MTE

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$$\text{Rotation matrix: } \mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \text{Translation vector: } \mathbf{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & q_x \\ \sin(\theta) & \cos(\theta) & q_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R^T} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{1}$$

$$\mathbf{0_{1\times 2}} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{2}$$

$$-\mathbf{R^T} \cdot \mathbf{q} = -\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} -\sin(\theta)q_y - \cos(\theta)q_x \\ -\cos(\theta)q_y + \sin(\theta)q_x \end{bmatrix} \tag{3}$$

$$\mathbf{G^{-1}} = \frac{\text{adj}(\mathbf{G})}{\det(\mathbf{G})}$$

The determinant of $\mathbf{G}$ can be evaluated using cofactor expansion along the last row:

$$\det(\mathbf{G}) = 0 \cdot \det \begin{bmatrix} -\sin(\theta) & q_x \\ \cos(\theta) & q_y \end{bmatrix} - 0 \cdot \det \begin{bmatrix} \cos(\theta) & q_x \\ \sin(\theta) & q_y \end{bmatrix} + 1 \cdot \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Since the first two terms are multiplied by 0, the determinant simplifies to:

$$\det(\mathbf{G}) = \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = 1$$

Thus:

$$\mathbf{G^{-1}} = \text{adj}(\mathbf{G})$$

Adjoint of $\mathbf{G}$

$$\operatorname{adj}(\mathbf{G}) = \begin{bmatrix} +\det\begin{bmatrix}\cos(\theta) & q_y \\ 0 & 1\end{bmatrix} & -\det\begin{bmatrix}-\sin(\theta) & q_x \\ 0 & 1\end{bmatrix} & +\det\begin{bmatrix}-\sin(\theta) & q_x \\ \cos(\theta) & q_y\end{bmatrix} \\ -\det\begin{bmatrix}\sin(\theta) & q_y \\ 0 & 1\end{bmatrix} & +\det\begin{bmatrix}\cos(\theta) & q_x \\ 0 & 1\end{bmatrix} & -\det\begin{bmatrix}\cos(\theta) & q_x \\ \sin(\theta) & q_y\end{bmatrix} \\ +\det\begin{bmatrix}\sin(\theta) & \cos(\theta) \\ 0 & 0\end{bmatrix} & -\det\begin{bmatrix}\cos(\theta) & -\sin(\theta) \\ 0 & 0\end{bmatrix} & +\det\begin{bmatrix}\cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{bmatrix} \end{bmatrix}$$

$$\operatorname{adj}(\mathbf{G}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & -\sin(\theta)q_y - \cos(\theta)q_x \\ -\sin(\theta) & \cos(\theta) & -\cos(\theta)q_y + \sin(\theta)q_x \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting (1), (2), and (3) into the adjoint of $\mathbf{G}$

$$\mathbf{G^{-1}} = \operatorname{adj}(\mathbf{G}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & -\sin(\theta)q_y - \cos(\theta)q_x \\ -\sin(\theta) & \cos(\theta) & -\cos(\theta)q_y + \sin(\theta)q_x \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R^T} & -\mathbf{R^T}\cdot\mathbf{q} \\ \mathbf{0_{1\times 2}} & 1 \end{bmatrix}$$

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(a)

$$\mathbf{G_e^s} = \mathbf{G_1^s} \cdot \mathbf{G_2^1} \cdot \mathbf{G_3^2} \cdot \mathbf{G_e^3}$$

$$\mathbf{G_e^3} = \begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) & l_3 \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & l_3 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_3^2} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & l_2 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_2^1} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_1^s} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix}$$

To evaluate $\mathbf{G_e^s}$, we compute the product of the matrices from right to left:

