Oblig 4 - FYS2160

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1 Introduction

We have been asked by a manufacturer that makes elastic membranes to analyse their membranes physical properties when used in balloons in earths atmosphere. We will do so by first getting a picture of the microscopic structure of the membrane so that we can find its entropy (a measure of how disordered the membrane is). With the entropy we can find an expression for the surface tension of the membrane which will be useful when determining when the balloon pops. Since the pressure of the atmosphere decreases the higher we go, so does the pressure in the balloon as it rises up towards the sky. A function for the pressure of the atmosphere determined by the height will therefor derived. With all this, we will try and find the maximum height of which the balloon can reach in the atmosphere. This is useful to know if the balloons for example carry expensive equipment as weather measurement instruments. At last we will also see how the radiation of the sun affects the maximum height of the balloon.

2 Microscopic model of an elastic membrane

We imagine that the membrane consist of N elements connected to each other, where each i element has energy ϵ_0. The elements can be in one of three states, state 0, state 1 or state 2, where elements of state 0 has length $b_0 = b$ and elements of state 1 and state 2 has length $b_1 = b_2 = bc$. The constant c is a measure of compression valued between 0 and 1. This system is illustrated in Figure 1. The total length of a membrane which contains N_0 state 0 elements and $N - N_0$ state 1 or 2 elements is then:

$$L = N_0 b + (N - N_0)bc \tag{1}$$

We will use this later. We also assume that the probability for an element to be in state 0, 1 or 2 are equal.

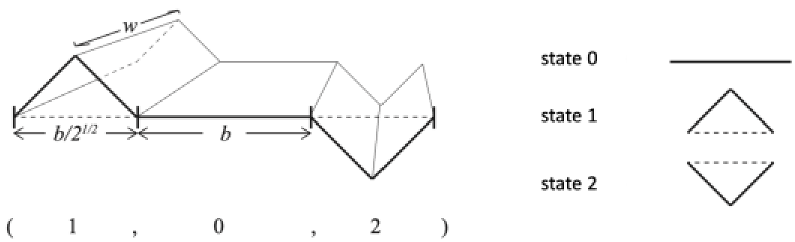

Figure 1: *An illustration of the membrane where the elements are in different states. Here c is set to* $1/\sqrt{2}$.

2.1 Multiplicity and entropy

Since each element can have one of three different states, then there are different ways for the membrane to be built up. The membrane with $N = 6$ can for example be set as (1, 2, 0, 2, 2, 0) or (0, 0 ,2, 1, 2, 0). The amount of different ways for the membrane to be set up is called its multiplicity Ω. Since the membrane contains N_0 of state 0 elements and $N - N_0$ of state 1 or 2 elements, the multiplicity of the membrane equals the different ways to sort N_0, Ω_{N_0}, with the different ways to sort $N - N_0$, Ω_{N-N_0}.

Lets first look at Ω_{N_0}. At first we have N elements in the membrane, therefor there are N different ways for one state 0 element to be placed:

$$\begin{gathered}(0, X, X, \ldots, X)\\(X, 0, X, \ldots, X)\\\vdots\\(X, X, X, \ldots, 0)\end{gathered}$$

where X represent undetermined states. Now there are only $N_0 - 1$ different ways to place a second state 0 element. If we take into account that each different pairs gets counted twice, then there are $N_0(N - N_0)/2$ different ways for two state 0 elements to be placed. In general we have that when we want to place the N_0-th state 0 element, we have $N - N_0$ places to choose from. This gives us the the different ways to sort N_0 state 0 elements:

$$\Omega_{N_0} = \frac{N(N-1)(N-2)\ldots(N-N_0)}{N_0(N_0-1)(N_0-2)\ldots(N_0-(N_0-1))} = \frac{N!}{N_0!(N-N_0)!}$$

Now we are left with $N - N_0$ places which can be of state 1 or 2. This is just the same as flipping a coin $N - N_0$ times, where there are two possible outcomes (heads or tails) that are equally probable. The number of different sets of outcomes, and thereby different ways to sort $N - N_0$, is then:

$$\Omega_{N-N_0} = 2^{N-N_0}$$

The total amount of different ways for the membrane to be set up, i.e. its multiplicity, is then:

$$\Omega = \Omega_{N_0} \cdot \Omega_{N-N_0} = \frac{N!}{N_0!(N-N_0)!} 2^{N-N_0} \tag{2}$$

