NST IB Mathematical Methods III

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Easter, 2023

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1 Normal modes

An n-particle system coupled by spring-like forces with a Lagrangian

$$\mathcal{L} = \frac{1}{2}T_{ij}\dot{\theta}_i\dot{\theta}_j - \frac{1}{2}V_{ij}\theta_i\theta_j = \frac{1}{2}\dot{\mathbf{q}}^\top\mathbf{T}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^\top\mathbf{V}\mathbf{q},$$

where T_{ij}, V_{ij} are constants, has an equation of motion

$$\mathbf{T}\ddot{\mathbf{q}} = -\mathbf{V}\mathbf{q}. \tag{1}$$

Eq.(1) admits n linearly independent normal modes $q(t)$ such that

$$q_i(t) = Q_i\sin(\omega_i t - \phi_i),$$

where Q_i and ϕ_i are undetermined constants while ω_i are the normal modes frequencies. From the form of the guessed solution, the angular frequency of modes ω_i can be found through the generalised eigenvalue problem

$$\boxed{(-\omega^2\mathbf{T} + \mathbf{V})\mathbf{Q} = \mathbf{0},}$$

where $Q^{(1)}, \dots Q^{(N)}$ are generalised eigenvectors satisfying the **orthogonal** relation

$$(\mathbf{Q}^{(i)})^\top\mathbf{T}Q^{(j)} = 0, \quad i \neq j.$$

The general solution of the system is

$$\mathbf{q}(t) = \sum_{m=1}^{N} A^{(m)}\mathbf{Q}^{(m)}\sin(\omega_m t - \phi_m),$$

where $A^{(m)}$ are undertermined coefficients from the initial conditions. From the orthogonality conditions, it can be shown that a linear combination of generalised coordinates $q_1, q_2, \dots$ oscillates at a single frequency ω_i of the i-th eigenvector $Q^{(i)}$

$$\boxed{\alpha^{(n)}(t) = q_i(t)T_{ij}Q_j^{(n)}} = A^{(n)}\sin(\omega_i t - \phi_i).$$

The proof is easy after expanding $\mathbf{q}$ in the eigenvector basis $q_i = \sum_m \alpha^{(m)}(t)Q_i^{(m)}$

2 Groups

2.1 Definitions about groups

Algebra is set with a binary operation defined for all its elements. ***Groups*** G is an algebra satisfying the group axioms

- Closure: the group action of any two elements maps to an element in the group
- Identity: there is a unique identity element in G s.t. $gI = Ig = g$.
- Inverse: there is a unique inverse for any $g \in G$ s.t. $gg^{-1} = g^{-1}g = I$
- Associativity

For example, $\mathbb{Z}$ is a group under addition. It can be proved by contradiction that the group identity and inverse is unique. A group is also ***Abelian*** if the group action commutes. The group action in a finite group can be summarised in a ***group table***. The uniqueness of inverse dictates that there is no repeating element in a row or a column of a group table.

A subset $H \subseteq G$ is said to ***generate*** G if any elements of G can be formed by composition of group elements in H. The ***order*** of a group $|G|$ is the number of distinct elements in a group. The ***order*** of a group element g is the smallest integer q such that

$$g^q = I.$$

Due to the closure of groups, the order of G and any $g \in G$ must follow

$$|g| \leq |G|\,.$$

A group H is a ***subgroup*** of G if it is a subset of G and obeys the group axioms, denoted $H \leq G$. A subgroup of G is ***proper*** if it is not the identity or G itself.

A mapping f from G to G' is said to be **1-1** if every $g' \in G'$ is mapped only once, or

$$f(g_1) = f(g_2) \implies g_1 = g_2.$$

A mapping f is said to be ***onto*** if any $g' \in G'$ is mapped to at least once. Two properties out of 1-1, onto, and $|G| = |G'|$ implies the other. Following this, two groups G and G' are said to be ***isomorphic*** if there *exists* a **1-1 onto** mapping f between G and G' that preserves the group structure, i.e.

$$f(g_2)f(g_1) = f(g_2 g_1) \qquad \forall g_1, g_2 \in G,$$

and the mapping f is the ***isomorphism*** of the two groups. Inverses and identities of two isomorphic groups are mapped onto each other.

2.2 Dihedral group

A ***dihedral group*** D_n is a group of symmetry transformation of regular n-gons. For a square (D_4), all such transformations include rotation clockwise by 90 deg and reflection about two sides and its diagonals m_1, m_2, m_3, m_4. Apparently, D_n is not Abelian. It has 8 distinct group elements $\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\}$. The generators of a dihedral group are $\{R, m\}$.

The dihedral group D_4 has 5 order-2 subgroups, each consisting I and a self-inversed group action R^2 or m_i. There are also 3 order-4 subgroups, one is the cyclic group $\{I, R, R^2, R^3\}$ and the other twos are Kelin four groups $\{I, R^2, m_1, m_2\}$ and $\{I, R^2, m_3, m_4\}$. A group with a single generator of order n is a ***cyclic group*** C_n. A cyclic group is isomorphic to multiplication of a pure phase $e^{2i\pi/n}$. The two Klein four-groups are isomorphic to each other, with a simple relabelling being the isomorphism. It can also be shown that all elements of D_4 are isomorphic to 2-by-2 matrices

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is called a ***faithful representation*** of D_4 because all $g \in D_4$ are represented by distinct matrices.

