Courbes de Bézier

Base de Bernstein de degré n

Fonctions de base des polynomes de degré n :

$$B_i^n(t) = C_n^i t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i} \qquad \forall t \in [0,1] \text{ et } 0 \leq i \leq n$$

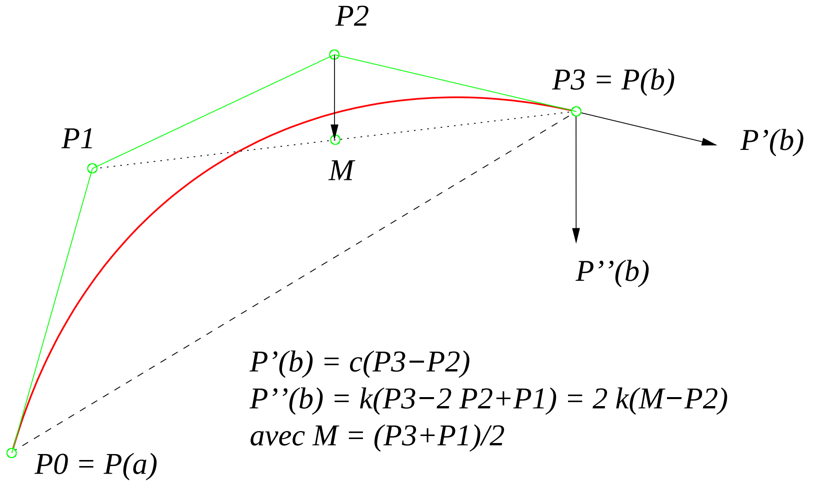
Courbe de Bézier

Courbe $\mathcal{C}$ paramétrée polynomiale de degré n : $\mathcal{C} = \left\{ P(t) = \sum_{i=0}^{n} P_i B_i^n(t), t \in [0,1] \right\}$

$\{P_0, P_1, \ldots, P_n\}$ polygone de contrôle de la courbe $\mathcal{C}$

$$\mathcal{C} = \left\{ P(u) = \sum_{i=0}^{n} P_i B_i^n \left(\frac{u-a}{b-a} \right), u \in [a,b] \right\} : t = \frac{u-a}{b-a} \in [0,1]$$

Quelques propriétés géométriques

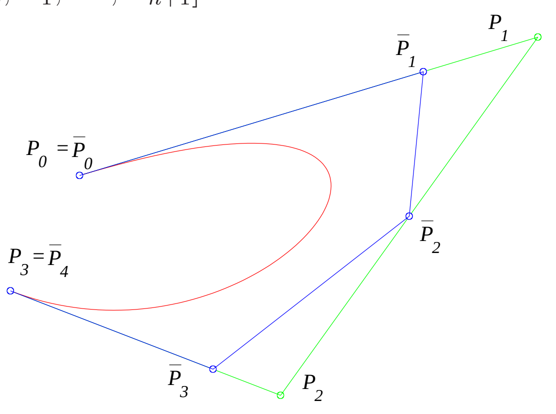


Elévation de degré

$$\overline{P}_i = \frac{i}{n+1} P_{i-1} + \frac{n+1-i}{n+1} P_i \text{ pour } 0 \leq i \leq n+1$$

Courbe $[P_0, P_1, \ldots, P_n]$ = Courbe $[\overline{P}_0, \overline{P}_1, \ldots, \overline{P}_{n+1}]$

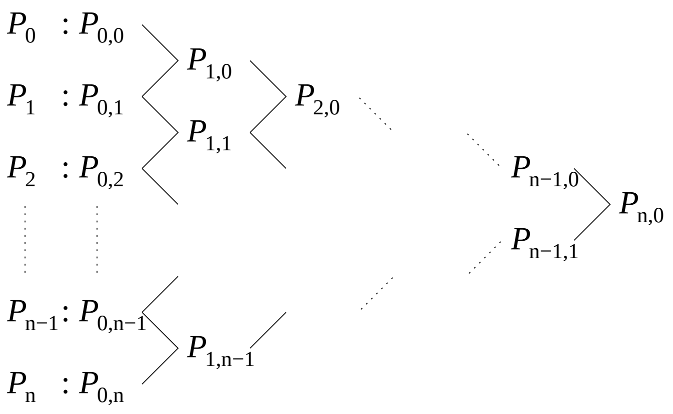
$$\begin{aligned}
\overline{P}_0 &= (\quad 4P_0 \quad)/4 = P_0 \\
\overline{P}_1 &= (1P_0 + 3P_1)/4 \\
\overline{P}_2 &= (2P_1 + 2P_2)/4 \\
\overline{P}_3 &= (3P_2 + 1P_3)/4 \\
\overline{P}_4 &= (4P_3 \quad)/4 = P_3
\end{aligned}$$

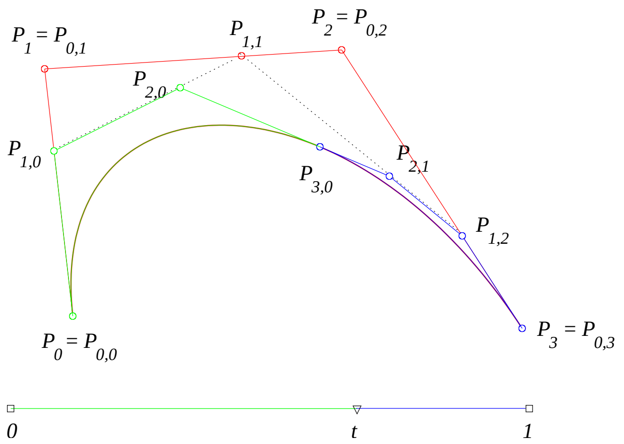


Algorithme de De Casteljau

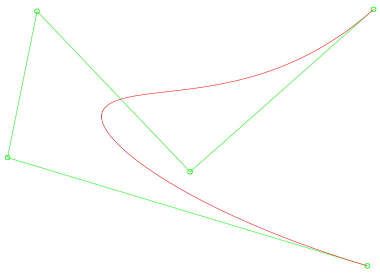
Données : Polygone de contrôle $\{P_0, P_1, \ldots, P_n\}$ et $t \in]0,1[$

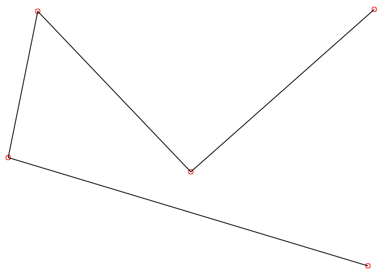
(1) $P_{0,i} = P_i$ pour $0 \leq i \leq n$
(2) $P_{j,i} = (1-t)\ P_{j-1,i} + t\ P_{j-1,i+1}$ pour $0 \leq i \leq n-j$ et $1 \leq j \leq n$

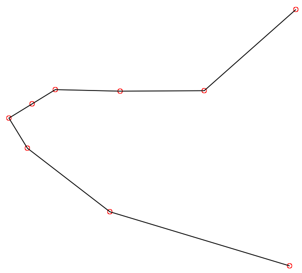


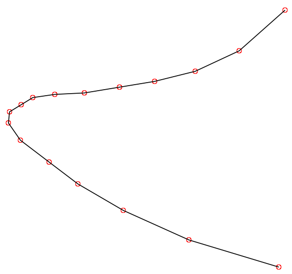


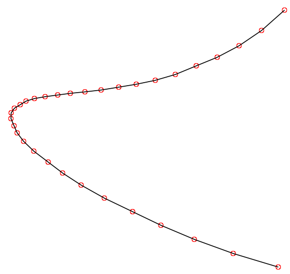
(a) $P_{n,0} = P(t)$
(b1) $\{P_{0,0}, P_{1,0}, \ldots, P_{n,0}\} = \{P_{i,0}\}_{0 \leq i \leq n}$:
polygone de contrôle de la partie de la courbe $\mathcal{C}$ correspondant à $s \in [0, t]$
(b2) $\{P_{0,n}, P_{1,n-1}, \ldots, P_{n,0}\} = \{P_{i,n-i}\}_{0 \leq i \leq n}$:
polygone de contrôle de la partie de la courbe $\mathcal{C}$ correspondant à $s \in [t, 1]$
(c) Subdivisions successives : convergence des polygones de contrôle vers la courbe $\mathcal{C}$

