UNIVERSIDADE FEDERAL DO RIO DE JANEIRO

DEFORMATION OF RESIDUAL INTERSECTIONS

Thesis Submitted for the Degree of Doctor of Philosophy in Mathematics

By

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UNDER THE DIRECTION OF PROFESSOR

S. HAMID HASSANZADEH

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Abstract

The first part of this work focuses on the construction of the generic version of the Koszul-Fitting ideals (Kitt). It is demonstrated that this ideal is independent of the choice of the system of generators of the basis ideal, up to a universal equivalence. Additionally, by using spectral sequence techniques, we establish conditions under which the ordinary Kitt ideal deforms into the generic Kitt ideal. A important consequence of the theory developed here is the proof that the generic linkage of an ideal I is a deformation of the arbitrary linkage of I in Cohen-Macaulay local settings.

The second part of this work aims to prove the openness of the Strongly Cohen-Macaulay locus of an ideal I in Cohen-Macaulay local rings that admit a canonical module. To achieve this, an elementary proof is provided for the openness of the Cohen-Macaulay locus of a finitely generated module over a Cohen-Macaulay local ring with a canonical module.

Keywords: Koszul-Fitting ideals, Kitt, Deformation of ideals, Residual intersections, Linkage.

Contents

Introduction

This thesis is structured into two distinct parts. In the initial part, our focus lies on the Residual Intersection Theory, constructing the Theory of Generic Kitt. Progressing into the second segment, we explore the opennness of the Cohen-Macaulay Locus and the Strongly Cohen-Macaulay Locus. In the opening chapter, we revisit classical tools and constructions of Commutative Algebra that serve as foundational elements throughout this thesis. Additionally, we demonstrate general results that we invoke when necessary.

The Linkage Theory was introduced by C. Peskine and L. Szpiro in [PS74] in order to classify varieties in the projective space $\mathbb{P}^n$. Two projective schemes $X, Y \subseteq \mathbb{P}^n$ are said linked if there exists a finite sequence of subvarieties $X = X_1, \ldots, X_n = Y$ such that $X_i \cup X_{i+1}$ is complete intersection for all $i = 1, \ldots, n-1$. Algebraically, one has

Definition. *Let R be a Noetherian ring, I, J ideals of R and g a non-negative integer*

- *One says that I and J are* ***directly linked*** *ideals if there exists a regular sequence $\mathbf{a} = a_1, \ldots, a_g$ contained in $I \cap J$ such that*

$$J = (\mathbf{a}) :_R I \qquad \textit{and} \qquad I = (\mathbf{a}) :_R J$$

  In this case, one denotes $I \sim_{\mathbf{a}} J$

- *One says that I and J are* ***linked*** *if there exists a finite family of ideals $I = I_1, \ldots, I_n = J$ such that I_i is directly linked to I_{i+1} for all $i = 1, \ldots, n-1$. In this case, one says that I and J are in the same linkage class.*

The Theory of Residual Intersections, or residual schemes, has its roots in Intersection Theory on the definition of the refined intersection products. In the Commutative Algebra point of view, the concept of Residual Intersection was established, but not explicitly, by M. Artin and M. Nagata [AN72] as a generalization of Linkage Theory. The modern definition of Residual Intersection is established by C. Huneke and B. Ulrich in the context of Cohen-Macaulay rings [HU88] and later extended to the class of Noetherian rings by Hassanzadeh and Naéliton in [HN16] as follows:

Definition. *Let R be a Noetherian ring, s a non-negative integer and I an ideal of R with $\operatorname{ht}(I) \leq s$.*

(i) An (algebraic) s-residual intersection of I is a proper ideal J of R such that $\operatorname{ht}(J) \geq s$ and it is of form $J = \mathfrak{a} :_R I$ for some ideal $\mathfrak{a} \subset I$ generated by s elements;

(ii) A geometric s-residual intersection of I is an algebraic s-residual intersection J of I such that $\operatorname{ht}(I + J) > s$.

Hassanzadeh and Naéliton [HN16] also introduce a new class of residual intersections called by *arithmetic residual intersections*.

Definition. *Let R be a Noetherian ring, s a non-negative integer and I an ideal of R with $\operatorname{ht}(I) \leq s$. An arithmetic s-residual intersection of I is an s-residual intersection $J = \mathfrak{a} :_R I$ of I such that $\mu_{R_\mathfrak{p}}((I/\mathfrak{a})_\mathfrak{p}) \leq 1$ for all $\mathfrak{p} \in \mathrm{V}(I + J)$ with $\operatorname{ht}(\mathfrak{p}) \leq s$, where μ denotes the minimum number of generators.*

It follows direct from definitions that every geometric s-residual intersection is an arithmetic s-residual intersection. Note that if R is Gorenstein ring, I is an unmixed ideal and $s = \operatorname{ht}(I)$, the (geometric) s-residual intersection of I corresponds to a (geometric) link of I. It is of high importance to see when the generic linkage is a deformation of an arbitrary linkage. Similarly, it is of high importance to see when the generic residual intersection is a deformation of an ordinary one. Recall that

Definition. *Let (R, I) and (S, J) be pairs, where R and S are rings, and $I \subset R$, $J \subset S$ are ideals or $I = R$ or $J = S$. One says that (S, J) is a deformation of (R, I) if there exists a sequence $\mathbf{x} \subset S$, which is regular over S and S/J such that there exists a ring isomorphism $\phi : S/(\mathbf{x}) \longrightarrow R$, with $\phi\big((J + (\mathbf{x}))/(\mathbf{x})\big) = I$.*

Studying residual intersections in his PhD, H. Hassanzadeh created a family of complexes, which he called them *residual approximation complexes*. From these complexes, he obtained an ideal, what he called *disguised residual intersection*. This term finds its justification in its close relation with the concept of residual intersection, as elucidated in theorems [HN16, Theorem 4.4] and [Has12, Theorem 2.11].

Studying disguised residual intersections in his PhD, V. Bouça provided a structure for these ideals as follows.

Theorem (Theorem 4.9 of [BH19])**.** *Let R be a ring and $\mathfrak{a} \subseteq I$ two finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Let $K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ be the Koszul complex equipped with structure of differential graded algebra. Let $\zeta_j = \sum_{i=1}^r c_{ij} e_i$ for $1 \leq j \leq s$, $\Gamma_\bullet = R\langle \zeta_1, \ldots, \zeta_s\rangle$ be the sub-algebra of $K_\bullet(\mathbf{f}; R)$ generated by the ζ's, and $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ be the algebra of Koszul cycles. Looking at the elements of degree r in the sub-algebra of $K_\bullet(\mathbf{f}; R)$ generated by the product of $\Gamma_\bullet$ and $Z_\bullet$, then one has

$$K(\mathbf{a}, \mathbf{f}, \Phi) = \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r.$$

Bouça noticed that the ideal $\langle \Gamma_\bullet \cdot Z_\bullet \rangle_r$ defined above remains invariant of the choice of the generators $\mathfrak{a} \subseteq I$ and the choice of the representation matrix Φ. This observation motivated the definition of the *Koszul-Fitting ideals*, which are given as follows:

Definition. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Let $K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ be the Koszul complex equipped with structure of differential graded algebra. Let $\zeta_j = \sum_{i=1}^r c_{ij} e_i$ for $1 \leq j \leq s$, $\Gamma_\bullet = R\langle \zeta_1, \ldots, \zeta_s\rangle$ be the sub-algebra of $K_\bullet(\mathbf{f}; R)$

generated by the ζ's and $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ the algebra of Koszul cycles. The Koszul-Fitting ideal of I with respect to $\mathfrak{a}$ or, simply, Kitt ideal of I with respect to $\mathfrak{a}$ is the ideal

$$\operatorname{Kitt}(\mathfrak{a}, I) := \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r.$$

In particular, he proved that $K(\mathbf{a}, \mathbf{f}, \Phi) = \operatorname{Kitt}(\mathfrak{a}, I)$ is independent from the choice of the system of generators $\mathbf{f}$, $\mathbf{a}$ of I, $\mathfrak{a}$, respectively and from the representation matrix Φ.

In the third chapter of this thesis, we extend the construction of residual approximation complexes to the generic case, resulting in what we call the *Generic Residual Approximation Complexes*. Similarly, we establish the *Generic Disguised Residual Intersection* and elucidate its relationship with the Kitt ideal, what led us to define the *Generic Kitt*. Unlike the ordinary case, we observe that the generic Kitt depends from the choice of the system of generators of the ideal I. Nonetheless, we demonstrate that this dependency is controlled, indicating that the generic Kitt of an ideal I is uniquely determined up to universal extension. To establish this, we explore the well-behavior of Kitt ideals under flat ring extensions, as demonstrated in Corollary 2.2.2, and their commutativity with ring isomorphisms, as proven in Proposition 2.3.2.

Additionally, we establish in Proposition 3.1.8 that the generic Kitt always specializes to the ordinary Kitt. In the second section of this chapter, one studies conditions under which the ordinary Kitt deforms to the generic one. We start this section by proving the existence of the following exact sequence

Theorem 3.2.2.

$$\begin{tikzcd}
H_2(\mathcal{Z}^+_\bullet(\mathbf{a},\mathbf{f})) \arrow[r] & H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \arrow[r] & H_1(\mathcal{Z}^{+\mathfrak{g}}_\bullet(s,\mathbf{f})) \otimes_S S/(\mathbf{x}) \arrow[dl] & \\
& H_1(\mathcal{Z}^+_\bullet(\mathbf{a},\mathbf{f})) \arrow[r] & H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \arrow[r] & 0.
\end{tikzcd}$$

Here one has: $\mathbf{f} = f_1, \ldots, f_r$, $\mathbf{a} = a_1, \ldots, a_s$ are systems of generators of I, $\mathfrak{a}$, respectively; $\mathcal{Z}^+_\bullet(\mathbf{a}, \mathbf{f})$, $\mathcal{Z}^{+\mathfrak{g}}_\bullet(s, \mathbf{f})$ are the residual approximation complex with respect the systems of generators $\mathbf{a}$ and $\mathbf{f}$ and s-generic residual approximation complex with respect the system of generators $\mathbf{f}$, respectively; $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ is a sequence in $S := R[U_{ij}]$ such that $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. From this sequence, we derive several consequences, such that

Corollary 3.2.3. *If $\mathcal{Z}^{+\mathfrak{g}}_\bullet(s, \mathbf{f})$ is an acyclic complex, then for all $0 \le i \le rs$,*

$$H_i(\mathcal{Z}^+_\bullet(\mathbf{a}, \mathbf{f})) = H_i(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})).$$

In particular, this corollary tells us that the aciclicity of $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})$ implies on the rigidity of $\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})$ when R is a Noetherian ring. The following result is immediate consequence of Theorem 3.2.2.

Proposition 3.2.5. *Suppose that R is a Noetherian ring. If $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})$ is acyclic and $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ is regular on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$, then $\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})$ is acyclic. The converse holds if $(R, \mathfrak{m})$ is Noetherian local and $\mathfrak{a} \subseteq \mathfrak{m}I$.*

By using [Has12, Corollary 2.9 (b)], one can deduce.

Corollary 3.2.7. *Let R be a Cohen-Macaulay local ring and $J = \mathfrak{a} :_R I$ be an s-residual intersection of I. Under any of the following conditions:*

(i) $s \leq \operatorname{ht}(I) + 1$;

(ii) *R Gorenstein, $s = \operatorname{ht}(I) + 2$ and I^{unm} Cohen-Macaulay ideal.*

The sequence $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ is regular on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$. In particular, $\operatorname{Kitt}(\mathfrak{a}, I)$ deforms to $\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$.

In the fourth chapter of this thesis, we extend the concept of generic s-residual intersections, as defined in [HU88, Definition 3.1], by relaxing the requirement of the G_s condition. We introduce a new type of ideal, termed *generic s-residual*, instead generic s-residual intersection, and denote it by $\mathrm{R}(s, \mathbf{f})$, where $\mathbf{f}$ represents a system of generators for I. Like the generic residual intersections, the generic residual also exhibits uniqueness up to a universal extension.

The second section of this chapter presents an example demonstrating the possibility of achieving $\operatorname{ht}(\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) < \operatorname{ht}(\operatorname{Kitt}(\mathfrak{a}, I))$, where $\mathfrak{a}$ is a subideal of I generated by s elements. Nevertheless, we establish that in certain types of rings, such as Cohen-Macaulay local rings and affine domains, it is indeed possible to find maximal ideals of S that yield a generic Kitt ideal with a greater height compared to the ordinary Kitt ideal.

Proposition 4.2.5. *Let R be a Cohen-Macaulay local ring (or an affine domain), $\mathfrak{a} \subseteq I$ ideals of R and s a positive integer. Consider $\mathbf{a} = a_1, \ldots, a_s$, $\mathbf{f} = f_1, \ldots, f_r$ systems of generators of the ideals $\mathfrak{a}$, I, respectively and $c_{11}, \ldots, c_{rs} \in R$ such that $a_i = \sum_{k=1}^{r} c_{ki} f_k$. Let $S = R[U_{ij}\,;\, 1 \leq i \leq r, 1 \leq j \leq s]$ be the polynomial extension of R in rs indeterminates and $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$. Then,* *given a maximal ideal* $\mathfrak{n}$ *of* S *containing* $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})$ *and denoting* $\mathfrak{m} := \mathfrak{n} \cap R$*, one has that*

$$\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}).$$

In particular, if $\mathfrak{a} :_R I$ *is an* s*-residual intersection of* I*, then* $\mathrm{R}(s, \mathbf{f})_{\mathfrak{n}}$ *is an* s*-residual intersection for all maximal ideals* $\mathfrak{n}$ *of* S *containing* $\mathrm{R}(s, \mathbf{f}) + (\mathbf{x})$*.*

In their seminal paper [HU87], Huneke and Ulrich determine the structure of linkage (especially licci ideals) through the study of generic linkage. One of the key tools in this study is the following result [HU87, Proposition 2.14].

Theorem. *Let* $(R, \mathfrak{m})$ *be a local Gorenstein ring, and* I*,* J *two Cohen-Macaulay ideals of* R *which are linked and of positive grade. Considering any first generic linkage* $L_1(I)$ *there exists a prime ideal* $\mathfrak{q} \in \mathrm{Spec}(R[\underline{\boldsymbol{x}}])$ *that contains* $\mathfrak{m}$ *such that* $(R[\underline{\boldsymbol{x}}]_{\mathfrak{q}}, L_1(I)_{\mathfrak{q}})$ *is a deformation of* (R, J)*.*

While it may seem initially that the Cohen-Macaulayness on ideals I and J is not overly restrictive, the evolution of the theory, in the aforementioned work and subsequent publications like [HU88] and [HU90], where the concept of generic linkage extended to generic residual intersections, led the authors to refine their theory by focusing on Strongly Cohen-Macaulay ideals instead of Cohen-Macaulay ideals.

Surprisingly, we found out that for generic linkage to be a deformation of an arbitrary linked ideal, say J, no conditions are needed to be imposed on I. Furthermore, this fact is true for any Cohen-Macaulay local ring $(R, \mathfrak{m})$. Additionally, we show that the same result extends to the class of $(\mathrm{ht}(I) + 1)$-residual intersections too.

Corollary 4.2.6. *Let* R *be a Cohen-Macaulay local ring and* $J = \mathfrak{a} :_R I$ *an* s*-residual intersection of* I*. Assume* $s \leq \mathrm{ht}(I) + 1$*. Let* $\mathbf{f} = f_1, \ldots, f_r$ *and* $\mathbf{a} = a_1, \ldots, a_s$ *be systems of generators for* I *and* $\mathfrak{a}$*, respectively, with* $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$*. Let* $S = R[U_{ij}]$ *the polynomial extension of* R *in* rs *indeterminates. Then, for any maximal ideal* $\mathfrak{n}$ *containing* $\mathrm{R}(s, \mathbf{f}) + (U_{ij} - c_{ij})$*, one has that*

$$(S_{\mathfrak{n}}, \mathrm{R}(s, \mathbf{f})_{\mathfrak{n}}) \text{ is a deformation of } (R, J).$$

So it concludes the first and foremost part of this thesis.

In the second part of this thesis, we embark on proving that the Strongly Cohen-Macaulay locus of a given ideal constitutes an open subspace of $\mathrm{Spec}(R)$, and then we derive some consequences from this assertion. Recognizing that the Strongly Cohen-Macaulay locus is essentially a finite intersection of Cohen-Macaulay locus, we explore conditions that ensure the openness of this locus.

The openness of the Cohen-Macaulay locus in Cohen-Macaulay settings has already been established by K. Kimura [Kim23, Corollary 5.5]. By utilizing the Schebeck formula, we are able to offer another proof of the openness of the Cohen-Macaulay locus in cases where the ring is Cohen-Macaulay with a canonical module. Our proof essentially consists of two parts. In the first part, we establish that the Cohen-Macaulay locus is open for what we call *equidimensional modules.*

Definition. *Let R be a Noetherian ring and M a finite R-module. One says that M is equidimensional if every minimal associated prime of M has the same dimension.*

The second part of the proof involves constructing a finite partition of $\mathrm{CM}(R)$ into open subsets, such that each element of this partition is the intersection of an open set with the Cohen-Macaulay locus of an equidimensional R-module, yielding

Theorem 5.2.4. *Let R be a Cohen-Macaulay local ring and M a finite R-module. If R admits canonical module, then $\mathrm{CM}(M)$ is open in $\mathrm{Spec}(R)$.*

We refer to this theorem as a weaker version because the requirement of a canonical module is unnecessary when we invoke [Kim23, Corollary 5.5]. Thus we set the stage to prove the main desired result of part two of this thesis

Theorem 5.3.6. *Let R be a Cohen-Macaulay local ring which admits canonical module and I an ideal of R. Then $\mathrm{SCM}(I)$ is a nonempty open subset of R.*

As an open subset of $\mathrm{Spec}(R)$, the non-Strongly Cohen-Macaulay locus of I, denoted by $\mathrm{nonSCM}(I)$, forms a Zariski variety. We demonstrate that it is indeed a proper subvariety of $\mathrm{V}(I)$.

Proposition 5.3.9. *Let R be a Cohen-Macaulay local ring and I an ideal of R, then $\mathrm{nonSCM}_R(I) \subseteq (\mathrm{V}(I) \setminus \mathrm{MinV}(I))$. In particular, one has*

$$\dim(\mathrm{nonSCM}_R(I)) < \dim(I).$$

We conclude part two of this thesis by studying the regular locus of ideals within the context of quotient of regular local rings by complete intersection. It is demonstrated that $\mathrm{nonSCM}(I)$ is contained within $\mathrm{Sing}(R/I)$.

Proposition 5.3.11. *Let S be a regular local ring and $\mathfrak{a}$ a complete intersection ideal of S. Consider $R = S/\mathfrak{a}$ and let I be an ideal of R, then*

$$\operatorname{Reg}(R/I) \subseteq \operatorname{SCM}(I).$$

In addition, we show through a example that Proposition 5.3.11 is not valid even if the defining ideal $\mathfrak{a}$ of S is a Gorenstein Strongly Cohen-Macaulay ideal.

Chapter 1

Preliminaries

In this thesis, R will always denote a commutative ring with unity, a finite R-module will always mean a finitely generated R-module and the zero module will always be Cohen-Macaulay. The objective of this chapter is to recall some concepts and constructions of Commutative Algebra. Unexplained notations are taken from the standard books [Mat89] and [BH93].

1.1 Koszul Complex

The Koszul complex stands out as a fundamental construction in Commutative Algebra. In this section, our focus is to provide a concise overview of its construction and recall some results that will be used in subsequent developments in this thesis.

Let R be a ring and $\mathbf{f} = f_1, \ldots, f_r$ a sequence of elements of R. Let F be a free R-module with rank r and basis $e_1, \ldots, e_r$. Recall that $\bigwedge^k F$ is a free R-module with basis

$$\mathcal{B}_k = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \ ; \ 1 \leq i_1 < \cdots < i_k \leq r\}$$

for each $0 \leq k \leq r$. Consider the map $F^k \longrightarrow \bigwedge^{k-1} F$ defined as following

$$(e_{i_1}, \ldots, e_{i_k}) \longrightarrow \sum_{j=1}^{k} (-1)^{j+1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k},$$

where the notation $\widehat{\ }$ indicates that the term beneath the hat is omitted. One observes that it defines an alternating k-linear map. By universal property of k-th exterior algebra, there exists an

R-linear map $d^{(k)} : \bigwedge^k F \longrightarrow \bigwedge^{k-1} F$ such that

$$d^{(k)}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}.$$

Setting $d = \bigoplus_{k=0}^{r} d^{(k)}$ and

$$\bigwedge F = \bigoplus_{k=0}^{r} \Big(\bigwedge^{k} F \Big),$$

we construct a homomorphism

$$d : \bigwedge F \longrightarrow \bigwedge F.$$

Moreover, since one can easily check that $d^2 = d \circ d = 0$, we have the following complex

$$K_\bullet(\mathbf{f}; R) := \quad 0 \longrightarrow \bigwedge^r F \xrightarrow{\ d\ } \bigwedge^{r-1} F \xrightarrow{\ d\ } \cdots \longrightarrow \bigwedge^2 F \xrightarrow{\ d\ } F \xrightarrow{\ d\ } R \longrightarrow 0.$$

The complex $(K_\bullet(\mathbf{f}; R), d)$ is called the *Koszul complex on the sequence* $\mathbf{f}$ *with coefficients in* R, more explicitly:

Definition 1.1.1. *Let R be a ring and $\mathbf{f} = f_1, \ldots, f_r$ a sequence of elements of R. The complex constructed above is called the Koszul complex on the sequence $\mathbf{f}$ with coefficients in R and it is denoted by $K_\bullet(\mathbf{f}; R)$. Moreover, given an R-module M, the Koszul complex on the sequence $\mathbf{f}$ with coefficients in M, denoted by $K_\bullet(\mathbf{f}; M)$, is the complex $K_\bullet(\mathbf{f}; R) \otimes_R M$ and its differential is $d_M := d_M \otimes_R 1_M$. The i-th homology of $K_\bullet(\mathbf{f}; M)$ is denoted $H_i(\mathbf{f}; M)$ and it is called the i-th Koszul homology of the sequence $\mathbf{f}$ with coefficients in M.*

As proved in [BH93, Proposition 1.6.2], the Koszul complex of a sequence $\mathbf{f}$ has structure of Differential Graded Algebra with structure induced by the one of the exterior algebra $\bigwedge F$, its differential d is an antiderivation with $\deg(d) = -1$ and, for any R-module M, the Koszul complex on the sequence $\mathbf{f}$ and coefficients in M is a $K_\bullet(\mathbf{f}; M)$-algebra. Moreover, setting

$$Z_\bullet(\mathbf{f}; R) = \mathrm{Ker}(d), \qquad Z_\bullet(\mathbf{f}; M) = \mathrm{Ker}(d_M),$$
$$B_\bullet(\mathbf{f}; R) = \mathrm{Im}(d), \qquad B_\bullet(\mathbf{f}; M) = \mathrm{Im}(d_M),$$

one can check that

$$Z_\bullet(\mathbf{f}; R) \cdot Z_\bullet(\mathbf{f}; M) \subseteq Z_\bullet(\mathbf{f}; M), \qquad B_\bullet(\mathbf{f}; R) \cdot Z_\bullet(\mathbf{f}; M) \subseteq B_\bullet(\mathbf{f}; M),$$
$$B_\bullet(\mathbf{f}; R) \cdot B_\bullet(\mathbf{f}; M) \subseteq B_\bullet(\mathbf{f}; M).$$

Thus we have the following proposition [BH93, Proposition 1.6.4].

Proposition 1.1.2. *Let R be a ring, $\mathbf{f}$ a sequence in R and M an R-module.*

(i) The set $Z_\bullet(\mathbf{f};R)$ is a subalgebra of $K_\bullet(\mathbf{f};R)$ and is called the algebra of Koszul cycles of the sequence $\mathbf{f}$. The set $Z_\bullet(\mathbf{f};M)$ is a $Z_\bullet(\mathbf{f};R)$-module;

(ii) The set $B_\bullet(\mathbf{f};R)$ is an ideal of $Z_\bullet(\mathbf{f};R)$ and is called the ideal of Koszul boundaries of the sequence $\mathbf{f}$;

(iii) The homology $H_\bullet(\mathbf{f};R) = Z_\bullet(\mathbf{f};R)/B_\bullet(\mathbf{f};R)$ carries the structure of associative graded alternating algebra. Moreover, the homology $H_\bullet(\mathbf{f};M) = Z_\bullet(\mathbf{f};M)/B_\bullet(\mathbf{f};M)$ has a structure of $H_\bullet(\mathbf{f};R)$-module.

Let R, R' be rings and $\phi : R \longrightarrow R'$ a ring homomorphism. Given $\mathbf{f} = f_1, \dots, f_r$ a sequence of elements of R, consider $\mathbf{g} = g_1, \dots, g_r$ the sequence of elements of R' such that $g_i = \phi(f_i)$ for all $i = 1, \dots, r$. For each $k = 0, \dots, r$, the map ϕ induces a natural map

$$\Phi_k : \bigwedge^k R^r \longrightarrow \bigwedge^k (R')^r$$

such that $\Phi_k(a_{i_1,\dots,i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}) = \phi(a_{i_1,\dots,i_k}) \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}$. Setting $\Phi := \bigoplus_{k=0}^r \Phi_k$, one obtains a natural map $\Phi : K_\bullet(\mathbf{f};R) \longrightarrow K_\bullet(\mathbf{g};R')$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^r R^r \arrow[r, "d"] \arrow[d, "\Phi_r"] & \bigwedge^{r-1} R^r \arrow[r, "d"] \arrow[d, "\Phi_{r-1}"] & \cdots \arrow[r] & \bigwedge^2 R^r \arrow[r, "d"] \arrow[d, "\Phi_2"] & R^r \arrow[r, "d"] \arrow[d, "\Phi_1"] & R \arrow[r] \arrow[d, "\phi"] & 0. \\
0 \arrow[r] & \bigwedge^r (R')^r \arrow[r, "d'"] & \bigwedge^{r-1} (R')^r \arrow[r, "d'"] & \cdots \arrow[r] & \bigwedge^2 (R')^r \arrow[r, "d'"] & (R')^r \arrow[r, "d'"] & R' \arrow[r] & 0.
\end{tikzcd}$$

is commutative. In other words, ϕ induces a morphism $\Phi : K_\bullet(\mathbf{f};R) \longrightarrow K_\bullet(\mathbf{g};R')$ of Koszul complexes. Next we will outline crucial results that will be invoked in the development of this thesis

Theorem 1.1.3. *Let R be a Noetherian ring, $\mathbf{f} = f_1, \dots, f_r$ a sequence in R and M a finite R-module.*

(i) Let $I = (\mathbf{f})$ be the ideal generated by $\mathbf{f}$. If I contains a sequence M-regular of length m, then $H_i(\mathbf{f};M) = 0$ for all $i \geq r - m + 1$ [BH93, Theorem 1.6.16]*;*

(ii) *All modules the* $H_i(\mathbf{f}; M)$ *vanish if and only if* $M = (\mathbf{f})M$*; If* $H_i(\mathbf{f}; M) \neq 0$ *for some* i*, letting*

$$h = \max\{j \in \mathbb{N} \ ; \ H_j(\mathbf{f}; M) \neq 0\},$$

then $\mathrm{grade}((\mathbf{f}); M) = r - h$. [BH93, Theorem 1.6.17]*;*

(iii) *Suppose that* $(R, \mathfrak{m})$ *is a local ring and* $M \neq 0$*, then* $\mathrm{grade}((\mathbf{f}); M) = r$ *if and only if* $\mathbf{f}$ *is an* M*-regular sequence* [BH93, Corollary 1.6.19]*;*

(iv) *Suppose that* $M \neq 0$ *and set* $g = \mathrm{grade}((\mathbf{f}); M)$*. Then* $H_i((\mathbf{f}); M) \neq 0$ *for all* $i = 0, \ldots, r - g$ *and* $H_i((\mathbf{f}); M) = 0$ *for all* $i = r - g + 1, \ldots, r$ [BH93, Exercise 1.6.31].

Definition 1.1.4. *Let* R *be a ring. A complex of* R*-modules*

$$\mathcal{C}_\bullet = \cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is said rigid if, whenever $H_i(\mathcal{C}_\bullet) = 0$ *for some* $i \in \mathbb{N}_0$*, then* $H_j(\mathcal{C}_\bullet) = 0$ *for all* $j \geq i$.

In particular Theorem 1.1.3 (iv) says that the Koszul complex is rigid. The following proposition is other way to see the rigidity of the Koszul Complex. It is clear that if Proposition 1.1.5 holds, then the Koszul complex is rigid.

Proposition 1.1.5. *Let* R *be a Noetherian ring,* I *an ideal and* M *a finite* R*-module. Consider* $\mathbf{f} = f_1, \ldots, f_r$ *a system of generators of* I*. Then*

$$\mathrm{Supp}(H_r(\mathbf{f}; M)) \subseteq \mathrm{Supp}(H_{r-1}(\mathbf{f}; M)) \subseteq \cdots \subseteq \mathrm{Supp}(H_1(\mathbf{f}; M)) \subseteq \mathrm{Supp}(H_0(\mathbf{f}; M)).$$

Proof: It is enough to prove that

$$\mathrm{Supp}(H_{i+1}(\mathbf{f}; M)) \subseteq \mathrm{Supp}(H_i(\mathbf{f}; M))$$

for all $i = 0, \ldots, r-1$. Let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Supp}(H_i(\mathbf{f}; M))$ be a chosen prime. If $\mathfrak{p} \notin \mathrm{Supp}(M)$, it is clear that $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Supp}(H_{i+1}(\mathbf{f}; M))$. Thus one can assume without loss of generality that $\mathfrak{p} \in \mathrm{Supp}(M)$. Since $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Supp}(H_i(\mathbf{f}; M))$, one has

$$H_i(\mathbf{f}; M_\mathfrak{p}) = H_i(\mathbf{f}; M)_\mathfrak{p} = 0.$$

Hence, by Koszul Rigidity, we get $H_{i+1}(\mathbf{f}; M)_\mathfrak{p} = H_{i+1}(\mathbf{f}_\mathfrak{p}; M_\mathfrak{p}) = 0$, which implies that $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Supp}(H_{i+1}(\mathbf{f}; M))$ and so $\mathrm{Supp}(H_{i+1}(\mathbf{f}; M)) \subseteq \mathrm{Supp}(H_i(\mathbf{f}; M))$. $\square$

1.2 Čech Complex and Local Cohomology Modules

Another crucial concept in Commutative Algebra extensively utilized in this thesis is the Čech Complex. Its i-th cohomology is called i-th Local Cohomology. To introduce this complex, it is necessary to establish certain notations.

Let $k,\ r \in \mathbb{N}$ such that $1 \leq k \leq r$, we shall write

$$\mathcal{I}(k,r) := \{I = (i(1), \dots, i(k)) \in \mathbb{N}^k \ ; \ 1 \leq i(1) < \cdots < i(k) \leq r\}$$

the set of all strictly increasing sequences of length k of positive integers taken from the set $[r] := \{1, \dots, r\}$. Now suppose that $k < r$, let $s \in \mathbb{N}$ such that $1 \leq s \leq k+1$ and $J \in \mathcal{I}(k+1, r)$. We denote $\{j(1), \dots, \widehat{j(s)}, \dots j(k+1)\}$ or, simply $J^{\hat{s}}$, the element of $\mathcal{I}(k,r)$ obtained by omitting the s-th component of J.

Example 1.2.1. Considering $J = \{1, 4, 7, 10, 11\} \in \mathcal{I}(5, 12)$, the set $J^{\hat{3}} = \{1, 4, 10, 11\} \in \mathcal{I}(4, 12)$ is obtained by omitting the third element of J.

Let R be a ring, $\mathbf{f} = f_1, \dots, f_r$ a sequence of elements of R. For each $I \in \mathcal{I}(k,r)$, denote $f_I = \prod_{j=1}^{k} f_{i(j)}$ and R_{f_I} the localization of R on the multiplicative closed subset $\{(f_I)^n \ ; \ n \in \mathbb{N}_0\} \subset R$. Next define $\check{C}^0_{\mathbf{f}}(R) = R$ and, for each $1 \leq k \leq r$,

$$\check{C}^k_{\mathbf{f}}(R) = \bigoplus_{I \in \mathcal{I}(k,r)} R_{f_I}.$$

Moreover, define the maps $d^0 : \check{C}^0_{\mathbf{f}}(R) \longrightarrow \check{C}^1_{\mathbf{f}}(R)$ such that

$$d^0(x) = (x/1, \dots, x/1) \in \check{C}^1_{\mathbf{f}}(R) = \bigoplus_{k=1}^{r} R_{f_k}$$

and, for $1 \leq k \leq r-1$, the map $d^k : \check{C}^k_{\mathbf{f}}(R) \longrightarrow \check{C}^{k+1}_{\mathbf{f}}(R)$ which works on the following way: Given $I \in \mathcal{I}(k,r)$ and $J \in \mathcal{I}(k+1,r)$, the composition

$$R_{f_I} \xrightarrow{\ i\ } \check{C}^k_{\mathbf{f}}(R) \xrightarrow{\ d^k\ } \check{C}^{k+1}_{\mathbf{f}}(R) \xrightarrow{\ p\ } R_{f_J}$$

is the natural map multiplied by $(-1)^{s-1}$ if $I = J^{\hat{s}}$ for some $1 \leq s \leq k+1$, or 0 otherwise. It is straightforward to check that $d^{k+1} \circ d^k = 0$ for all $k = 0, \dots, r-1$. Thus we have constructed the following complex

$$\check{C}^\bullet_{\mathbf{f}}(R):\ 0 \longrightarrow R \xrightarrow{d^0} \check{C}^1_{\mathbf{f}}(R) \xrightarrow{d^1} \check{C}^2_{\mathbf{f}}(R) \longrightarrow \cdots \longrightarrow \check{C}^{r-1}_{\mathbf{f}}(R) \xrightarrow{d^{r-1}} \check{C}^r_{\mathbf{f}}(R) \longrightarrow 0.$$

The complex $(\check{C}^\bullet_{\mathbf{f}}, d)$ is called the *Čech complex* of R with respect to the sequence $\mathbf{f}$. More explicitly:

Definition 1.2.2. *Let R be a ring and $\mathbf{f} = f_1, \ldots, f_r$ a sequence in R. The complex*

$$\check{C}^\bullet_{\mathbf{f}}(R):\ 0 \longrightarrow R \xrightarrow{d^0} \check{C}^1_{\mathbf{f}}(R) \xrightarrow{d^1} \check{C}^2_{\mathbf{f}}(R) \longrightarrow \cdots \longrightarrow \check{C}^{r-1}_{\mathbf{f}}(R) \xrightarrow{d^{r-1}} \check{C}^r_{\mathbf{f}}(R) \longrightarrow 0\ .$$

is called ***Čech complex*** *of R with respect the sequence $\mathbf{f}$. More generally, given M an R-module, the complex $\check{C}^\bullet_{\mathbf{f}}(M) := \check{C}^\bullet_{\mathbf{f}}(R) \otimes_R M$ is called the* ***Čech complex*** *of M with respect to the sequence $\mathbf{f}$.*

Let R be a ring, I a finitely generated ideal of R and M an R-module. Consider $\mathbf{f}$, $\mathbf{f}'$ systems of generators of I. [BS13, Exercise 5.1.17] shows that $H^i(\check{C}^\bullet_{\mathbf{f}}(M)) = H^i(\check{C}^\bullet_{\mathbf{f}'}(M))$ for all $i \in \mathbb{N}_0$. Thus we have the following definition.

Definition 1.2.3. *Let R be a ring, $I = (\mathbf{f})$ a finitely generated ideal of R and M an R-module. The i-th cohomology of the Čech complex $\check{C}^\bullet_{\mathbf{f}}(M)$ is called the i-th local cohomology of M with respect to the ideal I and it is denoted by $H^i_I(M)$.*

It follows directly from Definition 1.2.3 that if I can be generated by r elements, then $H^i_I(M) = 0$ for all $i > r$ and any R-module M. We finish this section outlining crucial results that will be used in the development of this thesis.

