OMMC Shortlist 2022-2023

OMMC Team

2022-2023

Guidelines

All problems should be original, thought-provoking, and around AMC 10 to AIME level. Answers and solution sketches are mandatory for your problem to be considered. Please claim your problems, with your full name preferably but online aliases if you really need to. Any problem not satisfying the conditions outlined above is subject to deletion. Place problems in their appropriate categories. Some problems fit into multiple categories, for those just choose any category that fits.

Example problem (not this block, only its contents)

Problem 0.0.1. Problem Text Here

Answer: $\boxed{69420}$

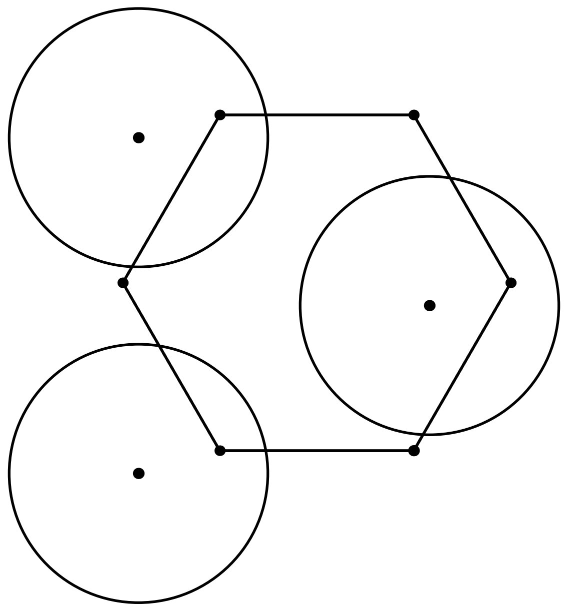
Solution: Solution Sketch Here

Contents

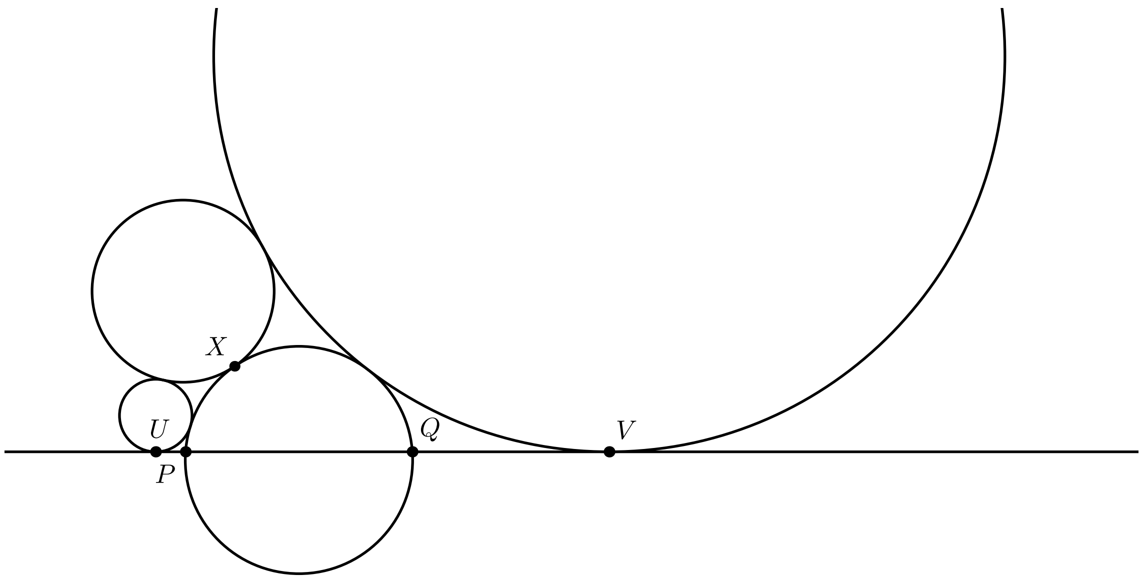
1 Shortlist, by subject

1.1 Geometry

Problem G1 (squareman). Regular hexagon $ABCDEF$ has area 13. Circles with area 6 centered at points O_1, O_2, O_3 intersect the hexagon. O_1ACO_2 and O_1AEO_3 are parallelograms. Find the area inside $ABCDEF$ outside any of the three circles.



Problem G2 (squareman). Define circles $\omega_1, \omega_2, \omega_3, \omega_4$ so ω_i is externally tangent to ω_{i+1} for all $1 \le i \le 4$ where $\omega_5 = \omega_1$. Furthermore, ω_1 and ω_3 are externally tangent at X. A common external tangent to ω_2, ω_4 at U, V respectively intersects ω_1 at P, Q so that U, P, Q, V lie in that order. If $PQ = 9, UV = 18, PX = 4, XQ = 8$, find UP.



Solution 1, by inversion (nyan) Let $PU = 9x$, so that $PV = 9(2-x)$. Invert at P with radius 3, and let * denote image under this inversion. By inversion distance formula we may compute $Q^*X^* = 2$. Also, $Q^*V^* = 1 - \frac{1}{2-x}$ and $Q^*U^* = 1 + \frac{1}{x}$.

Problem inverted

Lines ℓ_1 and $\ell_2 = \omega_4^*$ meet at Q^* with angle $\cos^{-1}\frac{1}{64}$. Circles ω_1^*, ω_3^* touch ℓ_1, ℓ_2, and respectively touch the former at U^*, V^*. A third circle ω_2^* (sic) touches $\omega_1^*, \omega_3^*, \ell_2$. Given that $Q^*X^* = 2$, $Q^*V^* = 1 - \frac{1}{2-x}$, and $Q^*U^* = 1 + \frac{1}{x}$, what is $9x$?

Consider a second inversion, at Q^* preserving ω_2^* and thus has power Q^*X^{*2}. As it swaps ω_1^*, ω_3^*, it also swaps their tangency points U^*, V^* on ℓ_1, so $Q^*X^{*2} = Q^*U^* \cdot Q^*V^*$. Plugging in their expressions in x yields

$$\left(1 + \frac{1}{x}\right)\left(1 - \frac{1}{2-x}\right) = 4 \Rightarrow 3x^2 - 8x + 1 = 0 \Rightarrow 9x = \boxed{3(4 - \sqrt{13}).}$$

Note. We take the smaller root because. . . wait why thonk

Problem G3 (nyan). In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.

Solution: For brevity let ℓ be the perpendicular bisector of $\overline{AC}$ aka the perpendicular from P to $\overline{AC}$.

Claim – $\angle BOD = 120°$.

Proof. Because P lies on $\overline{BD}$ and the perpendicular bisector of $\overline{OC}$ which are supposed to be parallel, they must be coincident, directly implying the claim. □

Claim – X is the orthocenter of $\triangle ABD$.

Proof. Proceed by phantom points, letting H be the mentioned orthocenter. Then, assuming $ABCD$ is oriented clockwise, $\measuredangle BHD = 60° = \measuredangle BOD$ so $H \in (OBD)$;
Now we use the lemma that in a triangle with an $60°$ angle, that vertex is equidistant from the circumcenter and orthocenter. As C, A are the circumcenter and orthocenter of $\triangle BHD$, this means $HC = HA$ and $H \in \ell$ whence $H = X$, as needed. □

Now, to the answer extraction... in some horrible notation, let $BH = u < v = HD$.

Then using the fact that $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$ in a general triangle ABC with $R = a/\sqrt{3}$ and $a^2 = b^2 - bc + c^2$, it follows that $AC = |u - v|$ in the actual problem. We get the following system (second equation by law of cosines):

$$\begin{cases} (u - v)^2 = AC^2 = 78; \\ u^2 - uv + v^2 = BD^2 = 169. \end{cases}$$

Homogenizing gives

$$7u^2 - 20uv + 7v^2 = 0 \Rightarrow \frac{u}{v} = \frac{10 \pm \sqrt{51}}{7} \Rightarrow \boxed{68}.$$

Problem G4 (nyan). Tetrahedron $ABCD$ has the property that $\overline{DA}, \overline{DB}, \overline{DC}$ are mutually perpendicular. Define the **A-exsphere** to be the sphere tangent to face BCD and to the extensions of the other faces. We also define the other exspheres similarly. Let the radius of such a sphere be called an exradius. It is known that $DA^{-2} + DB^{-2} + DC^{-2} = 9409$. If the A-,B-, C-exradii are $1/138, 1/168, 1/152$, respectively, then the D-exradius may be expressed as a/b for coprime positive integers a, b. Find $a + b$.

Solution: Let the 'legs' be $DA, DB, DC = a, b, c$, and we may let the vertices be $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c), D = (0, 0, 0)$. Hence plane BCD has equation $x/a + y/b + z/c = 1$. Let the exradii be r_a, etc.

Claim – $1/r_a = -1/a + 1/b + 1/c + \sqrt{\sum_{\text{cyc}} a^{-2}}$.

