UNIVERSITY OF MINES AND TECHNOLOGY

FACULTY OF ENGINEERING

DEPARTMENT OF MATHEMATICAL SCIENCES

LECTURE NOTES ON

CALCULUS (EL/MC/CE/RN 166)

Course Instructors

Dr Lewis Brew, Dr Eric N. Wiah, Dr Paul Boye and K. Agyarko

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CHAPTER ONE

INTRODUCTION TO CALCULUS

1.0 LIMITS

In calculus and its applications, we investigate the how the quantities vary, and whether they approach specific values under certain conditions. The quantities usually involve function values. The definition of derivative depends on the notion of the limit of a function. The concept of limit of a function is the one of the fundamental ideas that distinguishes calculus from the areas of mathematics such as algebra, trigonometry, etc.

Let a function f be defined throughout an open interval containing a real number *a*, except possibly at *a* itself. Let consider a function value *f(x)* which get closer to some number *L* when *x* gets closer and closer to *a* but not necessarily equal to *a*. We say that *f(x)* approaches *L* as *x* approaches *a* or *f(x)* has the limit *L* as *x* approaches *a* and represent by the notation

$$\lim_{x \to a} f(x) = L$$

Definition: Let $f(x)$ be defined in an open interval about a except possibly a itself. We say that the limit of $f(x)$ as x approaches a is L and denote by

$$\lim_{x \to a} f(x) = L$$

If for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$. The rational function *f* defined by $f(x) = \frac{1}{x}$ has no limit as *x* approaches 0.

Theorem 1: If *m*, *a* and *b* are any real numbers, then $\lim_{x \to a}(mx+b) = ma+b$

Example: If $f(x) = \frac{3x^2 - 8x + 3}{5x+1}$ what is $\lim_{x \to 2} f(x)$?

$$f(2) = \frac{3(2)^2 - 8(2) + 3}{5(2) + 1} = \frac{-1}{11}$$

Practice problems

Find 1. $\lim_{x \to 3} \left(4x^2 + 6x - 2\right)$

2. $\lim_{x \to 1/2} \frac{4x^2 - 6x + 3}{16x^3 + 8x - 7}$

3. $\lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$

4. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

5. $\lim_{x \to 2} \frac{x - 2}{x^3 - 8}$

6. $\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3}$

If f is a rational function, then the limits as $x \to \infty$ or $x \to -\infty$ may be found by first dividing numerator and denominator of $f(x)$ by suitable power of x and then applying the limit.

Example: Find $\lim_{x \to \infty} \frac{3x^2 - 9}{5x^2 + 2x - 6}$

$$\lim_{x \to \infty} \frac{\frac{3x^2}{x^2} - \frac{9}{x^2}}{\frac{5x^2}{x^2} + \frac{2x}{x^2} - \frac{6}{x^2}} = \lim_{x \to \infty} \frac{3 - \frac{9}{x^2}}{5 + \frac{2}{x} - \frac{6}{x^2}} = \lim_{x \to \infty} \frac{3 - 0}{5 + 0 - 0} = 3/5$$

Practice Problems

Find the limit

$$\lim_{x \to \infty} \frac{7x^2 - 4x + 1}{6x^2 + 2x - 5}$$

$$\lim_{x \to \infty} \frac{3x^2 - x + 7}{x^3 + 2}$$

One – Sided Limits

i. Left-hand Limit

Let $f(x)$ be defined in an open interval. The statement $\lim\limits_{x \to a^-} f(x) = L_1$ is the left hand limit and x approaches a from the left with $x < a$.

ii. Right-hand Limit

Let $f(x)$ be defined in an open interval. The statement $\lim\limits_{x \to a^+} f(x) = L_2$ is the right hand limit and x approaches a from the right with $x > a$.

Example: If $f(x) = \begin{cases} 5 - x & for \quad x < 1 \\ 2x^2 + 1 & for \quad x > 1 \end{cases}$ Find $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1^-} f(x)$

Solution: $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \left(2x^2 + 1\right) = 3$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (5 - x) = 4$

If $f(x) = \frac{|x|}{x}$ show that the $\lim\limits_{x \to 0} f(x) = exists$

Solution: If $x > 0$, then $\lim\limits_{x \to 0} \frac{|x|}{x} = \frac{x}{x} = 1$

If $x < 0$, then $\lim\limits_{x \to 0} \frac{|x|}{x} = \frac{-x}{x} = -1$

Since the left-hand and right-hand limits are different, it follows that the limit does not exist.

Practice Problems

Find the limit, if it exists

1. $\lim\limits_{x \to 4^-} \dfrac{|x-4|}{x-4}$

2. $\lim\limits_{x \to 4^+} \dfrac{|x-4|}{x-4}$

3. $\lim\limits_{x \to 4} \dfrac{|x-4|}{x-4}$

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ then:

i. $\lim\limits_{x \to a} \left[f(x) + g(x)\right] = L + M$

ii. $\lim\limits_{x \to a} \left[f(x) \cdot g(x)\right] = L \cdot M$

iii. $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{M}$ provided $M \neq 0$

iv. $\lim\limits_{x \to a} \left[f(x) - g(x)\right] = L - M$

v. $\lim\limits_{x \to a} \left[cf(x)\right] = cL$ for every real c

Limits of Trigonometric Functions

1. $\lim\limits_{x \to 0} \sin x = 0$

2. $\lim\limits_{x \to 0} \cos x = 1$

3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or $\lim_{x \to 0} \frac{x}{\sin x} = 1$

Example: Find $\lim_{x \to 0} \frac{\sin 7x}{3x}$

Solution: $\lim_{x \to 0} \frac{\sin 7x}{3x} = \lim_{x \to 0} \frac{1}{3} \frac{\sin 7x}{x} = \lim_{x \to 0} \frac{7}{3} \frac{\sin 7x}{7x} = \frac{7}{3} \lim_{x \to 0} \frac{\sin 7x}{7x} = \frac{7}{3} x 1 = \frac{7}{3}$

Practice Problems

Find the limit, if it exists

1. $\lim_{x \to 0} \frac{1 - \cos 3x}{x}$

2. $\lim_{x \to 0} \frac{2 + \sin 3x}{3 + x}$

3. $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

4. $\lim_{x \to 0} \frac{2x + 1 - \cos x}{3x}$

CHAPTER TWO

DIFFERENTIATION

The process of determining the derivative of a function or how quickly or slowly a function varies as the quantity on which it depends, the argument changed is known as differentiation. Specifically, it is the procedure for obtaining an expression (numerical or algebraic) for the rate of change of the function with respect to its argument. Familiar examples of rates of change include acceleration (the rate of change of velocity) and the rate of chemical reaction (rate of change of chemical composition). These two examples give a measure of the change of a quantity with respect to time. However, differentiation may also be applied to changes with respect to other quantities, for example change in pressure with respect to a change in temperature. Although it will not be apparent from what we have said so far, differentiation is in fact a limiting process. That is, it deals only with the infinitesimal change in one quantity resulting from an infinitesimal change in another.

Definition: If a function $f(x)$ is defined over an interval $[a, b]$, then taking a point $x = x_0$ in the interval, the change in value of $f(x)$ at a different point $x = x_1$ with respect to the value at $x = x_0$ divided by the change in $x = x_1$ with respect to $x = x_0$ is called the average rate of change of the function $f(x)$ with respect to x_0. That is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

where $\Delta f = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$ and $\Delta x = x_1 - x_0 = (x_0 + \Delta x) - x_0$

The exact rate of change at $x = x_0$ is given by:

$$f^{/}(x_0) = \frac{df(x)}{dx}$$

$$= \lim_{\Delta x \to 0} it \left(\frac{\Delta f}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} it \left[\frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right]$$

Provided that the limit exists where $f^{/}(x), \frac{df(x)}{dx}$ are the notations for the derivatives.

By computation, one may calculate $\frac{\Delta f}{\Delta x}$ to a given decimal and obtain the values

$$\frac{\Delta f_1}{\Delta x_1}, \frac{\Delta f_2}{\Delta x_2}, \frac{\Delta f_3}{\Delta x_3}, \frac{\Delta f_4}{\Delta x_4}, \cdots, \cdots, \cdots, \frac{\Delta f_n}{\Delta x_n} \text{ as } \Delta x_n \to 0$$

The limiting value to the decimal is $\frac{df}{dx}$ at $x = x_0$. The actual rate of change of $f(x)$ with respect to $x = x_0$ is called the derivative of $f(x)$ with respect to $x = x_0$. The use of this process in determining the derivative is called the first principle.

2.1 DIFFERENTIATION FROM FIRST PRINCIPLE

The derivative of a function $f(x) = x^n$ where n is a real number can be obtained from the first principle by the following considerations:

$$\frac{\Delta f}{\Delta x} = \left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

Taking the limits, we obtain

$$\frac{df}{dx} = \lim_{\Delta x \to 0} it\left(\frac{\Delta f}{\Delta x}\right) = \lim_{\Delta x \to 0} it\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

$$\frac{\Delta f}{\Delta x} = \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)\,\mathrm{x}^{n-3}(\Delta\,\mathrm{x})^3}{3!} + \cdots\cdots\cdots + \frac{n(n-1)(n-2)\cdots\cdots\cdots 1}{n!}(\Delta x)^n - x^n}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)\,\mathrm{x}^{n-3}(\Delta\,\mathrm{x})^3}{3!} + \cdots\cdots\cdots + \frac{n(n-1)(n-2)\cdots\cdots\cdots 1}{n!}(\Delta x)^n}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(\Delta x) + \frac{n(n-1)(n-2)\,\mathrm{x}^{n-3}(\Delta\,\mathrm{x})^2}{3!} + \cdots\cdots\cdots + \frac{n(n-1)(n-2)\cdots\cdots\cdots 1}{n!}(\Delta x)^{n-1}$$

Taking the limits, we obtain

$$\lim_{\Delta x \to 0} it\left(\frac{\Delta f}{\Delta x}\right) = \lim_{\Delta x \to 0} it\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(\Delta x) + \cdots\cdots\cdots + \frac{n(n-1)(n-2)\cdots\cdots\cdots 1}{n!}(\Delta x)^{n-1}\right]$$

But $\lim_{\Delta x \to 0} it\left(\frac{\Delta f}{\Delta x}\right) = \frac{df}{dx}$

$$\Rightarrow \frac{df}{dx} = \lim_{\Delta x \to 0} it\left(\frac{\Delta f}{\Delta x}\right) = nx^{n-1}$$

In general, it could be deduced that for a function $f(x) = x^n$

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

For a negative index, thus if $n = -m$, then the function $f(x) = x^{-m}$

$$\therefore \quad \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \left[\frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x}\right]$$

Expanding and taking limits as $\Delta x \to 0$, we obtain:

$$\frac{df}{dx} = -mx^{-m-1} = \frac{-m}{x^{m+1}}$$

Note that for $f(x) = kx^n$ where k is a constant, the derivative

$$\frac{d}{dx} f(x) = k\frac{d}{dx}\left[x^n\right] = knx^{n-1}$$

Example. Find from first principle the derivative with respect to x of $f(x) = x^2$.

Solution: By definition $f^{/}(x) = \lim_{\Delta x \to 0} it\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} it\left[\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right] = \lim_{\Delta x \to 0} it\left[\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[2x + \Delta x\right]$$

As Δx tends to zero, $2x + \Delta x$ tends towards $2x$. Hence $f^{/}(x) = 2x$

There are other forms of functions where application from first principle is possible. These include the product of two or more functions or the quotient of two factors.

2.1.1 Differentiation of Products from First Principle

Theorem: Given the differentiable functions $f(x)$ and $g(x)$, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx}$$

Proof: Let $h(x) = f(x)g(x)$ then $h(x+\Delta x) = f(x+\Delta x)g(x+\Delta x)$

By definition $h^{/}(x) = \lim_{\Delta x \to 0} it\left[\frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} it \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} \right] + \lim_{\Delta x \to 0} it \left[\frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[f(x+\Delta x) \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \right] + \lim_{\Delta x \to 0} it \left[g(x) \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \right]$$

$$= \lim_{\Delta x \to 0} it\, f(x+\Delta x) \lim_{\Delta x \to 0} it \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] + g(x) \lim_{\Delta x \to 0} it \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

In the limit as $\Delta x \to 0$, the factors in the square brackets become $\frac{dg}{dx}$ and $\frac{df}{dx}$ (by the definitions of these quantities) and $f(x+\Delta x)$ simply becomes $f(x)$. Consequently, we obtain:

$$\frac{dh}{dx} = \frac{d}{dx}\left[f(x)g(x) \right] = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx} \quad \cdots\cdots\cdots\cdots \quad (1)$$

This is the general results obtained without making any assumptions about the specific forms h, f and g other than the fact that $h(x) = f(x)g(x)$. In words, the results above could be read as:

The derivative of the product of two functions is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.

The product rule may readily be extended to the product of three or more functions. Considering the function $f(x) = u(x)v(x)w(x)$. By equation (1), we obtain

$$\frac{df(x)}{dx} = u(x)\frac{d}{dx}\left[v(x)w(x) \right] + v(x)w(x)\frac{d}{dx}u(x)$$

Using equation (1) to expand the first term of the right-hand side, we obtain:

$$\frac{df(x)}{dx} = u(x)\left[v(x)\frac{d}{dx}w(x) + w(x)\frac{d}{dx}v(x) \right] + v(x)w(x)\frac{d}{dx}u(x)$$

$$= u(x)v(x)\frac{d}{dx}w(x) + u(x)w(x)\frac{d}{dx}v(x) + v(x)w(x)\frac{d}{dx}u(x)$$

$$\therefore \quad \frac{df(x)}{dx} = \frac{d}{dx}\left[u(x)v(x)w(x) \right] = u(x)v(x)\frac{d}{dx}w(x) + u(x)w(x)\frac{d}{dx}v(x) + v(x)w(x)\frac{d}{dx}u(x)$$

This method can be extended to products containing any number of factors n and that the expression for the derivative will consist of n terms with the prime appearing in the successive terms on each of the n-factors in turn.

Example. Find from first principles the derivative with respect to x of the function $f(x)=(x^2+1)(2x-1)$

Solution: If $f(x)=(x^2+1)(2x-1)$, then $f(x+\Delta x)=\left[(x+\Delta x)^2+1\right]\left[2(x+\Delta x)-1\right]$

By definition

$$f^{/}(x)=\lim_{\Delta x\to 0} it\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\left\{(x+\Delta x)^2+1\right\}\left\{2(x+\Delta x)-1\right\}-(x^2+1)(2x-1)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\left\{\left(x^2+2x\Delta x+(\Delta x)^2\right)+1\right\}\left\{2x+2\Delta x-1\right\}-\left(2x^3-x^2+2x-1\right)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\left\{2x^2\Delta x+4x^2\Delta x+4x(\Delta x)^2-2x\Delta x+2x(\Delta x)^2+2(\Delta x)^3-(\Delta x)^2+2\Delta x\right\}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[2x^2+4x^2+4x\Delta x-2x+2x\Delta x+2(\Delta x)^2-\Delta x+2\right]$$

As Δx tends to zero, $\left[2x^2+4x^2+4x\Delta x-2x+2x\Delta x+2(\Delta x)^2-\Delta x+2\right]$ tends to $\left(2x^2+4x^2-2x+2\right)$.

Hence $f^{/}(x)=6x^2-2x+2$.

This is analogous to the application of $\frac{dh}{dx}=\frac{d}{dx}\left[f(x)g(x)\right]=g(x)\frac{d}{dx}\left[f(x)\right]+f(x)\frac{d}{dx}\left[g(x)\right]$

$$=\left(x^2+1\right)\frac{d}{dx}(2x-1)+(2x-1)\frac{d}{dx}\left(x^2+1\right)$$

$$=\left(x^2+1\right)(2)+(2x-1)(2x)$$

$$=2x^2+2+4x^2-2x$$

$$=6x^2-2x+2$$

2.1.2 Differentiation of Quotients from First Principle

Theorem: Given the differentiable functions $f(x)$ and $g(x)$ then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

Proof: Let $h(x)=\frac{f(x)}{g(x)}$ then $h(x+\Delta x)=\frac{f(x+\Delta x)}{g(x+\Delta x)}$

By definition $h^{/}(x)=\lim_{\Delta x\to 0}it\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]$

$$=\lim_{\Delta x\to 0}it\left[\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}it\left[\frac{\frac{g(x)f(x+\Delta x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}it\left[\frac{g(x)f(x+\Delta x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}it\left[\frac{g(x)f(x+\Delta x)-f(x)g(x)+f(x)g(x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}it\left[\frac{g(x)\{f(x+\Delta x)-f(x)\}-f(x)\{g(x+\Delta x)-g(x)\}}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}it\left[\frac{g(x)\{f(x+\Delta x)-f(x)\}}{g(x+\Delta x)g(x)\Delta x}\right]-\lim_{\Delta x\to 0}it\left[\frac{f(x)\{g(x+\Delta x)-g(x)\}}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$=\frac{\lim_{\Delta x\to 0}it\left[\frac{g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x}\right]}{\lim_{\Delta x\to 0}it\,g(x+\Delta x)g(x)}-\frac{\lim_{\Delta x\to 0}it\left[\frac{f(x)\{g(x+\Delta x)-g(x)\}}{\Delta x}\right]}{\lim_{\Delta x\to 0}it\,g(x+\Delta x)g(x)}$$

$$=\frac{g(x)\lim_{\Delta x\to 0}it\left[\frac{\{f(x+\Delta x)-f(x)\}}{\Delta x}\right]}{\lim_{\Delta x\to 0}it\,g(x+\Delta x)g(x)}-\frac{f(x)\lim_{\Delta x\to 0}it\left[\frac{\{g(x+\Delta x)-g(x)\}}{\Delta x}\right]}{\lim_{\Delta x\to 0}it\,g(x+\Delta x)g(x)}$$

$$=\frac{g(x)\frac{d}{dx}f(x)}{[g(x)]^2}-\frac{f(x)\frac{d}{dx}g(x)}{[g(x)]^2}=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

Hence $$\frac{d}{dx}h(x)=\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

In a situation where $f(x)=1$ then $h(x)=\frac{f(x)}{g(x)}=\frac{1}{g(x)}$

Hence $$\frac{d}{dx}h(x)=\frac{d}{dx}\left[\frac{1}{g(x)}\right]=\frac{g(x)\frac{d}{dx}(1)-(1)\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

But $\frac{d}{dx}[k]=0$ where k is a constant

$$\Rightarrow \quad \frac{d}{dx}h(x)=\frac{d}{dx}\left[\frac{1}{g(x)}\right]=\frac{g(x)[0]-(1)\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

$$=-\frac{\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

Example. Find from first principle the derivative with respect to x of the function $h(x)=\frac{5x^2-x+7}{x^2+x+1}$

Solution: If $h(x)=\frac{f(x)}{g(x)}$ then $h(x+\Delta x)=\frac{f(x+\Delta x)}{g(x+\Delta x)}$

By definition $h^{/}(x)=\lim_{\Delta x\to 0} it\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]$

$$=\lim_{\Delta x\to 0} it\left[\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\frac{5(x+\Delta x)^2-(x+\Delta x)+7}{(x+\Delta x)^2+(x+\Delta x)+1}-\frac{5x^2-x+7}{x^2+x+1}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{\dfrac{5(x^2 + 2x\Delta x + (\Delta x)^2) - (x + \Delta x) + 7}{(x^2 + 2x\Delta x + (\Delta x)^2) + (x + \Delta x) + 1} - \dfrac{5x^2 - x + 7}{x^2 + x + 1}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{(x^2 + x + 1)(5x^2 + 10x\Delta x + 5(\Delta x)^2 - x - \Delta x + 7) - (5x^2 - x + 7)(x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1)}{(x^2 + x + 1)(x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1)\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{6x^2\Delta x + 6x(\Delta x)^2 - 4x\Delta x - 2(\Delta x)^2 - 8\Delta x}{(x^2 + x + 1)(x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1)\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{6x^2 + 6x\Delta x - 4x - 2\Delta x - 8}{(x^2 + x + 1)(x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1)} \right]$$

As Δx tends to zero $\left[(x^2 + x + 1)(x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1)\right]$ tends towards $\left[(x^2 + x + 1)(x^2 + x + 1)\right]$ and $\left[6x^2 + 6x\Delta x - 4x - 2\Delta x - 8\right]$ tends towards $\left[6x^2 - 4x - 8\right]$. Hence,

$$h^{/}(x) = \frac{6x^2 - 4x - 8}{(x^2 + x + 1)^2}$$

This is analogous to the application of $\dfrac{d}{dx} h(x) = \dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$

$$\frac{d}{dx} h(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx} f(x) - f(x)\dfrac{d}{dx} g(x)}{\left[g(x)\right]^2}$$

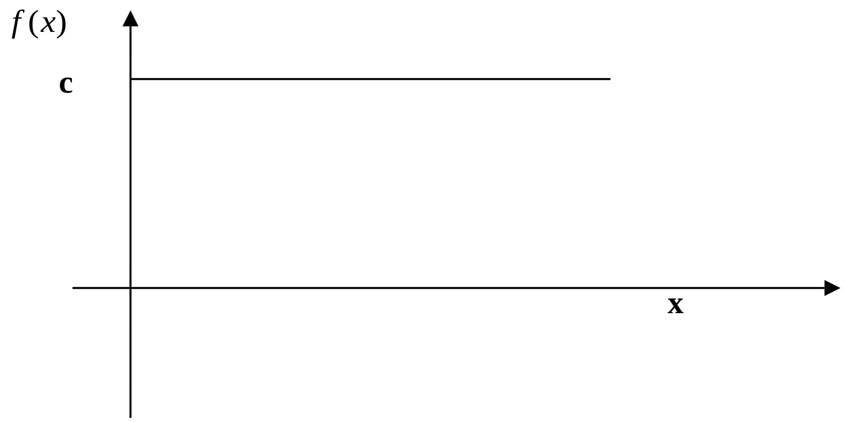
$$= \frac{(x^2 + x + 1)\dfrac{d}{dx}(5x^2 - x + 7) - (5x^2 - x + 7)\dfrac{d}{dx}(x^2 + x + 1)}{\left[(x^2 + x + 1)\right]^2}$$

$$= \frac{(x^2 + x + 1)(10x - 1) - (5x^2 - x + 7)(2x + 1)}{\left[(x^2 + x + 1)\right]^2}$$

$$= \frac{(6x^2 - 4x - 8)}{\left[(x^2 + x + 1)\right]^2}$$

Theorem: The derivative of any constant is zero.