Theorem 1.2.4. *Let R be a Noetherian ring and I an ideal of R.*

(i) **(Vanishing Grothendieck Theorem)**: *Given an R-module M, one has $H^i_I(M) = 0$ for all $i > \dim(M)$* [BS13, Theorem 6.1.2]*;*

(ii) Let M be a finitely generated R-module. If $M \neq IM$, then

$$\operatorname{grade}(I, M) = \inf\{n \in \mathbb{N}_0\ ;\ H^i_I(M) \neq 0\}$$

[BS13, Theorem 6.2.7].

1.3 Koszul-Čech Complex

Let R be a ring and $\phi : R^s \longrightarrow R^r$ a linear mapping represented by the matrix $\Phi = [c_{ij}]$ and $S = R[T_1, \ldots, T_r]$ the standard polynomial extension of R in r indeterminates. Consider the ideal

$\mathfrak{t} = (T_1, \ldots, T_r) \subset S$ and the sequence $\underline{\gamma} = \gamma_1, \ldots, \gamma_s \in S$ such that

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Consider the double complex $E^{\bullet,\bullet} = K_\bullet(\underline{\gamma}; S) \otimes_S \check{C}^\bullet_{\mathfrak{t}}(S)$, where $K_\bullet = K_\bullet(\underline{\gamma}; S)$ is the Koszul complex of the sequence $\underline{\gamma}$ in S and $\check{C}^\bullet = \check{C}^\bullet_{\mathfrak{t}}(S)$ is the Čech complex of the ideal $\mathfrak{t}$ in S. We display this double complex in the third quadrant as follows.

$$\begin{array}{ccccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_s \otimes_S \check{C}^0 & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_S \check{C}^0 & \longrightarrow & K_0 \otimes_S \check{C}^0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_s \otimes_S \check{C}^1 & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_S \check{C}^1 & \longrightarrow & K_0 \otimes_S \check{C}^1 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_s \otimes_S \check{C}^r & \longrightarrow & \cdots & \longrightarrow & K_1 \otimes_S \check{C}^r & \longrightarrow & K_0 \otimes_S \check{C}^r & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}.$$

With the usual grading of S, the differentials of this double complex are homogeneous with degree 0. Thus, for each $d \in \mathbb{Z}$, one has a double complex on the d-th homogeneous components of these these modules. Our objective is to use spectral sequences to construct a new family of complexes. Let's calculate the first page of vertical spectral sequence $E^{\bullet,\bullet}_{\text{ver}}$. Since each K_i is a flat module and $H^i_{\mathfrak{t}}(S) = 0$ for all $i \neq r$, then ${}^1E^{\bullet,\bullet}_{\text{ver}}$ is

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \vdots & & & & \vdots & & \vdots & & \\
0 & \longrightarrow & H^r_{\mathfrak{t}}(K_s) & \longrightarrow & \cdots & \longrightarrow & H^r_{\mathfrak{t}}(K_1) & \longrightarrow & H^r_{\mathfrak{t}}(K_0) & \longrightarrow & 0
\end{array}.$$

Since ${}^1E^{\bullet,\bullet}_{\text{ver}}$ collapses, then ${}^2E^{\bullet,\bullet}_{\text{ver}} = {}^{\infty}E^{\bullet,\bullet}_{\text{ver}}$. Recall that $K_i = S^{\binom{s}{i}}[-i]$ for all $0 \leq i \leq s$ and that

$$H^r_{\mathfrak{t}}(S) = \bigoplus_{i_1,\ldots,i_r \geq 0} \frac{1}{T_1^{i_i+1} \cdots T_r^{i_r+1}} \cdot R$$

by [BS13, Example 13.5.3]. Thus $\left(H^r_{\mathfrak{t}}(S)\right)_{(d)} = 0$ if $d > -r$ and so

$$(H^r_{\mathfrak{t}}(K_i))_{(d)} = \left(\bigoplus^{\binom{s}{i}} R[T_1^{-1}, \ldots, T_r^{-1}][-i] \right)_{(d)} = 0$$

whenever $i < d + r$. Hence, for a fixed degree d, the complex $\left({}^1E^{\bullet,r}_{\text{ver}}\right)_{(d)}$ is given by

$$0 \longrightarrow H^r_{\mathfrak{t}}(K_s)_{(d)} \longrightarrow \cdots \longrightarrow H^r_{\mathfrak{t}}(K_{d+r+1})_{(d)} \xrightarrow{\psi_d} H^r_{\mathfrak{t}}(K_{d+r})_{(d)} \longrightarrow 0. \tag{1.1}$$

Thus $\left({}^2E^{-d-r,-r}_{\text{ver}}\right)_{(d)} = \left({}^{\infty}E^{-d-r,-r}_{\text{ver}}\right)_{(d)} \cong \operatorname{Coker}(\psi_d)$. Now we are going to another part of the complex (1.1) by analyzing the horizontal spectral sequence. The first page of horizontal spectral sequence ${}^{1}E^{\bullet,\bullet}_{\text{hor}}$ is written as

$$\begin{array}{cccc}
0 & \cdots & 0 & 0 \\
\downarrow & & \downarrow & \downarrow \\
\check{C}^0_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & \check{C}^0_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & \check{C}^0_{\mathfrak{t}}(H_0(\underline{\gamma};S)) \\
\downarrow & & \downarrow & \downarrow \\
\check{C}^1_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & \check{C}^1_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & \check{C}^1_{\mathfrak{t}}(H_0(\underline{\gamma};S)) \\
\downarrow & & \downarrow & \downarrow \\
\vdots & \cdots & \vdots & \vdots \\
\downarrow & & \downarrow & \downarrow \\
\check{C}^r_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & \check{C}^r_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & \check{C}^r_{\mathfrak{t}}(H_0(\underline{\gamma};S)) \\
\downarrow & & \downarrow & \downarrow \\
0 & \cdots & 0 & 0
\end{array}.$$

The second page of horizontal spectral sequence ${}^{2}E^{\bullet,\bullet}_{\text{hor}}$ is written as

$$\begin{array}{cccc}
H^0_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & H^0_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & H^0_{\mathfrak{t}}(H_0(\underline{\gamma};S)) \\
 & & & \\
H^1_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & H^1_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & H^1_{\mathfrak{t}}(H_0(\underline{\gamma};S)) \\
 & & & \\
\vdots & \ddots & \vdots & \vdots \\
 & & & \\
H^r_{\mathfrak{t}}(H_s(\underline{\gamma};S)) & \cdots & H^r_{\mathfrak{t}}(H_1(\underline{\gamma};S)) & H^r_{\mathfrak{t}}(H_0(\underline{\gamma};S))
\end{array}.$$

Notice that, since $K_i = S^{\binom{s}{i}}[-i]$, one has that

$$(K_i)_{(d)} = \big(S^{\binom{s}{i}}[-i]\big)_{(d)} = 0$$

for all $d < i$. Hence one obtains the following complex

$$0 \longrightarrow (K_d)_{(d)} \xrightarrow{\mu_d} \cdots \longrightarrow (K_1)_{(d)} \longrightarrow (K_0)_{(d)} \longrightarrow 0. \tag{1.2}$$

Therefore $\big({}^{2}E^{-d,0}_{\text{hor}}\big)_{(d)} = \big(H^0_{\mathfrak{t}}(H_d(\underline{\gamma};S)\big)_{(d)} \hookrightarrow H_d(\underline{\gamma};S)_{(d)} = \ker(\mu_d) \subseteq (K_d)_{(d)}$. Returning to the vertical spectral sequence, the convergence theorem tells that

$$\big(H^d(\mathcal{D}^\bullet)\big)_{(d)} = \big({}^{\infty}E^{-d-r,-r}_{\text{ver}}\big)_{(d)},$$

where $\mathcal{D}^\bullet = \mathrm{Tot}\left(K_\bullet(\underline{\gamma}; S) \otimes_S \check{C}^\bullet_{\mathfrak{t}}\right)$ is the totalization complex. On the other hand by analysis of the convergence of the horizontal spectral sequence, one obtains a filtration

$$H^d(\mathcal{D}^\bullet)_{(d)} = \mathcal{F}_{d,0} \supseteq \mathcal{F}_{d,1} \supseteq \cdots \supseteq \mathcal{F}_{d,d}$$

such that

$$\frac{\mathcal{F}_{d,i}}{\mathcal{F}_{-d,-i-1}} \cong \left({}^{\infty}E^{-d-i,-i}_{\mathrm{hor}}\right)_{(d)}.$$

In particular, one has

$$\frac{H^d(\mathcal{D}^\bullet)_{(d)}}{\mathcal{F}_{d,1}} = \frac{\mathcal{F}_{d,0}}{\mathcal{F}_{d,1}} \cong \left({}^{\infty}E^{-d,0}_{\mathrm{hor}}\right)_{(d)}.$$

Hence we have a natural surjection

$$H^d(\mathcal{D}^\bullet)_{(d)} \longrightarrow \left({}^{\infty}E^{-d,0}_{\mathrm{hor}}\right)_{(d)} = \left({}^{2}E^{-d,0}_{\mathrm{hor}}\right)_{(d)} = (H^0_{\mathfrak{t}}(H_d(\underline{\gamma}, S)))_{(d)}.$$

Then we define $\tau_d : H^r_{\mathfrak{t}}(K_{d+r})_{(d)} \longrightarrow (K_d)_{(d)}$ as the composition of the following chain of R-homomorphisms

$$\begin{tikzcd}
H^r_{\mathfrak{t}}(K_{d+r})_{(d)} \arrow[r, "\mathrm{nat}"] & H^d(\mathcal{D}^\bullet)_{(d)} \arrow[r, two heads] & \frac{\mathrm{Coker}(\psi_d)}{\mathcal{F}_{d,1}} \arrow[dll, "\cong"'] & & \\
\left({}^{\infty}E^{-d,0}_{\mathrm{hor}}\right)_{(d)} \arrow[r, "1-1"] & \left(H^0_{\mathfrak{t}}(H_d(\underline{\gamma}; S)\right)_{(d)} \arrow[r, "1-1"] & H_d(\underline{\gamma}; S)_{(d)} \arrow[r, "1-1"] & (K_d)_{(d)}
\end{tikzcd}.$$

Since one can prove that $\tau_d \circ \psi_d = 0$ and $\mu_d \circ \tau_d = 0$, we define the complex $\mathcal{K}_d(\Phi)$ by gluing the complexes (1.1) and (1.2).

$$0 \longrightarrow H^r_{\mathfrak{t}}(K_s)_{(d)} \longrightarrow \cdots \xrightarrow{\psi_d} H^r_{\mathfrak{t}}(K_{d+r})_{(d)} \xrightarrow{\tau_d} (K_d)_{(d)} \xrightarrow{\mu_d} \cdots \longrightarrow (K_0)_{(d)} \longrightarrow 0.$$

Definition 1.3.1. *Let R be a ring, d a non-negative integer and $\phi : R^s \longrightarrow R^r$ a linear mapping represented by the matrix $\Phi = [c_{ij}]$. Let $S = R[T_1, \ldots, T_r]$ be the standard polynomial extension in r indeterminates, the ideal $\mathfrak{t} = (T_1, \ldots, T_r)$ of S and the sequence $\underline{\gamma} = \gamma_1, \ldots, \gamma_s \in S$ such that*

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

*We call the complex $\mathcal{K}_d(\Phi)$ as constructed above by **Koszul-Čech complex** with respect to the matrix Φ and of degree d.*

1.4 Approximation Complexes

In this section, we introduce the approximation complexes developed by Jürgen Herzog, Aron Simis, and Wolmer Vasconcelos in [HSV83]. These complexes, particularly the $\mathcal{Z}$-complexes, will be employed to construct the residual approximation complexes in Section 2.1.

Let R be a ring and I a finitely generated ideal of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I and let $\phi : R^r \longrightarrow R$ be a R-module homomorphism such that $\phi(e_i) = r_i$ for all $1 \leq i \leq r$. Let $S = R[T_1, \ldots, T_r]$ be the standard graded polynomial extension of R in r indeterminates. We now construct the double complex, which Herzog, Simis and Vasconcelos in [HSV83] referred it as the *Double Koszul Complex* associated to the sequence $\mathbf{f}$. Given $1 \leq k \leq r$ and $t \in \mathbb{N}_0$, consider the differential ∂ of the graded Koszul complex $K_\bullet(\mathbf{f}; S)$ with coefficients in S

$$\partial : \bigwedge^k (R^r) \otimes S_t \longrightarrow \bigwedge^{k-1} (R^r) \otimes S_t$$

such that

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes_R w) = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k} \otimes f_{i_j} w$$

and the differential ∂' of the graded Koszul complex $K_\bullet(T_1, \ldots, T_r; S)$ with coefficients in S

$$\partial' : \bigwedge^k (R^r) \otimes S_t \longrightarrow \bigwedge^{k-1} (R^r) \otimes S_{t+1}$$

such that

$$\partial'(e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes_R w) = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k} \otimes T_{i_j} w.$$

An straightforward verification shows that the maps ∂ and ∂' (skew) commute. Consequently, we obtain a differential graded algebra $\mathcal{L} = \mathcal{L}(\mathbf{f})$. Below, we display a portion of the complex $\mathcal{L}$.

$$\begin{tikzcd}
\bigwedge^{k+2}(R^r) \otimes S_{t-1} \arrow[r, "\partial'"] \arrow[d, "\partial"] & \bigwedge^{k+1}(R^r) \otimes S_t \arrow[r, "\partial'"] \arrow[d, "\partial"] & \bigwedge^{k}(R^r) \otimes S_{t+1} \arrow[d, "\partial"] \\
\bigwedge^{k+1}(R^r) \otimes S_{t-1} \arrow[r, "\partial'"] \arrow[d, "\partial"] & \bigwedge^{k}(R^r) \otimes S_t \arrow[r, "\partial'"] \arrow[d, "\partial"] & \bigwedge^{k-1}(R^r) \otimes S_{t+1} \arrow[d, "\partial"] \\
\bigwedge^{k}(R^r) \otimes S_{t-1} \arrow[r, "\partial'"] & \bigwedge^{k-1}(R^r) \otimes S_t \arrow[r, "\partial'"] & \bigwedge^{k-2}(R^r) \otimes S_{t+1}
\end{tikzcd}$$

Definition 1.4.1. *Let R be a ring and I a finitely generated ideal of R. Considering $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I, the differential graded algebra $\mathcal{L} = \mathcal{L}(\mathbf{f})$ constructed above is called the* ***double Koszul complex*** *associated to the sequence $\mathbf{f}$.*

We note that the complex $\mathcal{L}(\partial)$ is simply the Koszul complex of the sequence $\mathbf{f} = f_1, \ldots, f_r$ with coefficients in S and the complex $\mathcal{L}(\partial')$ is the Koszul complex of the sequence $\mathbf{T} = T_1, \ldots, T_r$ with coefficients in S. Furthermore, one has a grading of $\mathcal{L}(\partial')$ by subcomplexes of R-modules $\mathcal{L}(\partial') = \sum \mathcal{L}_k$, where

$$\mathcal{L}_k = \sum_{u+v=k} \bigwedge^u (R^r) \otimes S_v$$

or, more explicitly, $\mathcal{L}_k$ is

$$0 \longrightarrow \bigwedge^r (R^r) \otimes S_{k-r} \longrightarrow \bigwedge^{r-1} (R^r) \otimes S_{k-r+1} \longrightarrow \cdots \longrightarrow \bigwedge^1 (R^r) \otimes S_{k-1} \longrightarrow S_k \longrightarrow 0$$

for all $k \in \mathbb{N}_0$. Since the sequence $\mathbf{T}$ is regular on S, one has that $\mathcal{L}_k$ is an exact complex for $k > 0$.

Denote by $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ and $H_\bullet = H_\bullet(\mathbf{f}; R)$ the algebra of Koszul cycles and the Koszul homologies of $K_\bullet(\mathbf{f}; R)$, respectively. The commutativity of the differentials ∂ and ∂' allows to derive the complexes

$$\mathcal{Z}(\mathbf{f}; R) = (Z_\bullet \otimes S, \partial') \qquad \text{and} \qquad \mathcal{M}(\mathbf{f}; R) = (H_\bullet \otimes S, \partial'),$$

which are called the *approximation complexes* associated to the sequence $\mathbf{f}$. The $\mathcal{Z}$-complex and $\mathcal{M}$-complex are graded complexes over the polynomial S. The t-th component of $\mathcal{Z}$-complex and $\mathcal{M}$-complex are

$$0 \longrightarrow Z_r \otimes S_{t-r} \xrightarrow{\partial'} Z_{r-1} \otimes S_{t-r+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} Z_1 \otimes S_{t-1} \xrightarrow{\partial'} S_t \longrightarrow 0,$$

$$0 \longrightarrow H_r \otimes S_{t-r} \xrightarrow{\partial'} H_{r-1} \otimes S_{t-r+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} H_1 \otimes S_{t-1} \xrightarrow{\partial'} H_0 \otimes S_t \longrightarrow 0\ ,$$

respectively. Thus, one can also realize $\mathcal{Z}$-complex and $\mathcal{M}$-complex as

$$0 \longrightarrow Z_r \otimes S[-r] \xrightarrow{\partial'} Z_{r-1} \otimes S[-r+1] \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} Z_1 \otimes S[-1] \xrightarrow{\partial'} S \longrightarrow 0,$$

$$0 \longrightarrow H_r \otimes S[-r] \xrightarrow{\partial'} H_{r-1} \otimes S[-r+1] \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} H_1 \otimes S[-1] \xrightarrow{\partial'} H_0 \otimes S \longrightarrow 0,$$

respectively. Herzog, Simis and Vasconcelos shows in [HSV83] that

$$H_0(\mathcal{Z}(\mathbf{f}; R)) = \mathrm{Sym}(I) \qquad \text{and} \qquad H_0(\mathcal{M}(\mathbf{f}; R)) = \mathrm{Sym}(I/I^2),$$

so the $\mathcal{Z}$-complex and the $\mathcal{M}$-complex are candidates for resolution of the modules $\mathrm{Sym}(I)$ and $\mathrm{Sym}(I/I^2)$, respectively. This justifies their names as approximation complexes.

1.5 Hilbert Polynomial and Multiplicity

In section 2.4, we shall establish results of Kitt ideals exploring the Multiplicity Theory. The primary goal of this section is to precisely articulate and revisit the definition of the multiplicity of an R-module in relation to a given ideal.

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded finite R-module. Notice that R_0 is itself a Noetherian ring and that M_n is finitely generated as R_0-module. In particular, if we assume that R_0 is also an Artinian ring, then M_n has finite length as R_0-module for all $n \in \mathbb{N}_0$. These observations lead to define the *Hilbert Function of M* as follows.

Definition 1.5.1. *Let R_0 be a Artinian ring, $R = \bigoplus_{n=0}^{\infty} R_n$ a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded finite R-module. The Hilbert Function of M is the map*

$$\begin{aligned} \mathrm{HF}_M \colon\ \mathbb{N}_0 &\longrightarrow \mathbb{N}_0 \\ n &\longmapsto \ell_{R_0}(M_n) \end{aligned},$$

where $\ell_{R_0}(\_\_)$ denotes the length function as R_0-module.

Supposing that R is standard graded ring, [Mat89, Theorem 13.2] proves that there exists a polynomial $f(X) \in \mathbb{Q}[X]$ such that

$$\mathrm{HF}_M(n) = f(n)$$

for all $n \gg 0$. This polynomial is called the *Hilbert Polynomial of M* and it is denoted by $\mathrm{HP}_M(X)$. Furthermore, [Mat89, Matsumura 13.4] asserts that its degree is exactly $\dim(M) - 1$, where $\dim(M)$ is the Krull dimension of M.

Example 1.5.2. Let k be a field and consider the k-algebra $R = k[x, y, z]/(x^2, y^3, z^4)$ with $\deg(\overline{x}) = \deg(\overline{y}) = \deg(\overline{z}) = 1$. Observe that

$$R_0 = \mathrm{Span}_k\{1\}, \quad R_1 = \mathrm{Span}_k\{\overline{x}, \overline{y}, \overline{z}\}, \quad R_2 = \mathrm{Span}_k\{\overline{x}\,\overline{y}, \overline{x}\,\overline{z}, \overline{y}\,\overline{z}, \overline{y}^2, \overline{z}^2\}$$
$$R_3 = \mathrm{Span}_k\{\overline{x}\,\overline{y}^2, \overline{z}^3, \overline{y}^2\overline{z}, \overline{y}\,\overline{z}^2, \overline{x}\,\overline{z}^2, \overline{x}\,\overline{y}\,\overline{z}\}, \quad R_4 = \mathrm{Span}_k\{\overline{y}^2\overline{z}^2, \overline{y}\,\overline{z}^3, \overline{x}\,\overline{z}^3, \overline{x}\,\overline{y}\,\overline{z}^2, \overline{x}\,\overline{y}^2\,\overline{z}\}$$
$$R_5 = \mathrm{Span}_k\{\overline{y}^2\overline{z}^3, \overline{x}\,\overline{y}\,\overline{z}^3, \overline{x}\overline{y}^2\overline{z}^2\}, \quad R_6 = \mathrm{Span}_k\{\overline{x}\,\overline{y}^2\overline{z}^3\}$$

and $R_i = 0$ for all $i \geq 7$. Since $\mathrm{HF}_R(n) = \ell_k(R_n) = \dim_k(R_n)$, one concludes the Hilbert Polynomial of R is the zero polynomial.

Next we will prove a fact that we are going to use on Chapter 2.

Proposition 1.5.3. *Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian graded ring with R_0 Artinian and $\mathcal{C}_\bullet$ an exact graded complex of finitely generated graded R-modules*

$$\mathcal{C}_\bullet : \quad 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow 0,$$

(i) Then

$$\sum_{k=1}^{n}(-1)^k \mathrm{HF}_{C_k} = 0.$$

(ii) In addition, if $R = R_0[R_1]$, then

$$\sum_{k=1}^{n}(-1)^k \mathrm{HP}_{C_k} = 0.$$

Proof: (i) Denoting $C_i = \bigoplus_{m\in\mathbb{N}_0} C_{i,m}$, observe that $\ell_{R_0}(C_{i,m}) < \infty$ for all $1 \leq i \leq n$ and $m \in \mathbb{N}_0$. Since this complex is graded, for each $m \in \mathbb{N}_0$

$$0 \longrightarrow C_{n,m} \longrightarrow C_{n-1,m} \longrightarrow \cdots \longrightarrow C_{2,m} \longrightarrow C_{1,m} \longrightarrow 0$$

is an exact complex of R_0-modules. Hence, by additive property of length, one concludes

$$\left(\sum_{k=1}^{n}(-1)^k \mathrm{HF}_{C_k}\right)(m) = \sum_{k=1}^{n}(-1)^k \mathrm{HF}_{C_k}(m) = \sum_{k=1}^{n}(-1)^k \ell_{R_0}(C_{k,m}) = 0$$

for all $m \in \mathbb{N}_0$ and the result follows.

(ii) It follows directly from the part (i). $\square$

The following corollary shows that the Hilbert function is well-behaved with direct sum of R-modules.

Corollary 1.5.4. *Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian graded ring with R_0 Artinian. Given $M_1, \ldots, M_n$ graded finitely generated R-modules, then*

$$\mathrm{HF}_{\bigoplus_k^n M_k} = \sum_{k=1}^{n} \mathrm{HF}_{M_k}.$$

Proof: Using the graded exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{k-1} M_i \longrightarrow \bigoplus_{i=1}^{k} M_i \longrightarrow M_k \longrightarrow 0,$$

it is enough to prove for $n = 2$ and proceed by induction. For $n = 2$, we consider the sequence

$$0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0.$$

Applying Proposition 1.5.3, the statement follows. $\square$

From this point we are going to define the multiplicity of R-modules. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal of R and M be a finite R-module. Consider the associated graded algebra of M with respect to the ideal I

$$\operatorname{gr}_I(M) = \bigoplus_{k=0}^{\infty} \frac{I^k M}{I^{k+1} M} =: M'.$$

Observe that $\operatorname{gr}_I(M)$ has a natural structure of graded module over the graded ring

$$\operatorname{gr}_I(R) = \bigoplus_{k=0}^{\infty} \frac{I^k}{I^{k+1}}.$$

Furthermore, observe that $(\operatorname{gr}_I(R))_0 = R/I$ is an Artinian ring and that $\operatorname{gr}_I(R)$ is a Noetherian ring, which implies that the Hilbert Function and the Hilbert polynomial are well-defined for every graded finite $\operatorname{gr}_I(R)$-module.

Definition 1.5.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal of R and M be a finite R-module. The Samuel Function of M, denoted by χ_M^I, is the function*

$$\begin{aligned} \chi_M^I \colon\ \mathbb{N}_0 &\longrightarrow \mathbb{N}_0 \\ n &\longmapsto \ell(M/I^{n+1}M) \end{aligned},$$

where $\ell(\_)$ denotes the length function as R/I-module.

Notice that this function is well-defined. Indeed, considering the $\operatorname{gr}_I(R)$-module $\operatorname{gr}_I(M)$, it is straightforward to check that

$$\ell\left(\frac{M}{I^{n+1}M}\right) = \sum_{k=0}^{n} \ell\left(\frac{I^k M}{I^{k+1}M}\right) = \sum_{k=0}^{n} \operatorname{HF}_{M'}(k) < \infty$$

Moreover, since $\operatorname{gr}_R(M)$ is a finite $\operatorname{gr}_I(R)$-module generated by elements of degree one, using the formula above and [Mat89, Theorem 13.2], it is possible to prove that there are $a_0, \ldots, a_s \in \mathbb{Z}$ and $s \in \mathbb{N}_0$ such that

$$\chi_M^I(n) = a_0\binom{d+n}{n} + a_1\binom{d+n-1}{n} + \cdots + a_s\binom{d+n-s}{n},$$

for all $n \gg 0$, where $d = \dim(\operatorname{gr}_I(M)) = \dim(M)$. Hence χ_M^I is a polynomial function on n for $n \gg 0$ and has the following form

$$\chi_M^I(n) = \left(\frac{a_0}{d!}\right)n^d + \text{lower degree terms} =: \left(\frac{e}{d!}\right)n^d + \text{lower degree terms}.$$

The previous discussion led to the following definition

Definition 1.5.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal of R and M be a finite R-module. The Hilbert-Samuel Multiplicity of M with respect to the ideal I, denoted by $e(I, M)$, is the integer e which appears on Samuel function of M with respect to the ideal I*

$$\chi_M^I(n) = \left(\frac{e}{d!}\right)n^d + \textit{lower degree terms}.$$

In particular, if $I = \mathfrak{m}$, one simply denotes $e(M) := e(\mathfrak{m}, M)$.

In other words, $e(I, M)$ is the product of the leader coefficient of χ_M^I by $d!$, where d is the Krull dimension of M. We finish this section summarizing some results on Multiplicity Theory which will be used on this thesis.

Theorem 1.5.7. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring and I an $\mathfrak{m}$-primary ideal of R.*

(i) Given M, M', M'' finite R-modules and the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

one has $e(I, M) = e(I, M') + e(I, M'')$ [Mat89, Theorem 14.6]*;*

(ii) Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the set of all minimal primes of R such that $\dim(R/\mathfrak{p}_i) = d$, then

$$e(I, M) = \sum_{i=1}^{n} e(\overline{I}_i, R/\mathfrak{p}_i)\ell_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}),$$

where $\overline{I}_i$ denotes the image of I in $R/\mathfrak{p}_i$ and $\ell_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})$ stands for the length of $M_{\mathfrak{p}_i}$ as $R_{\mathfrak{p}_i}$-module [Mat89, Theorem 14.7].

1.6 Dimensional Unmixedness

In this section, we introduce the concepts of the *equidimensional part* and *dimensional unmixed ideals*, which can be seen as the dual notion of the unmixed part and unmixed ideals, respectively. We demonstrate the convergence of these notions for several classes of rings, as exemplified in Example 1.6.6. In Section 2.4, we use these notions to establish interesting results on Kitt ideals.

Definition 1.6.1. *Let R be a Noetherian ring and I an ideal of R.*

(i) The unmixed part of I, denoted by I^{unm}, is the intersection of all primary components $\mathfrak{q}$ of I such that $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(I)$. *An ideal of R is said unmixed if it coincides with its unmixed part. A quotient ring R/I is said unmixed if I is unmixed.*

(ii) The equidimensional part of I, denoted by I^{eq}, is the intersection of all $\mathfrak{p}$-primary components $\mathfrak{q}$ of I such that $\dim(R/\mathfrak{p}) = \dim(R/I)$. *An ideal is said dimensional unmixed if it coincides with its equidimensional part. A quotient ring R/I is said dimensional unmixed if I is dimensional unmixed.*

Observe that the definitions of I^{unm} and I^{eq} are well-defined. Indeed, given a $\mathfrak{p}$-primary component $\mathfrak{q}$ of I such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$ and $\dim(R/\mathfrak{p}) = \dim(R/I)$, respectively, then $\mathfrak{p}$ is minimal prime over I, which implies that $\mathfrak{p}$ is a minimal element in $\mathrm{ass}(I)$. Hence $\mathfrak{q}$ will always appear on the primary decomposition of I by the uniqueness in the Primary Decomposition Theorem.

Furthermore, note that the classes of equidimensional rings[1] and dimensional unmixed rings are distinct. In dimensional unmixed rings R/I, the property $\mathrm{Ass}(R/I) = \mathrm{MinAss}(R/I)$ always holds. However, in the case of equidimensional rings, it is possible for the ideal I to admit embedded associated primes as we see in the following example.

Example 1.6.2. Let k be a field and consider $R/I = k[x,y,z]/(x^2y^2z, x^5z)$. Observe that $\mathrm{ass}(I) = \{(x),\ (x,y),\ (z)\}$. Notice that R/I is an equidimensional ring, because

$$\dim\left(\frac{R}{I}\right) = \dim\left(\frac{R/I}{(x)/I}\right) = \dim\left(\frac{R/I}{(z)/I}\right)$$

However R/I is not a dimensional unmixed ring, because I is strictly contained in its equidimensional part $I^{\mathrm{eq}} = (x^2z)$.

[1]A ring R is said equidimensional if every minimal prime ideal has the same dimension.

Now it will be proved some basic properties about the equidimensional part of an ideal.

Proposition 1.6.3. *Let R be a Noetherian ring and I an ideal of R.*

(i) $\dim(R/I) = \dim(R/I^{\mathrm{eq}})$*;*

(ii) $\sqrt{I^{\mathrm{eq}}} = (\sqrt{I})^{\mathrm{eq}}$*;*

(iii) Given an ideal $\mathfrak{a}$ contained in I, then $(I/\mathfrak{a})^{\mathrm{eq}} = I^{\mathrm{eq}}/\mathfrak{a}$;

(iv) I^{eq} is a dimensional unmixed ideal, that is, $(I^{\mathrm{eq}})^{\mathrm{eq}} = I^{\mathrm{eq}}$;

(v) $\dim(R/I^{\mathrm{eq}}) = \dim(R/J^{\mathrm{eq}})$ *for any ideal J of R with* $\mathrm{V}(I) = \mathrm{V}(J)$.

Proof: (i) Let $\mathfrak{p}$ be a prime ideal containing I^{eq} such that $\dim(R/\mathfrak{p}) = \dim(R/I^{\mathrm{eq}})$. By definition of equidimensional ideal, one has that $I^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal with $\dim(R/\mathfrak{p}_i) = \dim(R/I)$. Since I^{eq} is contained in $\mathfrak{p}$, one has that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $1 \le i \le n$. Thus

$$\dim(R/\mathfrak{p}_i) \ge \dim(R/\mathfrak{p}).$$

On the other hand, since $I^{\mathrm{eq}} \subseteq \mathfrak{p}_i \subseteq \mathfrak{p}$, we have that $\dim(R/\mathfrak{p}) = \dim(R/I^{\mathrm{eq}}) \ge \dim(R/\mathfrak{p}_i)$, which implies that $\mathfrak{p} = \mathfrak{p}_i$ and so

$$\dim(R/I) = \dim(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}) = \dim(R/I^{\mathrm{eq}}).$$

(ii) Recall that $\dim(R/I) = \dim(R/\sqrt{I})$ since $\mathrm{V}(I) = V(\sqrt{I})$. Let

$$\mathrm{MinV}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{p}_{n+1}, \ldots, \mathfrak{p}_{n+m}\}$$

be the minimal primes over I and suppose that $\dim(R/\mathfrak{p}_i) = \dim(R/I)$ if $1 \le i \le n$ and $\dim(R/\mathfrak{p}_i) < \dim(R/I)$ if $i > n$. By definition of radical ideal, one has that $\sqrt{I} = \bigcap_{k=1}^{n+m} \mathfrak{p}_k$. Hence, denoting $I^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ a primary decomposition of I^{eq}, one gets

$$(\sqrt{I})^{\mathrm{eq}} = \bigcap_{k=1}^{n} \mathfrak{p}_k = \bigcap_{k=1}^{n} \sqrt{\mathfrak{q}_k} = \sqrt{\bigcap_{k=1}^{n} \mathfrak{q}_k} = \sqrt{I^{\mathrm{eq}}}.$$

(iii) Recall that if $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal containing $\mathfrak{a}$, then $\mathfrak{q}/\mathfrak{a}$ is a $\mathfrak{p}/\mathfrak{a}$-primary ideal of $R/\mathfrak{a}$. Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a primary decomposition of I with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ such that $\dim(R/\mathfrak{p}_i) = \dim(R/I)$ for $1 \le i \le m$ and $\dim(R/\mathfrak{p}_i) < \dim(R/I)$ for $i > m$, where $m \le n$. Then

$$I/\mathfrak{a} = \big(\mathfrak{q}_1/\mathfrak{a}\big) \cap \cdots \cap \big(\mathfrak{q}_n/\mathfrak{a}\big)$$

is a primary decomposition of $I/\mathfrak{a}$. Now observe that

$$\dim\left(\frac{R/\mathfrak{a}}{I/\mathfrak{a}}\right) = \dim\left(\frac{R/\mathfrak{a}}{\mathfrak{p}_i/\mathfrak{a}}\right) \quad \text{if and only if} \quad \dim\left(\frac{R}{I}\right) = \dim\left(\frac{R}{\mathfrak{p}_i}\right).$$

Hence we get that

$$(I/\mathfrak{a})^{\mathrm{eq}} = \bigcap_{k=1}^{m}(\mathfrak{q}_k/\mathfrak{a}) = \left(\bigcap_{k=1}^{m}\mathfrak{q}_k\right)/\mathfrak{a} = I^{\mathrm{eq}}/\mathfrak{a}.$$

(iv) By definition of equidimensional part ideal, one has that

$$I^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n,$$

where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary ideal with $\dim(R/\mathfrak{p}_i) = \dim(R/I)$ for all $1 \leq i \leq n$. Note that $I^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is a primary decomposition for I^{eq}. Thus, since

$$\dim(R/\mathfrak{p}_i) = \dim(R/I) = \dim(R/I^{\mathrm{eq}})$$

for all $1 \leq i \leq n$, we conclude that $(I^{\mathrm{eq}})^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = I^{\mathrm{eq}}$.

(v) Notice that $\dim(R/I) = \dim(R/J)$ when I and J have the same radical. Hence

$$\dim(R/I^{\mathrm{eq}}) = \dim(R/I) = \dim(R/J) = \dim(R/J^{\mathrm{eq}}).$$

□

Definition 1.6.4. *Let R be a ring. One says that R satisfies the dimension equality if*

$$\dim(R) = \mathrm{ht}(I) + \dim(R/I)$$

for any ideal I of R.

In order to verify the validity of this property, it is necessary and sufficient to check that the equality holds only for prime ideals of the ring R as one can see in the following proposition.

