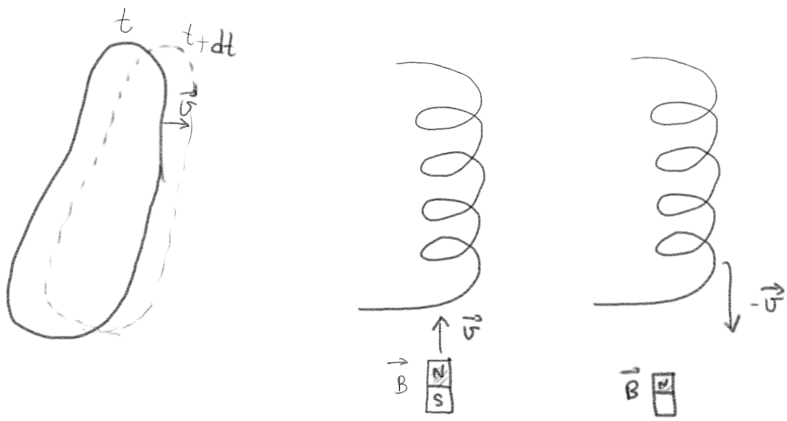


Fig. 13.4 (a) Illustration of a circuit moving with a velocity $\mathbf{v}$. (b,c) Illustrations of a circuit moving with a velocity $\mathbf{v}$ in a stationary magnetic field or a magnetic field moving with a velocity $-\mathbf{v}$ through a stationary circuit.

Fig. 13.5 illustrates the circuit and a smaller part of the circuit, all moving with the same velocity $\mathbf{v}$ through a stationary magnetic field $\mathbf{B}$. What happens to a point charge Q on an element $\mathrm{d}\mathbf{l}$ of the circuit? This charge is moving with the velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$ and experiences a force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$. This is an external force on a charge in the circuit, and the force per unit charge due to this interaction is then

$$\mathbf{f} = \frac{Q\mathbf{v} \times \mathbf{B}}{Q} = \mathbf{v} \times \mathbf{B} \, . \tag{13.6}$$

What is the emf in the circuit due to this external interaction?

$$e = \oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l} \, . \tag{13.7}$$

As illustrated in Fig. 13.5, the velocity $\mathbf{v}$ corresponds to a displacement $\mathrm{d}\mathbf{r}$ of the circuit in the time $\mathrm{d}t$:

$$e = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l} = \oint_C \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{B} \right) \cdot \mathrm{d}\mathbf{l} = \frac{1}{\mathrm{d}t} \oint_C (\mathrm{d}\mathbf{r} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l} \, . \tag{13.8}$$

Now, we will introduce a trick, and rewrite $(\mathrm{d}\mathbf{r} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$ using the vector forula $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$, which gives us:

$$e = \frac{1}{\mathrm{d}t} \oint_C \mathbf{B} \cdot (\mathrm{d}\mathbf{l} \times \mathrm{d}\mathbf{r}) = -\frac{1}{\mathrm{d}t} \oint_C \mathbf{B} \cdot (\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{l}) \ . \tag{13.9}$$

From Fig. 13.5 we see that $\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{l}$ is a surface element $\mathrm{d}\mathbf{S}$, which is the change in the surface S enclosed by the path C due to the motion of the element $\mathrm{d}\mathbf{l}$. The integral of this along all the elements $\mathrm{d}\mathbf{l}$ making up the path C gives the change in the integral of $\mathbf{B} \cdot \mathrm{d}\mathbf{S}$ for the surface, that is, the change in the flux of the magnetic field:

$$e = -\frac{1}{\mathrm{d}t} \oint_C \mathbf{B} \cdot (\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{l}) = -\frac{1}{\mathrm{d}t}\mathrm{d} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_S \ . \tag{13.10}$$

In this case we proved Faraday's law for what we call motional emf, that is, when the circuit is moving through a stationary field. However, experiments (and theoretical arguments) demonstrate that the law also is true if the change in flux is due to a time varying magnetic field. It does not matter what the origin of the change in flux through a circuit in, a change in flux will induce an emf in the circuit. This law is called *Faraday's law.*

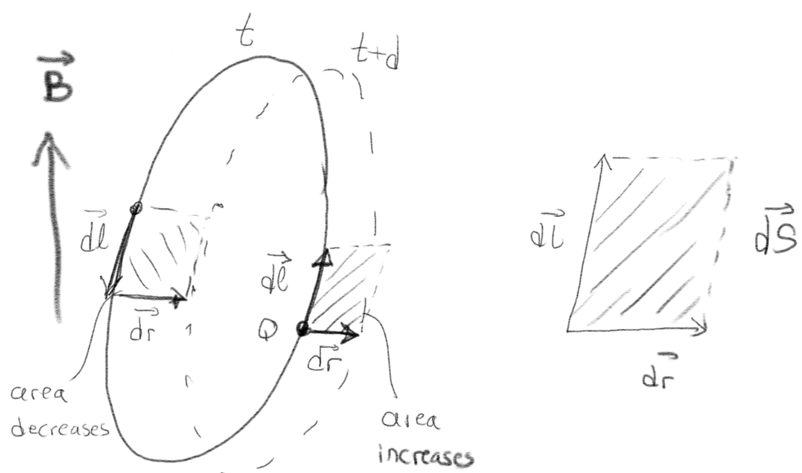


Fig. 13.5 Illustration of a circuit moving through a magnetic field. The element $\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{l} = \mathrm{d}\mathbf{S}$ corresponds to a change in the surface enclosed by the curve C.

Faraday's law

Faraday's law states that an emf is induced in a circuit C is the magnetic flux changes:

$$e = \oint_C (\mathbf{f} + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \Phi \ . \tag{13.11}$$

Faraday's law is valid independently of the source of change in the flux, and is true when the circuit is moving through a stationary magnetic field and when the magnetic field is varying in time through a stationary circuit.

We call the emf generated by this principle an *induced emf.*

The underlying symmetry in Faraday's law ensures that the emf induced in a circuit is the same if you take the circuit with you in a car and drive through a magnetic field which is localized in space or if you keep the circuit stationary and move the magnetic field. Inside the car, the circuit will be stationary and the magnetic field will be changing in time. The behavior in these two situations must be the same if the physics is to be the same in all inertial systems. (We will address these questions in more detail in the exercises).

13.2.1 Emf in a circuit

How do we include the magnetically induced emf in a circuit and how does it affect how we use Kirchoff's voltage law? Fig. 13.6 illustrates a circuit with a battery V_b, which represents one or more voltage sources, and a resistor R. There is a magntic field $\mathbf{B}$ going through the circuit with a resulting flux Φ, which varies in time, so that an emf $e = -\mathrm{d}\Phi/\mathrm{d}t$ is induced in the circuit. How do we include this emf in the sum of emfs and voltage drops around the circuit?

The sum of the emfs in the circuit is:

$$\sum e = \oint_C (\mathbf{f}_b + \mathbf{f}_m + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} \ . \tag{13.12}$$

Here, $\mathbf{f}_b$ is the force per unit charge inside the voltage sources in the circuit, such as inside a battery. We did not include this effect in our calculations above, only the magnetic interaction $\mathbf{f}_m$, for which we found that

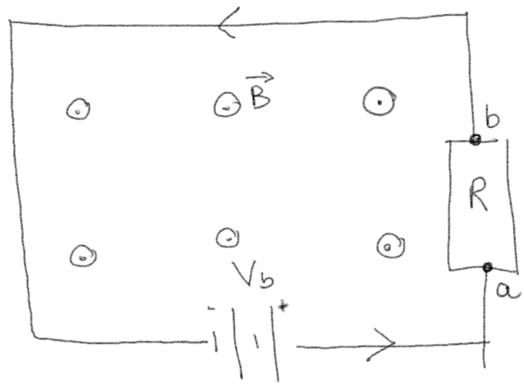


Fig. 13.6 Illustration of a circuit with a voltage source V_b, a resistor R and a magnetic field $\mathbf{B}$ is a flux Φ through the circuit.

$$\oint_C (\mathbf{f}_m + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{13.13}$$

The emf of the other voltage sources is:

$$\oint_C \mathbf{f}_b \cdot \mathrm{d}\mathbf{l} = V_b \ , \tag{13.14}$$

so that the total emf is:

$$\sum e = \oint_C (\mathbf{f}_b + \mathbf{f}_m + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} = V_b + \left(-\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) \ . \tag{13.15}$$

Everywhere except inside the resistor, we have that $\mathbf{f}_b + \mathbf{f}_m = -\mathbf{E}$, so that it is only across the resistor that the integral on the right hand side is non-zero. Therefore:

$$\sum e = \int_a^b (\mathbf{f}_b + \mathbf{f}_m + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} \ , \tag{13.16}$$

across the resistor (from a to b). Now, we will assume that $\mathbf{f}_m$ is small inside the resistor. This is an approximation, just like any circuit is an approximation of the real system, and it is reasonable as long as the resistor is small compared with the whole circuit. If we also assume that there are no other sources inside the resistor, so that $\mathbf{f}_b$ is zero inside the resistor, we get that

$$\sum e = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} \ , \tag{13.17}$$

across the resistor. This argument will be the same if we replace the resistor with any other component i with a voltage drop ΔV_i. If we have a circuit with several components, we can therefore sum the voltage drops of all the components in the integral on the right of (13.17):

$$\sum e = \sum_i \Delta i \ , \tag{13.18}$$

which we can rewrite as:

$$\sum_i \Delta i - \sum e = 0 \ . \tag{13.19}$$

If we introduce voltage drops for all the voltage sources and a voltage drop $-\mathrm{d}\Phi/\mathrm{d}t$ for the emf due to the change in flux through the circuit we get Kirchoff's voltage law:

$$\sum_i \Delta V_i = 0 \ , \tag{13.20}$$

where $\Delta V_1 = -\mathrm{d}\Phi/\mathrm{d}t$ is due to the change in flux, and e.g. $\Delta V_2 = V_b$ is due to the battery. Kirchoff's voltage law for the circuit in Fig. 13.6 then becomes:

$$\sum_i \Delta V_i = \left(-\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) + V_b - RI = 0 \ . \tag{13.21}$$

Notice that we have done some approximations when we have converted a real electromagnetic system into a circuit representation. For example, we have here assumed that we can ignore the effect of the magnetic field inside the resistor, so that we can calculate the properties of the resistor independently of the magnetic field. This is often an acceptable approximation, but it is useful to remember that this is indeed an *approximation* or a *model* of the real electromagnetic system.

13.2.2 Direction of the induced emf

What is the direction of the induced emf and the resulting induced current? Fig. 13.7 illustrates a circuit and a uniform magnetic field pointing into the plane. When we want to calculate the flux of $\mathbf{B}$ through a surface enclosed by circuit C, we need to determine an orientation of the surface. The flux integral is:

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \ , \tag{13.22}$$

where the direction of the surface is given by the direction of $\mathrm{d}\mathbf{S}$, which is given by a normal vector to the surface. For simplicity, let us assume that the circuit is in a plane and we choose a surface in the same plane — the plane of the paper. We can then choose the normal vector to point either into the plane or out of the plane. The choice of the direction of the normal vector also determines the positive direction of the circuit. The direction of the normal vector is determined by the right hand rule. For the circuit in Fig. 13.7a, you curve the four fingers from the index finger to your little finger on your right in the direction of the circuit and your thumb points in the direction of the corresponding surface normal. If the circuit is oriented in the positive direction, the normal vector points out of the plane. Whereas, if you choose the circuit to have the opposite direction, the surface normal points into the plane.

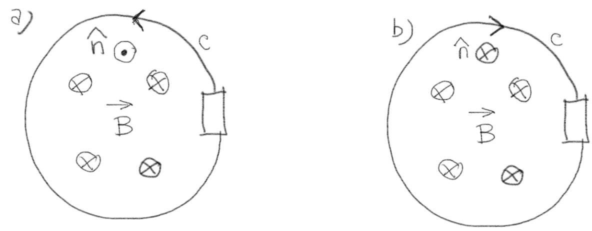


Fig. 13.7 Illustration of a circuit moving through a magnetic field. The element $\mathrm{d}\mathbf{r} \times \mathrm{d}\mathbf{l} = \mathrm{d}\mathbf{S}$ corresponds to a change in the surface enclosed by the curve C.

Let us assume that we choose alternative a. The the flux is then

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -BS \; , \tag{13.23}$$

If the magnetic field increases with time, $\mathrm{d}B/\mathrm{d}t > 0$, we see that the flux decreases with time, $\mathrm{d}\Phi/\mathrm{d}t < 0$, which means that the emf will be $e = -\mathrm{d}\Phi/\mathrm{d}t > 0$. We find the associated current I in the circuit by applying Kirchoff's voltage law: $e - RI = 0$, which gives $e = RI$ and $I = e/R$. Consequentlt, the current will be positive, that is, in the direction the arrow is points.

What would have happended if we instead chose alternative b? The flux would then be $\Phi = BS$. If the magnetic field increases with time, we would get $\mathrm{d}B/\mathrm{d}t > 0$, and $\mathrm{d}\Phi/\mathrm{d}t > 0$, which would give $e = -\mathrm{d}\Phi/\mathrm{d}t < 0$. Again, Kirchoff's voltage law gives $e - RI = 0$ and $I = e/R$, but now e is negative and the current is negative, which means that it is in the opposite direction to the arrow in figure b.

As expected, we get the same result for the direction of the emf and the direction of the current, independently of our choice of coordinate system and our choice of the positive direction for the circuit. We just have to be consistent and keep to our choices.

13.2.3 Lenz's law

What happens if you move a permanent magnet towards an electric circuit? We have illustrated the field from a permenent magnet in Fig. 13.8. When the magnet is moved towards the circuit, the magnetic field through the surface of the circuit increases, and therefore the magnitude of the flux increases. This means that we induce a current in the circuit. However, this current will also set up a magnetic field. What is the direction of the magnetic field set up by the current, which is induced by the moving magnet?

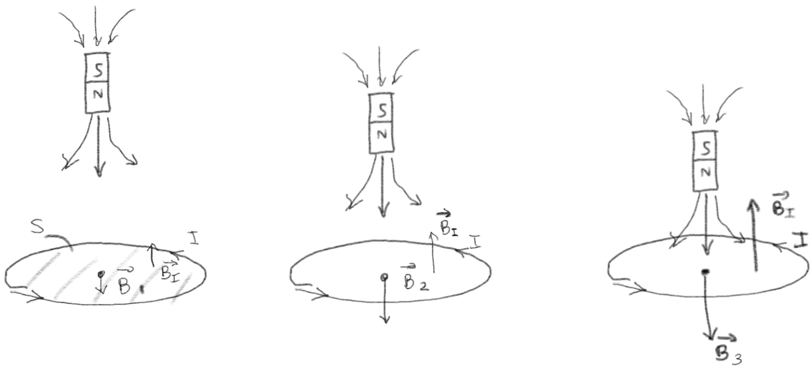


Fig. 13.8 Illustration of Lenz law. What is the direction fo the magnetic field set up by the current induced by a permenent magnet moving towards the circuit?

That was a complicated question, but it has a surprisingly simple answer. Let us analyze the behavior in the system in detail. Let us assume the permanent magnet and the circuit is oriented as in the figure. When the perment magnetic is moved towards the circuit, the magnitude of the magnetic field on the surface S encompassed by the circuit increases with time. Because the flux is negative, this means that the flux decreases with time. The induced emf is $e = -\mathrm{d}\Phi/\mathrm{d}t$ and it increases with time. Therefore, the current I increases with time. The magnetic field $\mathbf{B}_I$ from this current is oriented in the direction opposite of the magnetic field from the permanent magnet and the magnitude of B_I increases with time. Therefore, the magnetic field set up by the current induced by the change in flux acts in a direction opposite of the change in the field that caused the current.

This result is general and is called *Lenz law*: The direction of the current in the circuit is such that the magnetic field generated by the current will counteract the change in magnetic field that caused the current. This was a rather complex formulation. It can be simplified to "Nature counteracts changes in magnetic flux". If the flux is changed, the system will set up a current to generate a magnetic field, that will counteract the change. This is a very useful tool the check the sign of Faraday's law.

13.2.4 Example: Deformed circuit

Fig. 13.9 illustrates a circuit which has an edge consisting of a moving rod. The rod moves with a constant velocity v. There is a uniform magnetic field $\mathbf{B}$ pointing into the plane. What is the current I in the circuit, the power P consumed and the force F acting on the moving rod?

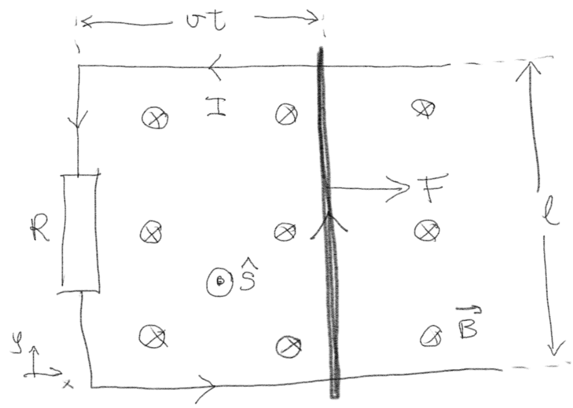


Fig. 13.9 Illustration a circuit consisting of a wired circuit with a moving rod forming one edge of the circuit.

Solution. First, we plan to find the emf due to the change in flux and use this to find the induced current. The flux is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = -BS = -Blvt \; , \tag{13.24}$$

where the negative sign is because with the given orientation of the circuit, the surface normal $\hat{\mathbf{n}}$ points upwards while the magnetic field points downward into the plane. The induced emf is then

$$e = -\frac{d\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(-Blvt\right) = Blv \; . \tag{13.25}$$

We find the induced current from Kirchoff's voltage law. For the circuit there is a voltage increase e and a voltage drop RI across the resistor, so that $e - RI = 0$ and $I = e/R$. The current is therefore:

$$I = \frac{e}{R} = \frac{Blv}{R} \; . \tag{13.26}$$

The direction of the current is in the direction of the arrows in the figure. (Check that the direction of the current is consistent with Lenz law).

The dissipated power is

$$P = RI^2 = R\frac{(Blv)^2}{R^2} = \frac{(Blv)^2}{R} \; . \tag{13.27}$$

Where does this energy come from? It comes from the work done by the external force F pulling on the moving edge of the circuit. We can find F by comparing the power exerted by the force F to the power dissipated in the circuit:

$$P_{\text{mechanical}} = Fv = P_{\text{electric}} = \frac{(Blv)^2}{R} \; , \tag{13.28}$$

which gives us that

$$F = \frac{B^2l^2v}{R} \tag{13.29}$$

We can check this result by comparing with the magnetic force on the moving rod:

$$\mathbf{F}_B = \int_l I\mathrm{d}\mathbf{l} \times \mathbf{B} = -\hat{\mathbf{x}}IBl = -\frac{B^2l^2v}{R}\hat{\mathbf{x}} \; . \tag{13.30}$$

When the rod is moving with constant velocity, the net force on the rod in the x-direction must be zero. Hence the mechanical force $\mathbf{F}$ applied to the rod must equal the magnetic force $\mathbf{F}_B$: $\mathbf{F} = -\mathbf{F}_B$, which is the same as we found by comparing the mechanical and electrical power dissipated.

13.2.5 Example: Rotating circuit in static magnetic field

A circuit rotates with a constant angular velocity ω in a uniform, stationary magnetic field $\mathbf{B}$ as illustrated in Fig. 13.10. The circuit is rectangular with sides a and b and has a resistance R. Find the induced emf and current in the circuit and the time-averaged power dissipated in the circuit. You can ignore the effect of the magnetic field generated by the current in the circuit.

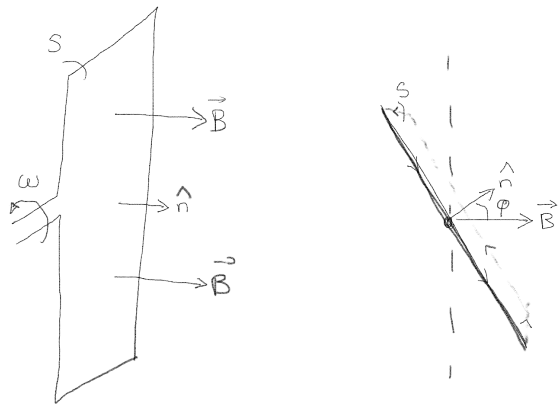


Fig. 13.10 Illustration a circuit rotating in an external magnetic field.