We use this to find the expression for the entropy of the membrane:

$$\begin{aligned}
S &= k \ln \Omega \\
&= k \ln \Big(\frac{N!}{N_0!(N-N_0)!} 2^{N-N_0} \Big) \\
&= k \Big[\ln(N!) - \ln(N_0!) - \ln((N-N_0)!) + \ln 2^{N-N_0} \Big] \\
&\approx k \Big[N \ln N - N - (N_0 \ln N_0 - N_0) - \big((N-N_0) \ln(N-N_0) \\
&\qquad - (N-N_0) \big) + (N-N_0) \ln 2 \Big] \\
&= k \Big[N \ln N - N - N_0 \ln N_0 + N_0 - (N-N_0) \ln(N-N_0) \\
&\qquad + N - N_0 + (N-N_0) \ln 2 \Big] \\
&= k \Big[N \ln N - N_0 \ln N_0 - (N-N_0) \ln(N-N_0) + (N-N_0) \ln 2 \Big]
\end{aligned}$$

where we used Sterling's approximation $\ln N! \approx N \ln N - N$ at the fourth line. To repeat, the entropy of the membrane is approximately:

$$S \approx k \Big[N \ln N - N_0 \ln N_0 - (N-N_0) \ln(N-N_0) + (N-N_0) \ln 2 \Big] \tag{3}$$

2.2 Membrane tension

Now that we have an expression for the entropy of the membrane can we find go forwards to find its surface tension σ. But first we have from the first law of thermodynamics that change in the energy of the system, dE, equals the heat transfer plus the work done on the system:

$$dE = Q + W$$

If we assume that the work done on the system is quasistatic, where the pressure P holds itself uniform, then the work done equal $W = -PdV$ and the heat transfer becomes $Q = TdS$. We then have that:

$$dE = TdS - PdV$$

Since σ is the surface tension at the ends of the membrane and the membrane has length L, it can be written as $\sigma = F/L = PL$ which gives $P = \sigma/L$. The change in energy can then be written as:

$$dE = TdS - \frac{\sigma}{L}dV = TdS - \sigma dA$$

Solving this for TdS we get the thermodynamic identity for the system to be:

$$TdS = dE - \sigma dA \tag{4}$$

By solving equation (4) for σ we get that the surface tension is:

$$\sigma = \frac{dE - TdS}{dA}$$

If we assume that the energy E is constant we get:

$$\sigma = -\frac{TdS}{dA} = -\frac{T\partial S}{\partial A} \tag{5}$$

We have that the area of the membrane equals $A = wL$ where w is the with of each element and L is the length we found with equation (1). The area can then be written as:

$$\begin{aligned} A &= w(N_0 b + (N - N_0)bc) \\ &= w(N_0 b - N_0 bc + Nbc) \\ &= wb(N_0 - N_0 c + Nc) \end{aligned}$$

$$\Downarrow$$

$$A = wb(1-c)N_0 + wbcN \tag{6}$$

Since N is constant, the change in area $dA = \partial A = wb(1-c)\partial N_0$. We plug this into equation (5) and get:

$$\sigma = -\frac{T\partial S}{wb(1-c)\partial N_0} = -T\frac{1}{wb(1-c)}\left(\frac{\partial S}{\partial N_0}\right)_{E,N} \tag{7}$$

We get the partial derivative factor by differentiating equation (3) in regards to N_0:

$$\begin{aligned}\frac{\partial S}{\partial N_0} &= \frac{\partial}{\partial N_0}\Big(k\Big[N\ln N - N_0\ln N_0 \\ &\qquad\qquad - (N-N_0)\ln(N-N_0) + (N-N_0)\ln 2\Big]\Big) \\ &= k\Big(-(\ln N_0 + \frac{N_0}{N_0}) - (-\ln(N-N_0) - \frac{N-N_0}{N-N_0}) - \ln 2\Big) \\ &= k\Big(-\ln N_0 - 1 + \ln(N-N_0) + 1 - \ln 2\Big) \\ &= k\Big(\ln(N-N_0) - \ln N_0 - \ln 2\Big) \\ &= k\ln\Big(\frac{N-N_0}{2N_0}\Big)\end{aligned}$$