Proposition 1.6.5. *Let R be a ring. If $\dim(R) = \dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p})$ for all prime ideal $\mathfrak{p}$ of R, then, for all ideal I of R, one has*

$$\dim(R) = \dim\left(\frac{R}{I}\right) + \mathrm{ht}(I).$$

Proof: Let I be an ideal of R. If either $\mathrm{ht}(I) = \infty$ or $\dim(R/I) = \infty$, it is clear that $\dim(R) = \dim(R/I) + \mathrm{ht}(I) = \infty$. Hence we can suppose without lost of generality that $\mathrm{ht}(I) < \infty$ and $\dim(R/I) < \infty$. Let $\mathfrak{p} \in \mathrm{MinV}(I)$ such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$. Then

$$\mathrm{ht}(I) + \dim(R/I) \geq \mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R).$$

Since the reverse inequality always holds, the statement follows. $\square$

We can present the following classes of rings which satisfy this property.

Example 1.6.6. The following classes of rings satisfy the dimension equality

(i) Cohen-Macaulay local rings [BH93, Corollary 2.1.4]*;*

(ii) Finitely generated algebras over a field [Mat80, Corollary 14.3.3]*;*

(iii) Local catenary domains [Mat89, Page 31].

Observe that if R is finite dimensional ring which satisfies the dimension equality, then, for a prime ideal $\mathfrak{p}$ of R, one has $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$ if and only if $\dim(R/\mathfrak{p}) = \dim(R/I)$. Hence

Proposition 1.6.7. *Let R be a finite-dimensional Noetherian ring which satisfies the dimension equality.*

(i) The unmmixed part of any ideal I coincides with its equidimensional part;

(ii) An ideal I is unmixed if and only if I is dimensional unmixed.

Proof: (i) Since R satisfies the dimension equality, then

$$\{\mathfrak{p} \in \mathrm{ass}_R(I) \ ; \ \dim(R/I) = \dim(R/\mathfrak{p})\} = \{\mathfrak{p} \in \mathrm{ass}_R(I) \ ; \ \mathrm{ht}(I) = \mathrm{ht}(\mathfrak{p})\}.$$

Thus the equality $I^{\mathrm{unm}} = I^{\mathrm{eq}}$ follows by definition.

(ii) If I is unmixed, then $I = I^{\mathrm{unm}} = I^{\mathrm{eq}}$, so I is dimensional unmixed. The converse is similar. $\square$

Definition 1.6.8. *Let R be a Noetherian ring and I an ideal of R. An associated prime $\mathfrak{p}$ of I is called height associated prime of I if* $\dim(R/I) = \dim(R/\mathfrak{p})$*. The set of height associated primes of I is denoted by* $\mathrm{assh}_R(I)$ *or* $\mathrm{Assh}_R(R/I)$*. Thus*

$$\mathrm{Assh}_R(R/I) = \mathrm{assh}_R(I) = \{\mathfrak{p} \in \mathrm{ass}_R(I) \ ; \ \dim(R/I) = \dim(R/\mathfrak{p})\}.$$

It follows directly from definitions that $\operatorname{assh}(I) = \operatorname{assh}(I^{\mathrm{eq}})$. The following proposition establishes that the multiplicity of R/I coincides with the multiplicity of R/I^{eq}.

Proposition 1.6.9. *Let R be a Noetherian local ring and I an ideal of R. Then*

$$e(R/I) = e(R/I^{\mathrm{eq}}).$$

In particular, if R satisfy the dimension equality, then $e(R/I) = e(R/I^{\mathrm{unm}})$.

Proof: Observe that $\operatorname{assh}(I) = \operatorname{assh}(I^{\mathrm{eq}})$ for any ideal I and that $I_{\mathfrak{p}} = I^{\mathrm{eq}}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{assh}(I)$. By Theorem 1.5.7, one gets

$$e(R/I) = \sum_{\mathfrak{p}\in\operatorname{assh}(I)} e(R/\mathfrak{p})\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I_{\mathfrak{p}}) = \sum_{\mathfrak{p}\in\operatorname{assh}(I^{\mathrm{eq}})} e(R/\mathfrak{p})\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I^{\mathrm{eq}}_{\mathfrak{p}}) = e(R/I^{\mathrm{eq}}).$$

□

Lemma 1.6.10. *Let R be ring and $\mathfrak{a} \subseteq I$ ideals of R. If I is a finitely generated ideal of R, then*

$$\operatorname{Ass}(R/(\mathfrak{a} :_R I)) \subseteq \operatorname{ass}(R/\mathfrak{a}).$$

Proof: Considering $\mathbf{f} = f_1, \dots, f_r$ a system of generators of I and the R-module monomorphism $\phi : R/(\mathfrak{a} :_R I) \longrightarrow \bigoplus_{k=1}^r R/\mathfrak{a}$ such that

$$\phi(\overline{x}) = (\overline{f_1 x}, \dots, \overline{f_r x}),$$

one has the following exact sequence

$$0 \longrightarrow R/(\mathfrak{a} :_R I) \xrightarrow{\ \phi\ } \textstyle\bigoplus_{k=1}^r R/\mathfrak{a} \longrightarrow R/I \longrightarrow 0.$$

Hence

$$\operatorname{Ass}\big(R/(\mathfrak{a} :_R I)\big) \subseteq \operatorname{Ass}\left(\bigoplus_{k=1}^r R/\mathfrak{a}\right) = \operatorname{Ass}(R/\mathfrak{a}).$$

□

Proposition 1.6.11. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq I$ ideals of R. Considering the ideal $J = \mathfrak{a} :_R I$, then*

$$\dim(R/J) = \dim(R/\mathfrak{a}) \qquad \textit{if and only if} \qquad \operatorname{assh}(J) \subseteq \operatorname{assh}(\mathfrak{a}).$$

Proof: Suppose that $\dim(R/J) = \dim(R/\mathfrak{a})$ and let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \in \operatorname{assh}(J)$. Thus $\mathfrak{p} \in \operatorname{ass}(J)$ and $\dim(R/\mathfrak{p}) = \dim(R/J)$. Since $\dim(R/J) = \dim(R/\mathfrak{a})$, we have

$$\dim\left(\frac{R}{\mathfrak{p}}\right) = \dim\left(\frac{R}{J}\right) = \dim\left(\frac{R}{\mathfrak{a}}\right).$$

Since we always have $\operatorname{ass}(J) \subseteq \operatorname{ass}(\mathfrak{a})$, then $\mathfrak{p} \in \operatorname{assh}(\mathfrak{a})$ and we get the desired inclusion.

Conversely suppose that $\operatorname{assh}(J) \subseteq \operatorname{assh}(\mathfrak{a})$. Let $\mathfrak{p} \in \operatorname{MinV}(J)$ such that $\dim(R/\mathfrak{p}) = \dim(R/J)$. Thus $\mathfrak{p} \in \operatorname{assh}(J) \subseteq \operatorname{assh}(\mathfrak{a})$, which implies that

$$\dim(R/J) = \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{a}).$$

□

Proposition 1.6.12. *Let R be a Noetherian ring and I, J ideals of R such that $\mathrm{V}(I) = \mathrm{V}(J)$.*

(i) $\operatorname{assh}(I) = \operatorname{assh}(J)$*;*

(ii) If $I \subseteq J$, then $I^{\mathrm{eq}} \subseteq J^{\mathrm{eq}}$;

(iii) If $I \subseteq J$, then $I^{\mathrm{unm}} \subseteq J^{\mathrm{unm}}$.

Proof: (i) Recall that if $\mathrm{V}(I) = \mathrm{V}(J)$, then $\dim(R/I) = \dim(R/J)$. Hence

$$\mathfrak{p} \in \operatorname{assh}(I) \iff \dim(R/\mathfrak{p}) = \dim(R/I) = \dim(R/J) = \dim(R/\mathfrak{p}) \iff \mathfrak{p} \in \operatorname{assh}(J),$$

which implies $\operatorname{assh}(I) = \operatorname{assh}(J)$.

(ii) Observe that $\operatorname{MinAss}(R/I) = \operatorname{MinAss}(R/J) = \operatorname{MinV}(I) = \operatorname{MinV}(J)$ and $\operatorname{ht}(I) = \operatorname{ht}(J)$. Denote

$$\operatorname{assh}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \operatorname{assh}(J).$$

Next consider the primary decompositions of I and J

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{q}_{n+1} \cap \cdots \cap \mathfrak{q}_{n+t} \qquad \text{and} \qquad J = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_n \cap \mathfrak{q}'_{n+1} \cap \cdots \cap \mathfrak{q}'_{n+t'},$$

where $\sqrt{\mathfrak{q}_i} = \sqrt{\mathfrak{q}'_i} = \mathfrak{p}_i$ for all $i = 1, \ldots, n$. Now, given $1 \leq i \leq n$, one has

$$(\mathfrak{q}_i)_{\mathfrak{p}_i} = I_{\mathfrak{p}_i} \subseteq J_{\mathfrak{p}_i} = (\mathfrak{q}'_i)_{\mathfrak{p}_i}$$

Thus $\mathfrak{q}_i = (\mathfrak{q}_i)_{\mathfrak{p}_i} \cap R \subseteq (\mathfrak{q}'_i)_{\mathfrak{p}_i} \cap R = \mathfrak{q}'_i$. Finally, since, by definition one has that

$$I^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \qquad \text{and} \qquad J^{\mathrm{eq}} = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_n,$$

one gets the desired inclusion. $\square$

(iii) The proof is similar. Notice that

$$\{\mathfrak{p} \in \operatorname{ass}(I) \ ; \ \operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p})\} = \{\mathfrak{p} \in \operatorname{ass}(J) \ ; \ \operatorname{ht}(J) = \operatorname{ht}(\mathfrak{p})\}$$

and denote this set by $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Next consider the primary decompositions of I and J

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{q}_{n+1} \cap \cdots \cap \mathfrak{q}_{n+t} \qquad \text{and} \qquad J = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_n \cap \mathfrak{q}'_{n+1} \cap \cdots \cap \mathfrak{q}'_{n+t'},$$

where $\sqrt{\mathfrak{q}_i} = \sqrt{\mathfrak{q}'_i} = \mathfrak{p}_i$ for all $i = 1, \ldots, n$. Notice that, given $1 \leq i \leq n$, one has

$$(\mathfrak{q}_i)_{\mathfrak{p}_i} = I_{\mathfrak{p}_i} \subseteq J_{\mathfrak{p}_i} = (\mathfrak{q}'_i)_{\mathfrak{p}_i}$$

Thus $\mathfrak{q}_i = (\mathfrak{q}_i)_{\mathfrak{p}_i} \cap R \subseteq (\mathfrak{q}'_i)_{\mathfrak{p}_i} \cap R = \mathfrak{q}'_i$. Hence the inclusion follows. $\square$

Part I

Generic Kitt and Deformation of Residual Intersections

Chapter 2

Residual Intersections and Kitt Ideals

2.1 Definitions and Known Results

The theory of Liaison of algebraic varieties is a fundamental topic in the classification of algebraic varieties, with its origins dating back to the 19th century in algebraic geometry. Over time, the theory has undergone significant developments, becoming a central theme in the works of Gaeta and Apery during the 1950s. Later, Peskine and Szpiro, [PS74] established the theory in the modern language during the 1970s, leading to its extension by Artin and Nagata [AN72] to the concept of residual intersections.

The theory of s-residual intersections has been further developed by Huneke, Ulrich [HU87, Hun83, HU88] and other researchers [CEU15] and continues to be an active research field.

Definition 2.1.1. *Let R be a Noetherian ring, s a non-negative integer and I an ideal of R with $\mathrm{ht}(I) \leq s$.*

(i) An (algebraic) s-residual intersection of I is a proper ideal J of R such that $\mathrm{ht}(J) \geq s$ and it is of form $J = \mathfrak{a} :_R I$ for some ideal $\mathfrak{a} \subset I$ generated by s elements;

(ii) A geometric s-residual intersection of I is an algebraic s-residual intersection J of I such that $\mathrm{ht}(I + J) > s$;

(iii) An arithmetic s-residual intersection of I is an algebraic s-residual intersection $J = \mathfrak{a} :_R I$ of I such that $\mu_{R_\mathfrak{p}}((I/\mathfrak{a})_\mathfrak{p}) \leq 1$ for all $\mathfrak{p} \in \mathrm{V}(I+J)$ with $\mathrm{ht}(\mathfrak{p}) \leq s$, where μ denotes the minimum number of generators.

It follows directly from definition that every geometric residual intersection is an arithmetic residual intersection. One of the main tools in our study of residual intersections is the *Residual Approximation Complexes*, that we now describe.

Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Let $S = R[T_1, \ldots, T_r]$ be the standard polynomial extension of R in r indeterminates and

$$\gamma_j = \sum_{i=1}^{r} c_{ij} T_i \in S_1$$

for $1 \leq j \leq s$. Then we consider the $\mathcal{Z}$-complex, $\mathcal{Z}_\bullet(\mathbf{f}; R)$, defined in Section 1.4.

$$0 \to Z_r(\mathbf{f}; R) \otimes_R S[-r] \to Z_{r-1}(\mathbf{f}; R) \otimes_R S[-r+1] \to \cdots \to Z_0(\mathbf{f}; R) \otimes_R S \to 0,$$

where $Z_i(\mathbf{f}; R)$ is the i-th module of Koszul cycles of the sequence $\mathbf{f}$. In the sequel we denote the sequence $\underline{\gamma} = \gamma_1, \ldots, \gamma_s$ and consider the new complex given by

$$\mathcal{D}_\bullet = \mathrm{Tot}\left(\mathcal{Z}_\bullet(\mathbf{f}; R) \otimes_S K_\bullet(\underline{\gamma}; S)\right),$$

where $K_\bullet = K(\underline{\gamma}; S)$ is the Koszul complex of the sequence $\underline{\gamma}$. Notice that the i-th component of this complex is given by

$$\mathcal{D}_i = \bigoplus_{k+j=i} \left(Z_k(\mathbf{f}; R) \otimes_R S(-k)\right) \otimes_S S^{\binom{s}{j}}(-j) \simeq \bigoplus_{k+j=i} \left(Z_k(\mathbf{f}; R) \otimes_R S^{\binom{s}{j}}\right)(-i).$$

Consider the ideal $\mathfrak{t} = (T_1, \ldots, T_r) \subseteq S$. Tensoring the complex $\mathcal{D}_\bullet$ with the Čech complex $\check{C}^\bullet_\mathfrak{t}(S)$, one can proceed as on the construction of the Koszul-Čech complexes and create a $\mathbb{Z}$-indexed family of complexes ${}_k\mathcal{Z}^+_\bullet(\mathbf{f}, \mathbf{a}, \Phi)$, where for each $k \in \mathbb{Z}$, one has

$$ {}_k\mathcal{Z}^+_\bullet(\mathbf{f}, \mathbf{a}, \Phi) : \qquad 0 \to H^r_{\mathfrak{t}}(\mathcal{D}_{r+s})_{(k)} \to \cdots \to H^r_{\mathfrak{t}}(\mathcal{D}_{r+k})_{(k)} \xrightarrow{\tau_k} (\mathcal{D}_k)_{(k)} \to \cdots \to (\mathcal{D}_0)_{(k)} \to 0. $$

Hence we have the following definition.

Definition 2.1.2. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively and let $\Phi = [c_{ij}]$ be an $r \times s$ matrix as above. The complex ${}_k\mathcal{Z}^+_\bullet = {}_k\mathcal{Z}^+_\bullet(\mathbf{f}, \mathbf{a}, \Phi)$ constructed above is called the k-th residual approximation complex with respect to the system of generators $\mathbf{f}$ and $\mathbf{a}$ of I and $\mathfrak{a}$, respectively and the matrix Φ.*

Among the whole family, we focus on $k = 0$ and we denote this complex simply as $\mathcal{Z}^+_\bullet = \mathcal{Z}^+_\bullet(\mathbf{f}, \mathbf{a}, \Phi)$.

$$\begin{tikzcd} 0 \arrow[r] & H^r_{\mathfrak{t}}(\mathcal{D}_{r+s})_{(0)} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & H^r_{\mathfrak{t}}(\mathcal{D}_r)_{(0)} \arrow[r, "\tau_0"] \arrow[d, equal] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{Z}^+_{s+1} \arrow[r] & \cdots \arrow[r] & \mathcal{Z}^+_1 \arrow[r, "\tau_0"] & \mathcal{Z}^+_0 \arrow[r] & 0 \end{tikzcd} .$$

In particular $\mathcal{Z}^+_{s+1}(\mathbf{a}, \mathbf{f}, \Phi) = 0$ if $\operatorname{grade}(I) > 0$, because $Z_r(\mathbf{f}; R) = 0$. Notice

$$ H_0(\mathcal{Z}^+_\bullet(\mathbf{f}, \mathbf{a}, \Phi)) = R/K(\mathbf{a}, \mathbf{f}, \Phi), $$

where $K(\mathbf{a}, \mathbf{f}, \Phi)$ is the image of the map τ_0 and it is an ideal of R. This ideal is called the *disguised residual intersection with respect to the systems of generators* $\mathbf{a}, \mathbf{f}$ *and the representation matrix* Φ. This term finds its justification in its close relation with the concept of residual intersection as elucidated in the following proposition

Proposition 2.1.3. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{a} = a_1, \ldots, a_s$ and $\mathbf{f}$ systems of generators of $\mathfrak{a}$ and I, respectively, $K = K(\mathbf{a}, \mathbf{f}, \Phi)$ the disguised residual intersection with respect to the systems of generators $\mathbf{a}, \mathbf{f}$ and the representation matrix Φ and $J = \mathfrak{a} :_R I$.*

(i) One has that $K \subseteq J$ and $\mathrm{V}(K) = \mathrm{V}(J)$ [Has12, Theorem 2.11]*;*

(ii) One has the equality $K = J$ if $\mathbf{f} = a_1, \ldots, a_s, b$ for any $b \in R$ [HN16, Theorem 4.4]*;*

(iii) If R is a Cohen-Macaulay local ring, J is an s-residual intersection and I satisfies SD_1 condition, then $K = J$ and it is a Cohen-Macaulay ideal [BH19, Theorem 5.1].

Until before 2019, the disguised residual intersection ideal was somewhat mysterious. The scenario began to change with Hassanzadeh and Bouça [BH19], where they provide an alternative characterization of this ideal as follows:

Theorem 2.1.4 (Theorem 4.9 of [BH19]). *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Let $K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ be the Koszul complex equipped with structure of differential graded algebra. Let $\zeta_j = \sum_{i=1}^r c_{ij} e_i$ for $1 \leq j \leq s$, $\Gamma_\bullet = R\langle \zeta_1, \ldots, \zeta_s\rangle$ be the sub-algebra of $K_\bullet(\mathbf{f}; R)$ generated by the ζ's and $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ the algebra of Koszul cycles. Looking at the elements of degree r in the sub-algebra of $K_\bullet(\mathbf{f}; R)$ generated by the product of $\Gamma_\bullet$ and $Z_\bullet$, one has

$$K(\mathbf{a}, \mathbf{f}, \Phi) = \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r.$$

Furthermore, still in [BH19, Propositions 4.11, 4.12, 4.15], they establish that the ideal $\langle \Gamma_\bullet \cdot Z_\bullet \rangle_r$ is independent of the choice of the representation matrix Φ and the generating sets of I and $\mathfrak{a}$. This invariance led them to define the notion of *Koszul-Fitting Ideals*.

Definition 2.1.5. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Let $K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$ be the Koszul complex equipped with structure of differential graded algebra. Let $\zeta_j = \sum_{i=1}^r c_{ij} e_i$ for $1 \leq j \leq s$, $\Gamma_\bullet = R\langle \zeta_1, \ldots, \zeta_s\rangle$ be the sub-algebra of $K_\bullet(\mathbf{f}; R)$

generated by the ζ's and $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ the algebra of Koszul cycles. The Koszul-Fitting ideal of I with respect to $\mathfrak{a}$ or, simply, Kitt ideal of I with respect to $\mathfrak{a}$ is the ideal

$$\operatorname{Kitt}(\mathfrak{a}, I) := \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r.$$

In particular, they prove that disguised residual intersections are independent of the choice of generating sets for the pair of ideals and of the representation matrix relating these generators. In what follows, we list some of the properties of Kitt ideals. Some of these properties can be proved easily, while for others we cite the appropriate reference.

Theorem 2.1.6. *Let R be a Noetherian ring, $\mathfrak{a} \subseteq I$ two finitely generated ideals of R and s a non-negative integer. Suppose that I and $\mathfrak{a}$ are generated by r and s elements, respectively.*

(i) *The ideal $\operatorname{Kitt}(\mathfrak{a}, I)$ does not depend of the choice of generators of $\mathfrak{a}$ and I or the representative matrix* [BH19, Propositions 4.11, 4.12 and 4.15]*;*

(ii) *Considering $\mathbf{f}$ a system of generators of I and denoting by $\tilde{H}_\bullet$ the sub-algebra of $K_\bullet(\mathbf{f}; R)$ generated by representatives of the Koszul homologies, then*

$$\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} + \langle \Gamma_\bullet \cdot \tilde{H}_\bullet \rangle_r.$$

Furthermore, one has $\operatorname{Kitt}(\mathfrak{a}, I) \supseteq \operatorname{Fitt}_0(I/\mathfrak{a})$ [BH19, Theorem 4.23]*;*

(iii) *One always has that $\operatorname{Kitt}(\mathfrak{a}, I) \subseteq \mathfrak{a} :_R I$ and they determine the same Zariski closed set. In addition if $\mu(I/\mathfrak{a}) \leq 1$, then $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$* [HN16, Theorem 4.4], [Has12, Theorem 2.11]*;*

(iv) *One always has that $\operatorname{Kitt}(0, I) = 0 :_R I$, $\operatorname{Kitt}(\mathfrak{a}, \mathfrak{a}) = (1)$ and $\operatorname{Kitt}(\mathfrak{a}, (1)) = \mathfrak{a}$;*

(v) *Given ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}$ and $\mathfrak{a} \subseteq I_1 \subseteq I$, then $\operatorname{Kitt}(\mathfrak{a}, I) \subseteq \operatorname{Kitt}(\mathfrak{a}, I_1)$ and $\operatorname{Kitt}(\mathfrak{a}_1, I) \subseteq \operatorname{Kitt}(\mathfrak{a}, I)$* [BH19, Theorem 4.23]*;*

(vi) *The Kitt ideal localizes. In other words, $\operatorname{Kitt}(\mathfrak{a}, I)_\mathfrak{p} = \operatorname{Kitt}(\mathfrak{a}_\mathfrak{p}, I_\mathfrak{p})$ for any prime ideal $\mathfrak{p}$;*

(vii) *Kitt specializes. In other words, for any regular element $f_0 \in \mathfrak{a}$*

$$\frac{\operatorname{Kitt}(\mathfrak{a}, I)}{(f_0)} = \operatorname{Kitt}\left(\frac{\mathfrak{a}}{(f_0)}, \frac{I}{(f_0)}\right)$$

[BH19, Theorem 4.27]*;*

(viii) Let $\mathrm{grade}(I) = g$ *and assume that* $\mathrm{grade}(\mathfrak{a} :_R I) \geq s$. *If* $s \leq g+1$, *then* $\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$ [BH19, Proposition 5.10]*;*

(ix) If $r \leq s$, *then*

$$\mathrm{Kitt}(\mathfrak{a}, I) = \mathfrak{a} + \mathrm{Fitt}_0(I/\mathfrak{a}) + \sum_{\{i_1,\ldots,i_{r-2}\} \subset \{1,\ldots,s\}} \mathrm{Kitt}((a_{i_1}, \ldots, a_{i_{r-2}}), I)$$

[Tar21, Proof of Theorem 4.7];

(x) If $r \geq s$, *then* $\mathrm{Kitt}(\mathfrak{a}, I) = \sum_{i=1}^{s} \mathrm{Kitt}((a_1, \ldots, \hat{a_i}, \ldots, a_s), I) + \zeta_1 \cdot \ldots \cdot \zeta_s \cdot Z_{r-s}$*;*

(xi) Let $\mathfrak{b}$ *be any ideal of* R *with* $I \cap \mathfrak{b} = 0$. *Then* $\overline{\mathrm{Kitt}_R(\mathfrak{a}, I)} = \mathrm{Kitt}_{\bar{R}}(\bar{\mathfrak{a}}, \bar{I})$, *where* $^{-}$ *denotes the image of canonical epimorphism of* R *to* $R/\mathfrak{b}$ [Tar21, Proof of Theorem 4.11]*;*

(xii) Suppose that R *is Cohen-Macaulay and* $J := \mathfrak{a} :_R I$ *is an* s*-residual intersection of* I. *If* I *satisfies* SD_1, *then* $\mathrm{Kitt}(\mathfrak{a}, I) = J$ *and it is a Cohen-Macaulay ideal* [BH19, Theorem 5.1]*;*

(xiii) Suppose that R *is Cohen-Macaulay and* $J = \mathfrak{a} :_R I$ *is an* s*-residual intersection of* I. *If* I *satisfies the* G_s *condition and is weakly* $(s-2)-$*residually* S_2, *then* $\mathrm{Kitt}(\mathfrak{a}, I) = J$. [Tar21, Theorem 4.11].

There are a few definitions in the last theorem that need to be remembered.

Definition 2.1.7. *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional Cohen-Macaulay local ring,* I *an ideal of* R *with* $\mathrm{grade}(I) = g$ *and* s *a integer*

(i) One says that I *satisfies the* G_s *condition if* $\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p})$ *for any prime ideal* $\mathfrak{p} \in \mathrm{V}(I)$ *of height at most* $s-1$*;*

(ii) One says that I *is weakly* $(s-2)-$*residually* S_2 *if, for all* $g \leq i \leq s-2$ *and every geometric* $i-$*residual intersection* $J = \mathfrak{a} :_R I$ *of* I, *the ring* R/J *satisfies the Serre's condition* S_2, *that is* $\mathrm{depth}\left((R/J)_\mathfrak{p}\right) \geq \min\{2, \dim\left((R/J)_\mathfrak{p}\right)\}$;

(iii) Let $\mathbf{f} = f_1, \ldots, f_r$ *be a system of generators of* I *and* k *an integer. One says that* I *satisfies the* SD_k *condition if* $\mathrm{depth}(H_i(\mathbf{f}; R)) \geq \min\{d-g, d-r+i+k\}$ *for all* $i \geq 0$*; also* SD *stands for* SD_0. *Similarly, one says that* I *satisfies the sliding depth condition on cycles,* SDC_k, *at level* t, *if* $\mathrm{depth}(Z_i(\mathbf{f}; R)) \geq \min\{d-r+i+k, d-g+2, d\}$ *for all* $r-g-t \leq i \leq r-g$.

In [BH19] it is conjectured that, for an $s-$residual intersection $J = \mathfrak{a} :_R I$ in a Cohen-Macaulay ring, one always has $J = \mathrm{Kitt}(\mathfrak{a}, I)$. However L. Busé and V. Bouça found a counterexample for that conjecture in December 2022.

Example 2.1.8. Consider $R = (\mathbb{Z}/101\mathbb{Z})[x_1, \ldots, x_4]$ and $I = (x_1, x_2^2 - x_3x_4)(x_3, x_4)$. Let M be any random matrix with entries of degrees $\{2, 2, 1, 1\}$ and define $\mathfrak{a}$ the ideal generated by the entries of $\mathrm{gens}(I) \cdot M$. Although the ideal $J = \mathfrak{a} :_R I$ is a $4-$residual intersection of I, one has that $J \neq \mathrm{Kitt}(\mathfrak{a}, I)$.

2.2 Flatness Behaviour of $\mathcal{Z}_\bullet^+$-complex and Kitt Ideal

The objective of this section is to present the well-behavior of Kitt ideals with flat ring extensions. In essence, we establish the proof that Kitt ideals commute with flat ring extensions. Actually, we are going the following stronger result:

Proposition 2.2.1. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Let $\Phi = [c_{ij}]$ be an $r \times s$ matrix in R such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Given a flat R-algebra A and denoting $\mathbf{a}A, \mathbf{f}A$ and ΦA the extensions in A of the sequences $\mathbf{a}$, $\mathbf{f}$ and the coefficients of the matrix Φ in A, one has

$$\mathcal{Z}_\bullet^+(\mathbf{a}, \mathbf{f}, \Phi) \otimes_R A = \mathcal{Z}_\bullet^+(\mathbf{a}A, \mathbf{f}A, \Phi A).$$

Proof: Firstly observe that $\mathfrak{a}A$ and IA are finitely generated ideals of A and $\mathfrak{a}A \subseteq IA$. Recall that $\mathcal{Z}_\bullet^+ = \mathcal{Z}_\bullet^+(\mathbf{a}, \mathbf{f}, \Phi)$ is the complex

$$\mathcal{Z}_\bullet^+ : \qquad 0 \longrightarrow \mathcal{Z}_{s+1}^+ \longrightarrow \cdots \longrightarrow \mathcal{Z}_1^+ \xrightarrow{\tau_0} \mathcal{Z}_0^+ = R \longrightarrow 0,$$

where

$$\mathcal{Z}_i^+ = \begin{cases} \bigoplus_{l+j=r+i-1} (K_l(\underline{\gamma}; S) \otimes_S H_{\mathfrak{t}}^r(S))_{[0]} \otimes_R Z_j(\mathbf{f}; R), & \text{if } i > 0; \\ R, & \text{if } i = 0, \end{cases} ,$$

and $S = R[T_1, \dots, T_r]$ and $\mathbf{t} = (T_1, \dots, T_r) \subseteq S$. Applying the exact functor $\_ \otimes_R A$ to $\mathcal{Z}^+_\bullet$, we obtain the complex $\mathcal{Z}^+_\bullet \otimes_R A$

$$0 \longrightarrow \mathcal{Z}^+_{s+1} \otimes_R A \longrightarrow \cdots \longrightarrow \mathcal{Z}^+_1 \otimes_R A \xrightarrow{\tau_0 \otimes_R 1_A} \mathcal{Z}^+_0 \otimes_R A = A \longrightarrow 0.$$

Denoting $B = A[T_1, \dots, T_r]$, since A is a flat A-algebra, we have

$$\mathcal{Z}^+_i \otimes_R A \cong \bigoplus_{l+j=r+i-1} (K_l(\underline{\gamma} A; B) \otimes_B H^r_{\mathbf{t}B}(B))_{[0]} \otimes_A Z_j(\mathbf{f}A; A) = \mathcal{Z}^+_i(\mathbf{a}A, \mathbf{f}A, \Phi A),$$

for $0 \leq i \leq s$ and the statement follows. $\square$

Corollary 2.2.2. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \dots, f_r$ and $\mathbf{a} = a_1, \dots .a_s$ systems of generators of I and $\mathfrak{a}$, respectively. Given a flat R-algebra A, then*

$$\mathrm{Kitt}_R(\mathfrak{a}, I)A = \mathrm{Kitt}_A(\mathfrak{a}A, IA).$$

Proof: By Theorem 2.1.4, ones has

$$\mathrm{Kitt}_R(\mathfrak{a}, I)A = K(\mathbf{a}, \mathbf{f}, \Phi)A = K(\mathbf{a}, \mathbf{f}, \Phi) \otimes_R A = K(\mathbf{a}A, \mathbf{f}A, \Phi A) = \mathrm{Kitt}_A(\mathfrak{a}A, IA).$$

$\square$

Corollary 2.2.2 gives us a series of immediate corollaries, which will be presented below.

Corollary 2.2.3. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R.*

(i) Given $S \subseteq R$ a multiplicatively closed subset, the Kitt ideal commutes with localization in S:

$$\mathrm{Kitt}_R(\mathfrak{a}, I)R_S = \mathrm{Kitt}_{R_S}(\mathfrak{a}_S, I_S).$$

In particular, $\mathrm{Kitt}_R(\mathfrak{a}, I)_\mathfrak{p} = \mathrm{Kitt}_{R_\mathfrak{p}}(\mathfrak{a}_\mathfrak{p}, I_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

(ii) Suppose that R is a Noetherian ring and let $\mathfrak{q}$ be a ideal of R. The ideal Kitt commutes with $\mathfrak{q}$-adic completion:

$$\mathrm{Kitt}_R(\mathfrak{a}, I)\hat{R}^\mathfrak{q} = \mathrm{Kitt}_{\hat{R}^\mathfrak{q}}(\hat{\mathfrak{a}}^\mathfrak{q}, \hat{I}^\mathfrak{q}).$$

(iii) Given $\{X_1, \dots, X_n\}$ a set of indeterminates over R, the Kitt ideal commutes with polynomial and formal extensions:

$$\mathrm{Kitt}(\mathfrak{a}[X_1, \dots, X_n], I[X_1, \dots, X_n]) = \mathrm{Kitt}(\mathfrak{a}, I)[X_1, \dots, X_n]$$
$$\mathrm{Kitt}(\mathfrak{a}[\![X_1, \dots, X_n]\!], I[\![X_1, \dots, X_n]\!]) = \mathrm{Kitt}(\mathfrak{a}, I)[\![X_1, \dots, X_n]\!].$$

Proof: Just note that R_S, $R[X_1, \ldots, X_n]$, $R[[X_1, \ldots, X_n]]$ and $\hat{R}^{\mathfrak{q}}$ (in the case when R is a Noetherian ring) are flat R-algebras. $\square$

Corollary 2.2.4. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R and $S \subset R$ a multiplicative subset of R. If $I_S/\mathfrak{a}_S$ is a principal ideal of $R_S/\mathfrak{a}_S$, then*

$$\operatorname{Kitt}_R(\mathfrak{a}, I)_S = \mathfrak{a}_S :_{R_S} I_S$$

Proof: Indeed, by Theorem 2.1.6 (iii), one has

$$\operatorname{Kitt}_R(\mathfrak{a}, I)_S = \operatorname{Kitt}_{R_S}(\mathfrak{a}_S, I_S) = \mathfrak{a}_S :_{R_S} I_S.$$

$\square$

An immediate consequence of the following corollary is that, to prove the equality of $\operatorname{Kitt}(\mathfrak{a}, I)$ and $\mathfrak{a} :_R I$ in a Noetherian local ring, we can assume without loss of generality that the ring R is complete.

Corollary 2.2.5. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Given a faithfully flat R-algebra A, then $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$ if and only if $\operatorname{Kitt}(\mathfrak{a}A, IA) = \mathfrak{a}A :_A IA$. In particular, if $(R, \mathfrak{m})$ is a local Noetherian ring, then*

$$\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I \qquad \textit{if and only if} \qquad \operatorname{Kitt}(\hat{\mathfrak{a}}^{\mathfrak{m}}, \hat{I}^{\mathfrak{m}}) = \hat{\mathfrak{a}}^{\mathfrak{m}} :_{\hat{R}^{\mathfrak{m}}} \hat{I}^{\mathfrak{m}}.$$

Proof: Since I is finitely generated ideal and A is flat R-algebra, then we have $\mathfrak{a}A :_A IA = (\mathfrak{a} :_R I)A$ by [Mat89, Theorem 7.4]. Thus, if $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$, Corollary 2.2.2 says that

$$\operatorname{Kitt}(\mathfrak{a}A, IA) = \operatorname{Kitt}(\mathfrak{a}, I)A = (\mathfrak{a} :_R I)A = \mathfrak{a}A :_A IA.$$

Conversely, define $M = (\mathfrak{a} :_R I)/\operatorname{Kitt}(\mathfrak{a}, I)$. If $\operatorname{Kitt}(\mathfrak{a}A, IA) = \mathfrak{a}A :_A IA$, then

$$M \otimes_R A = \frac{\mathfrak{a} :_R I}{\operatorname{Kitt}(\mathfrak{a}, I)} \otimes_R A = \frac{\mathfrak{a}A :_A IA}{\operatorname{Kitt}(\mathfrak{a}A, IA)} = 0.$$

Since A is a faithfully flat R-algebra, then $M = 0$ by [Mat89, Theorem 7.2], which implies $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$.

We conclude this section with an immediate consequence of Proposition 2.2.1, which is important and useful. Therefore, it has been explicitly included in the following.

