HW1 Suggested Answer

Problem 1

(a)

No. The variance of y_t is

$$Var(y_t) = \begin{cases} \sigma_1^2, & t < t' \\ \sigma_2^2, & t \geq t' \end{cases} .$$

Since $Var(y_t)$ depends on t, it is not weakly stationary.

(b)

No. The time series is not weakly stationary unless $\beta_0 = 0$. If $\beta_0 \neq 0$,

$$E(y_t) = \beta_0 t$$

It depends on t when $\beta_0 \neq 0$, thus, the time series is not weakly stationary when $\beta_0 \neq 0$. If $\beta_0 = 0$, $y_t = \varepsilon_t$, it is apparent that the series is weakly stationary.

(c)

Yes.

$$\begin{aligned} E(y_t^i) &= E[E[\varepsilon_t^i|\sigma_i^2 = \eta_i^2]] = 0 \\ Var(y_t^i) &= E[E[(\varepsilon_t^i)^2|\sigma_i^2 = \eta_i^2]] = E[\eta_i^2] = 1 \\ E[y_t^i, y_{t-j}^i] &= E[E[\varepsilon_t^i \varepsilon_{t-j}^i|\sigma_i^2 = \eta_i^2]] = 0 \qquad \text{for } j \geq 1 \end{aligned}$$

All of them does not depend on t, thus, the series is weakly stationary.

Problem 2

(a)

Below is ACF of raw growth rates with lag 48. There is a clear seasonal pattern in the ACF. For instance, Lag 12 and Lag 24 have large correlation.

Date: 01/14/24 Time: 09:59
Sample: 2000M01 2023M10
Included observations: 285

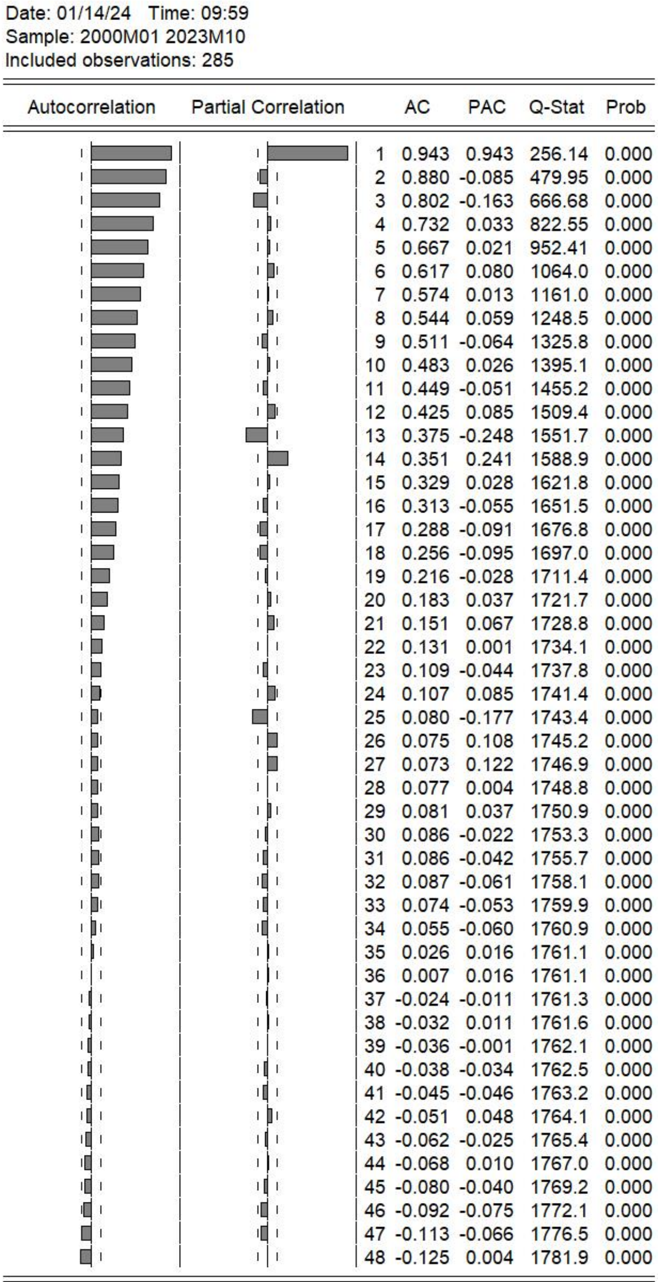
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.913	0.913	239.89	0.000
		2	0.726	-0.640	392.24	0.000
		3	0.514	0.117	468.94	0.000
		4	0.331	0.080	500.85	0.000
		5	0.203	0.042	512.83	0.000
		6	0.141	0.130	518.64	0.000
		7	0.150	0.181	525.24	0.000
		8	0.220	0.155	539.57	0.000
		9	0.333	0.169	572.38	0.000
		10	0.466	0.217	637.00	0.000
		11	0.574	-0.078	735.45	0.000
		12	0.596	-0.328	841.84	0.000
		13	0.506	-0.222	918.95	0.000
		14	0.349	0.122	955.78	0.000
		15	0.183	0.046	965.87	0.000
		16	0.044	-0.040	966.47	0.000
		17	-0.046	-0.015	967.12	0.000
		18	-0.080	-0.014	969.10	0.000
		19	-0.058	0.012	970.12	0.000
		20	0.013	0.030	970.17	0.000
		21	0.119	0.051	974.55	0.000
		22	0.237	0.014	992.08	0.000
		23	0.328	-0.005	1025.6	0.000
		24	0.341	-0.072	1062.1	0.000
		25	0.258	-0.166	1083.0	0.000
		26	0.123	0.124	1087.8	0.000
		27	-0.012	0.033	1087.8	0.000
		28	-0.116	-0.038	1092.1	0.000
		29	-0.169	0.101	1101.2	0.000
		30	-0.166	0.049	1110.0	0.000
		31	-0.110	-0.002	1113.9	0.000
		32	-0.021	-0.055	1114.1	0.000
		33	0.084	-0.011	1116.4	0.000
		34	0.184	-0.020	1127.4	0.000
		35	0.250	0.033	1147.9	0.000
		36	0.241	-0.044	1167.1	0.000
		37	0.155	-0.037	1175.0	0.000
		38	0.031	0.021	1175.3	0.000
		39	-0.091	-0.078	1178.0	0.000
		40	-0.188	-0.105	1189.8	0.000
		41	-0.243	-0.027	1209.6	0.000
		42	-0.242	0.098	1229.4	0.000
		43	-0.192	-0.035	1241.8	0.000
		44	-0.105	0.078	1245.6	0.000
		45	-0.002	-0.015	1245.6	0.000
		46	0.094	-0.049	1248.6	0.000
		47	0.148	-0.038	1256.1	0.000
		48	0.133	0.011	1262.3	0.000

