Online Monmouth Math Competition Solutions Manual

The OMMC Team

April 2021

1 Round 1

All problems must be done in 50 minutes. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. No discussion is allowed.

1. Find the remainder when

$$20^{20} + 21^{21} - 21^{20} - 20^{21}$$

is divided by 100.

Answer: $\boxed{20}$

Solution: Obviously we can ignore the first and last terms. The last digits of 21^{21} and 21^{20} must both be 1. Let $21^{20} = 10a + 1 \pmod{100}$. Then, $21^{21} \equiv 210a + 21 \equiv 10a + 20 + 1 \pmod{100}$. So, $21^{21} - 21^{20} = 20 \pmod{100}$ and we have $\boxed{20}$ as our answer.

2. There are a family of 5 siblings. They have a pile of candies, and are trying to split them up among themselves. If the 2 oldest siblings share the candy, they will have 1 candy left over. If the 3 oldest siblings share the candy, they will also have 1 left over. If all 5 siblings share the candy, they will also have 1 left over. What is the minimum amount of candy required for this to be true?

Answer: $\boxed{31}$

Solution: It is not hard to see that $2 \cdot 3 \cdot 5 + 1 = \boxed{31}$ is the desired minimum.

3. Define $f(x)$ as $\frac{x^2-x-2}{x^2+x-6}$. $f(f(f(f(1))))$ can be expressed as the common fraction $\frac{p}{q}$. Find $10p + q$.

Answer: $\boxed{211}$

Solution: We notice that

$$\frac{x^2-x-2}{x^2+x-6} = \frac{(x+1)(x-2)}{(x+3)(x-2)} = \frac{x+1}{x+3}.$$

Therefore,

$$f(f(f(f(1)))) = f(f(f\left(\frac{1}{2}\right))) = f(f\left(\frac{3}{7}\right)) = f\left(\frac{5}{12}\right) = \frac{17}{41}.$$

Therefore, $p = 17$ and $q = 41$ so $10p + q = 211$.

4. Robert tiles a 420 x 420 square grid completely with 1 x 2 blocks, then notices that the two diagonals of the grid pass through a total of n blocks. Find the sum of all possible values of n.

Answer: $\boxed{2517}$

Solution: The diagonals pass through a total of 840 squares, and if a diagonal passes through a block that means it passes through a square covered by that block. Call these 840 squares "diagonal squares," these squares must be covered by n blocks. With the exception of the 4 "center" squares, each 1 by 2 block can pass through at most one diagonal square, and each diagonal square has exactly one 1 by 2 block over it. It can be verified that the 4 center squares can then either be covered by 4, 3, or 2 blocks. So, we either have $836 + 2$, $836 + 3$, or $836 + 4$ squares covered by the diagonals yielding a total of $\boxed{2517}$.

5. Two points A, B are randomly chosen on a circle with radius 100. Given that the probability that the length of AB is less than x is greater than $\frac{2}{3}$ for some positive integer x, find the smallest possible value of x.

Answer: $\boxed{174}$

Solution: Fix the point A. Consider the probability that the length of AB is less than $100\sqrt{3}$. Let O be the center of the circle. By Law of Cosines on the isosceles triangle AOB, this is actually the probability that $\angle AOB \leq 120°$, which can be verified to be the desired $\frac{2}{3}$. So, x just has to be greater than $100\sqrt{3}$ to satisfy the given condition, so $x^2 \geq 30,000$ and the minimum integer value of x is $\boxed{174}$.

6. Jason and Jared take turns placing blocks within a gameboard with dimensions 3 x 300, with Jason going first, such that no two blocks can overlap. The player who cannot place a block within the boundaries loses. Jason can only place 2 x 100 blocks, and Jared can only place 2 x n blocks where n is some positive integer greater than 3. Find the smallest value of n that still guarantees Jason a win (if both players are playing optimally).

Answer: $\boxed{51}$

Solution: Notice that the length of both blocks is 2 and the length of the gameboard is 3, so two blocks cannot fit in one row. With that, we can essentially discard the length dimension and focus solely on the width.

If $n = 51$, Jason can place a block squarely in the middle, and Jared has to place his block in either of the two spaces to the right or left of Jason's block, both with length 100. Then, Jason can place his block in the other space, leaving no room for Jared.

If $n < 51$, let Jason's first move split the board into spaces of width a, b where a, b are nonnegative integers. If $a \neq b$, one of the spaces has width greater than 100, but at most 200. Jared can place his block right in the middle of that space, leaving 3 spaces all less than 100 in width. If $a = b = 100$, Jared can place his block at the very left, leaving two spaces of width $100 - n$ and 100. Jason can only place his block in the space of width 100, and Jared can place his block in the space of width $100 - n$ since $n < 51$, leaving no room for Jason. So, our answer is $\boxed{51}$.

7. Derek fills a square 10 by 10 grid with 50 1s and 50 2s with each of the 100 grid squares occupied by a 1 or a 2. He takes the product of the numbers in each of the 10 rows. He takes the product of the numbers in each of the 10 columns. He then sums these 20 products up to get an integer N. Find the minimum possible value of N.

Answer: $\boxed{640}$

Solution: The sum of the 10 products resulting from the rows can be represented as

$$2^{a_1} + 2^{a_2} + \cdots 2^{a_{10}}$$

Where the sum $a_1 + a_2 + \cdots a_{10} = 50$. Notice that this sum is greater than or equal to $10 \cdot 2^5$ by the AM-GM inequality. Similarly, the sum of the 10 products resulting from the columns is greater than or equal to $10 \cdot 2^5$ by the AM-GM inequality. Adding this up gets a minimum possible value of $20 \cdot 2^5 = \boxed{640}$. This can be achieved by filling the 2s in via a checkerboard pattern, such that there are 5 instances of 2s in each row and column.

8. The function $g(x)$ is defined as $\sqrt{\frac{x}{2}}$ for all positive x.

$$g\left(g\left(g\left(g\left(g\left(\frac{1}{2}\right)+1\right)+1\right)+1\right)+1\right)$$

can be expressed as $\cos(b)$ using degrees, where $0^\circ < b < 90^\circ$. $b = p/q$ for some relatively prime positive integers p, q. Find $p + q$.