Proof:



Let $f(x)=c$ then $f(x+\Delta x)=c$

By definition $f^{/}(x)=\lim_{\Delta x\to 0} it\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$

$$=\lim_{\Delta x\to 0} it\left[\frac{c-c}{\Delta x}\right]=\lim_{\Delta x\to 0} it\left[\frac{0}{\Delta x}\right]=0$$

Theorem: If $f(x)$ and $g(x)$ are given functions of x then,

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

Proof: Let $h(x)=f(x)+g(x)$ then $h(x+\Delta x)=f(x+\Delta x)+g(x+\Delta x)$

By definition

$$h^{/}(x)=\lim_{\Delta x\to 0} it\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\{f(x+\Delta x)+g(x+\Delta x)\}-\{f(x)+g(x)\}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\{f(x+\Delta x)-f(x)\}+\{g(x+\Delta x)-g(x)\}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{\{f(x+\Delta x)-f(x)\}}{\Delta x}+\frac{\{g(x+\Delta x)-g(x)\}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0} it\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]+\lim_{\Delta x\to 0} it\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$

In the limit as $\Delta x \to 0$, the expression in the square brackets becomes $\frac{d}{dx} f(x)$ and $\frac{d}{dx} g(x)$.

Consequently, we obtain:

$$h^{/}(x) = \frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Example. Find the derivative with respect to x of the function $h(x) = f(x) + g(x)$ if $f(x) = 3x^4$ and $g(x) = 5x^2$

Solution: $h(x) = f(x) + g(x) \quad \Rightarrow \quad \frac{d}{dx} h(x) = \frac{d}{dx}\left[f(x) + g(x)\right]$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$= \frac{d}{dx}(3x^4) + \frac{d}{dx}(5x^2)$$

$$= 12x^3 + 10x$$

2.2 CHAIN RULE

Theorem: If $y = f(u)$ is a composition on $u = u(x)$ and if y is differentiable with respect to u and u is differentiable with respect to x, then y is differentiable with respect to x and the derivative is given by:

$$y^{/}(x) = f^{/}\left[u(x)\right]u^{/}(x). \text{ Thus, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This theorem is called the **<u>chain</u>** rule.

Proof: $\lim_{\Delta u \to 0} it\left[\frac{\Delta y}{\Delta u}\right] = \frac{dy}{du}$ and $\lim_{\Delta x \to 0} it\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}$

If Δu is finite, then the difference $\frac{\Delta y}{\Delta u} - \frac{dy}{du} = \in > 0$. $\in$ is finite positive such that $\in \to 0$ as $\Delta u \to 0$.

$$\frac{\Delta y}{\Delta u} - \frac{dy}{du} = \in$$

$$\frac{\Delta y}{\Delta u} = \frac{dy}{du} + \in$$

$$\Delta y = \frac{dy}{du} \times \Delta u + \in \times \Delta u$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{du} \times \frac{\Delta u}{\Delta x} + \in \times \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta u \to 0} it\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \to 0} it\left[\frac{dy}{du} \times \frac{\Delta u}{\Delta x}\right] + \lim_{\Delta x \to 0} it\left[\in \times \frac{\Delta u}{\Delta x}\right]$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \lim_{\Delta x \to 0} it\left[\frac{\Delta u}{\Delta x}\right] + \in \times \lim_{\Delta x \to 0} it\left[\frac{\Delta u}{\Delta x}\right]$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} + \in \times \frac{du}{dx}$$

As $\in \to 0$, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

The chain rule is what must be applied when differentiating a function of a function. It is also useful for calculating the derivative of a function f with respect to x when both x and f are written in terms of a variable or parameter say t.

Example. Given that $y = 3u^2 + 1$ and $u = 4x^2 + 1$. Find the following:

(i). $\dfrac{dy}{du}$ (ii). $\dfrac{du}{dx}$ (iii). $\dfrac{dy}{dx}$

Solution: (i). $\dfrac{dy}{du} = 3\dfrac{d}{du}\left[u^2\right] + \dfrac{d}{du}[1] = 3[2u] + 0 = 6u$

(ii). $\dfrac{du}{dx} = 4\dfrac{d}{dx}\left[x^2\right] + \dfrac{d}{dx}[1] = 4[2x] + 0 = 8x$

(iii). $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = [6u] \times [8x] = 48xu = 48x(4x^2 + 1)$

2.3 IMPLICIT DIFFERENTIATION

Given the mapping $y \to f(x)$ if x and y are independent variables, then $f(x)$ is a function of a single variable. For example, consider the relation $y = x^2 + 3x - 5$; y is completely defined in terms of x, therefore y is called an explicit function of x. Where the relationship between x and y is more involved, it may not be possible to separate y completely on the left hand side, eg. $3xy + 5y^3 = 40$. In such a case, this y is called an implicit function of x, because a relationship of the form $y = f(x)$ is implied in the given equation. That is, it cannot be made explicit by writing either x in terms of y or y in terms of x. However, by differentiating term by term with respect to x, we can find the derivative of the implicit function. In differentiating y^2 which is a function of x

(i) $f(y) = y^2$ and this implies

$$\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$$

Hence, in general, if $f(y) = y^n$, then $\frac{d}{dx}\left[y^n\right] = ny^{n-1} \times \frac{dy}{dx}$

Example. Find $\frac{dy}{dx}$ if $x^3 - 3xy + y^3 = 2$,

Solution: Differentiating each term in the equation with respect to x, we obtain:

$$\frac{d}{dx}\left(x^3\right) - \frac{d}{dx}(3xy) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(2)$$

$$\Rightarrow \quad 3x^2 - 3\left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 3x\frac{dy}{dx} - 3y + 3y^2\frac{dy}{dx} = 0$$

where the derivative of $3xy$ has been found using the product rule. Hence, rearranging for $\frac{dy}{dx}$, we obtain $\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$.

Note: $\frac{dy}{dx}$ is a function of both x and y and cannot be expressed as a function of x only.

2.4 DIFFERENTIATION OF LOGARITHMIC FUNCTIONS

Circumstances in which the variable with respect to which we are differentiating is an exponent, taking the logarithms and differentiating implicitly is the simplest way to find the derivative. The natural logarithm function is denoted by **ln** or $\log_e$**.** The expression $\ln x$ is called the natural logarithm of x.

Consider an exponential function $y = a^x$. When this function is put in a logarithmic form, we obtain:

$$\ln y = x \ln a$$

$$\Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln a)$$

$$\Rightarrow \quad \frac{1}{y} \times \frac{dy}{dx} = \ln a$$

$$\Rightarrow \quad \frac{dy}{dx} = y \ln a = a^x \ln a$$

$$\therefore \quad \frac{dy}{dx} = a^x \ln a$$

Thus, in general if $y = \ln\left[f(x)\right]$, *then* $\frac{dy}{dx} = \frac{f^1(x)}{f(x)}$

Examples.

1. If $y = \ln\left(5x^2 + 6x - 8x^4\right)$, then $\frac{dy}{dx} = \frac{10x + 6 - 32x^3}{5x^2 + 6x - 8x^4}$.

2. Differentiate with respect to x simplifying your answer as far as possible the function $y = In(x^2 + 5x - 3)$.

Solution: Let $u = x^2 + 5x - 3$ then $y = Inu$

$$\Rightarrow \frac{du}{dx} = 2x + 5 \quad \text{and} \quad \frac{dy}{du} = \frac{1}{u}$$

But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (2x+5)$

$$= \frac{1}{x^2 + 5x - 3} \times (2x+5)$$

$$\therefore \frac{dy}{dx} = \frac{2x+5}{x^2 + 5x - 3}$$

3. Differentiate with respect to x of the function $y = e^x$.

Solution: $y = e^x \Rightarrow Iny = xIne = x$

$$\frac{d}{dx}(Iny) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = e^x$$

Hence, $\frac{d}{dx}(e^x) = e^x$

2.5 PARAMETRIC DIFFERENTIATION

Some of the functions whose derivatives are sought are sometimes so complicated that, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable.

For example, if

$y = \frac{3+2t}{1+t}$ and $x = \frac{2-3t}{1+t}$ except for $t = -1$, any value given to t will produce a pair of values for x and y which could if necessary be plotted to provide one point of the curve $y = f(x)$. This third variable t is called a **parameter** and the two expressions for x and y is called **parametric equations**. If the derivative of this curve $y = f(x)$ is to be found with respect to x, then from the example $y = \frac{3+2t}{1+t}$;

$$\frac{dy}{dt} = \frac{(1+t)\frac{d}{dt}(3+2t) - (3+2t)\frac{d}{dt}(1+t)}{(1+t)^2} = \frac{-1}{(1+t)^2}$$

Similarly,

$$\frac{dx}{dt} = \frac{(1+t)\frac{d}{dt}(2-3t) - (2-3t)\frac{d}{dt}(1+t)}{(1+t)^2} = \frac{-5}{(1+t)^2}$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-1}{(1+t)^2} \times \frac{(1+t)^2}{-5} = \frac{1}{5}$$

Examples.

1. Find $\frac{dy}{dx}$ in terms of t for the curve whose parametric equations are given by

$$x = t^2 + t + 1, \qquad y = 5t.$$

Solution: $x = t^2 + t + 1$ and $y = 5t$

$$\Rightarrow \frac{dx}{dt} = 2t + 1 \qquad \Rightarrow \frac{dy}{dt} = 5$$

But $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 5 \times \frac{1}{2t+1} = \frac{5}{2t+1}$

Therefore, $\frac{dy}{dx} = \frac{5}{2t+1}$.

2. The parametric equations of a curve are $x = \frac{3t}{1+t}, y = \frac{t^2}{1+t}$. Find the value of $\frac{dy}{dx}$ at the part for which $t = 2$.

Solution:
$$\frac{dx}{dt} = \frac{(1+t)\frac{d}{dt}(3t) - (3t)\frac{d}{dt}(1+t)}{(1+t)^2} = \frac{3}{(1+t)^2}$$

$$\frac{dy}{dt} = \frac{(1+t)\frac{d}{dt}(t^2) - (t^2)\frac{d}{dt}(1+t)}{(1+t)^2} = \frac{2t+t^2}{(1+t)^2}$$

Therefore $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t+t^2}{(1+t)^2} \times \frac{(1+t)^2}{3} = \frac{2t+t^2}{3}$

At $t = 2$, we have $\left.\frac{dy}{dx}\right|_{t=2} = \frac{2(2)+2^2}{3} = \frac{8}{3}$

3. If $x = 2\theta - Sin2\theta$ and $y = 1 - Cos2\theta$. Show that $\frac{dy}{dx} = Cot\theta$.

Solution: $x = 2\theta - Sin2\theta$ and $y = 1 - Cos2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2 - 2Cos2\theta \qquad \Rightarrow \frac{dy}{d\theta} = 2Sin2\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2Sin2\theta \times \frac{1}{2(1-Cos2\theta)} = \frac{Sin2\theta}{1-Cos2\theta}$$

But $Sin2\theta = 2Sin\theta Cos\theta$ and $Cos2\theta = 1 - 2Sin^2\theta$, so we have

$$\frac{dy}{dx} = \frac{2Sin\theta Cos\theta}{1-(1-2Sin^2\theta)} = \frac{2Sin\theta Cos\theta}{2Sin^2\theta} = \frac{Cos\theta}{Sin\theta} = Cot\theta \,.$$

2.6 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

We differentiate sine, cosine, tangent and other trigonometric functions by adopting the radian measure for all angles unless otherwise stated.

(i) Let $f(x) = \sin x$ then $f(x + \Delta x) = \sin(x + \Delta x)$

By definition $f^{/}(x) = \lim_{\Delta x \to 0} it\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} it\left[\frac{\{\sin(x+\Delta x) - \sin x\}}{\Delta x}\right] = \lim_{\Delta x \to 0} it\left[\frac{\{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\{\sin x(\cos\Delta x - 1) + \cos x\sin\Delta x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\sin x(\cos\Delta x - 1)}{\Delta x}\right] + \lim_{\Delta x \to 0} it\left[\frac{\cos x\sin\Delta x}{\Delta x}\right]$$

$$= \sin x \lim_{\Delta x \to 0} it\left[\frac{(\cos\Delta x - 1)}{\Delta x}\right] + \cos x \lim_{\Delta x \to 0} it\left[\frac{\sin\Delta x}{\Delta x}\right]$$

By the theorem of calculus on limits, $\lim_{\Delta x \to 0} it\left[\frac{(\cos\Delta x - 1)}{\Delta x}\right] = 0$ and $\lim_{\Delta x \to 0} it\left[\frac{\sin\Delta x}{\Delta x}\right] = 1$

Hence $f^{/}(x) = \sin x \bullet 0 + \cos x \bullet 1$

$$\therefore \quad \frac{d}{dx}[\sin x] = \cos x$$

(ii). Let $h(x) = \cos x$ then $h(x + \Delta x) = \cos(x + \Delta x)$

By definition $h^{/}(x) = \lim_{\Delta x \to 0} it\left[\frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} it\left[\frac{\{\cos(x+\Delta x) - \cos x\}}{\Delta x}\right] = \lim_{\Delta x \to 0} it\left[\frac{\{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\{\cos x(\cos\Delta x - 1) - \sin x\sin\Delta x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\cos x(\cos\Delta x - 1)}{\Delta x}\right] + \lim_{\Delta x \to 0} it\left[\frac{-\sin x\sin\Delta x}{\Delta x}\right]$$

$$= \cos x \lim_{\Delta x \to 0} it\left[\frac{(\cos \Delta x - 1)}{\Delta x}\right] - \sin x \lim_{\Delta x \to 0} it\left[\frac{\sin \Delta x}{\Delta x}\right]$$

By the theorem of calculus on limits $\lim_{\Delta x \to 0} it\left[\frac{(\cos \Delta x - 1)}{\Delta x}\right] = 0$ and $\lim_{\Delta x \to 0} it\left[\frac{\sin \Delta x}{\Delta x}\right] = 1$

Hence $h^{/}(x) = \cos x \bullet 0 + -\sin x \bullet 1$

$$\therefore \quad \frac{d}{dx}[\cos x] = -\sin x$$

(iii) Let $g(x) = \tan x$ then $g(x + \Delta x) = \tan(x + \Delta x) = \frac{\sin(x + \Delta x)}{\cos(x + \Delta x)}$

By definition $g^{/}(x) = \lim_{\Delta x \to 0} it\left[\frac{g(x + \Delta x) - g(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} it\left[\frac{\tan(x + \Delta x) - \tan x}{\Delta x}\right] = \lim_{\Delta x \to 0} it\left[\frac{\dfrac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \dfrac{\sin x}{\cos x}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\cos x \sin(x + \Delta x) - \sin x \cos(x + \Delta x)}{\Delta x \cos x \cos(x + \Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\cos x\{\sin x \cos \Delta x + \cos x \sin \Delta x\} - \sin x\{\cos x \cos \Delta x - \sin x \sin \Delta x\}}{\Delta x \cos x \cos(x + \Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\cos x \sin x \cos \Delta x + \cos x \cos x \sin \Delta x - \sin x \cos x \cos \Delta x + \sin x \sin x \sin \Delta x}{\Delta x \cos x \cos(x + \Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\cos^2 x \sin \Delta x + \sin^2 x \sin \Delta x}{\Delta x \cos x \cos(x + \Delta x)}\right] = \lim_{\Delta x \to 0} it\left[\frac{\sin \Delta x\{\cos^2 x + \sin^2 x}{\Delta x \cos x \cos(x + \Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} it\left[\frac{\sin \Delta x}{\Delta x \cos x \cos(x + \Delta x)}\right] = \frac{1}{\cos x} \bullet \lim_{\Delta x \to 0} it\left[\frac{1}{\cos(x + \Delta x)}\right] \bullet \lim_{\Delta x \to 0} it\left[\frac{\sin \Delta x}{\Delta x}\right]$$

$$= \frac{1}{\cos x} \bullet \left[\frac{1}{\cos x}\right] \bullet 1 = \frac{1}{\cos^2 x} = \sec^2 x$$

Hence $\frac{d}{dx}[\tan x] = \sec^2 x$

Example. Find the derivative with respect to x of the function $f(x) = \sec^2 x$

Solution: $\sec^2 x = \frac{1}{\cos^2 x}$

$$\frac{d}{dx}\left[\sec^2 x\right] = \frac{d}{dx}\left[\frac{1}{\cos^2 x}\right] = \frac{-\frac{d}{dx}\left[\cos x \cos x\right]}{\left[\cos^2 x\right]^2}$$

$$= \frac{-\left\{\cos x \frac{d}{dx}\left[\cos x\right] + \cos x \frac{d}{dx}\left[\cos x\right]\right\}}{\left[\cos^2 x)\right]^2}$$

$$= \frac{-\left\{\cos x\left[-\sin x\right] + \cos x\left[-\sin x\right]\right\}}{\left[\cos^2 x)\right]^2} = \frac{2\sin x \cos x}{\left[\cos^2 x\right]^2}$$

$$= \frac{2\sin x}{\cos^3 x} = 2 \bullet \frac{\sin x}{\cos x} \bullet \frac{1}{\cos^2 x} = 2\sec^2 x \tan x$$

Hence $\frac{d}{dx}\left[\sec^2 x\right] = 2\sec^2 x \tan x$

Example. Differentiate with respect to x simplifying as far as possible the function $f(x) = x^2 \sin x$

Solution: $\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[x^2 \sin x\right]$

$$= x^2 \frac{d}{dx}\left[\sin x\right] + \sin x \frac{d}{dx}\left[x^2\right] = x^2\left[\cos x\right] + \sin x\left[2x\right]$$

$$= x^2 \cos x + 2x \sin x$$

2.7 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

The function $y = \sin^{-1} x = \arcsin x$ where $\sin^{-1} x \neq \frac{1}{\sin x}$ is the value of the argument y when the variable x is given implying that $x = \sin y$. The function is multiple-value since there are infinite values of y for each value of x between -1 and $+1$.

The principal values of this is chosen as those between $-\pi/2$ and $\pi/2$

(i) Now let $y = \sin^{-1} x \Rightarrow x = \sin y$

$$\Rightarrow \frac{d}{dx}\left[x\right] = \frac{d}{dx}\left[\sin y\right]$$

$$\Rightarrow 1 = \cos y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1 - x^2}}$$

The positive part of the function is chosen since $\cos y > 0$

Hence $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$

(ii) Again, let $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$

$$\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[\cos y]$$

$$\Rightarrow \quad 1 = -\sin y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{-\sin y} = \frac{1}{\pm\sqrt{1-x^2}}$$

The negative part of the function is chosen since $\sin y > 0$ where y belongs to the close interval $[0, \pi]$. That is, $y \in [0, \pi]$ and $-\sin y < 0$.

$$\text{Hence } \frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$$

(iii) Finally, let $y = \tan^{-1} x \quad \Rightarrow \quad x = \tan y$

$$\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[\tan y]$$

$$\Rightarrow \quad 1 = \sec^2 y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$

To establish other results, we make use of the results in (i) and (ii) as well as (iii).

Example. Find the derivative with respect to x of $\sec^{-1} x$

Solution: Let $y = \sec^{-1} x \quad \Rightarrow \quad x = \sec y$

$$\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[\sec y]$$

$$\Rightarrow \quad 1 = \frac{\sin y \frac{dy}{dx}}{\cos^2 y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2 y}{\sin y} = \frac{1}{\pm x\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$$

2.8 LEIBNITZ'S FORMULA FOR NTH (REPEATED) DIFFERENTIATION OF A PRODUCT

The first derivatives of functions in the form such as product, quotient, logarithmic, implicit and many others have been dealt with in the previous section. Corresponding results for higher derivatives of the forms stated above could be obtained by the use of Leibnitz's Theorem.

Consider the function $f(x) = u(x)v(\text{x})$; from the product rule of differentiation,

$$\frac{df}{dx} = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

Applying the rule again on each of the product, we obtain

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(u(x)\frac{d}{dx}v(x)\right) + \frac{d}{dx}\left(v(x)\frac{d}{dx}u(x)\right)$$

$$\Rightarrow \frac{d^2 f}{dx^2} = \left[u(x)\frac{d^2}{dx^2}v(x) + \left(\frac{d}{dx}v(x)\bullet\frac{d}{dx}u(x)\right)\right] + \left[v(x)\frac{d^2}{dx^2}u(x) + \left(\frac{d}{dx}u(x)\bullet\frac{d}{dx}v(x)\right)\right]$$

$$= u(x)\frac{d^2v(x)}{dx^2} + \left[2\bullet\left(\frac{dv(x)}{dx}\right)\bullet\left(\frac{du(x)}{dx}\right)\right] + v(x)\frac{d^2u(x)}{dx^2}$$

Similarly, differentiating again to its third order, we obtain

$$\frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) = \frac{d}{dx}\left[u(x)\frac{d^2v(x)}{dx^2}\right] + 2\frac{d}{dx}\left[\left(\frac{dv(x)}{dx}\right)\bullet\left(\frac{du(x)}{dx}\right)\right] + \frac{d}{dx}\left[v(x)\frac{d^2u(x)}{dx^2}\right]$$

$$\Rightarrow \frac{d^3 f}{dx^3} = \left[u(x)\frac{d^3}{dx^3}v(x) + \frac{d^2v(x)}{dx^2}\frac{du(x)}{dx}\right] + 2\left[\frac{dv(x)}{dx}\bullet\frac{d^2}{dx}u(x) + \frac{du(x)}{dx}\bullet\frac{d^2}{dx}\text{v}(x)\right] +$$

$$\left[\text{v}(x)\frac{d^3}{dx^3}\text{u}(x) + \frac{d^2\,\text{u}(x)}{dx^2}\frac{dv(x)}{dx}\right]$$

$$= u(x)\frac{d^3v(x)}{dx^3} + \left[3\bullet\left(\frac{d^2v(x)}{dx^2}\right)\bullet\left(\frac{du(x)}{dx}\right)\right] + \left[3\bullet\left(\frac{d^2\,\text{u}(x)}{dx^2}\right)\bullet\left(\frac{dv(x)}{dx}\right)\right] + v(x)\frac{d^3u(x)}{dx^3}$$

Differentiating for the fourth time, we obtain

$$\frac{d^4 f}{dx^4} = u(x)\frac{d^4v(x)}{dx^4} + \left[4\bullet\left(\frac{du(x)}{dx}\right)\bullet\left(\frac{d^3v(x)}{dx^3}\right)\right] + \left[6\bullet\left(\frac{d^2u(x)}{dx^2}\right)\bullet\left(\frac{d^2v(x)}{dx^2}\right)\right] + \left[4\bullet\left(\frac{d^3u(x)}{dx^3}\right)\bullet\left(\frac{dv(x)}{dx}\right)\right] + v(x)\frac{d^4u(x)}{dx^4}$$

From the pattern that has emerged, the results generalize to a form given by:

$$f^n(x)=\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}[u(x)]^r\,[v(x)]^{n-r}$$

$$=\sum_{r=0}^{n}{}^nC_r\left[\frac{d^r}{dx^r}u(x)\frac{d^{n-r}}{dx^{n-r}}v(x)\right]=\sum_{r=0}^{n}{}^nC_r\left[\left(\frac{d^r u(x)}{dx^r}\right)\left(\frac{d^{n-r}v(x)}{dx^{n-r}}\right)\right]$$

The fraction $\dfrac{n!}{r!(n-r)!}$ is the binomial coefficient ${}^nC_r$. This leads to the Leibnitz's Theorem.