Solution: The magnetic flux through the circuit depends on the angle ϕ between the plane of the loop and a plane normal to the magnetic field as illustrated in the figure:

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mathbf{B} \cdot \mathbf{S} = BS \cos \phi \; , \tag{13.31}$$

where $\phi = \omega t$. The induced emf is therefore

$$e = -\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \left(BS \cos \omega t \right) = BS\omega \sin \omega t \; . \tag{13.32}$$

This means that in order to induce a large emf we need to have a large magnetic field B or a large surface area S of the circuit. A simple way to increase the surface area is to increase the number of windings, N, in the circuit. This is a simple model for a generator that generates an alternating voltage/current with a sinus-shape. To make a generator, you also have to find a smart way to connect two wires to the circuit.

This is often done using brushes, which will also turn the polarity of the outgoing wire after a rotation of 180°.

From Kirchoff's voltage law, we find that $e - IR = 0$ and $I = e/R$. The dissipated power is:

$$P = RI^2 = R\left(\frac{BS\omega \sin \omega t}{R}\right)^2 = \frac{\omega^2 B^2 S^2}{R} \sin^2 \omega t \; . \tag{13.33}$$

This is the instanteneous power. In order to find the time-averaged dissipated power we need to average over one rotational cycle. The average of the $\sin^2 \omega t$ term is

$$\langle \sin^2 \omega t \rangle = \frac{1}{T} \int_0^T \sin^2 \omega t \mathrm{d}t \; , \tag{13.34}$$

where $T = 2\pi/\omega$. We change variable to $u = \omega t$ so that the integral is from $u = 0$ to $u = T\omega = 2\pi$, where $\mathrm{d}u = \omega \mathrm{d}t$

$$\langle \sin^2 \omega t \rangle = \frac{1}{T\omega} \int_0^{2\pi} \sin^2 u \mathrm{d}u \tag{13.35}$$

This integral is simple to solve, and you may indeed remember it by heart, but we use `Sympy`:

```
import sympy as sp
u = sp.Symbol('u')
f = sp.sin(u)**2/(2*sp.pi)
sp.integrate(f,(u,0,2*sp.pi))
```

```
1/2
```

We therefore conclude that

$$\langle \sin^2 \omega t \rangle = \frac{1}{2} \; , \tag{13.36}$$

and that

$$\langle P \rangle = \frac{\omega^2 B^2 S^2}{R} \langle \sin^2 \omega t \rangle = \frac{\omega^2 B^2 S^2}{2R} \; . \tag{13.37}$$

13.2.6 Example: Multiple windings

Fig. 13.11 illustrates a circular circuit of radius a in the xy-plane that has been wound N times. The magnetic field $\mathbf{B} = B_z \hat{\mathbf{z}}$ varies in time. What is the emf and current in the circuit?

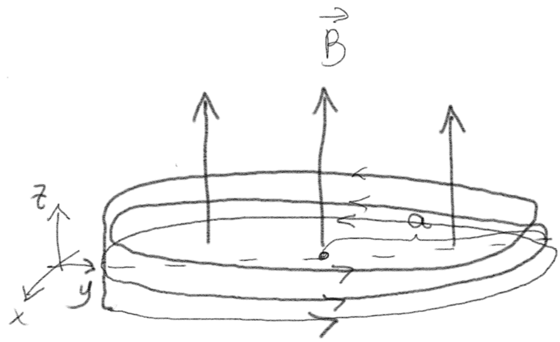


Fig. 13.11 Illustration a circular circuit with $N = 3$ windings in a magnetic field. rotating in an external magnetic field.

Solution: The flux Φ of the magnetic field is

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{13.38}$$

The flux of a single winding is $\Phi_1 = \mathbf{B} \cdot \mathbf{S} = B_z \pi a^2$, so that the total flux is N times the flux of a single winding as long as the windings are so close together that they all have the same flux $\Phi_N = N B_z \pi a^2$. The induced emf is then

$$e_N = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(N B_z \pi a^2\right) = -N\pi a^2 \frac{\mathrm{d}B_z}{\mathrm{d}t} = -N\frac{\mathrm{d}Phi_1}{\mathrm{d}t} \ . \tag{13.39}$$

and the corresponding current is found from $e - RI = 0$ and $I = e/R$.

13.3 Faraday's law on differential form

Faraday's law relates the emf to the rate of change of flux the a circuit

$$e = \oint_C (\mathbf{f}_m + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \ . \tag{13.40}$$

This is really two different laws: One law that describes a moving circuit and the corresponding motional emf, and one law that describes a time-varying magnetic field. We call both of these laws Faraday's law, even though there are two different mechanisms underlying the laws. These mechanisms can be united through the concepts of special relativity, but this is beyond the scope of this text.

If the circuit is stationary, then there are no magnetic forces on the charges, $\mathbf{f}_m = 0$, and we can move the derivative in (13.40) inside the integral, to get:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} . \tag{13.41}$$

where we can apply Stoke's theorem to get:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} . \tag{13.42}$$

This is Faraday's law on differential form.

Faraday's law on differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} . \tag{13.43}$$

13.3.1 Two types of electric field

Faraday's law shows that there are two types of electric field: the electric field set up by (stationary) charge distributions, and the electric field associated with magnetic fields.

If we write down Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} . \tag{13.44}$$

We see that it is very similar to Ampere's law for the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} . \tag{13.45}$$

Indeed, this similarity is even stronger because if the electric field is associated with the magnetic field alone, that is, if there are no charges present, then $\nabla \cdot \mathbf{E} = 0$. Thus we have two equations for the electric field, which is very similar to the equations for the magnetic field in Ampere's law. We can therefore use the intuition and the solution methods we have developed for Ampere's law also to find the electric field for a given rate of change of the magnetic field.

13.3.2 Example: Circular currents

A uniform, time-varying magnetic field, $\mathbf{B} = B_z(t)\hat{\mathbf{z}}$, passes through a circular conducting disc in the xy-plane. Find the associated electric field and current in the plane. You can assume that the field set up by the currents in the plane is negligible compared to $B_z(t)$.

Solution: We will attempt two types of solutions to this problem. First, we will solve the differential equation and second we will use the integral formulation inspired by Ampere's law.

First, we start from the partial differential equation for the electric field, Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial B_z}{\partial t}\hat{\mathbf{z}} \, . \tag{13.46}$$

We rewrite this in cylindrical coordinates, keeping only the z-component of the equations:

$$\frac{1}{r}\left(\frac{r\partial E_\phi}{\partial r} - \frac{\partial E_r}{\partial \phi}\right) = -\frac{\partial B_z}{\partial t} \, . \tag{13.47}$$

From the symmetry of the system, we realize that E_r cannot have any ϕ-dependence because the system is rotationally symmetric around the z-axis, therefore $\partial E_r/\partial\phi = 0$. We are therefore left with:

$$\frac{1}{r}\frac{r\partial E_\phi}{\partial r} = -\frac{\partial B_z}{\partial t} \, , \tag{13.48}$$

which gives

$$\frac{r\partial E_\phi}{\partial r} = -r\frac{\partial B_z}{\partial t} \, . \tag{13.49}$$

With solutions:

$$rE_\phi = -\frac{1}{2}r^2\frac{\partial B_z}{\partial t} + C \, , \tag{13.50}$$

and

$$E_\phi = -\frac{1}{2}r\frac{\partial B_z}{\partial t} + \frac{C}{r} \, . \tag{13.51}$$

The electric field must be finite at $r = 0$, therefore $C = 0$, and we have found the solution:

$$\mathbf{E} = -\frac{1}{2}r\frac{\partial B_z}{\partial t}\hat{\boldsymbol{\phi}} \, . \tag{13.52}$$

This is a field curling around the direction of the magnetic field, just like we have found from Ampere's law that the magnetic field is curling around the direction of the current.

For a conducting material, we know from Ohm's law that $\mathbf{J} = \sigma \mathbf{E}$. The circular electric field induced by the time-varying magnetic field, will therefore induce circular currents. Inside a conducting volume exposed to a varying magnetic field, there will often be a combination of many such circular currents called *eddy currents*.

We want to visualize the electric field. We rewrite the expression for the electric field using that $\hat{\boldsymbol{\phi}} = (-y, x)/r$ where $\mathbf{r} = (x, y)$:

$$\mathbf{E} = -\frac{1}{2} r \frac{\partial B_z}{\partial t} \hat{\boldsymbol{\phi}} = -\frac{1}{2} \frac{\partial B_z}{\partial t} \frac{r(-y, x)}{r} = -\frac{1}{2} \frac{\partial B_z}{\partial t} (-y, x) \; . \tag{13.53}$$

This is implemented in the following Python program and visualized in Fig. 13.12.

```
import numpy as np
import matplotlib.pyplot as plt
L = 5
NL = 25
dBdt = 1.0
x = np.linspace(-L,L,NL)
y = np.linspace(-L,L,NL)
rx,ry = np.meshgrid(x,y)
Ex = -(dBdt/2)*(-ry)
Ey = -(dBdt/2)*(rx)
plt.quiver(rx,ry,Ex,Ey)
```

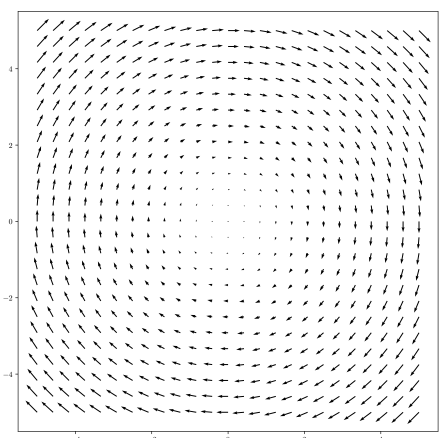


Fig. 13.12 Plot of the electric field (or the current density) associated with a time-varying magnetic field along the z-axis.

We can also solve this by applying an approach similar to Ampere's law by realizing that for a circular path of radius r around the center of the cylinder we have

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -B_z \pi r^2 \; . \tag{13.54}$$

Due to rotational symmetry, we expect E not to have any dependence on ϕ, and therefore:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 2\pi r E_\phi = -B_z \pi r^2 \tag{13.55}$$

and

$$E_\phi = -\frac{r}{2} B_z \; , \tag{13.56}$$

which is identical to what we found above. This method is considered a curiosity, demonstrating how the methods we develop can be transfered to new areas if the equations are similar.

13.4 Summary

The **electromotoric force**, **emf**, is defined as

$$e = \oint_C (\mathbf{f}_{NC} + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} \; .$$

The universal flux rule or **Faraday's law** states that

$$e = \oint_C (\mathbf{f}_{NC} + \mathbf{E}) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \Phi \; .$$

This law is valid both when the change in flux is due to a change in the circuit and when the change in flux is due to a time-varying magnetic field.

We can include the emf from Faraday's law in the voltage drops in **Kirchoff's voltage law** so that the sum of voltage drops along a circuit is

$$\sum_i \Delta V_i = 0$$

where we include $\Delta V_i = -\mathrm{d}\Phi/\mathrm{d}t$ for Faraday's law, $\Delta V_i = V_b$ for other sources such as a battery, and $\Delta V_i = -RI$ for a resistor.

Faraday's law on differential form states:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ .$$

13.5 Exercises

Learning outcomes. (1) Understand the concept of emf and how various processes can contribute to emf, (2) be able to reason qualitatively using Lenz law, (3) Recognize symmetries to apply Faraday's law to stationary and moving circuits, (4) Use line integrals to find the non-coloumbic **E**-field.

13.5.1 Test yourself

13.5.2 Discussion exercises

Exercise 13.1: Rotasjon av en kvadratisk krets

En kvadratisk krets er i et område med et uniformt (i rommet) og konstant (i tiden) magnetfelt. Kan kretsen roteres om en akse langs en av sidene på kretsen uten at en emf dannes i kretsen? Diskuter for forskjellige orienteringer av rotasjonsaksen i forhold til retningen til det magnetiske feltet.

Exercise 13.2: Force on copper plate

A copper plate is placed between the poles of an electromagnet with a magnetic field perpendicular to the plate. When the plate is pulled out a noticeable force is necessary and the force required increases with the velocity of the plate. Explain this phenomenon.

Exercise 13.3: Ring på linjen

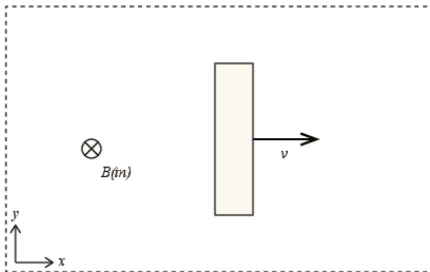
En lang, rett leder passerer gjennom sentrum av en metallring. Lederen står normalt på planet til ringen. Hva skjer hvis strømmen i lederen endres? Forklar!

13.5.3 Tutorials

Exercise 13.4: Motional emf

(*Based on a tutorial by Steven Pollock*)

A neutral metal bar is being pulled at constant velocity, speed v, to the right through a uniform magnetic field of magnitude B, as shown. The bar has been moving for some long time, and has achieved a dynamic steady-state.

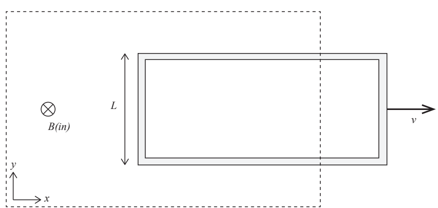


a) What is the magnetic force on charges in the bar (direction and magnitude)?

b) In the diagram, sketch the distribution of charges in the bar.

c) What is the electric field in the bar (direction and magnitude)?

Hint. Remember that the bar has reached a dynamic steady-state.
Now consider a rectangular metal loop of height L, moving to the right with speed v, which is exiting a region with a constant magnetic field, magnitude B.



d) The emf around any loop, C, is defined as $e = \oint_C \mathbf{f} \cdot d\mathbf{l}$, where $\mathbf{f}$ is the force per charge. What is the emf around the metal loop? (You should do this without using Faraday's law).

e) What is the magnetic flux, Φ, through the metal loop? (Define any new symbols used.)

f) Compute the time derivative of the flux through the loop $d\Phi/dt$ and compare with your computed emf. There is a +/- sign that you should worry about.

Exercise 13.5: Faraday's law

Faraday's law on differential form is

$$\nabla \times E = -\frac{\partial \mathbf{B}}{\partial t} \; . \tag{13.57}$$

a) Consider a very long solenoid of radius R with n turns per length and current I. Compute the **B**-field everywhere. (You can assume that the **B**-field is zero outside the solenoid.)

b) Suppose the current I in the solenoid is increasing at a steady rate $I(t) = Ct$, where C is a constant. Where do you think there is an **E**-field? (Inside the solenoid? Outside? Everywhere? Nowhere?) What do you think the **E**-field looks like? For now, just use your intuition, we'll check with calculations later.

c) Use Faraday's Law in integral form to compute the electric field inside the solenoid. Specify the loop you chose for the integral.

d) Use Faraday's law to compute the **E**-field outside the solenoid.

Exercise 13.6: Self-inductance

In this tutorial we will focus on the strategy and procedure used to find the self-inductance L of a component. The strategy is similar to the strategies we have used to find the resistance and the capacitance of a component. The general strategy is that (1) we assume that a current I runs through the component, (2) we find the magnetic field $\mathbf{B}$ *due to this current, (3) we find the flux Φ of this magnetic field through the curve C with the current I, and (4) we find the self-inductance $L = \Phi/I$.*
We will address a long cylindrical solenoid of length h, radius R and N turns of the wire.

a) Make a sketch of the solenoid as seen from above and from the side. Mark the direction of the current I through the solenoid in each of the drawings.

b) Assume that a current I runs through the wire. Use Amperes law for a well-chosen path to find the magnetic field $\mathbf{B}$ inside the solenoid. (You can assume that the magnetic field $\mathbf{B}$ is zero outside the solenoid). The field is a vector and should have a magnitude and a direction!

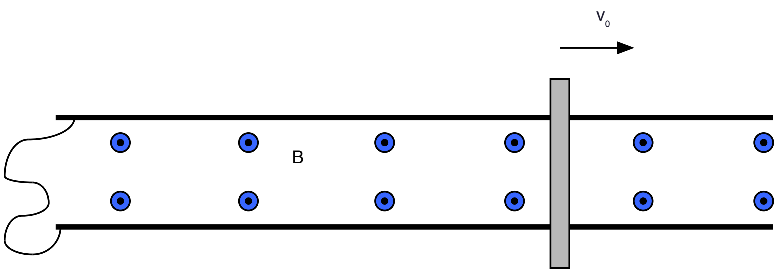
c) Find the flux of the magnetic field $\mathbf{B}$ through the loop(s) of the wire. We recall that there are N loops of the wire, hence the relevant area is $N\pi R^2$:

d) Show that the self-inductance is $L = \frac{\mu_0 \pi R^2 N^2}{h}$.

13.5.4 Homework

Exercise 13.7: Stav på skinner i magnetfelt

En stav med masse m går på skinner i et magnetfelt. Skinnene er koblet sammen med en ledning ved $x = 0$, slik at staven og skinnene sammen med denne ledningen danner en lukket krets. Avstanden mellom skinnene er l, og motstanden i staven er R.



Vi skal foreløpig anta at skinnene og ledningen er ideelle ledere, slik at den totale motstanden i kretsen er R. Ved tiden $t = 0$ beveger staven seg i positiv x-retning med hastighet v_0.

a) Beregn emf'en i kretsen med Lorentz' kraftlov, og indiker bidraget fra de forskjellige delene av kretsen.

b) Beregn emf'en med Faradays lov. Stemmer denne med emf'en du regnet ut i forrige oppgave?

c) Finn størrelse og retning på strømmen i kretsen.

d) Bestem bevegelsen til staven for $t > 0$. Kan du sjekke om svaret stemmer ved å bruke energibevaring?

Vi skal i resten av oppgaven anta at det er motstand i skinnene, og at disse skinnene har resistivitet ρ og tverrsnittareal A. Staven og ledningen mellom skinnene antas nå å være ideelle ledere. Vi kan få dette til å bli en god tilnærming ved å velge et litt dårlig ledende materiale for skinnene.

e) Hva er motstanden i kretsen som funksjon av posisjonen x til staven?

f) Lag et program som beregner bevegelsen til staven når den starter ved $x = 1$m med hastighet v_0.

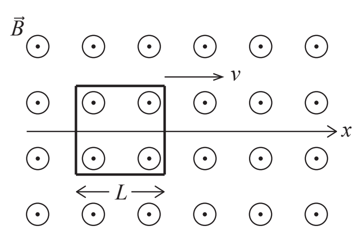
Du skal nå lage en bremsemekanisme for en rollercoaster. Du har tilgjengelig elektromagneter som lar deg sette opp et vilkårlig felt $B(x)$ langs de siste 100 meter av rollercoasteren, og du kan velge selv hvordan du vil plassere kabelen som kobler skinnene sammen. Anta at akslingen mellom bakhjulene på vognene er ledende, mens de øvrige akslingene er isolerende.

g) Gjør simuleringer for å avgjøre hva som vil være et fornuftig B-felt, og hvor det er lurt å plassere kabelen. (Denne deloppgaven er større enn de andre deloppgavene. Vi stiller som absolutt krav for å få obligen godkjent at du har gjort et ærlig forsøk på denne deloppgaven).

Hint. Du trenger å finne eller gjette på verdier for massen og hastigheten til vognene når oppbremsingen starter.

Exercise 13.8: Inhomogeneous field

You pull a quadratic current loop with side L with a constant velocity v_0 along the x-axis as illustrated in the figure. The current loop is oriented in the xy-plane. There is a magnetic field $\mathbf{B}(x, y, z) = B(x, y, z)\hat{\mathbf{z}}$. The current loop has a resistance R. You may ignore the magnetic field generated by the current in the loo. At the time $t = 0$, the left side of the loop is in the position $x = 0$.



a) Assume that the magnetic field is $\mathbf{B}(x, y, z) = B_0\hat{\mathbf{z}}$ where B_0 is a constant. What is the induced current I in the loop at a time t?

b) Assume instead that the magnetic field is $\mathbf{B}(x, y, z) = B_0(x/L)\hat{\mathbf{z}}$. What i the induced current I in the loop at a time t?

c) Assume that the magnetic field has the general form $\mathbf{B}(x, y, z) = B(x)\hat{\mathbf{z}}$. What is the induced current I in the loop at a time t?

