

# MTE 544 Assignment 1

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1

Rotation matrix:  $\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ , Translation vector:  $\mathbf{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$

$$\mathbf{G} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & q_x \\ \sin(\theta) & \cos(\theta) & q_y \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\mathbf{R}^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \quad (1)$$

$$\mathbf{0}_{1 \times 2} = [0 \quad 0] \quad (2)$$

$$-\mathbf{R}^T \cdot \mathbf{q} = - \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} -\sin(\theta)q_y - \cos(\theta)q_x \\ -\cos(\theta)q_y + \sin(\theta)q_x \end{bmatrix} \quad (3)$$

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$$\mathbf{G}^{-1} = \frac{\text{adj}(\mathbf{G})}{\det(\mathbf{G})}$$

The determinant of  $\mathbf{G}$  can be evaluated using cofactor expansion along the last row:

$$\det(\mathbf{G}) = 0 \cdot \det \begin{bmatrix} -\sin(\theta) & q_x \\ \cos(\theta) & q_y \end{bmatrix} - 0 \cdot \det \begin{bmatrix} \cos(\theta) & q_x \\ \sin(\theta) & q_y \end{bmatrix} + 1 \cdot \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Since the first two terms are multiplied by 0, the determinant simplifies to:

$$\det(\mathbf{G}) = \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = 1$$

Thus:

$$\mathbf{G}^{-1} = \text{adj}(\mathbf{G})$$

Adjoint of  $\mathbf{G}$

$$\text{adj}(\mathbf{G}) = \begin{bmatrix} +\det \begin{bmatrix} \cos(\theta) & q_y \\ 0 & 1 \end{bmatrix} & -\det \begin{bmatrix} -\sin(\theta) & q_x \\ 0 & 1 \end{bmatrix} & +\det \begin{bmatrix} -\sin(\theta) & q_x \\ \cos(\theta) & q_y \end{bmatrix} \\ -\det \begin{bmatrix} \sin(\theta) & q_y \\ 0 & 1 \end{bmatrix} & +\det \begin{bmatrix} \cos(\theta) & q_x \\ 0 & 1 \end{bmatrix} & -\det \begin{bmatrix} \cos(\theta) & q_x \\ \sin(\theta) & q_y \end{bmatrix} \\ +\det \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ 0 & 0 \end{bmatrix} & -\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ 0 & 0 \end{bmatrix} & +\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \end{bmatrix}$$

$$\text{adj}(\mathbf{G}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & -\sin(\theta)q_y - \cos(\theta)q_x \\ -\sin(\theta) & \cos(\theta) & -\cos(\theta)q_y + \sin(\theta)q_x \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting (1), (2), and (3) into the adjoint of  $\mathbf{G}$

$$\mathbf{G}^{-1} = \text{adj}(\mathbf{G}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & -\sin(\theta)q_y - \cos(\theta)q_x \\ -\sin(\theta) & \cos(\theta) & -\cos(\theta)q_y + \sin(\theta)q_x \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T \cdot \mathbf{q} \\ \mathbf{0}_{1 \times 2} & 1 \end{bmatrix}$$

2

(a)

$$\mathbf{G}_e^s = \mathbf{G}_1^s \cdot \mathbf{G}_2^1 \cdot \mathbf{G}_3^2 \cdot \mathbf{G}_e^3$$

$$\mathbf{G}_e^3 = \begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) & l_3 \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & l_3 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_3^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & l_2 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_2^1 = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_1^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix}$$