Answer: $\boxed{19}$

Solution: Notice that $g\left(\frac{1}{2}\right) = \frac{1}{2} = \cos 60^\circ$. Then, recount the half angle formula for cosine, and see that

$$\cos\left(\frac{A}{2}\right) = g(\cos A + 1)$$

for all angles $A < 90^\circ$. Applying this shows that

$$g\left(g\left(g\left(g\left(\cos 60^\circ + 1\right) + 1\right) + 1\right) + 1\right) = \cos\left(\frac{60}{16}\right)^\circ$$

and our final answer is $\frac{60}{16} = \frac{15}{4}$ yielding $\boxed{19}$.

9. The difference between the maximum and minimum values of

$$2\cos 2x + 7\sin x$$

over the real numbers equals $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

Answer: $\boxed{241}$

Solution: Let $\sin x = a$. It is not hard to see that via the double angle formula, the expression is equivalent to $-(4a+1)(a-2)$ which has maximum $\frac{81}{16}$ and minimum -9 on the interval $[-1, 1]$. The desired difference is $\frac{225}{16} \implies \boxed{241}$.

10. How many ways are there to arrange the numbers 1 through 8 into a 2 by 4 grid such that each grid square contains exactly one number, each number appears exactly once, the sum of the numbers in each of the two rows are all multiples of 6, and the sum of the numbers in each of the four columns are all multiples of 3?

Answer: $\boxed{288}$

Solution: Notice that the numbers 1 through 8 are $1, 2, 0, 1, 2, 0, 1, 2$ modulo 3. Notice that in each of the four columns, a number that is 1 mod 3 must be adjacent to a number that is 2 mod 3. So, we must have 3 of the four columns containing a number 1 mod 3 must be adjacent to a number that is 2 mod 3, and one of the four columns containing two numbers that are 0 modulo 3.

In addition, upon inspection each of the two rows must be one of $1, 1, 1, 0$ or $2, 2, 2, 0$ modulo 3 in some order in order for the sum to be 0 mod 3. That is, we cannot have a number that is 1 modulo 3 and a number that is 2 modulo 3 in the same row.

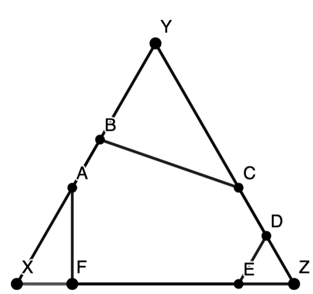
There are 2 ways to decide which of the two rows are $1, 1, 1, 0$ or $2, 2, 2, 0$ modulo 3 in some order. Then, there are 4 ways to choose the column that has the two numbers that are 0 modulo 3. Then, we can just order the numbers that are 1 mod 3, and order those that are 2 mod 3 as well at no effect to any of the row or column sums modulo 3, yielding 6^2 possibilities.

Then the numbers that are 0 mod 3 are simply fixed in order to ensure the even parity of the two rows - one of them is odd, and one of them is even. In fact, it can be verified that 6 must be in the row with $1, 4, 7$, and 3 must be in the row with $2, 5, 8$. Our final answer is $2 \cdot 4 \cdot 6 \cdot 6 = \boxed{288}$.

11. In equilateral triangle XYZ with side length 10, define points A, B on XY, points C, D on YZ, and points E, F on ZX such that $ABDE$ and $ACEF$ are rectangles. The area of hexagon $ABCDEF$ can be written as $\sqrt{x}$ for some positive integer x. Find x.

Answer: $\boxed{768}$

Solution: $AE \perp XY$ so $\angle AEF = \angle AEX = 30°$. Its not hard to verify from 30-60-90 triangle properties that $EF = 3 \cdot XF = 3 \cdot EZ$ so $EF = 6$, and $AX = BY = 2 \cdot XF = 4$. Subtracting the areas of triangles AXF, BYC, and DZE from XYZ yields a final answer of $16\sqrt{3} \implies \boxed{768}$.



12. Let $P(x) = x^3 + 8x^2 - x + 3$ and let the roots of P be a, b, and c. The roots of a monic polynomial $Q(x)$ are $ab - c^2, ac - b^2, bc - a^2$. Find $Q(-1)$.

Answer: $\boxed{1536}$

Solution: Since the roots of $Q(x)$ are $ab - c^2, ac - b^2, bc - a^2$, we know that

$$Q(x) = (x - ab + c^2)(x - ac + b^2)(x - bc + a^2).$$

We know that $abc = -3$ with Vieta's formulas. Therefore,

$$\begin{aligned} Q(-1) &= (-1 - ab + c^2)(-1 - ac + b^2)(-1 - bc + a^2) \\ &= (-1 + \frac{3}{c} + c^2)(-1 + \frac{3}{b} + b^2)(-1 + \frac{3}{a} + a^2) \\ &= \frac{1}{abc}(c^3 - c + 3)(b^3 - b + 3)(a^3 - a + 3) \\ &= -\frac{1}{3}(-8c^2)(-8b^2)(-8a^2) \\ &= \frac{512a^2b^2c^2}{3} \\ &= 1536. \end{aligned}$$

13. Find the number of nonnegative integers $n < 29$ such that there exists positive integers x, y such that $x^2 + 5xy - y^2$ has remainder n when divided by 29.

Answer: $\boxed{15}$

Solution: We can multiply the equation by 4 to get

$$4x^2 + 20xy - 4y^2 \equiv 4n \pmod{29}.$$

This is equivalent to

$$(2x + 5y)^2 - 29y^2 \equiv 4n \pmod{29},$$

so

$$(2x + 5y)^2 \equiv 4n \pmod{29}.$$

The number $4n$ is a quadratic residue if and only if n is a quadratic residue, so the possible values of n are the quadratic residues $\pmod{29}$. It is well known that there are $\frac{29+1}{2} = \boxed{15}$ quadratic residues $\pmod{29}$ so that is our answer.

14. There exist positive integers N, M such that N's remainders modulo the four integers $6, 36, 216$, and M form an increasing nonzero geometric sequence in that order. Find the smallest possible value of M.