If we plug this into equation (7) we get the following expression for the surface tension:

$$\sigma \approx -\frac{kT}{wb(1-c)}\ln\Big(\frac{N-N_0}{2N_0}\Big) \tag{8}$$

2.3 Membrane strain

Now that we have a function for the surface tension, we will attempt to find the equilibrium area A_0^*. We define this area is defined as the area of the membrane when there are no surface tension, i.e. $\sigma = 0$. Our first step to find A_0^* is then to set equation (8) equals zero and find what conditions that are necessary:

$$-\frac{kT}{wb(1-c)}\ln\Big(\frac{N-N_0}{2N_0}\Big) = 0$$

The left factor $\frac{kT}{wb(1-c)}$ only contains constants and the temperature T, which we assume are greater than zero, so this factor cant become zero. The only way for this to be zero is then by $\ln\Big(\frac{N-N_0}{2N_0}\Big)$. This equals zero when the argument in the parentheses equals one:

$$\begin{aligned}\frac{N-N_0}{2N_0} &= 1 \\ N - N_0 &= 2N_0 \\ N &= 3N_0\end{aligned}$$

This means that the surface tension on the membrane is zero when a third of the elements that make up the membrane is of state 0. This makes sense if we think that the most probable macrostate of the membrane is where the first third of the elements are of state 0, the second third are of state 1 and the last third of state 2. If plug in the condition for no surface tension $N = 2N_0$ into equation (6), then we get the equilibrium area A_0^*:

$$\begin{aligned} A_0^* &= wb(1-c)N_0 + wbc(3N_0) \\ &= wbN_0(1-c+3c) \\ &= wbN_0(1+2c) \end{aligned}$$

We see that the equilibrium area of the membrane then are the area of N_0 state 0 elements plus the area of $2N_0$ state 1 and 2 elements, which makes sense.

A_0^* can also be seen as the initial area of the membrane, where the membrane strain $\varepsilon = \frac{A-A_0^*}{A_0^*}$ is a measure of how different the area is now A compared to its initial area A_0^*. We can rewrite A_0^* in terms of the density of relaxed membrane ρ_0 and the mass m of the membrane:

$$\rho_0 = \frac{m}{A_0^*} \qquad \Longrightarrow \qquad A_0^* = \frac{m}{\rho_0}$$

With the membrane strain ε the surface tension can be written as:

$$\sigma(\varepsilon) = \frac{kT}{wb(1-c)} \ln\Big(\frac{1+\beta\varepsilon}{1-\beta\varepsilon/2}\Big) \tag{9}$$

where $\beta = \frac{1+2c}{1-c}$. We also have that the maximum strain of the membrane is $\varepsilon = 2/\beta$, which is when the membrane breaks apart. We assume that the our membrane is used as a balloon that is spherical and that has the membrane strain $\varepsilon = (\frac{r}{r_0^*})^2 - 1$ where r is the radius of the balloon and r_0^* is the equilibrium radius (where $\sigma = 0$). The equilibrium area for the balloon is then $A_0^* = 4\pi(r_0^*)^2$, which gives us the r_0^* if we know ρ_0 and m:

$$\frac{m}{\rho_0} = 4\pi(r_0^*)^2 \qquad \Longrightarrow \qquad r_0^* = \sqrt{\frac{m}{4\pi\rho_0}}$$

We can also find the maximum radius r of which the balloon can have before it breaks apart by assuming that the strain $\varepsilon = (\frac{r}{r_0^*})^2 + 1$ equals the maximum strain $\varepsilon = 2/\beta$:

$$\Big(\frac{r}{r_0^*}\Big)^2 - 1 = 2/\beta \qquad \Longrightarrow \qquad r = r_0^*\sqrt{2/\beta + 1} = r_{max}$$