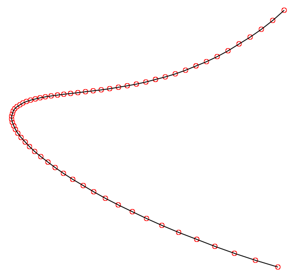

Polygone initial et courbe


Subdivision 0


Subdivision 1

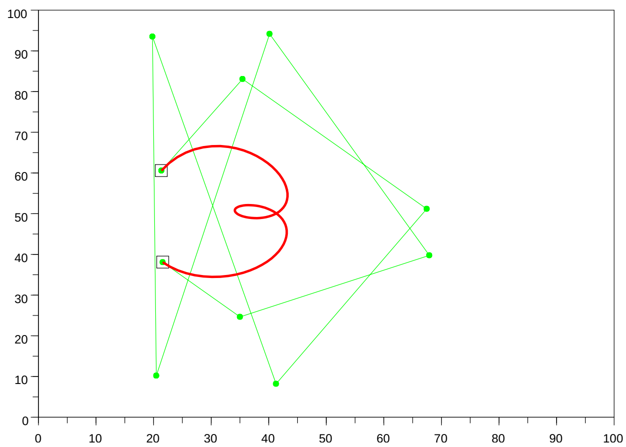

Subdivision 2


Subdivision 3

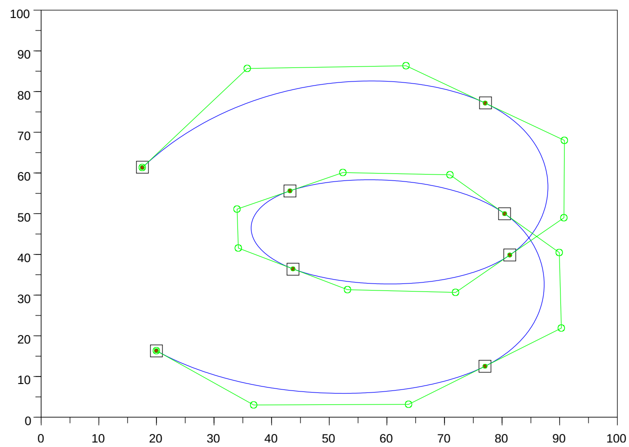

Subdivision 4

Courbes composites

Modélisation de formes complexes



Une seule courbe de Bézier (degré 9)



Plusieurs courbes de Bézier (degré 3)

Raccordement de courbes de Bézier

$$
\begin{array}{lll}
[P_0, P_1, \ldots, P_n] & \to & \mathcal{C}_1 = \left\{P(t) = \displaystyle\sum_{i=0}^{n} P_i B_i^n\left(\frac{t-a}{b-a}\right), t \in [a,b]\right\} \\
[Q_0, Q_1, \ldots, Q_n] & \to & \mathcal{C}_2 = \left\{Q(t) = \displaystyle\sum_{i=0}^{n} Q_i B_i^n\left(\frac{t-b}{c-b}\right), t \in [b,c]\right\}
\end{array}
\quad a < b < c :
$$

$$
r_1 = b-a,\ r_2 = c-b,\ \lambda = \frac{r_1}{r_1+r_2} = \frac{b-a}{c-a},\ 1-\lambda = \frac{r_2}{r_1+r_2} = \frac{c-b}{c-a}
$$

Raccordement C^0

$P(b) = Q(b) \Rightarrow P_n = Q_0$

Raccordement C^1

Raccordement C^0 $+$

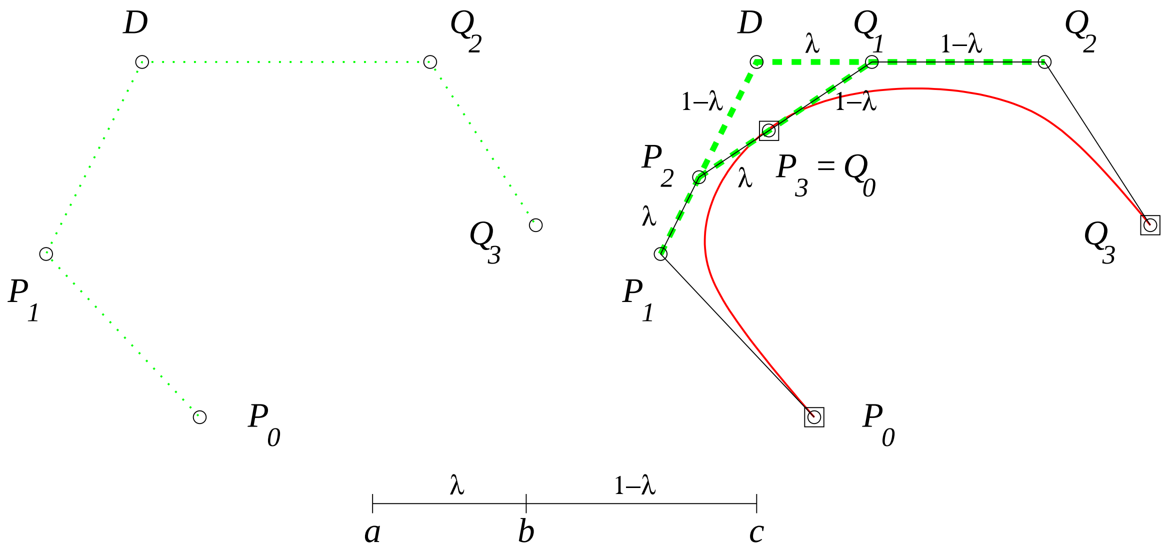
$P'(b) = Q'(b) \Rightarrow P_n = (1-\lambda)P_{n-1} + \lambda Q_1$

Raccordement C^2

Raccordement C^1 $+$

$P''(b) = Q''(b) \Rightarrow D = P_{n-1} + \frac{1-\lambda}{\lambda}(P_{n-1} - P_{n-2}) = Q_1 + \frac{\lambda}{1-\lambda}(Q_1 - Q_2)$

$$
\iff \left\{\begin{array}{rcrcl}
P_{n-1} & = & (1-\lambda)P_{n-2} & + & \lambda D \\
Q_1 & = & (1-\lambda)D & + & \lambda Q_2
\end{array}\right.
$$



B-splines

Base B-spline

Données : noeuds (abscisses) t_i avec $t_i < t_{i+1}$, $i \in \mathbb{Z}$

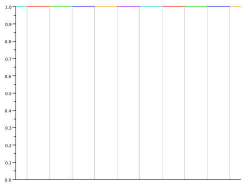
Ordre 1 (degré $d = 0$)

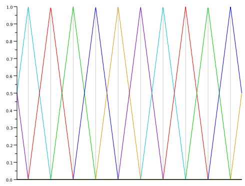
$$N_i^1(t) = \begin{cases} 1 \text{ si } t \in [t_i, t_{i+1}[\\ 0 \text{ sinon} \end{cases}$$

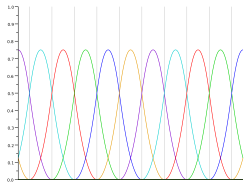
Ordre $k > 1$ (degré $d > 0$)

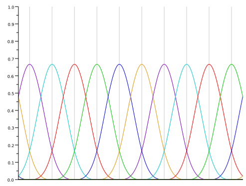
$$N_i^k(t) = \begin{cases} \dfrac{t - t_i}{t_{i+k-1} - t_i} N_i^{k-1}(t) + \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t) \text{ si } t \in [t_i, t_{i+k}[\\ 0 \text{ sinon} \end{cases}$$

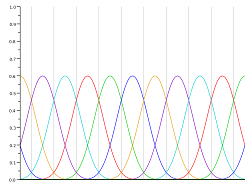
Cas noeuds équidistants


$k = 1$

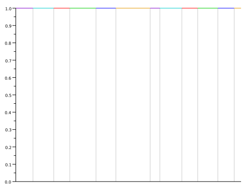

$k = 2$

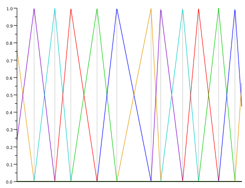

$k = 3$

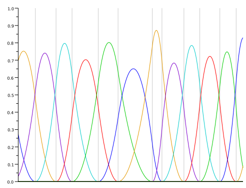

$k = 4$

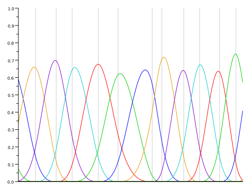

$k = 5$

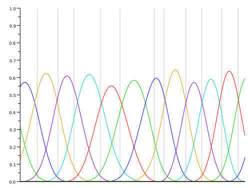
Cas noeuds quelconques


$k = 1$


$k = 2$


$k = 3$


$k = 4$


$k = 5$

Base B-spline périodique

Données : $n + 1$ noeuds (abscisses) $t_0 < t_1 < \cdots < t_n$

Période $T = t_n - t_0 \to t_{i+p\,n} = t_i + p\,T$, $0 \leq i \leq n - 1$, $p \in \mathbb{Z}$