Answer: $\boxed{2001}$

Solution: Let $N \equiv n \pmod{216}$. Write n as $36c + 6b + a$ where $a, b, c < 6$. Then, we must have

$$\frac{6b + a}{a} = \frac{6b + 36c + a}{6b + a}.$$

This arranges to $36b^2 + a^2 + 12ba = 6ba + 36ca + a^2$, which then becomes $36b^2 + 6ba = 36ca$. We therefore have $6ba$ as a multiple of 36, so one of a, b is either 2 or 4, and the other is 3.

Suppose that $b = 3$, then we have $324 + 18a = 36ca$ or $18 + a = 2ca$. If $a = 2$ then we have $c = 5$, if $a = 4$ we do not have c as an integer.

Suppose that $a = 3$, then $36b^2 + 18b = 108c$ or $2b^2 + b = 6c$. If $b = 2$ then c is not an integer, and if $b = 4$ then $c = 6$, which is too much.

So, $b = 3$, $a = 2$, $c = 5$, and $N \equiv 200 \pmod{216}$. So, N's remainder mod M must be 2000, and $M \geq \boxed{2001}$. It is not hard to see that there exists a number $2000 \pmod{2001}$ and $200 \pmod{216}$ as desired.

15. A point X exactly $\sqrt{2} - \frac{\sqrt{6}}{3}$ away from the origin is chosen randomly. A point Y less than 4 away from the origin is chosen randomly. The probability that a point Z less than 2 away from the origin exists such that $\triangle XYZ$ is an equilateral triangle can be expressed as $\frac{a\pi+b}{c\pi}$ for some positive integers a, b, c with a and c relatively prime. Find $a + b + c$.

Answer: $\boxed{34}$

Solution: Let O be the origin. Due to symmetry, we can fix X at some arbitrary point exactly $\sqrt{2} - \frac{\sqrt{6}}{3}$ away from the origin without loss of generality. Notice that the problem is asking the probability that a rotation around X 60° clockwise or counterclockwise brings Y to a point Z within the circle with radius 2 centered at O. The key insight is that if this is true, then Y lies within either of the circles with radius 2 centered at O_1 and O_2, where O_1 and O_2 are the images of O when rotated clockwise and counterclockwise around X respectively. We compute this area by subtracting the union of the two circles from $2 \cdot 2^2 \cdot \pi = 8\pi$, then divide that by the area of the space Y is bounded to, which is just $4^2\pi = 16\pi$.

Let the circles intersect at points A and B. It is clear that A, B, X, and O are collinear. It can be calculated that the distance from O_1 to AB equals the distance from O_1 to XO which is

$$(\sqrt{2} - \frac{\sqrt{6}}{3}) \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6} - \sqrt{2}}{2}.$$

Let P be the foot of O_1 to AB, it follows that $\angle AO_1P = \angle BO_1P = 75^\circ$ and $\angle AO_1B = 150^\circ$. It can be verified that the area of the union of the two circles is just the area of circle sections AO_1B and AO_2B, minus the area of the two triangles AO_1B and AO_2B. The former is $\frac{10\pi}{3}$ and the latter is just $2 \cdot 2 \cdot \sin 150^\circ = 2$. So, the area of the union of the two circles is just $8\pi - (\frac{10\pi}{3} - 2) = \frac{14\pi}{3} + 2$.

Putting this area over 16π yields $\frac{7\pi+3}{24\pi}$ as our final answer. $7 + 3 + 24 = \boxed{34}$ and we are done.

2 Round 2

All problems must be done in 2 hours. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. No discussion is allowed until discussion is opened.

1. There are 20 people in a group. Each person follows exactly 2 others in this group, and also has 2 people following him/her. What is the maximum possible number of people that can be placed into a subset of the main group such that no one in this subset follows someone else in the subset?

Answer: $\boxed{10}$

Solution: We first need to find a case such that 10 people in the subset is possible. We can have two groups: A and B where A is the subset. Each group has 10 people. The first person in A(A1) follows B1 and B2, A2 follows B2 and B3, and so on, until A10 follows B10 and B1. We then do the same thing for group B, i.e. B1 follows A1 and A2, and so on.

This is a valid example, so 10 is possible.

A group with 11 people and above is impossible because if the number of people in the subset is denoted x, then the number of follows that can be accounted for by people not in the subset is $2 \cdot (20 - x)$. However, there are x people in the subset, and thus $2x$ follows that must be accounted for by people not in the subset.

$2 \cdot (20 - x) < 2x$ for all $x \geq 11$, so we have shown that 11 and above is an impossible number of people in a group. Therefore, the answer is $\boxed{10}$.

2. Sequences a_n and b_n are defined for all positive integers n such that $a_1 = 5$, $b_1 = 7$,

$$a_{n+1} = \frac{\sqrt{(a_n + b_n - 1)^2 + (a_n - b_n + 1)^2}}{2},$$

and

$$b_{n+1} = \frac{\sqrt{(a_n + b_n + 1)^2 + (a_n - b_n - 1)^2}}{2}.$$

How many integers n from 1 to 1000 satisfy the property that a_n, b_n form the legs of a right triangle with a hypotenuse that has integer length?

Answer: $\boxed{24}$

Solution: Notice that

$$a_{n+1}^2 + b_{n+1}^2 = \frac{(a_n + b_n - 1)^2 + (a_n - b_n + 1)^2 + (a_n + b_n + 1)^2 + (a_n - b_n - 1)^2}{4} = a_n^2 + b_n^2 + 1.$$

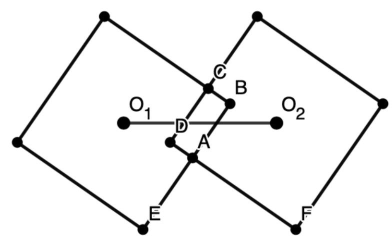
It follows that the quantity $a_n^2 + b_n^2$ traverses the integers from 74 to 1073 as n varies from 1 to 1000. The number of squares in this interval is $\boxed{24}$, as the squares range from $9^2 = 81$ to $32^2 = 1024$.

3. Two squares with perimeter 8 intersect in a rectangle with diagonal length 1. Given that the distance between the centers of the two squares is 2, the perimeter of the rectangle can be expressed as P. Find $10P$.