(b)

No. AR(1) model cannot be a good fit for the raw data because of the clear seasonal pattern of the data. The ACF of stationary AR(1) model should decreases exponentially.

(c)

Below is the ACF of adjusted rates. The seasonal pattern disappears since this data is seasonally adjusted. The season factor is excluded in the data.



Date: 01/14/24 Time: 09:59
Sample: 2000M01 2023M10
Included observations: 285

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.943	0.943	256.14	0.000
		2	0.880	-0.085	479.95	0.000
		3	0.802	-0.163	666.68	0.000
		4	0.732	0.033	822.55	0.000
		5	0.667	0.021	952.41	0.000
		6	0.617	0.080	1064.0	0.000
		7	0.574	0.013	1161.0	0.000
		8	0.544	0.059	1248.5	0.000
		9	0.511	-0.064	1325.8	0.000
		10	0.483	0.026	1395.1	0.000
		11	0.449	-0.051	1455.2	0.000
		12	0.425	0.085	1509.4	0.000
		13	0.375	-0.248	1551.7	0.000
		14	0.351	0.241	1588.9	0.000
		15	0.329	0.028	1621.8	0.000
		16	0.313	-0.055	1651.5	0.000
		17	0.288	-0.091	1676.8	0.000
		18	0.256	-0.095	1697.0	0.000
		19	0.216	-0.028	1711.4	0.000
		20	0.183	0.037	1721.7	0.000
		21	0.151	0.067	1728.8	0.000
		22	0.131	0.001	1734.1	0.000
		23	0.109	-0.044	1737.8	0.000
		24	0.107	0.085	1741.4	0.000
		25	0.080	-0.177	1743.4	0.000
		26	0.075	0.108	1745.2	0.000
		27	0.073	0.122	1746.9	0.000
		28	0.077	0.004	1748.8	0.000
		29	0.081	0.037	1750.9	0.000
		30	0.086	-0.022	1753.3	0.000
		31	0.086	-0.042	1755.7	0.000
		32	0.087	-0.061	1758.1	0.000
		33	0.074	-0.053	1759.9	0.000
		34	0.055	-0.060	1760.9	0.000
		35	0.026	0.016	1761.1	0.000
		36	0.007	0.016	1761.1	0.000
		37	-0.024	-0.011	1761.3	0.000
		38	-0.032	0.011	1761.6	0.000
		39	-0.036	-0.001	1762.1	0.000
		40	-0.038	-0.034	1762.5	0.000
		41	-0.045	-0.046	1763.2	0.000
		42	-0.051	0.048	1764.1	0.000
		43	-0.062	-0.025	1765.4	0.000
		44	-0.068	0.010	1767.0	0.000
		45	-0.080	-0.040	1769.2	0.000
		46	-0.092	-0.075	1772.1	0.000
		47	-0.113	-0.066	1776.5	0.000
		48	-0.125	0.004	1781.9	0.000

(d)

Below is the summary of AR(1) model.

