Derivation of Logistic Equation

$\frac{dP}{dt} = \alpha P - \beta P^2$ for 3 cases: 1. $\alpha = 0.2$ $\beta = 0.0005$ $P(0) = 10.0$
2. $\alpha = 0.01$ $\beta = 0.0005$ $P(0) = 10.0$
3. $\alpha = 2.0$ $\beta = 0.0005$ $P(0) = 10.0$

$\Rightarrow \frac{dP}{dt} = \alpha P\left(1 - \frac{P}{\frac{\alpha}{\beta}}\right)$

Solving $\frac{dP}{dt} = \alpha P - \beta P^2$ by separation of variables:

$dP = (\alpha P - \beta P^2)\,dt \Rightarrow \int \frac{1}{\alpha P - \beta P^2}\,dP = \int dt \Rightarrow$

$\int \frac{1}{P(\alpha - \beta P)}\,dP = t + C$. Now use partial fraction decomposition for L.H.S:

$\frac{A}{P} + \frac{B}{\alpha - \beta P} \Rightarrow A(\alpha - \beta P) + B(P) \Rightarrow A\alpha - A\beta P + BP \Rightarrow$

$A\alpha = 1 \Rightarrow A = \frac{1}{\alpha}$, $P(B - A\beta) = 0$

$B = A\beta = \frac{\beta}{\alpha}$

$\int \frac{\frac{1}{\alpha}}{P}\,dP + \int \frac{\frac{\beta}{\alpha}}{\alpha - \beta P}\,dP = t + C \Rightarrow$

$\frac{1}{\alpha}\ln(P) + \frac{\beta}{\alpha}\int \frac{1}{\alpha - \beta P}\,dP = t + C \Rightarrow \frac{1}{\alpha}\ln(P) - \frac{1}{\alpha}\ln(\alpha - \beta P) = t + C \Rightarrow$

$\frac{1}{\alpha}\left(\ln(P) - \ln(\alpha - \beta P)\right) = t + C \Rightarrow \frac{1}{\alpha}\ln\left(\frac{P}{\alpha - \beta P}\right) = t + C \Rightarrow$

$\ln\left(\frac{P}{\alpha - \beta P}\right) = t\alpha + C \Rightarrow \frac{P}{\alpha - \beta P} = Ce^{t\alpha} \Rightarrow P = Ce^{t\alpha}(\alpha - \beta P)$

$\Rightarrow P = \alpha C e^{t\alpha} - C\beta P e^{t\alpha} \Rightarrow P + C\beta P e^{t\alpha} = \alpha C e^{t\alpha} \Rightarrow$

$P(t) = \frac{\alpha C e^{t\alpha}}{1 + C\beta e^{t\alpha}}$. If $P(0) = P_0$, then $P_0 = \frac{\alpha C}{1 + C\beta} \Rightarrow$

$P_0(1 + C\beta) = \alpha C \Rightarrow P_0 + C(P_0)\beta = \alpha C \Rightarrow P_0 = \alpha C - P_0 C\beta \Rightarrow$

$C = \frac{P_0}{\alpha - P_0\beta}$. Thus, $P(t) = \frac{\alpha\left(\frac{P_0}{\alpha - P_0\beta}\right)e^{t\alpha}}{1 + \left(\frac{P_0}{\alpha - P_0\beta}\right)\beta e^{t\alpha}}$

case 1:	case 2:	case 3:
$P(t) = \frac{10.2564 e^{0.2t}}{1 + 0.0256 e^{0.2t}}$	$P(t) = \frac{20 e^{0.01t}}{1 + e^{0.01t}}$	$P(t) = \frac{10.0251 e^{2t}}{1 + 0.0025 e^{2t}}$