Answer: $\boxed{25}$

Solution: For the two squares to intersect in a rectangle, it is clear that their sides must be parallel to each other. Label the points as shown in the following diagram, and notice that since a translation with length 2 maps one square to the other, $EF = 2$. Let x be AE, then $AF = \sqrt{4-x^2}$. But then $(2-x)^2 + (2-\sqrt{4-x^2})^2 = 1$.

Solving this equation yields that either $AE = 2 - \frac{(11+\sqrt{7})}{8}$ and $AF = 2 - \frac{(11-\sqrt{7})}{8}$, or vice versa. Either way, it can then be calculated that the perimeter of the rectangle equals $2(4 - AE - AF) = 5/2$, which when multiplied by 10 gives $\boxed{25}$ as desired.



4. The sum

$$\frac{1^2-2}{1!} + \frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{2021^2-2}{2021!}$$

can be expressed as a rational number N. Find the last 3 digits of $2021! \cdot N$.

Answer: $\boxed{977}$

Solution: See that for any positive integer n, we can express $\frac{n^2-2}{n!}$ as $\frac{1}{(n-2)!} + \frac{1}{(n-1)!} - \frac{2}{n!}$ when $n \geq 2$. From there, most of the terms in the sequence telescope and we are left with

$$\frac{1^2-2}{1!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{1!} - \frac{1}{2020!} - \frac{1}{2021!} - \frac{1}{2021!}.$$

Multiplying by 2021! yields some large multiple of 1000 minus 2021 and 2 so our answer is $1000 - 21 - 2 = \boxed{977}$.

5. Let N be an 3 digit integer in base 10 such that the sum of its digits in base 4 is half the sum of its digits in base 8. In base 10, find the largest possible value of N.

Answer: $\boxed{884}$

Solution: We can see that 20 in base 8 is 100 in base 4. This means that we can add 20 to any number that we have that is valid, and as long as it doesn't carry over, we will be fine. Now, we look at what the ending digits should be modulo 20. We just have to look at 1, 2, 3, 4, 5, 6, 7, 10, 11, ... 20, all in base 8. We see that the ratio between the sums of the digits of the base 4 representations and base 8 representations of the numbers (1, 2, 3, ... 20) are maximized when we choose 4. Now we just have to look at numbers ending in 04, and add 20 a couple of times when we have found a sufficient number. Let's start at 1704 in base 8, because that is the largest 3 digit number in base 10 that ends in 04 with a base 8 representation. The base 4 representation is 33010, which means that the ratio is less than 2. We see that we have to go smaller. Looking at 1604, we see that it does not work as well. We finally see that 1504 works. We just have to add 20 a couple of times, as long as we don't carry over. We get 1564 as our answer which is $\boxed{884}$ in base 10.

6. In square $ABCD$ with $AB = 10$, point P, Q are chosen on side CD and AD respectively such that $BQ \perp AP$, and R lies on CD such that $RQ \parallel PA$. BC and AP intersect at X, and XQ intersects the circumcircle of PQD at Y. Given that $\angle PYR = 105^\circ$, AQ can be expressed as $a\sqrt{b} - c$ where a, b, c are positive integers and b is square free. Find $a + b + c$.

Answer: $\boxed{23}$

Solution: Let AP and BQ intersect at Z. Since $BQ \perp AP$, it follows that $PCBZ$ is cyclic and the circumcircle of PQD passes through Z as well. X must lie on the radical axis of these circles, and so $XY \cdot XQ = XC \cdot XB$, by the converse of Power of a Point $BCYQ$ is cyclic. But since $\angle RQB = 90^\circ$ we know that $RCBQ$ is cyclic and R, Y, C, B, Q all lie on one circle.

Since $\angle PYQ = \angle PDQ = 90^\circ$, $\angle DCQ = \angle RYQ = 15^\circ$. So, $AQ = 10 - DQ = 10 - 10 \cdot \tan 15^\circ = 10 - (20 - 10\sqrt{3}) = 10\sqrt{3} - 10$ yielding $\boxed{23}$.

7. An infinitely large grid is filled such that each grid square contains exactly one of the digits $1, 2, 3, 4$, each digit appears at least once, and the digit in each grid square equals the digit located 5 squares above it as well as the digit located 5 squares to the right. A group of 4 horizontally adjacent digits or 4 vertically adjacent digits is chosen randomly, and depending on its orientation is read left to right or top to bottom to form an 4-digit integer. The expected value of this integer is also a 4-digit integer N. Find the last three digits of the sum of all distinct possible values of N.

Answer: $\boxed{555}$

Solution: First, see that the grid "repeats" in 5 by 5 blocks. Suppose a digit a in $\{1, 2, 3, 4\}$ appears n times in each 5 by 5 block. Suppose the digit 1 appears a times, the digit 2 appears b times, the digit 3 appears c times, the digit 4 appears d times. Then there is an $\frac{a}{25}$ chance of 1 appearing in the units digit, $\frac{a}{25}$ chance of 1 appearing in the tens digit, $\frac{a}{25}$ chance of 1 appearing in the hundreds digit, and $\frac{a}{25}$ chance of 1 appearing in the thousands digit. The same can be said for the other digits. By linearity of expectation we must have

$$\frac{1111 \cdot a + 2222 \cdot b + 3333 \cdot c + 4444 \cdot d}{25}$$

be our expected value, and clearly if this is an integer it must be a multiple of 1111, so it is either 2222 or 3333 (1111 and 4444 implies the grid being completely filled with 1s or 4s). Our final sum has its last three digits as $\boxed{555}$ and that is our answer.