Proof. The A-excenter is of the form $(-r_a, r_a, r_a)$, and is r_a from face BCD, so applying the (directed) distance formula gives:

$$\frac{(-r_a)/a + r_a/b + r_a/c - 1}{\sqrt{\sum_{\text{cyc}} a^{-2}}} = -r_a.$$

Solving for $1/r_a$ gives $1/r_a = -1/a + 1/b + 1/c + \sqrt{\sum_{\text{cyc}} a^{-2}}$ as claimed. $\square$

Note. A similar calculation gives

$$1/r_d = \sum_{\text{cyc}} a^{-1} - \sqrt{\sum_{\text{cyc}} a^{-2}}.$$

Summing the claim statement cyclically gives $\sum_{\text{cyc}}(1/r_a) = 1/a + 1/b + 1/c + 3\sqrt{\sum_{\text{cyc}} a^{-2}}$. As we are given $\sqrt{\sum_{\text{cyc}} a^{-2}} = \sqrt{9409} = 97$, we get

$$1/r_d = \sum_{\text{cyc}} 1/r_a - 4\sqrt{\sum_{\text{cyc}} a^{-2}} = 70;$$

$$r_d = 1/70 \Rightarrow \boxed{071}.$$

Remark. The original legs intended were $a, b, c = 1/63, 1/48, 1/56$.
also imagine coordbashing

Problem G5 (ryan (modified by squareman)). Define $\triangle ABC$ such that $\angle A = 90^\circ$ and $\angle B = 30^\circ$. Define point P inside $\triangle ABC$ so $BC \cdot PC = 10$ and $\triangle BPC$ has area 3. If $AP = 2$, then APC's area is $\sqrt{a} - \frac{b}{c}$ for positive integers a, b, c in simplest radical form. Find $a + b + c$

Solution: From $\triangle BPC$, we see that

$$3 = \frac{1}{2} BC \cdot CP \sin \angle BCP \implies \sin \angle BCP = \frac{3}{5}.$$

Furthermore, $\cos BCP = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$. We can then compute

$$\sin \angle PCA = \sin(60^\circ - \angle BCP) = \frac{\sqrt{3}}{2} \cos \angle BCP - \frac{1}{2} \angle BCP = \frac{2}{5}\sqrt{3} - \frac{3}{10}.$$

Then,

$$\begin{aligned}
[APC] &= \frac{1}{2} AC \cdot PC \sin \angle PCA \\
&= \frac{1}{2}\left(\frac{2}{5}\sqrt{3} - \frac{3}{10}\right) \cdot PC \cdot BC \cos 60^\circ \\
&= \frac{5}{2}\left(\frac{2}{5}\sqrt{3} - \frac{3}{10}\right) \\
&= \sqrt{3} - \frac{3}{4} \\
&\implies \boxed{10}.
\end{aligned}$$

Problem G6 (Cyclic). Let ABC be an acute triangle with circumcenter O. Let the circumcircle of $\triangle ABO$ intersect segment BC at $D \neq B$, segment AC at $F \neq A$, and the Euler line of $\triangle ABC$ at $P \neq O$. Let the circumcircle of $\triangle ACO$ intersect segment BC at $E \neq C$. Let $\overline{BC}$ and $\overline{FP}$ intersect at X, with C between B and X. If $BD = 13$, $EC = 8$, and $CX = 27$, find DE.

Solution: Let $\measuredangle$ denote directed angles $\bmod 180^\circ$. Let H be the orthocenter of ABC, and let $\overline{OH}$ intersect the circumcircle of ACO at $Q \neq O$ and $\overline{BC}$ at Y.

Since $\measuredangle AFP = \measuredangle AOP = \measuredangle ACQ$, $\overline{FP} \parallel \overline{CQ}$, so $\triangle YPX \sim \triangle YQC$. We have $\angle CDO = 180^\circ - \angle BDO = \angle BAO = 90^\circ - \angle ACB = \angle CBH$, so $\overline{DO} \parallel \overline{BH}$. Similarly, $\overline{EO} \parallel \overline{CH}$, so BCH and DEO are homothetic.

Since Y is the center of homothety sending DEO to BCH, we have $\frac{YB}{YC} = \frac{YD}{YE} = \frac{BD}{CE}$. By power of a point at Y, we have $YO \cdot YP = YB \cdot YD$ and $YO \cdot YQ = YC \cdot YE$, so $\frac{YP}{YQ} = (\frac{BD}{CE})^2$. Thus, we have $\frac{YX}{YC} = (\frac{BD}{CE})^2$, so $YX = YC \cdot (\frac{BD}{CE})^2$. Since $\frac{YC}{YB} = \frac{CE}{BD}$, we have $YC = BC \cdot \frac{CE}{BD+CE}$, so $YX = BC \cdot \frac{CE}{BD+CE} \cdot (\frac{BD}{CE})^2 = \frac{BC}{BD+CE} \cdot \frac{BD^2}{CE}$. Therefore,

$$CX = YX - CY = \left(\frac{BC}{BD + CE}\right)\left(\frac{BD^2}{CE} - CE\right)$$

We plug $BD = 13$, $CE = 8$, and $CX = 27$ into this equation to get $27 = \frac{BC}{21} \cdot (\frac{169}{8} - 8) = \frac{BC}{21} \cdot \frac{105}{8} = BC \cdot \frac{5}{8}$, so $BC = \frac{216}{5}$ and $DE = BC - BD - CE = \boxed{\frac{111}{5}}$.

Problem G7 (rocketsri). Let ℓ be the angle bisector of $\angle A$ in $\triangle ABC$. Point $D \in \ell$ satisfies $\angle DAC = \angle DCA$. Lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ meet at E and lines ℓ and $\overleftrightarrow{BC}$ meet at F. Denote the circumcircles of $\triangle ABC$ and $\triangle CDF$ as ω and ω_1, respectively and O and O_1 as their respective centers. $\overleftrightarrow{CO}$ intersects ω_1 at $X \neq C$. If O lies inside ω_1, $OX \cdot OC = \frac{3}{4}CO_1^2$, ω has radius 36 and $EO_1 = 30$, find the maximum value of the length of CD.

Problem G8 (rocketsri). Suppose cyclic quadrilateral $ABCD$ has AC bisecting $\angle BAD$, with the intersection of AC and BD as E and $BE < ED$. Given $BE = 11, CE = 20$ and $BC = 24$, the area of $\triangle ABD$ can be expressed as $\frac{m\sqrt{p}}{n}$, where m, n, p are positive integers such that p is square-free and $\gcd(m, n) = 1$. Find the sum of digits of $m + n + p$.

Solution: Define F as the A-extouch point, I as the incenter of $\triangle ABD$ and I_a as the A-excenter of $\triangle ABD$. By Law of Cosines on $\triangle BCE$, we see $\cos\angle CBE = \frac{9}{16}$, and since $BE \neq ED$, applying Law of Cosines onto $\triangle CDE$ gives $ED = 16$ (as $\angle CBE = \angle CDE$ by inscribed angles). By the Incenter-Excenter Lemma, we see C is the center of a circle ω passing through B, I, D and I_a. By Power of a Point with respect to E, we see $\text{Pow}_\omega(E) = BE \cdot ED = EI \cdot EI_a = 11 \cdot 16$. However, since $BC = CI$ as B and I lie on ω, we have $EI = CI - CE = BC - CE = 4$, hence $EI \cdot EI_a = 4 \cdot EI_a = 11 \cdot 16 \implies EI_a = 44$. Letting G be the projection of C onto BD gives $\triangle CEG \sim \triangle I_aEF$ and $EG = \frac{27}{2} - 11 = \frac{5}{2}$, hence $\frac{EI_a}{EC} = \frac{FI_a}{CG} \implies FI_a = \frac{33\sqrt{7}}{2}$, by the Pythagorean Theorem. Using the Angle Bisector Theorem, we have $\frac{AB}{AD} = \frac{BE}{ED} \implies AB = 11x, AD = 16x$, for some x. By the Shooting Lemma, $CE \cdot CA = BC^2 \implies AC = \frac{144}{5}$. From Ptolemy's Theorem on $ABCD$, we have $11x \cdot 24 + 24 \cdot 16x = \frac{144}{5} \cdot 27 \implies x = \frac{6}{5}$. Lastly, $I_aF = \frac{[ABD]}{s-a} = \frac{[ABD]}{27/10} \implies [ABD] = \frac{891\sqrt{7}}{20}$, and extracting gives $9 + 1 + 8 = \boxed{18}$.

> *Remark.* The following solution was written by Isaac Chen.

Solution: Trivially, we have $DC = BC = 24$. The Shooting Lemma implies

$$20 \cdot CA = CE \cdot CA = CB^2 = 24^2$$

so $CA = \frac{144}{5}$ and $AE = \frac{144}{5} - 20 = \frac{44}{5}$. Now, observe

$$11 \cdot ED = BE \cdot ED = Pow_{(ABCD)}(E) = CE \cdot EA = 20 \cdot \frac{44}{5}$$

so $ED = 16$ and $BD = 11 + 16 = 27$.