To evaluate  $\mathbf{G}_e^s$ , we compute the product of the matrices from right to left:

$$\mathbf{G}_e^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & -\sin(\theta) & l_2 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & l_3 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_e^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin(\theta) & \cos(\theta) & l_3 \cos(\theta) + l_2 \\ -\cos(\theta) & \sin(\theta) & l_3 \sin(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_e^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin(\gamma + \theta) & \cos(\gamma + \theta) & l_1 + l_2 \cos(\gamma) + l_3 \cos(\gamma + \theta) \\ -\cos(\gamma + \theta) & \sin(\gamma + \theta) & l_2 \sin(\gamma) + l_3 \sin(\gamma + \theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_e^s = \begin{bmatrix} \sin(\alpha + \gamma + \theta) & \cos(\alpha + \gamma + \theta) & l_1 \cos(\alpha) + l_2 \cos(\alpha + \gamma) + l_3 \cos(\alpha + \gamma + \theta) + x \\ -\cos(\alpha + \gamma + \theta) & \sin(\alpha + \gamma + \theta) & l_1 \sin(\alpha) + l_2 \sin(\alpha + \gamma) + l_3 \sin(\alpha + \gamma + \theta) + y \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\langle \alpha, \gamma, \theta \rangle = \left\langle \frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{3} \right\rangle, \quad l_1 = l_2 = l_3 = 1, \quad x = y = 1$$

Substituting the above values into the result of 2(a):

$$\mathbf{G}_e^s = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous coordinate representations:

$$\bar{\mathbf{p}}_1^e = \begin{bmatrix} \mathbf{p}_1^e \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \bar{\mathbf{p}}_2^e = \begin{bmatrix} \mathbf{p}_2^e \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solve for:  $\mathbf{p}_1^s$ :

$$\bar{\mathbf{p}}_1^s = \mathbf{G}_e^s \cdot \bar{\mathbf{p}}_1^e = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ \frac{5+\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{p}_1^s = \begin{bmatrix} 2.573132185 \\ 3.207106781 \end{bmatrix}$$

Solve for:  $\mathbf{p}_2^s$ :

$$\bar{\mathbf{p}}_2^s = \mathbf{G}_e^s \cdot \bar{\mathbf{p}}_2^e = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \sqrt{3} + \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} + 1 + \frac{5+\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{p}_2^s = \begin{bmatrix} 4.805182993 \\ 3.341081377 \end{bmatrix}$$

(c)

$$\bar{\mathbf{p}}^s = \mathbf{G}_2^s \cdot \bar{\mathbf{p}}^2$$

Given the fact that point  $\mathbf{p}$  is rigidly fixed to frame 2, the following holds true:

$$\dot{\bar{\mathbf{p}}}^s = \dot{\mathbf{G}}_2^s \cdot \bar{\mathbf{p}}^2$$

Where

$$\mathbf{G}_2^s = \mathbf{G}_1^s \cdot \mathbf{G}_2^1 = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting  $x = 0, y = 0, l_1 = 1$  and multiplying through:

$$\mathbf{G}_2^s = \begin{bmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & \cos(\alpha) \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & \sin(\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

Thus:

$$\dot{\mathbf{G}}_2^s = \begin{bmatrix} -\sin(\alpha + \gamma) [\dot{\alpha} + \dot{\gamma}] & -\cos(\alpha + \gamma) [\dot{\alpha} + \dot{\gamma}] & -\sin(\alpha) \dot{\alpha} \\ \cos(\alpha + \gamma) [\dot{\alpha} + \dot{\gamma}] & -\sin(\alpha + \gamma) [\dot{\alpha} + \dot{\gamma}] & \cos(\alpha) \dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting  $\langle \alpha, \gamma \rangle = \langle \frac{\pi}{4}, \frac{\pi}{4} \rangle$ :

$$\mathbf{G}_2^s = \begin{bmatrix} -[\dot{\alpha} + \dot{\gamma}] & 0 & -\frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & -[\dot{\alpha} + \dot{\gamma}] & \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix}$$