Leibnitz Theorem: It states that if f(x) is the product of two functions u(x) and v(x) possessing derivatives of the nth order such that f(x)=u(x)v(x), then

$$f^n(x)=\sum_{r=0}^{n}{}^nC_r\left[\left(\frac{d^r u(x)}{dx^r}\right)\left(\frac{d^{n-r}v(x)}{dx^{n-r}}\right)\right]$$

where ${}^nC_r$ is the binomial coefficient given by ${}^nC_r=\dfrac{n!}{r!(n-r)!}$.

Example. Given that $f(x)=x^4Sinx$, use Leibnitz theorem to find

(i) $\dfrac{d^2f}{dx^2}$ (ii) $\dfrac{d^3f}{dx^3}$ (iii) $\dfrac{d^4f}{dx^4}$

Solution:

By Leibnitz's theorem, $f^n(x)=\sum_{r=0}^{n}{}^nC_r\left[\left(\frac{d^r u(x)}{dx^r}\right)\left(\frac{d^{n-r}v(x)}{dx^{n-r}}\right)\right]$

We let $u(x)=x^4$ and $v(x)=Sinx$

(i) $$\frac{d^2f}{dx^2}={}^2C_0\left[u(x)\frac{d^2v(x)}{dx^2}\right]+{}^2C_1\left[\left(\frac{du(x)}{dx}\right)\bullet\left(\frac{dv(x)}{dx}\right)\right]+{}^2C_2\left[\left(\frac{d^2u(x)}{dx^2}\right)\bullet v(x)\right]$$

$$=x^4\frac{d^2}{dx^2}(Sinx)+2\left[\left(\frac{d}{dx}(x^4)\right)\bullet\left(\frac{d}{dx}(Sinx)\right)\right]+\left(\frac{d^2}{dx^2}(x^4)\right)\bullet Sinx$$

$$=x^4(-Sinx)+2\left[\left(4x^3\right)(Cosx)\right]+12x^2(Sinx)$$

$$=-x^4Sinx+8x^3Cosx+12x^2Sinx$$

(ii)

$$\frac{d^3 f}{dx^3} = {}^3C_0\left[u(x)\frac{d^3v(x)}{dx^3}\right] + {}^3C_1\left[\left(\frac{du(x)}{dx}\right)\bullet\left(\frac{d^2v(x)}{dx^2}\right)\right] + {}^3C_2\left[\left(\frac{d^2u(x)}{dx^2}\right)\bullet\left(\frac{dv(x)}{dx}\right)\right] +$$
$$ {}^3C_3\left[\left(\frac{d^3u(x)}{dx^3}\right)\bullet v(x)\right]$$

$$= x^4\frac{d^3}{dx^3}(Sinx) + 3\left[\left(\frac{d^2}{dx^2}(Sinx)\right)\bullet\left(\frac{d}{dx}(x^4)\right)\right] + 3\left[\left(\frac{d^2}{dx^2}(x^4)\right)\bullet\left(\frac{d}{dx}(Sinx)\right)\right] + Sinx\frac{d^3}{dx^3}(x^4)$$

$$= x^4(-Cosx) + 3\left[(-Sinx)\bullet(4x^3)\right] + 3\left[\left(12x^2\right)\bullet\left(\operatorname{Cos} x\right)\right] + Sinx(24x)$$

$$= -x^4Cosx - 12x^3Sinx + 36x^2Cosx + 24xSinx$$

(iii)
$$\frac{d^4 f}{dx^4} = {}^4C_0\left[u(x)\frac{d^4v(x)}{dx^4}\right] + {}^4C_1\left[\left(\frac{du(x)}{dx}\right)\bullet\left(\frac{d^3v(x)}{dx^3}\right)\right] + {}^4C_2\left[\left(\frac{d^2u(x)}{dx^2}\right)\bullet\left(\frac{d^2v(x)}{dx^2}\right)\right] +$$
$$ {}^4C_3\left[\left(\frac{d^3u(x)}{dx^3}\right)\bullet\left(\frac{dv(x)}{dx}\right)\right] + {}^4C_4\left[\left(\frac{d^4u(x)}{dx^4}\right)\bullet v(x)\right]$$

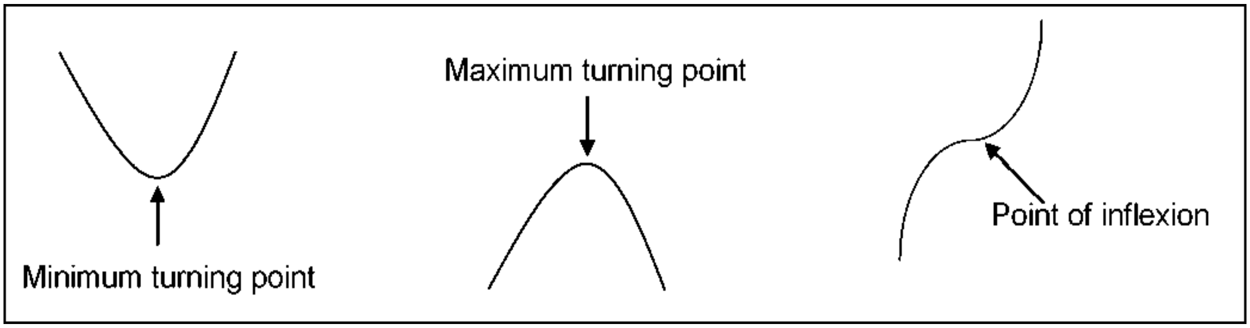
$$= x^4\frac{d^4}{dx^4}(Sinx) + 4\left[\left(\frac{d}{dx}(x^4)\right)\bullet\left(\frac{d^3}{dx^3}(\text{Sinx})\right)\right] + 6\left[\left(\frac{d^2}{dx^2}(x^4)\right)\left(\frac{d^2}{dx^2}(\text{Sinx})\right)\right] + 4\left[\left(\frac{d^3}{dx^3}(x^4)\right)\left(\frac{d}{dx}(Sinx)\right)\right] +$$
$$Sinx\frac{d^4}{dx^4}(x^4)$$

$$= x^4(Sinx) + 4\left[\left(4x^3\right)\left(-Cosx\right)\right] + 6\left[\left(12x^2\right)\left(-Sinx\right)\right] + 4\left[\left(24x\right)\left(Cosx\right)\right] + Sinx(24)$$

$$= x^4Sinx - 16x^3Cosx - 72x^2Sinx + 96xCosx + 24Sinx$$

2.9 STATIONARY POINTS

In this section, we show how differentiation can be used to define a stationary point on a curve and decide whether it is a turning point (maximum or minimum) or a point of inflexion. Because, the derivative provides information about the gradient or slope of the graph of a function, we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximizing power, profit or minimizing losses or costs.



Looking at the three (3) diagrams above, we see that at each of the points shown, the gradient is zero (i.e. the curve goes flat). Since differentiation gives us the gradient function, to find any stationary point, we need to differentiate first, then put the resulting gradient function to zero.

2.9.1 Finding the Stationary Points

When $\frac{dy}{dx}=0$ (i.e. the gradient is zero), we have a stationary point. In short, we find the **1st differential,** put it equal to zero and solve.

Examples.

1. Find the stationary point on the curve $y=3x^2-18x-7$.

Solution:

Step 1: Differentiate to give: $\frac{dy}{dx}=6x-18$

Step 2: Put this equal to zero: $6x-18=0$

Step 3: Solve: $6x=18$

$x=3$

We now know that the stationary point is when $x=3$

If we require the full co-ordinate, we need to find the value of y when $x=3$.

(We are looking for y so we need to use the original equation, $y=3x^2-18x-7$)

Original equation: $y=3x^2-18x-7$

Substitute x = 3: $y=3(3)^2-18(3)-7$

Solve: $y=-34$

Therefore, the stationary point is at $(3,-34)$.

2. Find the stationary point on the curve $y=\frac{1}{3}x^3+2x^2-12x+5$

Solution:

Step 1: Differentiate to give: $\frac{dy}{dx}=x^2+4x-12$

Step 2: Put this equal to zero: $x^2+4x-12=0$

Step 3: Solve: $(x+6)(x-2)=0$

$x=-6$ or $x=2$

We realize that we have two stationary points; one at $x=-6$ and the other at $x=2$.

Next, we find the corresponding y values using the equation $y=\frac{1}{3}x^3+2x^2-12x+5$

When $x=-6$: $y=\frac{1}{3}(-6)^3+2(-6)^2-12(-6)+5$

$y=77$

When $x=2$: $y=\frac{1}{3}(2)^3+2(2)^2-12(2)+5$

$y=-\frac{25}{3}$

Therefore, the stationary points are at $(-6,77)$ and $\left(2,-\frac{25}{3}\right)$

2.9.2 Type of Stationary Points

To find the type of stationary point, we need to differentiate again (i.e. find the 2nd differential, denoted by $\frac{d^2y}{dx^2}$). This measures the change in gradient and can help to decide whether a stationary point is maximum, minimum or point of inflexion.

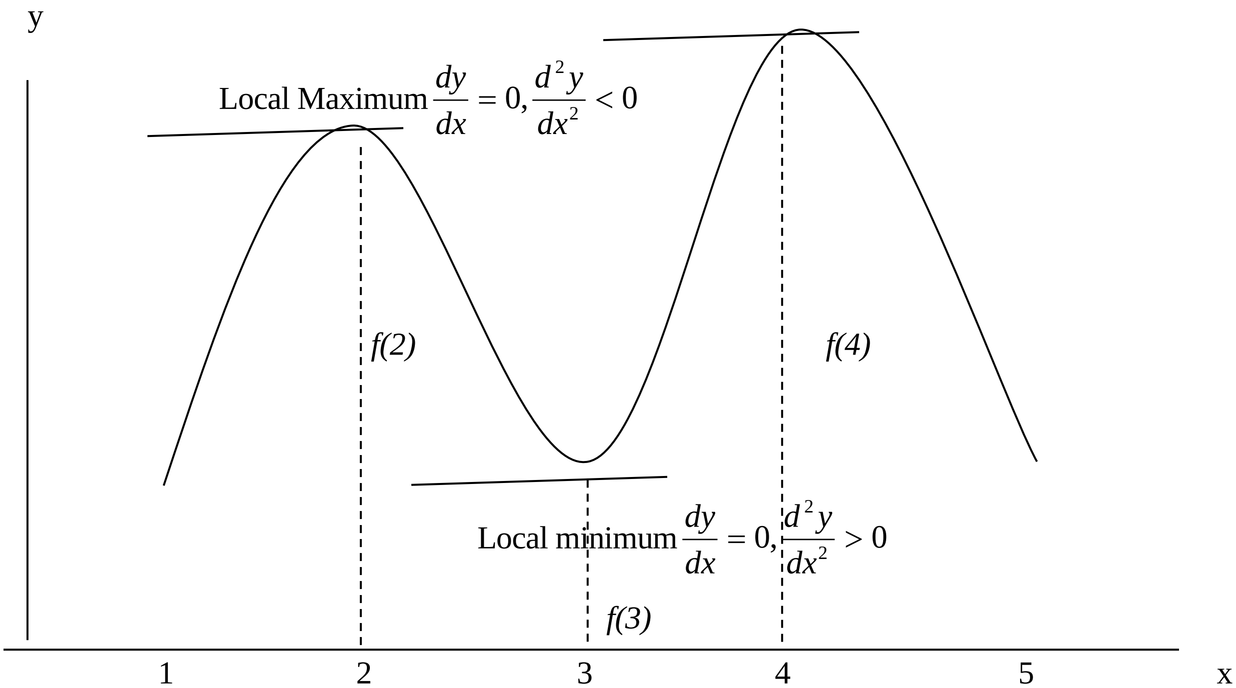
After finding where the stationary point is, we find the 2nd differential and if:

- $\frac{d^2y}{dx^2}>0$, then we have a minimum (thus positive is minimum).
- $\frac{d^2y}{dx^2}<0$, then we have a maximum (thus negative is maximum).
- $\frac{d^2y}{dx^2}=0$, then it could be maximum, minimum or point of inflexion.

So, we differentiate again and if:

- $\frac{d^3y}{dx^3} \neq 0$, then it is a point of inflexion.

Consider the graph below



The graph indicates that the quantity increased in the interval (1, 2) and decreased in (2, 3). Considering the interval (1, 5) the quantity had its highest or maximum values at 2 and 4 and its smallest value or minimum at 3. From the graph, we say that f takes on its maximum value at 2 and 4 and the points (2, *f(2)*) and (4, *f(4)*) are the highest or maximum points on the graph. Similarly, the function f takes on its minimum value at 3 and the point (3, f(3)) is the lowest or minimum point on the graph.

Examples.

1. Find the stationary point on the curve $y = x^2 - 4x + 2$ and determine the type.

Solution: We use the 1st differential to find where the stationary point is.

Step 1: Differentiate to give: $\frac{dy}{dx} = 2x - 4$

Step 2: Put this equal to zero: $2x-4=0$

Step 3: Solve: $2x=4$

$x=2$

We find the corresponding y value using $y=x^2-4x+2$

Substitute $x=2$: $y=(2)^2-4(2)+2$

Solve: $y=-2$

Therefore, the stationary point is at $(2,-2)$

We now use the 2nd differential to find the type of stationary point.

$\frac{d^2y}{dx^2}=2$. This is positive $(\succ 0)$ so the turning point must be a minimum.

2. Find the stationary point on the curve $f(x)=x^3-3x^2+4$ and determine the type.

Solution: We use the 1st differential to find where the stationary point is.

Step 1: Differentiate to give: $f^{/}(x)=3x^2-6x$

Step 2: Put this equal to zero: $3x^2-6x=0$

Step 3: Solve: $3x(x-2)=0$

$x=0$ or $x=2$

We find the corresponding y values using $f(x)=y=x^3-3x^2+4$

Substitute $x=0$: $y=(0)^3-3(0)^2+4$

$y=4$

Substitute $x=2$: $y=(2)^3-3(2)^2+4$

$y=0$

Therefore, the stationary points are at $(0,4)$ and $(2,0)$.

We now use the 2nd differential to find the type of stationary points.

$f^{//}(x)=6x-6$.

This looks a bit different from the 1st example since there is an x in the 2nd differential. We now need to substitute each value of x into the function separately to find the type of stationary point each one is.

When $x=0$: $f^{//}(x)=6(0)-6$

$f^{//}\left(x\right) = -6$. Thus, maximum

When $x = 2$: $f^{//}(x) = 6\left(2\right) - 6$

$f^{//}\left(x\right) = 6$. Thus, minimum

Therefore, $\left(0,4\right)$ is a maximum point and $\left(2,0\right)$ is a minimum point.

3. Find the stationary point on the curve $y = x^3$ and determine the type.

Solution: We use the 1st differential to find where the stationary point is.

Step 1: Differentiate to give: $\frac{dy}{dx} = 3x^2$

Step 2: Put this equal to zero: $3x^2 = 0$

Step 3: Solve: $x = 0$

We find the corresponding y value using $y = x^3$

Substitute $x = 0$: $y = \left(0\right)^3$

Solve: $y = 0$

Therefore, the stationary point is at $\left(0,0\right)$

We now use the 2nd differential to find the type of stationary point.

$$\frac{d^2y}{dx^2} = 6x$$

When $x = 0$: $\frac{d^2y}{dx^2} = 0$, so it could be a maximum, minimum or point of inflexion.

We use the 3rd differential to decide.

$$\frac{d^3y}{dx^3} = 6$$

Since, the 3rd differential is **not 0**, then it is a point of inflexion.

2.10 POWER SERIES AND INDETERMINATE FORMS

If a function can be expressed as a series of ascending powers of x, then the series is called a Power Series. Supposing a function of x is given by f(x), then the function f(x) can be represented as a power series of the infinite form as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots + = \sum_{n=0}^{\infty} a_n x^n \cdots\cdots\cdots(1)$$

where $a_0, a_1, a_2, a_3, \ldots\ldots$, etc. are constant coefficients. To establish this series, we must find the values of the constant coefficients. This could only be done by an appropriate substitution of the value of x that will render all the terms zeros except one.

Thus for x=0:

$$f(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots\cdots +$$
$$\Rightarrow f(0) = a_0$$

Since there is no other values of x for which other constants can be found, the only method is to differentiate both sides of the identity with respect to x to obtain the value of a_1. Thus, from equation (1), we obtain

$$f^{/}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots\cdots +$$

For x=0:

$$f^{/}(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + 4a_4(0)^3 + 5a_5(0)^4 + \cdots\cdots +$$
$$\Rightarrow f^{/}(0) = a_1 \Rightarrow a_1 = f^{/}(0)$$

Similarly, $f^{//}(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots\cdots +$

For x=0:

$$f^{//}(0) = 2a_2 + 6a_3(0) + 12a_4(0)^2 + 20a_5(0)^3 + \cdots\cdots +$$
$$\Rightarrow f^{//}(0) = 2a_2 \Rightarrow a_2 = \frac{f^{//}(0)}{2} = \frac{f^{//}(0)}{2!}$$

Again, $f^{///}(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + \cdots\cdots +$

For x=0:

$$f^{///}(0) = 6a_3 + 24a_4(0) + 60a_5(0)^2 + \cdots\cdots +$$
$$\Rightarrow f^{///}(0) = 6a_3 \Rightarrow a_3 = \frac{f^{///}(0)}{6} = \frac{f^{///}(0)}{3!}$$

In similar vein, $f^{/\mathrm{v}}(x) = 24a_4 + 120a_5 x + \cdots\cdots +$

For x=0:

$$f^{/\mathrm{v}}(0) = 24a_4 + 120a_5(0) + \cdots\cdots +$$
$$\Rightarrow f^{/\mathrm{v}}(0) = 24a_4 \Rightarrow a_4 = \frac{f^{/\mathrm{v}}(0)}{24} = \frac{f^{/\mathrm{v}}(0)}{4!}$$

Finally, $f^{\mathrm{v}}(x) = 120a_5 + \cdots\cdots +$

For x=0:

$$f^{v}(0) = 120a_5 + \cdots\cdots +$$

$$\Rightarrow f^{v}(0) = 120a_5 \Rightarrow a_5 = \frac{f^{v}(0)}{120} = \frac{f^{v}(0)}{5!}$$

Thus, $a_0 = f(0)$, $a_1 = f^{/}(0)$, $a_2 = \frac{f^{//}(0)}{2!}$, $a_3 = \frac{f^{///}(0)}{3!}$, $a_4 = \frac{f^{/v}(0)}{4!}$, $a_5 = \frac{f^{v}(0)}{5!}$,…….

Substituting the expressions for $a_0, a_1, a_2, a_3, ...$, etc. into equation (1), we obtain:

$$f(x) = f(0) + xf^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{/v}(0) + \frac{x^5}{5!}f^{v}(0) + \cdots\cdots +$$

This general series is known as Maclaurin's Series. Thus, in general, provided we can differentiate a given function over and over again, and find the values of the derivatives when x is put to zero, then this method would enable us to express any function as a series of ascending powers of x.

2.10.1 Limiting Values – Indeterminate Forms

It is sometimes necessary to find the limiting value of a function of x when $x \to 0$ or when $x \to a$

For example if we desire to determine $\lim_{x\to 0} it\left[\frac{x^2+5x-14}{x^2-5x+8}\right]$

then,

$$\lim_{x\to 0} it\left[\frac{x^2+5x-14}{x^2-5x+8}\right] = \frac{(0)^2+5(0)-14}{(0)^2-5(0)+8} = \frac{-14}{8} = -\frac{7}{4}$$

Thus, $-\frac{7}{4}$ is the limiting value of the function as $x \to 0$

Similarly, if for the function $\frac{x^2+5x-14}{x^2-5x+6}$, we wish to determine $\lim_{x\to 2} it\left[\frac{x^2+5x-14}{x^2-5x+6}\right]$

then, putting x = 2 in the function we obtain:

$$\lim_{x\to 2} it\left[\frac{x^2+5x-14}{x^2-5x+6}\right] = \frac{(2)^2+5(2)-14}{(2)^2-5(2)+6} = \frac{14-14}{10-10} = \frac{0}{0}$$

Thus, suppose we have two functions $f(x)$ and $g(x)$ such that $\lim_{x\to x_0} it\, f(x) = A$ and $\lim_{x\to x_0} it\, g(x) = B$ where A and B are either both zero or both infinite, then the ratio $\frac{f(x_0)}{g(x_0)}$ is an indeterminate quantity $\frac{0}{0}$ or $\frac{\infty}{\infty}$; however, the limit of $\frac{f(x_0)}{g(x_0)}$ as $x \to x_0$ may exist.

The series expansion can facilitate the evaluation of such limits.