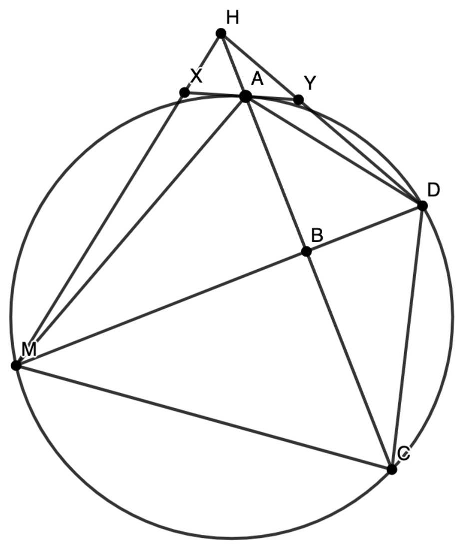
8. Let triangle MAD be inscribed in circle O with diameter 85 such that $MA = 68$ and $DA = 40$. The altitudes from M, D to sides AD and MA respectively intersect the tangent to circle O at A at X and Y respectively. $XA \times YA$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Answer: $\boxed{5453}$

Solution: By the extended Law of Sines, it can be found that $\sin\angle ADM = \frac{4}{5}$ and $\sin\angle AMD = \frac{8}{17}$. Let the foot of A to MD be B. It follows that $MD = MB + BD = 60 + 24 = 84$ and $AB = 32$. Extend AB to meet the circle O at C, as well as meet the altitudes from M, D to sides AD and MA respectively at H, the orthocenter of MAD. It can be computed by Power of a Point that $BC = 45$. It is well known that H's reflection over MD is C, so we have $HA = HB - BA = 45 - 32 = 13$.

$\angle HAX = \angle BAY = \angle CMA$ and $\angle XHA = \angle ACM$, so we have $\triangle HXA \sim \triangle CAM$. Similarly, $\triangle HYA \sim \triangle CAD$.

$\frac{XA}{12} = \frac{XA}{HA} = \frac{AM}{CM} = \frac{68}{75}$. $\frac{YA}{12} = \frac{YA}{HA} = \frac{AD}{CD} = \frac{40}{51}$. Multiplying XA and YA yields $\frac{5408}{45}$ yielding $\boxed{5453}$ as our final answer.



9. The infinite sequence of integers $a_1, a_2, \cdots$ is defined recursively as follows: $a_1 = 3$, $a_2 = 7$, and a_n equals the alternating sum $a_1 - 2a_2 + 3a_3 - 4a_4 + \cdots (n-1)a_{n-1}$ for all $n > 2$. Let a_x be the smallest positive multiple of 1090 appearing in this sequence. Find the remainder of a_x when divided by 113.

Answer: $\boxed{51}$

Solution: Notice that $a_n = a_{n-1} + (n-1)a_{n-1} = (n)a_{n-1}$ if n is even, and $a_n = a_{n-1} - (n-1)a_{n-1} = -(n-2)a_{n-1}$ if n is odd and greater than 1. Therefore, it can be seen that a_n must consist of either 1 or -1 then multiplied by the product of all the even numbers less than or equal to n and greater than 2, and all of the odd numbers less than or equal to $n-2$, multiplied by $a_3 = 3 - 2 \cdot 7 = -11$. It can be verified through simple induction that if n is 1 or 2 modulo 4 then a_n is positive, and if n is 0 or 3 modulo 4 then a_n is negative.

So, by analysis the first multiple of 109 appearing in this sequence must be a_{111} since 109 is prime, and it is obvious that it is a multiple of 10 as well. a_{111} equals 3 modulo 4 so it is negative, and thus we must instead go to a_{112} instead as our desired a_x, which then equals the positive integer $\frac{112!}{2} \cdot 11$. If this number equals m (mod 113), which is a prime, then $-11 \equiv 2m$ by Wilson's theorem and it follows that $m \equiv \boxed{51}$ (mod 113) which is our answer.

10. An *indivisible tiling* is a tiling of an $m \times n$ rectangular grid using only rectangles with a width and/or length of 1, such that nowhere in the tiling is a smaller complete tiling of a rectangle with more than 1 tile. Find the smallest integer a such that an indivisible tiling of an $a \times a$ square may contain exactly 2021 1×1 tiles.

Answer: $\boxed{91}$

Solution: We notice and prove two key facts.

1. No 1x1 tile can be on an edge.

2. No 1x1 tile can share a border or corner with another 1x1 tile.

Both of these follow from the requirement that every rectangle must have a length and/or a width of 1. Let a 1 x 1 square be on an edge. Then, we see that at least one of the rectangles bordering it must have its dimension of length 1 tangent to the 1 x 1 square. Thus, this rectangle combined with the 1 x 1 square will create a rectangular tiling of more than 1 tile. As for the second one, it follows in a similar way. If two unit tiles are adjacent, then obviously they both create a 2 x 1 rectangle consisting of 2 tiles. If they share a corner, then the rectangle that contains the square adjacent to both of them must have a dimension of length 1 that is tangent to at least one of the unit squares, which again as mentioned before will create a rectangular tiling of more than 1 tile.

So, we know that all 1x1 tiles must be at least a distance of 1 apart. The most space-efficient way to do this is to put every 1 x 1 tile in a rectangular or square grid, with each being exactly 1 unit apart. The diagram below shows a way in which this can be done:

We want to find the smallest possible value of a for which we can tile the 2021 1 x 1 tiles in this way. We can also see that the 1 x 1 tiles create a grid themselves. The smallest square greater than 2021 is 45^2, so with our knowledge from above, this 45 x 45 tiling of 1 x 1 squares spaced 1 unit apart yields $a = 91$. However, part of the tiling does not follow the rest of our pattern, as we must have *exactly* 2021 small tiles, and $45^2 = 2025$. We see below that four of the 'spots' in which small tiles can go can be replaced by a different configuration which satisfies the requirements. Thus, the smallest possible value of a is $\boxed{91}$

3 Round 3

All problems must be done in 60 minutes. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. Discussion is allowed ONLY among members of your team.

1. A man rows at a speed of 2 mph in still water. He set out on a trip towards a spot 2 miles downstream. He rowed with the current until he was halfway there, then turned back and rowed against the current for 15 minutes. Then, he turned around again and rowed with the current until he reached his destination. The entire trip took him 70 minutes. The speed of the current can be represented as $\frac{p}{q}$ mph where p, q are relatively prime positive integers. Find $10p + q$.