Given  $\bar{\mathbf{p}}^2 = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$ :

$$\dot{\bar{\mathbf{p}}}^s = \begin{bmatrix} -[\dot{\alpha} + \dot{\gamma}] & 0 & -\frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & -[\dot{\alpha} + \dot{\gamma}] & \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -x' [\dot{\alpha} + \dot{\gamma}] - \frac{\sqrt{2}}{2}\dot{\alpha} \\ -y' [\dot{\alpha} + \dot{\gamma}] + \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 \end{bmatrix}$$

### 3

(b)

Outlining the kinematic equations for a two-wheeled robot:

$$\omega = \frac{r}{T} \cdot (u_r - u_l)$$

$$v = \frac{r}{2} \cdot (u_r + u_l)$$

Solving these equations based on 3(a).i.1:

$$\omega = 0 \frac{rad}{s}, v = 1 \frac{m}{s}, T = 0.2 \text{ m}, r = 0.1 \text{ m}$$

$$u_r = u_l = 10 \frac{rad}{s}$$

Solving these equations based on 3(a).i.2:

$$\omega = 0.3 \frac{rad}{s}, v = 0 \frac{m}{s}, T = 0.2 \text{ m}, r = 0.1 \text{ m}$$

$$u_r = 0.3 \frac{rad}{s}$$

$$u_l = -0.3 \frac{rad}{s}$$

Solving these equations based on 3(a).i.3:

$$\omega = 0.3 \frac{rad}{s}, v = 1 \frac{m}{s}, T = 0.2 \text{ m}, r = 0.1 \text{ m}$$

$$u_r = 10.3 \frac{rad}{s}$$

$$u_l = 9.7 \frac{rad}{s}$$

If the input control parameters were changed from linear and angular velocities to wheel speeds, the state matrix would need to be changed in the implemented solution:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{r}{2} \cos(\theta) & \frac{r}{2} \cos(\theta) \\ \frac{r}{2} \sin(\theta) & \frac{r}{2} \sin(\theta) \\ \frac{r}{T} & -\frac{r}{T} \end{bmatrix} \cdot \begin{bmatrix} u_r \\ u_l \end{bmatrix}$$

## 4

I.

$$u_i = \frac{v_{drive}^i}{r_i} = \frac{1}{r_i}(v_x^i + v_y^i \tan \gamma_i) = \frac{1}{r_i} [1 \quad \tan \gamma_i] \mathbf{g}_i(\theta) \dot{\mathbf{q}}$$

$$\text{where } \mathbf{g}_i(\theta) = \begin{bmatrix} \cos(\theta + \beta_i) & \sin(\theta + \beta_i) & x_i \sin(\beta_i) - y_i \cos(\beta_i) \\ -\sin(\theta + \beta_i) & \cos(\theta + \beta_i) & x_i \cos(\beta_i) + y_i \sin(\beta_i) \end{bmatrix}, \quad \dot{\mathbf{q}} = [\dot{x} \quad \dot{y} \quad \dot{\omega}]^T$$

For a robot with 3 wheels:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{G}(\theta) \dot{\mathbf{q}} \quad (4)$$

Since all the Mecanum wheels have the same radius and roller orientation,

$$r_1 = r_2 = r_3 = r, \quad \gamma_1 = \gamma_2 = \gamma_3 = 0$$

Finding  $\beta_i, x_i$ , and  $y_i$  for all 3 wheels

$$\beta_1 = \frac{\pi}{2}, \beta_2 = \frac{7\pi}{6}, \beta_3 = \frac{11\pi}{6}$$

$$(x_1, y_1) = (l, 0) \quad (x_2, y_2) = (l \cos(\frac{2\pi}{3}), l \sin(\frac{2\pi}{3})) \quad (x_3, y_3) = (l \cos(\frac{4\pi}{3}), l \sin(\frac{4\pi}{3}))$$

$$(x_1, y_1) = (l, 0) \quad (x_2, y_2) = (-\frac{l}{2}, \frac{\sqrt{3}}{2}l) \quad (x_3, y_3) = (-\frac{l}{2}, -\frac{\sqrt{3}}{2}l)$$