Base B-spline avec noeuds confondus

Données : noeuds (abscisses) t_i avec $t_i \leq t_{i+1}$

Ordre $k > 1$

$$N_i^k(t) = \left\{ \begin{array}{lclll} \dfrac{t - t_i}{t_{i+k-1} - t_i} N_i^{k-1}(t) & + & \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t) & & \text{si } t \in [t_i, t_{i+k}[, t_{i+k-1} > t_i \text{ et } t_{i+k} > t_{i+1} \\ & & \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t) & & \text{si } t \in [t_i, t_{i+k}[, t_{i+k-1} = t_i \text{ et } t_{i+k} > t_{i+1} \\ \dfrac{t - t_i}{t_{i+k-1} - t_i} N_i^{k-1}(t) & & & & \text{si } t \in [t_i, t_{i+k}[, t_{i+k-1} > t_i \text{ et } t_{i+k} = t_{i+1} \\ & 0 & & & \text{sinon} \end{array} \right.$$

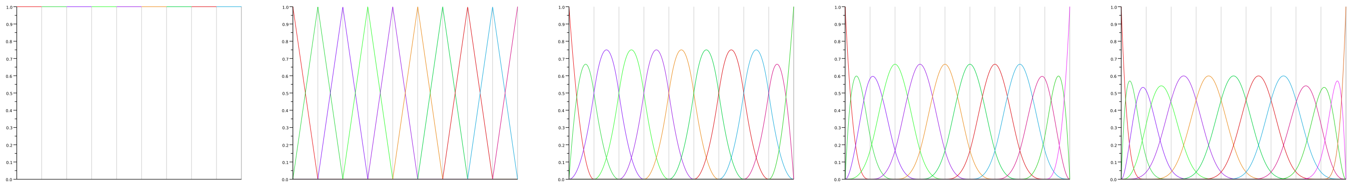
Base B-spline d'ordre k sur un intervalle borné $[a, b]$ et n intervalles

Noeuds $a = \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n = b$

$\rightarrow t_1 = t_2 = \cdots = t_k = \tau_0 < t_{k+1} = \tau_1 < \cdots < t_{k+n-1} = \tau_{n-1} < t_{k+n} = t_{k+n+1} = \cdots = t_{n+2k-1} = \tau_n$

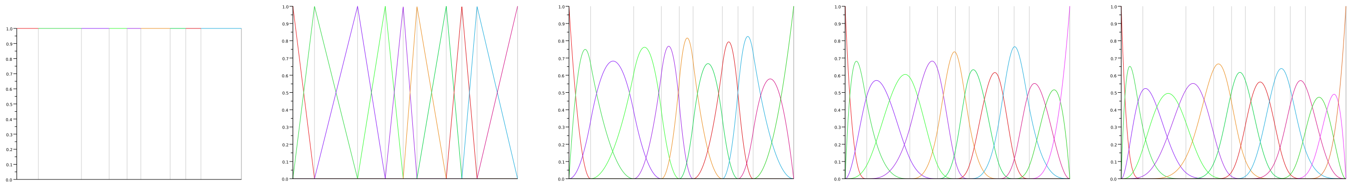
n intervalles et $n + k - 1$ fonctions de base N_i^k

– Cas noeuds équidistants



$k = 1$ $k = 2$ $k = 3$ $k = 4$ $k = 5$

– Cas noeuds quelconques



$k = 1$ $k = 2$ $k = 3$ $k = 4$ $k = 5$

Algorithme de DeBoor-Cox

Données : Points de contrôle D_i et noeuds associés t_i, une valeur t, k ordre de la B-spline

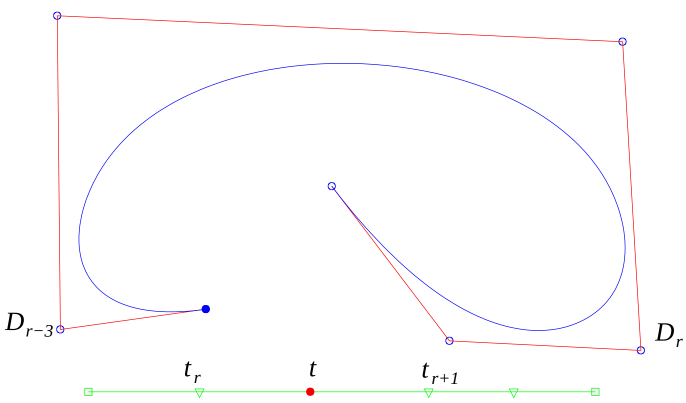
(1) trouver l'indice r tel que $t \in [t_r, t_{r+1}[$

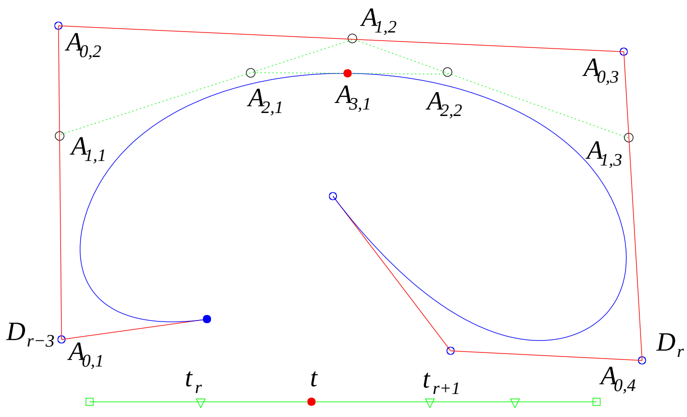
(2) $A_{0,i} = D_{r+i-k}$ pour $1 \leq i \leq k$

(3) $A_{j,i} = (1 - \lambda)A_{j-1,i} + \lambda A_{j-1,i+1}$ avec $\lambda = \dfrac{t - t_{r+i-k+j}}{t_{r+i} - t_{r+i-k+j}}$ pour $1 \leq i \leq k - j$ et $1 \leq j \leq k - 1$

$\rightarrow$ le point de la courbe B-Spline correspondant au paramètre t est $A_{k-1,1}$

Exemple avec $k = 4$





Insertion de noeuds

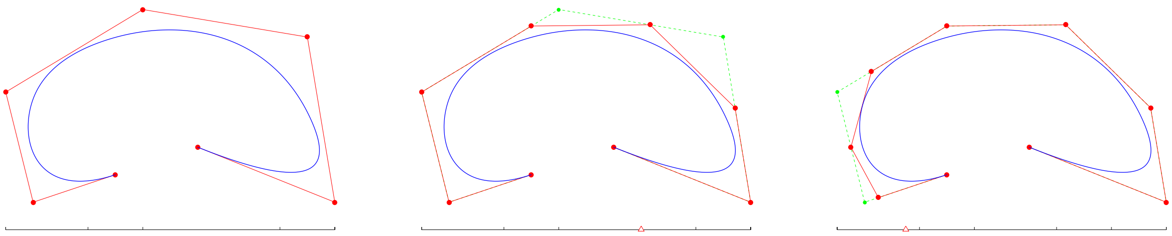
Données : Points de contrôle D_i et noeuds associés t_i, un noeud supplémentaire t, k ordre de la B-spline