Now, define the projections of A, C onto BD as P, Q respectively. It's easy to see $APE \sim CQE$. Now, since BCD is isosceles, we know

$$CQ = \sqrt{CB^2 - \left(\frac{BD}{2}\right)^2} = \sqrt{24^2 - \left(\frac{27}{2}\right)^2} = \frac{15\sqrt{7}}{2}.$$

Hence,

$$\frac{AP}{\frac{15\sqrt{7}}{2}} = \frac{AP}{CQ} = \frac{AE}{CE} = \frac{\frac{44}{5}}{20}$$

so $AP = \frac{33\sqrt{7}}{10}$. Thus,

$$[ABD] = \frac{1}{2} \cdot BD \cdot AP = \frac{1}{2} \cdot 27 \cdot \frac{33\sqrt{7}}{10} = \frac{891\sqrt{7}}{20}$$

so our final answer is $891 + 7 + 20 = \boxed{918}$.

Problem G9 (wolog). Let F_n denote the nth Fibonacci number with $F_0 = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$. Find the area of the polygon bound by the vertices

$$(F_1, F_2), (F_3, F_4), (F_5, F_6) \ldots (F_{13}, F_{14})$$

Answer: 69

Solution: Will write up in-full later, but essentially shoelace and lots of stuff simplifies. Or use trapezoids and triangles. Your choice. :)

Problem G10 (NH14). Right triangle ABC has leg lengths $AB = 3$ and $AC = 4$. Let the points I and O be the incenter and circumcenter of $\triangle ABC$, respectively. The line $\overleftrightarrow{OI}$ splits $\triangle ABC$ into two parts. The ratio of the smaller part to the larger one can be expressed as $\frac{m}{n}$. Find $m + n$.

Answer: 12

Solution: Note that the line $\overleftrightarrow{OI}$ will intersect side BC at O and intersect side AB at other point, say P. Then, note that the line splits the triangle into a triangle, $\triangle BOP$ and into a quadrilateral, $APOC$. We can easily find that

$$[\triangle BOP] = \frac{1}{2} BP \cdot BO \cdot \sin(\angle PBO) = \frac{1}{2} \cdot \left(\frac{5}{2}\right)^2 \cdot \frac{4}{5} = \frac{5}{2}.$$

Therefore, $[APOC] = [\triangle ABC] - [\triangle BOP] = 6 - \frac{5}{2} = \frac{7}{2}$. Therefore, the ratio of the two parts is $\frac{5}{7}$, and our requested answer is $\boxed{12}$.

Problem G11 (ryan). The base of an isosceles triangle is also the side of a square. The area of the intersection of the two shapes is 21. The area of the union of the two shapes is 34. Find the sum of all possible areas of the square.

1.2 Combinatorics

Problem C1 (squareman).

A frog is set at $(0,0,0)$ in 3 dimensional space. Each day, the frog does one of the following:

- hops 1 unit in the positive x direction
- hops 1 unit in the positive y direction
- hops 1 unit in the positive z direction

The probability the frog will hop in a given direction is not dependent on the day (the frog may be more likely to choose a direction over another). The probability the frog reaches one of $(1,1,0),(1,0,1),(0,1,1)$ but NOT $(1,1,1)$ is $\frac{2}{5}$. Find the probability the frog reaches one of $(2,0,0),(0,2,0),(0,0,2)$ but NOT one of $(3,0,0),(0,3,0),(0,0,3)$.

Solution (nyan)**:** Let a,b,c be the probabilities the frog moves in each direction. The given probability becomes $\sum_{\text{cyc}} 2ab(a+b) = 2/5$, while the desired is

$$\sum_{\text{cyc}} a^2(1-a) = \frac{1}{2}\sum_{\text{cyc}} 2ab(a+b) = \boxed{\frac{1}{5}}.$$

Problem C2 (Cyclic). Find the number of functions $f\colon \mathbb{Z}^2 \to \{0, 1, \dots, 5\}$ such that for any $x, y \in \mathbb{Z}$, $f(x, y) = f(x + 12, y) = f(x, y + 12)$ and exactly one of $f(x + 1, y)$ and $f(x, y + 1)$ is congruent to $f(x, y) + 1 \pmod 6$.

Solution: Consider $g(x) = f(x) - x - y$. It suffices to find the number of functions $g\colon \mathbb{Z}^2 \to \{0, 1, \dots, 5\}$ such that $g(x, y) = g(x + 12, y) = g(x, y + 12)$ and exactly one of $g(x + 1, y)$ and $g(x, y + 1)$ is congruent to $g(x, y) \pmod 6$.

Label each lattice point (x, y) in the coordinate plane with $g(x, y)$. For each lattice point (x, y), let its child $c(x, y)$ be the point out of $(x + 1, y)$ and $(x, y + 1)$ that is labeled with the same number as (x, y). Let a descendant be any point that can be found by iterating $c(x, y)$ finitely many times. Notice that the set of descendants of a point makes a path that goes up or to the right at every step. (unfinished)

Problem C3 (numberdude). Alice and Bob are each knowingly and secretly given a real number between 0 and 1 uniformly at random. Alice states, "My number is probably greater than yours." Bob repudiates, saying, "No, my number is probably greater than yours!" Alice concedes, muttering, "Fine, your number is probably greater than mine." What is the probability that Bob's number is actually greater than Alice's?

Note: probably means with probability at least $\frac{1}{2}$

Answer: $\frac{11}{12}$

Solution: From the first statement Alice's number must be at least $\frac{1}{2}$. From the second Bob's must be at least $\frac{3}{4}$. Finally, from the third Alice's is at most $\frac{7}{8}$. Then use geometric probability.

Problem C4 (nyan). On a deck of $3 \cdot 14 \cdot 15 \cdot 9 \cdot 26/30$ cards, for all divisors $n > 1$, we define a **n-shuffle** to be a rearrangement of the cards as follows:

(i) The deck is split into n [equal and originally contiguous] piles;

(ii) The piles are interleaved; more precisely, they are combined so that:

- The relative ordering of the top cards of the n piles is maintained;
- The relative ordering of the cards in each pile is maintained.

Let n_0 denote the smallest possible n so that an odd number M of moves are needed to return at least 1/10 of the cards in the deck to their original positions. What is $n_0 + M$?

Answer: 307

Solution: The 'at least 1/10' is a red herring. Treat it as 'all'. Let $N = 3 \cdot 14 \cdot 15 \cdot 9 \cdot 26/30 = 4914$ for brevity.

The last card doesn't move, and obscures the pattern, so we ignore it. Using zero-based indexing, the pith of this problem is:

> **Claim –** For any positive integer $0 \le k < N - 1$, card k is mapped to $nk \mod N - 1$.

Proof. Follows by casework on which pile the card in question is in. □

We're left with the following:

> **NT part**
>
> Find the smallest divisor of 4914 which has an odd order modulo $N - 1 = 4913 = 17^3$.

Because $17 - 1 = 2^4$, the only residue with odd order mod 17 is 1. If a number is to have odd order mod 4913, it must have odd order mod 17. We may see that the smallest divisor of $N = 4914$ exceeeding 1 congruent to $1 \pmod{17}$ is 18.

Obviously, $v_{17}(18^1 - 1) = 1$ and 18 has order $1 \pmod{17}$. Thus by LTE the order of 18 mod 17^3 is 17^2, leading to an answer of

$$18 + 289 = \boxed{307.}$$

> *Remark.* (nyan) **Ritwin** suggested the change from 1 to 0-based indexing; despite the fact that I used a Python program to simulate 2-shuffles, didn't notice fsr

Problem C5 (sscg13). Justin and Chris are playing Two Truths and a Lie. Justin comes up with the following three statements:

- 1: If statement 2 is true, statement 3 is false
- 2: If statement 3 is true, statement 1 is true
- 3: If statement 1 is true, statement 2 is false

Which one of Justin's statements is false?

Answer: 3

Solution: Check all three possible cases

Problem C6 (squareman). In a room, there are 100 men and 200 women. One by one, people are randomly chosen to leave the room, until the last man has left and only women are remaining. If the expected number of women remaining in the room can be expressed as $\frac{p}{q}$ for $\gcd(p, q) = 1$ and $p, q > 1$, find $p + q$.

Problem C7 (squareman). Let A and B be adjacent vertices of a cube (connected by an edge). An ant starts a walk at vertex A, and every minute he walks to one of the three vertices adjacent to his previous one. If there are N total 7 minute walks the ant can take which end up at either vertex A and B, find the sum of the digits of N.

Problem C8 (Sheldon). James the naked mole rat is hopping on the number line. He starts at 6 and jumps exactly 2^n either forward or backward at random at time n seconds, starting at time $n = 0$. What is the expected number of jumps James takes before he is on a number at least 9?

Answer: 4

Solution: Notice that after the first time James jumps forward, he will always be exactly at 7. I claim that after the second time he jumps forward, he is guaranteed to be past 9.

This is because if we change his first forwards jump to backwards, he will be at 7 after the second forwards jump. Since changing the forwards jump to backwards makes a difference of twice the length of the jump, this is at least 2 and thus he will be at at least 9.

