Online Monmouth Math Competition Round 3

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February 2021

This document contains Round 3 of the Online Monmouth Math Competition. All problems have positive integer answers. Scrap work or justification for your answers is neither required nor accepted. There are 10 problems in this round. All members of the team are required to have their camera on in the Zoom Meeting for the duration of the round.

All problems must be done in 1 hour (60 minutes). No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator (addition, subtraction, multiplication, division). In particular, Desmos, Wolfram Alpha, Geogebra, and other similar websites are forbidden. No discussion is allowed with other teams until discussion has opened. We have provided an extra 5 minute grace period for submitting answers. Keep in mind that once the time is up, your answers may be invalidated. Because of this, we ask that you submit early.

The problems approximate the difficulty of the AIME math competition. It is important to mention that these problems can be challenging to many, and failure to solve a problem does not and should not in any way signal inadequacy. Math competitions differ from normal tests in that there is no "passing" or "failing" score, rather any question solved is worth high commendation. Similarly, a team should not feel inclined to boast about their score to others, however high it may be.

After one member of the team records all of the team's answers on the timed Google Form, the team should check answers and submit before the timer is up, in which case the entire team can log off the call until individual members of the team decide to start another round. Only one set of answers from the team is required. Multiple members of the team can submit answers, in which case only the last recorded submission will be counted. For this reason, we recommend assigning one person in the team to submit answers for the round. Incorrect answers will not be penalized. Each correct answer in this round contributes 8 points to a team's Team Score.

Any questions on the above should be emailed to officialommc@gmail.com. With that, good luck! The OMMC team has spent a lot of time on this contest and we hope that you enjoy your OMMC experience.

From the OMMC staff

1 Round 3

All problems must be done in 2 hours. No aids are allowed besides scrap paper, writing utensils, a compass and a straightedge, and a four function calculator. No discussion is allowed besides among members of your team until discussion is opened.

1. A man rows at a speed of 2 mph in still water. He set out on a trip towards a spot 2 miles downstream. He rowed with the current until he was halfway there, then turned back and rowed against the current for 15 minutes. Then, he turned around again and rowed with the current until he reached his destination. The entire trip took him 70 minutes. The speed of the current can be represented as $\frac{p}{q}$ mph where p, q are relatively prime positive integers. Find $10p + q$.

2. The function $f(x)$ is defined on the reals such that

$$f\left(\frac{1-4x}{4-x}\right) = 4 - xf(x)$$

for all $x \neq 4$. There exists two distinct real numbers $a, b \neq 4$ such that $f(a) = f(b) = \frac{5}{2}$. $a + b$ can be represented as $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $10p + q$.

3. Two real numbers x, y are chosen randomly and independently on the interval $(1, r)$ where r is some real number between 1024 and 2048. Let P be the probability that $\lfloor \log_2 x \rfloor > \lfloor \log_2 y \rfloor$. The value of P is maximized when $r = \frac{p}{q}$ where p, q are relatively prime positive integers. Find $p + q$.

4. In 3-dimensional space, two spheres centered at points O_1 and O_2 with radii 13 and 20 respectively intersect in a circle. Points A, B, C lie on that circle, and lines O_1A and O_1B intersect sphere O_2 at points D and E respectively. Given that $O_1O_2 = AC = BC = 21$, DE can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers. Find $a + b + c$.

5. How many nonempty subsets of $1, 2, \ldots, 15$ are there such that the sum of the squares of each subset is a multiple of 5?

6. Find the minimum possible value of

$$\left(\sqrt{x^2+4} + \sqrt{x^2 + 7\sqrt{3}x + 49}\right)^2$$

over all real numbers.

7. Find the number of ordered triples of integers (a, b, c) such that

$$a^2 + b^2 + c^2 - ab - bc - ca - 1 \leq 4042b - 2021a - 2021c + 2021^2$$

and $|a|, |b|, |c| \leq 2021$.

8. Triangle ABC has circumcircle ω. The angle bisectors of $\angle A$ and $\angle B$ intersect ω at points D and E respectively. DE intersects BC and AC at X and Y respectively. Given $DX = 7$, $XY = 8$ and $YE = 9$, the area of $\triangle ABC$ can be written as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers, $\gcd(a, c) = 1$, and b is square free. Find $a + b + c$.

9. There is a 4×4 array of integers A, all initially equal to 0. An operation may be performed on the array for any row or column such that every number in that row or column has 1 added to it, and then is replaced with its remainder modulo 3. Given a random 4×4 array of integers between 0 and 2 not identical to A, the probability that it can be reached through a series of operations on A is $\frac{p}{q}$, where p, q are relatively prime positive integers. Find p.

10. Positive integers a, b, c exist such that $a+b+c+1$, $a^2+b^2+c^2+1$, $a^3+b^3+c^3+1$, and $a^4+b^4+c^4+7459$ are all multiples of p for some prime p. Find the sum of all possible values of p less than 1000.

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