14 Inductance

We have now introduced the basic law of magnetic induction, Faraday's law, which demonstrates how a time-varying magnetic field induces a emf in a circuit. For any circuit carrying a time-varying current, we know that the current will generate a time-varying magnetic field, and the magnetic field will generate a time-varying flux through the circuit, which again will induce an emf in the circuit. In this chapter, we will model this effect in a circuit by introducing the self-inductance of the circuit. The self-inductance is defined as the flux per unit current in a way similar to how we introduced capacitance as the charge per potential difference or the resistance as the potential difference per unit current. We will demonstrate how we can introduce a new element in a circuit, an inductor, to represent the inductance of the circuit, and we will show how you can calculate the inductance of a circuit based on the geometry of the current-carrying components and the magnetic materials. We will extend the concept of inductance to also include how one circuit can induce an emf in another circuit — through the mutual inductance of the two circuits, which again only depends on the geometry of the system. Finally, we will introduce the energy of the magnetic field in a capacitor in a way similar to how we introduced the energy of the electric field for a capacitor.

14.1 Inductance

Fig. 14.1 illustrates a circuit made of a conductor of finite thickness. If a current runs in the circuit, a magnetic field will be generated, and the magnetic field will have a flux through the circuit itself. This self-induced flux is called a *self-inductance* and we introduce the term L to describe this effect a circuit has on itself:

$$\Phi = L\,I \quad , \quad L = \frac{\Phi}{I} \; . \tag{14.1}$$

When we calculate the flux Φ in this expression, it is only the flux from the magnetic field generated by the circuit itself that is to be included:

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \; , \tag{14.2}$$

where $\mathbf{B}$ is generated by I. There may be other sources of magnetic fields, such as a permanent magnetic or another coil nearby, but these are not included in the flux Φ which is included in the self-inductance.

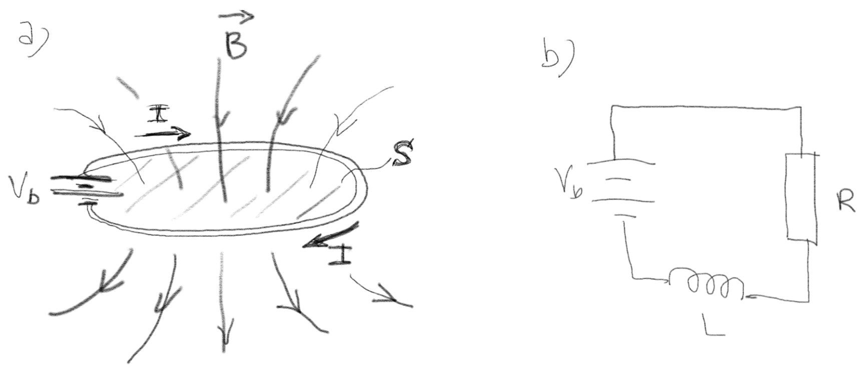


Fig. 14.1 (a) Illustration of a real circuit with a current I, the magnetic field $\mathbf{B}$ generated by I, and the resulting flux Φ through a surface S enclosed by the circuit C. (b) A simplified circuit diagram of the same system with the element L representing the self-inductance.

14.1.1 Definition of self-inductance

Self-inductance

The **self-inductance** L of a system/circuit is defined as

$$L = \frac{\Phi}{I} , \tag{14.3}$$

where the flux Φ is the flux through the circuit from the magnetic field generated by the current I.

14.1.2 What determines the self-inductance L?

For a circuit as illustrated in Fig. 14.1, the magnetic field $\mathbf{B}$ is generated by the current as described by Biot-Savart's law:

$$\mathbf{B} = \int_C \frac{\mu_0 I \mathrm{d}\mathbf{l} \times \hat{\mathbf{R}}}{4\pi R^2} , \tag{14.4}$$

and the flux is then

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} , \tag{14.5}$$

integrated over a surface S enclosed by the circuit/path C. This shows that

- The magnetic field is proportional to the current and the flux is proportional to the magnetic field, $\Phi \propto B \propto I$. Thus there is a *linear* relationship between Φ and I. This motivates the definition of $\Phi = LI$ so that L describes this constant of proportionality.
- The inductance L is then determined by the geometry of the system. It is the geometry of the circuit and the presence of magnetic materials that determine $\mathbf{B}$. And it is then again the geometry of the circuit that determines the flux.
- The definition of L is therefore very similar to our previous definitions of the capacitance $C = Q/V$ or the resistance $R = V/I$ of a system.
- This also suggests a *method to find the inductance*: We assume a given I, calculate the resulting $\mathbf{B}$ and the resulting flux Φ and find L from $L = \Phi/I$. This method is effectively identical to the method we introduced for the capacitance and the resistance.

Method: Calculate the inductance

- Assume that a current I runs through the circuit. (Often you will only have to look at the part of the circuit that acts as a coil with many windings).

- Calculate the magnetic field $\mathbf{B}$ from the current I using Biot-Savart's law or Ampere's law. Include effects of magnetic materials.
- Calculate the flux of the magnetic field $\mathbf{B}$ set up by the current I through a convenient surface S enclosed by the circuit path C
- Find the self-inductance L from $L = \Phi/I$.

14.1.3 How does the self-inductance enter the circuit equations?

The self-inductance of a circuit describes the relation between the flux Φ and the current I in a circuit, $\Phi = LI$. But how does this affect the behavior of the circuit? If the current is *time-varying*, then the current will induce a time-varying flux:

$$I = I(t) \quad \Rightarrow \quad \Phi(t) = LI = LI(t) \ . \tag{14.6}$$

This will again introduce an emf to the circuit according to Faraday's law:

$$e = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -\frac{\mathrm{d}}{\mathrm{d}t}LI(t) \ . \tag{14.7}$$

If the geometry of the circuit does not change, then $\mathrm{d}L/\mathrm{d}t = 0$ and we get:

$$e = -L\frac{\mathrm{d}I}{\mathrm{d}t} \ . \tag{14.8}$$

To include this effect in a circuit, we introduce an element L in the circuit as illustrated in Fig. 14.1b. This element is drawn like a coil in a circuit diagram.

This element is in principle a part of any circuit. For the circuit illustrated in Fig. 14.1, we may introduce an inductance L to model the effect of the geometry of the circuit. We must also include the effect of the emf in Kirchoff's voltage law for the circuit. We include the emf e in addition to the other voltage sources e_j in the circuit:

$$\sum e_j - \sum_i V_i = 0 \ , \tag{14.9}$$

where the V_i represents the voltage drops, such as $V_1 = RI$ for a resistor, whereas the emf's represents batteries, V_b and the emf due to inductance, $e = -\mathrm{d}\Phi/\mathrm{d}t$. For the illustrated circuit we have the emf $V_b + e$ and

$$V_b + e - RI = 0 \,. \tag{14.10}$$

All circuits have some inductance, often as an undesirable side effect. We represent this inductance by a coil (inductor) circuit element. However, we also design and include inductors that play important roles of circuits, such as a coil. We call such circuit elements *inductors*. Inductors can store magnetic energy, and play an important role in many circuits. In the next chapter, we will address how we construct circuits using the basic circuit elements resistor, capacitor and inductor, and how we can use circuits as models of model complex, real systems.

Notice that the model we have introduced for an inductor is a simplified model of the processes that goes on in the electromagnetic system, where we usually assume that the whole circuit can be described as a set of independent components. This is an approximation that sometimes breaks down. Then we have to apply the full machinery of an electromagnetic analysis.

14.1.4 Effect of self-inductance in a circuit

What is the effect of the self-inductance L in the circuit illustrated in Fig. 14.1? In this circuit, we found that

$$V_b + e - RI = V_b - L\frac{\mathrm{d}}{\mathrm{d}t}I - RI = 0 \,. \tag{14.11}$$

What happens if a switch in the circuit is suddenly turned on? In this case, we expect the current to increase rapidly. The effect of the self-inducance is to induce an emf $e = -L\mathrm{d}I/\mathrm{d}t$. If the current increases rapidly, this will generate a large emf, which will counteract the change: If the current is increasing, the emf will be negative, whereas if the current is decreasing, the emf will be positive. The self-inductance will therefore act to reduce changes in current.

Similarly, if we very rapidly turn off a circuit, for example by unplugging a wire, there will be a very large emf due to the rapid change in current. This is what induces sparks when you unplug an electric device. We will address the effect of inductance in circuits in the next chapter. In this chapter, we will focus on how to calculate the inductance for a given circuit geometry, just like we did for capacitors and resistors.

14.1.5 Example: Toroid

Fig. 14.2 illustrates a circuit wrapped N times around a toroid. The toroid has an inner radius a and an outer radius b, and a rectuangular cross section of height h. Find the self-inductance L of this circuit.

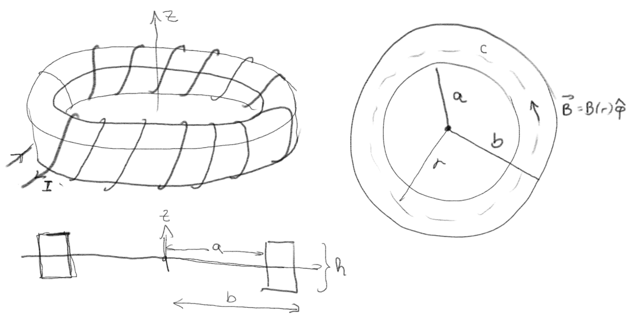


Fig. 14.2 Illustration of a toroid with rectangular cross section, inner radius a, outer radius b, heigh h and N windings.

Solution: Our plan is the use the method described above: We will assume a current I in the wire, find the resulting magnetic field $\mathbf{B}$ in the region enclosed by the circuit, calculate the resulting flux Φ and find the (self) inductance from $L = \Phi/I$.

Finding the magnetic field. We find the magnetic field using Ampere's law. First, we realize from symmetry and that the divergence of the magnetic field is zero that the magnetic field only has an azimuthal component, which only can depend on the distance r to the center of the toroid:

$$\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} = B(r)\hat{\boldsymbol{\phi}} \; . \tag{14.12}$$

We apply Ampere's law on a circular loop C or radius r inside the toroid. The magnetic field is constant along this circle is constant. Ampere's law gives that:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,2\pi r = \mu_0 I_{\mathrm{tot}} \; . \tag{14.13}$$

The total current going through the circle is I times the number of times the wire intersects the circle, which is N times, thus $I_{\mathrm{tot}} = NI$. We find that

$$\mathbf{B} = \frac{\mu_0 NI}{2\pi r}\hat{\boldsymbol{\phi}} \; . \tag{14.14}$$

Finding the flux. The flux through a single turn of the wire is

$$\Phi_1 = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \, , \tag{14.15}$$

where the integration is over a cross-sectional area from $r = a$ to $r = b$. Notice that $\mathbf{B}$ is not uniform across this area. It points in the same direction across the whole area, in a direction normal to the wire loop, but the magntide varies with r. A small surface element is $\mathrm{d}S = h\mathrm{d}r$. The flux of a single turn is therefore:

$$\Phi_1 = \int_a^b Bh\mathrm{d}r = \int_a^b \frac{\mu_0 NI}{2\pi r} h\mathrm{d}r \tag{14.16}$$

$$= h\frac{\mu_0 NI}{2\pi} = \int_a^b \frac{1}{r}\mathrm{d}r \tag{14.17}$$

$$= \frac{h\mu_0 NI}{2\pi} \ln \frac{b}{a} \, . \tag{14.18}$$

This is the flux through a single winding. There are N windings, so that the total flux is:

$$\Phi_N = N\Phi_N = \frac{h\mu_0 N^2 I}{2\pi} \ln \frac{b}{a} \, . \tag{14.19}$$

Notice that the number of windings enters this expression twice: Once in the magnetic field and once for the flux calculation.

Finding the inductance. We use this result to find the inductance:

$$L = \frac{\Phi}{I} = \frac{h\mu_0 N^2}{2\pi} \ln \frac{b}{a} \, . \tag{14.20}$$

We notice that this is indeed a geometric relation. The flux only depends on geometric properties of the circuit and not on the current I.

Approximation. What happens in the limit of a very thin toroid, where $b - a \ll a$? In this limit we can rewrite b as $b - a + a$, getting:

$$\ln \frac{b}{a} = \ln \frac{b - a + a}{a} = \ln \left(1 + \frac{b-a}{a}\right) \simeq \frac{b-a}{a} \, , \tag{14.21}$$

where we have used the Taylor expansion $\ln(1 + x) \simeq x$, which is valid when $x \ll 1$. This gives us the result:

$$L \simeq \frac{h\mu_0 N^2}{2\pi} \frac{d}{a} \, , \tag{14.22}$$

where $d = b - a$ is the thickness of the solenoid. We can rewrite this as $dh = S$ is the cross-sectional area of the toroid:

$$L \simeq \frac{S\mu_0 N^2}{2\pi a} . \tag{14.23}$$

14.1.6 Example: Coaxial cable

A coaxial cable with an inner conductor with radius a and an outer, thin conductor at radius b. Find the (self) inductance per unit length of the cable.

Solution: Our plan is to first assume a current I passes through the circuit, which is along the inner conductor and then back along the outer condutor. We find the magnetic field generated by the current, the flux of this current through the circuit and then use this to find the inductance.

Fig. 14.3 illustrates the system. However, we need to decide on what we want to consider the circuit. When a current I runs in the inner conductor, a corresponding current I must run in the opposite direction in the outer conductor. We assume these two are connected by a radially directed wire as illustrated. We assume that the outer conductor is thin and that the current is limited to the surface. Since the system has cylinder symmetry, we will assume that the current is uniformly distributed on the surface.

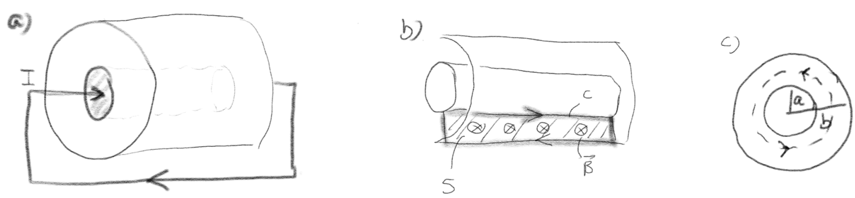


Fig. 14.3 Illustration of a coaxial cable. (a) The current loop from the inside the to the outside. (b) Illustration of the surface S for the flux. (c) Circuit used for Ampere's law calculation.

Finding the magnetic field. We find $\mathbf{B}$ using Ampere's law. The system has cylindrical symmetry. The field does not have any z-component, and cannot have a radial component, because this would lead to a finite flux of the magnetic field out of a cylindrical surface, and the flux must be zero. The field can therefore only have an azimuthal component. This component cannot depend on z because we assume the system to be long, and it cannot depend on ϕ, because the system is the same under any rotation around the cylinder axis. Therefore, $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} = B(r)\hat{\boldsymbol{\phi}}$.

We apply Ampere's law to the circuit illustrated in Fig. 14.3c:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,2\pi r = \mu_0 I \ , \tag{14.24}$$

when the circuit is within $a < r < b$. This gives

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} \ . \tag{14.25}$$

Finding the flux. We can then find the flux through the surface S, which is the flat surface spanned by the circuit in which the current I is running as illustrated in the figure:

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \ , \tag{14.26}$$

where a surface element goes from r to $r + \mathrm{d}r$ with length l, and area $\mathrm{d}S = l\mathrm{d}r$:

$$\Phi = \int_S B\mathrm{d}S = \int_a^b Bl\mathrm{d}r = \int_a^b \frac{\mu_0 I}{2\pi r} l\mathrm{d}r \tag{14.27}$$

$$= \frac{\mu_0 I}{2\pi}\int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_0 I}{2\pi} l \ln\frac{b}{a} \ . \tag{14.28}$$

Finding the inductance. Finally, we find the inductance L from

$$L = \frac{\Phi}{I} = \frac{\mu_0 I l}{2\pi I}\ln\frac{b}{a} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a} \tag{14.29}$$

and

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{b}{a} \tag{14.30}$$

14.2 Mutual inductance

We have so far defined inductance by how a circuit induces a flux on itself. However, a circuit can also incude a flux in another circuit. We therefore generalize the term inductance to *mutual inductance*. Fig. 14.4 illustrates two circuits C_1 and $_C2$. The current I_1 in generates a magnetic field $\mathbf{B}_1$, and the flux of the magnetic field $\mathbf{B}_1$ thorugh C_2 is called

$$\Phi_{1,2} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S} \ . \tag{14.31}$$

The first index refers to the source of the flux. Here it is circuit C_1. The second index refers to the circuit through which we measure the flux. Here it is a surface S_2 enclosed by C_2. This is the flux from circuit 1 in circuit 2. We define the mutual inductance $L_{1,2}$ as

$$L_{1,2} = \frac{\Phi_{1,2}}{I_1} \ , \tag{14.32}$$

using the same interpretation of the indexes as for the flux. We recognize the inductances $L_{1,1}$ and $L_{2,2}$ as the self inductances of circuits C_1 and C_2 respectively. There are in general four mutual inductances $L_{1,1}$, $L_{1,2}$, $L_{2,1}$, and $L_{2,2}$.

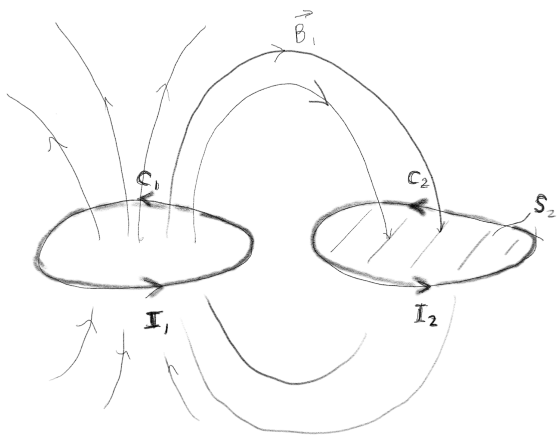


Fig. 14.4 Two circuits C_1 and C_2 interact through the magnetic fields generated by the currents in each circuit.

14.2.1 Definition of mutual inductance

Mutual inductance

We define the flux in a circuit C_j from a current I_i in circuit C_i to be

$$\Phi_{i,j} = \int_{S_j} \mathbf{B} \cdot d\mathbf{S} \tag{14.33}$$

where S_j is a surface enclosed by C_j. We define the **mutual inductanc** $L_{i,j}$ as

$$L_{i,j} = \frac{\Phi_{i,j}}{I_i} \; . \tag{14.34}$$

14.2.2 Symmetry of mutual inductances

A very practical result, which we will often use, relates the two mutual inductances of two circuits C_1 and C_2:

$$L_{1,2} = L_{2,1} \tag{14.35}$$

This nice symmetry provides us with a very powerful tool: If we want to calculate the mutual inductance, we can choose to calculate the magnetic field from the circuit where this is the simplest, and use this to find the flux through the other circuit. We will demonstrate how we use this principle below.

Proof. There is a very nice proof of this principle, which depends on the vector potential $\mathbf{A}$. You may skip this proof without loss of continuity.

The flux through loop C_2 from the magnetic field set up by circuit C_1 is:

$$\Phi_{1,2} = \int_S \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S} \; , \tag{14.36}$$

where we insert the vector potential $\mathbf{A}_1$ for $\mathbf{B}_1$:

$$\Phi_{1,2} = \int_S (\nabla \times \mathbf{A}_1) \cdot \mathrm{d}\mathbf{S} \; , \tag{14.37}$$

and we can then apply Stoke's theorem to get:

$$\Phi_{1,2} = \int_S (\nabla \times \mathbf{A}_1) \cdot \mathrm{d}\mathbf{S} = \int_{C_2} \mathbf{A}_1 \mathrm{d}\mathbf{l}_2 \; . \tag{14.38}$$

The vector potential $\mathbf{A}_1$ is found from:

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{\mathrm{d}\mathbf{l}_1}{R} \tag{14.39}$$

We insert this into the expression for the flux, and get:

$$\Phi_{1,2} = \frac{\mu_0 I_1}{4\pi} \oint_{C_2} \left(\oint_{C_1} \frac{\mathrm{d}\mathbf{l_1}}{R} \right) \cdot \mathrm{d}\mathbf{l}_2 \; . \tag{14.40}$$

We therefore find that the mutual inductance is

$$L_{1,2} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\mathbf{l}_1 \cdot \mathrm{d}\mathbf{l}_2}{R} \tag{14.41}$$

This is called the *Neumann formula*. The order in which we perform this double line integral is arbitrary. This proves that $L_{1,2} = L_{2,1}$. And it also demonstates that the mutual inductance is a purely geometry property: The line integrals depend on the shapes of the curves, but not on the currents in the circuits.