Since he needs exactly two forwards jumps, the expected time to get them is just $2(2) = \boxed{4}$.

Problem C9 (Calvin). Given that it is possible to place positive integers from 1 to 5, inclusive, in each of these cells such that each row and column contain each of $1, 2, 3, 4, 5$ exactly once, find the sum of the numbers in the yellow cells.

1	2			
3	4			

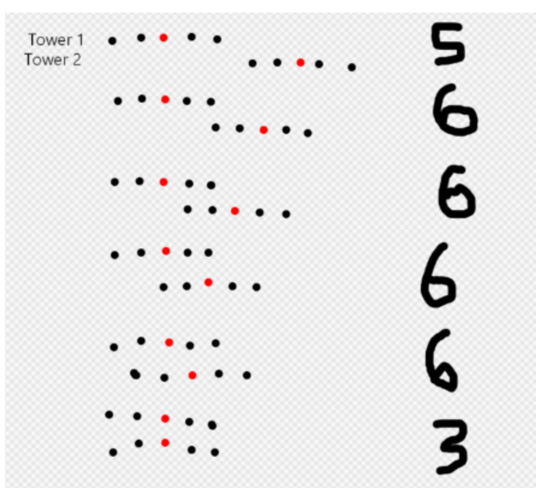
Answer: 25

Solution: The sum of the numbers in each row/column has to be exactly $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the 6 numbers below the 4 givens is equal to $15 \cdot 2 - 1 - 2 - 3 - 4 = 20$. Therefore, the sum of the 9 yellow numbers is equal to $15 \cdot 3 - 20 = \boxed{25}$.

Problem C10 (rjiangbz). There are 100 plots of land in a line, and a goat hiding in a plot of land chosen uniformly at random. You place down two radar towers (at the same time). If a radar tower is placed on the same plot as the goat, it will detect that the goat is there. If the goat is within three plots of land of the tower, the tower detects that the goat is "near." Otherwise, the tower detects that the goat is "far." After the two radar towers are placed, you may search one plot of land. If the towers are placed optimally, what is the probability of finding the goat?

Answer: $\frac{3}{50}$

Solution: Consider a set of spaces that all "look the same" to both towers, containing all such spaces. If there are n spaces, it contributes $\frac{1}{n} \cdot \frac{n}{100} = \frac{1}{100}$ to the total probability of finding the goat. Thus, we just need to maximize the total number of distinct possible readings. Now, we can just check cases: Thus, the answer is $\frac{6}{100} = \boxed{\frac{3}{50}}$.



1.3 Algebra

Problem A1 (squareman). Each of the numbers $a_1, a_2, \dots, a_{10001}$ is ± 1. What is the smallest possible value of

$$\left| \sum_{1 \le i < j \le 10001} a_i a_j \right| ?$$

Problem A2 (vsamc). Let $\omega = e^{2\pi i/3}$. If the quantity

$$\left| \prod_{p \text{ prime}} \left(1 - \frac{\omega}{p^2}\right) \right|$$

can be expressed as $\dfrac{\sqrt{a}}{b\pi^c}$ for positive integers a, b, c with $\gcd(a, b) = 1$, find $a + b + c$.

(Note that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$, and $\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}$.)

Answer: 634.

Solution: Let S be the requested value. Then, we have that

$$\begin{aligned} S^2 &= \prod_{p \text{ prime}} \left| \left(1 - \frac{\omega}{p^2}\right) \right|^2 \\ &= \prod_{p \text{ prime}} \left(1 - \frac{\omega}{p^2}\right)\left(1 - \frac{\omega^2}{p^2}\right), \end{aligned}$$

as ω and ω^2 are complex conjugates. Now, note that for all prime p,

$$\begin{aligned} \left(1 - \frac{\omega}{p^2}\right)\left(1 - \frac{\omega^2}{p^2}\right) &= 1 - \frac{\omega + \omega^2}{p^2} + \frac{\omega^3}{p^4} \\ &= 1 + \frac{1}{p^2} + \frac{1}{p^4} \\ &= \frac{1 - \frac{1}{p^6}}{1 - \frac{1}{p^2}}, \end{aligned}$$

since $\omega^3 = 1$, and as a consequence $\omega + \omega^2 = -1$. Thus, the product becomes

$$\begin{aligned} \prod_{p \text{ prime}} \frac{1 - \frac{1}{p^6}}{1 - \frac{1}{p^2}} &= \left(\prod_{p \text{ prime}} \frac{\left(1 - \frac{1}{p^6}\right)^{-1}}{\left(1 - \frac{1}{p^2}\right)^{-1}} \right)^{-1} \\ &= \left(\frac{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^6}\right)^{-1}}{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}} \right)^{-1}, \end{aligned}$$

since both individual products converge (as we will see shortly).

Now, note that for any $s > 1$, if we let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, then we have that

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \cdots\right)\left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \frac{1}{125^s} + \cdots\right) \\
&= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right) \\
&= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.
\end{aligned}$$

Thus, we have that our earlier expression (S^2) is equal to

$$\left(\frac{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^6}\right)^{-1}}{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}}\right)^{-1} = \left(\frac{\zeta(6)}{\zeta(2)}\right)^{-1} = \frac{\zeta(2)}{\zeta(6)}.$$

As per the problem, we know that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(6) = \frac{\pi^6}{945}$. Thus, we have that

$$S^2 = \frac{\zeta(2)}{\zeta(6)} = \frac{\frac{\pi^2}{6}}{\frac{\pi^6}{945}} = \frac{315}{2\pi^4},$$

whence

$$S = \sqrt{\frac{315}{2\pi^4}} = \frac{1}{\pi^2} \cdot \sqrt{\frac{315}{2}} = \frac{\sqrt{630}}{2\pi^2},$$

so $a + b + c = 634$.

Problem A3 (squareman). Find the number of positive integers $N < 1000$ where there exists positive integers a, b where

$$N = \frac{(a+b)^2 + 3a + b}{2}.$$

Solution: The answer is $999 - 44 - 43 = \boxed{912}$. Rewrite as $(a + b + 1)(a + b)/2 + a$. All numbers that are not triangular numbers or one less than a triangular number work.

Problem A4 (numberdude). For complex numbers x, consider the set

$$S = \{x^1 + 2022, x^2 + 2021, \ldots, x^{2022} + 1\}.$$

Given that $|S| = 2021$, find the sum of the (distinct) values of x^2.

Answer: 5558 (might be off by a small amount, don't trust)

Solution: Let the polynomials in the set be $P_1, P_2, \ldots, P_{2022}$. The condition $|S| = 2021$ is equivalent to exactly two of the polynomials $P_i - P_j$ to be equal. If the difference $|i - j|$ is greater than 2, this contributes nothing to the sum of x^2 by Vieta. Otherwise, there are 2021 cases $P_i - P_{i-1}$, each contributing 1 to the sum, and 2020 cases $P_i - P_{i-2}$, each contributing 2 to the sum. This is a total of 6061, but we're not quite done yet!

Consider the situation $P_i - P_{i-2}$, where i is even. There are a total of 1010 of these cases. In each of these cases, x and $-x$ both are solutions, but create the same values of x^2. This should not be counted, so we have to subtract an additional 1010.

Now consider the situation $P_{4i} - P_{4i-2}$. The values i and $-i$ are always a solution, but they are counted an extra 505 times! Therefore we have to add 505.

Finally, note that in the situation $P_2 - P_1$, there is a constant that adds two to the sum.

Adding, we get 5558.

Problem A5 (numberdude). Let $a_1, a_2, \ldots$ be a sequence such that $a_1 = a_2 = \frac{1}{5}$, and for $n \geq 3$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{1 + a_{n-1}a_{n-2}}.$$

Find the smallest integer n such that $a_n > 1 - 5^{-2022}$.

Answer: 21

Solution: Consider the substitution $a_n = \frac{5^{b_n}-1}{5^{b_n}+1}$. Then substituting, we get $b_n = b_{n-1} + b_{n-2}$! We are looking for the smallest integer n such that $n > 2022 + \varepsilon$ for some small ε, where $b_1 = b_2 \approx \frac{1}{4}$. The answer is 21.

Problem A6 (Cyclic). Find the sum of all integers a such that there are exactly 100 values of x that satisfy

$$\{x\} + \{2x\} + \cdots + \{10x\} + \{ax\} = x.$$

($\{x\}$ denotes the fractional part of x.)

Answer: -108

Solution: We claim that if $a_1 + a_2 + \cdots + a_n \neq 1$, then

$$\{a_1x\} + \{a_2x\} + \cdots + \{a_nx\} = x$$

has exactly $|a_1 + a_2 + \cdots + a_n - 1|$ solutions. Since $\{a_1x\} + \{a_2x\} + \cdots + \{a_nx\} - x = 0$ is an integer, $a_1x + a_2x + \cdots + a_nx - x = (a_1 + a_2 + \cdots + a_n - 1)x$ is also an integer, so $\{x\} = \frac{k}{|a_1+a_2+\cdots+a_n-1|}$ for some nonnegative integer $k < |a_1 + a_2 + \cdots + a_n - 1|$. Notice that for each possible value of $\{x\}$, $\{a_1x\} + \{a_2x\} + \cdots + \{a_nx\}$ is fixed and has the same fractional part as x, so there is one value of x that makes $\{a_1x\} + \{a_2x\} + \cdots + \{a_nx\} = x$ true. Since there are $|a_1 + a_2 + \cdots + a_n - 1|$ possible values of $\{x\}$, there are $|a_1 + a_2 + \cdots + a_n - 1|$ solutions.