Expanding  $\mathbf{g}_{1,2,3}(\theta)$

$$\mathbf{g}_1(\theta) = \begin{bmatrix} -\sin(\theta) & \cos(\theta) & l \\ -\cos(\theta) & -\sin(\theta) & 0 \end{bmatrix}$$

$$\mathbf{g}_2(\theta) = \begin{bmatrix} \cos(\theta + \frac{7\pi}{6}) & \sin(\theta + \frac{7\pi}{6}) & -\frac{l}{2} \sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l \cos(\frac{7\pi}{6}) \\ -\sin(\theta + \frac{7\pi}{6}) & \cos(\theta + \frac{7\pi}{6}) & -\frac{l}{2} \cos(\frac{7\pi}{6}) + \frac{\sqrt{3}}{2}l \sin(\frac{7\pi}{6}) \end{bmatrix}$$

$$\mathbf{g}_3(\theta) = \begin{bmatrix} \cos(\theta + \frac{11\pi}{6}) & \sin(\theta + \frac{11\pi}{6}) & -\frac{l}{2} \sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l \cos(\frac{11\pi}{6}) \\ -\sin(\theta + \frac{11\pi}{6}) & \cos(\theta + \frac{11\pi}{6}) & -\frac{l}{2} \cos(\frac{11\pi}{6}) - \frac{\sqrt{3}}{2}l \sin(\frac{11\pi}{6}) \end{bmatrix}$$

Evaluating  $\mathbf{G}(\theta)$ , each row being equivalent to the first row of the minor  $\mathbf{g}_{1,2,3}(\theta)$  matrix due to  $\tan(\gamma_{1,2,3}) = 0$

$$\mathbf{G}(\theta) = \frac{1}{r} \begin{bmatrix} -\sin(\theta) & \cos(\theta) & l \\ \cos(\theta + \frac{7\pi}{6}) & \sin(\theta + \frac{7\pi}{6}) & -\frac{l}{2} \sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l \cos(\frac{7\pi}{6}) \\ \cos(\theta + \frac{11\pi}{6}) & \sin(\theta + \frac{11\pi}{6}) & -\frac{l}{2} \sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l \cos(\frac{11\pi}{6}) \end{bmatrix}$$

Full state space equation:

$$\mathbf{u} = \frac{1}{r} \begin{bmatrix} -\sin(\theta) & \cos(\theta) & l \\ \cos(\theta + \frac{7\pi}{6}) & \sin(\theta + \frac{7\pi}{6}) & -\frac{l}{2}\sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l\cos(\frac{7\pi}{6}) \\ \cos(\theta + \frac{11\pi}{6}) & \sin(\theta + \frac{11\pi}{6}) & -\frac{l}{2}\sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l\cos(\frac{11\pi}{6}) \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix}$$

II.

1) The  $\dot{\mathbf{q}}$  of a straight line with a slope of 60 degrees is:

$$\dot{\mathbf{q}} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Using an initial state of  $x_0 = 0[cm]$ ,  $y_0 = 0[cm]$ ,  $\theta_0 = 0[rad]$ ,  $l = 25[cm]$ ,  $r = 10[cm]$  and solving for  $\mathbf{u}$

$$u_1 = u_2, \quad u_3 = 0$$

2) The  $\dot{\mathbf{q}}$  of a circular path with a diameter of 2 m = 200 cm is:

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 \\ \frac{d\pi}{T} \\ \frac{2\pi}{T} \end{bmatrix}$$

where  $d = 200 \text{ cm}$ ,  $T = \text{total simulation time}$

Using an initial state of  $x_0 = 0[cm]$ ,  $y_0 = 0[cm]$ ,  $\theta_0 = 0[rad]$ ,  $l = 25[cm]$ ,  $r = 10[cm]$  and solving for  $\mathbf{u}$

$$u_2 = u_3, \quad u_1 = -5u_2$$