Example. Show that (i) $\lim_{x\to 0} it\left[\frac{Sinx}{x}\right]=1$ (ii) $\lim_{x\to 0} it\left[\frac{Cosx-1}{x}\right]=0$

Solution:

(i) By the Maclaurin's series, the expansion of Sin x is given by:

$$Sinx = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\cdots$$

Substituting the expansion of Sin x into the given function, we obtain

$$\lim_{x\to 0} it\left[\frac{Sinx}{x}\right]=\lim_{x\to 0} it\left[\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\cdots}{x}\right]$$

$$=\lim_{x\to 0} it\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots+\cdots\right]$$

$$=\left[1-\frac{(0)^2}{3!}+\frac{(0)^4}{5!}-\frac{(0)^6}{7!}+\frac{(0)^8}{9!}-\cdots+\cdots\right]$$

$$= 1$$

(ii) By the Maclaurin's series, the expansion of Cos x is given by:

$$Cosx = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\cdots$$

Substituting the expansion of Cos x into the given function, we obtain

$$\lim_{x\to 0} it\left[\frac{Cosx-1}{x}\right]=\lim_{x\to 0} it\left[\frac{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\cdots-1}{x}\right]$$

$$=\lim_{x\to 0} it\left[-\frac{x}{2!}+\frac{x^3}{4!}-\frac{x^5}{6!}+\frac{x^7}{8!}-\cdots+\cdots\right]$$

$$=\left[-\frac{0}{2!}+\frac{(0)^3}{4!}-\frac{(0)^5}{6!}+\frac{(0)^7}{8!}-\cdots+\cdots\right]$$

$= 0$

The use of the series expansion may sometimes be avoided. For instances of such nature, the following theorems called L'Hospital's rule will facilitate the evaluation of such limits.

Theorem I: If *f (x)* and *g(x)* are differentiable in the open interval (a, b) except possibly at a point x_0 in this interval, and if $f(x_0) = g(x_0) = 0$, $g^{/}(x) \neq 0$ for $x \neq x_0$ then,

$$\lim_{x \to x_0} it\left[\frac{f(x)}{g(x)}\right] = \lim_{x \to x_0} it\left[\frac{f^{/}(x)}{g^{/}(x)}\right]$$

whenever the limit on the right hand side can be found. If $f^{/}(x)$ and $g^{/}(x)$ satisfy the same conditions as *f (x)* and *g(x)* in the interval (a, b), then the theorem can be applied to *f′ (x)* and *g′ (x)* and so on for higher derivatives.

Theorem II: If *f (x)* and *g(x)* are differentiable in the open interval (a, b) except possibly at a point x_0 in this interval, and if $f(x_0) = g(x_0) = \infty$, $g^{/}(x) \neq \infty$ for $x \neq x_0$ then,

$$\lim_{x \to x_0} it\left[\frac{f(x)}{g(x)}\right] = \lim_{x \to x_0} it\left[\frac{f^{/}(x)}{g^{/}(x)}\right]$$

Again, provided the limit on the right hand side exists.

Example. Use the algebraic technique of factorization to find the $\lim_{x \to 3} it\left[\frac{x^2 - 9}{x^2 - x - 6}\right]$

Solution:

$$\lim_{x \to 3} it\left[\frac{x^2 - 9}{x^2 - x - 6}\right] = \lim_{x \to 3} it\left[\frac{(x-3)(x+3)}{(x-3)(x+2)}\right] = \lim_{x \to 3} it\left[\frac{(x+3)}{(x+2)}\right]$$

$$= \frac{3+3}{3+2} \quad = \quad \frac{6}{5}$$

Example. Use L'Hospital's rule to show that:

(i) $\lim_{x \to 0} it\left[\frac{\sin x}{x}\right] = 1$

(ii) $\lim_{x \to 0} it\left[\frac{\cos x - 1}{x}\right] = 0$

Solution:

(i) $\lim_{x\to 0} it\left[\frac{\sin x}{x}\right] = \lim_{x\to 0} it\left[\frac{d/dx\{\sin x\}}{d/dx\{x\}}\right] = \lim_{x\to 0} it\left[\frac{\cos x}{1}\right] = 1$

(ii) $\lim_{x\to 0} it\left[\frac{\cos x - 1}{x}\right] = \lim_{x\to 0} it\left[\frac{d/dx\{\cos x - 1\}}{d/dx\{x\}}\right] = \lim_{x\to 0} it\left[\frac{-\sin x}{1}\right] = 0$

Example. Use L'Hospital's rule to find the $\lim_{x\to 3} it\left[\frac{x^2-9}{x^2-x-6}\right]$

Solution:

The limits have the $\frac{0}{0}$ form so by L'Hospital's rule:

$$\lim_{x\to 3} it\left[\frac{x^2-9}{x^2-x-6}\right] = \lim_{x\to 3} it\left[\frac{2x}{2x-1}\right] = \frac{2(3)}{2(3)-1} = \frac{6}{5}$$

Example. Find the $\lim_{x\to 0} it\left[\frac{\tan 2x}{\ln(1+x)}\right]$

Solution:

Both numerator and denominator have limits 0. Thus, the limit has the $\frac{0}{0}$ form so by L'Hospital rule,

$$\lim_{x\to 0} it\left[\frac{\tan 2x}{\ln(1+x)}\right] = \lim_{x\to 0} it\left[\frac{d/dx\{\tan 2x\}}{d/dx\{\ln(1+x)\}}\right] = \lim_{x\to 0} it\left[\frac{2\sec^2 2x}{1/(1+x)}\right] = 2$$

Example. Find the $\lim_{x\to\infty} it\left[\frac{x}{e^x}\right]$

Solution:

Both x and e^x tend to ∞ as $x\to\infty$. Hence by L'Hospital's rule

$$\lim_{x\to\infty} it\left[\frac{x}{e^x}\right] = \lim_{x\to\infty} it\left[\frac{1}{e^x}\right] = 0$$

Example. Show that if a is any positive real number, then $\lim_{x\to\infty} it\left[\frac{\ln x}{x^a}\right] = 0$

Solution:

Both $\ln x$ and x^a tends to ∞ as $x\to\infty$. Hence by one application of

L'Hospital's rule:

$$\lim_{x\to\infty} it\left[\frac{\ln x}{x^a}\right] = \lim_{x\to\infty} it\left[\frac{1/x}{ax^{a-1}}\right] = \lim_{x\to\infty} it\left[\frac{1}{ax^a}\right] = 0$$

2.10.2 Expansion of Functions

Let f(x) be any function of x and let a be a point provided $f(a)$ exists and that successive derivatives of f(x) all have finite values when $x = a$. f(x) may be expressed as an infinite series in ascending powers of n. We assume that it is in order to differentiate finite series term by term.

Let $f(\mathrm{x}) = \mathrm{c}_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$

Then, $f^{/}(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

$$f^{//}(x) = 2c_2 + 3\bullet 2c_3(x-a) + 4\bullet 3c_4(x-a)^2 + 5\bullet 4c_5(x-a)^3 + \cdots$$

$$f^{///}(x) = 3\bullet 2c_3 + 4\bullet 3\bullet 2c_4(x-a) + 5\bullet 4\bullet 3c_5(x-a)^2 + \cdots$$

$$f^{/v}(x) = 4\bullet 3\bullet 2c_4 + 5\bullet 4\bullet 3\bullet 2c_5(x-a) + \cdots$$

$$f^{v}(x) = 5\bullet 4\bullet 3\bullet 2c_5 + \cdots$$

Putting $x = a$ in each of the functions above, we obtain

$$c_0 = f(a); \qquad c_1 = f^{/}(a); \qquad c_2 = \frac{f^{//}(a)}{2!}; \quad c_3 = \frac{f^{///}(a)}{3!} \quad \text{and} \quad c_4 = \frac{f^{/v}(a)}{4!}$$

Thus,

$$f(x) = f(a) + (x-a)f^{/}(a) + \frac{(x-a)^2}{2!}f^{//}(a) + \frac{(x-a)^3}{3!}f^{///}(a) + \frac{(x-a)^4}{4!}f^{/v}(a) + \cdots$$

The above infinite series is called a Taylor's Series for f(x) about $x = a$. If $x = a + h$, then the series can be expressed in ascending powers of h, thus,

$$f(x) = f(a+h) = f(\mathrm{a}) + hf^{/}(a) + \frac{h^2}{2!}f^{//}(a) + \frac{h^3}{3!}f^{///}(a) + \frac{h^4}{4!}f^{/v}(a) + \cdots$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} h^n \frac{f^n(a)}{n!}$$

Example. Use Taylor's theorem to expand $Sin\left(\frac{\pi}{6} + h\right)$ in ascending powers of h as far as the term in h^4.

Solution:

$$f(x) = Sinx = Sin\left(\frac{\pi}{6} + h\right)$$

$$f\left(\frac{\pi}{6}\right) = Sin\frac{\pi}{6} = \frac{1}{2}$$

$$f^{/}(x) = Cosx \Rightarrow f^{/}\left(\frac{\pi}{6}\right) = Cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f^{//}(x) = -Sinx \Rightarrow f^{//}\left(\frac{\pi}{6}\right) = -Sin\frac{\pi}{6} = -\frac{1}{2}$$

$$f^{///}(x) = -Cosx \Rightarrow f^{///}\left(\frac{\pi}{6}\right) = -Cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$f^{/v}(x) = Sinx \Rightarrow f^{/v}\left(\frac{\pi}{6}\right) = Sin\frac{\pi}{6} = \frac{1}{2}$$

Now using the relation:

$$f(x) = f(a+h) = f(\mathrm{a}) + hf^{/}(a) + \frac{h^2}{2!} f^{//}(a) + \frac{h^3}{3!} f^{///}(a) + \frac{h^4}{4!} f^{/v}(a) + \cdots$$

we obtain

$$Sin\left(\frac{\pi}{6} + h\right) = \frac{1}{2} + hf^{/}\left(\frac{\pi}{6}\right) + \frac{h^2}{2!} f^{//}\left(\frac{\pi}{6}\right) + \frac{h^3}{3!} f^{///}\left(\frac{\pi}{6}\right) + \frac{h^4}{4!} f^{/v}\left(\frac{\pi}{6}\right) + \cdots$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} h - \frac{1}{4} h^2 - \frac{\sqrt{3}}{12} h^3 + \frac{1}{48} h^4 + \cdots$$

In a special case where $a = 0$, the Taylor's series reduces to the Maclaurin's Series.

2.11 ROLLLE'S THEOREM

If f(x) is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) and $f(a) = f(b)$, then there exists a point x^* in (a,b) such that $f^{/}(x^*) = 0$.

Extreme Value Theorem: If f, a continuous function has a maximum value $f(M)$ and a minimum value $f(m)$ on the closed interval $[\mathrm{a}, b]$, then either $f(M) = f(m)$ or $f(M) \neq f(m)$.

Theorem on Local Extrema: If $f(x^*)$ is a local extremum, then either f is not differentiable at x^* or $f^{/}(x^*) = 0$.

We will use these theorems to prove the Rolle's theorem.

PROOF OF ROLLE'S THEOREM

Case I

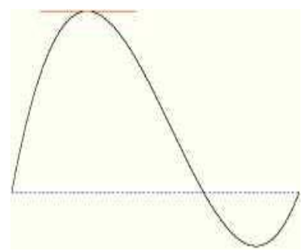
We suppose the maximum value is equal to the minimum value, thus $f(M) = f(m)$.

This implies all the values of f on $[a, b]$ are equal and f is constant on $[a, b]$, therefore $f'(x) = 0$ for all $x \in (a,b)$. So one may take x^* to be anything in (a,b); for example $x^* = \dfrac{a+b}{2}$ would suffice.

Case II

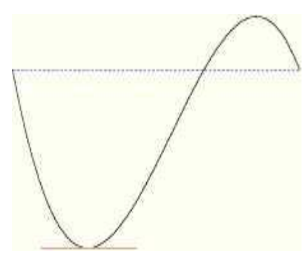
Now, we suppose $f(M) \neq f(m)$, so at least one of $f(M)$ and $f(m)$ is not equal to the value $f(a) = f(b)$

a. We first consider the case where the maximum value $f(M) \neq f(a) = f(b)$, that is M is neither a nor b. But M is in $[a, b]$ and not at the end points, thus M must be in the open interval (a,b). We have the maximum value $f(\mathrm{M}) \geq \mathrm{f(x)}$ for all $x \in [a,b]$ which contains the open interval (a,b), so we also have $f(\mathrm{M}) \geq \mathrm{f(x)}$ for every $x \in (a,b)$. Since M is also in the open interval (a,b), by definition, $f(M)$ is a local maximum.



Since, M is in the open interval (a,b), by hypothesis, we have that f is differentiable at M. Now by the Theorem on Local Extrema, we have that $f'(M) = 0$, so we take $x^* = M$ and that concludes this case.

b. We now consider the case where the minimum value $f(m) \neq f(\mathrm{a}) = f(b)$, that is m is neither a nor b. But m is in $[a, b]$ and not at the end points, thus m must be in the open interval (a,b). We have the minimum value $f(\mathrm{m}) \leq \mathrm{f(x)}$ for all $x \in [a,b]$ which contains the open interval (a,b), so we also have $f(\mathrm{m}) \leq \mathrm{f(x)}$ for every $x \in (a,b)$. Since m is also in the open interval (a,b), by definition, $f(\mathrm{m})$ is a local minimum.



Since, m is in the open interval (a,b), by hypothesis, we have that f is differentiable at m. Now by the Theorem on Local Extrema, we have that $f^{/}(\mathrm{m})=0$, so we take $x^{*}=m$ and that concludes this case.

Hence the proof.

Examples. 1. Verify the Rolle's theorem for $\mathrm{f}(\mathrm{x})=\mathrm{x}^3-4\mathrm{x}$ on $[-2,2]$.

Solution: At $x=-2;\ f(-2)=(-2)^3-4(-2)=0$

At $x=2;\ f(2)=(2)^3-4(-2)=0$

$\Rightarrow \mathrm{f}(2)=\mathrm{f}(-2)=0$

Now, $f(x)=x^3-4x \Rightarrow f^{/}(x)=3x^2-4$

Thus, $\mathrm{f}^{/}(\mathrm{x})=3\mathrm{x}^2-4$ must be zero at least once between -2 and 2

$\Rightarrow \mathrm{f}^{/}(\mathrm{x})=3\mathrm{x}^2-4=0 \quad \Rightarrow \mathrm{x}=\pm\frac{2}{3}\sqrt{3}$

Therefore $\mathrm{f}^{/}(\mathrm{x})=0$ at $\mathrm{x}_1^{*}=-\frac{2}{3}\sqrt{3}$ and $\mathrm{x}_2^{*}=\frac{2}{3}\sqrt{3}$ each of which is between -2 and 2. Hence, the theorem is verified.

2. Verify the Rolle's theorem for $f(x)=x^2(1-x)^2$ on $[0\le x\le 1]$.

Solution: At $x=0;\ f(0)=(0)^2(1-0)^2=0$

At $x=1;\ f(1)=(1)^2(1-1)^2=0$

$\Rightarrow f(0)=f(1)=0$

Now, $f(x)=x^2(1-x)^2 \Rightarrow f^{/}(x)=2x(1-x)^2-2x^2(1-x)$

Thus, $f^{/}(x)=2x(1-x)^2-2x^2(1-x)$ must be zero at least once between 0 and 1

$\Rightarrow f^{/}(x)=2x(1-x)^2-2x^2(1-x)=0 \Rightarrow 2x(1-x)(1-2x)=0$

$$\Rightarrow x = 0; x = 1; x = \frac{1}{2}$$

Therefore, $f^{/}(x^{*}) = 0$ at $x^{*} = \frac{1}{2}$ which is between 0 and 1. Hence, the theorem is verified.

2.12 MEAN VALUE THEOREM

Let $a \prec b$. If f is continuous on the closed interval $[a,b]$ and differentiable in (a,b), then there exists a point x^{*} in (a,b) such that $f^{/}(x^{*}) = \frac{f(b)-f(a)}{b-a}$.

Geometrically, the theorem states that a secant line drawn through two points on a smooth graph is parallel to the tangent line at some intermediate point on the curve. Thus, under appropriate smoothness conditions, the slope of the curve at some point between a and b is the same as the slope of the line joining $(a, f(a))$ to $(\mathrm{b}, f(b))$.

If f satisfies the hypotheses of the Rolle's theorem, then the Mean Value theorem also applies and $f(b)-f(a)=0$. For the x^{*} given by the Mean Value theorem, we have that $f^{/}(x^{*}) = \frac{f(b)-f(a)}{b-a} = 0$. So the Mean Value theorem says nothing new in this case but it does add information when $f(a) \neq f(b)$. The proof of the Mean Value theorem is accomplished by finding a way to apply Rolle's theorem. One considers the line joining the points $(a, f(a))$ and $(\mathrm{b}, f(b))$. The difference between f and that line is a function that turns out to satisfy the hypotheses of Rolle's theorem which then yields the desired result.

PROOF

The equation of the secant through $(a, f(a))$ and $(\mathrm{b}, f(b))$ is

$$y - f(a) = \frac{f(b)-f(a)}{b-a} \bullet (x-a)$$

which we can rewrite as $y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$

Let g be the difference between f and this line, that is

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b-a}(x-a) + f(a) \right]$$

g is the difference between two continuous functions so g is continuous on $[a,b]$. Also, g is the difference of two differentiable functions so g is differentiable on (a,b). Moreover, the derivative of g is the difference between the derivative of f and the derivative (slope) of the line. That is,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

Both f and the line go through the points $(a, f(a))$ and $(\mathrm{b}, f(b))$ so the difference between them is 0 at a and at b. Indeed,

$$g(a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b-a}(a-a) \right] = f(a) - \left[f(a) + 0 \right] = 0 \quad \text{and}$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b-a}(b-a) \right] = f(b) - \left[f(a) + f(b) - f(a) \right] = 0$$

Therefore, Rolle's theorem applies to g since $g(a) = g(b) = 0$, so there exist a point x^* in (a,b) such that $g'(x^*) = 0$. Using our calculated $g'(x)$ above, we have

$$g'(x^*) = f'(x^*) - \frac{f(b) - f(a)}{b-a} = 0$$

Hence, $f'(x^*) = \dfrac{f(b) - f(a)}{b-a}$ and that ends the proof.

Example. We illustrate the Mean Value theorem by considering $f(x) = x^3$ on the interval $[1,3]$.

Solution:

f is a polynomial so continuous everywhere. For any x, $f'(x)=3x^2$ so f is continuous on the $[1,3]$ and differentiable on $(1,3)$. Therefore, the Mean Value theorem applies to f and $[1,3]$.

$$\frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{27-1}{2}=13$$

$f'(x^*)=3(x^*)^2$, so we seek a point x^* in $[1,3]$ with $3(x^*)^2=13$.

$3(x*)^2=13$ iff $(x^*)^2=\frac{13}{3}\Rightarrow x^*=\pm\sqrt{\frac{13}{3}}$.

$-\sqrt{\frac{13}{3}}$ is not in the interval $(1,3)$ but $\sqrt{\frac{13}{3}}$ is in the interval since it is a little bigger than 2, thus approximately 2.0817.

Therefore, $x^*=\sqrt{\frac{13}{3}}$ is in the interval $(1,3)$ and

$$f'(x^*)=f'\left(\sqrt{\frac{13}{3}}\right)=13=\frac{f(3)-f(1)}{3-1}=\frac{f(b)-f(a)}{b-a}$$

Examples.

1. Verify the theorem of the mean for $f(x)=2\sqrt{x}$ on $[1,4]$.

Solution: At $x=1$; $f(1)=2\sqrt{1}=2$

At $x=4$; $f(4)=2\sqrt{4}=4$

$$\Rightarrow \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{4-2}{3}=\frac{2}{3}$$

Differentiating $f(x)=2\sqrt{x}$, we get

$$f'(x)=\frac{1}{\sqrt{x}}\Rightarrow f'(x^*)=\frac{1}{\sqrt{x^*}}$$

But $f'(x^*)=\frac{f(b)-f(a)}{b-a}\Rightarrow\frac{1}{\sqrt{x^*}}=\frac{2}{3}$

Therefore, $x^*=\frac{9}{4}$.

Hence, the theorem is verified since $1 < x^* = \frac{9}{4} < 4$.

2. Verify the theorem of the mean for $f(x) = 2x^2 - 7x + 10$ on $[2,5]$

Solution: At $x = 2$; $f(2) = 2(2)^2 - 7(2) + 10 = 4$

At $x = 5$; $f(5) = 2(5)^2 - 7(5) + 10 = 25$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} = \frac{f(5) - f(2)}{5-2} = \frac{25-4}{3} = 7$$

Differentiating $f(x) = 2x^2 - 7x + 10$, we get $f'(x) = 4x - 7 \Rightarrow f'(x^*) = 4x^* - 7$

By the theorem of the mean, we have

$$f'(x^*) = \frac{f(b) - f(a)}{b-a} \Rightarrow 4x^* - 7 = 7$$

$$\Rightarrow x^* = \frac{7}{2}$$

Hence, the theorem is verified since $2 < x^* = \frac{7}{2} < 5$.

Exercise

1. Differentiate from first principle the following functions.

(i) $y = 5x^2 + 2$ (iii) $y = 5x^3 + 2x^2 + 3x + 4$ (iv) $y = x^2 Sinx$

(ii) $h(x) = \frac{5x^2 - x + 7}{x^2 + x + 10}$

2. Find from first principle the derivative with respect to x of the function $f(x) = Inx$.

3. Differentiate the following with respect to x.

(i) $y = In(x^2 + 5x - 3)$ (iii) $y = xSin^{-1}x$

(ii) $y = \frac{x^{3/2}}{1 + x^{1/2}}$

4. If $x = a(Cos\theta + \theta Sin\theta)$ and $y = a(Sin\theta - \theta Cos\theta)$. Find $\frac{dy}{dx}$ in its simplest form.