Thus, we have $|1 + 2 + \cdots + 10 + a - 1| = |a + 54| = 100$, so the possible values of a are 46 and -154. This gives an answer of -108.

Problem A7 (wolog). Define the Fibonacci numbers with $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k > 2$. For any integers $p, q \geq 0$ let $f(p,q) = \sqrt{2023F_{2^p}^2 + q}$. For large positive integers n, the quantity

$$f(1, f(2, \ldots f(n-1, f(n,0)) \ldots))$$

approaches $\frac{a+\sqrt{b}}{c}$ for positive integers a, b, c where $\gcd(a, c) = 1$. Find $a + b + c$.

Answer: 8102

Solution: Let L_n denote the Lucas numbers, with $L_1 = 1, L_2 = 3$ and $L_h = L_{h-1} + L_{h-2}$ for $h \geq 2$. We have the identities

$$F_{2n} = F_n L_n$$

$$L_{2n} = \frac{L_n^2 + 5F_n^2}{2}$$

Lemma: $S_n = \frac{L_{2^n} + F_{2^n}\sqrt{(4)(2023)+5}}{2}$

Proof: Note that

$$S_n^2 = 2023F_{2^n}^2 + S_{n+1} \implies 4S_n^2 = 4(2023F_{2^n}^2 + S_{n+1})$$

Problem A8 (sscg13). How many digits does the number

$$N = \sum_{n=0}^{2022} \binom{2022}{n} (n - 1011)^2$$

have?

Answer: 612

Solution: $N = 2^{2020} \cdot 2022$

Problem A9 (ryan). Let $x, x_1, x_2, \ldots$ be the recursive sequence defined by $x_1 = 1$, $x_2 = 5$, and

$$x_n = 4x_{n-1} + x_{n-2}.$$

Prove that

$$\sum_{i=1}^{n} 2x_i = F_{3n+1} - 1$$

where $F_n = F_{n-1} + F_{n-2}$ and $F_1 = F_2 = 1$.

Proof. We start by proving a claim.

Claim — The closed form for the recursion is $x_n = F_{3n-1}$.

Proof. We induct to winduct. For our base case x_1, x_2 both work. Assume this holds for x_{k-1}, x_{k-2}.

We now show it holds for x_k. This is just algebra with fibonacci numbers:

$$\begin{aligned}
x_k &= 4x_{k-1} + x_{k-2}, \\
&= 4F_{3k-4} + F_{3k-7}, \\
&= 2F_{3k-4} + F_{3k-4} + F_{3k-5} + F_{3k-6} + F_{3k-7}, \\
&= 2F_{3k-4} + F_{3k-4} + 2F_{3k-5}, \\
&= 2F_{3k-3} + F_{3k-4}, \\
&= F_{3k-2} + F_{3k-3}, \\
&= F_{3k-1}
\end{aligned}$$

which completes our induction. □

Now we deal with the summation. Note $2x_i = F_{3i-1} + F_{3i-2} + F_{3i-3}$ and $2x_{i-1} = F_{3i-4} + F_{3i-5} + F_{3i-6}$. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} 2x_i &= \sum_{i=1}^{n} 2F_{3i-1} \\
&= \sum_{i=1}^{n} F_{3i-1} + F_{3i-2} + F_{3i-3} \\
&= \sum_{k=1}^{3n-1} F_k \\
&= \sum_{k=1}^{3n-1} F_{k+2} - F_{k+1} \\
&= F_{3n+1} - 1
\end{aligned}$$

as desired. □

Problem A10 (wolog & sscg13). Alice writes the numbers $1, 0, 0$ on a blackboard. At every second, Alice can perform one of three operations on the numbers: subtract the first number from the second, subtract the second number from the third, or subtract the third number from the first. After a sufficiently large number of seconds, find which triples are eventually reachable

Problem A11 (Calvin). Let S be the set of primitive 9991th roots of unity. Find

$$\sum_{z \in S} \sum_{k=1}^{103} z^k.$$

Answer: -97

Solution: Note that $9991 = 100^2 - 3^2 = 97 \cdot 103$. Then

$$\begin{aligned}
&\sum_{z\in S}\sum_{k=1}^{103} z^k \\
=&\sum_{k=1}^{103}\sum_{z\in S} z^k \\
=&101\left(\sum_{z\in S} z\right) + \sum_{z\in S} z^{97} + \sum_{z\in S} z^{103} \\
=&101\,(1) - 102 - 96 \\
=&\boxed{-97}.
\end{aligned}$$

Problem A12 (sscg13). Let u be the positive root of $x^4 - 320x^2 - 2206x - 1975$, and let v be the positive root of $x^4 - 50x^2 - 4x\sqrt{1103} - 15$. Let a, b, c be the roots of the polynomial $x^3 - ux^2 + vx - \frac{\sqrt{1103}}{2}$. Triangle ABC has side lengths a, b, c. If the area of triangle ABC is S, then $S^2 = \frac{m}{n}$, where m, n are coprime positive integers. Find $m + n$.

Answer: 31

Solution:

1.4 Number Theory

Problem N1 (ApraTrip).

Find the number of pairs (x, y) with $0 \le x, y \le 40$ where 41 divides $x^2 - xy^2 + 1$.

Solution: Let $p = 41$ and N be the desired value. One can easily see that

$$N = \sum_{x=1}^{p-1} \left(\frac{x + \frac{1}{x}}{p} \right) + 1.$$

Clearly, $N < 2p$. Consider N's remainder when divided by 2 and p.

Firstly, notice that $\left(\frac{x}{p}\right) + 1 \equiv 0 \pmod 2$ when $x \not\equiv 0 \pmod p$ and $\left(\frac{x}{p}\right) + 1 \equiv 1 \pmod 2$ when $x \equiv 0 \pmod p$. Thus (noting $x + x^{-1} \equiv 0 \pmod p$ has two solutions), $N \equiv 2 \equiv 0 \pmod 2$.

Now, since $\left(\frac{x}{p}\right) \equiv x^{\frac{p-1}{2}} \pmod p$,

$$\sum_{x=1}^{p-1} \left(\frac{x + \frac{1}{x}}{p} \right) + 1 \equiv -1 + \sum_{x=1}^{p-1} \left(x + \frac{1}{x} \right)^{\frac{p-1}{2}} \equiv -1 + \sum_{x=1}^{p-1} \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} x^{2y - \frac{p-1}{2}}$$
$$\equiv -1 + \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} \sum_{x=1}^{p-1} x^{2y - \frac{p-1}{2}} \pmod p.$$

Noting that $\sum_{x=1}^{p-1} x^k \equiv 0 \pmod p$ when $p \nmid k$, we see

$$-1 + \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} \sum_{x=1}^{p-1} x^{2y - \frac{p-1}{2}} \equiv -1 + \sum_{x=1}^{p-1} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} x^0 \equiv -1 - \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p.$$

Finally, since $p = 41$, $N \equiv -1 - \binom{20}{10} \equiv 30 \pmod{41}$. Since $N \equiv 0 \pmod 2$, $N \equiv 30 \pmod{41}$, and $N < 82$, N must be $\boxed{30}$.

Problem N2 (ApraTrip).

How many times does the digit 0 appear in the binary representation of $\frac{2^{100}-1}{101}$?

Answer: 44

Solution: Firstly, note that $101 \equiv 5 \not\equiv \pmod 8$, so $2^{\frac{101-1}{2}} \equiv -1 \pmod{101}$. Thus, $\frac{2^{50}+1}{101}$ is an integer.

Now, let the binary representation of $\frac{2^{50}+1}{101}$ be $\overline{(x_n x_{n-1} \ldots x_1)}_2$, where each $x_i \in \{0, 1\}$ (with $x_1 = 1$) and $n < 100$. Then

$$(2^{50} - 1)\frac{2^{50}+1}{101} = 2^{50} \cdot \frac{2^{50}+1}{101} - \frac{2^{50}+1}{101} = \overline{(x_n x_{n-1} \ldots x_1 0 \ldots 000)}_2 - \overline{(x_n x_{n-1} \ldots x_1)}_2,$$

where the first term has 50 ending zeros.

Simplifying, we see

$$\frac{2^{100}-1}{101} = \overline{(x_n x_{n-1} \dots x_2 01 \dots 111 \dots 1(1-x_n)(1-x_{n-1}) \dots (1-x_2)1)_2},$$

where the middle streak of ones contains $50-n$ ones. Finally, noticing that exactly one of x_i and $1-x_i$ is 1, we see that $\frac{2^{100}-1}{101}$ has 50 ones.

