NST IB Mathematical Methods III

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1 Normal modes

An n-particle system coupled by spring-like forces with a Lagrangian

$$\mathcal{L} = \frac{1}{2}T_{ij}\dot{\theta}_i\dot{\theta}_j - \frac{1}{2}V_{ij}\theta_i\theta_j = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{T}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^\top \mathbf{V}\mathbf{q},$$

where T_{ij}, V_{ij} are constants, has an equation of motion

$$\mathbf{T}\ddot{\mathbf{q}} = -\mathbf{V}\mathbf{q}. \tag{1}$$

Eq.(1) admits n linearly independent normal modes $q(t)$ such that

$$q_i(t) = Q_i \sin(\omega_i t - \phi_i),$$

where Q_i and ϕ_i are undetermined constants while ω_i are the normal modes frequencies. From the form of the guessed solution, the angular frequency of modes ω_i can be found through the generalised eigenvalue problem

$$\boxed{(-\omega^2\mathbf{T} + \mathbf{V})\mathbf{Q} = \mathbf{0},}$$

where $Q^{(1)}, \ldots Q^{(N)}$ are generalised eigenvectors satisfying the **orthogonal** relation

$$(\mathbf{Q}^{(i)})^\top \mathbf{T} Q^{(j)} = 0, \quad i \neq j.$$

The general solution of the system is

$$\mathbf{q}(t) = \sum_{m=1}^{N} A^{(m)}\mathbf{Q}^{(m)} \sin(\omega_m t - \phi_m),$$

where $A^{(m)}$ are undertermined coefficients from the initial conditions. From the orthogonality conditions, it can be shown that a linear combination of generalised coordinates $q_1, q_2, \ldots$ oscillates at a single frequency ω_i of the i-th eigenvector $Q^{(i)}$

$$\boxed{\alpha^{(n)}(t) = q_i(t)T_{ij}Q_j^{(n)}} = A^{(n)} \sin(\omega_i t - \phi_i).$$

The proof is easy after expanding $\mathbf{q}$ in the eigenvector basis $q_i = \sum_m \alpha^{(m)}(t)Q_i^{(m)}$

2 Groups and representations