Corollary 2.2.6. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots .a_s$ systems of generators of I and $\mathfrak{a}$, respectively and let Φ be the presentation matrix $[\mathbf{a}] = [\mathbf{f}] \cdot \Phi$. Given a faithfully flat R-algebra A, then $\mathcal{Z}_\bullet^+(\mathbf{f}, \mathbf{a}, \Phi)$ is an acyclic complex if and only if $\mathcal{Z}_\bullet^+(\mathbf{f}A, \mathbf{a}A, \Phi A)$ is an acyclic complex.*

Proof: Just observe that

$$H_i(\mathcal{Z}_\bullet^+(\mathbf{f}A, \mathbf{a}A, \Phi A)) = H_i(\mathcal{Z}_\bullet^+(\mathbf{f}, \mathbf{a}, \Phi) \otimes_R A) = H_i(\mathcal{Z}_\bullet^+(\mathbf{f}, \mathbf{a}, \Phi)) \otimes_R A = 0$$

if and only if $H_i(\mathcal{Z}_\bullet^+(\mathbf{f}, \mathbf{a}, \Phi)) = 0$. □

2.3 Commutativity of Kitt Ideal with Ring Isomorphisms

Other interesting property of Kitt ideals is their commutativity with ring isomorphisms. More explicitly, let R, R' be rings, $\mathfrak{a} \subseteq I$ finitely generated ideals of R and $\phi : R \longrightarrow R'$ a ring isomorphism, the objective of this section is to prove that

$$\phi\big(\operatorname{Kitt}(\mathfrak{a}, I)\big) = \operatorname{Kitt}(\phi(\mathfrak{a}), \phi(I)).$$

Since the Koszul complex has functorial property and the chain maps induce functorial maps in the Koszul cycles, one has the following result.

Lemma 2.3.1. *Let R, R' be rings and $\phi : R \longrightarrow R'$ a ring homomorphism. Consider $\mathbf{f} = f_1, \ldots, f_r$ a sequence in R and $\mathbf{g} = g_1, \ldots, g_r$ the sequence in R' such that $g_i = \phi(f_i)$ for all $i = 1, \ldots, r$. Let*

$$\Phi :\ (K_\bullet(\mathbf{f}; R), \partial) \longrightarrow (K_\bullet(\mathbf{g}; R'), \partial')$$

be the map induced by ϕ on Koszul complexes. Denoting $Z_\bullet(\mathbf{f}; R)$ and $Z_\bullet(\mathbf{g}; R')$ the algebras of cycles of the Koszul complexes $K_\bullet(\mathbf{f}; R)$ and $K_\bullet(\mathbf{g}; R')$, respectively, then $\Phi\big(Z_\bullet(\mathbf{f}; R)\big) \subseteq Z_\bullet(\mathbf{g}; R')$. Additionally, if ϕ is an isomorphism, then $\Phi\big(Z_\bullet(\mathbf{f}; R)\big) = Z_\bullet(\mathbf{g}; R')$.

Using this lemma, we can now establish the key result of this section. This theorem holds primordial role in Chapter 3 as it serves as a crucial step in demonstrating the uniqueness of the generic Kitt up to *universal equivalence*.

Proposition 2.3.2. *Let R, R' be rings and $\phi : R \longrightarrow R'$ a ring homomorphim. Given $\mathfrak{a} \subseteq I$ finitely generated ideals of R*

(i) Then $\phi(\mathrm{Kitt}_R(\mathfrak{a}, I)) \subseteq \mathrm{Kitt}_{R'}((\phi(\mathfrak{a})), (\phi(I)))$*, where* $(\phi(\mathfrak{a}))$*,* $(\phi(I))$ *are the ideals generated by* $\phi(\mathfrak{a})$ *and* $\phi(I)$*, respectively;*

(ii) If $\phi : R \longrightarrow R'$ *is an isomorphism, then* $\phi(\mathrm{Kitt}_R(\mathfrak{a}, I)) = \mathrm{Kitt}_{R'}(\phi(\mathfrak{a}), \phi(I))$.

Proof: (i) Let $\mathbf{a} = a_1, \ldots, a_s$ and $\mathbf{f} = f_1, \ldots, f_r$ be systems of generators of $\mathfrak{a}$ and I, respectively. One knows that there are $c_{11}, \ldots, c_{rs} \in R$ such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Notice that $\phi(\mathfrak{a})$ and $\phi(I)$ are finitely generated ideals of R' generated by $\mathbf{b} = b_1, \ldots, b_s$ and $\mathbf{g} = g_1, \ldots, g_r$, respectively. Moreover, these generators are related by

$$\begin{bmatrix} b_1 & b_2 & \cdots & b_s \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & \cdots & g_s \end{bmatrix} \begin{bmatrix} \phi(c_{11}) & \phi(c_{12}) & \cdots & \phi(c_{1s}) \\ \phi(c_{21}) & \phi(c_{22}) & \cdots & \phi(c_{2s}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(c_{r1}) & \phi(c_{r2}) & \cdots & \phi(c_{rs}) \end{bmatrix}.$$

Set

$$\gamma_j = \sum_{k=1}^{r} c_{kj} e_k \in K_1(\mathbf{f}; R) \subseteq K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r \,;\, \partial(e_i) = f_i \rangle$$

for $1 \leq j \leq s$ and let $\Gamma = \langle \gamma_1, \ldots, \gamma_s \rangle \subseteq K_\bullet(\mathbf{f}, R)$. Similarly, set

$$\lambda_j = \sum_{k=1}^{r} \phi(c_{kj}) e_k \in K_1(\mathbf{g}; R') \subseteq K_\bullet(\mathbf{g}; R') = R'\langle e_1, \ldots, e_r \,;\, \partial'(e_i) = g_i \rangle$$

for $1 \leq j \leq s$ and let $\Lambda = \langle \lambda_1, \ldots, \lambda_s \rangle \subseteq K_\bullet(\mathbf{g}; R')$. Let Φ be the map induced by ϕ on the Koszul complex $K_\bullet(\mathbf{f}; R)$. Let $z = re_1 \wedge \cdots \wedge e_r \in \mathrm{Kitt}(\mathfrak{a}, I)$. Then there are $n \in \mathbb{N}$, $L_{1_j} = \{i_{1_j}, \ldots, i_{k_j}\} \subseteq \{1, \ldots, s\}$ with $|L_{1_j}| = k_j$ and L_{2_j} such that $|L_2| = r - k_j$ such that

$$re_1 \wedge \cdots \wedge e_r = \sum_{j=1}^{n} \gamma_{i_{1_j}} \wedge \cdots \wedge \gamma_{i_{k_j}} \wedge z_{L_2, j},$$

where $z_{L_2,j} \in Z_{r-k_j}(\mathbf{f}; R)$ for all $1 \leq j \leq n$. Thus

$$\phi(z) = \Phi\left(\sum_{j=1}^{n} \gamma_{i_{1_j}} \wedge \cdots \wedge \gamma_{i_{k_j}} \wedge z_{L_2,j} \right) = \sum_{j=1}^{n} \lambda_{i_{1_j}} \wedge \cdots \wedge \lambda_{i_{k_j}} \wedge \Phi(z_{L_2,j}).$$

As Φ is homogeneous and $\Phi(z_{L_2,j}) \in Z_\bullet(\mathbf{g}; R)$, one gets that $\phi(z) \in \mathrm{Kitt}(\phi(\mathfrak{a}), \phi(I))$ and this proves the part *(i)*.

(ii) Now suppose that ϕ is a ring isomorphism and let $z = re_1 \wedge \cdots \wedge e_r \in \mathrm{Kitt}(\phi(\mathfrak{a}), \phi(I))$. One knows that there are $n \in \mathbb{N}$, $L_{1_j} = \{i_{1_j}, \ldots, i_{k_j}\} \subseteq \{1, \ldots, s\}$ with $|L_{1_j}| = k_j$ and L_{2_j} such that $|L_2| = r - k_j$ such that

$$z = \sum_{j=1}^{n} \lambda_{i_{1_j}} \wedge \cdots \wedge \lambda_{i_{k_j}} \wedge z_{L_2,j},$$

where $z_{L_2,j} \in Z_{r-k_j}(\mathbf{g}; R')$ for all $1 \leq 1 \leq n$. As $\Phi(\gamma_i) = \lambda_i$ for all $1 \leq i \leq s$ and there are $z'_{L_2,j} \in Z_{r-k_j}(\mathbf{f}; R)$ such that $\Phi(z'_{L_2,j}) = z_{L_2,j}$ because ϕ is isomorphism, one concludes that

$$\begin{aligned} z = \sum_{j=1}^{n} \lambda_{i_{1_j}} \wedge \cdots \wedge \lambda_{i_{k_j}} \wedge z_{L_2,j} &= \sum_{j=1}^{n} \Phi(\gamma_{i_{1_j}}) \wedge \cdots \wedge \Phi(\gamma_{i_{k_j}}) \wedge \Phi(z'_{L_2,j}) \\ &= \Phi\left(\sum_{j=1}^{n} \gamma_{i_{1_j}} \wedge \cdots \wedge \gamma_{i_{k_j}} \wedge z'_{L_2,j}\right). \end{aligned}$$

Hence $z \in \Phi(\mathrm{Kitt}(\mathfrak{a}, I))$ and the part *(ii)* follows. $\square$

2.4 Multiplicity and Kitt Ideals

In this section, we use the concepts of multiplicity and dimensional unmixedness (Section 1.6) to establish conditions wherein the Kitt ideal coincides with its corresponding quotient ideal.

Lemma 2.4.1. *Let R be a Noetherian local ring and $\mathfrak{a} \subseteq I$ be ideals of R. Suppose that* $\dim(R/\mathfrak{a}) = \dim(R/I)$ *and $e(R/\mathfrak{a}) = e(R/I)$. If $\mathfrak{a}$ is dimensional unmixed, then $I = \mathfrak{a}$.*

Proof: Notice that the Samuel polynomials of $R/\mathfrak{a}$ and R/I have the same leader terms. Hence, since one has the following exact sequence

$$0 \longrightarrow I/\mathfrak{a} \longrightarrow R/\mathfrak{a} \longrightarrow R/I \longrightarrow 0, \tag{2.1}$$

the additive property of the Samuel polynomials gives that

$$\dim\left(\frac{I}{\mathfrak{a}}\right) < \dim\left(\frac{R}{\mathfrak{a}}\right).$$

Suppose by contradiction that $\mathfrak{a}$ is strictly contained in I or, equivalently, that $\operatorname{Ass}(I/\mathfrak{a})$ is not the empty set and let

$$\mathfrak{p} \in \operatorname{MinAss}(I/\mathfrak{a}) = \operatorname{MinSupp}(I/\mathfrak{a}).$$

Using the sequence (2.1), one notes that $\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{a})$. Since $\mathfrak{a}$ is a dimensional unmixed ideal, then $\dim(R/\mathfrak{p}) = \dim(R/\mathfrak{a})$, which gives a contradiction, because there must be $\mathfrak{p} \in \operatorname{MinAss}(I/\mathfrak{a})$ such that

$$\dim\left(\frac{R}{\mathfrak{p}}\right) = \dim\left(\frac{I}{\mathfrak{a}}\right) < \dim\left(\frac{R}{\mathfrak{a}}\right).$$

$\square$

Proposition 2.4.2. *Let R be a Noetherian local ring and $\mathfrak{a} \subseteq I$ be ideals of R. Consider the quotient ideal $J = \mathfrak{a} :_R I$. If $\operatorname{Kitt}(\mathfrak{a}, I)$ is dimensional unmixed and $e(R/J) = e(R/\operatorname{Kitt}(\mathfrak{a}, I))$, then*

$$\operatorname{Kitt}(\mathfrak{a}, I) = J.$$

Proof: Since $\operatorname{Kitt}(\mathfrak{a}, I) \subseteq \mathfrak{a} :_R I$ and $\dim(\operatorname{Kitt}(\mathfrak{a}, I)) = \dim(R/(\mathfrak{a} :_R I))$, the conclusion follows from previous lemma. $\square$

Recall from Proposition 1.6.7 that unmixedness is equivalent to dimensional unmixedness when the ring satisfies the dimension equality. Thence

Corollary 2.4.3. *Let R be a Noetherian local ring which satisfies the dimension equality and $\mathfrak{a} \subseteq I$ ideals of R. Consider the quotient ideal $J = \mathfrak{a} :_R I$. If $\operatorname{Kitt}(\mathfrak{a}, I)$ is unmixed and $e(R/J) = e(R/\operatorname{Kitt}(\mathfrak{a}, I))$, then*

$$\operatorname{Kitt}(\mathfrak{a}, I) = J.$$

In the following we prove that the equidimensional part of the Kitt ideal is always contained in the equidimensional part of the quotient ideal. Furthermore, we will provide a condition that ensures their equality.

Proposition 2.4.4. *Let R be Noetherian local ring and $\mathfrak{a} \subseteq I$ ideals of R.*

(i) Considering the ideal $J = \mathfrak{a} :_R I$, then $\operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} \subseteq J^{\mathrm{eq}}$;

(ii) In addition, if $e(R/J) = e(R/\operatorname{Kitt}(\mathfrak{a}, I))$, then $\operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} = J^{\mathrm{eq}}$.

Proof: (i) Since $\mathrm{Kitt}(\mathfrak{a}, I)$ and J have the same radical, it follows from Proposition 1.6.12 that $\mathrm{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} \subseteq J^{\mathrm{eq}}$.

(ii) Observe that $\dim(R/J^{\mathrm{eq}}) = \dim(R/\mathrm{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}})$. Furthermore, by Proposition 1.6.9, one has $e(R/J^{\mathrm{eq}}) = e(R/J) = e(R/\mathrm{Kitt}(\mathfrak{a}, I)) = e(R/\mathrm{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}})$. Since $\mathrm{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}}$ is a dimensional unmixed ideal, then it follows from Proposition 2.4.2 that $\mathrm{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} = J^{\mathrm{eq}}$. $\square$

Finishing this section, we give a condition under which the saturation of Kitt coincides with the saturation of its associated quotient ideal. Recall that the saturation of an ideal I with respect to an ideal J, denoted by $I :_R J^{\infty}$, is the set of all elements $f \in R$ such that $J^n f \subseteq I$ for some $n \in \mathbb{N}$.

$$I :_R J^{\infty} := \bigcup_{n \in \mathbb{N}} (I :_R J^n) = \varinjlim_{n \in \mathbb{N}} (I :_R J^n).$$

In addition, if $(R, \mathfrak{m})$ is a local ring, one defines the *saturation of the ideal* I as $I^{\mathrm{sat}} := I :_R \mathfrak{m}^{\infty}$. Finally, if $(R, \mathfrak{m})$ is a *local ring and I a homogeneous ideal of R, one defines the *saturation of the ideal* I as $I^{\mathrm{sat}} := I :_R \mathfrak{m}^{\infty}$.

Lemma 2.4.5. *Let R be a Noetherian ring and I, J ideals of R. Consider $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ an irredundant primary decomposition of I. Suppose $J \not\subseteq \sqrt{\mathfrak{q}_i}$ for all $1 \leq i \leq k \leq n$ and $J \subseteq \sqrt{\mathfrak{q}_i}$ for all $k < i \leq n$. Then*

$$I :_R J^{\infty} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k.$$

Proof: Notice that, since there exists $m \in \mathbb{N}$ such that $J^m \subseteq \mathfrak{q}_i$ for all $k < i \leq n$, then $J^m \subseteq \mathfrak{q}_{k+1} \cap \cdots \cap \mathfrak{q}_n$. Thus, given $x \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$, one has

$$xJ^m \subseteq (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k)(\mathfrak{q}_{k+1} \cap \cdots \cap \mathfrak{q}_n) \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = I,$$

which implies that $x \in I :_R J^{\infty}$ and so $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k \subseteq I :_R J^{\infty}$. Conversely, let $x \in I :_R J^{\infty}$ and $m \in \mathbb{N}$ such that $xJ^m \subseteq I$. Thus, for all $1 \leq i \leq k$, one has

$$xJ^m \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \subseteq \mathfrak{q}_i.$$

Let $y_i \in J^m$ be chosen such that $y_i \notin \sqrt{\mathfrak{q}_i}$. Hence $xy_i \in \mathfrak{q}_i$, but no power of y_i belongs to $\mathfrak{q}_i$. As $\mathfrak{q}_i$ is a primary ideal, one concludes that $x \in \mathfrak{q}_i$. Hence $I :_R J^{\infty} \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ and the equality follows. $\square$

Proposition 2.4.6. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring (Noetherian *local ring, respectively) and $\mathfrak{a} \subseteq I$ ideals (homogeneous ideals, respectively) such that* $\operatorname{ht}(J) = d - 1$, *where* $J = \mathfrak{a} :_R I$. *If* $e(R/J) = e(R/\operatorname{Kitt}(\mathfrak{a}, I))$, *then*

$$J^{\mathrm{sat}} = \operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{sat}}.$$

Proof: Since J is an ideal with $\operatorname{ht}(J) = \dim(R) - 1$, then there exist essentially two possibilities for the irredundant primary decomposition of J

$$J = \begin{cases} \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{q}, & \text{if } J \text{ contains an } \mathfrak{m}\text{-primary component;} \\ \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n, & \text{otherwise.} \end{cases},$$

where $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i \in \mathrm{V}(J)$ with $\operatorname{ht}(\mathfrak{p}_i) = d-1$ for all $1 \leq i \leq n$ and $\sqrt{\mathfrak{q}} = \mathfrak{m}$. Observe that $\dim(R/\mathfrak{p}_i) = 1 = \dim(R/J)$ for every $1 \leq i \leq n$. By Lemma 2.4.5, one has $J^{\mathrm{eq}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = J^{\mathrm{sat}}$. Similarly, applying the same argument to $\operatorname{Kitt}(\mathfrak{a}, I)$, one also concludes that $\operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} = \operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{sat}}$. Finally, since $e(R/J) = e(R/\operatorname{Kitt}(\mathfrak{a}, I))$, Proposition 2.4.4 gives

$$J^{\mathrm{sat}} = J^{\mathrm{eq}} = \operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{eq}} = \operatorname{Kitt}(\mathfrak{a}, I)^{\mathrm{sat}}.$$

□

2.5 An interesting behavior of Kitt ideals

Let R be a Noetherian ring and $\mathfrak{a} \subseteq I$ ideals of R. Considering an R-regular element $h \in I$, we demonstrate that, if $\operatorname{grade}(I) \geq 2$, the quotient ideal $\mathfrak{a} :_R I$ exhibits the following property:

Proposition 2.5.1. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq I$ ideals of R. Let $h \in I$ be a non-zero divisor of R. If* $\operatorname{grade}(I) \geq 2$, *then*

$$h(\mathfrak{a} :_R I) = (h\mathfrak{a}) :_R I.$$

Proof: One inclusion is clear: let $z = hr \in h(\mathfrak{a} :_R I)$ for some $r \in \mathfrak{a} :_R I$. Hence, given $x \in I$, one has $zx = (hr)x = h(rx) \in h\mathfrak{a}$, so $z \in (h\mathfrak{a}) :_R I$ and $h(\mathfrak{a} :_R I) \subseteq (h\mathfrak{a}) :_R I$. Conversely let $h_2 \in I$ such that $\mathbf{h} = h, h_2$ is an R-regular sequence and let $z \in (h\mathfrak{a}) :_R I$. In particular, one has that $zh_2 \in h\mathfrak{a}$. By regularity of the sequence $\mathbf{h}$, one has that $z = hr$ for some $r \in R$. Now it is enough to show that $r \in \mathfrak{a} :_R I$. Given $x \in I$, one has $hrx = zx \in h\mathfrak{a}$. Since h is a nonzero divisor of R,

$rx \in \mathfrak{a}$ and the statement follows. $\square$

Unexpectedly, the same behavior does not occur with Kitt ideals, as illustrated by the following example.

Example 2.5.2. Let $R = k[x, y, z]$ be the polynomial ring with three indeterminates over a field k and consider the ideals

$$\mathfrak{a} = (xz, yz) \subset (xy,\ xz,\ yz) = I.$$

Set $h = xy$. Using *Macaulay2*[1], we obtain

$$h\,\mathrm{Kitt}(\mathfrak{a}, I) = (xyz) \neq (xy^2z,\ x^2yz) = \mathrm{Kitt}(h\mathfrak{a}, I).$$

However, using the grade of I, it is possible to obtain containment relations between $\mathrm{Kitt}(h\mathfrak{a}, I)$ and $h\,\mathrm{Kitt}(\mathfrak{a}, I)$.

Proposition 2.5.3. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq I$ ideals of R. Suppose that $g = \mathrm{grade}(I) > 0$ and let $h \in I$ be a non-zero divisor of R. Then*

$$\mathrm{Kitt}(h\mathfrak{a}, I) \subseteq h^g\,\mathrm{Kitt}(\mathfrak{a}, I) + h\mathfrak{a} \subseteq h\,\mathrm{Kitt}(\mathfrak{a}, I).$$

Proof: Let $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ be systems of generators of I and $\mathfrak{a}$, respectively. Consider $\Phi = [a_{ij}]_{r\times s}$ a matrix such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Recall that $\mathrm{Kitt}(\mathfrak{a}, I) = \sum_{k=0}^{r} \Gamma_k \cdot Z_{r-k}$, where $\Gamma_\bullet = R\langle \zeta_1, \cdots, \zeta_s\rangle \subseteq K_\bullet(\mathbf{f}; R) = R\langle e_1, \ldots, e_r; \partial(e_i) = f_i\rangle$, with $\zeta_j = \sum_{i=1}^{r} c_{ij} e_i$ for $1 \leq j \leq s$ and $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ is the algebra of Koszul cycles. On the other hand, observe that $h\mathfrak{a}$ is generated by $h\mathbf{a} = ha_1, \ldots, ha_s$ and

$$\begin{bmatrix} ha_1 & ha_2 & \cdots & ha_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} hc_{11} & hc_{12} & \cdots & hc_{1s} \\ hc_{21} & hc_{22} & \cdots & hc_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ hc_{r1} & hc_{r2} & \cdots & hc_{rs} \end{bmatrix}.$$

[1] The calculation is done by using the package `Kitt` available in Hassanzadeh's Homepage

Thus, by definition of Kitt and properties of wedge product, one concludes

$$\operatorname{Kitt}(h\mathfrak{a}, I) = \Gamma_0 \cdot Z_r + h\Gamma_1 \cdot Z_{r-1} + \cdots + h^{g-1}\Gamma_{g-1} \cdot Z_{r-g+1} + h^g\Gamma_g \cdot Z_{r-g} + \cdots + h^r\Gamma_r \cdot Z_0.$$

Since $\operatorname{grade}(I) = g > 0$, then $Z_i(\mathbf{f}; R) = B_i(\mathbf{f}; R)$ for all $i > r - g$ and $Z_r = 0$. Hence, by [BH19, Lemma 4.22], one has

$$\begin{aligned}\operatorname{Kitt}(h\mathfrak{a}, I) &\subseteq h\mathfrak{a} + \cdots + h^{g-1}\mathfrak{a} + h^g\Gamma_g \cdot Z_{r-g} + \cdots + h^r\Gamma_r \cdot Z_0 \subseteq h\mathfrak{a} + h^g(\Gamma_g \cdot Z_{r-g} + \cdots + \Gamma_r \cdot Z_0)\\ &\subseteq h\mathfrak{a} + h^g \operatorname{Kitt}(\mathfrak{a}, I).\end{aligned}$$

Finally, as $\mathfrak{a} \subseteq \operatorname{Kitt}(\mathfrak{a}, I)$, one concludes

$$\operatorname{Kitt}(h\mathfrak{a}, I) \subseteq h\mathfrak{a} + h^g \operatorname{Kitt}(\mathfrak{a}, I) \subseteq h \operatorname{Kitt}(\mathfrak{a}, I).$$

□

The following corollary provides the necessary and enough condition under which one has $\operatorname{Kitt}(h\mathfrak{a}, I) = h \operatorname{Kitt}(\mathfrak{a}, I)$.

Corollary 2.5.4. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq I$ ideals of R. Suppose that $g = \operatorname{grade}(I) > 0$ and let $h \in I$ be a non-zero divisor of R. If $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a}$, then $\operatorname{Kitt}(h\mathfrak{a}, I) = h \operatorname{Kitt}(\mathfrak{a}, I)$. The converse holds if $g \geq 2$ and $h \in \operatorname{Rad}(R)$, where $\operatorname{Rad}(R)$ denotes the Jacobson radical of R.*

Proof: If $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a}$, then

$$h \operatorname{Kitt}(\mathfrak{a}, I) = h\mathfrak{a} \subseteq \operatorname{Kitt}(h\mathfrak{a}, I) \subseteq h \operatorname{Kitt}(\mathfrak{a}, I).$$

Conversely suppose that $g \geq 2$. If $\operatorname{Kitt}(h\mathfrak{a}, I) = h \operatorname{Kitt}(\mathfrak{a}, I)$, then $h \operatorname{Kitt}(\mathfrak{a}, I) = h^g \operatorname{Kitt}(\mathfrak{a}, I) + h\mathfrak{a}$. As h is a non-zero divisor of R, one has

$$\operatorname{Kitt}(\mathfrak{a}, I) = h^{g-1} \operatorname{Kitt}(\mathfrak{a}, I) + \mathfrak{a}.$$

Passing the quotient modulo $\mathfrak{a}$, one gets

$$\frac{\operatorname{Kitt}(\mathfrak{a}, I)}{\mathfrak{a}} = h^{g-1}\left(\frac{\operatorname{Kitt}(\mathfrak{a}, I)}{\mathfrak{a}}\right).$$

By Nakayama Lemma, one concludes $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a}$. □

Corollary 2.5.5. *Let R be a Noetherian local ring and $\mathfrak{a} \subseteq I$ ideals of R. Suppose that $g = \operatorname{grade}(I) \geq 2$ and that $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I$. If there exists a non-zero divisor $h \in I$ such that $\operatorname{Kitt}(h\mathfrak{a}, I) = (h\mathfrak{a}) :_R I$, then $\operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a} :_R I = \mathfrak{a}$.*

Proof: Let $h \in I$ be a non-zero divisor such that $\operatorname{Kitt}(h\mathfrak{a}, I) = h\mathfrak{a} :_R I$. By Proposition 2.5.1, one has

$$h \operatorname{Kitt}(\mathfrak{a}, I) = h(\mathfrak{a} :_R I) = (h\mathfrak{a}) :_R I = \operatorname{Kitt}(h\mathfrak{a}, I).$$

Since R is local, the Corollary 2.5.4 implies that $I :_R \mathfrak{a} = \operatorname{Kitt}(\mathfrak{a}, I) = \mathfrak{a}$. □

2.6 The graded structure of the $\mathcal{Z}_\bullet^+$-complex

The main objective of this section is to calculate the graded structure of $\mathcal{Z}_\bullet^+$-complex. Let $R = \bigoplus_{k=0}^{\infty} R_k$ be a Noetherian graded ring and $\mathfrak{a} \subseteq I$ homogeneous ideals of R. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of homogeneous generators of I and $\mathfrak{a}$, respectively. Denote $\deg_R(a_i) = d_i$ for all $1 \le i \le s$ and $\deg_R(f_i) = \delta_i$ for all $1 \le i \le r$. Notice that there is an $r \times s$ matrix $\Phi = [c_{ij}]$ in R such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}.$$

Since each a_i and f_j are homogeneous elements, we can construct Φ such that each c_{ij} is a homogeneous element. In addition, one has that

$$\deg_R(c_{ij}) = \deg_R(a_j) - \deg_R(f_i) = d_j - \delta_i$$

for all $1 \le i \le r$ and $1 \le j \le s$.

Next let $S = R[T_1, \ldots, T_r]$ be the polynomial extension of R in r indeterminates and $\mathfrak{t} = (T_1, \ldots, T_r)$ the irrelevant ideal of S. We consider S with $\mathbb{N}_0^2$-graded structure by defining $\deg_S(x) = (\deg_R(x), 0)$ for all $x \in R$ and $\deg_S(T_i) = (\deg_R(f_i), 1) = (\delta_i, 1)$ for all $1 \le i \le r$. At first glance, this graded structure might not appear natural; however, it was selected for the following reason: considering

$$\gamma_j := \sum_{i=1}^{r} c_{ij} T_i \in S$$

for each $1 \le j \le s$, then each γ_j is an homogeneous element of S with

$$\deg_S(\gamma_j) = (\deg_R(a_j), 1) = (d_j, 1).$$

Thus the Koszul complex $K_\bullet(\underline{\gamma}; S)$ has the following graded structure

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & S(-\sum_{i=1}^s d_i, -s) \arrow[r] & \cdots \arrow[r] & \bigoplus_{1\le i<j\le s} S(-d_i-d_j,-2) \arrow[dl] & \\
& & \bigoplus_{i=1}^s S(-d_i,-1) \arrow[r] & S \arrow[r] & 0
\end{tikzcd}.$$

Let $Z_\bullet = Z_\bullet(\mathbf{f}; R)$ be the graded algebra of cycles of the Koszul complex $K_\bullet(\mathbf{f}; R)$. Then the graded complex $\mathcal{Z}_\bullet = \mathcal{Z}_\bullet(\mathbf{f}, R)$ is given by

$$\mathcal{Z}_\bullet : \quad 0 \longrightarrow Z_r \otimes_R S(0,-r) \longrightarrow \cdots \longrightarrow Z_1 \otimes_R S(0,-1) \longrightarrow S \longrightarrow 0.$$

Recall that $\mathcal{D}_\bullet = \operatorname{Tot}(K_\bullet(\underline{\gamma}; S) \otimes_S \mathcal{Z}_\bullet)$, hence

$$\begin{aligned}
\mathcal{D}_i &= \bigoplus_{t+l=i} \big(K_t(\underline{\gamma}; S) \otimes_S (Z_l \otimes_R S(0,-l))\big) \\
&= \bigoplus_{t+l=i} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} S(-(d_{i_1}+\cdots+d_{i_t}), -t) \otimes_S (Z_l \otimes_R S(0,-l)) \right) \\
&\cong \bigoplus_{t+l=i} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} \big(Z_l \otimes_R S(-(d_{i_1}+\cdots+d_{i_t}), -t-l)\big) \right) \\
&= \bigoplus_{t+l=i} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} \big(Z_l \otimes_R S(-(d_{i_1}+\cdots+d_{i_t}), -i)\big) \right).
\end{aligned}$$

By definition of the complex $\mathcal{Z}^+_\bullet = \mathcal{Z}^+_\bullet(\mathbf{a}, \mathbf{f}, \Phi)$, we have

$$\mathcal{Z}^+_0 = \bigoplus_{m=0}^{\infty} \big((\mathcal{D}_0)_{[(m,0)]}\big) = \bigoplus_{m=0}^{\infty} (S_{[(m,0)]}) = \bigoplus_{m=0}^{\infty} R_m = R.$$

and, for all $1 \le i \le r$, we have

$$\begin{aligned}
\mathcal{Z}^+_i &= \bigoplus_{m=-\infty}^{\infty} \big(H^r_{\mathbf{t}}(\mathcal{D}_{r+i-1})_{[(m,0)]}\big) \\
&= \bigoplus_{m=-\infty}^{\infty} \left[H^r_{\mathbf{t}} \left[\bigoplus_{t+l=r+i-1} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} \big(Z_l \otimes_R S(-(d_{i_1}+\cdots+d_{i_t}), -r-i+1)\big) \right) \right]_{[(m,0)]} \right] \\
&\cong \bigoplus_{m=-\infty}^{\infty} \left[\bigoplus_{t+l=r+i-1} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} H^r_{\mathbf{t}} \big(Z_l \otimes_R S(-(d_{i_1}+\cdots+d_{i_t}), -r-i+1)\big) \right)_{[(m,0)]} \right] \\
&\cong \bigoplus_{m=-\infty}^{\infty} \left[\bigoplus_{t+l=r+i-1} \left(\bigoplus_{1\le i_1<\cdots<i_t\le s} H^r_{\mathbf{t}} \big(Z_l[T_1,\ldots,T_r](-(d_{i_1}+\cdots+d_{i_t}), -r-i+1)\big) \right)_{[(m,0)]} \right].
\end{aligned}$$

Notice that

$$H^r_{\mathbf{t}}\big(Z_l[T_1,\dots,T_r](-(d_{i_1}+\cdots+d_{i_t}),-r-i+1)\big)_{[(m,0)]} = \big(H^r_{\mathbf{t}}\big(Z_l[T_1,\dots,T_r]\big)(-(d_{i_1}+\cdots+d_{i_t}),-r-i+1)\big)_{[(m,0)]}$$
$$= H^r_{\mathbf{t}}(Z_l[T_1,\dots,T_r])_{[(-(d_{i_1}+\cdots+d_{i_t})+m,-r-i+1)]}$$

and recall that

$$H^r_{\mathbf{t}}(Z_l[T_1,\dots,T_r]) \cong \bigoplus_{b_1,\dots,b_r\geq 0} \frac{Z_l}{T_1^{1+b_1}\cdots T_r^{1+b_r}}.$$

Let $\gamma \in$ be a homogeneous element of $H^r_{\mathbf{t}}(Z_l[T_1,\dots,T_r])$, then

$$\gamma = \frac{\eta}{T_1^{1+b_1}\cdots T_r^{1+b_r}}$$

for some homogeneous element $\eta \in Z_l$ and $b_1,\dots,b_r \geq 0$. Note that

$$\gamma \in H^r_{\mathbf{t}}(Z_l[T_1,\dots,T_r])_{[(-(d_{i_1}+\cdots+d_{i_t})+m,-r-i+1)]}$$

if and only if the following equations hold

$$\begin{cases} \sum_{k=1}^r b_k = i-1; \\ -\sum_{k=1}^r(\delta_k + b_k\delta_k) + \deg_R(\eta) = -\sum_{k=1}^t d_{i_k} + m. \end{cases}$$

Thus, given $b_1,\dots,b_r \geq 0$ such that $\sum_{k=1}^r b_k = i-1$, $\gamma \in H^r_{\mathbf{t}}(Z_l[T_1,\dots,T_r])_{[(-(d_{i_1}+\cdots+d_{i_t})+m,-r-i+1)]}$ if and only if

$$\deg_R(\eta) = m + \sum_{k=1}^r(\delta_k + b_k\delta_k) - \sum_{k=1}^t d_{i_k}.$$

Finally we conclude that

$$\mathcal{Z}_i^+ = \bigoplus_{m=-\infty}^{\infty}\Bigg[\bigoplus_{t+l=r+i-1}\Bigg(\bigoplus_{1\leq i_1<\cdots<i_t\leq s}\Bigg(\bigoplus_{\sum_{k=1}^r b_k=i-1}(Z_l)_{[m+\sum_{k=1}^r(\delta_k+b_k\delta_k)-\sum_{k=1}^t d_{i_k}]}\Bigg)\Bigg)\Bigg].$$

That is,

$$(\mathcal{Z}_i^+)_{[m]} = \Bigg[\bigoplus_{t+l=r+i-1}\Bigg(\bigoplus_{1\leq i_1<\cdots<i_t\leq s}\Bigg(\bigoplus_{\sum_{k=1}^r b_k=i-1}(Z_l)_{[m+\sum_{k=1}^r(\delta_k+b_k\delta_k)-\sum_{k=1}^t d_{i_k}]}\Bigg)\Bigg)\Bigg].$$

for all $1 \leq i \leq r$ and $m \in \mathbb{Z}$.

In particular if $d_1 = \cdots = d_s = d$ and $\delta_1 = \cdots = \delta_r = \delta$, then

$$(\mathcal{Z}_i^+)_{[m]} = \Bigg[\bigoplus_{t+l=r+i-1}\Bigg(\bigoplus_{k=1}^{\binom{s}{t}\binom{i+r-2}{r-1}}\Big((Z_l)_{[m-td+\delta(t+l)]}\Big)\Bigg)\Bigg]$$

for all $1 \leq i \leq r$ and $m \in \mathbb{Z}$.