Dependent Variable: R1
Method: Least Squares
Date: 01/14/24 Time: 11:08
Sample (adjusted): 2000M03 2023M10
Included observations: 284 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.000214	0.000179	1.196775	0.2324
R1(-1)	0.943315	0.019567	48.21049	0.0000

R-squared	0.891799	Mean dependent var	0.004040
Adjusted R-squared	0.891415	S.D. dependent var	0.008188
S.E. of regression	0.002698	Akaike info criterion	-8.985419
Sum squared resid	0.002053	Schwarz criterion	-8.959722
Log likelihood	1277.929	Hannan-Quinn criter.	-8.975116
F-statistic	2324.251	Durbin-Watson stat	1.837601
Prob(F-statistic)	0.000000		

(e)

Below is the ACF of residuals.

Date: 01/14/24 Time: 11:16
Sample: 2000M01 2023M10
Included observations: 284

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.081	0.081	1.8918	0.169
		2	0.147	0.141	8.1011	0.017
		3	-0.040	-0.064	8.5705	0.036
		4	-0.024	-0.039	8.7421	0.068
		5	-0.101	-0.084	11.718	0.039
		6	-0.042	-0.022	12.236	0.057
		7	-0.091	-0.064	14.674	0.040
		8	0.047	0.061	15.331	0.053
		9	-0.041	-0.036	15.837	0.070
		10	0.069	0.045	17.234	0.069
		11	-0.083	-0.091	19.297	0.056
		12	0.248	0.246	37.665	0.000
		13	-0.210	-0.254	50.942	0.000
		14	-0.001	-0.019	50.943	0.000
		15	-0.034	0.054	51.296	0.000
		16	0.068	0.069	52.683	0.000
		17	0.061	0.072	53.816	0.000
		18	0.072	0.008	55.407	0.000
		19	-0.052	-0.060	56.247	0.000
		20	-0.015	-0.080	56.316	0.000
		21	-0.085	0.000	58.548	0.000
		22	0.025	0.030	58.746	0.000
		23	-0.167	-0.110	67.456	0.000
		24	0.205	0.164	80.604	0.000
		25	-0.187	-0.136	91.576	0.000
		26	-0.036	-0.119	91.981	0.000
		27	-0.048	0.000	92.708	0.000
		28	-0.013	-0.026	92.760	0.000
		29	-0.000	0.016	92.760	0.000
		30	0.048	0.044	93.493	0.000
		31	0.008	0.069	93.514	0.000
		32	0.114	0.034	97.684	0.000
		33	0.064	0.054	99.012	0.000
		34	0.078	-0.030	101.00	0.000
		35	-0.080	-0.012	103.11	0.000
		36	0.102	-0.014	106.51	0.000

(f)

No. The residuals are possibly correlated. Lag 12 and Lag 13 are highly correlated. Hence, AR(1) model is not a good fit.

(g)

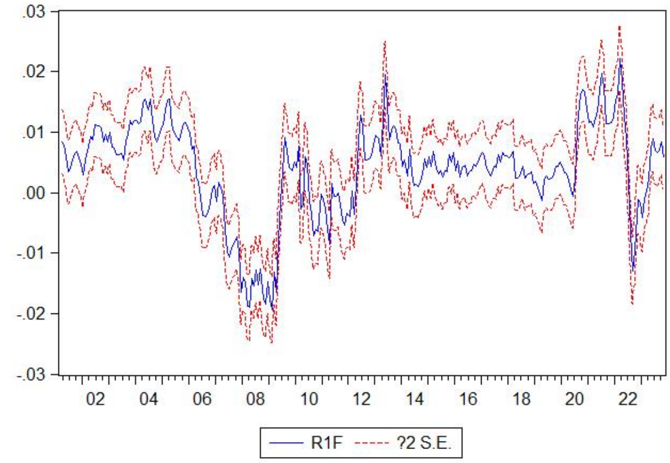
We fit an AR(13) model. Below is the ACF of residuals. We can see that only Lag 24 has slightly larger correlation. Based on residual diagnostics, AR(13) model fits the data better.