14.2.3 Example: Toroid with rectangular cross section

Fig. 14.5 shows a toroid with a rectangular cross section with a wire wrapped around the whole toiroid (C_1) and a single current loop C_2. What is the mutual inductance $L_{2,1}$ of this system?

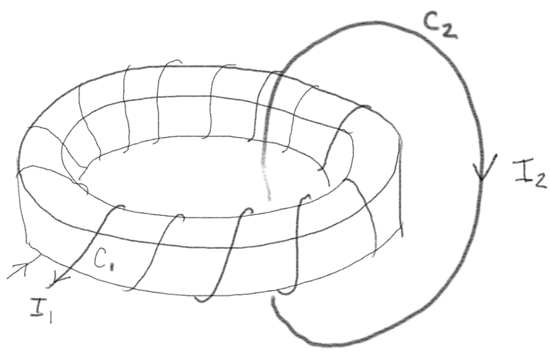


Fig. 14.5 A toroid circuit C_1 with a write wrapped around a torus with rectangular cross section, interacts with a circuit C_2.

Solution: In order to find $L_{2,1}$ we would need to find the magnetic field $\mathbf{B}_2$ generated by the current I_2 in circuit C_2. This is not simple. However, we can use that $L_{2,1} = L_{1,2}$. Finding $L_{1,2}$ is simpler, because it is simpler to find the magnetic field from the toroid circuit C_1.

We assume that the current I_1 is in the toroid. What is the magnetic field inside the toroid. We have previously found that

$$\mathbf{B}_1 = \frac{\mu N_1 I_1}{2\pi r} \hat{\boldsymbol{\phi}} \, , \tag{14.42}$$

where N_1 is the number of times the wire is wrapped around the toroid and μ is the permeability of the magnetic material inside the toroid.

What is the flux of $\mathbf{B}_1$ through S_2? We notice that the magnetic field from the toroid is only non-zero inside the toroid and zero outside. It is therefore only the intersection of the surface S_2 and the toroid that contributes to the integral. We choose a surface which is normal to the toroid, so that the intersection is a rectangular intersection from $r = a$ to $r = b$ and with height h. (We can choose such a surface, because we are free to choose any surface S_2 which is enclosed by C_2.)

The of $\mathbf{B}_1$ through C_2 is therefore:

$$\Phi_{1,2} = \int_{C_2} \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S} = \int_a^b Bh\mathrm{d}r = \int_a^b \frac{h\mu N_1 I_1}{2\pi r} \mathrm{d}r = \frac{\mu h N_1 I_1}{2\pi} \ln \frac{b}{a}. \quad (14.43)$$

If instead circuit C_2 consisted of N_2 windings, we would get N_2 times the flux from one winding:

$$\Phi_{1,2} = \frac{\mu h N_1 I_1 N_2}{2\pi} \ln \frac{b}{a} \ . \quad (14.44)$$

14.3 Applications of mutual inductance

If a time-varying current I_1 is sent through circuit C_1 in Fig. 14.4, an emf $e_{1,2}$ will be induced in circuit C_2. This emf will be:

$$e_{1,2} = -\frac{\mathrm{d}\Phi_{1,2}}{\mathrm{d}t} = -\frac{\mathrm{d}\left(L_{1,2} I_1\right)}{\mathrm{d}t} \ . \quad (14.45)$$

If the loops do not change with time, then $\mathrm{d}L_{1,2}/\mathrm{d}t = 0$, and

$$e_{1,2} = -L_{1,2} \frac{\mathrm{d}I_1}{\mathrm{d}t} \ . \quad (14.46)$$

This opens for a wireless transmission of voltage, current and energy between circuits. We will here address a few examples of this interaction such as transformers, but this type of interaction is the basis of many technologies.

14.3.1 Example: Cross-talk

Fig. 14.6 illustrates two wire pairs. Each pair form loops with wires that are far away and outside the sketch. Loop C_1 carries a current I_1 and loop C_2 carries a current I_2. Find the mutual inductance of this system and use this to discuss the cross-talk between the two lines.

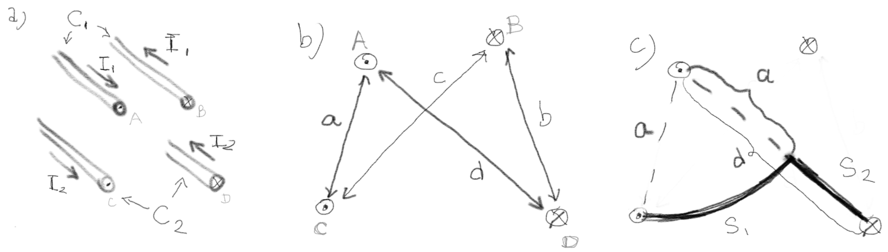


Fig. 14.6 (a) Two pairs of wires with two current I_1 and I_2 interact through a magnetic field. (b) Geometric aspects of the system. (c) The flux integral of the field B_1 from A toroid circuit C_1 with a write wrapped around a torus with rectangular cross section, interacts with a circuit C_2.

Solution: We follow our standard method for calculating the mutual inductance. We assume current I_1 is running in C_1 and find the flux through C_2. We realize that the magnetic field from C_1 is due to the superposition of two fields, one field from each of the two wires in C_1. We find the fluxes from each magnetic field and sum them to find the total flux from C_1 through a surface spanned by C_2.

Finding the mutual conductance. We start by finding the field from the current coming out of the plane in the top left conductor in Fig. 14.4. This is a current-carrying, long linear wire. We find the magnetic field by applying Ampere's law on a circular path of radius r around the wire:

$$\oint_C \mathbf{B}_1 \cdot \mathrm{d}\mathbf{l} = 2\pi r B_1 = \mu_0 I_1 \quad \Rightarrow \quad B_1 = \frac{\mu_0 I_1}{2\pi r} \,, \tag{14.47}$$

where r is the distance to the wire.

Second, we want to find the flux of B_1 through a surface enclosed by the two wires in C_2. We are free to choose any surface that is convenient for the integration. We therefore choose to divide the surface into two parts, S_1 and S_2 as shown in the figure. The surface S_1 is a cylindrical surface around the wire. Because the magnetic field is azimuthal to the wire, it curls around the wire, the magnetic field is tangential to this surface and the flux is zero. Surface S_2 is directed in the radial direction. The magnetic field is therefore normal to S_2 and the flux integral is simple to find and very similar to the integral we found for the toroid system:

$$\Phi_A = \underbrace{\int_{S_1} \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S}}_{=0} + \int_{S_2} \mathbf{B}_1 \cdot \mathrm{d}\mathbf{S} = \int_{r=a}^{r=d} \frac{\mu_0 I_1}{2\pi r} l \mathrm{d}r = \frac{\mu_0 I_1 l}{2\pi} \ln \frac{d}{a} \, , \quad (14.48)$$

where l is the length into the plane of the part of the wire we analyze. We find a similar result for the flux from the other wire, but the direction of the current is in the opposite direction:

$$\Phi_B = -\frac{\mu_0 I_1 l}{2\pi} \frac{b}{c} \, . \quad (14.49)$$

The total flux is the sum:

$$\Phi = \frac{\mu_0 I_1 l}{2\pi} \ln \frac{d}{a} - \frac{\mu_0 I_1 l}{2\pi} \frac{b}{c} = \frac{\mu_0 I_1 l}{2\pi} \ln \frac{dc}{ab} \, . \quad (14.50)$$

The mutual inductance per unit length is therefore

$$L_{1,2} = \frac{\Phi}{l I_1} = \frac{\mu_0}{2\pi} \ln \frac{dc}{ab} \quad (14.51)$$

Understanding cross-talk. We can use this formula to understand the inductive coupling between two wires, such as between two wires in an cable or a an electric devices. The current in one wire pair, will induce a emf in the other wire pair, which will impact the signales carried in the wires. Your internet cable consists of pairs of wires carrying high-frequency currents, and inductive cross-talk should be reduce to reduce interaction between signals in different wires.

14.3.2 Example: Voltage and current transformation

Mutual inductance allows the coupling between two circuits and can be used to transform voltages or currents. Here, we will study a typical system used for transformation and how it can be used to transform voltage and current.

Fig. 14.7 illustrates a typical geometry for a transformer. The system consists of a magnetic circuit formed by a thin toroid with a ferromagnetic core and two windings, that is, two current loops, C_1 and C_2 with N_1 and N_2 windings respectively.

The emf induced in circuit 1 from I_1 is called $e_{1,1}$ and the emf induced in circuit C_2 from current I_1 is $e_{1,2}$. We follow the same indexing convention as before: The first index refers to the circuit that generates the magnetic

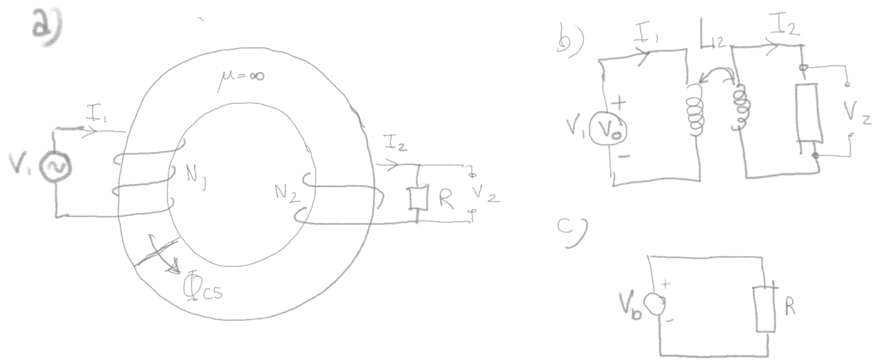


Fig. 14.7 (a) Illustration of a transformer consisting of two circuits wrapped around a ferromagnetic toroid. (b) Circuit diagram model of the same system. (c) Simplified circuit diagram for one circuit.

field and the second index refers to the circuit where the flux is generated or the emf induced. We know that

$$\frac{e_{1,2}}{e_{1,1}} = \frac{-\frac{\mathrm{d}\Phi_{1,2}}{\mathrm{d}t}}{-\frac{\mathrm{d}\Phi_{1,1}}{\mathrm{d}t}} = \frac{L_{1,1}\frac{\mathrm{d}I_1}{\mathrm{d}t}}{L_{1,1}\frac{\mathrm{d}I_1}{\mathrm{d}t}} = \frac{L_{1,2}}{L_{1,1}} \, . \tag{14.52}$$

This is a general result which is not based on any assumptions for arbitrary circuits. We can use this result to address the ratio of the induced emfs based on calculated or measured values for the mutual and self inductances.

Ideal transformer. We start by addressing an ideal transformer where $\mu = \infty$ so that all the flux remains in the ferromagnet. We want to find the ratio between the voltages V_1 on the signal generator and the voltage V_2 across the resistor in circuit C_2.

In these systems it is often tricky to find the correct signs for the voltages. Let us look at the simplified circuit in Fig. 14.7c. For this circuit we know that $\sum e = RI$, which is

$$V_b + \left(-\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) = RI \, . \tag{14.53}$$

In the case when $R = 0$, which is the case on the left in the figure (circuit C_1), we get that $V_b = V_1 = \mathrm{d}\Phi/\mathrm{d}t$. In the case when $V_b = 0$, which is the case on the right in the figure (circuit C_2), we get that $RI = -\mathrm{d}\Phi/\mathrm{d}t$.

Voltage transformation. This ratio between V_2 and V_1 is therefore:

$$\frac{V_2}{V_1} = \frac{RI_2}{V_1} = \frac{-\frac{\mathrm{d}\Phi_2}{\mathrm{d}t}}{+\frac{\mathrm{d}\Phi_1}{\mathrm{d}t}} \tag{14.54}$$

What are the fluxes Φ_1 and Φ_2. These depend on the cross-sectional flux, Φ_{CS}, through a cross-section of the ferromagnetic materials. We therefore have that $|\Phi_2| = N_2\Phi_{CS}$ and $|\Phi_1| = N_1\Phi_{CS}$, because each of the circuits wraps N_1 and N_2 times around the ferromagnet respectively. Let us also check that the signs of these fluxes. The cross-sectional flux is positive as indiciated in Fig. 14.7a. The flux Φ_1 should be positive, so that $\Phi_1 = N_1\Phi_{CS}$. However, we see that $\Phi_2 < 0$, so that $\Phi_2 = -N_2\Phi_{CS}$.

We insert this in the ratio and find

$$\frac{V_2}{V_1} = \frac{-\frac{\Phi_2}{\mathrm{d}t}}{+\frac{\mathrm{d}\Phi_1}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}N_2\Phi_{CS}}{\mathrm{d}t}}{\frac{\mathrm{d}N_1\Phi_{CS}}{\mathrm{d}t}} = \frac{N_2}{N_1} \; . \tag{14.55}$$

This system therefore acts as a transformer and can change one voltage into another voltage, while transmitting the signal.

Current transformation. We can find the ratio between the currents in the two currents by applying Ampere's law to a circuit C as illustrated in Fig. 14.8. Ampere's law gives that:

$$\int_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = N_1 I_1 - N_2 I_2 \; . \tag{14.56}$$

What is the value of the path integral? It is zero! However, we need a result we have not yet derived to demonstrate this. In Sec. 14.4.3 we show that the energy density in space in a magnetic material is $\mu H^2/2$. In the case when $\mu \to \infty$ we must therefore have that $H \to 0$, because the energy density cannot be zero. This is of course only an approxiation, because the real μ is not infinity, but still a large number. Notice that this does not imply that $\mathbf{B}$ is zero. We then get that:

$$N_1 I_1 = N_2 I_2 \; , \tag{14.57}$$

$$\frac{I_2}{I_1} = \frac{N_1}{N_2} \; . \tag{14.58}$$

Notice that this is the oppostive ratio of the voltages: If we transform up the voltage, we will transform down the currents. This is related to energy conservation: The power consumption must be the same on both sides of the magnetic circuit. The power consumption in the resistor is

$$P_2 = V_2 I_2 = \left(V_1 \frac{N_2}{N_1}\right)\left(I_1 \frac{N_1}{N_2}\right) = V_1 I_1 = P_1 \, . \tag{14.59}$$

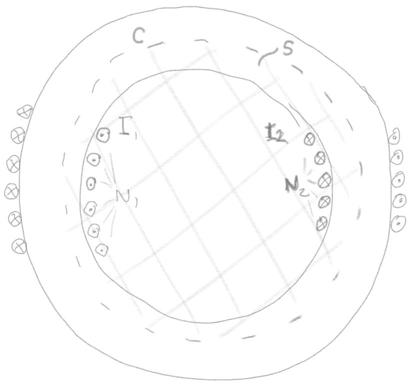


Fig. 14.8 Illustration of the integration path C and the surface S used for flux calculations. Only the currents on the *inside* of C constributes to the net current.

What is the resistance of the circuit as seen from the source V_1? We find that:

$$\frac{V_1}{I_1} = \frac{V_2 \frac{N_1}{N_2}}{I_2 \frac{N_2}{N_1}} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = R\left(\frac{N_1}{N_2}\right)^2 \, . \tag{14.60}$$

Notice that we have not made any assumptions about the time-dependence of the current. In principle, it is valid for both direct current and for alternating currents, but transformers only work for alternating currents.

14.4 Energy in magnetic fields

For capacitors, we developed a theory for the energy needed to charge a capacitor to a given charge and used this to define the energy density of the electric field. Here, we will follow a similar approach to define the energy density of the magnetic field.

14.4.1 Energy in a single circuit

What is the energy needed to get a current I flowing in a circuit such as the circuit in Fig. 14.9? You may think it is zero, because you can just turn a switch, and then the current is running. However, if you start with zero current, you need to gradually change the current until you reach a current I. As you do this, you will experience an emf that will counteract the change in current. For the circuit in the figure, Kirchoff's voltage law gives:

$$V_b = RI - e \; , \tag{14.61}$$

where V_b is the emf of the battery and e is the emf from the self-inductance of the circuit. The power P consumed in this circuit is

$$P = V_b I = RI^2 - eI \; , \tag{14.62}$$

and the work done in a short time interval $\mathrm{d}t$ is

$$\mathrm{d}W = P\mathrm{d}t = V_b I\mathrm{d}t = RI^2\mathrm{d}t - eI\mathrm{d}t \; . \tag{14.63}$$

The term $V_b I \mathrm{d}t$ represents the energy supplied by the battery. Some of this energy is dissipated in the resistor. This is the term $RI^2\mathrm{d}t$. The remaining energy is used as work against the emf due to the self inductance of the circuit. This is the energy that is stored in the magnetic field in the circuit. The work done by external sources — the battery — to the magnetic field of the circuit is

$$\mathrm{d}W_m = -eI\mathrm{d}t = -\left(-\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) I\mathrm{d}t \tag{14.64}$$

We insert $\Phi = LI$, so that $\mathrm{d}\Phi/\mathrm{d}t = L\mathrm{d}I/\mathrm{d}t$ (when L is not changing with time), getting

$$\mathrm{d}W_m = LI\frac{\mathrm{d}I}{\mathrm{d}t} = LI\mathrm{d}I \; . \tag{14.65}$$

This is the work done by the external source to change the current by an amount $\mathrm{d}I$. If we bring the system from zero current, where the magnetic energy is zero, up to a current I, the work done on the magnetic field is:

$$W_m = \int_0^I LI\mathrm{d}I = \frac{1}{2}LI^2 \; . \tag{14.66}$$

We can rewrite this using the flux $\Phi = LI$, getting:

$$W_m = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I = \frac{\Phi^2}{2L} \; . \tag{14.67}$$

This is similar to the expressions we found for a capacitor, where $W_e = QV/2 = CV^2/2$.

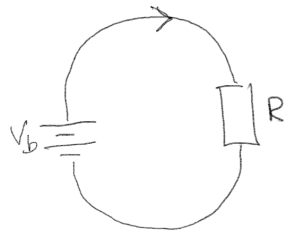


Fig. 14.9 Illustration of a simple circuit. What is the energy needed to get a current I flowing in this circuit?

Energy stored in an inductor

The energy W_m stored in the magnetic field of an inductor with inductance L and a current I when it does not interact magnetically with any other circuit is:

$$W_m = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi \; . \tag{14.68}$$

14.4.2 Energy in coupled circuits

This result can be generalized to a system with N magnetically coupled circuit loops. The total energy is

$$W_m = \frac{1}{2}\sum_{i=1}^{N} I_i \Phi_i \; , \tag{14.69}$$

where the flux Φ_i through circuit i is

$$\Phi_i = \sum_{j=1}^{N} \Phi_{j,i} \; , \tag{14.70}$$

is the sum of the fluxes from a circuit $j = 1, \ldots, N$ on a circuit i, where $\Phi_{j,i} = L_{j,i}I_i$. The magnetic energy is therefore:

$$W_m = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{j,i}I_iI_j \; . \tag{14.71}$$

The magnetic energy of two coupled loops can therefore be written as:

$$W_m = \frac{1}{2}\left(\underbrace{L_{1,1}I_1I_1 + L_{2,1}I_2I_1}_{\Phi_1} + \underbrace{L_{1,2}I_1I_2 + L_{2,2}I_2I_2}_{\Phi_2}\right) \quad (14.72)$$

Notice that $L_{1,1}$ and $L_{2,2}$ are positive, but that $L_{1,2}$ can be both positive or negative. The total energy of a coupled system can therefore both be larger or smaller than the sum of the energies of the individual systems.