2.3 Conjugacy relations

Note that action of g_1 followed by action of g_2 is denoted $g_2 g_1$ (i.e. group actions are read from right to left, like matrix multiplication). It can be easily shown that $(g_2 g_1)^{-1} = g_1^{-1} g_2^{-1}$.

Two group elements $g_1, g_2 \in G$ are ***conjugacy*** of each other $(g_1\ \ g_2)$if there *exists* $g \in G$ such that

$$\boxed{g_2 = g g_1 g^{-1}} \Leftrightarrow g_2 g = g g_1.$$

Conjugacy is a form of equivalency relation, and all equivalency relations observe

- reflexivity: $g_1\ \ g_1$
- symmetry: $g_1\ \ g_2 \Leftrightarrow g_2\ \ g_1$
- transitivity: $g_1\ \ g_2$, $g_2\ \ g_3 \implies g_1\ \ g_3$.

Any group G can be partitioned into disjoint ***conjugacy classes***, where all elements within the same classes are conjugate to each other and any two elements from different classes are not. The disjointness of conjugacy classes is obvious following transitivity. It can also be shown that all elements in an Abelian group form their own conjugacy classes. All elements conjugate to g_1 can be found by forming $g g_1 g^{-1}$ for any $g \in G$

A subgroup $H \leq G$ is said to be a ***normal subgroup*** of G if it contains complete conjugacy classes of G, i.e

$$\boxed{g h g^{-1} \in H \qquad \forall h \in H, \forall g \in G,}$$

denoted $H \trianglelefteq G$.

The three order-4 subgroups of D_4 are normal. Normality of a subgroup also depends on G. For example, while I, m_1 is not a normal subgroup of D_4, it is a normal subgroup of the Klein four-group.

A vector that is left invariant (up to multiplication of $\lambda \in \mathbb{R}$)under any operation $g \in G$ is said to be in the ***invariant subspace*** of G. For example, the Klein four-group leaves vectors (anti)parallel to $(1, 0)$ and $(0, 1)$ invariant.

2.4 Cosets

A ***left coset*** of a subgroup $H \leq G$ by a group element $g \in G$ is a *set*

$$gH = \{g, gh_1, gh_2, \ldots\}.$$

Similarly, we may also form the right coset

$$Hg = \{g, h_1 g, h_2 g, \ldots\}.$$

Noting that $h \in H$ is equivalent to $hH = Hh = H$, we can prove that two cosets $g_1 H$ and $g_2 H$ are identical iff $g_1 g_2^{-1} \in H$ and all the other cosets are disjoint. That means a group is *partitioned* into cosets of equal order $|H|$ by any subgroup $H \leq G$. Furthermore, if $H \trianglelefteq G$ is a normal subgroup, any left coset gH is identical to the right coset Hg.

This gives the Lagrange's theorem

Theorem 1 *Lagrange's theorem Let G be a finite group and let $H \leq G$, not necessarily normal, then*

$$\frac{|G|}{|H|} \in \mathbb{Z}.$$

Similarly, for any $g \in G$,

$$\frac{|G|}{|g|} \in \mathbb{Z}.$$

Lagrange's theorem follows from partitioning G into disjoint cosets of H and the corollary follows from generating a cyclic subgroup from g. As a result, for example, every order-4 group is isomorphic to C_4 or the Klein four-group.

We also define the ***product*** of two cosets as the *set* of all products of two elements from each coset. It can be proved that cosets form a group under this product.

2.5 Homomorphism

Two groups G and G' are said to be ***homomorphic*** if there is a group ***homomorphism*** $\Phi : G \to G'$, not necessarily 1-1 or onto, that preserves group structure. G is called the ***source,*** G' is the ***target,*** and the set $\{\Phi(g)\}$ is called the ***image***.

$$\Phi(g_1)\Phi(g_2) = \Phi(g_1)\Phi(g_2).$$

Isomorphism is a special case of homomorphism. Just like isomorphism, homomorphism maps identities and inverses to each other. If Φ is not 1-1, then there exists a non-trivial subgroup $H \leq G$, called the ***kernel***, that maps to the identity of G'. It can be shown that the kernel H is further a ***normal subgroup***.

For any *normal* subgroup $H \trianglelefteq G$, the products of any two cosets

$$(g_1H)(g_2H) = (g_1g_2)H.$$

This is equivalent to saying that the products of two cosets of a normal subgroup H must also have a size of $|H|$.

Any normal subgroup $H \trianglelefteq G$ and its cosets in G form a ***quotient group*** G/H under coset products. The identity of G/H is H. As a corollary, if K is the kernel of an *onto* homomorphism from G to G', then the quotient group G/K is *isomorphic* to G'.

2.6 Permutation group

The permutation group S_N is group of rearrangement transformation of N distinct objects. ***Cayley's theorem*** states that every finite group of order N is isomorphic to a subgroup of S_N.

A permutation group can be expressed as

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

meaning putting the first object to the second position, the second to the first, and leaving the third unchanged. It is evident that switching the rows makes an inverse while switching the columns doesn't do anything

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

3 Representations