Answer: $\boxed{47}$

Solution: Let x be the speed of the current. The man rowed with the current for $\frac{1}{2+x}$ hours, then spent $\frac{1}{4}$ hours rowing back, covering $\frac{1}{4}(2-x)$ miles along the way. So, it takes him $\frac{1+\frac{1}{4}(2-x)}{2+x}$ hours to finish the last leg. This entire trip takes him $\frac{7}{6}$ hours, solving

$$\frac{1}{2+x}+\frac{1}{4}+\frac{1+\frac{1}{4}(2-x)}{2+x}=\frac{7}{6}$$

gives us $x=\frac{4}{7}$ yielding $\boxed{47}$.

2. The function $f(x)$ is defined on the reals such that

$$f\left(\frac{1-4x}{4-x}\right)=4-xf(x)$$

for all $x \neq 4$. There exists two distinct real numbers $a, b \neq 4$ such that $f(a)=f(b)=\frac{5}{2}$. $a+b$ can be represented as $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $10p+q$.

Answer: $\boxed{180}$

Solution: Notice that $\frac{1-4x}{4-x}$ has itself as its inverse, so we substitute $\frac{1-4x}{4-x}$ in the given expression to find that

$$f\left(x\right)=4-\frac{1-4x}{4-x}f\left(\frac{1-4x}{4-x}\right).$$

We treat these two equations as a system in two variables $f(x)$ and $f\left(\frac{1-4x}{4-x}\right)$ in terms of x, and solve for $f(x)$ to get

$$f(x)=\frac{6\left(x+1\right)}{2x^2-x+2}.$$

The equation

$$f(x)=\frac{6\left(x+1\right)}{2x^2-x+2}=\frac{5}{2}$$

can be simplified to the quadratic $10x^2-17x-2$ which has roots that sum to $\frac{17}{10}$, so our answer is $\boxed{180}$.

3. Two real numbers x, y are chosen randomly and independently on the interval $(1, r)$ where r is some real number between 1024 and 2048. Let P be the probability that $\lfloor \log_2 x \rfloor > \lfloor \log_2 y \rfloor$. The value of P is maximized when $r=\frac{p}{q}$ where p, q are relatively prime positive integers. Find $p+q$.

Answer: $\boxed{4100}$

Notice that the probability that $\lfloor \log_2 x \rfloor > \lfloor \log_2 y \rfloor$ equals the probability that $\lfloor \log_2 x \rfloor < \lfloor \log_2 y \rfloor$. So, we try to minimize the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$.

Observe that if $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$, then x and y are both between 2^p and 2^{p+1} for some nonnegative integer $p \leq 10$. To find this probability, we split into two cases. The first case is when x, y are both between 1024 and n, which we will let happen with probability b^2, where $b=\frac{r-1024}{r-1}$. In this case, obviously

$\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = 10$ is always true so we have $1 \cdot b^2 = b^2$.

If x, y are both between 1024 and 1, which will happen with probability $(1-b)^2$, then the probability that x and y are both between 2^p and 2^{p+1} for any nonnegative integer $p < 10$ is just

$$\frac{(2^{p+1}-2^p)^2}{(2^{10}-1)^2} = \frac{4^p}{(2^{10}-1)^2},$$

summing this up over all p yields the quadratic

$$\frac{4^{10}-1}{3(2^{10}-1)(2^{10}-1)}(1-b)^2 + b^2 = \frac{(4^{10}-1)(b^2-2b+1)+3(2^{10}-1)(2^{10}-1)b^2}{3(2^{10}-1)(2^{10}-1)}$$

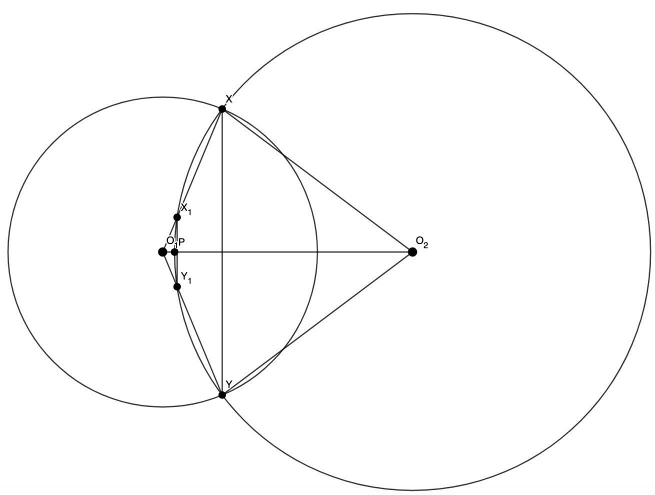
.

It is not hard to compute that the quadratic is minimized when $b = \frac{(4^{10}-1)}{(4^{10}-1)+3(1023)^2} = \frac{1025}{4094}$ and $r = \frac{4097}{3}$ yielding $\boxed{4100}$ as our final answer.

4. In 3-dimensional space, two spheres centered at points O_1 and O_2 with radii 13 and 20 respectively intersect in a circle. Points A, B, C lie on that circle, and lines O_1A and O_1B intersect sphere O_2 at points D and E respectively. Given that $O_1O_2 = AC = BC = 21$, DE can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers. Find $a+b+c$.

Answer: $\boxed{1552}$

Solution: First, take a cross section of the two spheres passing through O_1 and O_2.



We have two circles representing the two spheres, let them intersect at points X and Y. It is clear that XY is a diameter of the circle that is the intersection of spheres O_1 and O_2. Let O_1X and O_1Y intersect the circle centered at O_2 at X_1, Y_1 respectively. A dilation centered at O_1 maps the circle with diameter XY that A, B, C lie on to the circle with diameter X_1Y_1 that D and E lie on. Moreover, $\frac{DE}{AB}$ is the scale factor of the dilation which equals $\frac{O_1X_1}{O_1X}$.

Let P be the intersection of the circle centered O_2 and O_1O_2, it is clear that $O_1P = 21 - 20 = 1$. Thus, by power of a point $O_1X_1 = \frac{41}{13}$ and our desired scale factor equals $\frac{41}{169}$.

It remains to compute AB. It can be computed that X's distance to O_1O_2 equals 12 and thus $\triangle ABC$ has circumradius 12 and diameter 24. From there, it is not hard to see that $AB = \frac{21\sqrt{15}}{4}$ and $DE = \frac{861\sqrt{15}}{676}$ yielding a final answer of $\boxed{1552}$.