$$\mathbf{G_e^s} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & -\sin(\theta) & l_2 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & l_3 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_e^s} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin(\theta) & \cos(\theta) & l_3\cos(\theta) + l_2 \\ -\cos(\theta) & \sin(\theta) & l_3\sin(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_e^s} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin(\gamma+\theta) & \cos(\gamma+\theta) & l_1 + l_2\cos(\gamma) + l_3\cos(\gamma+\theta) \\ -\cos(\gamma+\theta) & \sin(\gamma+\theta) & l_2\sin(\gamma) + l_3\sin(\gamma+\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{G_e^s} = \begin{bmatrix} \sin(\alpha+\gamma+\theta) & \cos(\alpha+\gamma+\theta) & l_1\cos(\alpha) + l_2\cos(\alpha+\gamma) + l_3\cos(\alpha+\gamma+\theta) + x \\ -\cos(\alpha+\gamma+\theta) & \sin(\alpha+\gamma+\theta) & l_1\sin(\alpha) + l_2\sin(\alpha+\gamma) + l_3\sin(\alpha+\gamma+\theta) + y \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\langle \alpha, \gamma, \theta \rangle = \left\langle \frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{3} \right\rangle, \quad l_1 = l_2 = l_3 = 1, \quad x = y = 1$$

Substituting the above values into the result of 2(a):

$$\mathbf{G_e^s} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous coordinate representations:

$$\overline{\mathbf{p}}_{\mathbf{1}}^{\mathbf{e}} = \begin{bmatrix} \mathbf{p}_{\mathbf{1}}^{\mathbf{e}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \overline{\mathbf{p}}_{\mathbf{2}}^{\mathbf{e}} = \begin{bmatrix} \mathbf{p}_{\mathbf{2}}^{\mathbf{e}} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solve for: $\mathbf{p}_{\mathbf{1}}^{\mathbf{s}}$:

$$\overline{\mathbf{p}}_{\mathbf{1}}^{\mathbf{s}} = \mathbf{G}_{\mathbf{e}}^{\mathbf{s}} \cdot \overline{\mathbf{p}}_{\mathbf{1}}^{\mathbf{e}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ \frac{5+\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{p}_{\mathbf{1}}^{\mathbf{s}} = \begin{bmatrix} 2.573132185 \\ 3.207106781 \end{bmatrix}$$

Solve for: $\mathbf{p}_{\mathbf{2}}^{\mathbf{s}}$:

$$\overline{\mathbf{p}}_{\mathbf{2}}^{\mathbf{s}} = \mathbf{G}_{\mathbf{e}}^{\mathbf{s}} \cdot \overline{\mathbf{p}}_{\mathbf{2}}^{\mathbf{e}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5+\sqrt{2}}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \sqrt{3} + \frac{\sqrt{2}+\sqrt{3}}{2} + 1 \\ -\frac{\sqrt{3}}{2} + 1 + \frac{5+\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{p}_{\mathbf{2}}^{\mathbf{s}} = \begin{bmatrix} 4.805182993 \\ 3.341081377 \end{bmatrix}$$

(c)

$$\overline{\mathbf{p}}^{\mathbf{s}} = \mathbf{G}_{\mathbf{2}}^{\mathbf{s}} \cdot \overline{\mathbf{p}}^{\mathbf{2}}$$

Given the fact that point $\mathbf{p}$ is rigidly fixed to frame 2, the following holds true:

$$\dot{\overline{\mathbf{p}}}^{\mathbf{s}} = \dot{\mathbf{G}}_{\mathbf{2}}^{\mathbf{s}} \cdot \overline{\mathbf{p}}^{\mathbf{2}}$$

Where

$$\mathbf{G}_{\mathbf{2}}^{\mathbf{s}} = \mathbf{G}_{\mathbf{1}}^{\mathbf{s}} \cdot \mathbf{G}_{\mathbf{2}}^{\mathbf{1}} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & l_1 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting $x = 0, y = 0, l_1 = 1$ and multiplying through:

$$\mathbf{G}_{\mathbf{2}}^{\mathbf{s}} = \begin{bmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & \cos(\alpha) \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & \sin(\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

Thus:

$$\dot{\mathbf{G}}_{\mathbf{2}}^{\mathbf{s}} = \begin{bmatrix} -\sin(\alpha+\gamma)\left[\dot{\alpha}+\dot{\gamma}\right] & -\cos(\alpha+\gamma)\left[\dot{\alpha}+\dot{\gamma}\right] & -\sin(\alpha)\dot{\alpha} \\ \cos(\alpha+\gamma)\left[\dot{\alpha}+\dot{\gamma}\right] & -\sin(\alpha+\gamma)\left[\dot{\alpha}+\dot{\gamma}\right] & \cos(\alpha)\dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting $\langle \alpha, \gamma \rangle = \left\langle \frac{\pi}{4}, \frac{\pi}{4} \right\rangle$:

$$\mathbf{\dot{G}_2^s} = \begin{bmatrix} -\left[\dot{\alpha} + \dot{\gamma}\right] & 0 & -\frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & -\left[\dot{\alpha} + \dot{\gamma}\right] & \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix}$$