Our next objective revolves around deducing the Hilbert function of the R-modules $\mathcal{Z}_i^+(\mathbf{a}, \mathbf{f}, \Phi)$ in terms of the Hilbert functions of Koszul cycles $Z_i = Z_i(\mathbf{f}; R)$. Initially it is important to emphasize that both the ring R and each Z_i naturally have $\mathbb{Z}$-graded structure. This is achieved by setting $(Z_i)_{[m]} = R_{[m]} = 0$ for all $m < 0$. Thus, given $n \in \mathbb{Z}$, one has

Proposition 2.6.1. *Considering the construction above and supposing R_0 an Artinian ring, one has*

$$\mathrm{HF}_{\mathcal{Z}_0^+}(n) = \ell_{R_0}(R_n)$$

and, for each $1 \leq i \leq r$, one has

$$\mathrm{HF}_{\mathcal{Z}_i^+}(n) = \sum_{t+l=r+i-1} \left(\sum_{1 \leq i_1 < \cdots < i_t \leq s} \left(\sum_{\sum_{k=1}^r b_k = i-1} \mathrm{HF}_{Z_l} \left(n + \sum_{k=1}^{r} (\delta_k + b_k \delta_k) - \sum_{k=1}^{t} d_{i_k} \right) \right) \right).$$

Proof: For $i = 0$, one has $\mathrm{HF}_{\mathcal{Z}_0^+}(n) = \mathrm{HF}_R(n) = \ell_{R_0}(R_n)$. Next, using Corollary 1.5.4 for each $1 \leq i \leq r$, one obtains

$$\begin{aligned}
\mathrm{HF}_{\mathcal{Z}_i^+}(n) &= \ell_{R_0}((\mathcal{Z}_i^+)_{[n]}) \\
&= \ell_{R_0}\left[\bigoplus_{t+l=r+i-1} \left(\bigoplus_{1 \leq i_1 < \cdots < i_t \leq s} \left(\bigoplus_{\sum_{k=1}^r b_k = i-1} (Z_l)_{[n + \sum_{k=1}^r (\delta_k + b_k \delta_k) - \sum_{k=1}^t d_{i_k}]} \right) \right) \right] \\
&= \sum_{t+l=r+i-1} \left(\sum_{1 \leq i_1 < \cdots < i_t \leq s} \left(\sum_{\sum_{k=1}^r b_k = i-1} \ell_{R_0}\big((Z_l)_{[n + \sum_{k=1}^r (\delta_k + b_k \delta_k) - \sum_{k=1}^t d_{i_k}]}\big) \right) \right) \\
&= \sum_{t+l=r+i-1} \left(\sum_{1 \leq i_1 < \cdots < i_t \leq s} \left(\sum_{\sum_{k=1}^r b_k = i-1} \mathrm{HF}_{Z_l} \left(n + \sum_{k=1}^{r} (\delta_k + b_k \delta_k) - \sum_{k=1}^{t} d_{i_k} \right) \right) \right).
\end{aligned}$$

$\square$

In particular if $d_1 = \cdots = d_s = d$ and $\delta_1 = \cdots = \delta_r = \delta$, then

$$\mathrm{HF}_{\mathcal{Z}_i^+}(n) = \sum_{l=0}^{r+i-1} \left(\sum_{k=1}^{\binom{s}{r+i-1-l}\binom{i+r-2}{r-1}} \mathrm{HF}_{Z_l}(n - td + \delta(t+l)) \right).$$

for all $1 \leq i \leq r$.

We just discovered an explicit formula for computing the Hilbert function of $\mathcal{Z}_i^+$ in terms of the Hilbert functions of the Koszul cycles of $K_\bullet(\mathbf{f}; R)$. In the following Proposition, we calculate the

Hilbert polynomial of the cycles of Koszul of sequences which generate $\mathfrak{m}$-primary ideals in *local Noetherian rings. Recall that the Proj of a positively graded $R = R_0[R_1]$, denoted by $\mathrm{Proj}(R)$, is the family of all homogeneous prime ideals of R which do not contain the ideal $(\mathfrak{m}_0, R_1)$, where $\mathfrak{m}_0$ is the maximal ideal of R_0.

$$\mathrm{Proj}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ (\mathfrak{m}_0, R_1) \not\subset \mathfrak{p}, \ \mathfrak{p} \text{ homogeneous}\}.$$

Proposition 2.6.2. *Let* $R = R_0[R_1]$ *be a positively graded Noetherian* **local ring with* $(R_0, \mathfrak{m}_0)$ *Artinian local and* **local maximal ideal* $\mathfrak{m} = (\mathfrak{m}_0, R_1)$. *Let* $I = (f_1, \dots, f_r)$ *be a homogeneous* $\mathfrak{m}$-*primary ideal of* R. *Denoting* $\deg(f_i) = \delta_i$ *for all* $1 \le i \le r$ *and* $Z_i = Z_i(\mathbf{f}; R)$ *the* i-*th Koszul cycle of the ideal* I, *then*

$$\mathrm{HP}_{Z_k}(n) = \begin{cases} \mathrm{HP}_R(n), \ \text{if } k = 0; \\ \sum_{j=k}^{r-1} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_{j+1} \le r} \mathrm{HP}_R(n - (\delta_{i_1} + \cdots + \delta_{i_{j+1}})), \ \text{if } 1 \le k \le r-1; \\ 0, \ \text{if } k = r. \end{cases}$$

Proof: Let $K_\bullet = K_\bullet(\mathbf{f}; R)$ be the graded Koszul complex of the sequence $\mathbf{f} = f_1, \dots, f_r$.

$$\begin{tikzcd} 0 \arrow[r] & R(-\sum_{i=1}^r \delta_i) \arrow[r] & \cdots \arrow[r] & \bigoplus_{1 \le i < j \le r} R(-\delta_i - \delta_j) \arrow[dl] \\ & & \bigoplus_{i=1}^r R(-\delta_i) \arrow[r] & R \arrow[r] & 0 \end{tikzcd}.$$

Since I is $\mathfrak{m}$-primary, the complex $K_\bullet(\mathbf{f}, R)_\mathfrak{p} = K_\bullet(\mathbf{f}_\mathfrak{p}, R_\mathfrak{p})$ is exact for all $\mathfrak{p} \in \mathrm{Proj}(R)$, thus $H_i(K_\bullet)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Proj}(R)$, which implies that $\ell_{R_0}(H_i(K_\bullet)) < \infty$ for all $i = 1, \dots, r$. In particular, it implies that $H_i(K_\bullet)_{[n]} = 0$ for $n \gg 0$ and all $i = 1, \dots, r$. Since we have the following graded short exact sequence

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0,$$

where B_i is the i-th boundary of $K_\bullet$, one concludes that $(B_i)_{[n]} = (Z_i)_{[n]}$ for $n \gg 0$ and $i = 1, \dots, r$. Thence, for all $k = 1, \dots, r-1$, we have the following truncated Koszul complex

$$\begin{tikzcd} 0 \arrow[r] & R(-(\delta_1 + \cdots + \delta_r)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{1 \le i_1 < \cdots < i_{k+1} \le r} R(-(\delta_{i_1} + \cdots + \delta_{i_{k+1}})) \arrow[dl] \\ & & Z_k \arrow[r] & 0 \end{tikzcd},$$

which is exact in its n-th components for $n \gg 0$. Hence

$$\begin{aligned}
\mathrm{HP}_{Z_k}(n) = \ell_{R_0}\big((Z_k)_{[n]}\big) &= \sum_{j=k}^{r-1}(-1)^{j+1}\ell_{R_0}\Bigg(\Bigg(\bigoplus_{1\le i_1<\cdots<i_{j+1}\le r} R(-(d_{i_1}+\cdots+d_{i_{j+1}}))\Bigg)_{[n]}\Bigg)\\
&= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\le i_1<\cdots<i_{j+1}\le r}\ell_{R_0}\big(R(-(d_{i_1}+\cdots+d_{i_{j+1}}))_{[n]}\big)\\
&= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\le i_1<\cdots<i_{j+1}\le r}\ell_{R_0}\big(R_{[n-(d_{i_1}+\cdots+d_{i_{j+1}})]}\big)\\
&= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\le i_1<\cdots<i_{j+1}\le r}\mathrm{HP}_R(n-(d_{i_1}+\cdots+d_{i_{j+1}}))
\end{aligned}$$

for all $1 \le k \le r-1$. Furthermore it is clear that $\mathrm{HP}_{Z_0} = \mathrm{HP}_R$ and $\mathrm{HP}_{Z_r} = 0$, so the statement follows. $\square$

In particular, assuming that all generators of the ideal I have the same degree, one can simplify the formula obtained above.

Corollary 2.6.3. *Let $R = R_0[R_1]$ be a positively graded Noetherian *local ring with $(R_0, \mathfrak{m}_0)$ Artinian local and *local maximal ideal $\mathfrak{m} = (\mathfrak{m}_0, R_1)$. Let $I = (f_1, \dots, f_r)$ be a homogeneous $\mathfrak{m}$-primary ideal of R. Denote by $Z_i = Z_i(\mathbf{f}; R)$ the i-th Koszul cycle of the ideal I. If* $\deg(f_1) = \cdots = \deg(f_r) = \delta$, *then*

$$\mathrm{HP}_{Z_k}(n) = \begin{cases} \mathrm{HP}_R(n), & \text{if } k = 0;\\ \sum_{j=k}^{r-1}(-1)^{j+1}\binom{r}{j+1}\mathrm{HP}_R(n-(j+1)\delta), & \text{if } 1 \le k \le r-1;\\ 0, & \text{if } k = r. \end{cases}$$

Another class of ideals for which it is possible to determine the Hilbert polynomials of Koszul cycles is the family of the almost complete intersection ideals. These are ideals with deviation one. Recall that the deviation of an ideal I, denoted by $d(I)$, in a Noetherian local ring R is defined by $d(I) = \mu(I) - \mathrm{grade}(I)$.

Proposition 2.6.4. *Let R be a positively graded Gorenstein *local ring with R_0 Artinian local. Let $I = (f_1, \dots, f_r)$ be an homogeneous almost complete intersection ideal of R. Denoting* $\deg(f_i) = \delta_i$

for all $1 \leq i \leq r$ *and* $Z_i = Z_i(\mathbf{f}; R)$ *the* i*-th Koszul cycle of the ideal* I*, then*

$$\mathrm{HF}_{Z_k}(n) = \begin{cases} \mathrm{HF}_R(n), \ \ \textit{if } k = 0; \\ \mathrm{HF}_{\omega_{R/I}}(n) + \sum_{j=1}^{r-1}(-1)^{j+1}\sum_{1\leq i_1<\cdots<i_{j+1}\leq r}\mathrm{HF}_R(n-(\delta_{i_1}+\cdots+\delta_{i_{j+1}})), \ \textit{if } k=1; \\ \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\leq i_1<\cdots<i_{j+1}\leq r}\mathrm{HF}_R(n-(\delta_{i_1}+\cdots+\delta_{i_{j+1}})), \ \ \textit{if } 2\leq k\leq r-1; \\ 0, \ \ \textit{if } k = r. \end{cases}$$

Proof: Since I is almost complete intersection, then $\mathrm{grade}(I) = r-1$. Let $K_\bullet = K_\bullet(\mathbf{f}; R)$ be the graded Koszul complex of I.

$$K_\bullet: \quad 0 \longrightarrow R(-(\delta_1+\cdots+\delta_r)) \longrightarrow \cdots \longrightarrow \bigoplus_{k=1}^r R(-\delta_k) \longrightarrow R \longrightarrow 0.$$

As $\mathrm{grade}(I) = r-1$, one has $H_i(K_\bullet) = 0$ for all $i = 2, \ldots, r$. Thus $Z_r = 0$ and, for all $k = 2, \ldots, r-1$, one has the following exact sequence

$$0 \longrightarrow R(-(\delta_1+\cdots+\delta_r)) \longrightarrow \cdots \longrightarrow \bigoplus_{1\leq i_1<\cdots<i_{k+1}\leq r} R(-(\delta_{i_1}+\cdots+\delta_{i_{k+1}})) \longrightarrow Z_k \longrightarrow 0.$$

Hence $H_{Z_r} = 0$ and

$$\begin{aligned} \mathrm{HF}_{Z_k}(n) = \ell_{R_0}\big((Z_k)_{[n]}\big) &= -\sum_{j=k}^{r-1}(-1)^j \ell_{R_0}\Bigg(\Bigg(\bigoplus_{1\leq i_1<\cdots<i_{j+1}\leq r} R(-(\delta_{i_1}+\cdots+\delta_{i_{j+1}}))\Bigg)_{[n]}\Bigg) \\ &= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\leq i_1<\cdots<i_{j+1}\leq r}\ell_{R_0}\big((R(-(\delta_{i_1}+\cdots+\delta_{i_{j+1}})))_{[n]}\big) \\ &= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\leq i_1<\cdots<i_{j+1}\leq r}\ell_{R_0}\big(R_{[n-(\delta_{i_1}+\cdots+\delta_{i_{j+1}})]}\big) \\ &= \sum_{j=k}^{r-1}(-1)^{j+1}\sum_{1\leq i_1<\cdots<i_{j+1}\leq r}\mathrm{HF}_R(n-(\delta_{i_1}+\cdots+\delta_{i_{j+1}})) \end{aligned}$$

for all $k = 2, \ldots, r-1$. Finally we have to calculate the Hilbert Function of Z_1. Notice that $H_1(K_\bullet) \cong \mathrm{Ext}_R^{r-1}(R/I, R)$ by [BH93, Theorem 1.6.16]. Moreover, as R/I is homomorphic image of the Gorenstein R and

$$\dim(R) - \dim(R/I) = \mathrm{ht}(I) = \mathrm{grade}(I) = r-1,$$

by [BS13, Remarks 12.1.3 (iii)], one has $H_1(K_\bullet) \cong \omega_{R/I}$. Since one has the following exact sequences

$$0 \longrightarrow B_1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow 0,$$

$$0 \longrightarrow R(-(\delta_1 + \cdots + \delta_r)) \longrightarrow \cdots \longrightarrow \bigoplus_{1 \le i < j \le r} R(-(\delta_i + \delta_j)) \longrightarrow B_1 \longrightarrow 0,$$

then

$$\begin{aligned} \mathrm{HF}_{Z_1}(n) &= \mathrm{HF}_{H_1}(n) + \mathrm{HF}_{B_1}(n) \\ &= \mathrm{HF}_{\omega_{R/I}}(n) + \sum_{j=1}^{r-1} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_{j+1} \le r} \mathrm{HF}_R(n - (\delta_{i_1} + \cdots + \delta_{i_{j+1}})). \end{aligned}$$

□

In general, it is not always possible to compute the Hilbert function of all Koszul cycles directly from the Hilbert function of R. If we assume that R is a positively graded Noetherian *local ring, with R_0 Artinian local ring, and $I = (f_1, \ldots, f_r)$ a homogeneous ideal with $\mathrm{ht}(I) = g > 0$, then, by repeating the argument above, one concludes

Corollary 2.6.5. *Let R be a positively graded Noetherian *local ring with R_0 Artinian local and $I = (f_1, \ldots, f_r)$ with $\mathrm{ht}(I) > 0$. Denoting $\deg(f_i) = \delta_i$ for all $1 \le i \le r$ and $Z_i = Z_i(\mathbf{f}; R)$ the i-th Koszul cycle of the ideal I, then*

$$\mathrm{HF}_{Z_k}(n) = \begin{cases} \mathrm{HF}_R(n), & \text{if } k = 0; \\ \sum_{j=k}^{r-1} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_{j+1} \le r} \mathrm{HF}_R(n - (\delta_{i_1} + \cdots + \delta_{i_{j+1}})), & \text{if } r - g + 1 \le k \le r - 1; \\ 0, & \text{if } k = r. \end{cases}$$

Chapter 3

Generic Kitt and Generic Residual Intersections

3.1 Generic Residual Aproximation Complexes and Generic Kitt

Let R be a ring and I a finitely generated ideal of R. In this section we will generalize the construction of the k-th residual approximation complex to the generic case. Let s be a positive integer, $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I and $S = R[U_{ij}\ ;\ 1 \leq i \leq r,\ 1 \leq j \leq s]$ the polynomial extension of R in rs indeterminates. Define $S^{\mathfrak{g}} := S[T_1, \ldots, T_r]$ and $\underline{\gamma} = \gamma_1, \ldots, \gamma_s \in S^{\mathfrak{g}}$ such that

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix}.$$

Notice that $\mathbf{f}$ can be seen as a sequence of elements in S. Hence consider the generic $\mathcal{Z}$-approximation complex $\mathcal{Z}^{\mathfrak{g}}_{\bullet}(\mathbf{f}) := \mathcal{Z}(\mathbf{f}, S)$

$$\mathcal{Z}^{\mathfrak{g}}_{\bullet}(\mathbf{f}) : \quad 0 \longrightarrow Z_r(\mathbf{f}; S) \otimes_S S^{\mathfrak{g}}[-r] \longrightarrow \cdots \longrightarrow Z_1(\mathbf{f}; S) \otimes_S S^{\mathfrak{g}}[-1] \longrightarrow S^{\mathfrak{g}} \longrightarrow 0,$$

where $Z_\bullet(\mathbf{f};S)$ is the algebra of cycles of the Koszul complex $K_\bullet(\mathbf{f};S)$. Let $K_\bullet^{\mathfrak{g}}$ denote the Koszul complex $K_\bullet(\underline{\gamma};S^{\mathfrak{g}})$ and define $\mathcal{D}_\bullet^{\mathfrak{g}} = \mathrm{Tot}(K_\bullet^{\mathfrak{g}} \otimes_{S^{\mathfrak{g}}} \mathcal{Z}_\bullet^{\mathfrak{g}}(\mathbf{f}))$ the totalization complex of $K_\bullet^{\mathfrak{g}} \otimes_{S^{\mathfrak{g}}} \mathcal{Z}_\bullet^{\mathfrak{g}}(\mathbf{f})$. Thus

$$\mathcal{D}_\bullet^{\mathfrak{g}}: \quad 0 \longrightarrow \mathcal{D}_{r+s}^{\mathfrak{g}} \longrightarrow \mathcal{D}_{r+s-1}^{\mathfrak{g}} \longrightarrow \cdots \longrightarrow \mathcal{D}_1^{\mathfrak{g}} \longrightarrow \mathcal{D}_0^{\mathfrak{g}} = S^{\mathfrak{g}} \longrightarrow 0,$$

where

$$\mathcal{D}_i^{\mathfrak{g}} = \bigoplus_{\substack{l+j=i \\ 0\le l\le s,\, 0\le j\le r}} K_l^{\mathfrak{g}} \otimes_{S^{\mathfrak{g}}} (Z_j(\mathbf{f},S)\otimes_S S^{\mathfrak{g}}[-j]) = \bigoplus_{\substack{l+j=i \\ 0\le l\le s,\, 0\le j\le r}} \big(Z_j(\mathbf{f},S)\otimes_S (S^{\mathfrak{g}})^{\binom{s}{l}}\big)[-i].$$

Considering the ideal $\mathbf{t} = (T_1,\ldots,T_r) \subseteq S^{\mathfrak{g}}$, one can apply the very same procedure of the construction of the Koszul-Čech complex (Section 1.2) to the double complex

$$\mathcal{D}_\bullet^{\mathfrak{g}} \otimes \mathcal{C}_{\mathbf{t}}^\bullet(S^{\mathfrak{g}})$$

and we will get a $\mathbb{Z}-$indexed family of complexes ${}_k\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})$, where, for each $k\in\mathbb{Z}$, the complex ${}_k\mathcal{Z}_\bullet^{+\mathfrak{g}} := {}_k\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})$ is of the form

$$0 \longrightarrow H_{\mathbf{t}}^r(\mathcal{D}_{r+s}^{\mathfrak{g}})_{(k)} \longrightarrow \cdots \xrightarrow{\psi_k^{\mathfrak{g}}} H_{\mathbf{t}}^r(\mathcal{D}_{k+r}^{\mathfrak{g}})_{(k)} \xrightarrow{\tau_k^{\mathfrak{g}}} (\mathcal{D}_k^{\mathfrak{g}})_{(k)} \xrightarrow{\mu_k^{\mathfrak{g}}} \cdots \longrightarrow (\mathcal{D}_0^{\mathfrak{g}})_{(k)} \longrightarrow 0.$$

Analyzing each component of complex ${}_k\mathcal{Z}_\bullet^{+\mathfrak{g}}$, one concludes that

$$\begin{aligned} {}_k\mathcal{Z}_i^{+\mathfrak{g}} &= \bigoplus_{\substack{l+j=i \\ 0\le l\le s,\, 0\le j\le r}} (Z_j\big(\mathbf{f};S)\otimes_S (S^{\mathfrak{g}})^{\binom{s}{l}}\big)_{(k-i)}, \quad \text{if } i\le k; \\ {}_k\mathcal{Z}_i^{+\mathfrak{g}} &= \bigoplus_{\substack{l+j=r+i-1 \\ 0\le l\le s,\, 0\le j\le r}} (Z_j(\mathbf{f};S)\otimes_S H_{\mathbf{t}}^r(S^{\mathfrak{g}}))_{(k-l-j)}^{\binom{s}{l}}, \quad \text{if } i>k. \end{aligned}$$

Definition 3.1.1. *Let R be a ring, I a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1,\ldots,f_r$ a system of generators of I. Given k an integer, the complex ${}_k\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})$ as constructed above is called the s-generic k-th residual approximation complex with respect to $\mathbf{f}$.*

In particular if $k=0$, one has the complex

$${}_0\mathcal{Z}_\bullet^{+\mathfrak{g}}: \quad 0 \longrightarrow H_{\mathbf{t}}^r(\mathcal{D}_{r+s}^{\mathfrak{g}})_{(0)} \longrightarrow \cdots \xrightarrow{\psi_0^{\mathfrak{g}}} H_{\mathbf{t}}^r(\mathcal{D}_r^{\mathfrak{g}})_{(0)} \xrightarrow{\tau_0^{\mathfrak{g}}} (\mathcal{D}_0^{\mathfrak{g}})_{(0)} \cong S \longrightarrow 0,$$

which will be denoted simply by $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})$. Note that image of the map $\tau_0^{\mathfrak{g}}$ is an ideal of S. This ideal is called *s-generic disguised residual intersection* and is denoted by $K(s,\mathbf{f})$.

Invoking [BH19, Theorem 4.9], we obtain the following characterization of s-generic disguised residual intersection.

Theorem 3.1.2. *Let R be a ring, I a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1, \dots, f_r$ a system of generators of I. Let $S = R[U_{11}, \dots, U_{rs}]$ be the polynomial extension of R in rs indeterminates and $\underline{\alpha} = \alpha_1, \dots, \alpha_s \in S$ such that*

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \dots & U_{1s} \\ U_{21} & U_{22} & \dots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \dots & U_{rs} \end{bmatrix}.$$

Then the s-generic disguised residual intersection of I with respect the system of generators $\mathbf{f}$

$$K(s, \mathbf{f}) = \mathrm{Kitt}_S((\underline{\alpha}), IS).$$

Having in mind the discussion above, we define the *s-generic Kitt of I with respect the respect the sequence* $\mathbf{f}$ as following:

Definition 3.1.3. *Let R be a ring, I a finitely generated ideal and s a positive integer. Consider $\mathbf{f} = f_1, \dots, f_r$ a system of generators of I, $S = R[U_{11}, \dots, U_{rs}]$ the polynomial extension of R in rs indeterminates and the sequence $\underline{\alpha} = \alpha_1, \dots, \alpha_s$ in S such that*

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \dots & U_{1s} \\ U_{21} & U_{22} & \dots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \dots & U_{rs} \end{bmatrix},$$

Let $K_\bullet(\mathbf{f}; S) = S\langle e_1, \dots, e_r; \partial(e_i) = f_i \rangle$ be the Koszul complex equipped with structure of differential graded algebra. Let $\zeta_j = \sum_{i=1}^{r} U_{ij} e_i$ for $1 \leq j \leq s$, $\Gamma_\bullet = S\langle \zeta_1, \dots, \zeta_s \rangle$ be the sub-algebra of $K_\bullet(\mathbf{f}; S)$ generated by the ζ's and $Z_\bullet = Z_\bullet(\mathbf{f}; S)$ the algebra of Koszul cycles. The s-generic Kitt of I with respect to the sequence $\mathbf{f}$ *is the ideal*

$$\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) = \langle \Gamma_\bullet \cdot Z_\bullet \rangle_r = \mathrm{Kitt}_S((\underline{\alpha}), IS).$$

Differently from ordinary Kitt, the generic Kitt depends on the system of generators of I as illustrated in the following example:

Example 3.1.4. Let k be a field, $R = k[x, y]$ and I an ideal of R with two distinct systems of generators $\mathbf{f} = x^2 + y,\ x^5$ and $\mathbf{f}' = x^2 + y,\ x^5 + x^2 + y$. Using *Macaulay2*, one obtains

$$\mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}) = (x^5U_{22} + x^2U_{12} + yU_{12}, x^5U_{21} + x^2U_{11} + yU_{11}, U_{12}U_{21} - U_{11}U_{22}),$$

$$\mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}') = (x^5U_{22} + x^2U_{12} + x^2U_{22} + yU_{12}, x^5U_{21} + x^2U_{11} + x^2U_{21} + yU_{11} + yU_{21}, U_{12}U_{21} - U_{11}U_{22}).$$

Hence $\mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}) \neq \mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}')$.

In Example 3.1.4, if we consider $U = [U_{ij}]$, $[\mathbf{a}] = [\mathbf{f}][U]$ and $[\mathbf{a}'] = [\mathbf{f}'][U]$, we note that the ideals $\mathfrak{a} = (\mathbf{a})$ and $\mathfrak{a}' = (\mathbf{a}')$ are distinct. Hence it is not surprising that we end up with

$$\mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}) = \mathrm{Kitt}(\mathfrak{a}, IS) \neq \mathrm{Kitt}(\mathfrak{a}', IS) = \mathrm{Kitt}^{\mathfrak{g}}(2, \mathbf{f}').$$

Although Example 3.1.4 above demonstrates the sensitivity of the s-generic Kitt to the choice of generators of I, it is important to note that this sensitivity is controlled. In other words, one can establish that the s-generic Kitt is unique up to *universal equivalence*, which will be defined in the following.

Definition 3.1.5. *Let (R, I) and (S, K) be two pairs, where R and S are rings and $I \subset R$, $K \subset S$ are ideals or $I = R$ or $K = S$. One says that (R, I) and (S, K) are universally equivalent if there are finite sets of indeterminates X over R and Z over S and a ring isomorphism $\phi : R[X] \longrightarrow S[Z]$ such that $\phi(IR[X]) = KS[Z]$.*

It is straightforward to check that the universal equivalence is indeed an equivalence relation. Drawing inspiration from Huneke and Ulrich's proof establishing the uniqueness up to universal equivalence of the generic s-residual intersection concerning the choice of the system of generators of I presented in [HU90], we show that this kind of uniqueness also holds for the generic Kitt. The proof uses the property that Kitt ideals commute with ring isomorphisms, demonstrated in Proposition 2.3.2. First, we need to prove that, for any permutation of the elements of the system of generators, the generic Kitt is unique up to universal equivalence.

Lemma 3.1.6. *Let R be a ring, I be a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I and $\sigma \in S_r$ a permutation on $[r]$. Let $R[U]$ and $R[V]$ be polynomial extensions of R in rs indeterminates. Denoting $\sigma(\mathbf{f}) := f_{\sigma(1)}, \ldots, f_{\sigma(r)}$, then the pairs*

$$(R[U], \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \qquad \textit{and} \qquad (R[V], \mathrm{Kitt}^{\mathfrak{g}}(s, \sigma(\mathbf{f})))$$

are universally equivalent.

Proof: Since every permutation can be written as finite composition of transposition between successive elements, it is enough to consider the case $r = 2$. Moreover, since S_2 has only two permutations, it is enough to prove that $(R[U], \mathrm{Kitt}^{\mathfrak{g}}(s; f_1, f_2))$ and $(R[V], \mathrm{Kitt}^{\mathfrak{g}}(s; f_2, f_1))$ are universally equivalent. Let $\mathbf{a} = a_1, \ldots, a_s \in R[U]$, $\mathbf{a}' = a'_1, \ldots, a'_s \in R[V]$ such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \end{bmatrix},$$

$$\begin{bmatrix} a'_1 & a'_2 & \cdots & a'_s \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1s} \\ V_{21} & V_{22} & \cdots & V_{2s} \end{bmatrix}.$$

By definition of generic Kitt, one has

$$\mathrm{Kitt}^{\mathfrak{g}}(s; f_1, f_2) = \mathrm{Kitt}((\mathbf{a}), (\mathbf{f})R[U]) \qquad \text{and} \qquad \mathrm{Kitt}^{\mathfrak{g}}(s; f_2, f_1) = \mathrm{Kitt}((\mathbf{a}'), (\mathbf{f})R[V]).$$

Next consider the R-algebra homomorphism $\phi : R[U, V] \longrightarrow R[U, V]$ such that

$$\phi(U_{1j}) = V_{2j}, \quad 1 \le j \le s,$$
$$\phi(U_{2j}) = V_{1j}, \quad 1 \le j \le s,$$
$$\phi(V_{ij}) = U_{ij}, \quad 1 \le i \le 2 \text{ and } 1 \le j \le s.$$

Notice that ϕ is an isomorphism and $\phi(a_i) = a'_i$ for all $1 \le i \le s$. Thus $\phi((\mathbf{a})R[U, V]) = (\mathbf{a}')R[U, V]$. By the flatness property of Kitt and Proposition 2.3.2, one concludes that

$$\begin{aligned} \phi\big(\mathrm{Kitt}^{\mathfrak{g}}(s; f_1, f_2)R[U, V]\big) &= \phi\big(\big(\mathrm{Kitt}((\mathbf{a}), (\mathbf{f})R[U]\big)R[U, V]\big) = \phi\big(\mathrm{Kitt}((\mathbf{a})R[U, V], (\mathbf{f})R[U, V])\big) \\ &= \mathrm{Kitt}(\phi((\mathbf{a})R[U, V]), \phi((\mathbf{f})R[U, V])) = \mathrm{Kitt}((\mathbf{a}')R[U, V], (\mathbf{f})R[U, V]) = \mathrm{Kitt}((\mathbf{a}'), (\mathbf{f})R[V])R[U, V] \\ &= \mathrm{Kitt}^{\mathfrak{g}}(s; f_2, f_1)R[U, V]. \end{aligned}$$

□

Proposition 3.1.7. *Let R be a ring, I a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{f}' = f'_1, \ldots, f'_{r'}$ systems of generators of I and let $R[U]$ and $R[V]$ be polynomial extensions of R in rs and $r's$ indeterminates, respectively. Then the pairs*

$$(R[U], \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \qquad \textit{and} \qquad (R[V], \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}'))$$

are universally equivalent.

Proof: Suppose first that $\mathbf{f}' = f_1, \ldots, f_r, g$ and let $\mathbf{a} = a_1, \ldots, a_s \in R[U]$, $\mathbf{a}' = a'_1, \ldots, a'_s \in R[V]$ such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix},$$

$$\begin{bmatrix} a'_1 & a'_2 & \cdots & a'_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r & g \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1s} \\ V_{21} & V_{22} & \cdots & V_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ V_{r1} & V_{r2} & \cdots & V_{rs} \\ V_{r+1,1} & V_{r+1,2} & \cdots & V_{r+1,s} \end{bmatrix}.$$

By definition of generic Kitt, one has that

$$\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) = \mathrm{Kitt}((\mathbf{a}), (\mathbf{f})R[U]) \qquad \text{and} \qquad \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}') = \mathrm{Kitt}((\mathbf{a}'), (\mathbf{f}, g)R[V]).$$

As g belongs to I, there are $c_1, \ldots, c_r \in R$ such that $g = \sum_{k=1}^{r} c_k f_k$. Thus consider the R-algebra homomorphism $\phi : R[U, V] \longrightarrow R[U, V]$ such that

$$\phi(U_{ij}) = V_{ij} + c_i V_{r+1,j} \quad \text{for all } 1 \leq i \leq r \text{ and } 1 \leq j \leq s;$$

$$\phi(V_{ij}) = \begin{cases} U_{ij}, & \text{if } 1 \leq i \leq r \text{ and } 1 \leq j \leq s \\ V_{r+1,j}, & \text{if } i = r+1. \end{cases}$$

Notice that ϕ is an isomorphism and

$$\phi(a_i) = \phi\left(\sum_{k=1}^{r} f_k U_{ki}\right) = \sum_{k=1}^{r} (f_k V_{ki}) + V_{r+1,i}\left(\sum_{k=1}^{r} c_k f_k\right) = \sum_{k=1}^{r} f_k V_{ki} + g V_{r+1,i} = a'_i,$$

which implies that $\phi((a)R[U, V]) = (a')R[U, V]$. Furthermore, one has that

$$\phi((\mathbf{f})R[U, V]) = (\mathbf{f})R[U, V] = (\mathbf{f}, g)R[U, V],$$

because $(\mathbf{f})R[U, V] = (\mathbf{f}, g)R[U, V]$. By independence of generators property on the ordinary Kitt, flatness property and Proposition 2.3.2, one concludes that

$$
\begin{aligned}
&\phi\big(\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})(R[U])[V]\big) = \phi\big(\operatorname{Kitt}((\mathbf{a}),(\mathbf{f})R[U])(R[U])[V]\big) \\
&= \phi\big(\operatorname{Kitt}((\mathbf{a})R[U,V],(\mathbf{f})R[U,V])\big) = \operatorname{Kitt}(\phi((\mathbf{a})R[U,V]),\phi((\mathbf{f})R[U,V])) \\
&= \operatorname{Kitt}((\mathbf{a}')R[U,V],(\mathbf{f},g)R[U,V]) = \operatorname{Kitt}((\mathbf{a}'),(\mathbf{f},g)R[V])(R[V])[U] \\
&= \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}')((R[V])[U]).
\end{aligned}
$$

Hence $(R[U],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))$ and $(R[V],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}'))$ are universally equivalent. More generally, let $\mathbf{f}$ and $\mathbf{f}'$ be two systems of generators of I. Set $\mathbf{f}+\mathbf{f}' := f_1,\ldots,f_r,f'_1,\ldots,f'_{r'}$ and $\mathbf{f}'+\mathbf{f} := f'_1,\ldots,f'_{r'},f_1,\ldots,f_r$. Using the previous argument repeated times and the fact that universal equivalence is an equivalence relation, one concludes that

$$
\begin{aligned}
(R[U],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) &\equiv (R[W],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}+\mathbf{f}')), \\
(R[V],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}')) &\equiv (R[W'],\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}'+\mathbf{f})).
\end{aligned}
$$

The statement follows by using Lemma 3.1.6. $\square$

A noteworthy property of the generic Kitt is its ability to specialize into the ordinary Kitt, as elucidated in the upcoming proposition.