We are now able to plot the surface tension σ against the membrane strain ε. We assume quantities $\rho_0 = 0.05$ kg/m^2, $wb = 6 \cdot 10^{-26}$ m^2, $c = 0.1$ and $m = 1$ kg and use the following code to plot σ over ε:

```python
import numpy as np
import matplotlib.pyplot as plt

# Define quantities:
wb = 6 * 10**(-26)            # m2
c = 0.1                       # *
rho = 0.05                    # kg/m2
m = 1                         # kg
k = 1.380649 * 10**(-23)      # Boltzmann

beta = (1 + 2*c) / (1 - c)

# Find the equilibrium radius and maximum radius:
r_0 = np.sqrt(m / (4*np.pi*rho))
r_max = r_0*np.sqrt(2/beta + 1)

# Makes a list of radii from initial to maximum:
r = np.linspace(r_0, r_max, 100)

# Makes list the eqvivalent strain for each radii:
epsilon = (r/r_0)**2 - 1

# Finds the surface tension at temperature T:
T = 300
sigma = k*T / (wb*(1-c)) * np.log((1+beta*epsilon) / (1-beta*epsilon/2))

# Plots
plt.plot(epsilon, sigma)
plt.plot(epsilon[-2], sigma[-2], marker="o",
            markersize=2, markeredgecolor="red", markerfacecolor="red",
            label="The balloon pop")
plt.xlabel("$\epsilon$ [*]")
plt.ylabel("$\sigma$  [N/m]")
plt.title("Surface tension $\sigma$ over membrane strain $\epsilon$")
plt.grid(color='black', linestyle = '--', linewidth = 0.1)
plt.legend()
plt.show()
```

Figure 2 shows the resulting plot. We observe that the surface tension increases gradually with the membrane strain before it suddenly rise quite fast when the strain reaches a certain value, $\varepsilon = 1.50$. This value should correspond to the maximum strain given by $\varepsilon = 2/\beta$. We let $c = 0.1$ which gives us $\beta = \frac{1+2\cdot 0.1}{1-0.1} = \frac{4}{3}$ which again gives us $\varepsilon = \frac{2\cdot 3}{4} = 3/2 = 1.5$, which correlate. This plot makes sense considering that the pressure inside will not be able to change the volume when the membrane reaches its maximum strain and therefor causes extra "stress" on the surface of the membrane so that the surface tension increases rapidly.

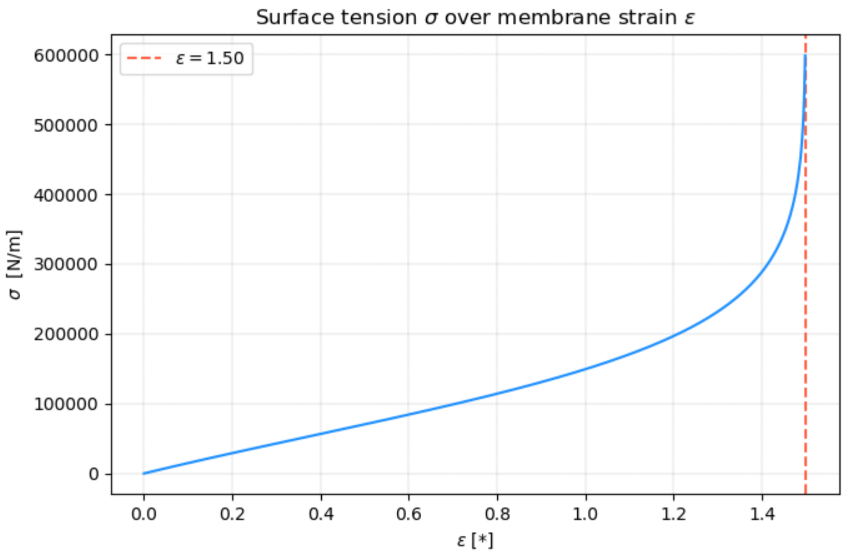

Figure 2: *A plot for the surface tension σ as a function of strain ε at temperature $T = 300K$.*