Given $\mathbf{\overline{p}^2} = \begin{bmatrix} x^{'} \\ y^{'} \\ 1 \end{bmatrix}$:

$$\mathbf{\dot{\overline{p}}^s} = \begin{bmatrix} -\left[\dot{\alpha} + \dot{\gamma}\right] & 0 & -\frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & -\left[\dot{\alpha} + \dot{\gamma}\right] & \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x^{'} \\ y^{'} \\ 1 \end{bmatrix} = \begin{bmatrix} -x^{'}\left[\dot{\alpha} + \dot{\gamma}\right] - \frac{\sqrt{2}}{2}\dot{\alpha} \\ -y^{'}\left[\dot{\alpha} + \dot{\gamma}\right] + \frac{\sqrt{2}}{2}\dot{\alpha} \\ 0 \end{bmatrix}$$

3

(a)

(b)

Outlining the kinematic equations for a two-wheeled robot:

$$\omega = \frac{r}{T} \cdot (u_r - u_l)$$

$$v = \frac{r}{2} \cdot (u_r + u_l)$$

Solving these equations based on 3(a).i.1:

$$\omega = 0\ \frac{rad}{s}, v = 1\ \frac{m}{s}, T = 0.2\ m, r = 0.1\ m$$

$$u_r = u_l = 10\ \frac{rad}{s}$$

Solving these equations based on 3(a).i.2:

$$\omega = 0.3\ \frac{rad}{s}, v = 0\ \frac{m}{s}, T = 0.2\ m, r = 0.1\ m$$

$$u_r = 0.3\ \frac{rad}{s}$$

$$u_l = -0.3\ \frac{rad}{s}$$

Solving these equations based on 3(a).i.3:

$$\omega = 0.3\ \frac{rad}{s}, v = 1\ \frac{m}{s}, T = 0.2\ m, r = 0.1\ m$$

$$u_r = 10.3\ \frac{rad}{s}$$

$$u_l = 9.7\ \frac{rad}{s}$$

If the input control parameters were changed from linear and angular velocities to wheel speeds, the state matrix would need to be changed in the implemented solution:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{r}{2}\cos(\theta) & \frac{r}{2}\cos(\theta) \\ \frac{r}{2}\sin(\theta) & \frac{r}{2}\sin(\theta) \\ \frac{r}{T} & -\frac{r}{T} \end{bmatrix} \cdot \begin{bmatrix} u_r \\ u_l \end{bmatrix}$$

4

I.

$$u_i = \frac{v^i_{drive}}{r_i} = \frac{1}{r_i}(v^i_x + v^i_y \tan\gamma_i) = \frac{1}{r_i}\begin{bmatrix}1 & \tan\gamma_i\end{bmatrix}\mathbf{g_i}(\theta)\dot{\mathbf{q}}$$

where $\mathbf{g_i}(\theta) = \begin{bmatrix}\cos(\theta+\beta_i) & \sin(\theta+\beta_i) & x_i\sin(\beta_i) - y_i\cos(\beta_i)\\ -\sin(\theta+\beta_i) & \cos(\theta+\beta_i) & x_i\cos(\beta_i) + y_i\sin(\beta_i)\end{bmatrix}, \quad \dot{\mathbf{q}} = \begin{bmatrix}\dot{x} & \dot{y} & \omega\end{bmatrix}^T$

For a robot with 3 wheels:

$$\mathbf{u} = \begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \mathbf{G}(\theta)\dot{\mathbf{q}} \tag{4}$$

