Online Monmouth Math Competition Round 2

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This document contains Round 2 of the Online Monmouth Math Competition. All problems have positive integer answers. Scrap work or justification for your answers is neither required nor accepted. There are 10 problems in this round. You are required to have your camera on in the Zoom Meeting for the duration of the round.

All problems must be done in 2 hours (120 minutes). No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator (addition, subtraction, multiplication, division). In particular, Desmos, Wolfram Alpha, Geogebra, and other similar websites are forbidden. No discussion is allowed until discussion has opened. Before then, do not discuss these problems with anyone, not even members of your own team. We have provided an extra 5 minute grace period for submitting answers. Keep in mind that once the time is up, your answers may be invalidated. Because of this, we ask that you submit early.

The problems approximate the difficulty of the earlier end of the AIME math competition. It is important to mention that these problems can be challenging to many, and failure to solve a problem does not and should not in any way signal inadequacy. Math competitions differ from normal tests in that there is no ”passing” or ”failing” score, rather any question solved is worth high commendation. Similarly, one should not feel inclined to boast about his/her score to others, however high it may be.

After recording all your answers on the timed Google Form, check your work and submit before the timer is up, in which case you can log off the call until you take another round. Incorrect answers will not be penalized. Each correct answer in this round contributes 6 points to your total individual score.

Any questions on the above should be emailed to officialommc@gmail.com. With that, good luck! The OMMC team has spent a lot of time on this contest and we hope that you enjoy your OMMC experience.

From the OMMC staff

1 Round 2

All problems must be done in 2 hours. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. No discussion is allowed until discussion is opened.

1. There are 20 people in a particular social network. Each person follows exactly 2 others in this network, and also has 2 people following her as well. What is the maximum possible number of people that can be placed into a subset of the network such that no one in this subset follows someone else in the subset?

2. Sequences a_n and b_n are defined for all positive integers n such that $a_1 = 5$, $b_1 = 7$,

$$a_{n+1} = \frac{\sqrt{(a_n + b_n - 1)^2 + (a_n - b_n + 1)^2}}{2},$$

and

$$b_{n+1} = \frac{\sqrt{(a_n + b_n + 1)^2 + (a_n - b_n - 1)^2}}{2}.$$

How many integers n from 1 to 1000 satisfy the property that a_n, b_n form the legs of a right triangle with a hypotenuse that has integer length?

3. The intersection of two squares with perimeter 8 is a rectangle with diagonal length 1. Given that the distance between the centers of the two squares is 2, the perimeter of the rectangle can be expressed as P. Find $10P$.

4. The sum

$$\frac{1^2 - 2}{1!} + \frac{2^2 - 2}{2!} + \frac{3^2 - 2}{3!} + \cdots + \frac{2021^2 - 2}{2021!}$$

can be expressed as a rational number N. Find the last 3 digits of $2021! \cdot N$.

5. Let N be an 3 digit integer in base 10 such that the sum of its digits in base 4 is half the sum of its digits in base 8. In base 10, find the largest possible value of N.

6. In square $ABCD$ with $AB = 10$, point P, Q are chosen on side CD and AD respectively such that $BQ \perp AP$, and R lies on CD such that $RQ \parallel PA$. BC and AP intersect at X, and XQ intersects the circumcircle of PQD at Y. Given that $\angle PYR = 105^\circ$, AQ can be expressed in simplest radical form as $b\sqrt{c} - a$ where a, b, c are positive integers. Find $a + b + c$.

7. An infinitely large grid is filled such that each grid square contains exactly one of the digits $\{1, 2, 3, 4\}$, each digit appears at least once, and the digit in each grid square equals the digit located 5 squares above it as well as the digit located 5 squares to the right. A group of 4 horizontally adjacent digits or 4 vertically adjacent digits is chosen randomly, and depending on its orientation is read left to right or top to bottom to form an 4-digit integer. The expected value of this integer is also a 4-digit integer N. Given this, find the last three digits of the sum of all possible values of N.

8. Let triangle MAD be inscribed in circle O with diameter 85 such that $MA = 68$ and $DA = 40$. The altitudes from M, D to sides AD and MA, respectively, intersect the tangent to circle O at A at X and Y respectively. $XA \times YA$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

9. The infinite sequence of integers $a_1, a_2, \cdots$ is defined recursively as follows: $a_1 = 3$, $a_2 = 7$, and a_n equals the alternating sum

$$a_1 - 2a_2 + 3a_3 - 4a_4 + \cdots (-1)^n \cdot (n-1)a_{n-1}$$

for all $n > 2$. Let a_x be the smallest positive multiple of 1090 appearing in this sequence. Find the remainder of a_x when divided by 113.

10. An *indivisible tiling* is a tiling of an $m \times n$ rectangular grid using only rectangles with a width and/or length of 1, such that nowhere in the tiling is a smaller complete tiling of a rectangle with more than 1 tile. Find the smallest integer a such that an indivisible tiling of an $a \times a$ square may contain exactly 2021 1×1 tiles.

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