NST IB Mathematical Methods II

Yu Lu

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1 Sturm-Liouville Theory

1.1 The operator

A Sturm-Liouville operator

$$\boxed{\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + \sigma(x), \qquad \rho(x) > 0,\, \sigma(x) \in \mathbb{R}}$$

can be shown to be self-adjoint under appropriate boundary conditions

$$\rho\left(vu^{*\prime} - u^* v'\right)\Big|_\alpha^\beta = 0,$$

meaning $\langle u|\mathcal{L}v\rangle = \langle \mathcal{L}u|v\rangle$ (equivalently, $\mathcal{L}^\dagger = \mathcal{L}$).

Such self-adjointness provides many convenient features. Just like Hermitian matrices, for example, in the eigenvalue equation for a self-adjoint operator $\mathcal{L}y_n = \lambda_n y_n$, the eigenvalues λ_n are real and **non-degenerate** eigenfunctions y_n are mutually orthogonal. Moreover, the eigenfunctions commonly form a complete basis, from which we could solve inhomogeneous linear differential equations.

1.2 A generalisation to any 2nd order linear ODE

Despite being seemingly restrictive, the Sturm-Liouville operator could express any second order linear differential operator ($a = -p, b = p' - q$)

$$\tilde{\mathcal{L}} = p(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}}{\mathrm{d}x} + r(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - b(x)\frac{\mathrm{d}}{\mathrm{d}x} + r(x)$$

could be converted into a Sturm-Liouville form by choosing $w(x)$ such that

$$w\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(aw\frac{\mathrm{d}}{\mathrm{d}x}\right) + (aw' - bw)\frac{\mathrm{d}}{\mathrm{d}x} + rw$$

has a vanishing coefficient for the first derivative term:

$$\boxed{w(x) = C\exp\left[\int^x \frac{b(x')}{a(x')}\mathrm{d}x'\right] > 0.}$$

This way, a Sturm-Liouville operator

$$\mathcal{L} = w(x)\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + \sigma(x), \qquad \rho = aw,\, \sigma = rw$$

could be constructed. From this, an eigenvalue problem $\tilde{\mathcal{L}}y_n = \lambda y_n$ involving $\tilde{\mathcal{L}}$ could be converted to a generalised eigenvalue problem $\mathcal{L}y_n = \lambda w(x) y_n$ of the Sturm-Liouville operator $\mathcal{L}$. Alternatively, we could stick with the original operator $\tilde{\mathcal{L}}$ and introduce the weight in the definition of inner product such that

$$\langle u|v\rangle_w = \int_\alpha^\beta w^* uv\,\mathrm{d}x \implies \|v\|_m^2 = \int_\alpha^\beta w^*\,|v|^2\,\mathrm{d}x,$$

and the self-adjoint condition becomes $\langle u|\mathcal{L}v\rangle_w = \langle \mathcal{L}u|v\rangle_w$. Under this convention, $\tilde{\mathcal{L}}$ satisfying the original eigenvalue equation is self-adjoint with weight $w(x)$ while $\mathcal{L}$ satisfying the generalised eigenvalue equation is self-adjoint with weight 1.