Energy stored in an set of coupled inductors

The energy W_m stored in the magnetic field of a set of N magnetically coupled inductors with mutual inductances $L_{i,j}$ and a current I_i isother circuit is:

$$W_m = \frac{1}{2}\sum_i \sum_j L_{i,j} I_i I_j \; . \quad (14.73)$$

Proof for a set of coupled circuits. For completeness, we provide a proof of (14.69). If we have N coupled circuits, then Kirchoff's voltage law for circuit j is

$$V_{b,j} + \left(-\frac{\mathrm{d}\Phi_j}{\mathrm{d}t}\right) = R_j I_j \; , \quad (14.74)$$

where $V_{b,j}$ is the emf of the voltage source in circuit j. The power dissipated in circuit j is then:

$$P_j = V_{b,j} I_j = \left(R I_j + \frac{\mathrm{d}\Phi_j}{\mathrm{d}t}\right) I_k = R I_j^2 + \frac{\mathrm{d}\Phi_j}{\mathrm{d}t} I_j \; , \quad (14.75)$$

where we recognize the first term as the power dissipation in the resistor and the second term is related to the energy stored in the magnetic field. The work delivered by voltage source j in a time interval $\mathrm{d}t$ is then:

$$\mathrm{d}W_j = R_j I_j^2 + I_j \mathrm{d}\Phi_j \; . \quad (14.76)$$

The total magnetic work done by all the voltage sources in a time interval $\mathrm{d}t$ is

$$\mathrm{d}W_m = \sum_j I_j \mathrm{d}\Phi_j \; , \quad (14.77)$$

where

$$\Phi_j = \sum_i \Phi_{i,j} = \sum_i L_{i,j} I_i \; . \quad (14.78)$$

When $L_{i,j}$ does not change with time we have that $\mathrm{d}(L_{i,j}I_i) = L_{i,j}\mathrm{d}I_i$:

$$\mathrm{d}W_m = \sum_i \sum_j L_{i,j} I_j \mathrm{d}I_i \ , \tag{14.79}$$

We now use the trick that $\mathrm{d}(I_i I_j) = I_i \mathrm{d}I_j + I_j \mathrm{d}I_i$ and that $L_{i,j} = L_{j,i}$ so that

$$\mathrm{d}\left(\sum_i \sum_j L_{i,j} I_i I_j\right) = \sum_i \sum_j L_{i,j} \left(I_j \mathrm{d}I_i + \mathrm{d}I_i I_j\right) \tag{14.80}$$

$$= \sum_i \sum_j L_{i,j} I_j \mathrm{d}I_i + \sum_i \sum_j L_{j,i} I_i \mathrm{d}I_j \tag{14.81}$$

$$= 2\sum_i \sum_j L_{i,j} I_j \mathrm{d}I_i \ , \tag{14.82}$$

We therefore see that with

$$W_m = \frac{1}{2} \sum_i \sum_j L_{i,j} I_i I_j \ . \tag{14.83}$$

we get the relation in (14.79), which proves that this is the magnetic energy of the system.

14.4.3 Energy density

What is the energy in a thin toroid with N windings expressed in terms of the magnetic field B, H? Fig. 14.10 illustrates the system. The energy stored in this system is

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} I \Phi \ . \tag{14.84}$$

We can relate the current I to the magnetic field in the toroid by applying Ampere's law on a circuit C around the center of the toroid, where the magnetic $\mathbf{H}$-vector is in the azimuthal direction due to symmetry, $\mathbf{H} = H(r)\hat{\boldsymbol{\phi}}$. Ampere's law gives

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = 2\pi r H = N I \quad \Rightarrow \quad I = \frac{2\pi r H}{N} \ . \tag{14.85}$$

The flux Φ of this magnetic field through the circuit C is N times the flux through a cross section with area S. If the toroid is thin compared to its radius, we can assume that the magnetic field is approximately uniform across the cross section and the flux through a cross-section S is:

$$\Phi_S = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = BS \tag{14.86}$$

and the total flux through N windings is $\Phi = N\Phi_S = NBS$. The magnetic energy stored in this system is therefore:

$$W = \frac{1}{2} I\Phi = \frac{1}{2}\left(\frac{2\pi r H}{N}\right) BSN = \frac{1}{2} 2\pi r S B H = \frac{1}{2} BH \, 2\pi r S \; . \tag{14.87}$$

Here, we recognize the term $BH/2$ as the energy per unit volume and $2\pi r S$ as the volume of the toroid. We can therefore introduce the energy density u

$$u = \frac{W}{V} = \frac{1}{2} BH \; , \tag{14.88}$$

This result is general, but we do not provide a general proof of this result here. (This proof typically requires the use of the vector potential for the magnetic field).

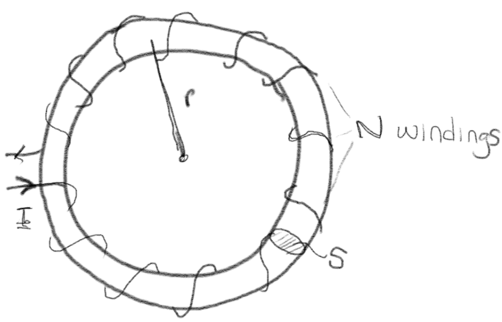


Fig. 14.10 A thin toroid with radius r, N windings and cross-sectional area S.

> **Energy density of a magnetic field**
>
> The **energy density** $u(\mathbf{r})$ of a magnetic field described by $\mathbf{B}$ and $\mathbf{H}$ is
>
> $$u = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \; . \tag{14.89}$$

14.4.4 Magnetic energy in a non-linear medium

What happens in a non-linear magnetic material? This is an important practical question, because we often use non-linear ferromagnetic cores in magnetic circuits and in various coils (toroids) in electric circuits.

The work performed by an external field (e.g. a voltage source or a battery) to change the magnetic flux in a magnetic circuit is:

$$\mathrm{d}W_m = I\mathrm{d}\Phi \; . \tag{14.90}$$

For a thin toroid of radius r we know from Ampere's law that the magnetic field H inside the toroid is

$$H = \frac{NI}{2\pi r} \; , \tag{14.91}$$

and that the total flux through the circuit wrapping around the toroid is $\Phi = NBS$. The contributions to the work is then:

$$\mathrm{d}W_m = \underbrace{\frac{2\pi r H}{N}}_{I} \underbrace{NS\mathrm{d}B}_{\mathrm{d}\Phi} = \underbrace{2\pi r S}_{\text{volume}} \; H\mathrm{d}B \; . \tag{14.92}$$

The change in energy density is the work done per unit volume, which is

$$\mathrm{d}u = H\mathrm{d}B \; . \tag{14.93}$$

So far, this is a general result. However, a non-linear medium has hysteresis. In Fig. 14.11 we illustrate how the magnetic fields B and H are related during a complete cycle for an alternating current. The work per unit volume done on the magnetic system if we change the field from B to $B + \mathrm{d}B$ is the area of a rectangle with height $\mathrm{d}B$ and length H. In part 1 of the path the changes in magnetic field are positive, but in part 2 the changes are negative, so that the work per unit volume is also negative. We therefore subtract the gray area. The work done on the magnetic field over a hole period is the area enclosed by the hysteresis loop. This work is energy which is dissipated as the orientations of the magnetic domains in the material and is transformed to thermal energy (heat) in the material.

The total area is the dissipated energy per cycle. If the cycle has a period T and a frequency $f = 1/T$, the dissipated energy per unit time (and unit volume), the dissipated power per unit volume, is

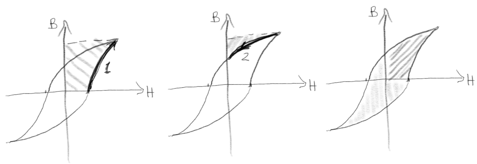


Fig. 14.11 Illustration of a hysteresis loop for the magnetic field in a circuit. (a) The integral of HdB along curve 1 is positive. (b) The integral of HdB along curve 2 is negative. (c) The net integral is the shaded area enclosed by the hysteresis curve.

$$p = \frac{P}{v} = \frac{\int H\mathrm{d}B}{vT} = A\,f \ , \tag{14.94}$$

where A is the area of the hysteresis curve and f is the frequency. In order to reduce the power dissipation, we should therefore try to design a system with as small an area of the hysteresis loop as possible (and have low frequencies).

14.4.5 Example: Coaxial cable

Find the energy per length stored in a coaxial cable carrying a current I. The inner radius of the cable is a and the outer radius is b.

Solution: We can find the energy either from $W_m = LI^2/2$, that is, by finding L, or from $W_m = \int_V \mathbf{B}\cdot\mathbf{H}\mathrm{d}v$. We demonstrate the second approach.

First, we find the magnetic field using Ampere's law as we have done many times before. Due to the cylindrical symmetry and that the divergence of the magnetic field is zero, $\mathbf{H}$ and $\mathbf{B}$ only has an azimuthal component, $\mathbf{H} = H(r)\hat{\boldsymbol{\phi}}$. Ampere's law states

$$\int_C \mathbf{H}\cdot\mathrm{d}\mathbf{l} = 2\pi r H = I \quad (a < r < b) \ , \tag{14.95}$$

We find the energy from the integral of the energy density over the volume between the inner and the outer radius, where the field is non-zero. The energy is:

$$
\begin{aligned}
W_m = & \int \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\mathrm{d}v = \frac{1}{2}\int \mu H^2 \mathrm{d}v && (14.96)\\
= & \frac{1}{2}\int \mu \frac{I^2}{(2\pi r)^2} r\mathrm{d}\phi\mathrm{d}r\mathrm{d} && (14.97)\\
= & l2\pi \int_a^b \frac{1}{2}\mu \frac{I^2}{(2\pi)^2}\frac{1}{r} && (14.98)\\
= & \frac{1}{2} l \frac{I^2}{2\pi}\mu \ln\frac{b}{a} && (14.99)\\
= & \frac{1}{2} l\mu \frac{I^2}{2\pi}\ln\frac{b}{a} && (14.100)\\
= & \frac{1}{2} l \underbrace{\frac{\mu}{2\pi}\ln\frac{b}{a}}_{L} I^2 \,. && (14.101)
\end{aligned}
$$

In principle, we can therefore find L by measuring the energy needed to charge the circuit and calculating L from $W_m = \frac{1}{2}LI^2$.

14.5 Magnetic forces

We have a way to find the force from a magnetic field on a current-carrying element, where we found that the contribution from a current element $I\mathrm{d}\mathbf{l}$ is $\mathrm{d}\mathbf{F} = I\mathrm{d}\mathbf{l}\times\mathbf{B}$. However, this is not practical in all situations. For example, if we want to find the force between two permament magnets we would need to know the magnetic field from one of them and the (bound) current density in the other to calculate the force. Here, we will introduce a more practical method based on energy considerations.

14.5.1 Magnetic force in a loss-free system

Fig. 14.12 illustrates a loss-free system with a permanent magnet moving relative to an ideal conductor. The sytem is closed, so that the energy is conserved. Assume we displace the magnet in the z-direction, $z \to z + \mathrm{d}z$. The work performed by the magnetic force is then:

$$
\mathrm{d}W = \mathbf{F}\cdot\mathbf{\Delta r} = \mathbf{F}_m \cdot \mathrm{d}z\hat{\mathbf{z}} = F_{m,z}\mathrm{d}z = -dW_m \,, \qquad (14.102)
$$

where W_m is the magnetic energy (and not a work). We interpret this to mean that the work done by the magnetic force must lead to a change in the magnetic energy. We can write this as:

$$F_{m,s} = -\frac{\partial W_m}{\partial z} \, . \tag{14.103}$$

We can introduce a similar argument for motion in the x and y directions. In general, we therefore get:

$$\mathbf{F}_m = -\nabla W_m \, , \tag{14.104}$$

where we must remember that W_m is an energy (and not work). This is a practical formula, because we may be able to find $W_m(x, y, z)$ for the system.

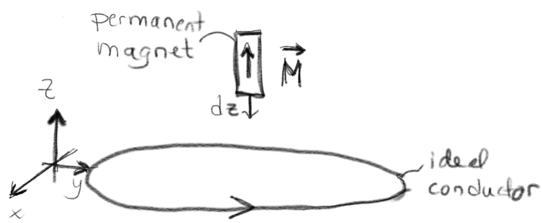


Fig. 14.12 An ideal conductor interacting with a permanent magnet.

14.5.2 Example: Carrying force of an electromagnet

Fig. 14.13 illustrates a horseshoe-shaped iron electromagnet with a wire wrapped around its top. When a current I is running in the wire, the magnet is able to lift a weight W made of iron. What is the lifting force of the electromagnet?

Solution: Our plan is to find an expression for the magnetic energy W_m of the system as function of the distance $\mathrm{d}z$ between the horseshoe magnet and the iron bar below it. We will assume that the permeability of the iron magnet is large ($\mu \to \infty$) so that all the flux remains inside the magnetic circuit consisting of the horseshoe magnet, the iron bar and the air gaps of thickness $\mathrm{d}z$. The total energy will consist of three part for the horseshoe, $W_{m,h}$, the air gaps, $W_{m,a}$ and the iron bar $W_{m,b}$. However, the energies of the horseshow and the iron bar will remain the same, it is only the magnetic energy in the air gaps that will depend on $\mathrm{d}z$.

$$W_m = W_{m,h} + W_{m,a} + W_{m,b} \tag{14.105}$$

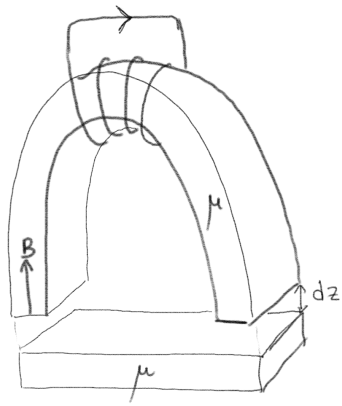


Fig. 14.13 A horseshoe shaped iron core electromagnet lifting an iron bar.

We find that the magnetic energy in the air gaps is the energy density in the air times the volume of the air gaps. Each air gap has a cross-sectional area S and a length $\mathrm{d}z$. The energy is therefore:

$$W_{m,a} = -\underbrace{\frac{B^2}{2\mu_0}}_{u}\underbrace{2S\mathrm{d}z}_{dv} = -\frac{B^2 S\mathrm{d}z}{\mu_0} \ . \tag{14.106}$$

The resulting force is then

$$\mathbf{F} = F_z\hat{\mathbf{z}} = -\frac{B^2 S}{\mu_0} \ . \tag{14.107}$$

The force is in the negative z-direction, that is, it is an attractive force pulling the two parts together. For a magnet with magnetic field $B = 1\mathrm{T}$ and a surface area of $1\mathrm{cm}^2$, we find that the force is $F \simeq 80\mathrm{N}$, which means that it can support about $80/g \simeq 8\mathrm{kg}$.

14.6 Summary

The **self inductance** of a system with a flux Φ when a current I is running through it is $L = \Phi/I$.

We find the **self inductance** by assuming a current I is running in the circuit, finding the resulting magnetic field $\mathbf{B}$ and the flux Φ of the magnetici field through the circuit. The self inductance L is then found from $L = \Phi/I$.

The flux *in* a circuit C_j *from* a current I_i in circuit C_i is $\Phi_{i,j} = \int_{S_1} \mathbf{B}_i \cdot d\mathbf{S}$, where $\mathbf{B}_i$ is the magnetic field from current I_i. The **mutual inductance** is defined as:

$$L_{i,j} = \frac{\Phi_{i,j}}{I_i} \; .$$

The inductance $L_{i,i}$ is the self-inductance of circuit i.

The **mutual inductances are symmetric**: $L_{i,j} = L_{j,j}$.

The **magnetic energy** W_m stored in an inductor with inductance L and current I is

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} I \Phi \; .$$

The **energy density** u_m of a magnetic field is

$$u_m = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \; .$$

14.7 Exercises

Learning outcomes. (0) Understand induction and be able to calculate the self induction for circuit, (1) Include the emf from an inductor in a Kirchoff's voltage law, (2) Understand and calculate the mutual inductance for a set of coupled circuits, (3) Define and apply magnetic energy density to find the energy stored in a circuit.

14.7.1 Discussion exercises

Exercise 14.1: Induktans

En tett spunnet toroide er en av noe få konfigurasjoner hvor det er enkelt å regne ut selvinduktansen. Hvilke egenskaper ved en solenoide er det som gjør det enkelt?

Exercise 14.2: Energitetthet i magnetiserbare materialer

For den samme magnetiske feltstyrken, B, er energitettheten høyere i vakuum eller i et magnetisk materiale? Forklar? Betyr likningen

$u = (1/2)BH$ at for en long solenoide med strømmen I så er den lagrede energien proporsjonal med $1/\mu$? Og betyr dette at for den samme strømmen er det mindre energi lagret når solenoiden er fylt med et ferromagnetisk materiale i stedet for luft? Forklar.

Exercise 14.3: Energi i en spole

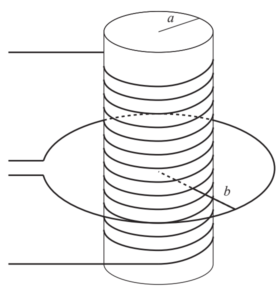
Først kobler du en spole til et batteri. Hvor mye energi er det i en spole med induktans L hvis det går en strøm I gjennom den? Du kobler hurtig av ledningen til batteriet slik at strømmen blir null. Hvor ble det av energien i spolen?

14.7.2 Tutorials

Exercise 14.4: Mutual inductance in a solenoid and a circular circuit

In this tutorial we will focus on establishing the skills needed to find the mutual inductance of a given system.

A solenoid consists of a wire wrapped N times around a cylinder of length L and radius a. A circular circuit of radius b is centered on the cylinder as illustrated in the figure.



a) First, what strategy would you choose to find the mutual inductance for this system? What symmetries would you use and how would you reason? In which system will you assume there is a current and in which system would you calculate the flux? Explain.

b) Based on your assumption, assume a current I_1 runs through your selected circuit 1. What is the magnetic field $\mathbf{B}_1$? Explain the steps in the argument used to find $\mathbf{B}_1$. How does the argument depend on what is inside the solenoid?

c) Given the magnetic field $\mathbf{B}_1$, what is the flux Φ_{12} of $\mathbf{B}_1$ through circuit 2? Explain your reasoning.

d) Finally, find the mutual inductance L_{12} for this system.

e) Can you extend your argument to the case where $N = 1$ and $b \gg a$?

Exercise 14.5: Energy densities

In this tutorial we will focus on developing the skills to find the energy density and to use the energy density to find the total energy in a magnetic system.
A solenoid consists of a wire wrapped N times around a cylinder of length h and radius a.

a) What is the magnetic fields $\mathbf{H}$ and $\mathbf{B}$ inside the solenoid when a current I is runnding through the wire? (You should go through the complete argument and not simly use a result).

b) What is the energy density inside the solenoid? And outside the solenoid?

c) What is the total energy stored in the solenoid when a current I is running through the wire?

d) Express the total energy in terms of the inductance L of the solenoid.

e) Explain how you can use a similar approach to find the energy density for a magnetic dipole in the form of a circular current circuit.

Exercise 14.6: Force on a permeable rod in a solenoid

(From Munsat)
A cylindrical rod (permeability μ, length L_0, radius $R \ll L_0$) is partially inserted into a long solenoid (length L_1, radius $R \ll L_1$, number of turns per unit length, n, fixed current I_0)

a) How large is the force on the rod?

b) Is the rod pulled into or pushed out of the solenoid? Does it matter whether the material is paramagnetic or diamagnetic?

Hint. Start by considering how much energy is contained in the magnetic fields.

14.7.3 Homework

14.7.4 Modeling projects

15 Electric circuits — part 2

With the introduction of inductance, we have completed the set of standard circuit components. We now have components to model the effect of capacitance, the local storage of charge, resistance, the finite conductivity of conductors and isolators, and inductance, the self-inducante due to interactions between the circuit and the magnetic field. These components provide the basic building blocks of the circuits we build and of the circuit models we use to represent complex, real systems. In this chapter, we will complete the description of circuits with the introduction of the inductor with inductance L, we will analyze and study the time dynamics of circuits by direct time integration. We will also introduce a simplified method to represent components and dynamics in systems driven by an alternating current.

15.1 Circuit components

We have already introduced the main components used in circuit diagrams. We now complete the description by introducing the *inductor* with inductance L. The inductor can either be thought of as modeling the effective inductance in a system, such as the inherent inductance in any current loop, or it can be thought of as a circuit element such as a coil with an iron core, which is used to damp out rapid changes in current. The circuit elements are illustrated in Fig. 15.1. The inductor is illustrated as a small spiral, imitating the look of a small solenoid.