Date: 01/17/20 Time: 07:47
Sample: 2000M02 2019M10
Included observations: 237

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.066	0.066	1.0462	0.306
		2	0.022	0.018	1.1676	0.558
		3	-0.001	-0.003	1.1677	0.761
		4	-0.017	-0.017	1.2360	0.872
		5	-0.028	-0.026	1.4248	0.922
		6	-0.012	-0.008	1.4601	0.962
		7	0.009	0.011	1.4797	0.983
		8	0.014	0.013	1.5307	0.992
		9	-0.006	-0.009	1.5396	0.997
		10	-0.013	-0.014	1.5837	0.999
		11	0.039	0.041	1.9661	0.999
		12	-0.067	-0.071	3.0794	0.995
		13	-0.114	-0.108	6.3826	0.931
		14	-0.028	-0.012	6.5767	0.950
		15	0.016	0.024	6.6426	0.967
		16	0.053	0.052	7.3588	0.966
		17	0.094	0.083	9.6230	0.919
		18	0.048	0.028	10.217	0.925
		19	-0.025	-0.037	10.374	0.943
		20	-0.051	-0.047	11.061	0.945
		21	-0.030	-0.016	11.293	0.957
		22	0.022	0.029	11.426	0.968
		23	-0.050	-0.048	12.080	0.969
		24	0.215	0.231	24.427	0.437
		25	-0.068	-0.118	25.678	0.425
		26	-0.123	-0.151	29.764	0.278
		27	-0.067	-0.058	30.983	0.272
		28	-0.032	-0.012	31.262	0.306
		29	0.034	0.081	31.586	0.338
		30	-0.007	0.027	31.601	0.386
		31	0.057	0.069	32.488	0.393
		32	0.089	0.058	34.698	0.341
		33	0.130	0.093	39.410	0.205
		34	0.032	0.008	39.702	0.231
		35	0.033	-0.012	40.014	0.257
		36	0.000	0.031	40.014	0.297
		37	-0.169	-0.118	48.095	0.105
		38	-0.039	-0.034	48.519	0.118
		39	0.063	0.036	49.657	0.118
		40	0.100	0.071	52.551	0.088
		41	-0.063	-0.115	53.704	0.088
		42	0.026	0.049	53.898	0.103
		43	-0.058	-0.020	54.880	0.106
		44	-0.048	-0.002	55.548	0.114
		45	-0.011	0.047	55.585	0.134
		46	0.069	0.076	56.999	0.128
		47	-0.012	-0.006	57.040	0.150
		48	-0.018	-0.100	57.137	0.172

(h)

We plot the fitted value against the real data.



Forecast: R1F	
Actual: R1	
Forecast sample: 2000M01 2023M11	
Adjusted sample: 2001M03 2023M11	
Included observations: 273	
Root Mean Squared Error	0.002557
Mean Absolute Error	0.001831
Mean Abs. Percent Error	64.47299
Theil Inequality Coefficient	0.143578
Bias Proportion	0.000000
Variance Proportion	0.025192
Covariance Proportion	0.974808
Theil U2 Coefficient	0.631151
Symmetric MAPE	39.87664

(i)

One step ahead out of sample forecast is 0.0058 and the 95% prediction interval is $[0.0005, 0.0111]$.

Problem 3

(a)