Proposition 3.1.8. *Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Consider $\mathbf{f} = f_1,\ldots,f_r$, $\mathbf{a} = a_1,\ldots,a_s$ systems of generators of I, $\mathfrak{a}$, respectively and $\Phi = [c_{ij}]_{r\times s}$ such that*

$$
\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \Phi.
$$

Let $S = R[U_{ij}\ ;\ 1 \le i \le r,\ 1 \le j \le s]$, $\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$ be the s-generic Kitt ideal with respect the generating set $\mathbf{f}$ and the sequence $\mathbf{x} = U_{11}-c_{11},\ldots,U_{rs}-c_{rs}$ in S, then

$$
\frac{\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})+(\mathbf{x})}{(\mathbf{x})} = \operatorname{Kitt}(\mathfrak{a},I).
$$

Proof: Define the R-algebra homomorphism $\sigma : S \longrightarrow R$ with $\sigma(U_{ij}) = c_{ij}$. Notice that $(\mathbf{x}) \subseteq$

$\ker(\sigma)$. Thus σ factors in $\phi : S/(\mathbf{x}) \longrightarrow R$ such that the below diagram

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "\pi"] & R \\ S/(\mathbf{x}) \arrow[ur, "\phi"] & \end{tikzcd}$$

is commutative. Recall from definition of Kitt that $\mathrm{Kitt}(\mathfrak{a}, I) = \langle Z_\bullet(\mathbf{f}; R) \cdot \Gamma \rangle_r$, where $\Gamma = \langle \gamma_1, \ldots, \gamma_s \rangle \subseteq K_1(\mathbf{f}; R)$ and $\gamma_j = \sum_{k=1}^r c_{kj} e_j$. Let $z = re_1 \wedge \cdots \wedge e_r \in \mathrm{Kitt}(\mathfrak{a}, I)$. Then there are $n \in \mathbb{N}$, $L_{1_j} = \{i_{1_j}, \ldots, i_{k_j}\} \subseteq \{1, \ldots, s\}$ with $|L_{1_j}| = k_j$ and L_{2_j} such that $|L_2| = r - k_j$ such that

$$re_1 \wedge \cdots \wedge e_r = \sum_{j=1}^{n} \gamma_{i_{1_j}} \wedge \cdots \wedge \gamma_{i_{k_j}} \wedge z_{L_2,j},$$

where $z_{L_2,j} \in Z_{r-k_j}(\mathbf{f}; R)$ for all $1 \leq j \leq n$. Thus, denoting $\gamma_i^{\mathfrak{g}} = \sum_{k=1}^r f_k U_{ki}$ for $1 \leq i \leq s$ and taking

$$Z = \sum_{j=1}^{n} \gamma^{\mathfrak{g}}_{i_{1_j}} \wedge \cdots \wedge \gamma^{\mathfrak{g}}_{i_{k_j}} \wedge z_{L_2,j} \in \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}),$$

we get $\phi(\overline{Z}) = z$. Thus

$$\mathrm{Kitt}(\mathfrak{a}, I) \subseteq \phi\left(\frac{\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})}{\mathbf{x}}\right).$$

Conversely, given $Z \in \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$, there are $n \in \mathbb{N}$, $L_{1_j} = \{i_{1_j}, \ldots, i_{k_j}\} \subseteq \{1, \ldots, s\}$ with $|L_{1_j}| = k_j$ and L_{2_j} such that $|L_2| = r - k_j$ such that

$$Z = \sum_{j=1}^{n} \gamma^{\mathfrak{g}}_{i_{1_j}} \wedge \cdots \wedge \gamma^{\mathfrak{g}}_{i_{k_j}} \wedge z_{L_2,j},$$

where $z_{L_2,j} \in Z_{r-k_j}(\mathbf{f}; R)$ for all $1 \leq j \leq n$. Now applying the map ϕ to z, one gets

$$\phi(z) = \sum_{j=1}^{n} \gamma_{i_{1_j}} \wedge \cdots \wedge \gamma_{i_{k_j}} \wedge z_{L_2,j} \in \mathrm{Kitt}(\mathfrak{a}, I)$$

and the statement follows. $\square$

Remark 3.1.9. Observe that ϕ in Proposition 3.1.8 is a ring isomorphism. It follows from general fact that, for any evaluation $\sigma : R[X_1, \ldots, X_n] \longrightarrow R$, the correspondent map ϕ obtained by specializing $R[X_1, \ldots, X_n]$ by the sequence $X_1 - \sigma(X_1), \ldots, X_n - \sigma(X_n)$ is a ring isomorphism.

Considering the notation of Proposition 3.1.8, let $K_\bullet(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))$ be the Koszul complex of the sequence $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ with coefficents in $S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$. We prove that the 0-th homology of $K_\bullet(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))$ is $R/\operatorname{Kitt}(\mathfrak{a}, I)$.

Corollary 3.1.10. *Considering the notation of Proposition 3.1.8, let* $\mathcal{C}_\bullet := K_\bullet(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))$ *be the Koszul complex of the sequence* $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ *with coefficients in* $A := S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$*, then* $H_0(\mathcal{C}_\bullet) = R/\operatorname{Kitt}(\mathfrak{a}, I)$ *as A-module.*

Proof: Notice that one can extend $\mathcal{C}_\bullet$ to the following exact sequence

$$0 \longrightarrow A \longrightarrow A^{rs} \longrightarrow \cdots \longrightarrow A^{rs} \xrightarrow{\partial_1} A \longrightarrow \operatorname{Coker}(\partial_1) \longrightarrow 0.$$

$$\begin{aligned} H_0(\mathcal{C}_\bullet) = \operatorname{Coker}(\partial_1) = \frac{A}{\mathbf{x}A} &= \frac{S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})}{(\mathbf{x})(S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))} = \frac{S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})}{((\mathbf{x}) + \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})} \\ &\cong \frac{S}{(\mathbf{x}) + \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})} \cong \frac{S/(\mathbf{x})}{(\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}) + (\mathbf{x}))/(\mathbf{x})}. \end{aligned}$$

Hence we get $H_0(\mathcal{C}_\bullet) \cong R/\operatorname{Kitt}(\mathfrak{a}, I)$ by Proposition 3.1.8. □

Next, let R, R' be rings and $\phi : R \longrightarrow R'$ a ring homomorphism. Consider $\mathbf{f} = f_1, \ldots, f_r$ a sequence in R and $\mathbf{g} = g_1, \ldots, g_r$ the sequence in R' such that $g_i = \phi(f_i)$ for all $i = 1, \ldots, r$. Given s a positive integer, we prove that ϕ induces a natural map

$$\operatorname{Kitt}^{\mathfrak{g}}(s, \phi) : \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}) \longrightarrow \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{g}).$$

More explicitly,

Proposition 3.1.11. *Let* R, R' *be rings,* $\phi : R \longrightarrow R'$ *a ring homomorphism and* s *a positive integer. Given* $\mathbf{f} = f_1, \ldots, f_r$ *a sequence of elements of* R*, consider* $\mathbf{g} = g_1, \ldots, g_r$ *the sequence in* R' *such that* $g_i = \phi(f_i)$ *for all* $i = 1, \ldots, r$*. Let* $U_{11}, \ldots, U_{rs}$ *and* $V_{11}, \ldots, V_{rs}$ *be* rs *indeterminates over* R *and* R'*, respectively and consider* $\Phi : R[U] \longrightarrow R'[V]$ *the natural map induced by* ϕ *such that*

$$\Phi\left(\sum_{i_1,\ldots,i_{rs}} a_{i_1,\ldots,i_{rs}} U_1^{i_1} \cdots U_{rs}^{i_{rs}}\right) = \sum_{i_1,\ldots,i_{rs}} \phi(a_{i_1,\ldots,i_{rs}}) V_1^{i_1} \cdots V_{rs}^{i_{rs}}.$$

Then

$$\operatorname{Kitt}^{\mathfrak{g}}(s, \phi) := \Phi\Big|_{\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})} : \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}) \longrightarrow \operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{g})$$

is well-defined.

Proof: It is enough to show the Φ takes $\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$ into $\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{g})$. Set $\gamma_j = \sum_{k=1}^{r} U_{kj}e_k \in K_1(\mathbf{f};R[U])$ for $1 \le j \le s$ and let $\Gamma_\bullet = \langle \gamma_1,\ldots,\gamma_s\rangle \subseteq K_\bullet(\mathbf{f};R[U])$. Similarly, $\gamma'_j = \sum_{k=1}^{r} V_{kj}e_k \in K_1(\mathbf{g};R[V])$ for $1 \le j \le s$ and let $\Gamma'_\bullet = \langle \gamma'_1,\ldots,\gamma'_s\rangle \subseteq K_\bullet(\mathbf{g};R[V])$. Recall that, if $z = re_1\wedge\cdots\wedge e_r \in \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$, then there are $n\in\mathbb{N}$, $L_{1_j} = \{i_{1_j},\ldots,i_{k_j}\} \subseteq \{1,\ldots,s\}$ with $|L_{1_j}| = k_j$ and L_{2_j} such that $|L_2| = r-k_j$ such that

$$re_1\wedge\cdots\wedge e_r = \sum_{j=1}^{n} \gamma_{i_{1_j}}\wedge\cdots\wedge\gamma_{i_{k_j}}\wedge z_{L_2,j},$$

where $z_{L_2,j}\in Z_{r-k_j}(\mathbf{f};R[U])$ for all $1\le j\le n$. Recall that Φ induces the morphism

$$\Phi' : K_\bullet(\mathbf{f};R[U]) \longrightarrow K_\bullet(\mathbf{g};R'[V])$$

of Koszul complexes. Since $\Phi'(Z_\bullet(\mathbf{f};R[U])) \subseteq Z_\bullet(\mathbf{g};R'[V])$ and

$$\Phi'(\gamma_j) = \sum_{k=1}^{r} V_{kj}e_k = \gamma'_j,$$

one concludes that

$$\Phi'(re_1\wedge\cdots\wedge e_r) = \sum_{j=1}^{n} \Phi'(\gamma_{i_{1_j}})\wedge\cdots\wedge\Phi'(\gamma_{i_{k_j}})\wedge\Phi'(z_{L_2,j})$$

$$= \sum_{j=1}^{n} \gamma'_{i_{1_j}}\wedge\cdots\wedge\gamma'_{i_{k_j}}\wedge\Phi'(z_{L_2,j}) \in \big\langle \langle\gamma'_1,\ldots,\gamma'_s\rangle\cdot Z_\bullet(\mathbf{g};R'[V])\big\rangle_r = \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{g}).$$

Thus $\Phi\big(\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})\big) \subseteq \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{g})$ and the map

$$\mathrm{Kitt}^{\mathfrak{g}}(s,\phi) = \Phi\Big|_{\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})} : \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f}) \longrightarrow \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{g})$$

is well-defined. $\square$

3.2 Deformation of Kitt ideals

In this section, we will establish one of the central results of this thesis. we investigate conditions under which the ordinary Kitt deforms to the generic one. To begin, let's revisit the definition of deformation.

Definition 3.2.1. *Let (R,I) and (S,J) be pairs, where R and S are rings, and $I\subset R$, $J\subset S$ are ideals or $I=R$ or $J=S$. One says that (S,J) is a deformation of (R,I) if there exists a sequence*

$\mathbf{x} \subset S$*, which is regular over* S *and* S/J *such that there exists a ring isomorphism* $\phi : S/(\mathbf{x}) \longrightarrow R$ *satisfying* $\phi\big((J + (\mathbf{x}))/(\mathbf{x})\big) = I$*.*

Roughly speaking, we can interpret deformation as the dual notion of specialization by regular sequences. When the rings R and S are understood, we simply say that the ideal I deforms to J. With Proposition 3.1.8 and its remark in mind, one observes that $(R, \operatorname{Kitt}(\mathfrak{a}, I))$ deforms to $(S, \operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f}))$ if and only if the sequence $\mathbf{x}$ is $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$-regular. In the following theorem, we derive an exact sequence that allows us to identify enough conditions for deforming the ordinary Kitt to the generic one.

Theorem 3.2.2. *Let* R *be a ring,* $\mathfrak{a} \subseteq I$ *finitely generated ideals of* R *and* s *a positive integer. Consider* $\mathbf{f} = f_1, \ldots, f_r$ *and* $\mathbf{a} = a_1, \ldots, a_s$ *systems of generators of* $\mathfrak{a}$ *and* I*, respectively, with* $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$*. Let* $S = R[U_{ij}]$ *be the polynomial extension of* R *in* rs *indeterminates. Considering* $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$*, one has the following exact sequence.*

$$\begin{tikzcd}
H_2(\mathcal{Z}^+_\bullet(\mathbf{a}, \mathbf{f})) \arrow[r] & H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \arrow[r] & H_1(\mathcal{Z}^{+\mathfrak{g}}_\bullet(s, \mathbf{f})) \otimes_S S/(\mathbf{x}) \arrow[dl] & \\
& H_1(\mathcal{Z}^+_\bullet(\mathbf{a}, \mathbf{f})) \arrow[r] & H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \arrow[r] & 0.
\end{tikzcd} \tag{3.1}$$

Proof: Let $S^{\mathfrak{g}} = S[T_1, \ldots, T_r]$ be the polynomial extension of S in r indeterminates and $\underline{\gamma} = \gamma_1, \ldots, \gamma_s \in S^{\mathfrak{g}}$ such that

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix} \Phi^{\mathfrak{g}}.$$

Consider the complex $\mathcal{Z}^{+\mathfrak{g}}_\bullet = \mathcal{Z}^{+\mathfrak{g}}_\bullet(s, \mathbf{f})$ and $K_\bullet = K_\bullet(\mathbf{x}; S)$ the Koszul complex of the sequence $\mathbf{x}$ in S. Consider $E^{\bullet,\bullet} = \mathcal{Z}^{+\mathfrak{g}}_\bullet \otimes_S K_\bullet$ the double complex obtained by tensor product of $\mathcal{Z}^{+\mathfrak{g}}_\bullet$ and $K_\bullet$.

One displays this double complex in the second quadrant as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{Z}_s^{+\mathfrak{g}} \otimes_S K_{rs} \arrow[r] \arrow[d] & \cdots \arrow[r] & & \mathcal{Z}_1^{+\mathfrak{g}} \otimes_S K_{rs} \arrow[r] \arrow[d] & \mathcal{Z}_0^{+\mathfrak{g}} \otimes_S K_{rs} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathcal{Z}_s^{+\mathfrak{g}} \otimes_S K_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & & \mathcal{Z}_1^{+\mathfrak{g}} \otimes_S K_1 \arrow[r] \arrow[d] & \mathcal{Z}_0^{+\mathfrak{g}} \otimes_S K_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Z}_s^{+\mathfrak{g}} \otimes_S K_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & & \mathcal{Z}_1^{+\mathfrak{g}} \otimes_S K_0 \arrow[r] \arrow[d] & \mathcal{Z}_0^{+\mathfrak{g}} \otimes_S K_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & & & 0 & 0 &
\end{tikzcd}$$

Observe that $(-i)$-th column of this double complex is the Koszul complex of sequence $\mathbf{x}$ with coefficients in $\mathcal{Z}_i^{+\mathfrak{g}}$. Moreover, by [Has12, Equation 2.5], one has $\mathcal{Z}_i^{+\mathfrak{g}} = \mathcal{Z}_i^{+}(\mathbf{a}, \mathbf{f})[U_{ij}]$. Hence $\mathbf{x}$ is regular on $\mathcal{Z}_i^{+\mathfrak{g}}$, which implies that $H_j(\mathbf{x}; \mathcal{Z}_i^{+\mathfrak{g}}) = 0$ for all $1 \leq j \leq rs$. Moreover $H_0(\mathbf{x}; \mathcal{Z}_i^{+\mathfrak{g}}) = \mathcal{Z}_i^{+\mathfrak{g}} \otimes_S S/(\mathbf{x}) \cong \mathcal{Z}_i^{+}$ for all $0 \leq i \leq s$. Thus the first page of vertical spectral ${}^1E_{\text{ver}}^{\bullet,\bullet}$ gives

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & & & & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & \mathcal{Z}_s^{+} & \longrightarrow \cdots \longrightarrow & \mathcal{Z}_1^{+} & \longrightarrow & \mathcal{Z}_0^{+} & \longrightarrow & 0
\end{array}.$$

and the second page of vertical spectral ${}^2E_{\text{ver}}^{\bullet,\bullet}$ gives

$$\begin{array}{cccccc}
0 & 0 & \cdots & 0 & 0 & 0 \\
 & & & & & \\
 & \vdots & & \vdots & \vdots & \\
 & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 \\
 & & & & & \\
0 & H_s(\mathcal{Z}_\bullet^{+}) & \cdots & H_1(\mathcal{Z}_\bullet^{+}) & H_0(\mathcal{Z}_\bullet^{+}) & 0
\end{array}.$$

Since the vertical spectral sequence collapses on second page, we have that ${}^2E^{\bullet,\bullet}_{\mathrm{ver}} = {}^{\infty}E^{\bullet,\bullet}_{\mathrm{ver}}$. Hence, by convergence theorem, we conclude that

$$H_{-i}(\mathrm{Tot}(E^{\bullet,\bullet})) = H_i(\mathcal{Z}^{+}_{\bullet})$$

for all $i \geq 0$. Now we are going to analyze the horizontal spectral sequence. As K_i is a flat S-module for all $0 \leq i \leq rs$, the first page of horizontal spectral ${}^1E^{\bullet,\bullet}_{\mathrm{hor}}$ is given by

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_{rs} & \cdots & H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_{rs} & & H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_{rs} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_1 & \cdots & H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_1 & & H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_1 \\
\downarrow & & \downarrow & & \downarrow \\
H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_0 & \cdots & H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_0 & & H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}) \otimes_S K_0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Since the $(-i)$-column of ${}^1E^{\bullet,\bullet}_{\mathrm{hor}}$ is the Koszul complex of the sequence $\mathbf{x}$ with coefficients in $H_i(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})$, the second page of horizontal spectral ${}^2E^{\bullet,\bullet}_{\mathrm{hor}}$ gives

$$\begin{tikzcd}
H_{rs}(\mathbf{x}; H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & \cdots & H_{rs}(\mathbf{x}; H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & H_{rs}(\mathbf{x}; H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) \\
\vdots & & \vdots & \vdots \\
H_2(\mathbf{x}; H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & \cdots & H_2(\mathbf{x}; H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & H_2(\mathbf{x}; H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) \arrow[ddl, "d_2^{0,2}"'] \\
H_1(\mathbf{x}; H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & \cdots & H_1(\mathbf{x}; H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & H_1(\mathbf{x}; H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) \\
H_0(\mathbf{x}; H_s(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & \cdots & H_0(\mathbf{x}; H_1(\mathcal{Z}^{+\mathfrak{g}}_{\bullet})) & H_0(\mathbf{x}; H_0(\mathcal{Z}^{+\mathfrak{g}}_{\bullet}))
\end{tikzcd} .$$

Note that

$$H_j(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}})) = H_j(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \quad \text{for all } 0 \le j \le rs;$$
$$H_0(\mathbf{x}; H_i(\mathcal{Z}_\bullet^{+\mathfrak{g}})) = H_i(\mathcal{Z}_\bullet^{+\mathfrak{g}}) \otimes_S S/(\mathbf{x}) \quad \text{for all } 0 \le i \le s.$$

As the differential d_2 has bidegree $(-1,-2)$, one has that

$$^2E^{0,0}_{\text{hor}} = {}^\infty E^{0,0}_{\text{hor}}, \qquad {}^2E^{0,1}_{\text{hor}} = {}^\infty E^{0,1}_{\text{hor}}, \qquad {}^\infty E^{-1,0}_{\text{hor}} = \operatorname{Coker}(d_2^{0,2}).$$

Thus we can construct the following exact sequence

$$0 \longrightarrow \ker(d_2^{0,2}) \longrightarrow H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \xrightarrow{d_2^{0,2}} H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}) \otimes_S S/(\mathbf{x}) \longrightarrow \operatorname{Coker}(d_2^{0,2}) \longrightarrow 0. \tag{3.2}$$

Observe that $\ker\left(d_2^{0,2}\right) \cong {}^3E^{0,2}_{\text{hor}} = {}^\infty E^{0,2}_{\text{hor}}$. By convergence theorem, there is a filtration of $H_{-2}(\operatorname{Tot}(E^{\bullet,\bullet}))$

$$0 \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 = H_{-2}(\operatorname{Tot}(E^{\bullet,\bullet}))$$

such that

$$\mathcal{F}_2/\mathcal{F}_1 = H_{-2}(\operatorname{Tot}(E^{\bullet,\bullet}))/\mathcal{F}_1 = {}^\infty E^{0,2}_{\text{hor}} = \ker(d_2^{0,2}),$$

which implies that there is the following exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow H_{-2}(\operatorname{Tot}(E^{\bullet,\bullet})) \longrightarrow \ker(d_2^{0,2}) \longrightarrow 0. \tag{3.3}$$

Since $H_{-2}(\operatorname{Tot}(E^{\bullet,\bullet})) \cong H_2(\mathcal{Z}_\bullet^+)$, splicing the sequences 3.2 and 3.3, we obtain

$$H_2(\mathcal{Z}_\bullet^+) \longrightarrow H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \xrightarrow{d_2^{0,2}} H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}) \otimes_S S/(\mathbf{x}) \longrightarrow \operatorname{Coker}(d_2^{0,2}) \longrightarrow 0. \tag{3.4}$$

As ${}^\infty E^{-1,0}_{\text{hor}} = {}^2E^{-1,0}_{\text{hor}} = \operatorname{Coker}\left(d_2^{0,2}\right)$ and ${}^\infty E^{0,1}_{\text{hor}} = H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f}))$, the convergence theorem tells us that there exists a filtration of $H_{-1}(\operatorname{Tot}(E^{\bullet,\bullet}))$

$$0 \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 = H_{-1}(\operatorname{Tot}(E^{\bullet,\bullet}))$$

such that

$$\mathcal{F}_0 = {}^\infty E^{-1,0}_{\text{hor}} = \operatorname{Coker}\left(d_2^{0,2}\right) \quad \text{and} \quad \frac{H_{-1}(\operatorname{Tot}(E^{\bullet,\bullet}))}{\operatorname{Coker}\left(d_2^{0,2}\right)} = {}^\infty E^{0,1}_{\text{hor}} = H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})).$$

Thus we are able to construct the following exact sequence

$$0 \longrightarrow \operatorname{Coker}(d_2^{0,2}) \longrightarrow H_{-1}(\operatorname{Tot}(E^{\bullet,\bullet})) \longrightarrow H_1(\mathbf{x}, S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \longrightarrow 0. \qquad (3.5)$$

Finally, since $H_{-1}(\operatorname{Tot}(E^{\bullet,\bullet})) = H_1(\mathcal{Z}_\bullet^+)$, by splicing the sequences 3.4 and 3.5, one obtains

$$\begin{tikzcd}
H_2(\mathcal{Z}_\bullet^+(\mathbf{a},\mathbf{f})) \arrow[r] & H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \arrow[r, "d_2^{0,2}"] & H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})) \otimes_S S/(\mathbf{x}) \arrow[dl] & \\
& H_1(\mathcal{Z}_\bullet^+(\mathbf{a},\mathbf{f})) \arrow[r] & H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \arrow[r] & 0
\end{tikzcd}.$$

□

The general construction method for the exact sequence (3.1) is commonly referred to in the literature as the five-term exact sequence. [God73, Thèorème 4.5.1] and [Wei94, Exercise 5.1.3]. As immediate consequence of the proof of Theorem 3.2.2, one has

Corollary 3.2.3. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of $\mathfrak{a}$ and I, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Let $S = R[U_{ij}]$ be the polynomial extension of R in rs indeterminates and consider $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$. If $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s,\mathbf{f})$ is an acyclic complex, then, for all $0 \le i \le rs$, one has*

$$H_i(\mathcal{Z}_\bullet^+(\mathbf{a},\mathbf{f})) = H_i(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})).$$

Proof: Indeed if $\mathcal{Z}_\bullet^{+\mathfrak{g}}$ is acyclic, then the second page of horizontal spectral of double complex $E^{\bullet,\bullet} = \mathcal{Z}_\bullet^{+\mathfrak{g}} \otimes_S K_\bullet$ collapses becoming

$$\begin{array}{cccc}
0 & \cdots & 0 & H_{rs}(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}})) \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & H_0(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}}))
\end{array}$$

Thus, by the convergence theorem, we get $H_{-i}(\operatorname{Tot}(E^{\bullet,\bullet})) \cong H_i(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}}))$ for all $0 \le i \le rs$. Since $H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})$, comparing with the vertical spectral sequence, we conclude

$$H_i(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s,\mathbf{f})) \cong H_i(\mathcal{Z}_\bullet^+)$$

for all $0 \leq i \leq rs$ $\square$

In particular, if R is a Noetherian ring and $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})$ is an acyclic complex, Corollary 3.2.3 says that the complex $\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})$ is rigid.

Corollary 3.2.4. *Let R be a ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of $\mathfrak{a}$ and I, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Let $S = R[U_{ij}]$ be the polynomial extension of R in rs indeterminates and consider $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$. If $H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})) = 0$, then*

$$H_1(\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})) \cong H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})).$$

Moreover, assuming that R is a Noetherian ring, one has

$$\operatorname{Kitt}(\mathfrak{a}, I) \text{ deforms to } \operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) \qquad \Longleftrightarrow \qquad H_1(\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})) = 0.$$

Proof: The first part follows directly from the exact sequence

$$\begin{tikzcd} H_2(\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})) \arrow[r] & H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \arrow[r] & H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})) \otimes_S S/(\mathbf{x}) \arrow[dl] & \\ & H_1(\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})) \arrow[r] & H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \arrow[r] & 0 \end{tikzcd}$$

and the second part follows from rigidity of Koszul complex. $\square$

Proposition 3.2.5. *Let R be a Noetherian ring, $\mathfrak{a} \subseteq I$ finitely generated ideals of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ and $\mathbf{a} = a_1, \ldots, a_s$ systems of generators of $\mathfrak{a}$ and I, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Let $S = R[U_{ij}]$ be the polynomial extension of R in rs indeterminates and consider $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$. If $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})$ is acyclic and $\mathbf{x}$ is regular on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$, then $\mathcal{Z}_\bullet^{+}(\mathbf{a}, \mathbf{f})$ is acyclic. The converse holds if $(R, \mathfrak{m})$ is Noetherian local and $\mathfrak{a} \subseteq \mathfrak{m}I$.*

Proof: Suppose that $\mathcal{Z}_\bullet^{+\mathfrak{g}}$ is acyclic and $\mathbf{x}$ is regular on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$. By Corollary 3.2.3, we have have $H_i(\mathcal{Z}_\bullet^{+}) \cong H_i(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) = 0$ for all $i \geq 1$, which implies that $\mathcal{Z}_\bullet^{+}$ is an acyclic complex.

Conversely suppose the complex $\mathcal{Z}_\bullet^{+}$ is acyclic. Firstly note that ${}^2E_{\text{ver}}^{p,q} = 0$ except if $(p, q) = (0, 0)$, which implies that $H_{-i}(\operatorname{Tot}(E^{\bullet,\bullet})) = 0$ for all $i \geq 1$. Analyzing the horizontal spectral sequence on infinite, we obtain

$$\begin{array}{cc} \vdots & \vdots \\ & \\ * & {}^2E_{\mathrm{hor}}^{0,1} \\ & \\ {}^{\infty}E_{\mathrm{hor}}^{-1,0} & H_0(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}})) \end{array}$$

By convergence theorem, one knows that there exists a filtration of $H_{-1}(\mathrm{Tot}(E^{\bullet,\bullet}))$

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 = H_{-1}(\mathrm{Tot}(E^{\bullet,\bullet})) = 0$$

such that ${}^2E_{\mathrm{hor}}^{0,1} = \mathcal{F}_1/\mathcal{F}_0$. However, since $\mathcal{F}_1 = \mathcal{F}_2 = 0$, one has that ${}^2E_{\mathrm{hor}}^{0,1} = 0$. As

$$H_1(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(\mathbf{x}, \mathbf{f})) = {}^2E_{\mathrm{hor}}^{0,1} = 0$$

and R is a Noetherian ring, the Koszul rigidity tells us that $\mathbf{x}$ is a regular sequence on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$. Now we will prove by induction on i that $H_i(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = 0$ for all $i \geq 1$. For $i = 1$, observe that

$${}^{\infty}E_{\mathrm{hor}}^{-1,0} = \mathrm{Coker}(d_2^{0,2}) = {}^2E_{\mathrm{hor}}^{-1,0}$$

because ${}^2E_{\mathrm{hor}}^{0,2} = H_2(\mathbf{x}; S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) = 0$. Thus there exists a filtration of $H_{-1}(\mathrm{Tot}(E^{\bullet,\bullet}))$

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 = H_{-1}(\mathrm{Tot}(E^{\bullet,\bullet})) = H_1(\mathcal{Z}_\bullet^{+}) = 0$$

such that $H_0(\mathbf{x}; H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})) = {}^2E_{\mathrm{hor}}^{-1,0} = \mathcal{F}_1 = 0$. Since

$$H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})/(\mathbf{x})H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = H_0(\mathbf{x}; H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})) = 0,$$

we have that $H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = (\mathbf{x})H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})$. Note $H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})$ is a finitely generated graded S-module, since S is Noetherian graded ring. Moreover, since $\mathfrak{a} \subseteq \mathfrak{m}I$, one can choose the elements $c_{ij} \in R$ such that $c_{ij} \in \mathfrak{m}$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$, which implies that the *maximal ideal $\mathfrak{m} + (\{U_{ij}\ ;\ 1 \leq i \leq r,\, 1 \leq j \leq s\})$ contains $(\mathbf{x})$. Finally, as

$$H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})_{(\mathfrak{m}, U_{ij})} = (\mathbf{x})H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})_{(\mathfrak{m}, U_{ij})},$$

the Graded Nakayama Lemma says that $H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}})_{(\mathfrak{m}, U_{ij})} = 0$, which means that $H_1(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = 0$.

Suppose that we have proved that $H_i(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = 0$ for all $1 \leq i < n \leq s$. Thus the second page of horizontal spectral ${}^2E_{\mathrm{hor}}^{\bullet,\bullet}$ becomes

$$\begin{array}{ccccccc}
* & * & H_{rs}(\mathbf{x}; H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}})) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & H_1(\mathbf{x}; H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}})) & 0 & \cdots & 0 & 0 \\
* & * & H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}})/(\mathbf{x})H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}}) & 0 & \cdots & 0 & H_0(\mathbf{x}; H_0(\mathcal{Z}_\bullet^{+\mathfrak{g}}))
\end{array}$$

By the convergence theorem, we conclude that

$$H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}})/(\mathbf{x})H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = H_{-n}(\mathrm{Tot}(E^{\bullet,\bullet})) = H_n(\mathcal{Z}_\bullet^{+}) = 0.$$

Finally, proceeding as in the case $i = 1$, we conclude that $H_n(\mathcal{Z}_\bullet^{+\mathfrak{g}}) = 0$. □

Remark 3.2.6. *Note that the local condition of R and $\mathfrak{a} \subseteq \mathfrak{m}I$ mentioned in the Proposition 3.2.5 is not required for establishing that the sequence $\mathbf{x}$ is regular on $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$.*

The combination of [Has12, Corollary 2.9(b)] with Proposition 3.2.5 and Remark 3.2.6 provides the following interesting criterion for deformation of the ordinary Kitt to the generic one.

Corollary 3.2.7. *Let R be a Cohen-Macaulay local ring and $\mathfrak{a} \subset I$ ideals of R such that $J = \mathfrak{a} :_R I$ is an s-residual intersection of I. Consider $\mathbf{f}$ and $\mathbf{a}$ systems of the generators of I and $\mathfrak{a}$, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Under any of the following conditions:*

(i) $s \leq \mathrm{ht}(I) + 1$*;*

(ii) R Gorenstein, $s = \mathrm{ht}(I) + 2$ and I^{unm} Cohen-Macaulay ideal;

the sequence $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ is regular over $S/\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$. In particular, $\operatorname{Kitt}(\mathfrak{a}, I)$ *deforms to* $\operatorname{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$.

Proof: In fact, by [Has12, Corollary 2.9 (b)], the complex $\mathcal{Z}_\bullet^{+}$ is acyclic. By Proposition 3.2.5 and its remark, we conclude that the sequence $\mathbf{x}$ is regular on $S/\operatorname{Kitt}_s^{\mathfrak{g}}(s, \mathbf{f})$. Since $\mathbf{x}$ also is an

S-regular sequence, then, by Proposition 3.1.8 and its remark, one concludes that $\mathrm{Kitt}(\mathfrak{a}, I)$ deforms to $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$. $\square$

The following corollary gives an enough condition which ensures the rigidity of $\mathcal{Z}_\bullet^+(\mathbf{a}, \mathbf{f})$ for any s-generated ideal $\mathfrak{a} = (\mathbf{a}) \subseteq (\mathbf{f}) = I$.

Corollary 3.2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R and s a positive integer. Consider $\mathbf{f}$ a system of generators of I and suppose that there exists an s-generated ideal $\mathfrak{b} = (\mathbf{b})$ with $\mathfrak{b} \subseteq \mathfrak{m}I$ such that $\mathcal{Z}_\bullet^+(\mathbf{b}, \mathbf{f})$ is acyclic. Then for any s-generated ideal $\mathfrak{a} = (\mathbf{a})$, the complex $\mathcal{Z}_\bullet^+(\mathbf{a}, \mathbf{f})$ is rigid.*

Proof: Since there exists an s-generated ideal $\mathfrak{b} = (\mathbf{b})$ with $\mathfrak{b} \subseteq \mathfrak{m}I$ such that $\mathcal{Z}_\bullet^+(\mathbf{b}, \mathbf{f})$ is acyclic, Proposition 3.2.5 tells that $\mathcal{Z}_\bullet^{+\mathfrak{g}}(s, \mathbf{f})$ is acyclic. As the Koszul complex $K_\bullet(\mathbf{x}; S/\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}))$ is rigid, the statement follows from Corollary 3.2.3. $\square$

Recall that the Cohen-Macaulayness and Gorensteiness are stable under specialization by regular sequences. In particular, these properties keep preserved when the generic Kitt specializes to the generic one, when the latter deforms to the generic Kitt. More explicitly

Proposition 3.2.9. *Let R be a Noetherian ring, I an ideal of R and s a non-negative integer. Consider $\mathbf{f}$ a system of generators of I and let $\mathfrak{a} \subseteq I$ be an ideal generated by s elements. Suppose that $\mathrm{Kitt}(\mathfrak{a}, I)$ deforms to $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$.*

(i) If $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$ is Cohen-Macaulay, then $\mathrm{Kitt}(\mathfrak{a}, I)$ is Cohen-Macaulay;

(ii) If $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$ is Gorenstein, then $\mathrm{Kitt}(\mathfrak{a}, I)$ is Gorenstein.

To finish this section, we prove that these properties on the generic Kitt, when they hold, do not depend from the system of generators of I.

Lemma 3.2.10. *Let R, S be Noetherian rings and I, J ideals of R and S, respectively. Suppose that the pairs (R, I) and (S, J) are universally equivalent.*

(i) Then I is a Cohen-Macaulay ideal if and only if J is a Cohen-Macaulay ideal;

(ii) Then I is a Gorenstein ideal if and only if J is a Gorenstein ideal.

In particular, given $\mathbf{f}$ and $\mathbf{f}'$ systems of generators of I, the ideal $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})$ is Cohen-Macaulay (Gorenstein) if and only if the ideal $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}')$ is Cohen-Macaulay (Gorenstein).

Proof: Let X and Y be finite sets of indeterminates over R and S, respectively such that there is a ring isomorphism $\phi : R[X] \longrightarrow S[Y]$ with $\phi(IR[X]) = JS[Y]$. Notice that ϕ induces the ring isomorphism

$$\psi : \frac{R[X]}{IR[X]} \longrightarrow \frac{S[X]}{JS[X]}. \tag{3.6}$$

(i) Suppose that I is a Cohen-Macaulay ideal. Since the quotient ring R/I is Cohen-Macaulay, by [BH93, Theorem 2.1.9], one has that $R[X]/IR[X] = R/I \otimes_R R[X]$ is Cohen-Macaulay. Hence the ring $S[Y]/JS[Y] = S/J \otimes_S S[Y]$ is Cohen-Macaulay and, applying [BH93, Theorem 2.1.9] again, one concludes that J is a Cohen-Macaulay ideal. Switching I by J and repeating the same argument, we prove the converse.

(ii) Suppose that I is a Gorenstein ideal. Since the quotient ring R/I is Gorenstein, $R[X]/IR[X] = (R/I)[X]$ and the polynomial extension of Gorenstein ring still is Gorenstein. Using the isomorphism (3.6), one concludes that the ring $S[Y]/JS[Y] = (S/J)[Y]$ is a Gorenstein ring. Since the sequence of the indeterminates Y is $(S/J)[Y]$-regular, by [BH93, Proposition 3.1.19], one concludes that

$$\frac{S}{J} \cong \frac{(S/J)[Y]}{(Y)(S/J)[Y]}$$

is a Gorenstein ring, which implies that J is a Gorenstein ideal. Switching I by J and repeating the same argument, we prove the converse. $\square$

3.3 Higher Order Generic Kitt Ideals

In this section, we introduce the notion of higher-order generic Kitt ideals through a recursive approach, drawing parallels with Ulrich and Huneke's definition of generic linkage defined in [HU87].