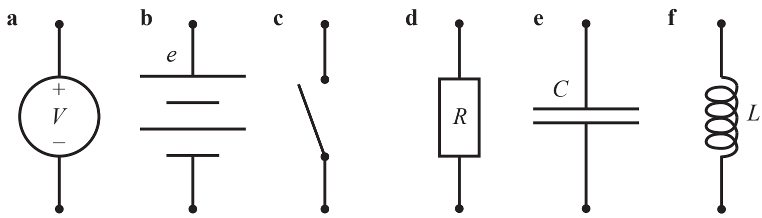


Fig. 15.1 Illustrations of the main circuit elements: (a) Voltage source, (b) battery, (c) switch, (d) resistor, (e) capacitor, (f) inductor.

An inductor is characterized by its inductance L. We know how to calculate or measure the inductance for a given circuit geometry, and we know that it only depend on the geometry of the system and the magnetic materials used. The flux through the circuit due to the inductor is

$$\Phi = LI \,, \tag{15.1}$$

and the emf associated with the inductor is

$$e = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi = -L\frac{\mathrm{d}I}{\mathrm{d}t} \,, \tag{15.2}$$

where L does not vary with time. The emf from the inductor is included in Kirchoff's voltage law for a circuit.

15.1.1 Description of circuit elements

Let us summarize the contributions to Kirchoff's voltage law of the various components in the circuit illustrated in Fig. 15.2.

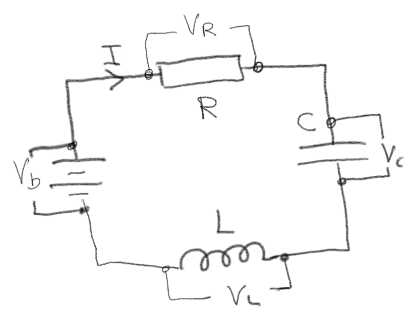


Fig. 15.2 Illustration of a circuit with a battery V_b, a resistor R, a capacitor C and an inductor L.

Kirchoff's voltage law for a complete loop around the circuit is

$$\sum_i e_i - \sum_j \Delta V_j = 0 , \tag{15.3}$$

which in this case is

$$\underbrace{V_b}_{\text{emf}} - \underbrace{L\frac{\mathrm{d}I}{\mathrm{d}t}}_{V_L} - \underbrace{RI}_{V_R} - \underbrace{\frac{1}{C}Q}_{V_C} = 0 \tag{15.4}$$

Let us address the individual components:

Battery. A battery or a voltage source contributes with an emf $V_b(t)$ to the circuit. This corresponds to a voltage increase where the other components corresponds to voltage drops. We recall that a battery may be through of as a mechanical battery lifting charges by an external, non-coulombic force against the electric field.

Resistor. The resistor R is drawn either as a small rectangle, illustrating the cylindrical shape often used, or as a zig-zag line. The voltage drop across the resistor is $\Delta V = RI$, where R is the resistance of the resistor. We recall that we combine resistors in series by adding them: $R = \sum_i R_i$ and that we combine resistors in parallel by adding their inverse: $1/R = \sum_i 1/R_i$. The power dissipation in a resistor is $P = RI^2$.

Capacitor. The capacitor C acts as a charge storage device and is characterized by its capacitance $C = Q/V$, where Q is the charge on each side of the capacitor when the voltage difference is V. The current flowing into a capacitor is related to the change in charge Q on the capacitor: $I = \mathrm{d}Q/\mathrm{d}t$. If we insert $Q = CV$, we get

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = I = C\frac{\mathrm{d}V}{\mathrm{d}t} , \tag{15.5}$$

when C does not vary with time. The capacitor cam be thought of as a component that resists changes in voltage in the system. We recall that we combine capacitors in series by adding their inverse capacitances: $1/C = \sum_i 1/C_i$ and that we combine capacitors in parallel by adding their capacitances: $C = \sum_i C_i$.

Inductor. The inductor L introduces an additional emf $e = -\mathrm{d}\Phi/\mathrm{d}t = -L\mathrm{d}I/\mathrm{d}t$. The emf is included as a voltage jump, similar to the battery, or we include $-e$ as a voltage drop in the circuit. The voltage drop across the inductor is then $V = -e = L\mathrm{d}I/\mathrm{d}t$.

15.1.2 Combing inductors

We have found rules for combining both resistors and capacitors in series and parallel, and this is actively used to analyze circuits. Can we find similar laws for the combination of inductors?

Inductors in series. Fig. 15.3a illustrates a series coupling of two inductors L_1 and L_2. Can we replace these two elements by a single inductor L? The same current I runs through both inductors. The voltages drop across each of the two inductors are $V_1 = L_1 \mathrm{d}I/\mathrm{d}t$ and $V_2 = L_2 \mathrm{d}I/\mathrm{d}t$. The total voltage drop across the two inductors are

$$V = V_1 + V_2 = L_1 \frac{\mathrm{d}I}{\mathrm{d}t} + L_2 \frac{\mathrm{d}I}{\mathrm{d}t} = \underbrace{(L_1 + L_2)}_{L} \frac{\mathrm{d}I}{\mathrm{d}t} = L \frac{\mathrm{d}I}{\mathrm{d}t} \,. \tag{15.6}$$

We can therefore replace the two inductors L_1 and L_2 with a single inductor with inductance $L = L_1 + L_2$. We therefore conclude that we *add inductances in series.*

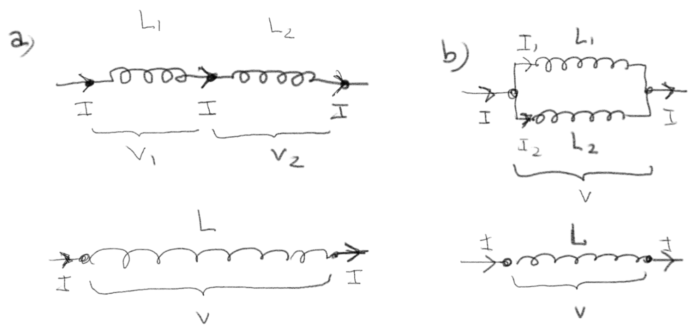


Fig. 15.3 (a) Illustration of a series coupling of two inductors. (b) Illustration of a parallel coupling of two inductors.

Inductors in parallel. Fig. 15.3b illustrates a parallel coupling of two inductors L_1 and L_2. The voltage drop over both inductors are V so that

$$V = L_1 \frac{\mathrm{d}I_1}{\mathrm{d}t} = L_2 \frac{\mathrm{d}I_2}{\mathrm{d}t} \,, \tag{15.7}$$

giving

$$\frac{V}{L_1} = \frac{\mathrm{d}I_1}{\mathrm{d}t} \quad \text{and} \quad \frac{V}{L_2} = \frac{\mathrm{d}I_2}{\mathrm{d}t} \,, \tag{15.8}$$

where $I = I_1 + I_2$ so that

$$\frac{V}{L} = \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{\mathrm{d}I_2}{\mathrm{d}t} = \frac{V}{L_1} + \frac{V}{L_2} = V\left(\frac{1}{L_1} + \frac{1}{L_2}\right) . \tag{15.9}$$

We can therefore replace the two inductors L_1 and L_2 with one inductor with inductance $1/L = 1/L_1 + 1/L_2$ and we conclude that we add the inverses of inductors in parallel.

15.2 Circuits in the time domain

In order to analyze the time dynamics of circuits, we apply Kirchoff's voltage and current laws to form a set of (differential) equations describing the currents and voltages in the circuits. Let us address some classical examples.

15.2.1 Example: The RC circuit

We already analyzed an example of an RC-circuit previously. Let us quickly repeat here. Fig. 15.4a illustrates a circuit consisting of a voltage source $V_b(t)$ connected in series with a resistor R and a capacitor C. We apply Kirchoff's voltage law to the complete loop:

$$V_b - V_R - V_C = 0 , \tag{15.10}$$

where V_b is the given voltage source, $V_R = RI(t)$ is the voltage drop across the resistor, and $V_C = Q/C$ is the voltage drop across the capacitor.

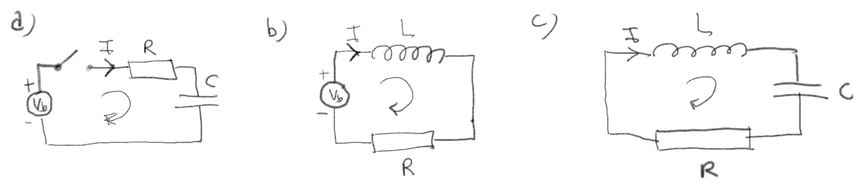


Fig. 15.4 (a) Illustration of a series coupling of two inductors. (b) Illustration of a parallel coupling of two inductors.

Approach 1. To solve this we take the time derviative of all the voltages

$$\frac{\mathrm{d}V_b}{\mathrm{d}t} - \frac{\mathrm{d}V_R}{\mathrm{d}t} - \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V_b}{\mathrm{d}t} - R\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{1}{C}\underbrace{\frac{\mathrm{d}Q}{\mathrm{d}t}}_{I} = 0 , \tag{15.11}$$

which gives a differential equation for the current $I(t)$:

$$\frac{\mathrm{d}V_b}{\mathrm{d}t} - R\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{1}{C}I = 0 \; , \tag{15.12}$$

Approach 2. Alternatively, we can use that $Q = V_C C$ so that

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}V_C}{\mathrm{d}t} \; , \tag{15.13}$$

we insert this into $V_R = RI(t)$, getting a differential equation in V_C:

$$V_b(t) - RC\frac{\mathrm{d}V_C}{\mathrm{d}t} - V_C = 0 \; . \tag{15.14}$$

Characteristic time τ. In both cases we introduce a characteristic time $\tau = RC$. The second approach can then be rewritten as:

$$V_b(t) - \tau\frac{\mathrm{d}V_C}{\mathrm{d}t} - V_C = 0 \; . \tag{15.15}$$

Analytical solution. If the V_b is a constant and the circuit is closed at $t = 0$, we expect $V_C(0) = 0$. After an infinite time, we expect that the system has reached a stationary state, so that the time derivative is zero and $V_b - V_C = 0$, that is, $V_C(\infty) = V_b$. The solution this equation is then

$$V_C(t) = V_b e^{-t/\tau} \; , \tag{15.16}$$

which you can check by insertion.

Numerical solution. The differential equation can also be solved by e.g. Euler's method for numerical integration of the differential equation

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} = \frac{1}{\tau}\left(V_b - V_C\right) \; . \tag{15.17}$$

which is done by the stepwise algorithm:

$$V_C(t + \Delta t) = V_C(t) + \frac{\Delta t}{\tau}\left(V_b(t) - V_C(t)\right) \; , \tag{15.18}$$

with initial condition $V_C(0) = V_0$. This is implemented in the following Python program

```
import numpy as np
import matplotlib.pyplot as plt
tau = 1.0
dt = 0.01
```

```
Vb = 1.0
time = 10.0
N = int(time/dt)
VC = np.zeros(N)
t = np.zeros(N)
VC[0] = 0.0
for i in range(N-1):
    VC[i+1] = VC[i] + dt/tau*(Vb-VC[i])
    t[i+1] = t[i] + dt
plt.plot(t,VC)
plt.xlabel('t')
plt.ylabel('V_C')
```

The resulting plot is shown in Fig. 15.5. It may seem unnecessary to solve this equation numerically, all the time we have an exact solution. However, the numerical solution opens for studying perturbations or more realistic variants of the system. For example, what if the voltage source $V_b(t)$ had a different shape, such as a square pulse:

$$V_b(t) = \begin{cases} 0 & t < 0 \\ V_0 & 0 \geq t < t_0 \\ 0 & t \geq t_0 \end{cases} , \tag{15.19}$$

or if the capacitor or resistor has a time dependent value or a non-linear behavior? These cases are easily solved by the same numerical approach, but generally cannot be solved by analytical methods.

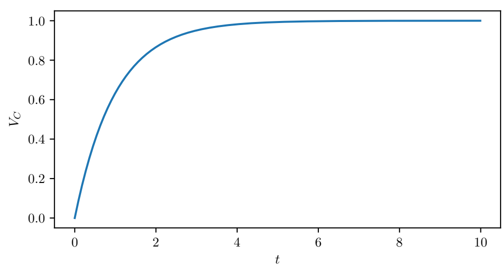


Fig. 15.5 Plot of $V_C(t)$ for the RC-circuit solved by Euler's method.

15.2.2 Example: The RL circuit

Fig. 15.4b illustrates a circuit consisting of a voltage source $V_b(t)$, an inductor L and a resistor R in series. We call such a circuit an RL-circuit.

Kirchoff's voltage law around the circuit gives:

$$V_b - V_L - V_R = V_b - L\frac{\mathrm{d}I}{\mathrm{d}t} - IR = 0 \;, \tag{15.20}$$

which gives a differential equation for the current I in the circuit:

$$L\frac{\mathrm{d}I}{\mathrm{d}t} = V_b - RI \;. \tag{15.21}$$

We introduce the characteristic time $\tau = L/R$:

$$\frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{V_b}{R} - I = \tau\frac{\mathrm{d}I}{\mathrm{d}t} \;. \tag{15.22}$$

If we turn the circuit on at $t = 0$ we expect the initial condition $I(0) = 0$. The exact solution to this equation is then

$$I(t) = \frac{V_b}{R}\left(1 - e^{-t/\tau}\right) \;. \tag{15.23}$$

The resulting plot is shown in Fig. 15.6 and shows that the effect of the inductor L is the dampen the rapid onset of current when the circuit is turned on with a characteristic time scale $\tau = L/R$, which depends on L: The larger L, the longer time it takes the current to reach its stationary value. This equation can also be solved numerically using Euler's scheme as we did above.

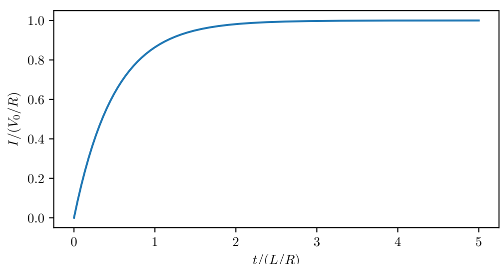


Fig. 15.6 Plot of $I(t)$ for the RL-circuit.

15.2.3 Example: The RLC circuit

Fig. 15.4c illustrates a circuit consisting of a capacitor C, an inductor L and a resistor R in series. We call such a circuit an RLC-circuit.

Kirchoff's voltage law around the circuit gives:

$$-L\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{Q}{C} - RI = 0\ . \tag{15.24}$$

We can relate Q and the current I by $I = \mathrm{d}Q/\mathrm{d}t$. Taking the time derivative of (15.24) therefore gives:

$$-L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} - \frac{1}{C}I - R\frac{\mathrm{d}I}{\mathrm{d}t} = 0\ . \tag{15.25}$$

We can rewrite this equation as

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{\tau}\frac{\mathrm{d}I}{\mathrm{d}t} + \omega_0^2 I = 0\ , \tag{15.26}$$

where $\tau = \frac{L}{R}$ and $\omega_0^2 = \frac{1}{LC}$. We recognize this equation from mechanics as the equation for damped, harmonic oscillations. We also need two initial conditions. The general solution to this equation is

$$I(t) = I_0 e^{-t/(2\tau)}\cos\left(\omega_0' t\right)\ , \tag{15.27}$$

where

$$\omega_0' = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}\ . \tag{15.28}$$

In the limit when $\omega_0^2 \gg 1/\tau^2$, that is, when R is small, we get that $\omega_0' \simeq \omega_0$. We therefore get damped oscillations with an angular frequence ω_0 and a damping with a characteristic time 2τ. Fig. 15.7 illustrates a few cases for the dynamics of the circuit. Notice both the damping and the oscillatory behavior. We will address the various types of damping and applications of such circuits in more detail when we study waves and oscillations.

15.2.4 Example: Periodic signal

Fig. 15.8 illustrates a simple circuit consisting of a voltage source $V_b(t) = V_0 \sin \omega t$, an inductor L and a resistor R. Assume that the system starts with $I(0) = 0$. Find the behavior of the system. Find $V(t)$ numerically in the case when $V_b(t) = V_0 \mathrm{sign} \sin \omega t$, that is, when $V_b(t)$ is s square pulse. Test your numerical solution by comparison with the exact solution for the sinus-pulse.

We apply Kirchoff's voltage law to a loop around the circuit:

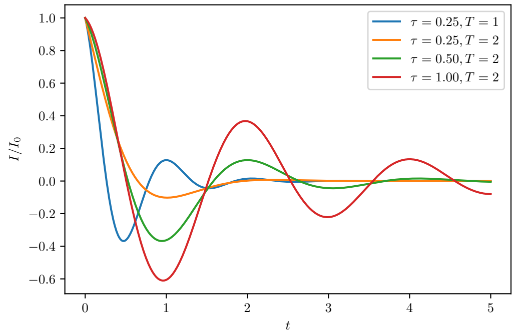


Fig. 15.7 Plot of $I(t)$ for the RLC-circuit.

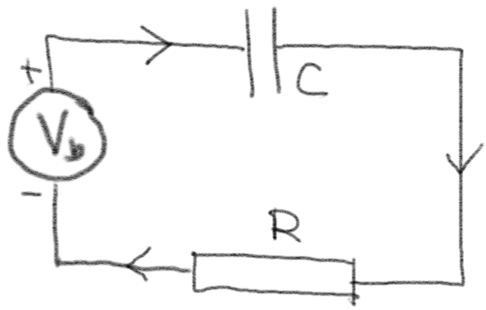


Fig. 15.8 Illustration of a circuit driven by a periodic voltage source $V_b(t)$.

$$V_b - V_C - IR = 0\,. \tag{15.29}$$

To relate V_C and I we use that $V_C = Q/C$ so that

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} = \frac{1}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{C}I \quad \Rightarrow \quad I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}\,. \tag{15.30}$$

We can therefore rewrite (15.29) by inserting this expression for I:

$$V_b(t) - V_C - RC\frac{\mathrm{d}V_C}{\mathrm{d}t} = 0\,. \tag{15.31}$$

Analytical solution. We look for the long-term solution after potential initial transients. We expect this solution to have the same periodicity as the driving voltage source: $V_C(t) = A\sin(\omega t + \phi_0)$, where A and ϕ_0 must be determined. We insert this and $V_b = V_0 \sin \omega t$ into (15.31):

$$V_0 \sin \omega t - A \sin(\omega t + \phi_0) - RCA\omega \cos(\omega t + \phi_0) \; . \tag{15.32}$$

We write $RC = \tau$. (15.32) must be true when $t = 0$:

$$- A \sin \phi_0 - \tau\omega A \cos \phi_0 = 0 \quad \Rightarrow \quad \tan \phi_0 = -\frac{1}{\omega\tau} \; . \tag{15.33}$$

And, simiarly, (15.32) must be true when $\omega t + \phi_0 = \pi/2$ so that

$$V_0 \sin\left(\frac{\pi}{2} - \phi_0\right) - A = 0 \quad \Rightarrow \quad V_0 = \frac{A}{\cos \phi_0} \; . \tag{15.34}$$

Numerical solution. We rewrite (15.31)

$$\frac{1}{\tau}\frac{\mathrm{d}V_C}{\mathrm{d}t} = V_b(t) - V_C(t) \; , \tag{15.35}$$

and introduce a dimensionless time $t' = t/\tau$ and a dimensionless potential $V'_C = V_C/V_0$ so that

$$\frac{1}{\tau}\frac{\mathrm{d}V'_C}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\mathrm{d}V'_C}{\mathrm{d}t'} = \frac{V_b(t'\tau)}{V_0} - V'_C \; . \tag{15.36}$$

We solve this equation numerically using a forward Euler integrator:

$$V'_C(t' + \Delta t') = V_C(t') + \Delta t' \left(V'_b(t'\tau) - V'_C(t')\right) \; . \tag{15.37}$$

The voltage source is

$$V'_b(t'\tau) = \frac{V_0 \sin \omega(t'\tau)}{V_0} = \sin \omega' t' \; , \tag{15.38}$$

where $\omega' = \omega\tau$. We rewrite the exact solution in terms of t' and V', getting

$$V(t) = V_0 \cos \phi_0 \sin(\omega t + \phi_0) \quad \Rightarrow \quad V'(t) = \cos \phi_0 \sin(\omega' t + \phi_0) \tag{15.39}$$

where $\phi_0 = \arctan(-1/\omega')$. This is implemented in the following program:

```
import numpy as np
import matplotlib.pyplot as plt
dt = 0.001
time = 20.0
omegam = 1.0 # omega*tau
N = int(time/dt)
V = np.zeros(N)
t = np.zeros(N)
for i in range(N-1):
```

```
    Vb = np.sin(omegam*t[i])
    V[i+1] = V[i] + dt*(Vb-V[i])
    t[i+1] = t[i] + dt
# Plot result
plt.plot(t,V)
# Compare with analytical solution
phi0 = np.arctan(-1/omegam)
A =  1.0*np.cos(phi0)
Va = A*np.sin(omegam*t+phi0)
plt.plot(t,Va,'-r')
```

The resulting plot in Fig. 15.9 shows that there is a short transiet time when the analytical solution and the numerical solution differ, but that the long term behavior coincides. This provides a test of the numerical solution method. We apply the method also to the square-pulse signal by replacing `Vb = np.sin(omegam*t[i])` with `Vb = np.sign(np.sin(omegam*t[i]))`. The resulting plot is also shown in Fig. 15.9. You can change the parameters to see what happens to the resulting signal.