5. How many nonempty subsets of $1, 2, ..., 15$ are there such that the sum of the squares of each subset is a multiple of 5?

Answer: $\boxed{7583}$

Solution: Consider the generating function

$$F(x) = \prod_{k=1}^{15}(1 + x^{k^2}),$$

we can apply Roots of Unity Filter on this product, then subtract 1 to get the desired answer. We simply compute:

$$F(1) = 2^{15}$$

$$F(e^{2\pi/5}) = (1 + e^{2\pi/5})^6(1 + e^{8\pi/5})^6 \cdot 2^3$$

$$F(e^{4\pi/5}) = (1 + e^{4\pi/5})^6(1 + e^{6\pi/5})^6 \cdot 2^3$$

$$F(e^{6\pi/5}) = (1 + e^{6\pi/5})^6(1 + e^{4\pi/5})^6 \cdot 2^3$$

$$F(e^{8\pi/5}) = (1 + e^{8\pi/5})^6(1 + e^{2\pi/5})^6 \cdot 2^3$$

We can compute this sum by noticing that $(1 + e^{2\pi/5})(1 + e^{8\pi/5}) = 2 + e^{2\pi/5} + e^{8\pi/5} = \frac{3}{2} + \frac{\sqrt{5}}{2}$ and $(1 + e^{4\pi/5})(1 + e^{6\pi/5}) = 2 + e^{4\pi/5} + e^{6\pi/5} = \frac{3}{2} - \frac{\sqrt{5}}{2}$. From there, see that

$$\begin{aligned}
\frac{F(1) + F(e^{2\pi/5}) + F(e^{4\pi/5}) + F(e^{6\pi/5}) + F(e^{8\pi/5})}{5} - 1 &= \frac{2^{15} + 2^4((\frac{3}{2} + \frac{\sqrt{5}}{2})^6 + (\frac{3}{2} - \frac{\sqrt{5}}{2})^6)}{5} - 1 \\
&= \frac{2^{15} + 16 \cdot 322}{5} - 1 \\
&= \frac{37920}{5} - 1 \\
&= \boxed{7583}
\end{aligned}$$

which is our final answer.

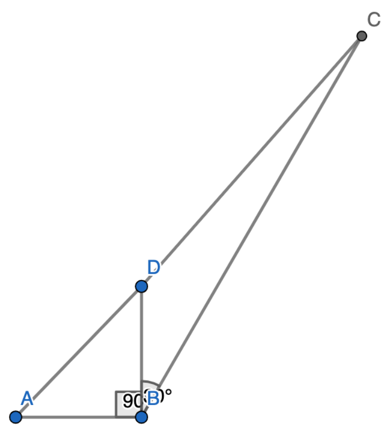
6. Find the minimum possible value of

$$\left(\sqrt{x^2 + 4} + \sqrt{x^2 + 7\sqrt{3}x + 49}\right)^2$$

over all real numbers.

Answer: $\boxed{67}$

Solution: Notice that since the only term that depends on the sign of x is $7\sqrt{3}x$, then the minimum is achieved when x is negative. Substitute $-a$ in for x for some positive real a. Now, notice that $\sqrt{a^2+4}$ is the hypotenuse of a right triangle with lengths 2 and x, and $\sqrt{a^2-7\sqrt{3}a+49}$ is the length of the third side of a triangle with side lengths a and 7, such that it is facing an angle with measure $30°$. We "join" these two triangles together by their side with length a, to get this following diagram:



Notice that we just have to minimize $(AD+DC)^2$ from here. This occurs when A, D, C are collinear. From there we just use Law of Cosines to get our final answer of $2^2+7^2-2\cdot 2\cdot 7\cdot(-1/2) = \boxed{67}$.

7. Find the number of ordered triples of integers (a, b, c) such that

$$a^2+b^2+c^2-ab-bc-ca-1 \leq 4042b-2021a-2021c-2021^2$$

and $|a|, |b|, |c| \leq 2021$.

Answer: $\boxed{14152}$

Solution: Rearrange the given to find that

$$\frac{(a-b+2021)^2}{2}+\frac{(c-b+2021)^2}{2}+\frac{(c-a)^2}{2} \leq 1.$$

Then we know either $a=c=b-2021$ causing the left hand side to equal 0, or $a=c+1=b-2021$, $a=c-1=b-2021$, $a+1=c=b-2021$, $a-1=c=b-2021$, $a=c=b-2021+1$, or $a=c=b-2021-1$ each causing the left hand side to equal 1.

Since $|a|, |b|, |c| \leq 2021$, it can be verified that these cases have $2022, 2022, 2021, 2022, 2021, 2023$ and 2021 solutions yielding a sum of $\boxed{14152}$.

Note: This problem was initially stated incorrectly, making it unreasonable to solve. All points have since been compensated for this incorrect problem.

8. Triangle ABC has circumcircle ω. The angle bisectors of $\angle A$ and $\angle B$ intersect ω at points D and E respectively. DE intersects BC and AC at X and Y respectively. Given $DX = 7$, $XY = 8$, and $YE = 9$, the area of $\triangle ABC$ can be written as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers, $\gcd(a, c) = 1$, and b is square free. Find $a + b + c$.

Answer: $\boxed{2398}$

Solution: Let M be the midpoint of XY and let N be the midpoint of DE. Let O be the circumcenter of $\triangle ABC$. By $\angle CXY = \angle CYX$ so $\triangle CXY$ is isosceles and $CM \perp DE$, and since DE is a chord, $ON \perp DE$ as well.

Let $ON = x$. By the inscribed angle formula it can be seen $\angle DOE = 180 - \angle ABC$. It follows that $\triangle ONE \sim \triangle XMC$ and since $XM = 4$ and $NE = 12$, $CM = \frac{48}{x}$. Since CO and OE are radii, and $MN = 1$, we have

$$\left(x + \frac{48}{x}\right)^2 + 1 = 144 + x^2$$

and solving this equation yields $ON = x = \frac{48}{\sqrt{47}}$ and $CM = \sqrt{47}$. By Pythagorean theorem $CX = CY = 3\sqrt{7}$.