3 Elastic balloons in the atmosphere

3.1 Pressures and buoyancy

We would like to determine the pressure in the atmosphere as a function of height z. If we assume diffusive equilibrium we know that atmosphere follows the ideal gas law:

$$PV = NkT = nRT$$

We solve this for the pressure and use the relation $\frac{n}{V} = \frac{m}{V}\frac{n}{m} = \rho\frac{1}{M}$ where ρ and M is the density and molar mass of the atmosphere respectively to get:

$$P = \frac{\rho RT}{M} \tag{10}$$

Now, with a change in height dz there is a corresponding change in volume $dV = Adz$. The gravitational force that pull on this volume then change by $dF = dm \cdot g = \rho dV \cdot g$. If we define upwards as positive, then the change in pressure equals:

$$dP = -\frac{dF}{A} = -\frac{\rho dV g}{A} = -\frac{\rho(Adz)g}{A} = -\rho g dz \tag{11}$$

We divide equation (11) with equation (10) and get:

$$\frac{dP}{P} = \frac{-\rho g dz}{\frac{\rho RT}{M}}$$

$$\frac{1}{P}dP = -\frac{Mg}{RT}dz$$

We then integrate from $z = 0$ to z on both sides:

$$\frac{1}{P}dP = -\frac{Mg}{RT}dz$$

$$\int_{P(z=0)}^{P(z)} \frac{1}{P}dP = \int_0^z -\frac{Mg}{RT}dz$$

$$\ln P(z) - \ln P(z=0) = -\frac{Mg}{R}\int_0^z \frac{1}{T}dz$$

We now assume that the temperature change linearly with the height z so that $T = Lz + T_0$ where $L = \partial T/\partial z$ is the slope and T_0 is the temperature at $z = 0$. We can then solve the temperature integral:

$$\int_0^z \frac{1}{T}dz = \int_0^z \frac{1}{Lz + T_0}dz$$

$$= \frac{1}{L}\Big[\ln(Lz + T_0) - \ln T_0\Big]$$

$$= \frac{1}{L}\ln\Big(\frac{Lz + T_0}{T_0}\Big)$$

Plugging this back into the equation above and then solves for $P(z)$:

$$\ln P(z) - \ln P_0 = -\frac{Mg}{R}\frac{1}{L}\ln\Big(\frac{Lz + T_0}{T_0}\Big)$$

$$\ln\Big(\frac{P(z)}{P_0}\Big) = \ln\Big(\frac{Lz + T_0}{T_0}\Big)^{-\frac{Mg}{RL}}$$

$$\frac{P(z)}{P_0} = \Big(\frac{Lz + T_0}{T_0}\Big)^{-\frac{Mg}{RL}}$$

$$P(z) = P_0\Big(\frac{Lz}{T_0} + 1\Big)^{-\frac{Mg}{RL}}$$

We thereby have that the pressure of the atmosphere at height z equals:

$$P(z) = P_0\left(\frac{Lz}{T_0} + 1\right)^{-\frac{Mg}{RL}} \tag{12}$$

[I couldt figure out how to go from the pressure difference to the force :(]

3.2 Sun radiating on the balloon

As the balloon is in the atmosphere it is exposed to the sun, which bombard the balloon with electromagnetic waves. If we assume the balloon acts as a perfect blackbody ($e = 1$), then it absorbs all of the radiation from the sun. As I am running out of time I wont be able to finish this, but I would assume that the temperature of the membrane of the balloon does increase due to the radiation from the sun. But the temperature of the gas inside the balloon also change, but not uniformly. I think that the temperature change of the gas would be larger close to the membrane compared to the center of the balloon where the change would be minimal. But overall the temperature would increase and that would also increase the surface tension as the surface tension is proportional with the temperature. This would then lower the height of which the balloon is able to reach without popping.

4 Conclusion

From out microscopic model of an elastic membrane, we were able to find out how the surface tension around the membrane would look like. We found that the relationship between the surface tension and membrane strain changed drastically when the strain reached its maximum value, which made sense with everyday observations. We also found that have a more realistic model of a balloon in the atmosphere we also should take the sun into consideration, which reduces the height a balloon can reach.