5. Differentiate with respect to x simplifying as far as possible the functions

(i) $f(x) = \tan 2x \cos x$ (ii) $f(x) = \dfrac{\cos 2x}{x^2}$

6. Find the derivative with respect to x of the following

(i) $\cos ec^{-1} x$ (ii) $\cot^{-1} x$

(iii) $\tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ (iv) $\cos^{-1}\left(\dfrac{x-2}{2}\right)$ (v) $\sin^{-1}\left(\dfrac{x}{2}\right)$

7. Find $\dfrac{dy}{dx}$ given that

(i) $y = (1-x^2)\sin^{-1} x$ (ii) $y = \tan^{-1}(2x-1)$

8. Determine the following limits:

(i). $\lim_{x\to 0} it\left[\dfrac{1-\cos x}{x^2+3x}\right]$ (ii). $\lim_{x\to 0} it\left[\dfrac{1-\cos(x^2)}{x^3 \sin x}\right]$

(iii). $\lim_{x\to(1/2)\pi} it\left[\dfrac{\cos x}{x-\frac{1}{2}\pi}\right]$ (iv). $\lim_{x\to 0} it\left[\dfrac{e^x - 1 - x - x^2/2 - x^3/6}{x^4}\right]$

(v). $\lim_{x\to\pi/2} it\left[\tan x \bullet \ln \sin x\right]$

9. Use Leibnitz's theorem to find the following

(a) $\dfrac{d^4}{dx^4}(x^2 Sinx)$ (b) $\dfrac{d^8}{dx^8}\left[x^2(3x+1)^{12}\right]$ (c) $\dfrac{d^n}{dx^n}\left(x^4 e^{-2x}\right)$

10. Find the stationary point on the following curves and determine the type.

(i) $f(x) = \dfrac{1}{3}x^3 - 2x^2 + 3x$ (ii) $y = x^3 - 3x + 2$ (iii) $y = \dfrac{(x-1)^2}{x}$

CHAPTER THREE

INTEGRATION

3.1 INTRODUCTION

Integration is the reverse process of differentiation. When we differentiate, we start with an expression and proceed to find its derivative. When we integrate, we start with the derivative and find the expression from which it has been derived. Thus, if f(x) is given, then any function F(x) such that $F'(\mathrm{x}) = \mathrm{f(x)}$ is called the anti-derivative or integral of f(x) with respect to x. It could be deduced that if F(x) is the integral of f(x), then $F(\mathrm{x}) + \mathrm{c}$, where c is any constant is also an integral of f(x) since $\left[F(x)+c\right]' = F'(x) = f(x)$. Thus, all indefinite integrals differ by a constant. The symbol $\int f(x)dx$ denotes the integral of $f(x)$ with respect to the variable x. The expression $f(x)$ to be integrated is called the integrand.

For example, $\frac{d}{dx}(x^4) = 4x^3$. Therefore the integral of $4x^3$ with respect to x is known to be x^4. Comparing the example with the information above implies $F(x) = x^4$ and $f(x) = 4x^3$.

3.2 ALGEBRAIC INTEGRATION

3.2.1 Power of x

Consider $\frac{d}{dx}(x^n) = nx^{n-1}$.

Replacing n by $(n+1)$, we obtain

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$\therefore \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \, , n \neq -1$. This is known as the Power Rule

3.2.2 Constant of Integration

Let us consider the following

$\frac{d}{dx}(x^4) = 4x^3$ $\qquad \therefore \int 4x^3 \, dx = x^4$

$\frac{d}{dx}(x^4+2) = 4x^3$ $\qquad \therefore \int 4x^3 \, dx = x^4 + 2$

$\frac{d}{dx}(x^4 - 5) = 4x^3$ $\qquad\qquad\qquad \therefore \int 4x^3\,dx = x^4 - 5$

In the three examples, we happen to know the expressions from which the derivative $4x^3$ was derived. But any constant term in the original expression becomes zero in the derivative and all trace of it is lost. This means that if the History of the derivative $4x^3$ is not known, then no evidence of the value of the constant term will be $0, +2, -2$ or any other value could be detected. It is therefore necessary to acknowledge the presence of such a constant term of some value by adding a symbol c to the results of the integration. That is,

$$\int 4x^3\,dx = x^4 + c$$

c is called the constant of integration and must always be included. Such an integral is called an indefinite integral. Thus, if $f(x)$ is given, then any function $F(x)$ such that $F'(x) = f(x)$ is called an indefinite integral of $f(x)$. In certain circumstances, however, the value of c might be found if further information about the integral is available.

Example. Determine I = $\int 4x^3\,dx$, given that $I = 3$ when $x = 2$.

Solution: I = $\int 4x^3\,dx = x^4 + c$

But when $x = 2$, $I = 3$.

$\Rightarrow$ I = $x^4 + c = 2^4 + c = 3$

$\Longrightarrow$ $c = 3 - 2^4 = 3 - 16 = -13$

$\therefore$ I = $x^4 - 13$.

Example. Solve $\int x^2\left(x^2 - \frac{1}{x^2}\right)dx$

Soluton: $\int x^2\left(x^2 - \frac{1}{x^2}\right)dx = \int\left(x^4 - \frac{1}{x^2}\bullet x^2\right)dx = \int x^4 dx - \int 1 dx$

$$= \frac{x^5}{5} + c_1 - x + c_2$$

Let $c = c_1 + c_2$, so we have

$$\int x^2\left(x^2 - \frac{1}{x^2}\right)dx = \frac{x^5}{5} - x + c$$

3.2.3 Standard Integrals

Every derivative written in the reverse gives an integral.

Example: $\frac{d}{dx}(Sinx) = Cosx$ $\qquad\qquad \therefore \int Cosx\,dx = Sinx + c$

$$\frac{d}{dx}(Cos\,x) = -Sin\,x \qquad\qquad \therefore \int Sin\,x\,dx = -Cos\,x + c$$

It therefore follows that the list of standard derivatives provides a source of standard integrals.

$$\int \tan x\,dx = \int \frac{Sin\,x}{Cos\,x}dx$$

$$Let\ u = Cos\,x \Rightarrow du = -Sin\,x\,dx \Rightarrow dx = -\frac{1}{Sin\,x}du$$

$$\Rightarrow \int \frac{Sin\,x}{Cos\,x}dx = \int \frac{Sin\,x}{u} \bullet - \frac{1}{Sin\,x}du = -\int \frac{1}{u}du$$

$$= -In\,u + c = -In\,Cos\,x + c$$

$$= In(Cos\,x)^{-1} + c = In\,Sec\,x + c$$

3.2.4 Integration of Exponential/Logarithmic

$$\int e^x dx = e^x + c$$

PROOF

Since we know that the derivative $\frac{d}{dx}e^x = e^x$. We can use the Fundamental Theorem of Calculus

$$\int e^x dx = \int \frac{d}{dx}(e^x)dx = e^x + c$$

Hence proved.

$$\int In\,x\,dx$$

We solve this using integration by parts.

Let $u = In\,x \Rightarrow du = \frac{1}{x}dx \qquad\qquad dv = dx \Rightarrow v = x$

$$\int u\,dv = uv - \int v\,du$$

$$= (In\,x)x - \int x \cdot \frac{1}{x}dx$$

$$= xIn\,x - \int 1dx$$

$$= xIn\,x - x + c$$

$$\int b^x dx$$

We use $\int e^x dx = e^x + c$

Since $e^{Inb} = b$, we have

$$\int b^x dx = \int \left(e^{In b}\right)^x dx$$
$$= \int e^{(In b)x} dx$$

Let $u = (In b)x \Rightarrow du = In b\, dx$

By substitution, we have

$$\int b^x dx = \int e^u \cdot \frac{du}{In b} = \frac{1}{In b}\int e^u du$$
$$= \frac{1}{In b} e^u + c$$

Substituting $u = (In b)x$ yields

$$\int b^x dx = \frac{1}{In b} e^{(In b)x} + c$$
$$= \frac{1}{In b}\left(e^{In b}\right)^x + c$$
$$= \frac{1}{In b} b^x + c = \frac{b^x}{In b} + c$$
$$\therefore \int b^x dx = \frac{b^x}{In b} + c$$

Example. $\int 2^x dx$

We know that $e^{In 2} = 2$

$$\int 2^x dx = \int \left(e^{In 2}\right)^x dx$$
$$= \int e^{(In 2)x} dx$$

Let $u = (In 2)x \Rightarrow du = In 2\, dx$

By substitution, we get

$$\int 2^x dx = \int e^u \cdot \frac{du}{In 2} = \frac{1}{In 2}\int e^u du$$
$$= \frac{1}{In 2} e^u + c$$

Substituting $u = (In 2)x$ yields

$$\int 2^x dx = \frac{1}{In 2} e^{(In 2)x} + c$$
$$= \frac{1}{In 2}\left(e^{In 2}\right)^x + c$$
$$= \frac{1}{In 2} 2^x + c = \frac{2^x}{In 2} + c$$
$$\therefore \int 2^x dx = \frac{2^x}{In 2} + c$$

3.2.5 Integration of Polynomial Expression

Differentiation of polynomial expressions are done term by term and hence polynomial expressions are also integrated term by term. The integral of any polynomial is the sum of the integrals of its terms. A general term of a polynomial can be written as ax^n and the indefinite integral of that term is

$$\int ax^n \, dx = a\frac{x^{n+1}}{n+1} + c$$

where a and c are constants.

The expression applies to both positive and negative values of n except for the special case of $n = -1$.

Example. 1. $\int(4x^3 + 5x^2 - 2x + 7)\,dx = x^4 + \frac{5x^3}{3} - x^2 + 7x + c$

2. $\int\left(x^5 - 2x^{-2} + 1\right)dx = \frac{x^6}{6} - 2\frac{x^{-1}}{-1} + x + c$

3.2.6 Integration of Functions of a Linear Function of x

When the variable x of a function is replaced by a linear expression in x which is of the form $+b$, then the example $y = \int(3x+2)^4\text{dx}$ is of the same structure as $y = \int x^4 dx$ except that x is replaced by a linear expression $3x + 2$. Integration of such forms is done by simple substitution.

Example. Find I = $\int(3x+2)^4\text{dx}$

Solution: Let u = $3x + 2 \Rightarrow \frac{du}{dx} = 3 \;\therefore\; dx = \frac{1}{3}du$

$$\Rightarrow I = \int(3x+2)^4 dx = \int u^4 \cdot \frac{1}{3}du$$

$$= \frac{1}{3}\cdot\frac{1}{5}u^5 + c$$

$$\Rightarrow I = \int(3x+2)^4 dx = \frac{1}{3}\cdot\frac{1}{5}(3x+2)^5 + c$$

$$= \frac{1}{15}(3x+2)^5 + c$$

Thus, to integrate a "function of a linear function of x"; simply replace x in the corresponding standard result by the linear expression and divide by the coefficient of x in the linear expression.

Example. Integrate the following functions:

(i) $(4x-3)^2$ (ii) $\text{Cos } 3x$ (iii) e^{5x+2}

Solution:

(i) Let $u = 4x - 3 \implies \frac{du}{dx} = 4 \therefore dx = \frac{1}{4}du$

$$\int (4x-3)^2\,dx = \int u^2 \cdot \frac{1}{4}du = \int \frac{1}{4}\,u^2 du$$

$$= \frac{1}{4}\cdot\frac{1}{3}\,u^3 + c = \frac{1}{12}\,u^3 + c$$

$$= \frac{1}{12}\,(4x-3)^3 + c$$

(ii) Let $u = 3x \Rightarrow \frac{du}{dx} = 3 \quad \therefore dx = \frac{1}{3}du$

$$\int Cos3xdx = \int Cosu.\frac{1}{3}du = \frac{1}{3}\int Cosudu$$

$$= \frac{1}{3}Sinu + c = \frac{1}{3}Sin3x + c$$

(iii) Let $u = 5x + 2 \Rightarrow \frac{du}{dx} = 5 \qquad \therefore dx = \frac{1}{5}du$

$$\int e^{5x+2}dx = \int e^u \bullet \frac{1}{5}du = \frac{1}{5}\int e^u du$$

$$= \frac{1}{5}e^u + c = \frac{1}{5}e^{5x+2} + c$$

3.3 INTEGRALS OF THE FORM $\int \frac{f'(x)}{f(x)}dx$

From our previous lessons on differentiation, we observed that

$$\frac{d}{dx}\left[Inf(x)\right] = \frac{f'(x)}{f(x)} \qquad \therefore \int \frac{f'(x)}{f(x)}dx = In\left[f(x)\right] + c$$

Consider the integral $\int \frac{2x+3}{x^2+3x-5}dx$. It could be noticed that if the denominator is differentiated, the expression in the numerator is obtained. Now, let u be the denominator, i.e. $u = x^2+3x-5$.

Then, $\frac{du}{dx} = 2x+3 \Rightarrow du = (2x+3)dx$.

The integral can then be written in terms of u as below;

$$\int \frac{2x+3}{x^2+3x-5}dx = \int \frac{1}{u}du = \int \frac{du}{u}$$

$$= Inu + c = In\left(x^2+3x-5\right)+c$$

Hence, any integral in which the numerator is the derivative of the denominator will be of this kind: $\int \frac{f^{/}(x)}{f(x)}dx = In\left[f(x)\right]+c$

Example. Find $\int \frac{3x^2}{x^3-4}dx$.

Solution: $\int \frac{3x^2}{x^3-4}dx = \int \frac{\frac{d}{dx}[x^3-4]}{x^3-4}\,dx$

$$= \ln\,(x^3-4)+c$$

Example. Find the integral of the following functions:

(i) $\frac{6x^2}{x^3-4}$ (ii) $\frac{2x^2}{x^3-4}$ (iii) tan x

Solution:

(i) $\int \frac{6x^2}{x^3-4}\,dx = 2\int \frac{3x^2}{x^3-4}dx$

$$= 2\int \frac{\frac{d}{dx}[x^3-4]}{x^3-4}\,dx$$

$$= 2[\ln(x^3-4)]+\ c$$

(ii) $\int \frac{2x^2}{x^3-4}\,dx = \frac{2}{3}\int \frac{3x^2}{x^3-4}\,dx$

$$= \frac{2}{3}\int \frac{\frac{d}{dx}[x^3-4]}{x^3-4}\,dx$$

$$= \frac{2}{3}\left[In\left(x^3-4\right)\right]+c$$

(iii) $\int \tan x dx = \int \frac{Sinx}{Cosx}dx = -\int \frac{(-Sinx)}{Cosx}dx$

$$= -In Cos x + c$$
$$= In Sec x + c$$

3.4 INTEGRALS OF THE FORM $\int f(x) \bullet f^{/}(x)\,dx$

We shall consider situations in which the function to be integrated is a product of two functions in which one function is the derivative of the other function. Let us consider the example $\int tanx\ sec^2 x\ dx$. The function to be integrated is a function of product of two functions of which the function ($sec^2 x$) of the product is the derivative of the other function (tanx).

Let u = $tanx$, then du = $sec^2 x\ dx$. Thus

$$\int tanx\ sec^2 x\ dx = \int u\,.du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{1}{2}(tanx)^2 + c$$

$$= \frac{1}{2}tan^2 x + c$$

Similarly, this can be represented as below:

$$\int tanx\ sec^2 x\ dx = \int \tan x \frac{d}{dx}[\tan\ x]\ dx$$

$$= \frac{1}{2}tan^2 x + c$$

Example. Find $\int \sin x\ \ cos\,x\ dx$

Solution: $\int \sin x\ \ cos\,x\ dx = \int \sin x \cdot \frac{d}{dx}[\sin x]\ \ dx$

$$= \frac{1}{2}\ Sin^2 x + c$$

Example. Find the following:

(i) $\int \frac{\ln x}{x} dx$ (ii) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution:

(i) $\int \frac{\ln x}{x} dx = \int lnx \cdot \frac{1}{x}\ dx$

$$= \int lnx \cdot \frac{d}{dx}[\ln\ x]\ dx$$

$$= \frac{1}{2}[\ln\ x]^2 + c\ \ .$$

Alternatively, let $u = \ln x \implies\ du = \frac{1}{x}dx$

$\implies \int \ln x \cdot \frac{1}{x}dx = \int u\ \cdot du$

$$= \frac{u^2}{2} + c$$

$$= \frac{1}{2}[Inx]^2 + c$$

(ii) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} dx$

$$= \int \sin^{-1} x \cdot \frac{d}{dx}[\sin^{-1} x]dx$$

$$= \frac{1}{2}(\sin^{-1} x)^2 + c$$

Alternatively, let $u = \sin^{-1} x \implies\ du = \frac{1}{\sqrt{1-x^2}}dx$

$$= \int \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} dx = \int u\ \cdot du$$

$$= \frac{1}{2}u^2 +\ c$$

$$= \frac{1}{2}(\sin^{-1} x)^2 + c$$

3.5 INTEGRATION OF PRODUCTS – INTEGRATION BY PARTS

There are instances where we need to integrate a function which is a product of two functions where either function is not the derivative of the other. For example, in $\int x^2 \ln x\ \ dx$, $\ln x$ is not the derivative of x^2 and x^2 is also not the derivative of ln x. There is therefore the need to find some other method of dealing with the integral.

Assuming u and v are functions of x, then from our knowledge in derivatives, we obtain the equation below:

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides of the above with respect to x, we obtain:

$$\int \frac{d}{dx}[uv]dx = \int u\frac{dv}{dx} \cdot dx + \int v\frac{du}{dx} \cdot dx$$

$$\Longrightarrow \quad uv = \int u\frac{dv}{dx} \cdot dx + \int v\frac{du}{dx} \cdot dx$$

$$\int u\frac{dv}{dx} \cdot dx = uv - \int v\frac{du}{dx} \cdot dx \; \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

It would be noted that on the left hand side, there are two factors to integrate in which one factor is chosen as a function u and the other as the derivative of the function v. To find the function v from its derivative, one must integrate the factor $\frac{dv}{dx}$ separately for v. The functions u and v obtained can be substituted in the right hand side of the equation to complete the routine.

For convenience, the equation (1) is written in the form: $\int udv = uv - \int v\,du$

Example. Find $\int x^2 \ln x \, dx$.

Solution: Let $u = ln\,x \Longrightarrow du = \frac{1}{x}dx$. $\frac{dv}{dx} = x^2 \Longrightarrow v = \frac{x^3}{3}$

$$\Longrightarrow \int x^2 \ln x \; dx = ln\,x\left(\frac{1}{3}x^3\right) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}dx$$

$$= \frac{1}{3}x^3\, ln\,x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left[\frac{1}{3}x^3\right] + c$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$$

Example. Evaluate the following integrals:

$$(i)\int x\,sin\;3x\,dx \quad (ii)\int x^2 e^{2x}dx \quad (iii)\int x^3 \cos 2x\,dx \quad (iv)\int x^2 \cos ax\,dx$$

Solution:

(i) Let $u = x \implies du = dx$; $\frac{dv}{dx} = \sin 3x \implies v = -\frac{1}{3}\cos 3x$

$$\int x\, sin\ 3x\, dx = -\frac{1}{3}\, x \cos 3x - \int -\frac{1}{3}\cos 3x \cdot dx$$

$$= -\frac{1}{3}\, x\cos 3x + \frac{1}{3}\int \cos 3x\ dx$$

$$= -\frac{1}{3}\, x\cos 3x + \frac{1}{3}\left[\frac{1}{3} Sin\ 3x\right] + c$$

$$= -\frac{1}{3}\, x\cos 3x + \frac{1}{9} Sin\ 3x + c$$

(ii) Let $u = x^{\,2} \implies du = 2xdx$; $\frac{dv}{dx} = e^{2x} \implies v = \frac{1}{2}e^{2x}$

$$\int x^2\, e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \int \frac{1}{2} e^{2x}.\, 2xdx$$

$$= \frac{1}{2}x^2 e^{2x} - \int xe^{2x} dx$$

Let $u = x \implies du = dx$; $\frac{dv}{dx} = e^{2x} \implies v = \frac{1}{2}e^{2x}$

$$\Rightarrow \int x^2 e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\, dx$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + c$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x} + c$$

(iii) Let $u = x^3 \implies du = 3x^2 dx$; $\frac{dv}{dx} = Cos\ 2x \implies v = \frac{1}{2}\ Sin\ 2x$

$$\int x^3\ Cos2xdx = \frac{1}{2}x^3 Sin2x - \int \frac{1}{2}\cdot 3x^2 Sin2xdx$$

$$= \frac{1}{2}x^3 Sin2x - \frac{3}{2}\int x^2 Sin2xdx$$

Let $u = x^2 \Rightarrow du = 2xdx$; $\frac{dv}{dx} = Sin\ 2x \Rightarrow v = -\frac{1}{2}\ Cos\ 2x$

$$\Rightarrow \int x^3 Cos2x\ dx = \frac{1}{2}x^3 Sin2x - \frac{3}{2}\left[-\frac{1}{2}x^2 Cos\ 2x - \int -\frac{1}{2}\cdot 2x\ Cos2xdx\right]$$

$$= \frac{1}{2}\ x^3 Sin\ 2x + \frac{3}{4}x^2 Cos\ 2x - \frac{3}{2}\int x\ Cos\ \ 2x\ dx$$

Let $u = x \Rightarrow du = dx$; $\frac{dv}{dx} = Cos\ 2x \Rightarrow v = \frac{1}{2}\ Sin\ 2x.$

$$\Rightarrow \int x^3 Cos2xdx = \frac{1}{2}x^3 Sin2x + \ \frac{3}{4}x^2 Cos\ 2x - \ \frac{3}{2}\left[\frac{1}{2}\ x\ Sin\ 2x - \int \frac{1}{2} Sin2xdx\right]$$

$$= \frac{1}{2}x^3 Sin2x + \frac{3}{4}x^2 Cos\ 2x - \frac{3}{4}x\ Sin\ 2x + \frac{3}{4}\left[-\frac{1}{2}Cos2x\right] + c$$

$$= \frac{1}{2}x^3 Sin2x + \ \frac{3}{4}x^2 Cos\ 2x - \frac{3}{4}x\ Sin\ 2x - \frac{3}{8}cos2x + c$$

(iv) Let u = $x^2 \Rightarrow du = 2xdx$; $\frac{dv}{dx} = Cos\ ax \ \Rightarrow v = \ \frac{1}{a}Sinax$

$$\int x^2 Cos\ ax\ dx = \frac{1}{a}x^2 Sin\ ax - \frac{2}{a}\left[-\frac{1}{a}\cdot x\ Cosax - \int -\frac{1}{a}\ Cos\ ax\ \ dx\right]$$

$$= \frac{1}{a}x^2 Sin\ ax + \frac{2}{a^2}x\ Cos\ ax - \frac{2}{a^2}\int Cos\ ax\ dx$$

$$= \frac{1}{a}x^2 Sin\ ax + \frac{2}{a^2}x\ Cos\ ax - \frac{2}{a^2}\left[\frac{1}{a}Sin\ ax\right] + c$$

$$= \frac{1}{a}x^2 Sin\ ax + \frac{2}{a^2}x\ Cos\ ax - \frac{2}{a^3}Sin\ ax + c$$

There are other instances where the two functions are neither a log factor nor a power of x, but of the form $e^{ax}\,\mathrm{Sin}\,bx$ or $e^{ax}Cosbx$. Part of the result in course of the evaluation appears as though we are back to where we started.

Example. Evaluate $\int e^{ax} Sinbx\ dx$.