Let OD intersect BC at F and let OE intersect AC at G. By similar triangle ratios $XF = \frac{4\sqrt{7}}{3}$ and $YG = \frac{12\sqrt{7}}{7}$. Since F and G are midpoints $BC = \frac{26\sqrt{7}}{3}$ and $AC = \frac{66\sqrt{7}}{7}$.

By the double angle formula $\sin \angle BCA = \frac{8\sqrt{47}}{63}$. From there it can computed that the area of $\triangle ABC$ is

$$\frac{2288\sqrt{47}}{63}$$

and our answer is $2288 + 47 + 63 = \boxed{2398}$ as desired.

9. There is a 4×4 array of nonnegative integers A, all initially equal to 0. An operation may be performed on the array for any row or column such that every number in that row or column is incremented by 1, then replaced with its remainder when divided by 3. Given a random 4×4 array of nonnegative integers between 0 and 2 not identical to A, the probability that it can be reached through a series of operations on A is $\frac{p}{q}$, where p, q are relatively prime positive integers. Find p.

Answer: $\boxed{1093}$

Solution: Let G be the group on valid operation of addition. Obviously G is abelian and has at least 8 elements (which are generators) corresponding to singular applications of each of the row and column operations. Let these operations be a, b, c, d for the rows and e, f, g, h for the columns. It's not hard to see that the following hold:

$$a^3 = b^3 = c^3 = d^3 = e^3 = f^3 = g^3 = h^3 = 1$$

$$abcd = efgh$$

Let G' be the largest abelian group with generating set a, b, c, d, e, f, g, h satisfying the above constraints. It's not hard to see that

$$|G'| = 3^7$$

as $h = abcde^-1f^-1g^-1$ implies that G' can be defined as teh group with generating set a, b, c, d, e, f, g satisfying

$$a^3 = b^3 = c^3 = d^3 = e^3 = f^3 = g^3 = h^3 = 1.$$

Claim: $G \cong G'$ Proof: Suppose otherwise. Then there exist distinct $(x_1, \dots x_8), (y_1, \dots y_8) \in (/3\mathbb{Z})^3$ with $x_8 = y_8$ such that

$$a^{x_1} \dots h^{x_8} = a^{y_1} \dots h^{y_8}$$

holds in G. But this implies that

$$a^{x_1-y_1} b^{x_2-y_2} \dots g^{x_7-y_7} = 1$$

so there must be a way of performing a set of row/column actions on the zero matrix to return it to its original state without using the first row and using at least one other action (considered mod 3). But this sequence of actions does not affect the first row, so none of the column actions are used. From there, it can be seen that no row actions are used either - this is a contradiction. Trivially, $|G| = |G'| = 3^7$. Obviously, the probability is trivially $\frac{3^7-1}{3^{16}-1}$. The common divisor between the 2 is 2, so we divide both the numerator and denominator by 2. Eventually, we get the answer to be $\boxed{1093}$.

10. Positive integers a, b, c exist such that $a+b+c+1$, $a^2+b^2+c^2+1$, $a^3+b^3+c^3+1$, and $a^4+b^4+c^4+7459$ are all multiples of p for some prime p. Find the sum of all possible values of p less than 1000.

Answer: $\boxed{59}$

Solution: The condition that a, b, c is positive is irrelevant as we can simply work modulo p. It is obvious that $p = 3$ works, simply substitute in $a = 1, b = 1, c = 0$.

Suppose $p \neq 3$, then we have $(a + b + c)^2 \equiv 1 \pmod p$. It follows that $ab + bc + ca \equiv 1 \pmod p$. We also have $(a + b + c)(a^2 + b^2 + c^2) \equiv 1 \pmod p \implies ab^2 + ba^2 + bc^2 + cb^2 + ac^2 + ca^2 \equiv 2 \pmod p$.
But

$$ab^2 + ba^2 + bc^2 + cb^2 + ac^2 + ca^2 = (ab + bc + ac)(a + b + c) - 3abc$$

So then we know that $abc \equiv -1 \pmod p$.

But then $-7459 \equiv a^4 + b^4 + c^4 = (a + b + c)(a^3 + b^3 + c^3) - (ab + bc + ac)(a^2 + b^2 + c^2) + abc(a + b + c) \equiv 3 \pmod p$. So, p divides 7462 and p is either 7, $13, 41$ or 2. 2 trivially works.

Notice that $5^2 \equiv (-5)^2 \equiv -1 \pmod{13}$. Motivated by this, we construct the following solution for $p = 13$:

$$5 + (-5) + -1 + 1 \equiv 0 \pmod{13}$$

$$5^2 + (-5)^2 + (-1)^2 + 1 \equiv 0 \pmod{13}$$

$$5^3 + (-5)^3 + (-1)^3 + 1 \equiv (-5) + 5 - 1 + 1 \equiv 0 \pmod{13}$$

$$5^4 + (-5)^4 + (-1)^4 + 673 \equiv 1 + 1 + 1 - 3 \equiv 0 \pmod{13}$$

So, $p = 13$ is valid. Using $9^2 \equiv (-9)^2 \equiv -1 \pmod{41}$, we can similarly generate a solution for $p = 41$.

Suppose we had a solution for $p = 7$. Since cubes can only be $-1, 0, 1 \pmod 7$ we have either two of a^3, b^3, c^3 are 0 and the other is -1, or two of a^3, b^3, c^3 are -1 and the other is 1 (mod 7). In the former, WLOG let $a^3 \equiv b^3 \equiv 0 \pmod 7$ and $c^3 \equiv -1$, then obviously for $a + b + c + 1$ to be a multiple of 7 $c = -1$ and $a^2 + b^2 + c^2 \equiv 1 \pmod 7$, not possible. In the latter, two of a, b, c must be one of $3, 5, 6$ and the other must be one of $1, 2, 4$ modulo 7.

Now, it is not hard to verify that the only way for $a + b + c + 1$ to be a multiple of 7 is for a, b, c to be $3, 6, 4$; $5, 6, 2$; or $6, 6, 1$ in some order modulo 7. In each case $a^2 + b^2 + c^2 + 1$ is not a multiple of 7. So, there do not exist solutions when $p = 7$. Our sum is thus $2 + 3 + 13 + 41 = \boxed{59}$.