Finally, noticing that $\left\lfloor \log_2 \left(\frac{2^{100}-1}{101} \right) \right\rfloor + 1 = 94$, our answer is $94 - 50 = \boxed{44}$.

Problem N3 (Sheldon).

There are two boards labeled A and B, with in A containing the numbers from 1 to n and B empty. I move by erasing three numbers in one whiteboard and writing the largest number of the three on the other board. For how many positive integer values $n < 1434$ is it possible that I cannot move and two numbers are on each of the boards?

Answer: 179

Solution: If we use a and b as the number of numbers in A and B respectively, we notice that $a+3b$ is invariant $\mod 8$. Two numbers on each of the boards represents $a+3b = 0 \mod 8$. Since there are only eight possible final scenarios where I can't move, and none of the other ones correspond to $0 \mod 8$, then all $n = 0 \mod 8$ works. Thus, the answer is simply $\lfloor \frac{1434}{8} \rfloor = \boxed{179}$.

Problem N4 (Calvin).

$2^{16}+1$ is a prime. Consider all sets S with positive integers less than $2^{16}+1$ such that no two distinct positive integers (a, b) in S satisfy

$$a^2 \equiv b \pmod{2^{16}+1}.$$

What is the maximum number of elements in one of these sets?

Answer: $\frac{2^{17}+1}{3}$

Solution: Consider the tree rooted at 1, with the only child of 1 being -1, and the children of everything else being its two square roots. We know this is the structure of such a tree because of the existence of a primitive root g mod 65537. Then it is easy to see that $|S|$ is maximized when you take the row with 2^{15} elements, then the row with 2^{13} elements, etc, until the row with 2 elements and 1. Our final answer is then

$$2^{15} + 2^{13} + \cdots + 2 + 1 = \frac{2^{17}-2}{3} + 1 = \frac{2^{17}+1}{3}.$$

2 Longlist, by author

2.1 Pigeon

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Problem N1 (Pigeon). What is the remainder when 63^4 is divided by 127?

Answer: $\boxed{8}$

Solution: Note that $63 \equiv -\frac{1}{2} \pmod{127}$.

Problem N2 (Pigeon). Let r be a real number for which $\lfloor r \rfloor = 42$. How many distinct values could $\lfloor 3r \rfloor$ possibly equal?

Answer: $\boxed{3}$

Solution: The possibilities are 126 to 128, giving 3 answers. (The number 42 is irrelevant.)

Problem N3 (Pigeon). A rectangular prism has a square base and a height of 4 inches. The diagonal connecting opposite vertices of this prism has length 10 inches. What is the volume of the prism, in cubic inches?

Answer: $\boxed{168}$

Solution: Let the square have side length s. Then:

$$s^2 + s^2 + 4^2 = 10^2 \implies s^2 = 42.$$

So the volume is $s \times s \times 4 = 4s^2 = 168$.

Problem N4 (Pigeon). What is the sum of the absolute values of the terms of the following arithmetic sequence?

$$-28, -25, -22, \ldots, 20, 23, 26$$

Answer: 271.

Solution: Note that the summation is equivalent to:

$$(1 + 2 + 3 + \cdots + 28) - (3 + 6 + 9 + \cdots + 27) = \frac{28 \times 29}{2} - \frac{3 \times (9 \times 10)}{2} = 406 - 135 = 271.$$

Problem N5 (Pigeon). What is the sum of the terms of the following arithmetic sequence?

$$-28, -25, -22, \ldots, 20, 23, 26$$

Answer: -21.

Solution: The average is $\frac{-28 + 26}{2} = -1$, and there are 21 terms in total, so the sum is -21.

Problem N6 (Pigeon). How many permutations of the first 5 positive integers exist for which:

- The number 1 comes before the numbers 2, 3, and 4.
- The number 2 comes before the number 5.

Answer: 12.

Solution: The number 1 clearly must come first. The problem is now reduced to finding the number of permutations of $(2, 3, 4, 5)$ with the 2 coming before the 5.

In any one of the 24 permutations of $(2, 3, 4, 5)$, we can associate it with another permutation with the 2 and 5 swapped. Then each pair of permutations contains precisely one permutation with the 2 before the 5. Therefore, the answer is half of 24, or 12.

Problem N7 (Pigeon). Greg ran from his house to the park at a constant walking pace. He then ran back to his house at a faster jogging pace. Given that 35% of Greg's run was spent walking, to the nearest percent, what percent faster was Greg's jog than his walk?

Answer: 86%.

Solution: WLOG Greg spent 7 seconds walking and 13 seconds jogging. Then the answer is $\frac{13}{7}$ increase, which is $\frac{6}{7} \approx 86\%$.

Greg walked from his house to the park at a constant walking pace. He then jogged back to his house at a faster jogging pace. Given that 65% of Greg's run was spent walking, to the nearest percent, what percent faster was Greg's jog than his walk?

Problem N8 (Pigeon). A certain machine takes in two positive integers and outputs 2 times one positive integer and 3 times the other. A certain pair of inputs results in the outputs 84 and 76. What is the sum of these inputs?

Answer: 66.

Solution: The number 76 could only be derived by $76 \div 2 = 38$, since 76 cannot be divided by 3 to give an integer. Then 84 must have been multiplied by 3, making the two inputs 38 and 28. Their sum is $38 + 28 = 66$.

Problem N9 (Pigeon). In rectangle $ABCD$ with $AB = 4$ and $BC = 6$, point M is the midpoint of side $\overline{AD}$. There exists a point T on segment $\overline{BM}$ for which the circumcircle of $\triangle DTC$ is tangent to $\overline{BM}$. What is the length of BT, expressed in simplest radical form?

Answer: $10 - 4\sqrt{2}$.

Solution: This is just AAMC 10A #25.

Problem N10 (Pigeon). The graphs of $y = |x - 29| + 16$ and $y = -|x - 61| + 94$ intersect at points P and Q. What is the sum of the coordinates of the midpoint of segment $\overline{PQ}$?

Answer: 100.

Solution: The "vertices" of the two graphs are $X = (29, 16)$ and $Y = (61, 94)$. Notice that $XPYQ$ is a rectangle, so the midpoint of diagonal $\overline{PQ}$ is the same as the midpoint of diagonal $\overline{XY}$, which is:

$$\left(\frac{29 + 61}{2}, \frac{16 + 94}{2}\right) = (45, 55) \implies \boxed{100.}$$

Problem N11 (Pigeon). A jar contains 52 coins in total, each either a penny or a nickel. (A penny is worth 1 cent, and a nickel is worth 5 cents.) If a coin is drawn out of this jar at random, the expected value of the coin's worth would be 4 cents. What is the total amount of money in the jar, in cents?

Answer: 208.

Solution: The distribution of pennies and nickels is irrelevant – if there are c cents in the jar, then the expected value is $\frac{c}{52}$. Thus, $\frac{c}{52} = 4$, so $c = 208$.

Problem N12 (Pigeon). Two ants are on the number line in opposite directions at the same, constant speed. At time $t = 0$, the ants are 14 units apart, and at time $t = 2$, the ants are 5 units apart. What is the sum of all possible shared speeds of the ants?

Answer: 7

Solution: The common speed could either have been $\frac{14-5}{4}$ or $\frac{14+5}{4}$, depending on whether the ants' paths crossed during the walk. The sum of these two speeds is just $14 \div 2 = \boxed{7.}$

Problem N13 (Pigeon). What is the sum of the base-ten digits of the product,

$$2,010,201,020,102,010 \times 32?$$

Answer: 60

Solution: No regrouping occurs, so the sum of the digits of the product is the product of the sum of the digits, or:

$$(3 \times 4) \times 5 = \boxed{60.}$$

Problem N14 (Pigeon). There exists positive integers $a, b, c \geq 3$ for which:

$$a! = b! \cdot c!$$

What is the least possible value of $a + b + c$?

Answer: 14

Solution: Notice that $6! = 3! \times 5!$. This gives a sum of $6 + 3 + 5 = 14$. Clearly no smaller values of a are possible.

Problem N15 (Pigeon). For what number n does $\sqrt{(n-7)^2}$ equal $n + 17$?

Answer: -5

Solution: This is equivalent to solving $|n - 7| = n + 17$. Clearly $n - 7$ cannot be positive, so $n - 7$ is negative, meaning:

$$7 - n = n + 17 \implies n = \boxed{-5.}$$

Problem N16 (Pigeon). Suppose $\mathcal{V}$ is a set of 30 vertices that make up a regular polygon with 30 sides. What is the sum of all positive integers n for which there exists a regular polygon with n vertices, all of which are in $\mathcal{V}$?

Answer: 70

Solution: The only possibilities are divisors of 30, excluding 1 and 2. The sum of these is $3 \times 4 \times 6 - 2 = \boxed{70.}$

Problem N17 (Pigeon). Miva and Niva begin running clockwise around a circular track at the same time. Miva completes a lap in 2 minutes, while Niva completes a lap in 5 minutes. For how long will Miva and Niva run, in seconds, before they first arrive at the same position?