From

$$Y_t = 0.1 + 0.99 Y_{t-1} + \varepsilon_t$$

we have

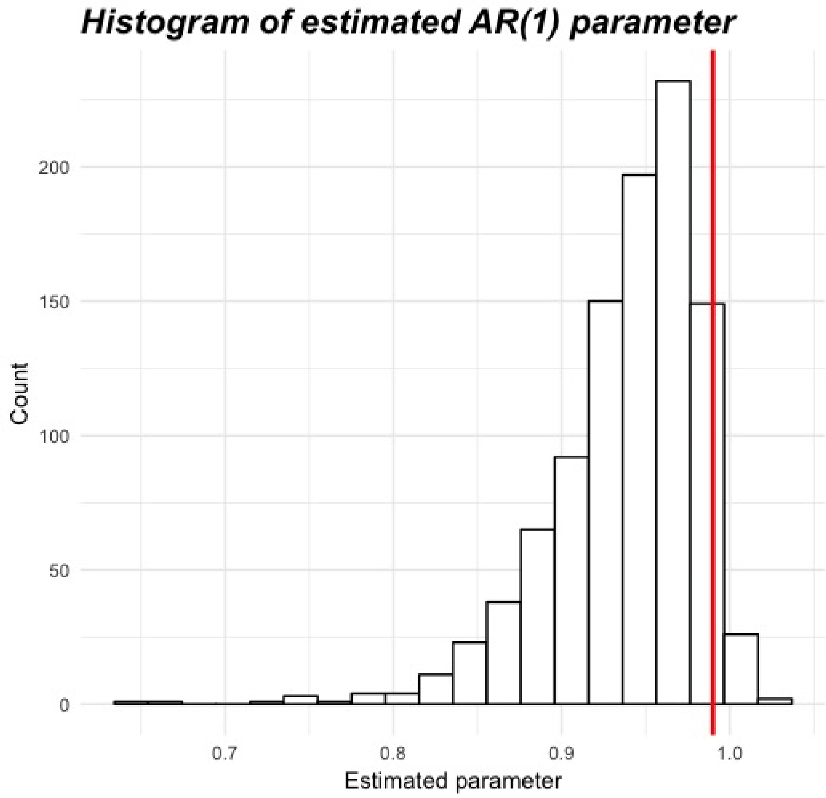
$$\begin{aligned} \mu_Y &= 0.1 + 0.99 \mu_Y \\ \sigma_Y^2 &= 0.99^2 \sigma_Y^2 + 0.5^2 \end{aligned}$$

Hence

$$\mu_Y = 10, \quad \sigma_Y^2 = \frac{2500}{199}$$

(b)

The histogram is shown below.



The red line represents the true value, 0.99.

(c)

The mean value of our estimated parameter is 0.9386, which is smaller than the true value (0.99). We can conclude that the least squares estimator is biased for AR(1) model and the finite sample bias makes the mean of estimator shrinks to zero in this case.

Problem 4

(a)

Take expectation and get

$$\mu = \frac{0.5}{1 - 1.3 + 0.45} = \frac{10}{3}$$

Write Yule–Walker equations

$$\begin{cases} \gamma_0 & = 1.3\gamma_1 - 0.45\gamma_2 + 0.5^2 \\ \gamma_1 & = 1.3\gamma_0 - 0.45\gamma_1 \\ \gamma_2 & = 1.3\gamma_1 - 0.45\gamma_0 \end{cases}$$

The solution is

$$\gamma_0 = \frac{580}{363}, \quad \gamma_1 = \frac{520}{363}, \quad \gamma_2 = \frac{415}{363}$$

(b)

The lag polynomial is

$$\phi(L) = 1 - 1.3L + 0.45L^2$$

The roots are

$$L = \frac{1}{9}(13 \pm \sqrt{11}i)$$

This process is covariance stationary as the inverse of roots lie inside the unit circle.

(c)

Yes. ε_t are Gaussian implies that y_t follows Gaussian distribution. Thus, for any $k > 0$, the joint distribution of $(y_t, \cdots, y_{t-k})$ follows a multivariate Gaussian distribution with mean $(\mu, \cdots, \mu)^T$ and covariance matrix Σ, where $\Sigma_{ij} = \gamma_{|i-j|}$. Hence, the process is strong stationary.

(d)

Let

$$B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1.3 & -0.45 \\ 1 & 0 \end{bmatrix}, \quad \nu_t = \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$$

Then, write companion form vector AR(1) representation

$$\xi_t = B + F\xi_{t-1} + \nu_t$$

(e)

The eigenvalues are roots of

$$\lambda^2 - 1.3\lambda + 0.45 = 0$$

Hence, eigenvalues of F are

$$\lambda_1 = \frac{1}{20}(13 + \sqrt{11}i), \quad \lambda_2 = \frac{1}{20}(13 - \sqrt{11}i) \tag{1}$$

Then, we can solve eigenvectors from $F - \lambda_i I = 0$ for $i = 1, 2$. The result is

$$u_1 = \begin{bmatrix} \frac{1}{20}(13 + \sqrt{11}i) \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} \frac{1}{20}(13 - \sqrt{11}i) \\ 1 \end{bmatrix}$$

(f)

From the result of (e), we have

$$F = U\Lambda U^{-1}$$

where

$$U = \begin{bmatrix} \frac{1}{20}(13 + \sqrt{11}i) & \frac{1}{20}(13 - \sqrt{11}i) \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The kth IRF is determined by the (1,1) element of F^k. Using $F^k = U\Lambda^k U^{-1}$, kth IRF is given by

$$F^k(1,1) = -\frac{10}{\sqrt{11}}i \times \lambda_1^{k+1} + \frac{10}{\sqrt{11}}i \times \lambda_2^{k+1}$$

λ_1, λ_2 are given in part (e).