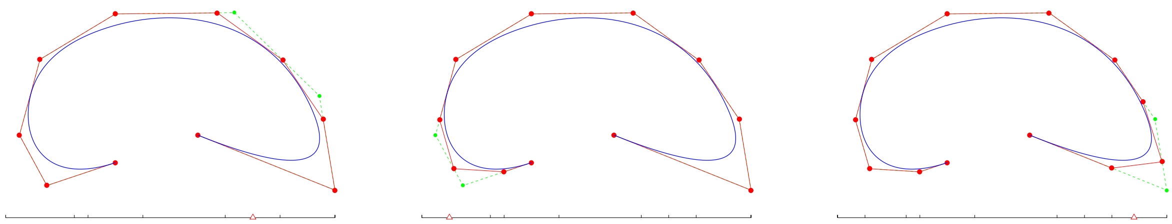
(1) trouver l'indice r tel que $t \in [t_r, t_{r+1}[$

(2) $A_{0,i} = D_{r+i-k}$ pour $1 \leq i \leq k$

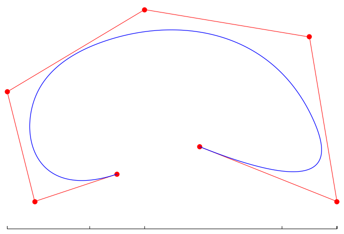
(3) $A_{1,i} = (1-\lambda)A_{0,i} + \lambda A_{0,i+1}$ avec $\lambda = \dfrac{t - t_{r+i-k+1}}{t_{r+i} - t_{r+i-k+1}}$ pour $1 \leq i \leq k-1$

$\rightarrow$ le polygone de contrôle correspondant à $\ldots, t_{r-1}, t_r, t, t_{r+1}, t_{r+2}, \ldots$ est $\ldots, D_{r-k}, D_{r-k+1}, A_{1,1}, \ldots, A_{1,k-1}, D_r, D_{r+1}, \ldots$

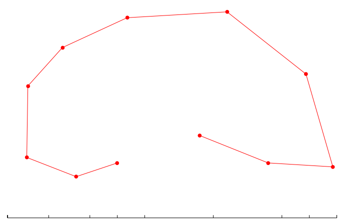




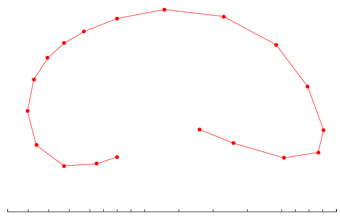
Subdivision



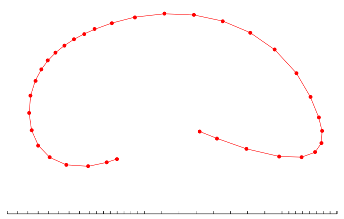
Polygone initial et courbe



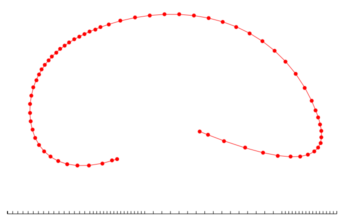
Polygone après 1 subdivision



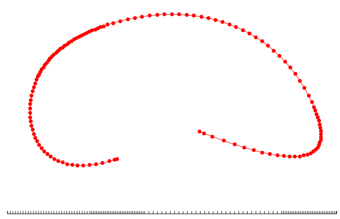
Polygone après 2 subdivisions



Polygone après 3 subdivisions



Polygone après 4 subdivisions



Polygone après 5 subdivisions

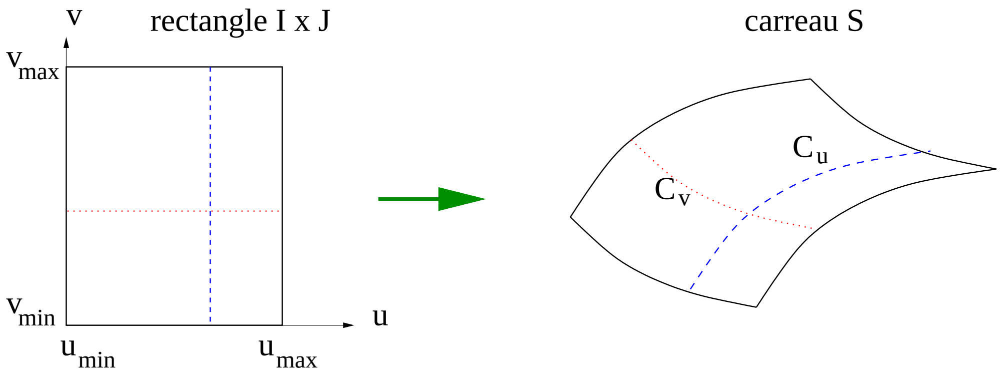
Surfaces produit tensoriel

Cas général

$$\text{Surface } \mathcal{S} = \left\{ P(u,v) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{i,j}\ f_i(u)\ g_j(v),\ u \in I = [u_{min}, u_{max}],\ v \in J = [v_{min}, v_{max}] \right\}$$

$$P(u,v) = \begin{pmatrix} f_1(u) & f_2(u) & \dots & f_m(u) \end{pmatrix} \begin{pmatrix} C_{1,1} & C_{1,2} & \dots & C_{1,n} \\ C_{2,1} & C_{2,2} & \dots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m,1} & C_{m,2} & \dots & C_{m,n} \end{pmatrix} \begin{pmatrix} g_1(v) \\ g_2(v) \\ \vdots \\ g_n(v) \end{pmatrix}$$

$\mathcal{C}_u = \{P(u,v),\ v \in J\}$ courbe à u constant
$\mathcal{C}_v = \{P(u,v),\ u \in I\}$ courbe à v constant

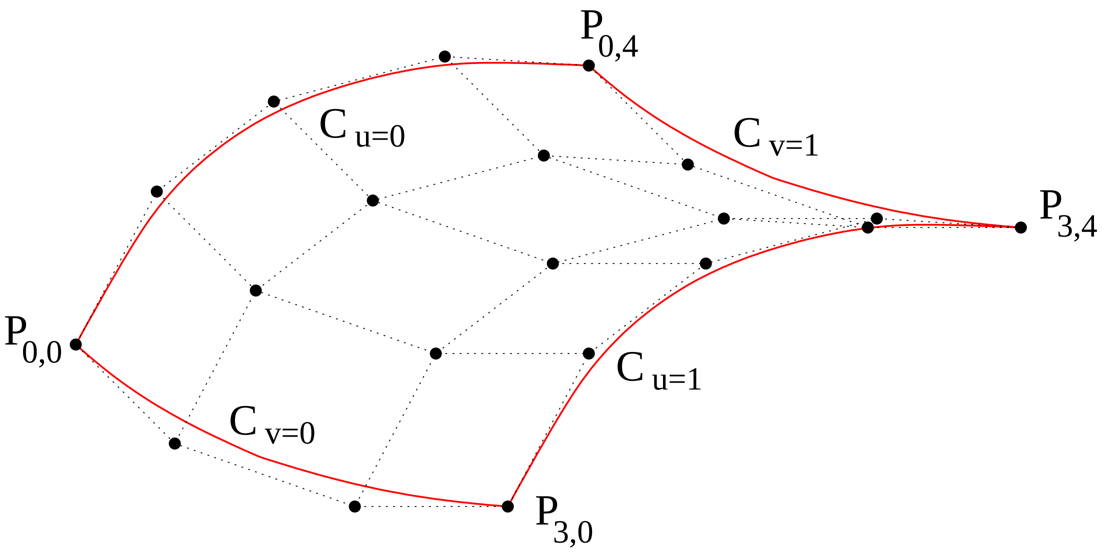


Carreau de Bézier

$$\text{Surface } \mathcal{S} = \left\{ P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j}\ B_i^m(u)\ B_j^n(v),\ u \in [0,1],\ v \in [0,1] \right\} \qquad \text{cas polynomial}$$

$$\text{Surface } \mathcal{S} = \left\{ P(u,v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} \omega_{i,j}\ P_{i,j}\ B_i^m(u)\ B_j^n(v)}{\sum_{i=0}^{m} \sum_{j=0}^{n} \omega_{i,j}\ B_i^m(u)\ B_j^n(v)},\ u \in [0,1],\ v \in [0,1] \right\} \qquad \text{cas rationnel}$$

polyèdre de contrôle : $\{P_{i,j}\}_{0 \le i \le m, 0 \le j \le n}$
(+ poids associés $\{\omega_{i,j}\}_{0 \le i \le m, 0 \le j \le n}$ pour le cas rationnel)