Since all the Mecanum wheels have the same radius and roller orientation,

$$r_1 = r_2 = r_3 = r, \quad \gamma_1 = \gamma_2 = \gamma_3 = 0$$

Finding $\beta_i, x_i,$ and y_i for all 3 wheels

$$\beta_1 = \frac{\pi}{2}, \beta_2 = \frac{7\pi}{6}, \beta_3 = \frac{11\pi}{6}$$

$$(x_1, y_1) = (l, 0) \quad (x_2, y_2) = (l\cos(\frac{2\pi}{3}), l\sin(\frac{2\pi}{3})) \quad (x_3, y_3) = (l\cos(\frac{4\pi}{3}), l\sin(\frac{4\pi}{3}))$$

$$(x_1, y_1) = (l, 0) \quad (x_2, y_2) = (-\frac{l}{2}, \frac{\sqrt{3}}{2}l) \quad (x_3, y_3) = (-\frac{l}{2}, -\frac{\sqrt{3}}{2}l)$$

Expanding $\mathbf{g_{1,2,3}}(\theta)$

$$\mathbf{g_1}(\theta) = \begin{bmatrix}-\sin(\theta) & \cos(\theta) & l\\ -\cos(\theta) & -\sin(\theta) & 0\end{bmatrix}$$

$$\mathbf{g_2}(\theta) = \begin{bmatrix}\cos(\theta+\frac{7\pi}{6}) & \sin(\theta+\frac{7\pi}{6}) & -\frac{l}{2}\sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l\cos(\frac{7\pi}{6})\\ -\sin(\theta+\frac{7\pi}{6}) & \cos(\theta+\frac{7\pi}{6}) & -\frac{l}{2}\cos(\frac{7\pi}{6}) + \frac{\sqrt{3}}{2}l\sin(\frac{7\pi}{6})\end{bmatrix}$$

$$\mathbf{g_3}(\theta) = \begin{bmatrix}\cos(\theta+\frac{11\pi}{6}) & \sin(\theta+\frac{11\pi}{6}) & -\frac{l}{2}\sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l\cos(\frac{11\pi}{6})\\ -\sin(\theta+\frac{11\pi}{6}) & \cos(\theta+\frac{11\pi}{6}) & -\frac{l}{2}\cos(\frac{11\pi}{6}) - \frac{\sqrt{3}}{2}l\sin(\frac{11\pi}{6})\end{bmatrix}$$

Evaluating $\mathbf{G}(\theta)$, each row being equivalent to the first row of the minor $\mathbf{g_{1,2,3}}(\theta)$ matrix due to $\tan(\gamma_{1,2,3}) = 0$

$$\mathbf{G}(\theta) = \frac{1}{r}\begin{bmatrix}-\sin(\theta) & \cos(\theta) & l\\ \cos(\theta+\frac{7\pi}{6}) & \sin(\theta+\frac{7\pi}{6}) & -\frac{l}{2}\sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l\cos(\frac{7\pi}{6})\\ \cos(\theta+\frac{11\pi}{6}) & \sin(\theta+\frac{11\pi}{6}) & -\frac{l}{2}\sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l\cos(\frac{11\pi}{6})\end{bmatrix}$$

Full state space equation:

$$\mathbf{u} = \frac{1}{r}\begin{bmatrix} -\sin(\theta) & \cos(\theta) & l \\ \cos(\theta + \frac{7\pi}{6}) & \sin(\theta + \frac{7\pi}{6}) & -\frac{l}{2}\sin(\frac{7\pi}{6}) - \frac{\sqrt{3}}{2}l\cos(\frac{7\pi}{6}) \\ \cos(\theta + \frac{11\pi}{6}) & \sin(\theta + \frac{11\pi}{6}) & -\frac{l}{2}\sin(\frac{11\pi}{6}) + \frac{\sqrt{3}}{2}l\cos(\frac{11\pi}{6}) \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix}$$

II.

1) The $\dot{\mathbf{q}}$ of a straight line with a slope of 60 degrees is:

$$\dot{\mathbf{q}} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Using an initial state of $x_0 = 0[cm],\ y_0 = 0[cm],\ \theta_0 = 0[rad],\ l = 25[cm],\ r = 10[cm]$ and solving for $\mathbf{u}$

$$u_1 = u_2, \quad u_3 = 0$$

2) The $\dot{\mathbf{q}}$ of a circular path with a diameter of 2 m (radius = 100 cm) is:

$$\dot{\mathbf{q}} = \begin{bmatrix} -100\omega\sin(\omega t) \\ 100\omega\cos(\omega t) \\ \omega \end{bmatrix}$$