Solution: Let $u = Sinbx \Rightarrow du = bCosbx$ and $\frac{dv}{dx} = e^{ax} \Rightarrow v = \frac{1}{a}e^{ax}$.

$$\int e^{ax} Sinbx\ dx = uv - \int vdu$$

$$\Rightarrow \int e^{ax} Sinbxdx = \frac{1}{a} e^{ax} Sinbx - \int \frac{b}{a} e^{ax} Cosbx \ dx$$

$$= \frac{1}{a} e^{ax} Sinbx - \frac{b}{a} \int e^{ax} Cosbx \ \ dx$$

Let $u = Cosbx \Rightarrow du = -bSinbx \ dx$ and $\frac{dv}{dx} = e^{ax} \Rightarrow v = \frac{1}{a} e^{ax}$

$$\Rightarrow \int e^{ax} Sinbxdx = \frac{1}{a} e^{ax} Sinbx - \frac{b}{a} \left[\frac{1}{a} e^{ax} Cosbx - \int -\frac{b}{a} e^{ax} Sinbxdx \right]$$

$$\Rightarrow \int e^{ax} Sinbxdx = \frac{1}{a} e^{ax} Sinbx - \frac{b}{a^2} e^{ax} Cosbx - \frac{b^2}{a^2} \int e^{ax} Sinbx \ dx$$

$$\Rightarrow \int e^{ax} Sinbxdx + \frac{b^2}{a^2} \int e^{ax} Sinbxdx = \frac{1}{a} e^{ax} Sinbx - \frac{b}{a^2} e^{ax} Cosbx$$

$$\Rightarrow \left(1 + \frac{b^2}{a^2} \right) \int e^{ax} Sinbxdx = \frac{1}{a} e^{ax} Sinbx - \frac{b}{a^2} e^{ax} Cosbx$$

$$\Rightarrow \left(\frac{a^2 + b^2}{a^2} \right) \int e^{ax} Sinbxdx = \frac{1}{a^2} \left[ae^{ax} Sinbx - be^{ax} Cosbx \right]$$

$$\Rightarrow \int e^{ax} Sinbxdx = \frac{a^2}{a^2 + b^2} \bullet \frac{1}{a^2} \left[ae^{ax} Sinbx - be^{ax} Cosbx \right]$$

$$\therefore \int e^{ax} Sinbx \ dx = \frac{1}{a^2 + b^2} \left[aSinbx - bCosbx \right] e^{ax} + c$$

Example. Evaluate $\int e^{3x} Sinx \ dx$

Solution: Let $u = e^{3x} \Rightarrow du = 3e^{3x} dx$ and $\frac{dv}{dx} = Sinx \Rightarrow v = -Cosx$.

$$\int e^{3x} Sinx \ dx = uv - \int vdu$$

$$\Rightarrow \int e^{3x} Sinxdx = -e^{3x} Cosx - \int -3e^{3x} Cosx \ dx$$

$$= -e^{3x} Cosx + 3 \int e^{3x} Cosx \ dx$$

Let $u = e^{3x} \Rightarrow du = 3e^{3x} dx$ and $\frac{dv}{dx} = Cosx \Rightarrow v = Sinx$.

$$\Rightarrow \int e^{3x} Sinxdx = -e^{3x} Cosx + 3 \left[e^{3x} Sinx - \int 3e^{3x} Sinxdx \right]$$

$$\Rightarrow \int e^{3x} Sinxdx = -e^{3x} Cosx + 3e^{3x} Sinx - 9 \int e^{3x} Sinx \ dx$$

$$\Rightarrow \int e^{3x} Sinxdx + 9 \int e^{3x} Sinxdx = -e^{3x} Cosx + 3e^{3x} Sinx$$

$$\Rightarrow 10\int e^{3x}Sinxdx = \left(3Sinx - Cosx\right)e^{3x} + c$$

$$\therefore \int e^{3x}Sinxdx = \frac{1}{10}\left(3Sinx - Cosx\right)e^{3x} + c$$

The examples considered provides us with priority order for u in functions involving In x, x^n and e^{kx}. Thus if:

(i) One factor is a log function, that must be taken as "u".

(ii) There is no log function but a power of x, then the power of x must be taken as "u".

(iii) There is neither a log function nor a power of x, then the exponential function is taken as "u".

Remembering this priority order will save a lot of false starts.

3.6 INTEGRATION BY PARTIAL FRACTIONS

Suppose one has $\int \frac{x+1}{x^2-3x+2}dx$ to evaluate, it would be noted that, this does not fall under the standard types that exist and at the same time, the numerator is not the derivative of the denominator.

In a case of this nature, one must first of all express the algebraic fraction in terms of its partial fractions which in most cases provide a number of simpler algebraic fractions which one would most likely be able to integrate separately without difficulty. The rules for consideration are as follows:

a. The numerator of the given function must be of lower degree than that of the denominator. If it is not, then first of all, divide out by long division.

b. Factorize the denominator into its prime factors since the factors obtained determine the shape of the partial fractions.

c. A linear factor $ax+b$ gives a partial fraction of the form $\frac{A}{ax+b}$.

d. Factors of the form $\left(ax+b\right)^n$ give a partial fraction of the form

$$\frac{A}{ax+b} + \frac{B}{\left(ax+b\right)^2} + \cdots\cdots + \frac{Z}{\left(ax+b\right)^n}$$

e. A quadratic factor ax^2+bx+c gives a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$

f. Factors of the form $\left(ax^2+bx+c\right)^n$ give a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{\left(ax^2+bx+c\right)^2}+\cdots\cdots+\frac{Yx+Z}{\left(ax^2+bx+c\right)^n}$$

Examples.

1. Evaluate $\displaystyle\int\frac{x+1}{x^2-3x+2}dx$.

Solution: $$\frac{x+1}{x^2-3x+2}=\frac{x+1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

$$\Rightarrow x+1=A(x-2)+B(x-1)$$

Substituting $x=1$, we get

$2=A(-1)+B(0)$ $\qquad \therefore A=-2$

Substituting $x=2$, we get

$3=A(0)+B(1)$ $\qquad \therefore B=3$

$$\Rightarrow\int\frac{x+1}{x^2-3x+2}dx=\int\frac{-2}{x-1}dx+\int\frac{3}{x-2}dx$$

$$=-2\int\frac{1}{x-1}dx+3\int\frac{1}{x-2}dx$$

$$=-2In(x-1)+3In(x-2)+c$$

2. Evaluate $\displaystyle\int\frac{4x^2+26x+5}{2x^2+9x+4}dx$

Solution:
$$\begin{array}{r|l} & \quad 2 \\ 2x^2+9x+4 & 4x^2+26x+5 \\ & \underline{-4x^2+18x+8} \\ & \qquad 8x-3 \end{array}$$

$$\Rightarrow\frac{4x^2+26x+5}{2x^2+9x+4}=2+\frac{8x-3}{2x^2+9x+4}$$

Now, $\dfrac{8x-3}{2x^2+9x+4}=\dfrac{8x-3}{(x+4)(2x+1)}=\dfrac{A}{x+4}+\dfrac{B}{2x+1}$

$$\Rightarrow 8x-3=A(2x+1)+B(x+4)$$

Substituting $x=-4$, we get

$$-35=A(-7)+B(0) \qquad \therefore A=5$$

Substituting $x=-\frac{1}{2}$, we get

$$-7=A(0)+B\left(\frac{7}{2}\right) \qquad \therefore B=-2$$

$$\Rightarrow \int \frac{4x^2+26x+5}{2x^2+9x+4}dx=\int\left[2+\frac{5}{x+4}-\frac{2}{2x+1}\right]dx$$

$$=\int 2dx+5\int\frac{1}{x+4}dx-2\int\frac{1}{2x+1}dx$$

$$=\int 2dx+5\int\frac{1}{x+4}dx-\int\frac{2}{2x+1}dx$$

$$=2x+5In(x+4)-In(2x+1)+c$$

3. Evaluate $\int \frac{x^2}{(x+1)(x-1)^2}dx$.

Solution: $$\frac{x^2}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$\Rightarrow x^2=A(x-1)^2+B(x-1)(x+1)+C(x+1)$$

Substituting $x=1$, we get

$$1=A(0)+B(0)+C(2) \qquad \therefore C=\frac{1}{2}$$

Substituting $x=-1$, we get

$$1=A(-2)(-2)+B(-2)(0)+C(0) \qquad \therefore A=\frac{1}{4}$$

Expanding our expression above, grouping like terms and setting coefficients equal yields

$$A+B=1$$

Substituting A into the equation, we get

$$\frac{1}{4}+B=1\Rightarrow B=1-\frac{1}{4} \qquad \therefore B=\frac{3}{4}$$

$$\Rightarrow \int \frac{x^2}{(x+1)(x-1)^2}\,dx = \int \left[\frac{1/4}{x+1} + \frac{3/4}{x-1} + \frac{1/2}{(x-1)^2} \right] dx$$

$$= \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx$$

$$= \frac{1}{4} In(x+1) + \frac{3}{4} In(x-1) - \frac{1}{2} \bullet \frac{1}{x-1} + c$$

$$= \frac{1}{4} In(x+1) + \frac{3}{4} In(x-1) - \frac{1}{2(x-1)} + c$$

We now consider a quadratic factor which will not factorize any further.

Example. Evaluate $\int \frac{x^2}{(x-2)(x^2+1)}\,dx$

Solution: $\frac{x^2}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$$\Rightarrow x^2 = A(x^2+1) + (Bx+C)(x-2)$$

Substituting $x=2$, we get

$4 = A(5) + (2B+C)(0)$ $\qquad \therefore A = \frac{4}{5}$

Expanding our expression above, grouping like terms and setting coefficients equal yields

$A+B=1$ $\qquad A-2C=0$

Substituting A into the equation $A+B=1$, we get

$\frac{4}{5} + B = 1 \Rightarrow B = 1 - \frac{4}{5}$ $\qquad \therefore B = \frac{1}{5}$

Substituting A into the equation $A-2C=0$, we get

$\frac{4}{5} - 2C = 0 \Rightarrow 2C = \frac{4}{5}$ $\qquad \therefore C = \frac{2}{5}$

$$\Rightarrow \int \frac{x^2}{(x-2)(x^2+1)}\,dx = \int \left[\frac{4/5}{x-2} + \frac{1/5\,x + 2/5}{x^2+1} \right] dx$$

$$= \frac{4}{5}\int \frac{1}{x-2}\,dx + \int \frac{1/5\,x}{x^2+1}\,dx + \int \frac{2/5}{x^2+1}\,dx$$

$$= \frac{4}{5}\int \frac{1}{x-2}dx + \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{2}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{4}{5}In(x-2) + \frac{1}{5}\bullet\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{2}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{4}{5}In(x-2) + \frac{1}{10}In(x^2+1) + \frac{2}{5}\tan^{-1} x + c$$

3.7 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

3.7.1 Powers of Sinx and of Cos x

From $\frac{d}{dx}(Sinx) = Cosx$, we have that $\int Cosxdx = Sinx + c$.

Similarly, $\frac{d}{dx}(Cosx) = -Sinx$, so we have

$$\int Sinxdx = -Cosx + c$$

To integrate $Sin^2 x$ and $Cos^2 x$, one must express the function in terms of the Cosine of the double angle:

$Cos2x = 1 - 2Sin^2 x \Rightarrow Sin^2 x = \frac{1}{2}(1 - Cos2x)$ and

$Cos2x = 2Cos^2 x - 1 \Rightarrow Cos^2 x = \frac{1}{2}(1 + Cos2x)$

$$\Rightarrow \int Sin^2 xdx = \int \left[\frac{1}{2}(1 - Cos2x)\right]dx$$

$$= \frac{1}{2}\int (1 - C\,os\,2x)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}Sin2x\right] + c$$

$$= \frac{1}{2}x - \frac{1}{4}Sin2x + c$$

Similarly, $\int Cos^2 xdx = \frac{1}{2}\int (1 + Cos2x)dx$

$$= \frac{1}{2}x + \frac{1}{4}Sin2x + c$$

To integrate Sin^3x and $\mathrm{Cos}^3 x$, one of the factors must be released from the power and the remaining converted using the identity $Sin^2x+Cos^2x=1$.

Example. Evaluate $\int Sin^3x\ dx$

Solution: $\int Sin^3xdx=\int Sin^2xSinx\ dx$

$$=\int\left(1-Cos^2x\right)Sinx\ dx$$

Let $u=Cosx \Rightarrow \frac{du}{dx}=-Sinx \Leftrightarrow -du=Sinxdx$

$$\Rightarrow \int Sin^3xdx=\int\left(1-u^2\right)\bullet -du=-\int\left(1-u^2\right)du$$

$$=-\left(u-\frac{1}{3}u^3\right)+c=\frac{1}{3}u^3-u+c$$

$$\therefore \int Sin^3xdx=\frac{1}{3}Cos^3x-Cosx+c$$

Similarly, $\int Cos^3xdx=\int Cos^2xCosx\ dx$

$$=\int\left(1-Sin^2x\right)Cosx\ dx$$

Let $u=Sinx \Rightarrow \frac{du}{dx}=Cosx \Leftrightarrow du=Cosxdx$

$$\Rightarrow \int Cos^3xdx=\int\left(1-u^2\right)du=u-\frac{1}{3}u^3+c$$

$$\therefore \int Cos^3xdx=Sinx-\frac{1}{3}Sin^3x+c$$

We now consider the integration of Sin^4x and Cos^4x.

Example. Evaluate $\int Sin^4x\ dx$

Solution: $\int Sin^4xdx=\int\left(Sin^2x\right)^2 dx$

Using the sine half angle, we know that $Sin^2x=\frac{1}{2}(1-Cos2x)$

$$\Rightarrow \int Sin^4xdx=\int\left[\frac{1}{2}(1-Cos2x)\right]^2 dx=\int\left(\frac{1}{2}-\frac{1}{2}Cos2x\right)^2 dx$$

$$= \int \frac{1}{4}\left(1 - 2Cos2x + Cos^2 2x\right)dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int 2Cos2xdx + \frac{1}{4}\int Cos^2 2xdx$$

Again, using the cosine half angle, $Cos^2 x = \frac{1}{2}\left(1 + Cos2x\right)$

$$\Rightarrow \int Sin^4 xdx = \frac{1}{4}\int dx - \frac{1}{4}\int 2Cos2xdx + \frac{1}{4}\int \left[\frac{1}{2}\left(1 + Cos4x\right)\right]dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int 2Cos2xdx + \frac{1}{8}\int \left(1 + Cos4x\right)dx$$

$$= \frac{1}{4}x - \frac{1}{4}Sin2x + \frac{1}{8}\left(x + Sin4x \bullet \frac{1}{4}\right) + c$$

$$= \frac{1}{4}x - \frac{1}{4}Sin2x + \frac{1}{8}x + \frac{1}{32}Sin4x + c$$

$$\therefore \int Sin^4 xdx = \frac{3}{8}x - \frac{1}{4}Sin2x + \frac{1}{32}Sin4x + c$$

NOTE: The $\int Cos^4 xdx$ is left as Exercise.

We consider the integration of $Sin^5 x$ and $Cos^5 x$.

Example. Evaluate $\int Sin^5 x\ dx$

Solution: $\int Sin^4 xdx = \int Sin^4 xSinx\ dx$

$$= \int \left(1 - Cos^2 x\right)^2 Sinx\ dx$$

$$= \int \left(1 - 2Cos^2 x + Cos^4 x\right)Sinx\ dx$$

$$= \int Sinxdx - 2\int Cos^2 xSinxdx + \int Cos^4 xSinx\ dx$$

$$\therefore \int Sin^4 xdx = -Cosx + \frac{2}{3}Cos^3 x - \frac{1}{5}Cos^5 x + c$$

NOTE: The $\int Cos^5 xdx$ is left as Exercise.

3.7.2 Products of Sines and Cosines

A. The integral of trigonometric functions of the form $Sin^n axCosax$ or $SinaxCos^n ax$ can be dealt with by the use of substitution. For instance, in dealing with $\int Sin^n axCosax\ dx$, the following substitution could be made:

Let $u = Sinax \Rightarrow du = aCosaxdx \Leftrightarrow Cosaxdx = \frac{1}{a}du$

$$\Rightarrow \int Sin^n axCosaxdx = \int u^n.\frac{1}{a}du = \frac{1}{a}\int u^n du$$

$$= \frac{1}{a}\left(\frac{1}{n+1}\bullet u^{n+1}\right) + c$$

$$= \frac{1}{a}\bullet\frac{1}{n+1}\bullet\left(Sinax\right)^{n+1} + c$$

$$= \frac{1}{a\left(n+1\right)}Sin^{n+1}ax + c\,;n \neq -1$$

Example. Evaluate the following

(i) $\int Sin^2 3xCos3x\ dx$ (ii) $\int Sin2xCos^3 2x\ dx$

Solution:

(i) Let $u = Sin3x \Rightarrow du = 3Cos3xdx \Leftrightarrow \frac{1}{3}du = Cos3xdx$

$$\Rightarrow \int Sin^2 3xCos3xdx = \int u^2\bullet\frac{1}{3}du\ dx$$

$$= \frac{1}{3}\int u^2 du$$

$$= \frac{1}{3}\bullet\frac{1}{3}u^3 + c$$

$$\therefore \int Sin^2 3xCos3xdx = \frac{1}{9}Sin^3 3x + c$$

(ii) Let $u = Cos2x \Rightarrow du = -2Sin2xdx \Leftrightarrow -\frac{1}{2}du = Sin2xdx$

$$\Rightarrow \int Sin2xCos^3 2xdx = \int u^3\bullet -\frac{1}{2}du$$

$$= -\frac{1}{2}\int u^3 du$$

$$= -\frac{1}{2}\bullet\frac{1}{4}u^4 + c$$

$$= -\frac{1}{2}\bullet\frac{1}{4}(Cos2x)^4 + c$$

$$\therefore \int Sin2xCos^3 2xdx = -\frac{1}{8}Cos^4 2x + c$$

B. The integral of trigonometric functions of the form $Sin^m axCos^n ax$ is easily performed if one or both m and n is **odd**. If m for example is odd, $Sin^{m-1}ax$ is even power of $Sinax$ and by the relationship $Sin^2 ax = 1 - Cos^2 ax$, the integrand can be expressed as a polynomial in $Cosax$. The other $Sinax$ is taken with dx to make the derivative of $Cosax$, and the integral is expressed as the sum powers of $Cosax$.

Example. Evaluate the following

(i) $\int Sin^2 xCos^3 x\ dx$ (ii) $\int Sin^3 xCos^4 x\ dx$ (iii) $\int \text{Sin}^3 x \text{Cos}^5 x\, dx$

Solution:

(i) $\int Sin^2 xCos^3 xdx = \int Sin^2 xCos^2 xCosxdx$

$$= \int Sin^2 x\left(1 - Sin^2 x\right)Cosx\ dx$$

$$= \int \left(Sin^2 x - Sin^4 x\right)Cosx\ dx$$

$$= \int Sin^2 xCosxdx - \int Sin^4 xCosx\ dx$$

$$\therefore \int Sin^2 xCos^3 xdx = \frac{1}{3}Sin^3 x - \frac{1}{5}Sin^5 x + c$$

(ii) $\int Sin^3 xCos^4 xdx = \int Sin^2 xCos^4 xSinx\ dx$

$$= \int \left(1 - Cos^2 x\right)Cos^4 xSinx\ dx$$

$$= \int Cos^4 xSinxdx - \int Cos^6 xSinx\ dx$$

$$\therefore \int Sin^2 xCos^3 xdx = -\frac{1}{5}Cos^5 x + \frac{1}{7}Cos^7 x + c$$

(iii) $\int \text{Sin}^3 x \,\text{Cos}^5 x \,dx = \int \text{Sin}^2 x \,\text{Sin}\, x \;\text{Cos}^5 x \,dx$

$$= \int \left(1 - \text{Cos}^2 x\right) \text{Sin}\, x \,\text{Cos}^5 x \,dx$$
$$= \int \text{Sin}\, x \,\text{Cos}^5 x \,dx - \int \text{Sin}\, x \,\text{Cos}^7 x \,dx$$

Using the substitution method, let $u = \text{Cos}\, x$

$$\frac{du}{dx} = -\text{Sin}\, x \Rightarrow du = -\text{Sin}\, x \,dx$$

$$\Rightarrow \int \text{Sin}^3 x \,\text{Cos}^5 x \,dx = \int u^5 \cdot -du - \int u^7 \cdot -du$$
$$= -\int \left(u^5 - u^7\right) du$$
$$= -\left(\frac{1}{6}u^6 - \frac{1}{8}u^8\right) + c$$
$$= -\frac{1}{6}u^6 + \frac{1}{8}u^8 + c$$
$$= -\frac{1}{6}\text{Cos}^6 x + \frac{1}{8}\text{Cos}^8 x + c$$

If the trigonometric functions under consideration is such that both m and n are **even**, then the integration is performed by expressing the integrand in terms of multiple angles.