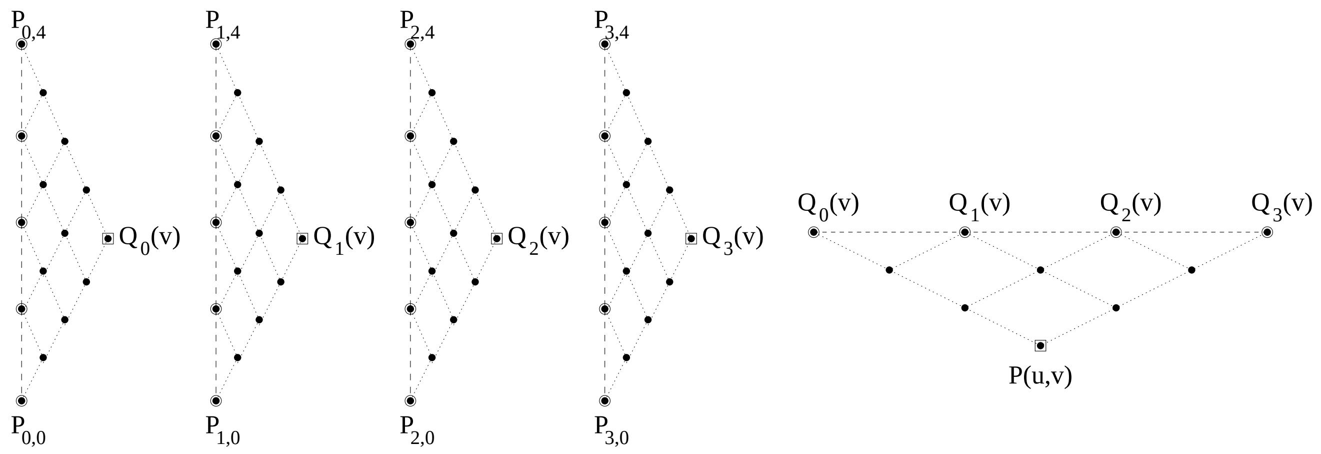


Carreau de Bézier de degré 3 en u et degré 4 en v

Propriétés (similaires aux courbes de Bézier)

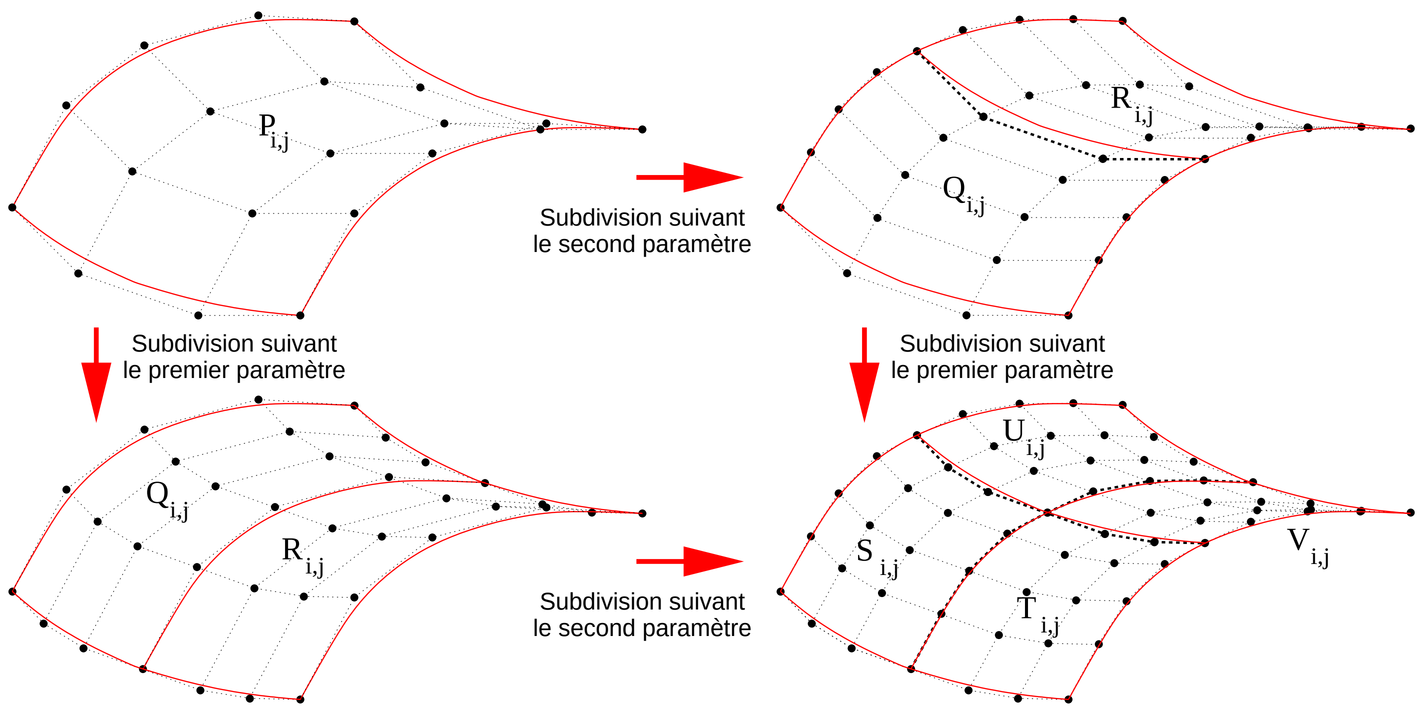
$$\mathcal{S} \in \text{conv}(\{P_{i,j}\})$$
$$P(0,0) = P_{0,0},\ P(1,0) = P_{m,0},\ P(0,1) = P_{0,n},\ P(1,1) = P_{m,n}$$
$$P'_u(u,v) = \frac{\partial P(u,v)}{\partial u} = \sum_{i=0}^{m-1} \sum_{j=0}^{n} m\ (P_{i+1,j} - P_{i,j})\ B_i^{m-1}(u)\ B_j^n(v)$$
$$P'_v(u,v) = \frac{\partial P(u,v)}{\partial v} = \sum_{i=0}^{m} \sum_{j=0}^{n-1} n\ (P_{i,j+1} - P_{i,j})\ B_i^{m}(u)\ B_j^{n-1}(v)$$
$$P''_{uv}(u,v) = \frac{\partial^2 P(u,v)}{\partial u \partial v} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m\ n\ (P_{i+1,j+1} - P_{i+1,j} - P_{i,j+1} + P_{i,j})\ B_i^{m-1}(u)\ B_j^{n-1}(v)$$
$$P'_u(0,0) = m\ (P_{1,0} - P_{0,0}) \qquad P'_v(0,0) = n\ (P_{0,1} - P_{0,0})$$
$$P''_{uv}(0,0) = m\ n\ (P_{1,1} - P_{1,0} - P_{0,1} + P_{0,0})\ \text{(vecteur twist en } (0,0))$$