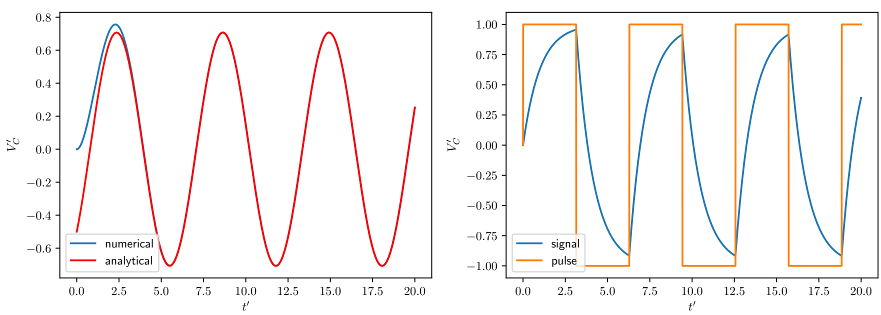


Fig. 15.9 Plots of the comparison of the analytical and numerical solution and comparison of the driving voltage pulse and the resulting voltage signal.

15.3 Circuits in the frequency domain

We have demonstrated how we can find the behavior of a circuit by applying Kirchoff's voltage law in combination with the description of individual components and that this leads to a set of (differential) equations for the current and voltages in the circuit. In the last example, we saw that if a circuit is driven by a harmonic voltage source, the voltage and current in the circuit also follows the same periodic behavior after a transient time. Here, we will introduce methods to effectively describe the stationary (non-transient) behavior of circuits driven by an harmonic voltage source. We will study circuits that are driven by periodic, harmonic signals with an angular frequency ω, and then look for solutions with the same frequency ω. This is a good approximation for the stationary behavior and will provide us with very efficient methods for describing their behavior and with new concepts that are useful to *understand* the behavior of circuits with alternating currents.

15.3.1 Complex representation of a signal

We represent a harmonic signal by two values, the frequency ω and the phase ϕ as illustrated in Fig. 15.10:

$$V(t) = V_0 \cos(\omega t + \phi) \ , \tag{15.40}$$

where ω is related to the period T through $\omega = 2\pi/T$. The phase ϕ determines how the signal is shifted. The sign of the phase can sometimes be confusing: A negative phase, $\phi < 0$ means that the curve $V(t)$ is shifted to the right, while a positive phase, $\phi > 0$, means that hte curve $V(t)$ is shifted to the left. However, a negative phase also means that the signal is lagging behind: It takes a small time ϕ/ω before the signal from $\cos \omega t + \phi$ has the same value as $\cos \omega t$.

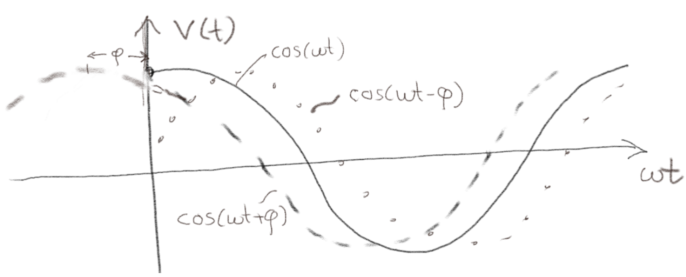


Fig. 15.10 Illustration of an harmonic signal $V(t) = V_0 \cos \omega t$ and two signals with two different phases, $V_0 \cos(\omega t - \phi)$ and $V_0 \cos(\omega t + \phi)$.

Mathematically, it is more convenient to work with complex numbers to represent the signal because this allows us to work with exponential functions instead of sines and cosines. We will therefore introduce a complex representation of a signal

$$V(t) = \hat{V} e^{i\omega t} \,, \tag{15.41}$$

where $\hat{V}$ is a complex number and

$$e^{i\omega t} = \cos \omega t + i \, \sin \omega t \,. \tag{15.42}$$

We will use this to simplify our calculations, realizing that it is only the real part of the expression that represents the physical part of the signal, so that

$$V(t) = \mathrm{Re} \left\{ \hat{V} e^{i\omega t} \right\} \,, \tag{15.43}$$

is the physical voltage. The complex amplitude $\hat{V}$ represents both the real amplitude and the phase of the signal. In general, we can write a complex number $\hat{V}$

$$\hat{V} = |\hat{V}| e^{i\phi} \,, \tag{15.44}$$

where ϕ is the phase. The physical voltage $V(t)$ is

$$V(t) = \mathrm{Re} \left\{ |\hat{V}| e^{i\phi} e^{i\omega t} \right\} = |\hat{V}| \cos(\omega t + \phi) \,. \tag{15.45}$$

We call $\hat{V}$ the complex amplitude or the *phasor* of the signal. Fig. 15.11 illustrates a signal with a phase $\phi = -\pi/4$ as a function of time and how the phasor is illustrated in the complex plane.

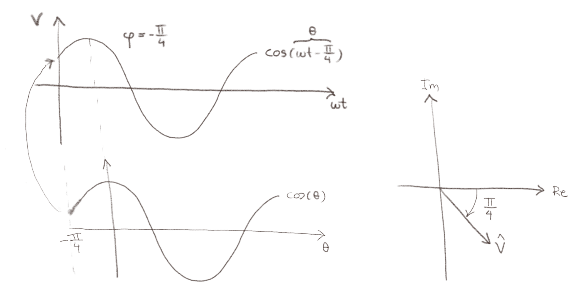


Fig. 15.11 Illustration of a signal with a phase $\phi = -\pi/4$.

15.3.2 Circuits: Complex representation of a resistor

How can we use this representation to describe a circuit with a signal generator, the voltage source V_b, and a resistor R? We represent the voltage and the current using the complex representation:

$$V(t) = \hat{V}e^{i\omega t} \quad , \quad I(t) = \hat{I}e^{i\omega t} \; . \tag{15.46}$$

Kirchoff's voltage law describes relations between the real components of the voltages and currents:

$$\mathrm{Re}\left\{V(t)\right\} = R\mathrm{Re}\left\{I(t)\right\} = \mathrm{Re}\left\{R\,I(t)\right\} \; , \tag{15.47}$$

and

$$\mathrm{Re}\left\{\hat{V}e^{i\omega t}\right\} = \mathrm{Re}\left\{R\,\hat{I}e^{i\omega t}\right\} \; . \tag{15.48}$$

This means that we have the relation

$$\boxed{\hat{V} = R\hat{I}} \; , \tag{15.49}$$

for a resistor. We can therefore use Ohm's law also the relate the complex amplitudes of the voltages and currents. To understand the impact of this relation, let us address how we can represent a capacitor.

15.3.3 Circuits: Complex representation of a capacitor

In a circuit consisting of a voltage source V_b and a single capacitor C, we know that the voltage drop over the capacitor is related to the charge Q on the capacitor, $C = Q/V$, so that $Q = CV$ and the current $I = \mathrm{d}Q/\mathrm{d}t$ is

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}V}{\mathrm{d}t} \; . \tag{15.50}$$

The current I is the real part of the complex current $\hat{I}e^{i\omega t}$ and the voltage is the real part of the complex voltage, $\hat{V}e^{i\omega t}$:

$$\mathrm{Re}\left\{\hat{I}e^{i\omega t}\right\} = C\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\left\{\hat{V}e^{i\omega t}\right\} = \mathrm{Re}\left\{C\hat{V}i\omega e^{i\omega t}\right\} \; . \tag{15.51}$$

This equation implies that the prefactors also must be equal:

$$\hat{I} = Ci\omega\hat{V} \; , \tag{15.52}$$

and

$$\hat{V} = \frac{1}{i\omega C}\hat{I} = \hat{Z}\hat{I} \; . \tag{15.53}$$

We call the term $\hat{Z}$ the **impedance**. The impedance acts like a complex resistance.

Interpretation of the complex impedance. The resistor R had a real impedance $\hat{Z} = R$, while the capacitor has a complex impedance $\hat{Z} = 1/(i\omega C)$. How does a complex impedance affect the current or voltage in a circuit? Let us assume that the current I is real, that is, that the phase of the current in zero and $\hat{I} = |\hat{I}|$:

$$I(t) = \mathrm{Re}\left\{\hat{I}e^{i\omega t}\right\} = \hat{I}\cos\omega t \ . \tag{15.54}$$

For a circuit with an impedance $\hat{Z}$, the voltage is $\hat{V} = \hat{Z}\hat{I}$:

$$V(t) = \mathrm{Re}\left\{\hat{Z}\hat{I}e^{i\omega t}\right\} \tag{15.55}$$

$$= |\hat{Z}|\hat{I}\mathrm{Re}\left\{e^{i\omega t+\phi}\right\} \tag{15.56}$$

$$= |\hat{Z}|\hat{I}\cos(\omega t + \phi) \ . \tag{15.57}$$

This allows us to interpret the role of a complex impedance $\hat{Z}$:

- The norm $|\hat{Z}|$ descibes the ratio between the voltage and current amplitudes.
- The phase ϕ describes the phase shift of the voltage relative to the current: It describes by how much the voltage is before (in time) the current.

For the capacitor the complex impedance is:

$$\hat{Z} = \frac{1}{i\omega C} = \frac{-i}{\omega C} = \frac{1}{\omega C}e^{-i\frac{\pi}{2}} \ . \tag{15.58}$$

which corresponds to a phase shift of $-\pi/2$. The corresponding $I(t)$ and $V(t)$ curves are illustrated in Fig. 15.12.

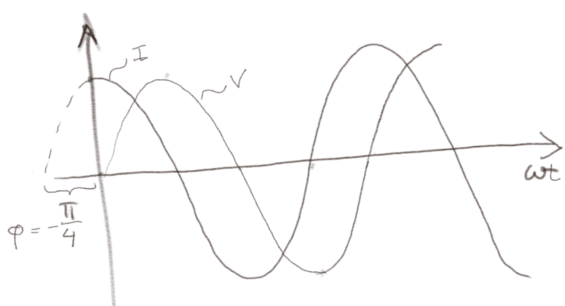


Fig. 15.12 Illustration of the current, $I(t)$, and voltage, $V(t)$, for a circuit with a capacitor C and an impedance $\hat{Z} = 1/(i\omega C)$.

15.3.4 Circuits: Complex representation of an inductor

For a circuit with a voltage source V_b and an inductor L, Kirchoff's law for the circuit gives

$$V - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \quad \Rightarrow \quad V = L\frac{\mathrm{d}I}{\mathrm{d}t} \; . \tag{15.59}$$

We rewrite this on complex form:

$$\mathrm{Re}\left\{\hat{V}e^{i\omega t}\right\} = L\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\left\{\hat{I}e^{i\omega t}\right\} = \mathrm{Re}\left\{i\omega L\hat{I}e^{i\omega t}\right\} \; . \tag{15.60}$$

This means that we again have the relation $\hat{V} = \hat{Z}\hat{I}$ where $\hat{Z} = i\omega L$ for the inductor.

15.3.5 Circuits with impedances

We have found that we can describe the relation between the complex amplitudes of the current and voltage with an impedance for resistors, capacitors, and inductors.

Impedances for the main components

We describe a harmonic signal in complex notation as

$$I(t) = \mathrm{Re}\left\{\hat{I}e^{i\omega t}\right\} \quad , \quad V(t) = \mathrm{Re}\left\{\hat{V}e^{i\omega t}\right\} \; , \tag{15.61}$$

where $\hat{I}$ and $\hat{V}$ are called the complex amplitudes or the **phasors** of the signals. The phasor includes both an amplitude and phase information:

$$\hat{I} = |\hat{I}|e^{\phi_I} \quad , \quad \hat{V} = |\hat{V}|e^{\phi_V} \tag{15.62}$$

For the main components resistor, capacitor and inductor, we find a generalized Ohm's law relating the complex amplitudes to the **impedance** $\hat{Z}$ of the component:

$$\hat{V} = \hat{Z}\,\hat{I} \; , \tag{15.63}$$

where the impedances are

- $\hat{Z} = R$ for a **resistor**
- $\hat{Z} = \frac{1}{i\omega C}$ for a **capacitor**

- $\hat{Z} = i\omega L$ for an **inductor**

We can prove[1] that we can use the same rules for combining impedances as we already know for resistors:

- **Two impedances** $\hat{Z}_1$ and $\hat{Z}_2$ **in series** can be replaced by a single component with impedance $\hat{Z} = \hat{Z}_1 + \hat{Z}_2$.
- **Two impedances** $\hat{Z}_1$ and $\hat{Z}_2$ **in parallel** can be replaced by a single component with impedance $1/\hat{Z} = 1/\hat{Z}_1 + 1/\hat{Z}_2$.

However, we must remember that impedances are *complex numbers* and we must use correct algebra for complex numbers when we perform these calculations.

15.3.6 Example: RC-circuit in the frequency domain

Fig. 15.13 illustrates two possible circuits consisting of an input signal $V_S(t)$ (a signal generator), a resistor R and a capacitor C. Determine how $\hat{V}$ measured as show in the figure is related to $\hat{V}_S$ in the two cases.

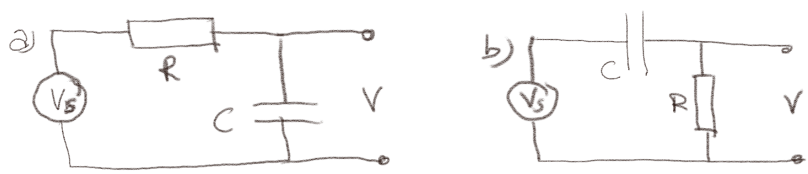


Fig. 15.13 Illustration of two circuits with a resistor R and a capacitor C. What is the signal V compared with the input signal V_S?

Solution: In both cases, we can combine the two components in series so that

$$\hat{V}_S = \hat{I}\left(\hat{Z}_C + \hat{Z}_R\right) , \tag{15.64}$$

and

$$\hat{I} = \frac{\hat{V}_S}{\hat{Z}_R + \hat{Z}_C} . \tag{15.65}$$

For the circuit in Fig. 15.13a we get:

[1] We leave this proof for the exercises.

$$\hat{V} = \hat{Z}_C\,\hat{I} = \frac{\hat{Z}}{R+\hat{Z}}\hat{V}_S \tag{15.66}$$

$$= \frac{\frac{1}{i\omega C}}{R+\frac{1}{i\omega C}}\hat{V}_S \tag{15.67}$$

$$= \frac{1}{1+i\omega RC}\hat{V}_S = \frac{1}{1+i\omega\tau}\hat{V}_S \ . \tag{15.68}$$

We interpret this by inserting $\hat{V} = |\hat{V}|e^{i\phi}$. We get that

$$|\hat{V}| = \frac{1}{\sqrt{1+(\omega\tau)^2}}|\hat{V}_S| \ , \tag{15.69}$$

and

$$\phi = -\arctan(\omega\tau) \ , \tag{15.70}$$

We see that in the limit when $(\omega\tau) \gg 1$ we get that

$$|\hat{V}| \simeq \frac{1}{\omega\tau}|\hat{V}_S| \ , \tag{15.71}$$

which implies that $|\hat{V}| \ll |\hat{V}_S|$. This means that this circuit acts as a low-pass filter: Frequencies much larger than $1/\tau$ as not let through, but frequencies smaller than this as lew through without any significant damping. We call such a circuit a *low-pass filter*.

15.3.7 Power dissipation in complex circuits

We recall that the dissipated power in a resistor is

$$P = RI^2 = V(t)I(t) \ . \tag{15.72}$$

Unfortunately, this expression is non-linear (it goes as I^2), and we have implicitely assumed that all expressions are linear. However, we can still calculate the power using a smart trick, where we write

$$\mathrm{Re}\,\{z\} = \frac{z+z^*}{2} \ . \tag{15.73}$$

We use this to rewrite the expression for the power:

$$
\begin{aligned}
P(t) = & \quad V(t)I(t) = \mathrm{Re}\left\{\hat{V}e^{i\omega t}\right\}\mathrm{Re}\left\{\hat{I}e^{i\omega t}\right\} && (15.74)\\
= & \quad \frac{1}{2}\left(\hat{V}e^{i\omega t}+\hat{V}^*e^{-i\omega t}\right)\frac{1}{2}\left(\hat{I}e^{i\omega t}+\hat{I}^*e^{-i\omega t}\right) && (15.75)\\
= & \quad \frac{1}{4}\left(\hat{V}\hat{I}e^{2i\omega t}+\hat{V}^*\hat{I}^*e^{-2i\omega t}+\hat{V}\hat{I}^*+\hat{V}^*\hat{I}\right) && (15.76)\\
= & \quad \frac{1}{2}\mathrm{Re}\left\{\hat{V}\hat{I}e^{2i\omega t}\right\}+\frac{1}{2}\mathrm{Re}\left\{\hat{V}\hat{I}^*\right\} \; . && (15.77)
\end{aligned}
$$

Here, we observe that the left term varies with $\omega' = 2\omega$, which means that if we average over a whole number of half periods this term becomes zero. The average power is therefore only given by the second term:

$$
\langle P \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{V}\hat{I}^*\right\} \; . \tag{15.78}
$$

What is the power dissipation of a component with impedance $\hat{Z} = |\hat{Z}|e^{i\phi} = R + iX$? We use that $\hat{V} = \hat{Z}\hat{I}$ and get:

$$
\langle P \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{Z}\hat{I}\hat{I}^*\right\} \; , \tag{15.79}
$$

where we recognize $\hat{I}\hat{I}^* = |\hat{I}|^2$, so that

$$
\langle P \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{Z}|\hat{I}|^2\right\} = \frac{1}{2}R|\hat{I}|^2 \; . \tag{15.80}
$$

We therefore conclude that only R contributes to the power dissipation, and that the imaginary component X does not lead to any dissipation, that is, the capacitor and inductor changes the phase of the signal, but does not contribute to the power dissipation.

15.4 Assumptions underlying circuit models

Previously, we discussed how our representation of circuits was an approximation to the physical system. We assume that we can describe the system as a set of separate components that do not interact. This is one of several underlying assumptions that we have implicitly made when we describe real systems as circuits, but also in our treatment of circuits we have built from circuit components. Modeling a system as a circuit is an *approximation* to the full electromagnetic theory described by Maxwell's equation — an approximation we use because it allows us to understand, predict and explain behavior, and to develop a simplified theory that may guide our intuition. What are the underlying assumptions, and how can we compensate or modify the model to accommodate these deviations?

- We assume that the system is quasistatic, which means that we assume that conductors are small compared with the wavelength $\lambda = c/f$. Long conductors or high frequencies may therefore lead to deviations from this assumption.
- We assume that the mutual inductor between circuits and the self induction of a single circuit can be described a set of decoupled inductors. This is often a good approximation, but it ignores how the magnetic field may affect other components in the circuit. We can sometimes compensate by introducing more inductors in the circuit to include these effects.
- Junctions and parts of conductors may in reality store some charge, even though we have assumed that they are ideal. This can be addressed by including capacitors in the circuit to model these effects.
- The electric or magnetic field from one component may impact another component in the circuit, even though we have assumed that the components are independent. To compensate for this effect, we introduce additions components to reflect these interactions.
- Real conductors have a finite resistivity and are not ideal condutors. We compensate for this effect by including resistors to represent the non-ideal aspect of wires.