Example. Evaluate the following integrals

(i) $\int Cos^2 x Sin^4 x \; dx$ (ii) $\int Sin^2 x Cos^4 x \; dx$

Solution:

$$\int Cos^2 x Sin^4 x dx = \int Cos^2 x Sin^2 x Sin^2 x \; dx$$

We know that $Sin 2x = 2 Sinx Cosx \Rightarrow Sinx Cosx = \frac{1}{2} Sin 2x$

$$\Rightarrow \left(Sinx Cosx\right)^2 = \left(\frac{1}{2}Sin 2x\right)^2 \Rightarrow Sin^2 x Cos^2 x = \frac{1}{4} Sin^2 2x$$

Thus, $\int Cos^2 x Sin^4 x dx = \int \left(\frac{1}{4} Sin^2 2x\right) Sin^2 x \; dx$

$$= \frac{1}{4} \int Sin^2 2x \left[\frac{1}{2}\left(1 - Cos 2x\right)\right] dx$$
$$= \frac{1}{8} \int Sin^2 2x \left(1 - Cos 2x\right) dx$$

$$=\frac{1}{8}\int Sin^2 2xdx-\frac{1}{8}\int Sin^2 2xCos2x\ dx$$

$$=\frac{1}{8}\int \frac{1}{2}(1-Cos4x)\,dx-\frac{1}{8}\int Sin^2 2xCos2x\ dx$$

$$=\frac{1}{16}\int (1-Cos4x)\,dx-\frac{1}{8}\int Sin^2 2xCos2x\ dx$$

$$=\frac{1}{16}\left(x-\frac{1}{4}Sin4x\right)-\frac{1}{8}\left(\frac{1}{3}\bullet\frac{1}{2}Sin^3 2x\right)+c$$

$$=\frac{1}{16}x-\frac{1}{64}Sin4x-\frac{1}{48}Sin^3 2x+c$$

$$\therefore \int Cos^2 xSin^4 xdx=\frac{1}{64}\left[4x-Sin4x-\frac{4}{3}Sin^3 2x\right]+c$$

(ii) $\int Sin^2 xCos^4 xdx=\int Sin^2 xCos^2 xCos^2 x\ dx$

$$=\int\left(\frac{1}{4}Sin^2 2x\right)Cos^2 x\ dx$$

$$=\frac{1}{4}\int Sin^2 2xCos^2 x\ dx$$

$$=\frac{1}{4}\int Sin^2 2x\left[\frac{1}{2}(1+Cos2x)\right]dx$$

$$=\frac{1}{8}\int Sin^2 2x(1+Cos2x)\,dx$$

$$=\frac{1}{8}\int Sin^2 2xdx+\frac{1}{8}\int Sin^2 2xCos2x\ dx$$

$$=\frac{1}{8}\int\left[\frac{1}{2}(1-Cos4x)\right]dx+\frac{1}{8}\int Sin^2 2xCos2x\ dx$$

$$=\frac{1}{16}\int (1-Cos4x)\,dx+\frac{1}{8}\int Sin^2 2xCos2x$$

$$=\frac{1}{16}\left(x-\frac{1}{4}Sin4x\right)+\frac{1}{8}\left(\frac{1}{3}\bullet\frac{1}{2}Sin^3 2x\right)+c$$

$$=\frac{1}{16}x-\frac{1}{64}Sin4x+\frac{1}{48}Sin^3 2x+c$$

$$\therefore \int Sin^2 xCos^4 xdx=\frac{1}{64}\left(4x-Sin4x+\frac{4}{3}Sin^3 2x\right)+c$$

C. The integral of trigonometric functions of the form $SinaxCosbx$ are easily performed by the use of the factor formulae to transform the integrand into a sum or difference of Sine and Cosine. The factor formulae are as follows:

- $2SinACosB = Sin(A+B) + Sin(A-B)$
- $2CosASinB = Sin(A+B) - Sin(A-B)$
- $2CosACosB = Cos(A+B) + Cos(A-B)$
- $2SinASinB = -Cos(A+B) + Cos(A-B)$

Example. Evaluate the following integrals

(i) $\int Sin3xCos5x\ dx$ (ii) $\int 2Sin6xCos2x\ dx$

Solution:

(i) $$\int Sin3xCos5xdx = \frac{1}{2}\int 2Sin3xCos5x\ dx$$

$$= \frac{1}{2}\int \left[Sin(3x+5x) + Sin(3x-5x)\right]dx$$

$$= \frac{1}{2}\int \left[Sin8x + Sin(-2x)\right]dx$$

$$= \frac{1}{2}\int \left[Sin8x - Sin2x\right]dx$$

$$= \frac{1}{2}\int Sin8xdx - \frac{1}{2}\int Sin2xdx$$

$$= \frac{1}{2}\left(-\frac{1}{8}Cos8x\right) - \frac{1}{2}\left(-\frac{1}{2}Cos2x\right) + c$$

$$= -\frac{1}{16}Cos8x + \frac{1}{4}Cos2x + c$$

$$\therefore \int Sin3xCos5xdx = \frac{1}{16}(4Cos2x - Cos8x) + c$$

(ii) $$\int 2Sin6xCos2xdx = \int \left[Sin(6x+2x) + Sin(6x-2x)\right]dx$$

$$= \int (Sin8x + Sin4x)dx$$

$$= \int Sin8xdx + \int Sin4xdx$$

$$= -\frac{1}{8}Cos8x - \frac{1}{4}Cos4x + c$$

$$\therefore \int 2Sin6xCos2xdx = -\frac{1}{8}\left(Cos8x + 2Cos4x\right) + c$$

3.8 REDUCTION FORMULA

Many integrals which are impossible to integrate may be found by means of a reduction formula. In a situation where the integrand involves a constant integer n, it may be possible to find a connection between this integral and the corresponding integral for the value $(n-1)$ or $(n-2)$.

Reduction formula stems from the idea of integration by parts. If an integral $\int x^n e^x dx$ is denoted by I_n, by the application of integration by parts, in the first stage the result below is obtained.

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx \dots\dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

It will be noticed that, the integral on the right is of exactly the same form as the one we started with except for the fact that n has now been replaced with $(n-1)$. Since the original integral is denoted by I_n, the integral $\int x^{n-1} e^x dx$ can be denoted by I_{n-1} and hence equation (1) can be written as:

$$\mathrm{I}_{\boldsymbol{n}} = x^n e^x - n\mathrm{I}_{\mathrm{n}-1} \dots\dots\dots\dots\dots\dots(2)$$

The relation (2) is called a **reduction formula** since it expresses an integral in n in terms of the same integral in $(n-1)$.

3.8.1 Integral of the form $\int x^n e^{ax} dx$:

By the application of integration by parts to the integral above, if we let

$$u = x^n \Rightarrow du = nx^{n-1}dx; \quad \frac{dv}{dx} = e^{ax} \Rightarrow v = \frac{1}{a}e^{ax}$$

$$\Rightarrow \int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \int \frac{1}{a} \cdot nx^{n-1} e^{ax} dx$$

$$= \frac{1}{a} x^n e^{ax} - \frac{1}{a} \bullet n \int x^{n-1} e^{ax} dx$$

Since the integral on the right is of exactly the same form as the one we started with except for the fact that n has now been replaced with $(n-1)$, then if we denote $\int x^n e^{ax} dx$ by $\mathrm{I}_{\boldsymbol{n}}$**,** then we can denote $\int x^{n-1} e^{ax} dx$ by I_{n-1} and so the result obtained can be written in the form:

$$\mathrm{I}_{\boldsymbol{n}} = \frac{1}{a} x^n e^{ax} - \frac{1}{a} n\, I_{n-1}$$

Example. Use the reduction formula to determine the integral $\int x^3 e^{2x} dx$

Solution: The reduction formula for the integral type above is given by:

$$\mathrm{I}_{\boldsymbol{n}} = \frac{1}{a} x^n e^{ax} - \frac{1}{a} n\, I_{n-1}$$

$$n = 3, a = 2 : \ \mathrm{I}_{\mathbf{3}} = \frac{1}{2} x^3 e^{2x} - \frac{1}{2} \cdot 3\mathrm{I}_{\mathbf{2}}$$

$$\mathrm{n} = 2, \quad a = 2 : \qquad \mathrm{I}_{\mathbf{2}} = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} \cdot 2\mathrm{I}_{\mathbf{1}}$$

$$n = 1, a = 2: \quad \mathrm{I}_{\mathbf{1}} = \frac{1}{2} x^1 e^{2x} - \frac{1}{2} \cdot 1 \cdot \mathrm{I}_{\mathbf{0}}$$

$$n = 0, a = 2: \ \mathrm{I}_{\mathbf{0}} = \frac{1}{2} \int x^0\, e^{2x} dx = \int e^{2x} dx$$

$$= \frac{1}{2} e^{2x} + c$$

$$\therefore\ \mathrm{I}_3 = \frac{1}{2} x^3 e^{2x} - \frac{1}{2} \cdot 3 \cdot \mathrm{I}_{\mathbf{2}}$$

$$= \frac{1}{2} x^3 e^{2x} - \frac{3}{2}\left[\frac{1}{2} x^2 e^{2x} - \frac{1}{2} \cdot 2\mathrm{I}_{\mathbf{1}}\right]$$

$$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + 2 \cdot \frac{3}{4} \mathrm{I}_{\mathbf{1}}$$

$$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{2}\left[\frac{1}{2} x e^{2x} - \frac{1}{2} \mathrm{I}_0\right]$$

$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\mathrm{I}_0$$

$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\left[\frac{1}{2}e^{2x}\right] + c$$

$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + c$$

$$= \frac{1}{8}[4x^3 - 6x^2 + 6x - 3]e^{2x} + c$$

3.8.2 Integral of the form $\int x^n Cosaxdx$ and $\int x^n Sinaxdx$

A. In determining the reduction formula for $\int x^n Cosaxdx,$ the application of integration by parts will lead to the following:

$$\text{Let } u = x^n \implies du = nx^{n-1}dx; \ \frac{dv}{dx} = Cosax \implies v = +\frac{1}{a}Sinax$$

$$\implies \int x^n Cosaxdx = +\frac{1}{a}x^n\, Sinax - \int \frac{1}{a}\cdot nx^{n-1}Sinax\, dx$$

$$= \frac{1}{a}x^n\, Sinax - \frac{1}{a}\cdot n \int x^{n-1}Sinax\, dx$$

$$\text{Let } u = x^{n-1} \implies du = (n-1)x^{n-2}dx; \ \frac{dv}{dx} = Sinax \implies v = -\frac{1}{a}Cosax$$

$$\implies \int x^n\, Cosaxdx = \frac{1}{a}x^n\, Sinax - \frac{1}{a}\cdot n$$

$$\left[-\frac{1}{a}\, x^{n-1}Cosax + \int \frac{1}{a}\cdot (n-1)x^{n-2}Cosaxdx]\right]$$

$$= \frac{1}{a}x^n\, Sinax + \frac{1}{a^2}\cdot nx^{n-1}\, Cosax - \frac{1}{a^2}\cdot n(n-1)\int x^{n-2}Cosax\, dx$$

Since the integral on the right is of exactly the same form as the one we started with except for the fact that n has now been replaced with $(n-2)$, then if we denote $\int x^n Cosaxdx$ by $I_n,$ then we can denote $\int x^{n-2}Cosaxdx$ by I_{n-2} and so the results obtained can be written in the form:

$$\mathrm{I}_n = \frac{1}{a}x^n\, Sinax + \frac{1}{a^2}\cdot nx^{n-1}\, Cosax - \frac{1}{a^2}\cdot n(n-1)I_{n-2} \ldots\ldots\ldots\ldots\ldots (1)$$

So equation (1) above is the reduction formula for the integral $\mathrm{I}_n = \int x^n Cos\ ax\ dx$

In a case where a = 1, then the integral becomes $\int x^n Cosxdx$ and the reduction formula reduces to the form below:

$$\mathrm{I}_n = x^n\, Sinx + nx^{n-1}\, Cosx - n(n-1)I_{n-2}$$

Example. Use the reduction formula to determine the integral $\int x^3\ Cos\ 2x\ dx$

Solution: The reduction formula for the integral type above is given by:

$$\mathrm{I}_n = \frac{1}{a}x^n Sinax + \frac{1}{a^2}\cdot nx^{n-1}Cosax - \frac{1}{a^2}\cdot n(n-1)\mathrm{I}_{n-2.}$$

n = 3, $a = 2$: $\mathrm{I}_3 = \frac{1}{2}x^3 Sin2x + \frac{1}{4}\cdot 3x^2 Cos2x - \frac{1}{4}\cdot 3\cdot 2\mathrm{I}_{1.}$

$$= \frac{1}{2}x^3 Sin2x + \frac{3}{4}\cdot x^2 Cos2x - \frac{3}{2}\mathrm{I}_{1.}$$

n = 1, $a = 2$: $\mathrm{I}_1 = \frac{1}{2}xSin2x + \frac{1}{4}\cdot 1\cdot Cos2x + c$

$$\therefore\ \mathrm{I}_3 = \frac{1}{2}x^3 Sin2x + \frac{3}{4}x^2 Cos2x - \frac{3}{2}\left[\frac{1}{2}x\ Sin2x + \frac{1}{4}Cos2x\right] + c$$

$$= \frac{1}{2}x^3 Sin2x + \frac{3}{4}x^2 Cos2x - \frac{3}{4}xSin2x - \frac{3}{8}\mathrm{Cos}\ 2x + c$$

B. In determining the reduction formula for $\int x^n Sinaxdx$, the procedure is the same as that applied in the (A) part of this section.

Thus if $u = x^n \Longrightarrow du = nx^{n-1}dx$; $\frac{dv}{dx} = Sin\ ax \Longrightarrow v = -\frac{1}{a}Cos\ ax$

$$\Rightarrow \int x^n Sin\ axdx = -\frac{1}{a}x^n Cosax - \int -\frac{1}{a}\cdot nx^{n-1}Cos\ axdx$$

$$= -\frac{1}{a}x^n Cosax + \frac{1}{a}\cdot n\int x^{n-1}Cos\ axdx$$

Let $u = x^{n-1} \Longrightarrow du = (n-1)x^{n-2}dx$; $\frac{dv}{dx} = Cosax \Longrightarrow v = \frac{1}{a}Sinax$

$$\Longrightarrow \int x^n Sin\,ax\,dx = \; = -\frac{1}{a}x^n Cosax + \frac{1}{a}\,n\left[\frac{1}{a}x^{n-1}Sin\,ax - \int \frac{1}{a}\cdot(n-1)x^{n-2}Sin\,ax\,dx\right]$$

$$\Longrightarrow \int x^n Sin\,ax\,dx = -\frac{1}{a}x^n Cosax + \frac{1}{a^2}\,n\,x^{n-1}Sinax - \frac{1}{a^2}n(n-1)\int x^{n-2}Sin\,axdx$$

As was observed in the (A) part of this section, if $\int x^n\ Sin\,axdx = I_n$, then $\int x^{n-2}Sin\,ax\,dx = I_{n-2}$

$$\Longrightarrow I_n \;= -\frac{1}{a}x^n Cosax + \frac{1}{a^2}n\,x^{n-1}Sinax - \frac{1}{a^2}\,n\,(n-1)I_{n-2}\ \ldots\ldots..(1)$$

Hence the equation (1) above is the reduction formula for the integral $I_n = \int x^n\,Sin\,ax\,dx$

In the case where $a=1$, the integral becomes $\int x^n Sinxdx$ and the reduction formula reduces to the form below:

$$\Longrightarrow I_n \;= -x^n Cosx + n\,x^{n-1}Sinx - \,n\,(n-1)I_{n-2}$$

Example. Use the reduction formula to determine the integral $\int x^3\,Sin\,2x\,dx$

Solution: The reduction formula for the integral type above is given by:

$$I_n \;= -\frac{1}{a}x^n Cosax + \frac{1}{a^2}n\,x^{n-1}Sinax - \frac{1}{a^2}\,n\,(n-1)I_{n-2}$$

n = 3, $a=2$: $\mathrm{I}_3 \;=\; -\frac{1}{2}x^3Cos2x + \frac{1}{4}\cdot 3x^2Sin2x - \frac{1}{4}\cdot 3\cdot 2\mathrm{I}_{1}.$

$$= -\frac{1}{2}x^3Cos2x + \frac{3}{4}x^2Sin2x - \frac{3}{2}\mathrm{I}_{1}.$$

n = 1, $a=2$: $\mathrm{I}_1 = -\frac{1}{2}x\,Cos\,2x + \frac{1}{4}Sin\,2x$

$$\therefore\ \mathrm{I}_3 \;=\; -\frac{1}{2}x^3Cos2x + \frac{3}{4}\cdot x^2Sin\,2x - \frac{3}{2}\left[-\frac{1}{2}x\,Cos\,2x + \frac{1}{4}Sin\,2x\right] + c$$

$$= -\frac{1}{2}x^3Cos2x + \frac{3}{4}\cdot x^2Sin\,2x + \frac{3}{4}x\,Cos2x - \frac{3}{8}Sin\,2x + c$$

It should be noted that a reduction fomula can be repeated until the value of n decreases to $n=1$ or $n=0$ when the final integral is determined by normal methods.

3.8.3 Integrals of the form $\int \boldsymbol{Sin^n\, ax\, dx}$ and $\int \boldsymbol{Cos^n\, ax\, dx}$

Since reduction formulas are based on integration by parts, reduction formulas for integrals of the form above are taken care of by writing the functions as product of two functions. For instance, for the integral $\int \boldsymbol{Sin^n\, ax\, dx}$, we write the integral as below:

$$\int Sin^n\, ax\, dx = \int Sin^{n-1}\, ax\, Sin\, ax\, dx$$

Let $u = Sin^{n-1} ax \Rightarrow du = (n-1)Sin^{n-2} ax \cdot a\, Cos\, ax\, dx$

$$= a(n-1)Sin^{n-2} ax\, Cos\, ax\, dx$$

$$\frac{dv}{dx} = Sin\, ax \Rightarrow v = -\frac{1}{a}Cos\, ax$$

$$\Rightarrow \int Sin^n\, ax\, dx = \int Sin^{n-1}\, ax\, Sin\, ax\, dx$$

$$= -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax - \int -\frac{1}{\cancel{a}} \cdot \cancel{a}(n-1)Sin^{n-2}ax\, Cos^2\, ax\, dx$$

$$= -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\int Cos^2\, ax\, Sin^{n-2}ax\, dx$$

$$= -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\int [1 - Sin^2 ax]\, Sin^{n-2}ax\, dx$$

$$= -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\int Sin^{n-2}ax\, dx - (n-1)\int Sin^n ax\, dx$$

Since the integral on the left is of exacly the same form as the two other integrals on the right except for the fact that for one of the integrals on the right n has been replaced by $n-2$. Denoting $\int Sin^n\, ax\, dx$ by I_n and $\int Sin^{n-2}ax\, dx$ by I_{n-2}, the result obtained can be written as:

$$\mathrm{I}_n = -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\mathrm{I}_{n-2} - (n-1)\mathrm{I}_n$$

$$\Rightarrow \mathrm{I}_n + (n-1)\mathrm{I}_n = -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\mathrm{I}_{n-2}$$

$$\Rightarrow n\mathrm{I}_n = -\frac{1}{a}Sin^{n-1}ax\, Cos\, ax + (n-1)\mathrm{I}_{n-2}$$

$$\therefore \ \mathrm{I_n} = -\ \frac{1}{an}\, Sin^{n-1}axCosax + \frac{(n-1)}{n}\ \mathrm{I}_{n-2}$$

The relation above is the reduction formula for an integral of the form $\int Sin^n\, ax\, dx$

In a case when $a = 1$, then the reduction above reduces to the form:

$$\Longrightarrow \mathrm{I_n} = -\frac{1}{n} Sin^{n-1}x\ Cos\ x + \frac{(n-1)}{n} \mathrm{I}_{n-2}$$

Example. Use the reduction formula to determine the integral $\int Sin^6 2x\ dx$

Solution: The reduction formula for the integral type above is given by:

$$\therefore \ \mathrm{I_n} = -\ \frac{1}{an}\, Sin^{n-1}axCosax + \frac{(n-1)}{n}\ \mathrm{I}_{n-2}$$

$n = 6,\ a = 2 \quad \mathrm{I}_6 = -\frac{1}{2.6}\ Sin^5 2x\ Cos2x + \frac{5}{6}\ \mathrm{I}_4$

$\mathrm{n} = 4,\ a = 2:\ \mathrm{I}_4 = -\frac{1}{2.4}\ Sin^3 2xCos2x + \frac{3}{4}\ \mathrm{I}_2$

$\mathrm{n} = 2,\ a = 2:\ \mathrm{I}_2 = -\frac{1}{2.2}\ Sin2xCos2x + \frac{1}{2}\ \mathrm{I}_0$

$\mathrm{n} = 0,\ \mathrm{a} = 2: \quad \mathrm{I}_0 = \int (\mathrm{Sin2x})^0 \mathrm{dx} = \int dx = x + c\mathrm{k}$

$$\therefore \ \mathrm{I}_6 = -\frac{1}{12}\ Sin^5 2xCos2x + \frac{1}{5}\left[-\frac{1}{8} Sin^3 2xCos2x + \frac{3}{4}\mathrm{I}_2\right]$$

$$= -\frac{1}{12}\ Sin^5 2xCos2x - \frac{5}{48} Sin^3 2xCos2x + \frac{5}{8}\mathrm{I}_2$$

$$= -\frac{1}{12}\ Sin^5 2xCos2x - \frac{5}{48} Sin^3 2xCos2x + \frac{5}{8}\left[-\frac{1}{4} Sin2xCos2x + \frac{1}{2}\ \mathrm{I}_0\right]$$

$$= -\frac{1}{12}\ Sin^5 2xCos2x - \frac{5}{48} Sin^3 2xCos2x - \frac{5}{32} Sin2xCos2x + \frac{5}{16}\ \mathrm{I}_0$$

$$= -\frac{1}{12}\ Sin^5 2xCos2x - \frac{5}{48} Sin^3 2xCos2x - \frac{5}{32} Sin2xCos2x + \frac{5}{16} x + c$$

The reduction formula for $\int Cos^n\, ax\, dx$ is also obtained in the same manner as was done in $\int Sin^n\, ax\, dx$. Thus,

$$\int Cos^n\, ax\, dx = \int Cosl^{n-1}\, ax\, Cos\, ax\, dx$$

Let u = $Cos^{\,n-1}ax \implies du = -(n-1)Cos^{\,n-2}ax \cdot a\,\mathrm{Sin}\,ax\,dx$

$$= -a\,(n-1)Cos^{\,n-2}ax \cdot \mathrm{Sin}\,ax\,dx$$

$$\frac{dv}{dx} = \mathrm{Cos}\,ax \implies v = \frac{1}{a}Sinax$$

$$\implies \int Cos^n\, ax\, dx = \int Cos^{\,n-1}ax\,\mathrm{Cos}\,ax\,dx$$

$$= \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}\,ax - \int -\frac{1}{a}\cdot a(n-1)Cos^{\,n-2}ax\,Sin^2ax\,dx$$

$$= \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}\,ax + (n-1)\int Cos^{\,n-2}ax\,Sin^2ax\,dx$$

$$= \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}\,ax + (n-1)\int(1 - Cos^2ax)Cos^{\,n-2}ax\,dx$$

$$\implies \int Cos^n\, axdx = \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}\,ax + (n-1)\int Cos^{\,n-2}axdx -$$

$(n-1)\int Cos^n\, axdx$

As was observed in the earlier situations, if

$\int Cos^n\, axdx = I_n\,, then\ \int Cos^{\,n-2}ax\,dx = I_{n-2}$ and hence

$I_n = \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}\,ax + (n-1)I_{n-2} - (n-1)I_n$

$\implies I_n + (n-1)I_n = \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}ax + (n-1)I_{n-2}$

$\implies nI_n = \frac{1}{a}\,Cos^{\,n-1}ax\,\mathrm{Sin}ax + (n-1)I_{n-2}$

$\therefore\ I_n = \frac{1}{an}\,Cos^{\,n-1}ax\,\mathrm{Sin}ax + \frac{(n-1)}{n}I_{n-2}$

The relation above is the reduction formula for an integral of the form $\int Cos^n\, axdx$

In a case when $a = 1$, the relation above reduces to the form:

$$I_n = \frac{1}{n}\, Cos^{\,n-1}x\ \mathrm{Sin}x + \frac{(n-1)}{n} I_{n-2}$$

Example. Use the reduction formula to determine the integral $\int cos^5\, 2xdx$

Solution: The reduction formula for the integral type above is given by:

$$I_n = \frac{1}{an}\, Cos^{\,n-1}ax\ \mathrm{Sin}ax + \frac{(n-1)}{n} I_{n-2}$$

$$n = 5\,, a = 2 :\ I_5 = \frac{1}{10} Cos^4 2x\ Sin2x + \frac{4}{5} I_3$$

$$n = 3,\ a = 2:\ \mathrm{I}_3 = \frac{1}{6} Cos^2 2x\ Sin2x + \frac{2}{3}\mathrm{I}_1$$

$$n = 1,\ a = 2:\ \mathrm{I}_1 = \frac{1}{2}\ Sin2x + c$$

$$\therefore\ \mathrm{I}_5 = \frac{1}{10} Cos^4 2x\ Sin2x + \frac{4}{5}\left[\frac{1}{6} Cos^2 2x\ Sin2x + \frac{2}{3}\mathrm{I}_1\right]$$

$$= \frac{1}{10} Cos^4 2x\ Sin2x + \frac{2}{15} Cos^2 2x\ Sin2x + \frac{8}{15}\mathrm{I}_1$$

$$= \frac{1}{10} Cos^4 2x\ Sin2x + \frac{2}{15} Cos^2 2x\ Sin2x + \frac{8}{15}\left[\frac{1}{2}\ Sin2x\right] + C$$

$$= \frac{1}{10} Cos^4 2x\ Sin2x + \frac{2}{15} Cos^2 2x\ Sin2x + \frac{4}{15}\ Sin2x + c$$

Exercise

Evaluate the following:

1. $\int \left(x^3 + 5x^2 - 7x\right) dx$

2. $\int \left(6x^3 + 10\right)^8 dx$

3. $\int \left(\dfrac{x^2 - 1}{x - 1}\right) dx$

4. $\int \left(1 - \cos 4x\right) dx$

5. $\int \left(5 \operatorname{Sin}^4 x\right) dx$

Find the integral of the following functions:

6. $\dfrac{Sec^2 x}{\tan x}$

7. $\dfrac{x - 3}{x^2 - 6x + 2}$

8. $Cot\, x$

Evaluate

9. $\int Cos^4 x\, dx$

10. $\int Cos^5 x\, dx$

Evaluate the following integrals

11. $\int Cos\, 6x\, Cos\, 4x\, dx$

12. $\int Sin\, 5x\, Sin\, x\, dx$

CHAPTER FOUR

PARTIAL DIFFERENTIATION

4.1 FUNCTIONS OF SEVERAL VARIABLES

Functions of the explicit form $y = f(x)$ or implicit form $f(x, y) = 0$ are functions of one variable. Such functions express relationships between two variables (x and y) and assume that the process being studied can be represented adequately in terms of only two variables. There are many cases in which such a representation is so inadequate and it becomes necessary to express a relationship in terms of several variables or to express one variable as a function of more other variables. Such representations give functions of several variables. The functions $w = f(x, y, z)$; $z = f(x, y)$; $z = f(x_{1,} x_2, ..., x_n)$, $p^2 = x^2 + y^2 + z^2$ are examples of functions of several variables.