Evaluation



1) Evaluation suivant v 2) Evaluation suivant u

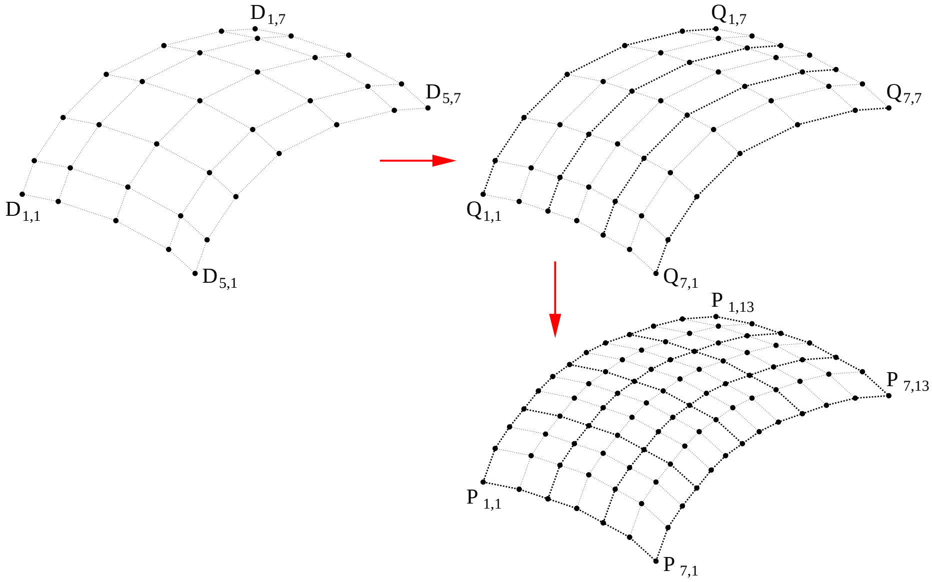
Subdivision



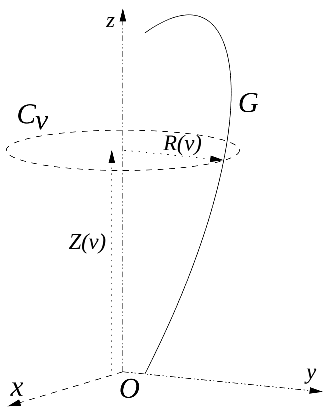
Carreau NURBS

Surface $\mathcal{S} = \left\{ P(u,v) = \dfrac{\displaystyle\sum_{i=1}^{M}\sum_{j=1}^{N} \omega_{i,j}\, D_{i,j}\, N_i^m(u)\, N_j^n(v)}{\displaystyle\sum_{i=1}^{M}\sum_{j=1}^{N} \omega_{i,j}\, N_i^m(u)\, N_j^n(v)} \right\}$	
ordre m (degré $du = m-1$) en u vecteur de noeuds en u : $u_0, u_1, \ldots, u_{nu}$	ordre n (degré $dv = n-1$) en v vecteur de noeuds en v : $v_0, v_1, \ldots, v_{nv}$
grille de $M \times N$ points $D_{i,j}$ et poids $\omega_{i,j}$ ($M = nu + du$, $N = nv + dv$)	

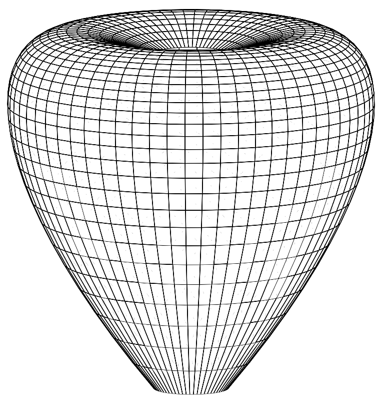
Passage à la forme Bézier



Surface de révolution



Révolution d'une courbe plane (génératrice) $\mathcal{G}$ autour de l'axe Oz



Surface de révolution $\mathcal{S}$

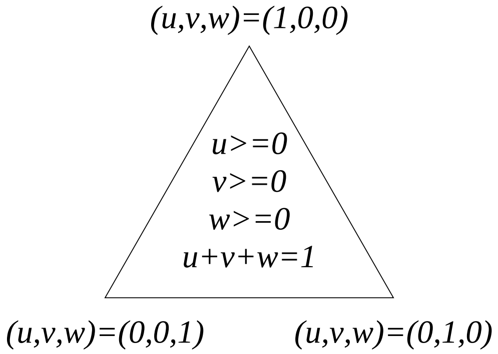
Cercle unité $\mathcal{C}$ $\mathcal{C} = \{(X(u), Y(u)), u \in [a,b]\}$	Génératrice $\mathcal{G}$ $\mathcal{G} = \{(R(v), Z(v)), v \in [c,d]\}$
Définition NURBS	
Points (X_i, Y_i) - poids W_i - vecteur de noeuds (u_k)	Points (R_j, Z_j) - poids Q_j - vecteur de noeuds (v_l)

Surface de révolution $\mathcal{S}$ $\mathcal{S} = \{(x(u,v) = X(u)R(v), y(u,v) = Y(u)R(v), z(u,v) = Z(v)), (u,v) \in [a,b] \times [c,d]\}$
Définition NURBS
Points $D_{i,j} = (X_iR_j, Y_iR_j, Z_j)$ - poids $\omega_{i,j} = W_iQ_j$ - vecteurs de noeuds (u_k) et (v_l)

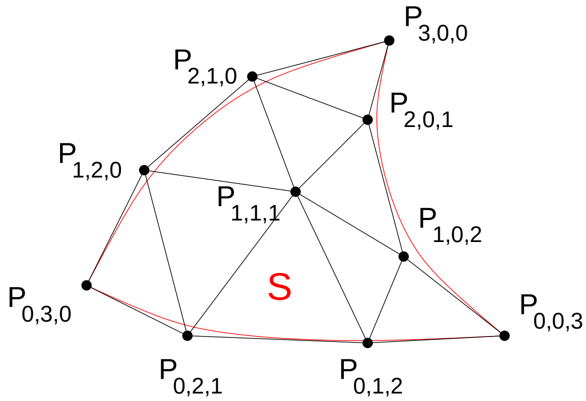
Patch de Bézier triangulaire

$$\text{Surface } \mathcal{S} = \left\{ P(u,v,w) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} P_{i,j,k}\, B^n_{i,j,k}(u,v,w),\ u,v,w \geq 0, u+v+w=1 \right\}$$

$$\left.\begin{array}{ll} \text{fonctions de base :} & B^n_{i,j,k}(u,v,w) = \dfrac{n!}{i!j!k!} u^i v^j w^k \\ \text{points de contrôle :} & \{P_{i,j,k}\} \end{array}\right\} \quad i,j,k \geq 0, i+j+k=n$$



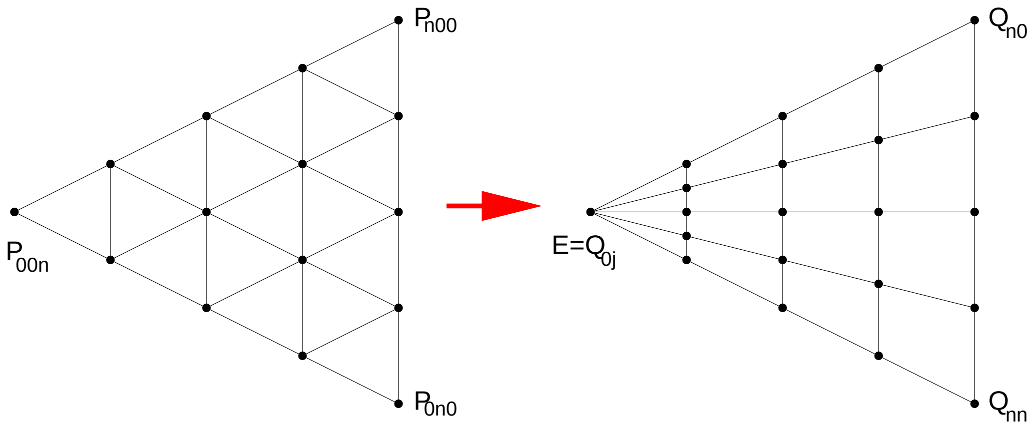
Triangle unité des paramètres



Polyèdre $\{P_{i,j,k}\}$ et patch correspondant

Passage à la forme Bézier rectangulaire

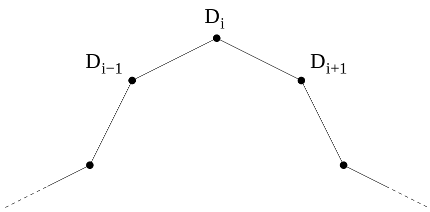
pour i variant de 0 à n : $\{P_{i-j,j,n-i}\}_{0\leq j\leq i} \longrightarrow \boxed{(n-i) \text{ élévations de degré}} \longrightarrow \{Q_{i,j}\}_{0\leq j\leq n}$



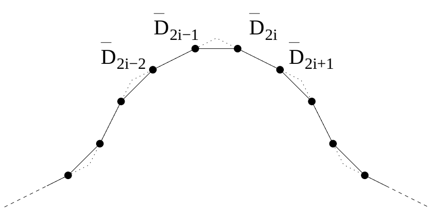
Passage d'une structure triangulaire à une structure rectangulaire dégénérée

Méthodes de subdivision

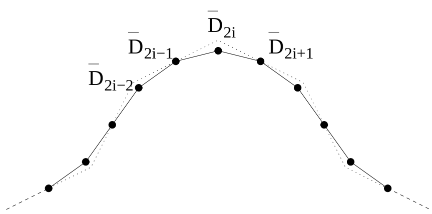
Courbes de subdivision - B-spline (noeuds équidistants)