15.5 Summary

15.6 Exercises

Learning outcomes. (1) Know the basic circuit components of emf, resistance, capacitance, and inductance; (2) Set up equations for circuits using Kirchoffs laws; (3) Solve circuit equations in the time domain; (4) Understand descriptions of current and voltage in the frequency domain; (5) Find complex impedances for simple circuits; (6) Understand and apply the laws for combinations of complex impedances.

15.6.1 Discussion exercises

Exercise 15.1: Snurrig motstand

Du skal lage en motstand ved å vikle en wire rundt en sylinder. For å gjøre induktansen så liten som mulig har du blitt foreslått å vikle halvparten av wiren den ene veien og halvparten den andre veien. Vil det gi ønsket effekt? Forklar hvorfor eller hvorfor ikke.

Exercise 15.2: En spole pluss en spole er fire spoler

En venninne kommer til deg med et spørsmål:

Hvis du plasserer to spoler med induktansene L etter hverandre i en krets vil de oppføre seg som en spole med induktansen $2L$. Men hvis du legger en spole med induktans L oppå en spole med induktans L blir den totale induktansen $4L$. Hvordan kan dette være mulig?

Hva vil du si til din venninne? Har hun regnet feil eller er det noe annet som er galt med argumentet hennes?

Exercise 15.3: Energioverføring

Strømmen i en vekselsstrømslinje endrer retning et visst antall ganger per sekund og gjennomsnittsverdien er null. Forklar hvordan det er mulig å transportere energi (effekt) i et slikt system.

Exercise 15.4: Kansellerende komponenter

En krets består av en vekselsstrømkilde, en lyspære, en kondensator og en spole, alle koblet i serie. Er det mulig at pæren lyser like sterkt hvis man fjerner både kondensatoren og spolen?

Beskriv oppførselen til kretsen ved hjelp av impedansen til kondensatoren ($\hat{Z} = 1/(i\omega C)$) og spolen ($\hat{Z} = i\omega L$).

15.6.2 Tutorials

Exercise 15.5: LR circuit

Consider a circuit consisting of a battery with emf V_0, a resistance R and an inductance L connected in series in a loop. A current I is flowing due to the emf.

a) Draw the circuit using standard symbols for the components. Illustrate the positive direction for I.

b) What is the voltage drop V_R across the resistor? (The voltage drop is a positive number in the direction of the current).

c) For the inductance L the flux is $\Phi = LI$. What is the emf across the inductance L?

d) Write down the sum of emf and potential drops around the circuit.

e) What is the potential drop V_L over the inductance L? Ensure that you get the same sum of emf and potential drops around the circuit when you represent the inductance as a component with a voltage drop as when you represented it by an emf.

f) Find a differential equation for the current, I.

g) First, find the solution to this equation when $V_0 = 0$ and $I(0) = I_0$.

h) Second, find the solution to this equation when $V_0 > 0$ and $I(0) = 0$.

Exercise 15.6: LC circuit

Consider a circuit with a capacitor and an inductor only. Suppose this LC circuit starts at time $t = 0$ with charge Q_0 on the capacitor and zero initial current.

a) What is the voltage drop V_C across the capacitor C expressed in terms of the charge Q on the capacitor?

b) What is the voltage drop V_L across the inductance L in terms of the current I. (Ensure that you have the sign correct by a careful reflection on the relation between emf and voltage drops.)

c) Find a differential equation for the current I in the circuit.

d) Find the solution to the equation for I. Use this to find $Q(t)$.

e) Describe an analogous mechanical system (i.e. one that obeys the same differential equation) and provide "translations" between all important variables.

Exercise 15.7: Complex warm-up

Adapted from Steven Pollock, University of Colorado-Boulder
In this tutorial we will focus on basic skills in complex numbers. If you master these elements, do them quickly and move on to the next tutorial in the set

Recall that a complex number z can be written in two ways: $z = a + ib$ or $z = Ae^{i\theta}$, where a, b, A and θ are real numbers such that $|z| = A = \sqrt{a^2 + b^2}$, $a = Re\{z\} = A\cos(\theta)$, $b = Im\{z\} = A\sin(\theta)$. When you multiply two complex numbers, the phase angles add: $z_1 = A_1 e^{i\theta_1}$, $z_2 = A_2 e^{i\theta_2}$, $z_1\, z_2 = A_1 A_2 e^{i(\theta_1+\theta_2)}$.

a) Rewrite the following complex numbers in the form $Ae^{i\theta}$:

$$-i = \qquad \frac{5}{i} = \tag{15.81}$$

$$1 + i = \qquad \frac{1}{1-i} = \tag{15.82}$$

b) Use the last two answers and the rules for multiplying complex exponentials to find

$$\frac{1+i}{1-i} = \tag{15.83}$$

What is the magnitude, phase and real part of your answer?

c) Draw the following complex numbers in the complex plane: $e^{i\pi/4}$, -1, $\cos(3\pi/4) - i\sin(3\pi/4)$, $e^{-i3\pi/4}$.

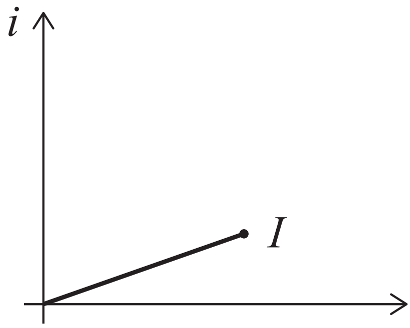
d) Draw the following complex numbers $e^{-i\omega t}$ in the complex plane for the times $\omega t_1 = \pi/4$, $\omega t_2 = \pi/2$, $\omega t_3 = 3\pi/4$. Would an arrow representing $e^{-i\omega t}$ in the complex plane rotate *clockwise* or *counter-clockwise* as time advances?

e) For a circuit with a resistor R and an AC source $V(t) = V_0 e^{i\omega t}$ (V_0 is real), what is the magnitude of the physical current through the resistor when $\omega t = \pi/3$.

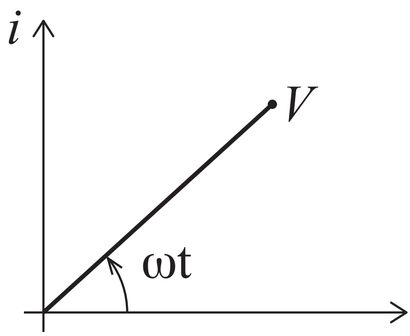
Exercise 15.8: Complex impedance

(*Based on a tutorial from Steven Pollock, University of Colorado-Boulder*)
In this tutorial we will focus on the complex impedance, Z, in AC circuits. The goal is for you to build a strong intuition for the concepts and basic skills that allow you to address longer and more elaborate exercises

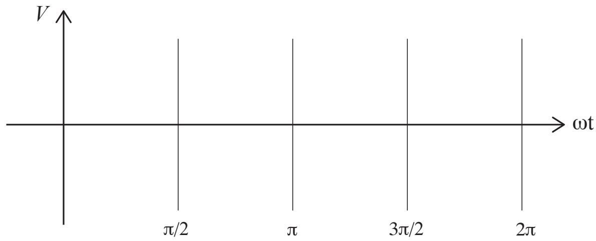
a) Given the impedance $Z = 2e^{i\pi/4}$ and the complex number I shown in the figure below, plot the complex number $V = I\,Z$.

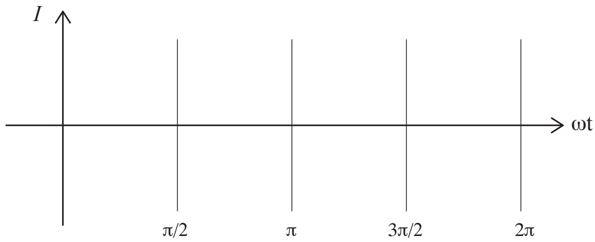


b) Given $V = V_0 e^{i\omega t}$ and $Z = 2e^{-i\pi/2}$, plot the complex number $I = V/Z$ at the instant in time shown in the figure below.



c) For the same situation as above, with $V = V_0 e^{i\omega t}$ and $Z = 2e^{-i\pi/2}$, sketch the real (physical) values of V and I as functions of time in the graphs below.

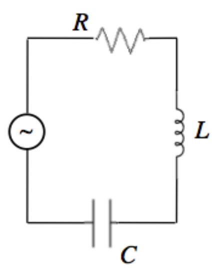




Does the current *lead* or *lag* the voltage? Make sure your answers on this page are consistent with the phasor diagram you drew on the previous page for the same situation.

Before you move on, make sure to check your answers to this part with an instructor!

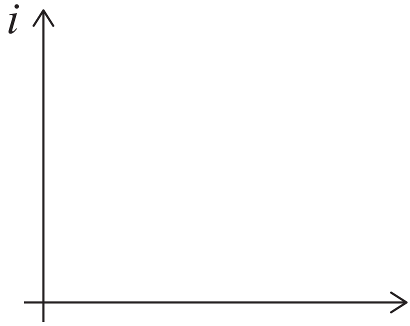
d) The complex impedances for the circuit elements in the figure below are: $Z_R = R$, $Z_L = i\omega L$, $Z_C = \frac{1}{i\omega C}$.



What Z_{TOTAL} for this circuit?

Write Z_{TOTAL} in the form $a + ib$.

e) For graphing purposes, assume that $\omega L > 1/(\omega C)$. Sketch Z_R, Z_C, Z_L, and show how they add as vectors to get Z_{TOTAL}.



Under what circumstances does the current *lead* the voltage?

Under what circumstances are the current and the voltage *in phase*?

15.6.3 Homework

Exercise 15.9: Complex capacitor

(*From Johannes Skaar*)

a) A capacitor C is first charged to the voltage V_0. After $t = 0$ it is discharged through the resistor R as illustrated in Fig. 15.14a. Sketch the voltage $V(t)$ across the capacitor as a function of time. How long time, approximately, does it take to charge or discharge a capacitor?

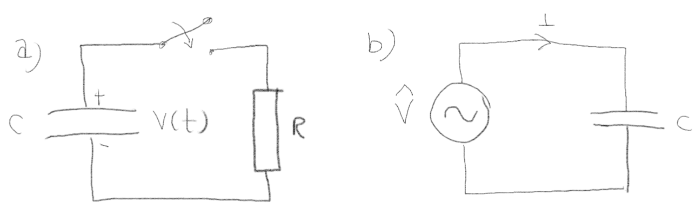


Fig. 15.14 Illustration of a circuit with a capacitor.

b) What is the impedance of the capacitor?

c) We now connect the capacitor directly to an AC source with complex amplitude $\hat{V}$ as illustrated in Fig. 15.14b. What is the complex current $\hat{I}$ through the capacitor?

d) Give an interpretation of the complex circuit equation (the connection between current and voltage) you found in the previous exercise. You should interpret both the absolute value and the imaginary unit i in the equation.

Hint. i is $e^{i\pi/2}$ in polar coordinates.

Exercise 15.10: Complex RL-circuit

(*From Johannes Skaar*)

An AC voltage source with angular frequency ω is connected in series with a resistance R and an inductance L as shown in Fig. 15.15a.

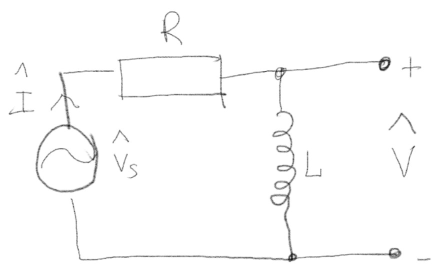


Fig. 15.15 Illustration of a circuit with an inductance.

a) What is the physical, time-varying voltage $V_s(t)$ that corresponds to the complex amplitude $\hat{V}_s$?

b) Find the complex voltage $\hat{V}$ expressed in terms of the source $\hat{V}_s$, the time constant $\tau = L/R$ and the angular frequency ω. Also find an expression for $|\hat{V}|/|\hat{V}_s|$, plot it and provide an interpretation.

Exercise 15.11: RL-krets

Vi ser på en modell for en krets som består av en spole med induktans L, en motstand med resistans R og et batteri med emf e koblet sammen i en sirkulær krets. Du kan anta at selvinduktansen for kretsen er ubetydelig sammenliknet med L.

a) Vis at strømmen gjennom kretsen kan beskrives med likningen

$$\frac{dI}{dt} = \frac{1}{L}(e - IR) \ . \tag{15.84}$$

b) Skriv et program som finner $I(t)$.

Maxwell's equations 16

We have so far almost completely introduced the general equations that describe electromagnetism — Maxwell's equation. Here, we will complete the description by introducing the last term needed to ensure that the equations are consistent. We will introduce an interpretation of the new term as a displacement current. We will formulate the complete set of Maxwell's equations and discuss how they are used to build theories for e.g. electromagenetic waves.

16.1 Fixing Maxwell's equation

16.1.1 There is something wrong with the equations

We have introduced a set of equations to describe the coupling between the electric and magnetic fields. On differential form, the equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ , \tag{16.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \ , \tag{16.2}$$

$$\nabla \cdot \mathbf{D} = \rho \ , \tag{16.3}$$

$$\nabla \cdot \mathbf{B} = 0 \ . \tag{16.4}$$

In addition, we have conservation of charge, which is formulated as:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \; . \tag{16.5}$$

Unfortunately, these equations are not consistent with each other. We can see this by realizing that the divergence of a curl always is zero. This means that

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \; . \tag{16.6}$$

Which would imply that $\nabla \cdot \mathbf{J} = 0$. But this is not always true, it is only true for static systems. In general we know from the conservation of charge that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \; . \tag{16.7}$$

This implies that there must be something wrong with or lacking from the equation $\nabla \times \mathbf{H} = \mathbf{J}$. In what kinds of situations would this problem appear, and how can we fix this?

16.1.2 Example: Flow into a capacitor

Let us examine a problem where charge conservation is essential — how charge flows into a capacitor. Fig. 16.1 illustrates a simple circuit with a battery, a resistor and a capacitor. When the battery is connected, there will be a current flowing into the capacitor. Over time, the current will decrease as charges are building up across the capacitor, increasing the potential drop $V = Q/C$ across the capacitor until the potential drop equals the emf of the battery.

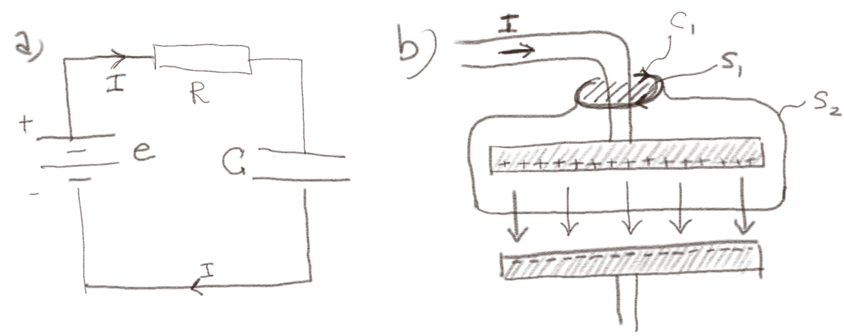


Fig. 16.1 (a) Illustration of a circuit with a capacitor, (b) Illustration of the capacitor.

If we apply Ampere's law to a loop C around the wire leading into the capacitor, as illustrated in the figure, we know that

$$\oint_S \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_S \tag{16.8}$$

Where the current I is the current through a surface S, which has the loop C as its limit. This equation should hold for any such surface. We have illustrated to surfaces in the figure. For surface S_1, the current I_{S_1} is the current $I(t)$ in the wire. However, if we choose a surface S_2 which passes through the gap between the capacitor plates, there is no current through this surface, and $I_{S_2} = 0$. Hmmm. This means that also Ampere's law is inconsistent. How can we fix this?

We know that the current $I(t)$ in the wire is related to the charge $Q(t)$ on the capacitor, which again is related to the electric field $\mathbf{D}$ in the space between the capacitors:

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} , \tag{16.9}$$

We can relate $Q(t)$ to the electric field through Gauss' law:

$$\int_S \mathbf{D}(t) \cdot \mathrm{d}\mathbf{S} = Q(t) . \tag{16.10}$$

The current I is therefore:

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{D}(t) \cdot \mathrm{d}\mathbf{S} . \tag{16.11}$$

If the surface S does not change with time, the time variation only comes from the time dependence of the electric field $\mathbf{D}(t)$, and we can move the time derivate into the integral:

$$I(t) = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S} . \tag{16.12}$$

It is useful to consider the right-hand side of this expression like a current that flows across the gap between the capacitor plates, even though there is no actual transport of charges across the gap. This allows us to consider $I(t)$ a current that flows continuously around the whole circuit, including through the capacitor gap.

16.1.3 Displacement current

We introduce the term displacement current based on this definition:

> **Displacement current**
>
> The displacement current I_d through a surface is defined as
>
> $$I_d = \int_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} \,. \tag{16.13}$$

Notice that the displacement current comes in addition the conductor current. For example, if a capacitor is filled with a dielectric with a very low conductivity, there would be both an ordinary (conduction) current I and a displacement current I_d.

16.1.4 Fixing Maxwell's equations

We can now fix Ampere's law by including both types of currents.

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I + I_d \tag{16.14}$$

We write the conductive current $I(t)$ as the surface integral of the current density $\mathbf{J}$:

$$I(t) = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \,, \tag{16.15}$$

where the surface S is a (any) surface with the loop C as its boundary. Similarly, we write the displacement current I_d as:

$$I_d = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S} \,. \tag{16.16}$$

Ampere's law therefore becomes:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathrm{d}\mathbf{S} \,. \tag{16.17}$$

This fixes the problem in the example above. But does it also fix the problem we discovered in Maxwell's equations on differential form? We use Stoke's theorem to rewrite Ampere's law to:

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \; . \tag{16.18}$$

This is valid for any surface S, and therefore, the equation also holds for the arguments of the integral:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \; . \tag{16.19}$$

Let us check if this now fixes the problem. The divergence of a curl is always zero, so the divergence of the left side is zero:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \; . \tag{16.20}$$

What about the right-hand side? We take the divergence and find:

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} \; . \tag{16.21}$$

We can now use Gauss' law on differential form, $\nabla \cdot \mathbf{D} = \rho$, giving:

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \; . \tag{16.22}$$

We recognize this as the conservation of charge from (16.5) and that this is always equal to zero. Hence, we have found that the new formulation at least provides us with a consistent set of the equations: Maxwell's equations, and a new formulation of Ampere's law, which is called Ampere-Maxwell's law:

Ampere-Maxwell's law

Ampere-Maxwell's law on *integral form* states that for any closed curve C which encloses a surface S:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \; . \tag{16.23}$$

Ampere-Maxwell's law on *differential form* states that:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \; . \tag{16.24}$$

We call the term

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} \,, \tag{16.25}$$

the *displacement current density.*

16.1.5 Maxwell's equations

We now have a complete set of equations, which are called Maxwell's equations, that provide a complete description of all electromagnetic phenomena:

Maxwell's equations

Maxwell's equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,, \tag{16.26}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \,, \tag{16.27}$$

$$\nabla \cdot \mathbf{D} = \rho \,, \tag{16.28}$$

$$\nabla \cdot \mathbf{B} = 0 \,. \tag{16.29}$$

These equations, together with the continuity equation and the definitions of **D** and **H** provide a complete description of the physics of electromagnetism:

Charge continuity

Charge continuity is formulated as

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \,. \tag{16.30}$$

Dielectric and magnetic materials

For **dielectric materials** we have defined **D** as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \,. \tag{16.31}$$

For *linear dielectric materials* we have that $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$ and

$$\mathbf{D} = \epsilon \mathbf{E} \,, \tag{16.32}$$

where $\epsilon = \epsilon_r \epsilon_0$.

For **magnetic materials** we have defined $\mathbf{H}$ as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \,. \tag{16.33}$$

For a *linear magnetic material* we have that $\mathbf{M} = \chi_m \mathbf{H}$ and

$$\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \,. \tag{16.34}$$

16.2 Exercises

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