Definition 3.3.1. *Let R be a ring, I a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I. The first order s-generic Kitt of I with respect to the generating set $\mathbf{f}$ is $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) \subseteq S_1 := R[U_{1,ij} \ ; \ 1 \leq i \leq r, \ 1 \leq j \leq s]$ and it is denoted by $\mathrm{K}^1(s, \mathbf{f})$. Once defined the n-th order s-generic Kitt $\mathrm{K}^n(s, \mathbf{f}) \subseteq S_n$ of I, the $(n+1)$-th order s-generic Kitt of I with respect to the generating set $\mathbf{f}$ is the ideal*

$$\mathrm{K}^{n+1}(s, \mathbf{f}) := \mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}S_n) \subseteq S_{n+1} := S_n[U_{n+1,ij} \ ; \ 1 \leq i \leq r, \ 1 \leq j \leq s],$$

where $\mathbf{f}S_n$ means the sequence $\mathbf{f}$ as a sequence of elements of the ring S_n.

Notice that, passing from $\mathrm{K}^n(s, \mathbf{f})$ to $\mathrm{K}^{n+1}(s, \mathbf{f})$, we are only increasing dimension of ambient ring by adding indeterminates. This fact implies that $\mathrm{K}^i(\mathbf{f}, s)$ and $\mathrm{K}^j(\mathbf{f}, s)$ are universally equivalent as we will see in next proposition.

Proposition 3.3.2. *Let R be a ring, I a finitely generated ideal of R and s a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I. For any i, $j \in \mathbb{N}$, the pairs*

$$(S_i, K^i(s, \mathbf{f})) \qquad \textit{and} \qquad (S_j, K^j(s, \mathbf{f}))$$

are universally equivalent.

Proof: Since the definition of higher-order generic Kitt is recursive and the universal equivalence is an equivalence relation, it is enough to show that the pairs $(S_1, K^1(s, \mathbf{f}))$ and $(S_2, K^2(s, \mathbf{f}))$ are universally equivalent. Let $U = \{U_{11}, \ldots, U_{rs}\}$ be indeterminates over R and $V = \{V_{11}, \ldots, V_{rs}\}$ indeterminates over $S_1 = R[U_{ij}\ ;\ 1 \leq i \leq r,\ 1 \leq j \leq s]$. Let $\mathbf{a}' = a'_1, \ldots, a'_s \in S_1$ and $\mathbf{b}' = b'_1, \ldots, b'_s \in S_2 = S_1[V_{ij}\ ;\ 1 \leq i \leq r,\ 1 \leq j \leq s]$ such that

$$\begin{bmatrix} a'_1 & a'_2 & \cdots & a'_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \ldots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix},$$

$$\begin{bmatrix} b'_1 & b'_2 & \cdots & b'_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1s} \\ V_{21} & V_{22} & \ldots & V_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ V_{r1} & V_{r2} & \cdots & V_{rs} \end{bmatrix}.$$

We have to prove that the pairs $(S_1, \mathrm{Kitt}((\mathbf{a}'), IS_1))$ and $(S_2, \mathrm{Kitt}((\mathbf{b}'), IS_2))$ are universally equivalent. Now consider $Z = \{Z_{11}, \ldots, Z_{rs}\}$ a set of indeterminates over S_1 and define the R-algebra homomorphism $\phi : S_1[Z] \longrightarrow S_2$ such that

$$U_{ij} \longmapsto V_{ij} \qquad\qquad Z_{ij} \longmapsto U_{ij}$$

Clearly ϕ is an R-algebra isomorphism. Furthermore, observe that $\phi((\mathbf{a}')S_1[Z]) = (\mathbf{b}')$ and

$\phi(IS_1[Z]) = IS_2$. Using Proposition 2.3.2, one concludes

$$\begin{aligned}\phi(K^1(s,\mathbf{f})S_1[Z]) &= \phi(\mathrm{Kitt}((\mathbf{a}'), IS_1)S_1[Z]) = \phi(\mathrm{Kitt}((\mathbf{a}')S_1[Z], IS_1[Z])) \\ &= \mathrm{Kitt}(\phi((\mathbf{a}')S_1[Z]), \phi(IS_1[Z])) = \mathrm{Kitt}((\mathbf{b}'), IS_2) = K^2(s,\mathbf{f}),\end{aligned}$$

which implies that the pairs $(S_1, K^1(s,\mathbf{f}))$ and $(S_2, K^2(s,\mathbf{f}))$ are universally equivalent. $\square$

Chapter 4

Generic Residual

4.1 Definitions and its Relation with the Generic Residual Intersection

In this section, we introduce the concept of **generic s-residual** of an ideal, which consists in a weaker notion of generic s-residual intersection introduced by Huneke and Ulrich [HU88, Definition 3.1]. The main difference arises from the exclusion of the G_{s+1} condition. Recall from Definition 2.1.7 that an ideal I satisfies the G_s condition if $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{V}(I)$ with $\mathrm{ht}(\mathfrak{p}) < s$.

Definition 4.1.1. *Let R be a Noetherian ring, $I \neq 0$ an ideal of R and $s \geq \max\{1, \mathrm{ht}(I)\}$ a positive integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I. Let $S = R[U_{ij};\ 1 \leq i \leq r, 1 \leq j \leq s]$ be the polynomial extension of R in rs indeterminates and $\mathbf{a} = a_1, \ldots, a_s \in S$ such that*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix}.$$

*The **generic s-residual** of the system of generators $\mathbf{f}$ is the ideal $(\mathbf{a}) :_S IS$ of S and it is denoted by $\mathrm{R}(s, \mathbf{f})$.*

Note that the generic s-residual is sensible to the choice of generators of I as illustrated in the following example.

Example 4.1.2. Let $I = (x_1^2 - x_0x_2,\ x_0x_1 - x_2,\ x_0^2 - x_1) =: (f_1, f_2, f_3) \subseteq k[x_0, x_1, x_2]$ be the definition ideal of the variety parameterized by

$$\begin{aligned} \varphi\colon \mathbb{A}^1 &\longrightarrow \mathbb{A}^3 \\ t &\longmapsto (t, t^2, t^3) \end{aligned}.$$

Let $\mathbf{f}' = f_1, f_2 - f_1, f_3$ be other system of generators of I, $S = R[U_{ij};\ 1 \le i \le 3, 1 \le j \le 2]$ the polynomial extension of R in six indeterminates and $\mathbf{a} = a_1, a_2$, $\mathbf{a}' = a_1', a_2' \in S$ such that

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{bmatrix}, \qquad \begin{bmatrix} a_1' & a_2' \end{bmatrix} = \begin{bmatrix} f_1 & f_2 - f_1 & f_3 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{bmatrix}.$$

Using *Macaulay2*, one checks that the ideals

$$\begin{gathered} \mathrm{R}(2, \mathbf{f}) = (x_1^2U_{11} - x_0x_2U_{11} + x_0x_1U_{21} + x_0^2U_{31} - x_2U_{21} - x_1U_{31}, \\ x_1^2U_{12} - x_0x_2U_{12} + x_0x_1U_{22} + x_0^2U_{32} - x_2U_{22} - x_1U_{32}), \\ \mathrm{R}(2, \mathbf{f}') = (x_1^2U_{11} - x_0x_2U_{11} + x_0x_1U_{21} - x_1^2U_{21} + x_0x_2U_{21} + x_0^2U_{31} - x_2U_{21} - x_1U_{31}, \\ x_1^2U_{12} - x_0x_2U_{12} + x_0x_1U_{22} - x_1^2U_{22} + x_0x_2U_{22} + x_0^2U_{32} - x_2U_{22} - x_1U_{32}) \end{gathered}$$

are distinct ideals of S.

Similarly to what occurs with the generic Kitt, it can be demonstrated that the generic s-residual of the same ideal with different generators is also unique up to universal equivalence. The proof follows the identical argument employed in the proof established by Huneke and Ulrich [HU90, Lemma 2.2], where they proved that the generic s-residual intersection of an ideal is unique up to universal equivalence.

Huneke and Ulrich [HU88, Theorem 3.3] proved that, when R is a Cohen-Macaulay local ring, for any non-zero strongly Cohen-Macaulay ideal I satisfying the condition G_{s+1}, its s-generic residual is a geometric s-residual intersection of $IR[U]$. Now we will prove that $\mathrm{ht}(\mathrm{R}(s, \mathbf{f})) \le s$ if R is formally equidimensional ring and I is a non-nilpotent ideal. Recall that a Noetherian ring R is formally equidimensional if, for any $\mathfrak{p} \in \mathrm{Spec}(R)$, its $\mathfrak{p}$-adic completion is an equidimensional ring.

Proposition 4.1.3. *Let R be a Noetherian formally equidimensional ring, I an ideal and $s \geq \max\{1, \mathrm{ht}(I)\}$ an integer. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I. If I is a non-nilpotent ideal, then* $\mathrm{ht}(\mathrm{R}(s, \mathbf{f})) \leq s$.

Proof: Suppose by contradiction that $\mathrm{ht}(\mathrm{R}(s, \mathbf{f})) > s$. Let $S = R[U_{ij}\ ;\ 1 \leq i \leq r, 1 \leq j \leq s]$ be the polynomial extension of R in rs indeterminates and $\mathbf{a} = a_1, \ldots, a_s \in S$ such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix}.$$

Denote $\mathfrak{a} = (\mathbf{a}) \subseteq S$ and let t be an indeterminate over S. Observe that

$$(\mathfrak{a}, t) :_{S[t]} (I, t) = ((\mathfrak{a} :_S IS), t).$$

Furthermore, note that $\mathrm{R}(s, \mathbf{f}) = \mathfrak{a} :_S IS$ and

$$\mathrm{ht}\left((\mathfrak{a}, t) :_{S[t]} (IS, t)\right) = \mathrm{ht}((\mathfrak{a} :_S (IS)), t) = \mathrm{ht}(\mathrm{R}(s, \mathbf{f})) + 1 > s + 1.$$

Since formal equidimensionality is stable under polynomial extension [HIO88, Theorem 18.13], $S[t]$ is a formally equidimensional ring. Hence, applying the same idea of the proof of [Ulr92, Proposition 3] for the non-local ring $S[t]$, one has that $(I, t)S \subseteq \overline{(\mathfrak{a}, t)}$, where $\overline{(\mathfrak{a}, t)}$ is the integral closure of the ideal $(\mathfrak{a}, t)$. Recall that $\mathfrak{a}$ is a generic subideal of IS. By doing specialization, one concludes that (I, t) is integral over the principal ideal (t). Since t is an indeterminate, one gets that I is a nilpotent ideal, giving a contradiction. Hence $\mathrm{ht}(\mathrm{R}(s, \mathbf{f})) \leq s$. $\square$

4.2 Height Comparison

In his Ph.D. thesis, Bouça [Bou19, Theorem 4.3.4] proved that $\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, (\mathbf{f})))$ for any s-generated ideal $\mathfrak{a} \subseteq (\mathbf{f})$ in Cohen-Macaulay local ring. However, we realized this conclusion may not always hold, as demonstrated in the following example:

Example 4.2.1. Let $R = k[x_0, x_1, x_2, x_3]$ and $T = k[s, t]$ be polynomial rings over a field k in four and two indeterminates, respectively. Consider the k-algebra homomorphism $\phi : R \longrightarrow T$ such that

$$x_0 \longmapsto s^4, \qquad x_1 \longmapsto s^3t, \qquad x_2 \longmapsto st^3, \qquad x_3 \longmapsto t^4.$$

Let $\mathfrak{a}$ be the 3-generated ideal $(x_2^3 - x_1x_3^2,\ x_0x_2^2 - x_1^2x_3,\ x_1^3 - x_0^2x_2) \subset \mathrm{Ker}(\phi)$ and

$I = \mathfrak{a} :_R (x_0, x_1, x_2, x_3) = (x_2^3 - x_1x_3^2,\ x_0x_2^2 - x_1^2x_3,\ x_1^3 - x_0^2x_2,\ x_1^2x_2^2 - x_0x_1x_2x_3) =: (f_1, f_2, f_3, f_4).$

Since $J = \mathfrak{a} :_R I = (x_0, x_1, x_2, x_3)$, one concludes that $\mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)) = \mathrm{ht}(J) = 4$. Now consider $S = R[U_1, \ldots, U_{12}]$ be a polynomial extension of R in twelve indeterminates and let $\mathbf{a} = a_1, a_2, a_3 \in S$ such that

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} U_1 & U_5 & U_9 \\ U_2 & U_6 & U_{10} \\ U_3 & U_7 & U_{11} \\ U_4 & U_8 & U_{12} \end{bmatrix}.$$

By using the *Macaulay2*, one gets that $\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(3, \mathbf{f})) = \mathrm{ht}(\mathrm{R}(3, \mathbf{f})) = 3$. Hence one obtained an specialization $\mathrm{Kitt}(\mathfrak{a}, I)$ of $\mathrm{Kitt}^{\mathfrak{g}}(3, \mathbf{f})$ such that $\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(3, \mathbf{f})) < \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I))$.

Taking this direction, we proved that, under more specific conditions on the ring, the following inequality holds

$$\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}})$$

for any maximal ideal $\mathfrak{n} \subset S$ containing $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})$, where $\mathfrak{m} = \mathfrak{n} \cap R$, $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$ and $c_{11}, \ldots, c_{rs} \in R$ are such that $a_i = \sum_{k=1}^{r} c_{ki} f_k$ for all $i = 1, \ldots, s$. In order to prove this result, two lemmas will be necessary.

Lemma 4.2.2. *Let R be a ring and $c_1, \ldots, c_n$ elements of R. Consider $S = R[X_1, \ldots, X_n]$ the polynomial extension of R in n indeterminates and let $\mathfrak{n}$ be an ideal of S containing $\mathfrak{c} := (X_1 - c_1, \ldots, X_n - c_n)$. Then*

(i) If $\mathfrak{n}$ is a proper ideal of S, then $\big(\{f(c_1, \ldots, c_n)\ ;\ f \in \mathfrak{n}\}\big)S \subset S$ also is a proper ideal of S;

(ii) If $\mathfrak{n}$ is a maximal ideal of S, then $\mathfrak{n} = \mathfrak{c} + \big(\{f(c_1, \ldots, c_n)\ ;\ f \in \mathfrak{n}\}\big)S$.

Proof: (i) Suppose by contradiction that $\big(\{f(c_1, \ldots, c_n)\ ;\ f \in \mathfrak{n}\}\big)S = S$. Let $f_1, \ldots, f_m \in \mathfrak{n}$ and $g_1, \ldots, g_m \in S$ such that

$$\sum_{k=1}^{m} f_k(c) g_k(X) = 1.$$

Applying the division algorithm, there are $h_1, \dots, h_k \in \mathfrak{c}$ such that $f_k = h_k + f_k(c)$ for all $k = 1, \dots, m$. Thus

$$1 = \sum_{k=1}^{m} (f_k(X) - h_k(X)) g_k(X) = \sum_{k=1}^{m} f_k(X) g_k(X) - \sum_{k=1}^{m} h_k(X) g_k(X) \in \mathfrak{n} + \mathfrak{c} = \mathfrak{n},$$

giving a contradiction. Then $\big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S$ is a proper ideal of S.

(ii) By using the division algorithm, one can see that $\mathfrak{n} \subseteq \mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S$. Hence, since $\mathfrak{n}$ is maximal, it is enough to show that $\mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S$ is a proper ideal of S. Suppose by contradiction that $\mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S = S$. Let $h \in \mathfrak{c}$, $f_1, \dots, f_m \in \mathfrak{n}$ and $g_1, \dots, g_m \in S$ such that

$$h(X) + \sum_{k=1}^{m} f_k(c) g_k(X) = 1.$$

Evaluating in $(c_1, \dots, c_n)$, one gets that $\sum_{k=1}^{m} f_k(c) g_k(c) = 1$, contradicting the item (i). Hence $\mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S$ is a proper ideal of S and

$$\mathfrak{n} = \mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f \in \mathfrak{n}\}\big)S.$$

$\square$

Lemma 4.2.3. *Let R be a Noetherian ring such that all maximal ideals have the same height. Let $S = R[x_1, \dots, x_n]$ be the polynomial extension of R in n indeterminates and $c_1, \dots, c_n \in R$. Then, for any maximal ideal $\mathfrak{n}$ of S containing the ideal $(x_1 - c_1, \dots, x_n - c_n)$, one has*

$$\mathrm{ht}(\mathfrak{n}) = \dim(S) = \dim(R) + n.$$

Proof: Let $\mathfrak{n}$ be a maximal ideal of S containing $\mathfrak{c} := (x_1 - c_1, \dots, x_n - c_n)$. Firstly notice that, since $\mathfrak{n}$ it contains $\mathfrak{c}$, then $\mathfrak{n} = \mathfrak{c} + (\mathfrak{n} \cap R)S$. Indeed, by Lemma 4.2.2, one has

$$\mathfrak{n} = \mathfrak{c} + \big(\{f(c_1, \dots, c_n) \ ; \ f(x_1, \dots, x_n) \in \mathfrak{n}\}\big)S =: \mathfrak{c} + \mathfrak{a}.$$

Thus $\mathfrak{n} \cap R = \mathfrak{a} \cap R$, so

$$\mathfrak{n} = \mathfrak{c} + (\mathfrak{a} \cap R)S = \mathfrak{c} + (\mathfrak{n} \cap R)S \subseteq \mathfrak{n} + \mathfrak{n} = \mathfrak{n}$$

and the equality follows. As $\mathfrak{n} \cap R$ is a prime ideal of R, then $\mathfrak{n} \subseteq \mathfrak{c} + \mathfrak{m}S$ for some maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{n} \cap R$. Note this inclusion actually is an equality. Indeed, if it was not, there would be $a_1, \ldots, a_m \in \mathfrak{m}$, $g_1, \ldots, g_m \in S$ and a polynomial $f \in S$ such that $f(c_1, \ldots, c_n) = 0$ and

$$f(x_1, \ldots, x_n) + \sum_{k=1}^{m} a_k g_k = 1.$$

Evaluating this expression in $(c_1, \ldots, c_n)$, one gets a contradiction. Thus $\mathfrak{n} = \mathfrak{c} + \mathfrak{m}S$ and $\mathfrak{n} \cap R = \mathfrak{m}$. Applying [Mat89, Theorem 15.1], one gets that

$$\mathrm{ht}_S(\mathfrak{n}) = \mathrm{ht}_R(\mathfrak{m}) + \dim\left(\frac{S_{\mathfrak{n}}}{\mathfrak{m}S_{\mathfrak{n}}}\right) = \dim(R) + \dim\left(\frac{S_{\mathfrak{n}}}{\mathfrak{m}S_{\mathfrak{n}}}\right)$$

As $S_{\mathfrak{n}}/\mathfrak{m}S_{\mathfrak{n}} \cong (R/\mathfrak{m})[x_1, \ldots, x_n]_{\mathfrak{c}}$ is localization of an affine integral domain at a maximal ideal, its dimension is n, hence one gets

$$\dim(S) \geq \mathrm{ht}_S(\mathfrak{n}) = \dim(R) + \dim\left(\frac{S_{\mathfrak{n}}}{\mathfrak{m}S_{\mathfrak{n}}}\right) = \dim(R) + n = \dim(S)$$

and the statement follows. $\square$

The following example shows the hypothesis that $\mathfrak{n}$ should contain some ideal of form $\mathfrak{c}$ in Lemma 4.2.3 cannot be avoided.

Example 4.2.4. Let $R = k[\![x]\!]$ be the ring of formal series over a field k. Consider $S = R[y]$ the polynomial extension of R in one indeterminate. Notice that R is a Noetherian one-dimensional integral domain, so every maximal ideal of R has the height one. Note the maximal ideal $\mathfrak{m} = (1 - xy) \subseteq S$ does not contain an element of form $y - c$ and $\mathrm{ht}(\mathfrak{m}) = 1 \neq 2 = \dim(S)$.

Finally we are ready to establish the promised result. Recall that a ring R is said to satisfy the dimension equality if

$$\dim(R) = \mathrm{ht}(I) + \dim\left(\frac{R}{I}\right)$$

for all ideal I of R.

Proposition 4.2.5. *Let R be a Cohen-Macaulay local ring (or an affine domain), $\mathfrak{a} \subset I$ ideals of R and s a positive integer. Consider $\mathbf{a} = a_1, \ldots, a_s$, $\mathbf{f} = f_1, \ldots, f_r$ systems of generators of the ideals $\mathfrak{a}$, I, respectively and $c_{11}, \ldots, c_{rs} \in R$ such that $a_i = \sum_{k=1}^{r} c_{ki} f_k$. Let $S = R[U_{ij}\,;\, 1 \leq i \leq r, 1 \leq j \leq s]$ be the polynomial extension of R in rs indeterminates and $\mathbf{x} = U_{11} - c_{11}, \ldots, U_{rs} - c_{rs}$. Then, given a maximal ideal $\mathfrak{n}$ of S containing $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})$ and denoting $\mathfrak{m} := \mathfrak{n} \cap R$, one has*

$$\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}).$$

for any s-generated subideal $\mathfrak{a} \subseteq I$. In particular, if $\mathfrak{a} :_R I$ is an s-residual intersection of I, then $\mathrm{R}(s, \mathbf{f})_{\mathfrak{n}}$ *is an s-residual intersection of $IS_{\mathfrak{n}}$ for all maximal ideal $\mathfrak{n}$ of S containing* $\mathrm{R}(s, \mathbf{f}) + (\mathbf{x})$.

Proof: Before starting the proof, we notice that R and S are both locally universally catenary Noetherian rings such that all maximal ideals have the same height and all local rings satisfies the dimension equality. Furthermore, if the ideal $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})$ is not a proper ideal, then $\mathrm{Kitt}(\mathfrak{a}, I) = (1)$ by Proposition 3.1.8. Therefore $\mathfrak{a} = I$ by Theorem 2.1.6 (iii), which is not the case. Then Lemma 4.2.3 can be applied. Let $\mathfrak{a} = (a_1, \ldots, a_s) \subseteq I$ be an ideal generated by s elements and $\Phi = [c_{ij}]_{r \times s}$ such that

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \Phi.$$

Now let $a'_1, \ldots, a'_s \in S$ such that

$$\begin{bmatrix} a'_1 & a'_2 & \cdots & a'_s \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & \cdots & f_r \end{bmatrix} \Psi.$$

Let $\mathfrak{n}$ be a maximal ideal of S which contains $\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x})$. Observe that $\mathfrak{n} \cap R = \mathfrak{m}$ is a maximal ideal of R. Furthermore, by Proposition 3.1.8, one has

$$\frac{S_{\mathfrak{n}}}{(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x}))_{\mathfrak{n}}} \cong \frac{R_{\mathfrak{m}}}{\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}}. \tag{4.1}$$

Since the catenary property is stable under localization and the rings $R_{\mathfrak{m}}$, $S_{\mathfrak{n}}$ satisfy the dimension equality [Mat89, page 31], [HIO88, Lemma 18.6], one has

$$\dim(R) - \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) - \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}) = \dim\left(\frac{R_{\mathfrak{m}}}{\mathrm{Kitt}(\mathfrak{a}, I)_{\mathfrak{m}}}\right)$$
$$= \dim\left(\frac{S_{\mathfrak{n}}}{(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f}) + (\mathbf{x}))_{\mathfrak{n}}}\right) \geq \dim\left(\frac{S_{\mathfrak{n}}}{\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}}\right) - rs = \dim(S_{\mathfrak{n}}) - \mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}) - rs$$
$$= \dim(S) - \mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}) - rs = \dim(R) - \mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s, \mathbf{f})_{\mathfrak{n}}),$$

which implies that $\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})_{\mathfrak{n}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a},I)_{\mathfrak{m}})$. In particular, if $\mathfrak{a} :_R I$ is an s-residual intersection, one has

$$\mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})_{\mathfrak{n}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a},I)_{\mathfrak{m}}) \geq \mathrm{ht}(\mathrm{Kitt}(\mathfrak{a},I)) = \mathrm{ht}(\mathfrak{a} :_R I) \geq s$$

for all maximal ideal $\mathfrak{n}$ of S containing the ideal $\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f}) + (\mathbf{x})$. $\square$

Combining Corollary 3.2.7 and Proposition 4.2.5, one obtains the following result.

Corollary 4.2.6. *Let R be a Cohen-Macaulay local ring, $\mathfrak{a} \subseteq I$ ideals of R and $J = \mathfrak{a} :_R I$ an s-residual intersection of I. Assume $s \leq \mathrm{ht}(I)+1$. Let $\mathbf{f} = f_1,\dots,f_r$ and $\mathbf{a} = a_1,\dots,a_s$ be systems of generators of I and $\mathfrak{a}$, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Let $S = R[U_{ij}]$ be the polynomial extension of R in rs indeterminates. Then, for any maximal ideal $\mathfrak{n}$ containing $\mathrm{R}(s,\mathbf{f}) + (U_{ij} - c_{ij})$, one has*

$$(S_{\mathfrak{n}}, \mathrm{R}(s,\mathbf{f})_{\mathfrak{n}}) \text{ is a deformation of } (R,J).$$

Proof: According to Theorem 2.1.6(iii) and Proposition 4.2.5, one has

$$\mathrm{ht}(\mathrm{R}(s,\mathbf{f})_{\mathfrak{n}}) = \mathrm{ht}(\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})_{\mathfrak{n}}) \geq s.$$

Thus $\mathrm{R}(s,\mathbf{f})_{\mathfrak{n}}$ is an s-residual intersection of $IS_{\mathfrak{n}}$. Since $s \leq \mathrm{ht}(I)+1$, Theorem 2.1.6(viii) implies that $J = \mathrm{Kitt}(\mathfrak{a},I)$ and $\mathrm{R}(s,\mathbf{f})_{\mathfrak{n}} = \mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})_{\mathfrak{n}}$ for any maximal ideal $\mathfrak{n}$ containing $(U_{ij} - c_{ij}) =: \mathbf{x}$. Applying Corollary 3.2.7(i), one concludes the sequence $\mathbf{x}_{\mathfrak{n}}$ is regular over $S_{\mathfrak{n}}$ and $S_{\mathfrak{n}}/\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})_{\mathfrak{n}}$. Hence, by Equation 4.1, one has

$$\frac{S_{\mathfrak{n}}/(\mathbf{x})_{\mathfrak{n}}}{(\mathrm{R}(s,\mathbf{f})+(\mathbf{x}))_{\mathfrak{n}}/(\mathbf{x})_{\mathfrak{n}}} = \frac{S_{\mathfrak{n}}}{(\mathrm{Kitt}^{\mathfrak{g}}(s,\mathbf{f})+(\mathbf{x}))_{\mathfrak{n}}} \cong \frac{R_{\mathfrak{m}}}{\mathrm{Kitt}(\mathfrak{a},I)_{\mathfrak{m}}} = \frac{R}{J}.$$

Hence the statement follows. $\square$

Here we show that for the first two classes of residual intersections, the generic residual is a deformation of arbitrary residual. The main point of the Corollary 4.2.6 is that the ideal I need not to satisfy any condition.

Corollary 4.2.7. *Let R be a Cohen-Macaulay local ring, $\mathfrak{a} \subset I$ ideals of R and $J = \mathfrak{a} :_R I$ an s-residual intersection of I. Consider $\mathbf{f} = f_1,\dots,f_r$ and $\mathbf{a} = a_1,\dots,a_s$ systems of generators of I and $\mathfrak{a}$, respectively, with $[\mathbf{a}] = [\mathbf{f}][c_{ij}]$. Suppose that I has height g and satisfies any of the following conditions*

- *I satisfies the G_s condition and* SDC_1 *at level* $\min\{s-g-2, r-g\}$ *and it is weakly $(s-2)$-residually S_2;*

- *I satisfies* SD_1.

Let $S = R[U_{ij}]$ the polynomial extension of R in rs indeterminates. Then, for any maximal ideal $\mathfrak{n}$ containing $\mathrm{R}(s, \mathbf{f}) + (U_{ij} - c_{ij})$, one has

$$(S_{\mathfrak{n}}, \mathrm{R}(s, \mathbf{f})_{\mathfrak{n}}) \textit{ is a deformation of } (R, J).$$

Part II

Loci openness problems: Cohen-Macaulayness and Strongly Cohen-Macaulayness

Chapter 5

Kitt Ideals and Open Loci Problems

Let R be a ring, M an R-module and $\mathbb{P}$ a property in R. The set of prime ideals $\mathfrak{p}$ of R such that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ satisfies the property $\mathbb{P}$ is called $\mathbb{P}$-*locus of* M and is denoted

$$\mathbb{P}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \ ; \ M_\mathfrak{p} \text{ satisfies the property } \mathbb{P} \text{ as } R_\mathfrak{p}\text{-module}\}.$$

Consider $\mathfrak{a} \subseteq I$ finitely generated ideals of R. Since the ideal $\operatorname{Kitt}(\mathfrak{a}, I)$ is always contained in $J = \mathfrak{a} :_R I$, mathematicians often seek conditions under which these two ideals are equal. However, one might go further and ask for which prime ideals $\mathfrak{p} \in \operatorname{Spec}(R)$ the following equality holds

$$\operatorname{Kitt}(\mathfrak{a}, I)_\mathfrak{p} = (\mathfrak{a} :_R I)_\mathfrak{p}.$$

Suppose that R is a Cohen-Macaulay ring and that $J = \mathfrak{a} :_R I$ is an s-residual intersection of I. Assume that I satisfies the property SD_1. Since the properties of being s-residual intersection and SD_1 are stable under localization [HSV84], Theorem 2.1.6(xii) says that

$$\operatorname{Kitt}(\mathfrak{a}, I)_\mathfrak{p} = \operatorname{Kitt}(\mathfrak{a}_\mathfrak{p}, I_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p} :_{R_\mathfrak{p}} I_\mathfrak{p} = (\mathfrak{a} :_R I)_\mathfrak{p}.$$

In particular, when $I_\mathfrak{p}$ is Strongly Cohen-Macaulay, one also has that this equality. The main objective of this chapter is to investigate what we define as the *Strongly Cohen-Macaulay locus of* I and demonstrate that this locus consists in a nonempty open subset of $\operatorname{Spec}(R)$, assuming R is a Cohen-Macaulay local ring that admits a canonical module. To achieve this, we first establish in Section 5.2 that the Cohen-Macaulay locus of a finitely generated module over a Cohen-Macaulay local ring with canonical module is open. The proof is divided into two parts: the first part focuses on a particular class of modules known as *equidimensional modules*, which will be introduced in the following section.

5.1 Equidimensionality

In this section, we introduce the class of *equidimensional modules* and establishes some properties that will be used in the subsequent sections.

Definition 5.1.1. *Let R be a Noetherian ring and M a finite R-module. One says that M is equidimensional if every minimal associated prime of M has the same dimension, that is*

$$\dim(M) = \dim(R/\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{MinSupp}(M)$. Moreover an ideal I of R is said equidimensional if and only if R/I is an equidimensional R-module.

In particular, a Noetherian ring R is equidimensional if and only if R is the zero ring or R is equidimensional as R-module with natural structure. For the sake of simplicity, it is often assumed that the zero module is equidimensional.

Example 5.1.2.

(i) Given any ideal I of R, the R-module R/I^{eq} is equidimensional;

(ii) Any Cohen-Macaulay module over a local ring is equidimensional.

Definition 5.1.3. *Let R be a ring. One says that R satisfies the dimension equality locally if R and $R_{\mathfrak{p}}$ satisfy the dimension equality for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Cohen-Macaulay local rings and local catenary domains are some examples of rings which satisfies the dimension equality locally. In the following proposition, we establish that the equidimensionality localizes when R satisfies the dimension equality locally.

Proposition 5.1.4. *Let R be ring which satisfies the dimension equality locally, M a finite R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$ a prime ideal of R. If M is an equidimensional R-module, then $M_\mathfrak{p}$ also is an equidimensional $R_\mathfrak{p}$-module.*

Proof: Suppose without loss of generality that $\mathfrak{p} \in \operatorname{Supp}(M)$. Recall from [Mat89, Theorem 6.2] that $\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Ass}(M)\ ;\ \mathfrak{q} \subseteq \mathfrak{p}\}$. Thus one has

$$\operatorname{MinAss}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{MinAss}(M)\ ;\ \mathfrak{q} \subseteq \mathfrak{p}\}.$$

By definition, one has

$$\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(R_\mathfrak{p}/(0 :_{R_\mathfrak{p}} M_\mathfrak{p})) = \dim(R_\mathfrak{p}/((0 :_R M)R_\mathfrak{p})) = \dim(R_\mathfrak{p}) - \operatorname{ht}((0 :_R M)R_\mathfrak{p})$$

Now it is enough to show that $\operatorname{ht}((0 :_R M)R_\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}R_\mathfrak{p})$ for all $\mathfrak{q}R_\mathfrak{p} \in \operatorname{MinAss}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Indeed, if it holds, then

$$\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(R_\mathfrak{p}) - \operatorname{ht}((0 :_R M)R_\mathfrak{p}) = \dim(R_\mathfrak{p}) - \operatorname{ht}(\mathfrak{q}R_\mathfrak{p}) = \dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}).$$

Let $\mathfrak{q} \in \operatorname{MinAss}(M)$. Since M is equidimensional, then

$$\operatorname{ht}(\mathfrak{q}) = \dim(R) - \dim(R/\mathfrak{q}) = \dim(R) - \dim(M),$$

which implies that

$$\begin{aligned}\operatorname{ht}((0 :_R M)R_\mathfrak{p}) &= \inf\{\operatorname{ht}(\mathfrak{q}R_\mathfrak{p})\ ;\ (0 :_R M)R_\mathfrak{p} \overset{min}{\subseteq} \mathfrak{q}R_\mathfrak{p}\} = \inf\{\operatorname{ht}(\mathfrak{q})\ ;\ \mathfrak{q} \in \operatorname{MinAss}(M),\ \mathfrak{q} \subseteq \mathfrak{p}\} \\ &= \dim(R) - \dim(M) = \dim(R) - \dim(R/\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}R_\mathfrak{p}).\end{aligned}$$

□

Proposition 5.1.5. *Let R be a Cohen-Macaulay local ring and M an equidimensional R-module. Let $\mathfrak{p} \in \operatorname{Supp}(M)$.*

(i) Given $\mathfrak{q}R_\mathfrak{p} \in \operatorname{MinAss}_{R_\mathfrak{p}}(M_\mathfrak{p})$, then $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) + \dim(M_\mathfrak{p})$;

(ii) Let $0 = M_1 \cap M_2 \cap \cdots \cap M_n$ be a primary decomposition of the zero submodule of M with $\mathfrak{p}_i = \sqrt{M_i :_R M}$ for all $1 \leq i \leq n$. If $\mathfrak{p}_i \in \operatorname{MinAss}(M)$ and $\dim(R/\mathfrak{p}_i) \neq \dim(R/\mathfrak{p}) + \dim(M_\mathfrak{p})$, then $(M_i)_\mathfrak{p} = M_\mathfrak{p}$.