Avant subdivision



Après subdivision
ordre 3 - Schéma de Chaikin



Après subdivision
ordre 4 - Schéma de Catmull-Clark

Surfaces de subdivision

Schéma de Catmull-Clark (B-spline d'ordre 4 - noeuds équidistants)

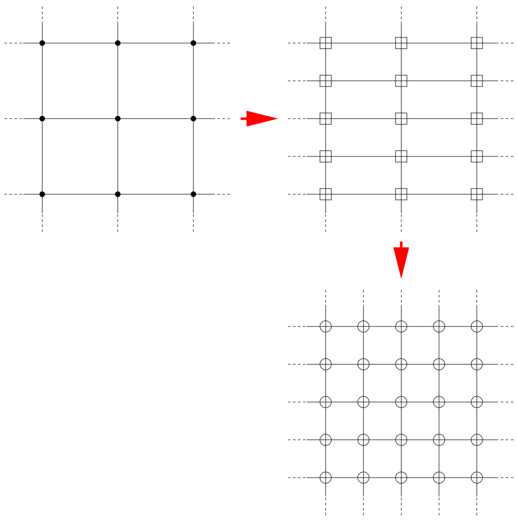


Schéma de subdivision "produit tensoriel"

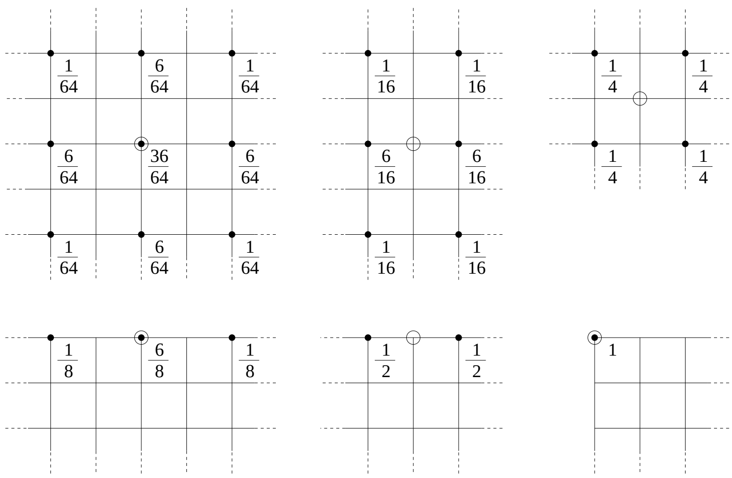
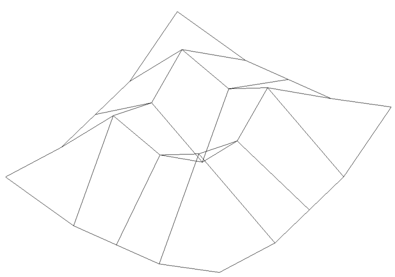
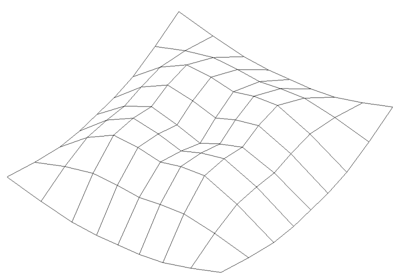


Schéma de Catmull-Clark (grille rectangulaire)
règles de subdivision

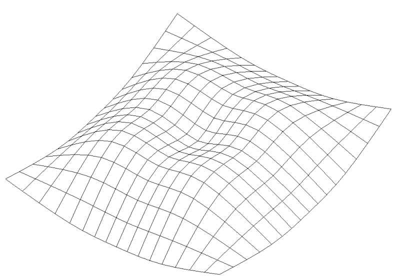
Un exemple :



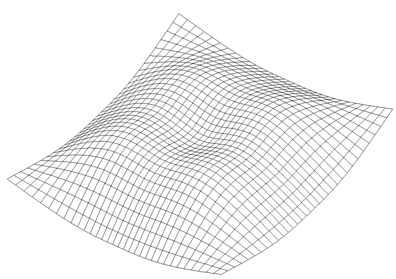
Grille initiale



Après 1 subdivision

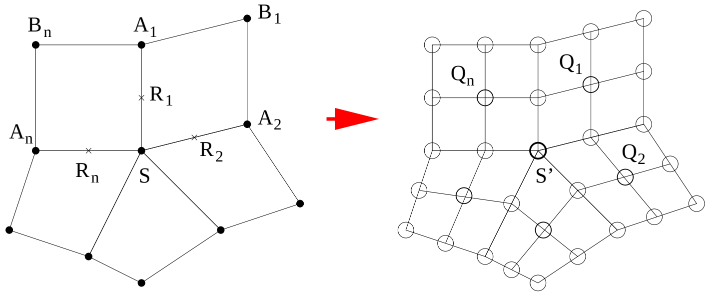


Après 2 subdivisions



Après 3 subdivisions

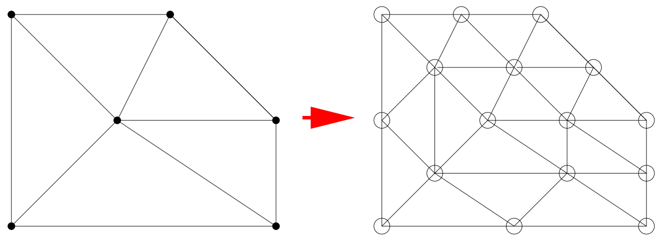
Schéma de Catmull-Clark (maillage quadrangulaire quelconque)

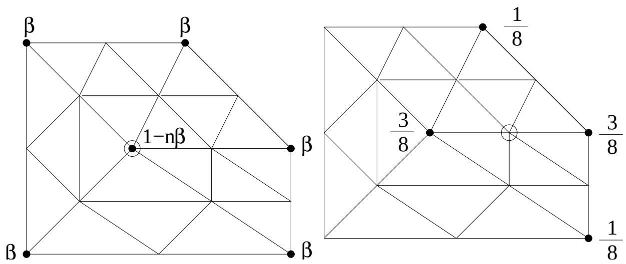


Règles pour les sommets internes S'

$$\left\{\begin{array}{ll} Q_i = \dfrac{1}{4}(A_i + B_i + A_{i+1} + S) & Q = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} Q_i \\ & \\ R_i = \dfrac{1}{2}(A_i + S) & R = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} R_i \end{array}\right\} \text{ et } S' = \frac{1}{n}\left(Q + 2R + (n-3)S\right)$$

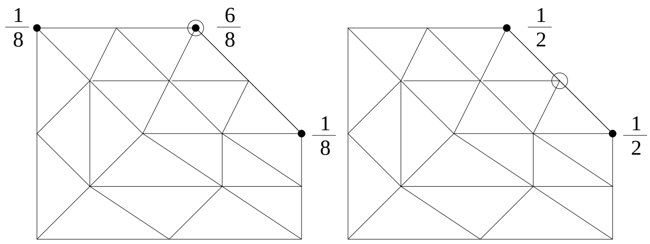
Schéma de Loop (maillage triangulaire quelconque)





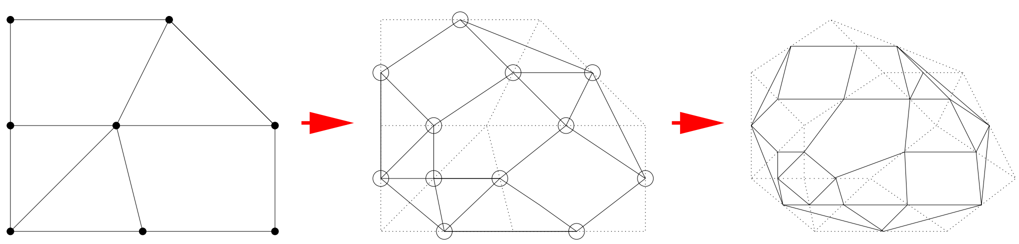
Règles pour les sommets internes

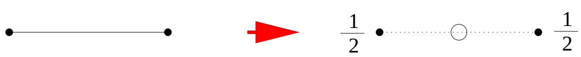
$$\beta = \frac{1}{n}\left(\frac{5}{8} - \left[\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right]^2\right)$$