Answer: 200

Solution: We need Miva to travel one lap more than Niva. After t minutes have passed, Miva has travelled $\frac{t}{2}$ laps and Niva has travelled $\frac{t}{5}$ laps. Thus, we want:

$$\frac{t}{2} - \frac{t}{5} = 1 \implies t = \frac{10}{3}.$$

This is the same as $\dfrac{10}{3} \times 60 = \boxed{200}$ seconds.

Problem N18 (Pigeon). Two congruent rectangles can be glued together to form two possible rectangles: a 20×22 rectangle and an $m \times n$ rectangle. What is the sum of all possible values of $(m+n)$?

Answer: 109

Solution: There are two cases:

- The two rectangles both have dimension 10×22, giving $m \times n = 10 \times 44$, and $(m+n) = 54$.
- The two rectangles both have dimension 11×20, giving $m \times n = 11 \times 40$ and $(m+n) = 55$.

The sum of these numbers is $54 + 55 = \boxed{109.}$

Problem N19 (Pigeon). What is the sum of all distinct possible values of mn, for positive integers m and n in the set $\{2, 3, 5, 7\}$?

Answer: 168

Solution: The sum of all possible values of mn (where the "distinct" condition is not needed) equals:

$$(2+3+5+7)(2+3+5+7) = 289.$$

By symmetry, all mn for $m \neq n$ are counted twice, whereas each mn for $m = n$ is counted once. Thus, the answer is:

$$\frac{289 - 2^2 - 3^2 - 5^2 - 7^2}{2} + 2^2 + 3^2 + 5^2 + 7^2 = \boxed{168.}$$

Problem N20 (Pigeon). Let $f(N)$ denote the sum of the first N digits after the decimal point in the decimal representation of $\frac{k}{99}$, where $k < 99$ is a positive integer. Given that $f(10) = 35$ and $f(11) = 41$, what is k?

Answer: 61

Solution: The tens digit of k has to be $f(11) - f(10) = 6$, and the fact that $f(10) = 35$ implies that the sum of the digits of k is $35 \div 5 = 7$. Thus, the ones digit must be $7 - 6 = 1$, and the number must be $\boxed{61.}$

Problem N21 (Pigeon). An arithmetic progression of two-digit positive integers contains 5 terms, exactly one multiple of 3, and exactly two multiples of 4. What is the greatest possible sum of the terms of this progression?

Answer: 450

Solution: The middle term must be the multiple of 3, and the first and last terms must be the multiples of 4. The greatest sequence satisfying this is $\{88, 89, 90, 91, 92\}$, with a sum of terms equalling $\boxed{450.}$

Problem N22 (Pigeon). Given that $x^2 - 1 = 5x$, what is the value of $x^2 + \frac{1}{x^2}$?

Answer: 27

Solution: The provided equation implies that $x - \frac{1}{x} = 5$. Squaring both sides gives $x^2 + \frac{1}{x^2} - 2 = 25$, so the answer is $25 + 2 = \boxed{27.}$

Problem N23 (Pigeon). Pogon has 13 distinct positive integers. What is the maximum possible number of unordered pairs of these integers whose average is not an integer?

Answer: $6 \times 7 = 42$.

Problem N24 (Pigeon). What is the sum of all positive integers b for which there exists a positive integer a satisfying:

$$|a + 2b| = 19 \text{ and } |a - 2b| = 15.$$

Answer: -17.

2.2 Billert

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Problem N1 (Bill Fei). Let $d_b(n)$ be the sum of digits of n when expressed in base b. Find the expected value of $d_{10}(x) - d_5(x)$ over all 2-digit numbers (2 digits in base 10).

Answer: $\frac{11}{3}$

Solution: Use Linearity of Expectation. The sum of all the digits in base 10 of these numbers is

$$45 \cdot 10 + 45 \cdot 9 = 45 \cdot 19 = 855$$

99 in base 5 is 344. 10 in base 5 is 20. The sum of the digits of the numbers 20 to 44 in base 5 is

$$(2 + 3 + 4) \cdot 5 + 3 \cdot 10 = 45 + 30 = 75$$

The sum of the digits of the numbers 100 to 144 is

$$1 \cdot 25 + 10 \cdot 5 + 10 \cdot 5 = 125$$

, and similarly for 200 to 244 is 150, and similarly for 300 to 344 is 175. The sum of all those numbers is

$$75 + 125 + 150 + 175 = 525$$

The difference between the two is 330, and since there are 90 total 2 digit numbers in base 10, our final answer should be $\frac{11}{3}$

Problem N2 (Billert). There are three regular polygons with a distinct number of sides a, b, and c. The number of diagonals of each form an arithmetic progression. What is the smallest possible value of $a + b + c$?

Answer: 23

Solution: $5 + 8 + 10 = 23$

Problem N3 (Billert). How many factors of two does the smallest number with 691 factors have?

Answer: 690

Solution: 691 is prime so $2^6 90$ is the smallest number.

Problem N4 (Billert). What is the length of the shortest string that contains all permutations of the characters a, b, and c?

Answer: 8

Solution: abcabacba works. I'm pretty sure it's not possible to have it be shorter.

Problem N5 (Billert). By replacing each of the $\circ$'s with either $+, -, \times$, or $\div$ (repetitions allowed), what is the minimum value of the following expression?

$$6 \circ 5 \circ 3 \circ 2 \circ 0$$

Answer: -24

Solution: $6 - 5 \times 3 \times 2 - 0$.

Problem N6 (Billert). Bill and Orz have birthdays. Their birthdays can be expressed as DD/MM, where DD is the date (from 1 to 31) and MM is the month (from 1 to 12). The sum of the day and month of Bill's birthday is 29, while the sum of the day and month of Orz's birthday is 6. This year (2022), Bill's birthday happens to land on a Monday while Orz's lands on a Sunday, and their birthdays are less than a week apart. What is the product of the day and the month of Bill's birthday?.

Answer: $\frac{4}{25}$

Solution: Just check it. There also shouldn't be any duplicates.

Problem N7 (Billert). What's $696969696969 + 6969696969 + \ldots + 69$?

Answer: 704009794914

Solution: 704009794914

Problem N8 (Billert). What are the smallest possible sum $a + b$, where a and b are distinct positive integers, such that $a + b$ is prime and $ab + 1$ is prime?

Answer: 3

Problem N9 (Billert). How many ways are there to arrange the letters $AMOGUS$ without any permutation of the letters MOG being existent in the arrangement?

Answer: 576

Problem N10 (Billert). Let $\phi(n)$ denote the number of positive integers relatively prime to an integer n. Find the largest possible value of n such that $\phi(\phi(\phi(n))) = 1$

Answer: 18

Problem N11 (Billert). A function $f(a, b)$ returns the value n, where n is the minimum nonzero value such that $a^n \equiv 1 \pmod{b}$. What is the value of

$$\sum_{i=1}^{6} f(i, 7)?$$

Answer: 20

Problem N12 (Billert). $a_n = a_{n-1} \cdot 2$, and $a_0 = 3$. $b_n = (b_{n-1} + 1) \cdot 2$, and $b_0 = 1$. What is $a_{10} - b_{10}$?

Answer: 2

2.3 Squareman

Problem N1 (squareman). A square of side length 6 and a square of side length 11 are conjoined as shown. Find the area of the shaded triangle.
(Image in Discord)

Answer: 18

Problem N2 (squareman). Find

$$\frac{9}{5}(100) + 32$$

Answer: 212

Problem N3 (squareman). Find the sum of the real solutions to $x^2 - 28x + 1 = \sqrt{x}(x + 1)$.

Answer: 34

Problem N4 (squareman). Find the smallest positive integer a where the LCM of a and 420 is greater than 2022.

Answer: 11

Problem N5 (squareman). A clock has a second, minute, and hour hand. A fly initially rides on the second hand of the clock starting at noon. Every time the hand the fly is currently riding crosses with another, the fly will then switch to riding the other hand. Once the clock strikes midnight, how many revolutions has the fly taken?

Answer:

2.4 ApraTrip

Problem N1 (ApraTrip). Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 3$, and $AC = 8$. Let P_1 and P_2 be two distinct points on line $\overline{AC}$ (in the order of A, P_1, C, P_2) and Q_1 and Q_2 be two distinct points on line $\overline{AB}$ (in the order of A, Q_1, B, Q_2) such that $BQ_1 = P_1Q_1 = P_1C$ and $BQ_2 = P_2Q_2 = P_2C$. Find the distance between the circumcenters of $\triangle BP_1P_2$ and $\triangle CQ_1Q_2$.

Answer: $\frac{7}{2}$

Problem N2 (ApraTrip). Let $ABCD$ be an isosceles trapezoid with $AB = 5$, $CD = 8$, and $BC = DA = 6$. There exists an angle θ such that there is only one point X satisfying $\angle AXD = 180^\circ - \angle BXC = \theta$. Find $\sin(\theta)$.