Proof: *(i)* Let $\mathfrak{q}R_\mathfrak{p} \in \mathrm{MinAss}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{MinAss}(M) \ ; \ \mathfrak{q} \subseteq \mathfrak{p}\}$. Since M is equidimensional, one obtains from Proposition 5.1.4 that $M_\mathfrak{p}$ also satisfies this property. Hence

$$\dim(M_\mathfrak{p}) = \dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = \dim(R_\mathfrak{p}) - \mathrm{ht}(\mathfrak{q}R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) - \mathrm{ht}(\mathfrak{q})$$

for all $\mathfrak{q}R_\mathfrak{p} \in \mathrm{MinAss}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Thus

$$\dim(R/\mathfrak{q}) = \dim(R) - \mathrm{ht}(\mathfrak{q}) = \dim(R) - (\mathrm{ht}(\mathfrak{p}) - \dim(M_\mathfrak{p})) = \dim(R/\mathfrak{p}) + \dim(M_\mathfrak{p}).$$

(ii) Let $1 \leq i \leq n$ and suppose that $\dim(R/\mathfrak{p}_i) \neq \dim(R/\mathfrak{p}) + \dim(M_\mathfrak{p})$. Then $\mathfrak{p}_iR_\mathfrak{p} \notin \mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ by item (i). Since $\mathfrak{p}_i \in \mathrm{Ass}_R(M)$ by hypothesis, one has that $\mathfrak{p}_i$ is not contained in $\mathfrak{p}$ by [Mat89, Theorem 6.2]. As $\mathfrak{p}_i = \sqrt{M_i :_R M}$, then

$$R_\mathfrak{p} = (M_i :_R M)R_\mathfrak{p} = (M_i)_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p},$$

which implies that $(M_i)_\mathfrak{p} = M_\mathfrak{p}$. □

5.2 Openness of Cohen-Macaulay Loci

In this section, we examine the Cohen-Macaulay locus of an R-module. Although the openness of the Cohen-Macaulay locus in Cohen-Macaulay settings has already been proved by K. Kimura in [Kim23, Corollary 5.5], we will provide an alternative proof assuming that R is Cohen-Macaulay local with canonical module. Let's start by defining and making some observations about the Cohen-Macaulay locus.

Definition 5.2.1. *Let R be a Noetherian ring and M a finite R-module. The Cohen-Macaulay locus of M is defined as*

$$\mathrm{CM}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ M_\mathfrak{p} \textit{ is a Cohen-Macaulay } R_\mathfrak{p}\textit{-module}\}.$$

It follows directly from the definition of Cohen-Macaulayness that an R-module M is Cohen-Macaulay if and only if $\mathrm{CM}(M) = \mathrm{Spec}(R)$. Recall that there exists an inclusion-preserving correspondence between the prime ideals of $R_\mathfrak{p}$ and the prime ideals of R contained in $\mathfrak{p}$. This allows us to conclude that the the topology of $\mathrm{Spec}(R_\mathfrak{p})$ is induced by $\mathrm{Spec}(R)$. The next proposition shows that the Cohen-Macaulay locus of $M_\mathfrak{p}$ is induced by the Cohen-Macaulay locus of M.

Proposition 5.2.2. *Let R be a Noetherian ring and M a finite R-module. Given $\mathfrak{p} \in \operatorname{Spec}(R)$, then*

$$\operatorname{CM}(M_{\mathfrak{p}}) = \operatorname{CM}(M) \cap \operatorname{Spec}(R_{\mathfrak{p}}).$$

In particular, if $\operatorname{CM}(M)$ is open in $\operatorname{Spec}(R)$, then $\operatorname{CM}(M_{\mathfrak{p}})$ is open in $\operatorname{Spec}(R_{\mathfrak{p}})$.

Proof: Let $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{CM}(M_{\mathfrak{p}})$, then $M_{\mathfrak{q}} = (M_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}$ is Cohen-Macaulay. Hence $\mathfrak{q} \in \operatorname{CM}(M) \cap \operatorname{Spec}(R_{\mathfrak{p}})$. Conversely suppose that $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{CM}(M) \cap \operatorname{Spec}(R_{\mathfrak{p}})$, then $M_{\mathfrak{q}} = (M_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}$ is Cohen-Macaulay, which implies that $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{CM}(M_{\mathfrak{p}})$. $\square$

Our next objective is to determine necessary conditions to ensure the openness of the Cohen-Macaulay locus of a finite R-module. Kaito Kimura [Kim23, Corollary 5.5] gives a non-constructive proof of the openness of $\operatorname{CM}(M)$ in Cohen-Macaulay ring-setting. Providing a constructive proof, we will establish that $\operatorname{CM}(M)$ is an open subspace of $\operatorname{Spec}(R)$ whenever R is a Cohen-Macaulay local ring that admits canonical module. To achieve this, we will first prove the result in the specific case where M is equidimensional.

Lemma 5.2.3. *Let R be a Cohen-Macaulay local ring which admits canonical module ω_R and M a finite R-module. If M is equidimensional, then*

$$\operatorname{CM}(M) = \bigcap_{j > \operatorname{codim}(M)} \operatorname{Supp}\left(\operatorname{Ext}^j_R(M, \omega_R)\right)^c$$

is open in $\operatorname{Spec}(R)$.

Proof: We can suppose without lost of generality that $M \neq 0$. Set $d := \operatorname{codim}(M) = \dim(R) - \dim(R/\mathfrak{q})$, where $\mathfrak{q} \in \operatorname{MinAss}(M)$. We will prove

$$\operatorname{CM}(M) = \bigcap_{j > d} \operatorname{Supp}\left(\operatorname{Ext}^j_R(M, \omega_R)\right)^c =: \mathcal{A}.$$

Notice $\mathcal{A}$ actually is a finite intersection. Indeed, by Ischebeck's formula [BH93, Exercise 3.1.24], one has

$$\sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}^i_R(M, \omega_R) \neq 0\} = \operatorname{depth}(R) - \operatorname{depth}(M) =: t < \infty.$$

This fact implies that $\operatorname{Supp}\left(\operatorname{Ext}^i_R(M, \omega_R)_{\mathfrak{p}}\right) = \emptyset$ for all $i > t$ and so $\operatorname{Supp}\left(\operatorname{Ext}^i_R(M, \omega_R)_{\mathfrak{p}}\right)^c = \operatorname{Spec}(R)$ for all $i > t$. Thus, if one proves the desired equality, one concludes that $\operatorname{CM}(M)$ is open in $\operatorname{Spec}(R)$.

Let $\mathfrak{p} \in \operatorname{CM}(M)$. If $\mathfrak{p} \notin \operatorname{Supp}(M)$, then $\operatorname{Ext}_R^j(M, \omega_R)_{\mathfrak{p}} = 0$ for all $j > d$, which implies that $\mathfrak{p} \notin \operatorname{Supp}\left(\operatorname{Ext}_R^j(M, \omega_R)\right)$ for all $j > d$ and so $\mathfrak{p} \in \mathcal{A}$. Now suppose that $\mathfrak{p} \in \operatorname{Supp}(M)$. By Ischebeck's formula, one has

$$\begin{aligned}\sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}_R^i(M, \omega_R)_{\mathfrak{p}} \neq 0\} = \sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}}) \neq 0\} &= \operatorname{depth}(R_{\mathfrak{p}}) - \operatorname{depth}(M_{\mathfrak{p}}) \\ &= \operatorname{ht}(\mathfrak{p}) - \dim(M_{\mathfrak{p}}).\end{aligned}$$

Let $\mathfrak{q} \in \operatorname{MinAss}(M)$ contained in $\mathfrak{p}$. By Proposition 5.1.4, $M_{\mathfrak{p}}$ is also an equidimensional module. Hence, as R is Cohen-Macaulay and $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{MinAss}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, one has

$$\dim(M_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}) - \operatorname{ht}(\mathfrak{q}R_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p}) - \operatorname{ht}(\mathfrak{q}).$$

Hence, since R satisfies the dimension equality, one has

$$\sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}_R^i(M, \omega_R)_{\mathfrak{p}} \neq 0\} = \operatorname{ht}(\mathfrak{p}) - (\operatorname{ht}(\mathfrak{p}) - \operatorname{ht}(\mathfrak{q})) = \operatorname{ht}(\mathfrak{q}) = \dim(R) - \dim(R/\mathfrak{q}) = d,$$

which implies that $\mathfrak{p} \in \operatorname{Supp}\left(\operatorname{Ext}_R^j(M, \omega_R)\right)^c$ for all $i > d$ and so $\mathfrak{p} \in \mathcal{A}$ and $\operatorname{CM}(M) \subseteq \mathcal{A}$.

Conversely let $\mathfrak{p} \in \mathcal{A}$ be a prime ideal chosen. One can suppose without loss of generality that $\mathfrak{p} \in \operatorname{Supp}(M)$. Since $\mathfrak{p} \in \mathcal{A}$, then $\operatorname{Ext}_R^i(M, \omega_R)_{\mathfrak{p}} = 0$ for all $i > d$. Invoking the Ischebeck's formula again, one obtains

$$\operatorname{ht}(\mathfrak{p}) - \operatorname{depth}(M_{\mathfrak{p}}) = \sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}_R^i(M, \omega_R)_{\mathfrak{p}} \neq 0\} \leq \sup\{i \in \mathbb{N} \ ; \ \operatorname{Ext}_R^i(M, \omega_R) \neq 0\} = d. \tag{5.1}$$

Consider $\mathfrak{q} \in \operatorname{MinAss}(M)$ with $\mathfrak{q} \subseteq \mathfrak{p}$, this choice is possible because $\mathfrak{p} \in \operatorname{Supp}(M)$. By inequality (5.1) and Proposition 5.1.5 (i), one has

$$\begin{aligned}\operatorname{depth}(M_{\mathfrak{p}}) \geq \operatorname{ht}(\mathfrak{p}) - d = \operatorname{ht}(\mathfrak{p}) - (\dim(R) - \dim(R/\mathfrak{q})) = \operatorname{ht}(\mathfrak{p}) - \operatorname{ht}(\mathfrak{q}) \\ = (\dim(R) - \dim(R/\mathfrak{p})) - (\dim(R) - \dim(R/\mathfrak{q})) = \dim(R/\mathfrak{q}) - \dim(R/\mathfrak{p}) = \dim(M_{\mathfrak{p}}).\end{aligned}$$

Therefore $M_{\mathfrak{p}}$ is Cohen-Macaulay $R_{\mathfrak{p}}$-module. □

We are now ready to establish the central result of this section. The fundamental idea of its proof involves constructing a finite partition of $\operatorname{CM}(M)$ in such a way that each element of this partition constitutes an open subset of $\operatorname{Spec}(R)$.

Theorem 5.2.4. *Let R be a Cohen-Macaulay local ring and M a finite R-module. If R admits canonical module, then* $\mathrm{CM}(M)$ *is open in* $\mathrm{Spec}(R)$*.*

Proof: Let $0 = P_1 \cap P_2 \cap \cdots \cap P_n$ be a primary decomposition of the zero submodule of M, with $\mathfrak{p}_i = \sqrt{P_i :_R M}$ for all $1 \leq i \leq n$. Denote $\mathfrak{M} = \{\dim(R/\mathfrak{p}_i) \ ; \ 1 \leq i \leq n\} = \{d_1 < \cdots < d_m\}$ and, for each $1 \leq j \leq m$, consider the R-module

$$N_j = \bigcap_{\dim(R/\mathfrak{p}_i)=d_j} P_i.$$

Observe $N_j = \bigcap_{\dim(R/\mathfrak{p}_i)=d_j} P_i$ is indeed a primary decomposition of N_j. By uniqueness of primary decomposition, one has that

$$\mathrm{Ass}(N_j) = \{\mathfrak{p}_i \ ; \ 1 \leq i \leq n, \dim(R/\mathfrak{p}_i) = d_j\},$$

whence N_j is an equidimensional R-module. Now define

$$U_j = \bigcap_{i \neq j} \mathrm{Supp}(M/N_i)^c$$

for each $1 \leq j \leq m$. Observe that U_j is open in $\mathrm{Spec}(R)$ for all $1 \leq j \leq m$. Let $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$. By Proposition 5.1.5, there is $1 \leq i_0 \leq m$ such that such that $(N_j)_\mathfrak{p} = M_\mathfrak{p}$ for all $j \neq i_0$. This fact implies that

$$\mathfrak{p} \in \bigcap_{i \neq i_0} \mathrm{Supp}(M/N_i)^c = U_{i_0}.$$

Furthermore, notice that

$$M_\mathfrak{p} = \frac{M_\mathfrak{p}}{(N_1)_\mathfrak{p} \cap \cdots \cap (N_m)_\mathfrak{p}} = \frac{M_\mathfrak{p}}{(N_{i_0})_\mathfrak{p}} = \left(\frac{M}{N_{i_0}}\right)_\mathfrak{p}.$$

Thus one gets that $\mathfrak{p} \in \mathrm{CM}(M/N_{i_0})$ and so

$$\mathfrak{p} \in U_{i_0} \cap \mathrm{CM}(M/N_{i_0}) \subseteq \bigcup_{i=1}^{m} \big(U_i \cap \mathrm{CM}(M/N_i)\big) := \mathcal{U}.$$

Observe that one still has $\mathfrak{p} \in \mathcal{U}$ if $\mathfrak{p} \notin \mathrm{Supp}(M)$. On the other hand let $\mathfrak{p} \in \mathcal{U}$, then $\mathfrak{p} \in U_t \cap \mathrm{CM}(M/N_t)$ for some $1 \leq t \leq m$. Since $\mathfrak{p} \in U_t$, one gets $M_\mathfrak{p} = (N_i)_\mathfrak{p}$ for all $i \neq t$, which implies that

$$M_\mathfrak{p} = \frac{M_\mathfrak{p}}{(N_1)_\mathfrak{p} \cap \cdots \cap (N_m)_\mathfrak{p}} = \frac{M_\mathfrak{p}}{(N_t)_\mathfrak{p}} = \left(\frac{M}{N_t}\right)_\mathfrak{p}.$$

As $\mathfrak{p} \in \mathrm{CM}(M/N_t)$, one gets that $\mathfrak{p} \in \mathrm{CM}(M)$ and so

$$\mathrm{CM}(M) = \bigcup_{i=1}^{m} \big(U_i \cap \mathrm{CM}(M/N_i)\big).$$

Finally, since $\mathrm{CM}(M/N_i)$ is open in $\mathrm{Spec}(R)$ by Lemma 5.2.3 for all $1 \leq i \leq m$, one concludes that $\mathrm{CM}(M)$ is open in $\mathrm{Spec}(R)$. □

5.3 Strongly Cohen-Macaulay Locus and its Openness

Let R be a Noetherian local ring, I an ideal of R and M an R-module. Consider $\mathbf{f} = f_1, \ldots, f_r$ a system of generators of I. Huneke [Hun83, Remark 1.2] proved that the property $H_i(\mathbf{f}; M)$ is Cohen-Macaulay for all $0 \leq i \leq n \leq r$ is independent from the choice of system of generators $\mathbf{f}$. In particular, the property $H_i(\mathbf{f}; M)$ is Cohen-Macaulay for all $i \geq 0$ is independent from the choice of the system of generators of I. This fact led to the following definition.

Definition 5.3.1. *Let R be a Noetherian local ring. An ideal I is said Strongly Cohen-Macaulay if, given a system of generators $\mathbf{f} = f_1, \ldots, f_r$ of I, the Koszul homologies $H_i(\mathbf{f}; R)$ are Cohen-Macaulay R-modules for all $i \geq 0$.*

Here are some examples of Strongly Cohen-Macaulay ideals.

Example 5.3.2.

(i) Any complete intersection ideal I of a Cohen-Macaulay local ring R is Strongly Cohen-Macaulay. Indeed, let $g = \mathrm{grade}(I, R)$ and denote $I = (f_1, \ldots, f_g)$. Since $\mathrm{grade}(I, R) = g$, by Theorem 1.1.3, one has

$$H_i(\mathbf{f}; R) = \begin{cases} 0, & \text{if } i > 0; \\ R/I, & \text{if } i = 0. \end{cases}$$

Thus, as Cohen-Macaulayness is stable under specialization via regular sequences, one concludes that I is Strongly Cohen-Macaulay.

(ii) Any $\mathfrak{m}$-primary ideal I of a Noetherian local ring $(R, \mathfrak{m})$ is Strongly Cohen-Macaulay. In fact, consider $\mathbf{f} = f_1, \ldots, f_n$ a system of generators of I. Notice that $H_0(\mathbf{f}; R)$ is Cohen-Macaulay, because $\dim(H_0(\mathbf{f}; R)) = 0$ since $\mathrm{Supp}(H_0(\mathbf{f}; R)) = \{\mathfrak{m}\}$. Moreover, by Proposition 1.1.5, one has $\mathrm{Supp}(H_i(\mathbf{f}; R)) \subseteq \{\mathfrak{m}\}$. Hence I is a Strongly Cohen-Macaulay ideal.

Now we are going to define the strongly Cohen-Macaulay locus of an ideal I.

Definition 5.3.3. *Let R be a Noetherian ring and I an ideal of R. The strongly Cohen-Macaulay locus of I, denoted by* $\mathrm{SCM}(I)$*, is the family of all primes ideals $\mathfrak{p}$ of R such that $I_{\mathfrak{p}}$ is Strongly Cohen-Macaulay ideal of $R_{\mathfrak{p}}$.*

$$\mathrm{SCM}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ I_{\mathfrak{p}} \textit{ is strongly Cohen-Macaulay}\}.$$

Considering the locus perspective, we observe that the Strongly Cohen-Macaulayness is a local property as one can see in the following proposition.

Proposition 5.3.4. *Let R be a Noetherian local ring and I an ideal of R. Then I is strongly Cohen-Macaulay if and only if* $\mathrm{SCM}(I) = \mathrm{Spec}(R)$*.*

Proof: Let $\mathbf{f} = f_1, \ldots, f_r$ be a system of generators of I and suppose that $\mathrm{SCM}(I) = \mathrm{Spec}(R)$. Consider $\mathfrak{p} \in \mathrm{Spec}(R)$. As $I_{\mathfrak{p}}$ is Strongly Cohen-Macaulay and $\mathbf{f}_{\mathfrak{p}}$ is a system of generators of $I_{\mathfrak{p}}$, one has that

$$H_i(\mathbf{f}; R)_{\mathfrak{p}} = H_i(\mathbf{f}_{\mathfrak{p}}; R_{\mathfrak{p}})$$

is Cohen-Macaulay for all $i \geq 0$. Hence $H_i(\mathbf{f}; R)$ is Cohen-Macaulay for all $i \geq 0$ and so I is a strongly Cohen-Macaulay ideal. Conversely suppose that I is a strongly Cohen-Macaulay ideal and let $\mathfrak{p} \in \mathrm{Spec}(R)$. For all $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Spec}(R_{\mathfrak{p}})$, one has that

$$H_i\big(((\mathbf{f})_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}; (R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}\big) = H_i(\mathbf{f}_{\mathfrak{p}}; R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}.$$

Since $H_i(\mathbf{f}_{\mathfrak{p}}; R_{\mathfrak{p}})$ is Cohen-Macaulay and Cohen-Macaulayness is stable under localization, one concludes that $H_i\big(((\mathbf{f})_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}; (R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}\big)$ is Cohen-Macaulay for all $i \geq 0$. As $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Spec}(R_{\mathfrak{p}})$ is arbitrary, one concludes that $I_{\mathfrak{p}}$ is strongly Cohen-Macaulay. Since $\mathfrak{p} \in \mathrm{Spec}(R)$ is arbitrary, one concludes that $\mathrm{SCM}(I) = \mathrm{Spec}(R)$. $\square$

The next proposition reveals that the topology of the Strongly Cohen-Macaulay locus of $I_{\mathfrak{p}}$ is induced by the topology of Strongly Cohen-Macaulay locus of I.

Proposition 5.3.5. *Let R be a Noetherian ring and I an ideal of R. Given $\mathfrak{p} \in \mathrm{V}(I)$, then*

$$\mathrm{SCM}(I_{\mathfrak{p}}) = \mathrm{Spec}(R_{\mathfrak{p}}) \cap \mathrm{SCM}(I).$$

Proof: Let $\mathbf{f} = f_1, \ldots, f_r$ be a system of generators of I and $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{SCM}(I_{\mathfrak{p}})$. Thus

$$H_i(\mathbf{f}_{\mathfrak{q}}; R_{\mathfrak{q}}) = H_i(\mathbf{f}; R)_{\mathfrak{q}} = H_i\big(((\mathbf{f})_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}; (R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}\big)$$

is Cohen-Macaulay for all $i \geq 0$, which implies that $I_{\mathfrak{q}}$ is a strongly Cohen-Macaulay ideal. Hence $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Spec}(R_{\mathfrak{p}}) \cap \mathrm{SCM}(I)$. Conversely suppose that $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Spec}(R_{\mathfrak{p}}) \cap \mathrm{SCM}(I)$. Note that $\mathfrak{q} \subseteq \mathfrak{p}$. Thus

$$H_i\big(((\mathbf{f})_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}; (R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}\big) = H_i(\mathbf{f}_{\mathfrak{q}}; R_{\mathfrak{q}}) = H_i(\mathbf{f}; R)_{\mathfrak{q}}$$

is Cohen-Macaulay for all $i \geq 0$, which implies that $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{SCM}(I_{\mathfrak{p}})$. $\square$

In particular, if $\mathrm{SCM}(I)$ is open in $\mathrm{Spec}(R)$, then $\mathrm{SCM}(I_{\mathfrak{p}})$ also is open in $\mathrm{Spec}(R_{\mathfrak{p}})$. We now establish the main result of this section. Specifically, we aim to prove that in the case when R is a Cohen-Macaulay local ring which admits canonical module, the strongly Cohen-Macaulay locus of any ideal in R is open in the spectrum of R.

Theorem 5.3.6. *Let R be a Cohen-Macaulay local ring which admits canonical module and I an ideal of R. Then* $\mathrm{SCM}(I)$ *is a nonempty open subset of* $\mathrm{Spec}(R)$*.*

Proof: Let $\mathbf{f} = f_1, \ldots, f_r$ be a system of generators of I. Notice that

$$\begin{aligned} \mathrm{SCM}(I) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ I_{\mathfrak{p}} \text{ is strongly Cohen-Macaulay}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ H_i(\mathbf{f}_{\mathfrak{p}}; R_{\mathfrak{p}}) \text{ is Cohen-Macaulay } R_{\mathfrak{p}}\text{-module for all } 0 \leq i \leq r\} \\ &= \bigcap_{i=0}^{r} \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ H_i(\mathbf{f}; R)_{\mathfrak{p}} \text{ is Cohen-Macaulay } R_{\mathfrak{p}}\text{-module}\} = \bigcap_{i=0}^{r} \mathrm{CM}(H_i(\mathbf{f}; R)). \end{aligned}$$

Since $\mathrm{CM}(H_i(\mathbf{f}, R))$ is open in $\mathrm{Spec}(R)$ by Theorem 5.2.4 for all $0 \leq i \leq r$, the openness of $\mathrm{SCM}(I)$ follows. Observe that $I = R$ always is Strongly Cohen-Macaulay, because $H_i(1; R) = 0$ and so Cohen-Macaulay for all $i \geq 0$, which implies that $\mathrm{SCM}(R) \neq \emptyset$. Suppose then that I is a proper ideal of R. Given $\mathfrak{p}$ a prime ideal containing minimally I, one has that $I_{\mathfrak{p}}$ is $\mathfrak{p}R_{\mathfrak{p}}$-primary ideal. Hence $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{SCM}(I_{\mathfrak{p}}) = \mathrm{SCM}(I) \cap \mathrm{Spec}(R_{\mathfrak{p}})$ by Example 5.3.2 (ii), which implies that $\mathrm{SCM}(I)$ is a nonempty open subset of $\mathrm{Spec}(R)$. $\square$

Remark 5.3.7. *When all Koszul homologies of I are equidimensional modules, it is possible to provide an explicit characterization of* $\mathrm{SCM}(I)$*. Indeed, suppose I an ideal of R generated by*

$\mathbf{f} = f_1, \ldots, f_r$ *and denote* $d_i = \operatorname{codim}(H_i(\mathbf{f}; R))$ *for all* $i = 0, \ldots, r$*. By Lemma 5.2.3, one concludes that*

$$\begin{aligned}\operatorname{SCM}(I) = \bigcap_{i=0}^{r} \operatorname{CM}(H_i(\mathbf{f}; R)) &= \bigcap_{i=0}^{r} \left(\bigcap_{j>d_i} \operatorname{Supp}\left(\operatorname{Ext}_R^j(H_i(\mathbf{f}; R), \omega_R)\right)^c \right) \\ &= \bigcap_{i=0}^{r} \bigcap_{j>d_i} \operatorname{V}\left(0 :_R \operatorname{Ext}_R^j(H_i(\mathbf{f}; R), \omega_R)\right)^c.\end{aligned}$$

Let R be a ring and $\mathfrak{a} \subseteq I$ finitely generated ideals of R. The *obstruction of* I *with respect to* $\mathfrak{a}$ is defined as the R-module

$$H(\mathfrak{a}, I) := \frac{\mathfrak{a} :_R I}{\operatorname{Kitt}(\mathfrak{a}, I)}.$$

Observe that

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \ ; \ (\mathfrak{a} :_R I)_{\mathfrak{p}} = \operatorname{Kitt}(\mathfrak{a}, I)_{\mathfrak{p}}\} = \operatorname{Supp}(H(\mathfrak{a}, I))^c.$$

is open in $\operatorname{Spec}(R)$. It follows immediately from Theorem 5.3.6 the following proposition.

Proposition 5.3.8. *Let* R *be a Cohen-Macaulay local ring which admits canonical module and* $\mathfrak{a} \subset I$ *ideals of* R*. Suppose that* $J = \mathfrak{a} :_R I$ *is an* s*-residual intersection of* I*. Then*

$$\operatorname{SCM}(I) \subseteq \operatorname{Supp}(H(\mathfrak{a}, I))^c$$

Proof: Consider a prime ideal $\mathfrak{p} \in \operatorname{SCM}(I)$. If $\mathfrak{p} \notin \operatorname{V}(I)$, then

$$H(\mathfrak{a}, I)_{\mathfrak{p}} = \left(\frac{\mathfrak{a} :_R I}{\operatorname{Kitt}(\mathfrak{a}, I)} \right)_{\mathfrak{p}} = \frac{\mathfrak{a}_{\mathfrak{p}} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}}}{\operatorname{Kitt}(\mathfrak{a}_{\mathfrak{p}}, I_{\mathfrak{p}})} = \frac{\mathfrak{a}_{\mathfrak{p}}}{\mathfrak{a}_{\mathfrak{p}}} = 0,$$

which implies that $\mathfrak{p} \in \operatorname{Supp}(H(\mathfrak{a}, I))^c$. Suppose now that $I \subseteq \mathfrak{p}$. Since the property of being an s-residual intersection is preserved under localization, Theorem 2.1.6(xii) tells us that

$$\operatorname{Kitt}(\mathfrak{a}, I)_{\mathfrak{p}} = \operatorname{Kitt}(\mathfrak{a}_{\mathfrak{p}}, I_{\mathfrak{p}}) = \mathfrak{a}_{\mathfrak{p}} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}} = (\mathfrak{a} :_R I)_{\mathfrak{p}}.$$

□

Let R be a Cohen-Macaulay local ring which admits canonical module and I be an ideal of R. Theorem 5.3.6 says us that $\operatorname{SCM}(I)$ is an open subset of $\operatorname{Spec}(R)$. Thus there is an ideal $\mathfrak{s}$ of R such that

$$\operatorname{SCM}(I) = \operatorname{Spec}(R) \setminus \operatorname{V}(\mathfrak{s}).$$

Hence the non-Strongly Cohen-Macaulay locus of I, denoted by $\operatorname{nonSCM}(I)$, is the variety $\operatorname{V}(\mathfrak{s})$. The following proposition shows that $\operatorname{nonSCM}(I)$ is indeed a proper subvariety of $\operatorname{V}(I)$.

Proposition 5.3.9. *Let R be a Cohen-Macaulay local ring which admits canonical module and I an ideal of R, then*

$$\mathrm{nonSCM}(I) \subseteq \mathrm{V}(I) \setminus \mathrm{MinV}(I).$$

In particular, one has $\dim(\mathrm{nonSCM}(I)) < \dim(I)$.

Proof: Firstly observe that $\mathrm{nonSCM}_R(I) \subseteq \mathrm{V}(I)$. Indeed, if $I \not\subset \mathfrak{p}$, then $I_{\mathfrak{p}} = R_{\mathfrak{p}}$, which implies that all Koszul homologies of a system of generators of I are 0 and so Cohen-Macaulay, implying that $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ is Strongly Cohen-Macaulay ideal. On the other hand, given $\mathfrak{p} \in \mathrm{MinV}(I)$, one has that $I_{\mathfrak{p}}$ is a $\mathfrak{p}R_{\mathfrak{p}}$-primary ideal. Thus, by Example 5.3.2, one sees that $I_{\mathfrak{p}}$ is Strongly Cohen-Macaulay, which implies that $\mathrm{nonSCM}(I) \subseteq \mathrm{V}(I) \setminus \mathrm{MinV}(I)$. Denote $\mathrm{nonSCM}(I) = \mathrm{V}(\mathfrak{s})$ and let $\mathfrak{p} \in \mathrm{V}(\mathfrak{s})$ such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{s})$. Since $\mathrm{nonSCM}(I) \subseteq \mathrm{V}(I) \setminus \mathrm{MinV}(I)$, then $\mathfrak{p} \in \mathrm{V}(I)$ and it does not contain I minimally, which implies that $\mathrm{ht}(I) < \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{s})$. Hence

$$\dim(\mathrm{nonSCM}(I)) = \dim\left(\frac{R}{\mathfrak{s}}\right) = \dim(R) - \mathrm{ht}(\mathfrak{s}) < \dim(R) - \mathrm{ht}(I) = \dim\left(\frac{R}{I}\right) = \dim(I).$$

□

Definition 5.3.10. *Let R be a Noetherian ring. The regular locus of R, denoted by* $\mathrm{Reg}(R)$*, is the family of all prime ideals $\mathfrak{p}$ of R for which $R_{\mathfrak{p}}$ is a regular local ring. More explicitly*

$$\mathrm{Reg}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \ ; \ R_{\mathfrak{p}} \textit{ is a regular local ring}\} =: \mathrm{Sing}(R)^c.$$

The following result proves that the regular locus of ideals of complete intersection rings resides within its strongly Cohen-Macaulay locus. Indeed, we prove a stronger statement.

Proposition 5.3.11. *Let S be a regular local ring and $\mathfrak{a}$ a complete intersection ideal of S. Consider $R = S/\mathfrak{a}$ and let I be an ideal of R, then*

$$\mathrm{Reg}(R/I) \subseteq \mathrm{SCM}(I).$$

Proof: Let $\mathbf{a} = a_1, \ldots, a_g$ and $\overline{\mathbf{f}} = \overline{f}_1, \ldots, \overline{f}_t$ be systems of generators of $\mathfrak{a}$ and I, respectively. Let $\mathfrak{Q} = \mathfrak{q}/I$ be a prime ideal of R/I such that $\mathfrak{Q} \in \mathrm{Reg}(R/I)$. Observe that $\mathfrak{q}$ is a prime ideal of R, which implies that $\mathfrak{q} = \mathfrak{p}/\mathfrak{a}$ for some prime ideal $\mathfrak{p}$ of S containing $\mathfrak{a}$. Since $\mathfrak{Q} \in \mathrm{Reg}(R/I)$, one has that

$$\left(\frac{R}{I}\right)_{\mathfrak{Q}} = \frac{R_{\mathfrak{q}}}{I_{\mathfrak{q}}}$$

is a regular local ring and so $I_{\mathfrak{q}}$ is a prime ideal of $R_{\mathfrak{q}}$, because $R_{\mathfrak{q}}/I_{\mathfrak{q}}$ is an integral domain [BH93, Proposition 2.2.3], thus there exists a prime ideal $\mathfrak{p}'$ of S with $\mathfrak{a} \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$ such that $I_{\mathfrak{q}} = \mathfrak{p}'S_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$. Hence

$$\frac{R_{\mathfrak{q}}}{I_{\mathfrak{q}}} = \frac{S_{\mathfrak{p}}}{\mathfrak{p}'S_{\mathfrak{p}}}.$$

Since $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}}/\mathfrak{p}'S_{\mathfrak{p}}$ are regular local rings, [BH93, Proposition 2.2.4] tells us that $\mathfrak{p}'S_{\mathfrak{p}}$ is generated by a part of a regular system of parameters $\mathbf{g}_{\mathfrak{p}} = g_{1\mathfrak{p}}, \ldots, g_{u\mathfrak{p}}$ of $S_{\mathfrak{p}}$. As $S_{\mathfrak{p}}$ is Cohen-Macaulay local, one has that $\mathfrak{p}'S_{\mathfrak{p}}$ is a complete intersection ideal of $S_{\mathfrak{p}}$ and so strongly Cohen-Macaulay. Observe that $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{p}'S_{\mathfrak{p}} = \mathfrak{p}'S_{\mathfrak{p}}$. Consequently, due to the property the Cohen-Macaulayness of homologies of the Koszul complex is independent of the generators of the ideal, incorporating the generators of $\mathfrak{a}_{\mathfrak{p}}$ into $\mathfrak{p}'S_{\mathfrak{p}}$, the resulting Koszul complex formed from the generators of both $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{p}'S_{\mathfrak{p}}$ still exhibits Cohen-Macaulay homologies. Now, applying [Hun83, Corollary 1.5], one then passes modulo $\mathfrak{a}_{\mathfrak{p}}$, and concludes that the $H_i(\overline{\mathbf{f}}_{\mathfrak{q}}; R_{\mathfrak{q}}) = H_i(\overline{\mathbf{g}}_{\mathfrak{p}}; S_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}})$ is Cohen-Macaulay for all $i \geq 0$, which implies that $\mathfrak{q} \in \mathrm{SCM}(I)$. $\square$

The following example created on *Macaulay2* shows that Proposition 5.3.11 is not valid even if the defining ideal $\mathfrak{a}$ of S is a Gorenstein strongly Cohen-Macaulay.

Example 5.3.12. Let $S = (\mathbb{Z}/3\mathbb{Z})[x_0, x_1, x_2, x_3]$ and M be the following skew-symmetric matrix

$$\begin{pmatrix} 0 & -x_1 & x_0 + x_1 & x_0 - x_1 + x_2 - x_3 & -x_0 + x_1 + x_2 - x_3 \\ x_1 & 0 & -x_2 - x_3 & x_0 + x_2 - x_3 & -x_0 + x_1 + x_2 \\ -x_0 - x_1 & x_2 + x_3 & 0 & x_2 & x_0 \\ -x_0 + x_1 - x_2 + x_3 & -x_0 - x_2 + x_3 & -x_2 & 0 & x_0 - x_1 + x_2 \\ x_0 - x_1 - x_2 + x_3 & x_0 - x_1 - x_2 & -x_0 & -x_0 + x_1 - x_2 & 0 \end{pmatrix}.$$

Consider the ideal $\mathfrak{a} = \mathrm{pfaff}(4, M)$ which minimal free resolution is

$$0 \longrightarrow S \longrightarrow S^5 \longrightarrow S^5 \longrightarrow S \longrightarrow S/\mathfrak{a} \longrightarrow 0.$$

Notice that $\mathfrak{a}$ is a perfect ideal, because

$$\mathrm{projdim}\left(\frac{S}{\mathfrak{a}}\right) = 3 = \mathrm{ht}(\mathfrak{a}) = \mathrm{grade}(\mathfrak{a}).$$

Hence $\mathfrak{a}$ is a Cohen-Macaulay ideal. Moreover, since $\tau(R/\mathfrak{a}) = 1$, one concludes that $\mathfrak{a}$ is a Gorenstein ideal. Calculating its Koszul complex, one checks that all its homologies are Cohen-Macaulay, which implies that $\mathfrak{a}$ is Strongly Cohen-Macaulay. As $\mu(\mathfrak{a}) = 5 > \operatorname{grade}(\mathfrak{a})$, one concludes that $\mathfrak{a}$ is a non-complete intersection, Gorenstein and Strongly Cohen-Macaulay ideal. Let $R = S/\mathfrak{a}$ and consider the ideal $I = (x_0+x_2, x_1+x_2, x_3)/\mathfrak{a} \subset R$. Clearly, the maximal ideal belongs to $\operatorname{Reg}(R/I)$. However, I is not a Strongly Cohen-Macaulay ideal in the ring R, since its first Koszul homology module is not Cohen-Macaulay. Therefore the maximal ideal does not belong to $\operatorname{SCM}(I)$.

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