Online Monmouth Math Competition Round 1

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This document contains Round 1 of the Online Monmouth Math Competition. All problems have positive integer answers. Scrap work or justification for your answers is neither required nor accepted. There are 15 problems in this round. You are required to have your camera on in the Zoom Meeting for the duration of the round.

All problems must be done in 50 minutes and are roughly ordered by difficulty. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator (addition, subtraction, multiplication, division). In particular, Desmos, Wolfram Alpha, Geogebra, and other similar websites are forbidden. No discussion is allowed until discussion has opened. Before then, do not discuss these problems with anyone, not even members of your own team. We have provided an extra 5 minute grace period for submitting answers. Keep in mind that once the time is up, your answers may be invalidated. Because of this, we ask that you submit early.

The problems approximate the difficulty of the AMC 10/12 math competitions. It is important to mention that these problems can be challenging to many, and failure to solve a problem does not and should not in any way signal inadequacy. Math competitions differ from normal tests in that there is no ”passing” or ”failing” score, rather any question solved is worth high commendation. Similarly, one should not feel inclined to boast about his/her score to others, however high it may be.

After recording all your answers on the timed Google Form, check your work and submit before the timer is up, in which case you can log off the zoom call until you take another round. Incorrect answers will not be penalized. Each correct answer in this round contributes 4 points to your total individual score.

Any questions on the above should be emailed to officialommc@gmail.com. With that, good luck! The OMMC team has spent a lot of time on this contest and we hope that you enjoy your OMMC experience.

From the OMMC staff

1 Round 1

All problems must be done in 50 minutes. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. No discussion is allowed until discussion is opened.

1. Find the remainder when

$$20^{20} + 21^{21} - 21^{20} - 20^{21}$$

is divided by 100.

2. There are a family of 5 siblings. They have a pile of at least 2 candies and are trying to split them up amongst themselves. If the 2 oldest siblings share the candy equally, they will have 1 piece of candy left over. If the 3 oldest siblings share the candy equally, they will also have 1 piece of candy left over. If all 5 siblings share the candy equally, they will also have 1 piece left over. What is the minimum amount of candy required for this to be true?

3. Define $f(x)$ as $\frac{x^2-x-2}{x^2+x-6}$. $f(f(f(f(1))))$ can be expressed as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $10p + q$.

4. Robert tiles a 420 x 420 square grid completely with 1 x 2 blocks, then notices that the two diagonals of the grid pass through a total of n blocks. Find the sum of all possible values of n.

5. Two points A, B are randomly chosen on a circle with radius 100. For a positive integer x, denote P(x) as the probability that the length of AB is less than x. Find the minimum possible integer value of x such that $\mathrm{P}(x) > \frac{2}{3}$.

6. Jason and Jared take turns placing blocks within a game board with dimensions 3 x 300, with Jason going first, such that no two blocks can overlap. The player who cannot place a block within the boundaries loses. Jason can only place 2 x 100 blocks, and Jared can only place 2 x n blocks where n is some positive integer greater than 3. Find the smallest integer value of n that still guarantees Jason a win (given both players are playing optimally).

7. Derek fills a square 10 by 10 grid with 50 1s and 50 2s. He takes the product of the numbers in each of the 10 rows. He takes the product of the numbers in each of the 10 columns. He then sums these 20 products up to get an integer N. Find the minimum possible value of N.

8. The function $g(x)$ is defined as $\sqrt{\dfrac{x}{2}}$ for all positive x.

$$g\left(g\left(g\left(g\left(g\left(\frac{1}{2}\right)+1\right)+1\right)+1\right)+1\right)$$

can be expressed as $\cos(b)$ using degrees, where $0° < b < 90°$ and $b = p/q$ for some relatively prime positive integers p, q. Find $p + q$.

9. The difference between the maximum and minimum values of

$$2\cos 2x + 7\sin x$$

over the real numbers equals $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

10. How many ways are there to arrange the numbers 1 through 8 into a 2 by 4 grid such that the sum of the numbers in each of the two rows are all multiples of 6, and the sum of the numbers in each of the four columns are all multiples of 3?

11. In equilateral triangle XYZ with side length 10, define points A, B on XY, points C, D on YZ, and points E, F on ZX such that $ABDE$ and $ACEF$ are rectangles, $XA < XB$, $YC < YD$, and $ZE < ZF$. The area of hexagon $ABCDEF$ can be written as $\sqrt{x}$ for some positive integer x. Find x.

12. Let $P(x) = x^3 + 8x^2 - x + 3$ and let the roots of P be a, b, and c. The roots of a monic polynomial $Q(x)$ are $ab - c^2, ac - b^2, bc - a^2$. Find $Q(-1)$.

13. Find the number of nonnegative integers $n < 29$ such that there exists positive integers x, y where

$$x^2 + 5xy - y^2$$

has remainder n when divided by 29.

14. There exist positive integers N, M such that N's remainders modulo the four integers $6, 36$, 216, and M form an increasing nonzero geometric sequence in that order. Find the smallest possible value of M.

15. A point X in the coordinate plane exactly $\sqrt{2} - \frac{\sqrt{6}}{3}$ away from the origin is chosen randomly. A point Y less than 4 away from the origin is chosen randomly. The probability that a point Z less than 2 away from the origin forms an equilateral triangle $\triangle XYZ$ with the points X, Y can be expressed as $\frac{a\pi + b}{c\pi}$ for some positive integers a, b, c with a and c relatively prime. Find $a + b + c$.

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