Règles pour les sommets externes

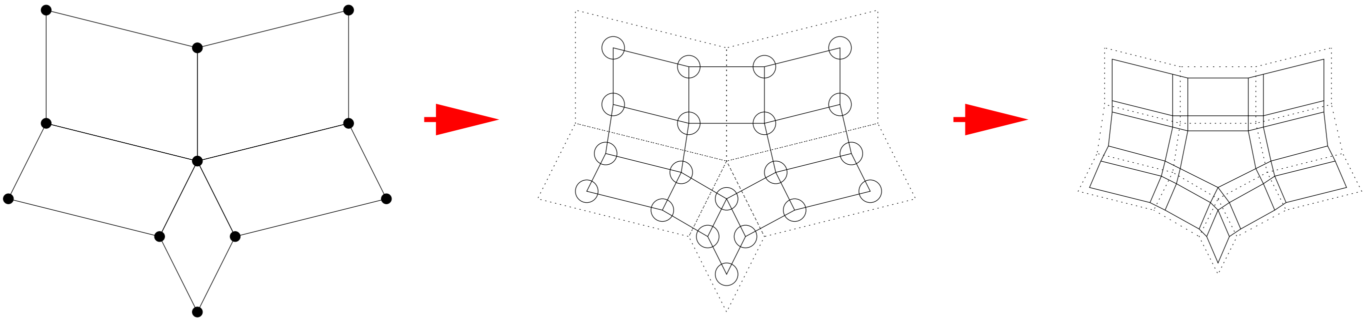
Schéma Mid-Edge (maillage quelconque)

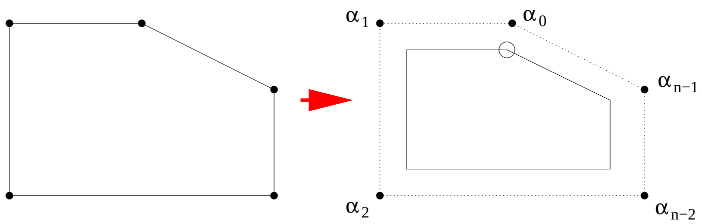




Règle de subdivision

Schéma Doo-Sabin (maillage quadrangulaire quelconque)

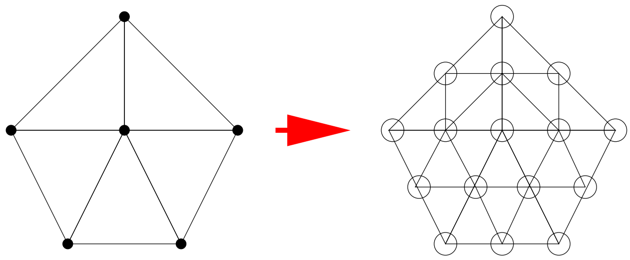


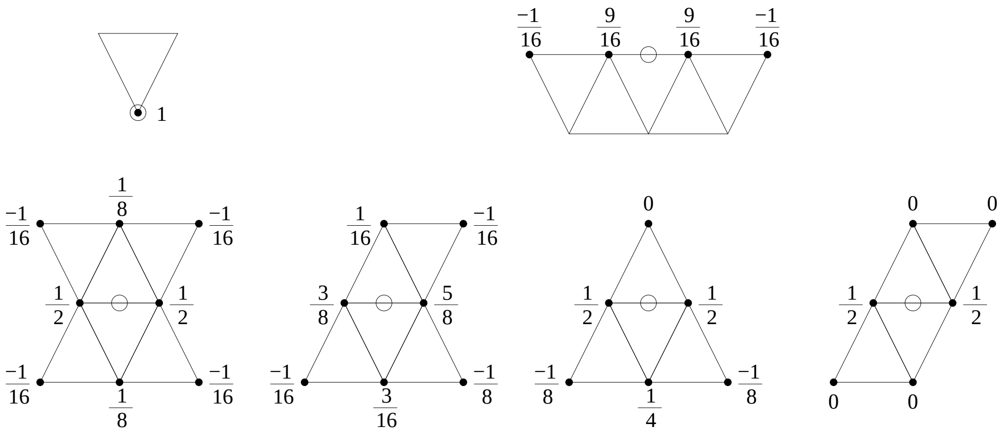


Règle pour les sommets d'une face avec n sommets

$$\alpha_0 = \frac{1}{4} + \frac{5}{4n} \text{ et } \alpha_i = \frac{3 + 2\cos(2i\pi/n)}{4n} \text{ pour } 1 \leq i \leq n-1$$

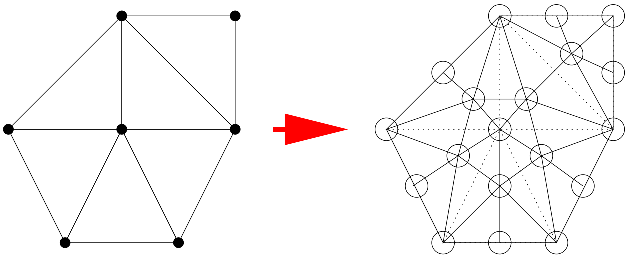
Schéma Butterfly (maillage triangulaire quelconque)

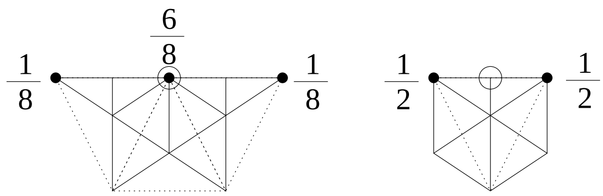




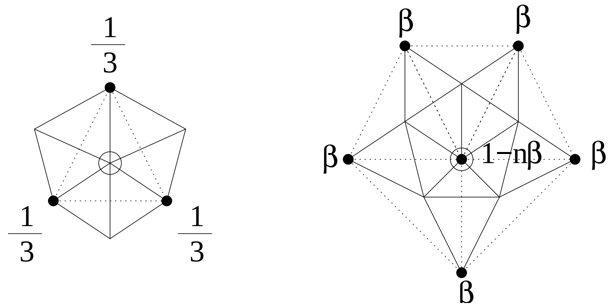
Règles de subdivision

Schéma $\sqrt{3}$ (maillage triangulaire quelconque)





Règles pour les sommets externes



Règles pour les sommets internes

$$\beta = \frac{4 - 2\cos(2\pi/n)}{9n}$$

Exemples

Schéma de Loop

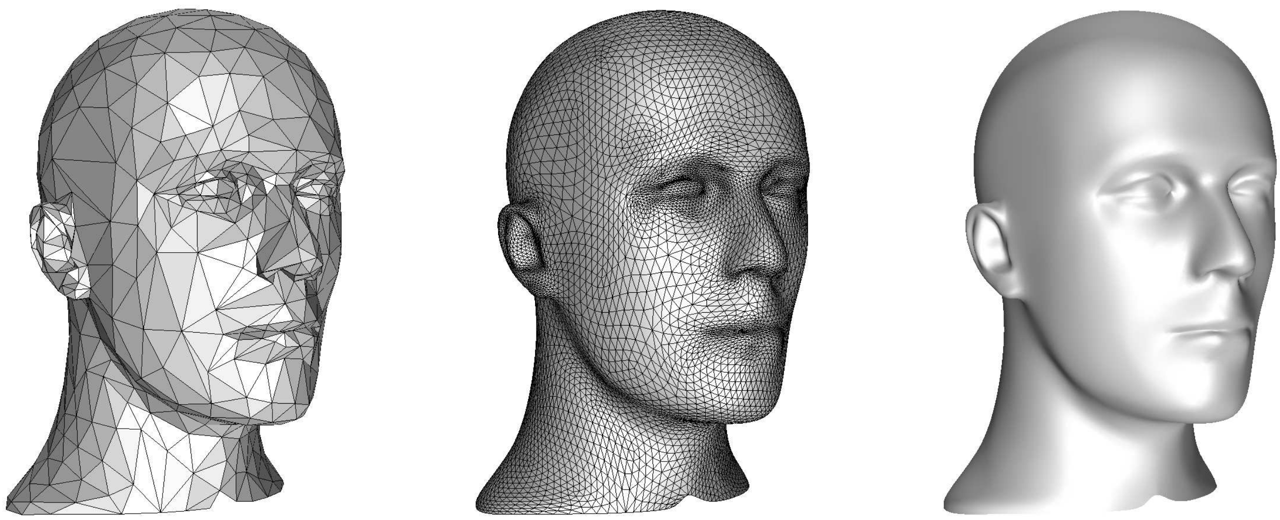


Schéma de Catmull-Clark