Answer: $\frac{3\sqrt{6}}{8}$

Problem N3 (ApraTrip). How many times does the digit 0 appear in the binary representation of $\frac{2^{100}-1}{101}$?

Answer: 44

Problem N4 (ApraTrip). Find the number of pairs (x, y) with $0 \le x, y \le 40$ where 41 divides $x^2 - xy^2 + 1$.

Answer: 30

Solution: Let $p = 41$ and N be the desired value. One can easily see that

$$N = \sum_{x=1}^{p-1} \left(\frac{x + \frac{1}{x}}{p} \right) + 1.$$

Clearly, $N < 2p$. Consider N's remainder when divided by 2 and p.

Firstly, notice that $\left(\frac{x}{p}\right) + 1 \equiv 0 \pmod 2$ when $x \not\equiv 0 \pmod p$ and $\left(\frac{x}{p}\right) + 1 \equiv 1 \pmod 2$ when $x \equiv 0 \pmod p$. Thus (noting $x + x^{-1} \equiv 0 \pmod p$ has two solutions), $N \equiv 2 \equiv 0 \pmod 2$.

Now, since $\left(\frac{x}{p}\right) \equiv x^{\frac{p-1}{2}} \pmod p$,

$$\sum_{x=1}^{p-1} \left(\frac{x + \frac{1}{x}}{p} \right) + 1 \equiv -1 + \sum_{x=1}^{p-1} \left(x + \frac{1}{x} \right)^{\frac{p-1}{2}} \equiv -1 + \sum_{x=1}^{p-1} \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} x^{2y - \frac{p-1}{2}}$$
$$\equiv -1 + \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} \sum_{x=1}^{p-1} x^{2y - \frac{p-1}{2}} \pmod p.$$

Noting that $\sum_{x=1}^{p-1} x^k \equiv 0 \pmod p$ when $p \nmid k$, we see

$$-1 + \sum_{y=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{y} \sum_{x=1}^{p-1} x^{2y - \frac{p-1}{2}} \equiv -1 + \sum_{x=1}^{p-1} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} x^0 \equiv -1 - \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p.$$

Finally, since $p = 41$, $N \equiv -1 - \binom{20}{10} \equiv 30 \pmod{41}$. Since $N \equiv 0 \pmod 2$, $N \equiv 30 \pmod{41}$, and $N < 82$, N must be $\boxed{30}$. ■

2.5 Others

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Problem N1 (Puddles_Penguin). Let P be a point in the interior equilateral triangle ABC so that $AP = 6$, $BP = 8$, and $CP = 10$. Find the area of the equilateral triangle.

Answer: $25\sqrt{3} - 24$

Solution: This is just a low qual rotation problem

Problem N2. Allan and Ryan are about to shake hands. Both Allan and Ryan each independently have a p probability of bringing out their dominant hand, and $1 - p$ probability of bringing their nondominant hand. Allan is a lefty and Ryan is a righty. If there is $\frac{4}{25}$ chance they both bring out their left hand to shake hands, find all possible values of p.

Answer: $1/5, 4/5$.

Problem N3. On a twelve hour analog clock, what is the smaller angle formed by the minute and second hands at 4 PM?

Answer: 0.

Problem N4. If $AP = 1$ find the area of polygon $ABCDEFGHIJKLMNOP$.

Answer: 14.
(see the discord for the image i used for this)

Problem N5. Evan has a 1/2 probability of pinging everyone in OMMC Community in the first hour of the day. Every hour following this, Evan has a 1/3 probability of pinging everyone in OMMC Community if he did not ping everyone in the previous hour. Otherwise, he has a 2/3 probability of pinging everyone in OMMC Community. Find the expected number of times he pings everyone during the day.

Answer: 12

Problem N6. The probability your answer to this question is correct is p for some real number $0 \le p \le 1$. Find p.

Answer: 1

Problem N7. If $a + 2b = 3$, $b + 3c = 4$, and $c + 4a = 5$, Find $a + b + c$.

Answer: 3

Problem N8. no. this is my problem. bad.

Answer: 69

Solution: trivial. ■

Problem N9 (Ritwin). Let S_n be the sum of digits of 2^n. Find the minimum positive integer b for which there exists a positive integer a such that $S_a = S_{a+b}$.

Answer: $\boxed{6}$

Solution: $b = 6$ works with $S_3 = S_9 = 8$, and $S_n \bmod 9$ implies $6 \mid b$.

Problem N10 (Puddles_Penguin). Suppose a, b, and c are distinct positive integers so that $a+b$, $a+c$, $b+c$ are perfect squares. Find the smallest possible value of $a+b+c$

Answer: 55

Solution: For a, b, and c to be positive, the squares need to satisfy the triangle inequality. The number of odd squares has to be even (or else $(a+b)+(a+c)+(b+c) = 2(a+b+c)$ will be odd. Under these conditions, the smallest triplet of squares we can get is $(25, 36, 49)$. This yields a sum of $\frac{25+36+49}{2} = \boxed{55}$.

Problem N11 (Puddles_Penguin). For positive integers a, b, and c, if $\operatorname{lcm}(a,b) = 140$ and $\operatorname{lcm}(b,c) = 75$, what is $\operatorname{lcm}(a,c)$?

Answer: 2100

Solution: Because you are epic, you notice $(a,b,c) = (140, 1, 75)$ just works. $\operatorname{lcm}(a,c) = \boxed{10500}$ so you become happy. By magic (or alternatively, boring nt alg stuff), it turns out that the lcm can not be anything else.

Problem N12. Evan and Kevin play rock paper scissors. On the first turn, each of them independently chooses rock, paper, or scissors with equal probability. On every turn after that, Evan and Kevin will each independently choose at random one of the options they did not choose on the previous turn. What is the probability that after 10 turns, they have always tied?

Answer: $\frac{1}{1536}$

Problem N13. If $31 \cdot a + 30 \cdot b + 28 \cdot 1 = 365$ and $a + b + 1 = 12$, find $a \cdot b$

Answer: 28

Problem N14. What value of x minimizes the expression $|5 * 3 + 4 - 7 + x|$?

Answer: -12

Problem N15. Simplify

$$(\log_2 60)^2 - (\log_2 20)^2 - (\log_2 30)^2 + (\log_2 10)^2.$$

Answer:

Problem N16. If a, b are the two solutions to $x^2 - 2022x + 2023$. Find $\tan^{-1} a + \tan^{-1} b$ in radians.

Answer:

Problem N17. Define a sequence $a_1, a_2, \ldots$ of real numbers where $a_1 = \sqrt{3}$, and

$$a_n = 1 + \frac{n-1}{a_{n-1}}$$

for all $n > 1$. Find $\lfloor a_{729} \rfloor$.

Answer:

Problem N18. Each unit square of a 10×10 unit square grid is initially assigned a different real number. Each second, each square's number changes to the number that is largest out of all the neighboring squares sharing at least one side with it (including itself). Find the difference between the maximum and minimum possible amount of time needed for all the squares to become the same number.

Answer:

Problem N19 (squareman). Find the minimal constant C where for any nonzero reals $a_1, a2, \ldots a2022$, it holds that

$$2a_1a_{2022} + \sum_{i=1}^{2022}(a_i - a_{i+1})^2 \le C \sum_{i=1}^{2022} a_i^2$$

where $a_{2023} = a_1$.

Problem N20 (squareman). Let $a_1, a_2, \ldots, a_n$ be positive real numbers summing to 1. Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\sqrt{1 - (a_1 + a_2 + \cdots + a_{i-1})^2}\right) > \frac{3}{5}.$$

Problem N21 (squareman). For positive integers $a_1 < a_2 < \cdots < a_n$ prove that

$$\frac{1}{\operatorname{lcm}(a_1, a_2)} + \frac{1}{\operatorname{lcm}(a_2, a_3)} + \cdots + \frac{1}{\operatorname{lcm}(a_{n-1}, a_n)} \leq 1 - \frac{1}{2^{n-1}}.$$

Problem N22 (DottedCalculator and ChatGPT). Let S be the set of positive integers greater than 9 that are divisible by the sum of their digits. Find the smallest possible value of $a^2 + b^2$ out of all pairs (a, b) of positive integers such that ab and $a + b$ are both in S.

Problem N23 (DottedCalculator). If X and Y are two distinct points, let ω_{XY} denote the circle with center X and radius XY. Let the radical center of ω_{AB}, ω_{BC}, and ω_{CA} be M, and let the radical center of ω_{BA}, ω_{CB}, and ω_{AC} be N. The midpoint of MN is $X(n)$ for some positive integer n. Find n.

Problem N24 (DottedCalculator). Given a permutation of the integers between 1 and n, you can replace any two adjacent numbers a and b with $a + b$ and $|a - b|$ in some order. For which n is it possible to turn $n - 1$ of the numbers equal to 0?

Problem N25 (DottedCalculator). Call a word bad if it contains the same number of consonants and vowels. Suppose y is always a consonant. Which of the following words are bad?

alecks, oren, ryen, arnev, bad

Problem N26 (rjiangbz).