4.2 PARTIAL DIFFERENTIATION FROM FIRST PRINCIPLE

A situation where the function is in the explicit form $y = f(x)$ or implicit form $f(x, y) = 0,$ then the derivative of one of the variables with respect to the other can be sought. We wish to consider here what the situation would be if the function is of several variables.

Consider the function $z = f(x, y)$. If y is held constant , then z is a function of only x and the derivative of z with respect to *x* can be found. The derivative obtained in this way is the **partial derivative** of z with respect to x. Partial derivatives of the function $z = f(x, y)$ with respect to x is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, $\frac{\partial}{\partial x} f(x, y)$, $f_x(x, y)$, f_x and z_x

Similarly, if x is held constant, the partial derivative of z with respect to y can be computed and denoted by $\frac{\partial z}{\partial y}$, $\frac{\partial f}{\partial y}$, $\frac{\partial}{\partial y} f(x, y)$, $f_y(x, y)$, f_y and z_y. Thus, the partial derivative of a function $f(x, y)$ can be defined as the ordinary derivative of a function of several variables with respect to one of the independent variables. Holding all other independent variables constant is called the **partial derivative** of the function with respect to that variable.

By definition $\dfrac{\partial f}{\partial x} = \lim_{\Delta x \to 0} it \left[\dfrac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \right]$

Similarly, $\dfrac{\partial f}{\partial y} = \lim_{\Delta y \to 0} it \left[\dfrac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \right]$

The derivatives evaluated at a particular point (x_o, y_o) are often indicated by

$\left.\dfrac{\partial f}{\partial x}\right|_{(x_o, y_o)} = f_x(x_o, y_o)$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(x_o, y_o)} = f_y(x_o, y_o)$ respectively.

Example. Using the definition of partial derivative, determine the partial derivative with respect to y and x given that $f(x, y) = 3xy^2 - 2y + 5x^2y^2$ and evaluate your answer when $x = -1$ and y = 2.

Solution:

By definition $\dfrac{\partial f}{\partial x} = \lim_{\Delta x \to 0} it \left[\dfrac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \right]$

$$\Rightarrow \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} it \left[\frac{\left\{3(x+\Delta x)y^2 - 2y + 5(x+\Delta x)^2 y^2\right\} - \left\{3xy^2 - 2y + 5x^2y^2\right\}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{\left\{3xy^2 + 3\Delta xy^2 - 2y + 5\left(x^2 + 2x\Delta x + (\Delta x)^2\right)y^2\right\} - 3xy^2 + 2y - 5x^2y^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{3xy^2 + 3\Delta xy^2 - 2y + 5x^2y^2 + 10x\Delta xy^2 + 5(\Delta x)^2 y^2 - 3xy^2 + 2y - 5x^2y^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it \left[\frac{3\Delta xy^2 + 10x\Delta xy^2 + 5(\Delta x)^2 y^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} it\left[3y^2 + 10xy^2 + 5\Delta xy^2\right]$$

$$\therefore \frac{\partial f}{\partial x} = 3y^2 + 10xy^2$$

Now $\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = 3(2)^2 + 10(-1)(2)^2 = 12 - 40 = -28$

By definition $\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} it\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$

$$\Rightarrow \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} it\left[\frac{\left\{3x(y+\Delta y)^2 - 2(y+\Delta y) + 5x^2(y+\Delta y)^2\right\} - \left\{3xy^2 - 2y^2 - 5x^2y^2\right\}}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} it\left[\frac{\left\{3x\left(y^2 + 2y\Delta y + (\Delta y)^2\right) - 2(y+\Delta y) + 5x^2\left(y^2 + 2y\Delta y + (\Delta y)^2\right)\right\} - \left\{3xy^2 - 2y^2 + 5x^2y^2\right\}}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} it\left[\frac{3xy^2 + 6xy\Delta y + 3x(\Delta y)^2 - 2y - 2\Delta y + 5x^2y^2 + 10x^2y\Delta y + 5x^2(\Delta y)^2 - 3xy^2 + 2y^2 - 5x^2y^2}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} it\left[\frac{6xy\Delta y + 3x(\Delta y)^2 - 2\Delta y + 10x^2y\Delta y + 5x^2(\Delta y)^2}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} it\left[6xy + 3x\Delta y - 2 + 10x^2y + 5x^2\Delta y\right]$$

$$\therefore \frac{\partial f}{\partial y} = 6xy - 2 + 10x^2y$$

Now $\left.\frac{\partial f}{\partial y}\right|_{(-1,2)} = 6(-1)(2) - 2 + 10(-1)^2(2) = -12 - 2 + 20 = 6$

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given that $z = 2x^2 + 3xy - 6y^2$

Solution: $z = 2x^2 + 3xy - 6y^2 \Rightarrow \dfrac{\partial z}{\partial x} = 4x + 3y$

$$z = 2x^2 + 3xy - 6y^2 \Rightarrow \frac{\partial z}{\partial y} = 3x - 12y$$

Example. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ given that $z = xy + \text{In}\, x$

Solution: $z = xy + \text{In}\, x \Rightarrow \dfrac{\partial z}{\partial x} = y + \dfrac{1}{x} = \dfrac{xy+1}{x}$

$$z = xy + \text{In}\, x \Rightarrow \frac{\partial z}{\partial y} = x$$

Example. Given that $z = \dfrac{x^3 - y^3}{xy}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

Solution: $\dfrac{\partial z}{\partial x} = \dfrac{xy\dfrac{\partial}{\partial x}\left(x^3 - y^3\right) - \left(x^3 - y^3\right)\dfrac{\partial}{\partial x}(xy)}{x^2y^2}$

$$= \frac{3x^2(xy) - y\left(x^3 - y^3\right)}{x^2y^2}$$

$$= \frac{3x^3y - x^3y + y^4}{x^2y^2}$$

$$= \frac{2x^3y + y^4}{x^2y^2} = \frac{y(2x^3 + y^3)}{x^2y^2}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{2x^3 + y^3}{x^2y}$$

Again, $\dfrac{\partial z}{\partial y} = \dfrac{xy\dfrac{\partial}{\partial y}\left(x^3 - y^3\right) - \left(x^3 - y^3\right)\dfrac{\partial}{\partial y}(xy)}{x^2y^2}$

$$= \frac{xy\left(-3y^2\right) - \left(x^3 - y^3\right)x}{x^2y^2}$$

$$= \frac{-3xy^3 - x^4 - xy^3}{x^2y^2}$$

$$= \frac{-2xy^3 - x^4}{x^2y^2} = \frac{x(-2y^3 - x^3)}{x^2y^2}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{-2y^3 - x^3}{xy^2}$$

Now $\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{2x^3 + y^3}{x^2y}\right) + y\left(\frac{-2y^3 - x^3}{xy^2}\right)$

$$= \frac{2x^3 + y^3}{xy} + \frac{-2y^3 - x^3}{xy}$$

$$= \frac{2x^3 + y^3 - 2y^3 - x^3}{xy}$$

$$= \frac{x^3 - y^3}{xy} = z$$

Hence, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ as proved.

Example. If $w(x, y, z) = xz + e^{y^2z} + \sqrt{xy^2z^3}$. Calculate the partial derivatives of $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$

Solution: $\quad \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial x}\left(e^{y^2z}\right) + \frac{\partial}{\partial x}\left[\left(xy^2z^3\right)^{1/2}\right]$

$$= z + \frac{1}{2}\left(xy^2z^3\right)^{-1/2} \bullet y^2z^3$$

$$= z + \frac{y^2z^3}{2\left(xy^2z^3\right)^{1/2}}$$

Again, $\quad \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial y}\left(e^{y^2z}\right) + \frac{\partial}{\partial y}\left[\left(xy^2z^3\right)^{1/2}\right]$

$$= 2yze^{y^2z} + \frac{1}{2}\left(xy^2z^3\right)^{-1/2} \bullet 2xyz^3$$

$$= 2yze^{y^2z} + \frac{2xyz^3}{2\left(xy^2z^3\right)^{1/2}}$$

$$\therefore \frac{\partial w}{\partial y} = 2yze^{y^2z} + \frac{xyz^3}{\left(xy^2z^3\right)^{1/2}}$$

Also, $\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial z}(xz) + \dfrac{\partial}{\partial z}\left(e^{y^2z}\right) + \dfrac{\partial}{\partial z}\left[\left(xy^2z^3\right)^{1/2}\right]$

$$= x + y^2e^{y^2z} + \frac{1}{2}\left(xy^2z^3\right)^{-1/2}.3xy^2z^2$$

$$\therefore \frac{\partial w}{\partial z} = x + y^2e^{y^2z} + \frac{3xy^2z^2}{2\left(xy^2z^3\right)^{1/2}}$$

4.3 HIGHER ORDER PARTIAL DERIVATIVES

If $f(x, y)$ has partial derivatives at each point (x, y) in a region, then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are themselves functions of x and y which may also have partial derivatives. These second derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Example. **F**ind the four partial derivatives of the function $f(x, y) = xe^y - \mathrm{Sin}\left(\dfrac{x}{y}\right) + x^3y^2$

Solution: $f_x(x, y) = \dfrac{\partial f}{\partial x} = e^y - \dfrac{1}{y}\mathrm{Cos}\left(\dfrac{x}{y}\right) + 3x^2y^2$

$$f_y(x,y)=\frac{\partial f}{\partial y}=xe^y+\frac{x}{y^2}\cos\left(\frac{x}{y}\right)+2x^3y$$

$$f_{xx}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left[e^y-\frac{1}{y}\mathrm{Cos}\left(\frac{x}{y}\right)+3x^2y^2\right]=\frac{1}{y^2}\mathrm{Sin}\left(\frac{x}{y}\right)+6xy^2$$

$$f_{yy}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left[xe^y+\frac{x}{y^2}\mathrm{Cos}\left(\frac{x}{y}\right)+2x^3y\right]=xe^y+\frac{x^2}{y^4}\mathrm{Sin}\left(\frac{x}{y}\right)-\frac{2x}{y^3}\mathrm{Cos}\left(\frac{x}{y}\right)+2x^3$$

$$f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left[e^y-\frac{1}{y}\mathrm{Cos}\left(\frac{x}{y}\right)+3x^2y^2\right]=e^y-\frac{x}{y^3}\mathrm{Sin}\left(\frac{x}{y}\right)+\frac{1}{y^2}\mathrm{Cos}\left(\frac{x}{y}\right)+6x^2y$$

$$f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left[xe^y+\frac{x}{y^2}\mathrm{Cos}\left(\frac{x}{y}\right)+2x^3y\right]=e^y-\frac{x}{y^3}\mathrm{Sin}\left(\frac{x}{y}\right)+\frac{1}{y^2}\mathrm{Cos}\left(\frac{x}{y}\right)+6x^2y$$

4.4 TOTAL DIFFERENTIAL

Suppose we simultaneously make the small changes, $\Delta x \text{ in } x$ and $\Delta y \text{ in } y$ and as a result the function f changes to $f+\Delta f$, then we must have

$$\begin{aligned}
f+\Delta f &= f(x+\Delta x, y+\Delta y)\\
\Rightarrow \Delta f &= f(x+\Delta x, y+\Delta y)-f(x,y)\\
&= f(x+\Delta x, y+\Delta y)-f(x,y+\Delta y)+f(x,y+\Delta y)-f(x,y)
\end{aligned}$$

$$=\left[\frac{f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)}{\Delta x}\right]\Delta x+\left[\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\right]\Delta y \quad \text{...........(1)}$$

It could be noted that the quantities in brackets are very similar to those involved in the definitions of partial derivatives. For them to be strictly equal to the partial derivatives, $\Delta x \text{ and } \Delta y$ would need to be infinitesimally small. It will be noticed that the first bracket in equation (1) approximates to $\frac{\partial f}{\partial x}(x, y+\Delta y)$ and could be replaced by $\frac{\partial f(x,y)}{\partial x}$ and similarly, the second bracket by $\frac{\partial f(x,y)}{\partial y}$ and hence the whole equation becomes

$$\Delta f \approx \frac{\partial f(x,y)}{\partial x} \bullet \Delta x + \frac{\partial f(x,y)}{\partial y} \bullet \Delta y \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Letting the small changes Δx and Δy in equation (2) become infinitesimal, one can define the total differential df of the function $f(x,y)$ without any approximation as

$$df = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy \ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

Equation (3) can easily be extended to the case of a function of n variables $f(x_1, x_{2,} x_3 \cdots\cdots x_n)$ in which case $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 + \cdots\cdots + \frac{\partial f}{\partial x_n} dx_n \ldots\ldots\ldots.(4)$

Example. Find the total differential of the function $f(x,y) = y\exp(x+y)$

Solution: Evaluating the first partial derivatives, we obtain

$$\frac{\partial f}{\partial x} = y\exp(x+y); \quad \frac{\partial f}{\partial y} = \exp(x+y) + y\exp(x+y)$$

Using $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, the total differential is obtained as

$$df = \left[y\exp(x+y)\right]dx + \left[(1+y)\exp(x+y)\right]dy$$

4.4.1 The Chain Rule of Functions of Several Variables

If $y = f(x)$ *and* $x = x(t)$ where both f *and* x are differentiable functions, then by Chain Rule for composite functions of one variable

$$\frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt}$$

Our aim in this section is to obtain generalization for functions of several variables.

FIRST VERSION: If $z = f(x,y)$ where x and y are functions of t , then by the definition of total differential, we obtain

$$dz = \frac{\partial z}{\partial x} \bullet dx + \frac{\partial z}{\partial y} \bullet dy$$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x} \bullet \frac{dx}{dt} + \frac{\partial z}{\partial y} \bullet \frac{dy}{dt}$$

Example. Suppose $z = x^3 y$ where x=2t and $y = t^2$. Find $\frac{dz}{dt}$

Solution: $\frac{\partial z}{\partial x} = 3x^2 y$ and $\frac{\partial z}{\partial y} = x^3$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} . \frac{dx}{dt} + \frac{\partial z}{\partial y} . \frac{dy}{dt} = 3x^2 y \frac{dx}{dt} + x^3 \frac{dy}{dt}$$

$$\frac{dz}{dt} = 3x^2 y(2) + x^3(2t) = 6x^2 y + 2x^3 t$$

$$= 6(2t)^2 (t^2) + 2(2t)^3 t = 24t^4 + 16t^4$$

$$\therefore \frac{dz}{dt} = 40t^4$$

Example. For a solid right circular cylinder, its surface area S increases as a result of its radius increment and an increase in height h. Suppose at the instant when r = 10 cm and h =100 cm, r is increasing at 0.2cm /hr and h is increasing at 0.5 cm/hr. How fast is S increasing?

Solution: The total surface area of a cylinder is given as $S = 2\pi r h + 2\pi r^2$.

Thus, $\frac{ds}{dt} = \frac{\partial s}{\partial r} \bullet \frac{dr}{dt} + \frac{\partial s}{\partial h} \bullet \frac{dh}{dt}$

$$= (2\pi h + 4\pi r)(0.2) + 2\pi r(0.5)$$

At r = 10 and h =100;

$$\frac{ds}{dt} = [2 \bullet \pi \bullet 100 + 4 \bullet \pi \bullet 10](0.2) + (2 \bullet \pi \bullet 10)(0.5)$$

$$= 240\pi(0.2) + 20\pi(0.5)$$

$$= 48\pi + 10\pi = 58\pi$$

$$= 58\pi \, \text{sq cm / hr}$$

Example. Suppose $W = x^2y + y + xz$ where $x = \operatorname{Cos}\theta$, $y = \operatorname{Sin}\theta$ and $z = \theta^2$.

Find $\dfrac{dw}{d\theta}$ and evaluate at $\theta = \dfrac{\pi}{3}$

Solution: $\dfrac{\partial w}{\partial x} = 2xy + z$, $\dfrac{\partial w}{\partial y} = x^2 + 1$ and $\dfrac{\partial w}{\partial z} = x$.

$$y = \operatorname{Sin}\theta \Rightarrow \frac{dy}{d\theta} = \operatorname{Cos}\theta$$
$$x = \operatorname{Cos}\theta \Rightarrow \frac{dx}{d\theta} = -\operatorname{Sin}\theta$$
$$z = \theta^2 \Rightarrow \frac{dz}{d\theta} = 2\theta$$

By chain rule, $\dfrac{dw}{d\theta} = \dfrac{\partial w}{\partial x}.\dfrac{dx}{d\theta} + \dfrac{\partial w}{\partial y}.\dfrac{dy}{d\theta} + \dfrac{\partial w}{\partial z}.\dfrac{dz}{d\theta}$

$$\Rightarrow \frac{dw}{d\theta} = [2xy + z](-\operatorname{Sin}\theta) + [x^2 + 1](\operatorname{Cos}\theta) + [x](2\theta)$$
$$= -2xySin\theta - zSin\theta + x^2Cos\theta + Cos\theta + 2x\theta$$

$$\therefore \frac{dw}{d\theta} = -2\operatorname{Cos}\theta\operatorname{Sin}^2\theta - \theta^2\operatorname{Sin}\theta + \operatorname{Cos}^3\theta + \operatorname{Cos}\theta + 2\theta\operatorname{Cos}\theta$$

At $\theta = \dfrac{\pi}{3}$;

$$\frac{dw}{d\theta} = -2\operatorname{Cos}\left(\frac{\pi}{3}\right)\operatorname{Sin}^2\left(\frac{\pi}{3}\right) - \left(\frac{\pi^2}{9}\right)\operatorname{Sin}\left(\frac{\pi}{3}\right) + \operatorname{Cos}^3\left(\frac{\pi}{3}\right) + \operatorname{Cos}\left(\frac{\pi}{3}\right) + 2\left(\frac{\pi}{3}\right)\operatorname{Cos}\left(\frac{\pi}{3}\right)$$

$$= -2\bullet\frac{1}{2}\bullet\frac{3}{4} - \frac{\pi^2}{9}\bullet\frac{\sqrt{3}}{2} + \frac{1}{8} + \frac{1}{2} + \frac{2\pi}{3}\bullet\frac{1}{2}$$

$$= -\frac{1}{8} - \frac{\pi^2\sqrt{3}}{18} + \frac{\pi}{3}$$

SECOND VERSION: Again suppose that $z = f(x, y)$ where $x = x(s,t)$ and $y = y(s,t)$, then it would not be possible to talk about total derivative with respect to s or t, but rather it will make sense to ask for the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$. This could be supported by the theorem below:

Theorem: Let $x = x(s,t)$ and $y = y(s,t)$ have first partial derivatives and let $z = f(x, y)$ be differentiable at $(x(s,t), y(s,t))$. Then, $z = f(x(s,t), y(s,t))$ has first partial derivatives given by:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\bullet\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\bullet\frac{\partial y}{\partial s} \text{.........................}\mathbf{(I)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\bullet\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\bullet\frac{\partial y}{\partial t} \text{..........................}\mathbf{(II)}$$

We note that if s is held fixed, then $x(s,t)$ and $y(s,t)$ become functions of t alone which means that the total derivative applies. When the theorem above is used, then ∂ replaces d to indicate that s is fixed and in this case, the formula (II) above is obtained. Thus, $\frac{\partial z}{\partial t}$ is obtained by holding s fixed and similarly, $\frac{\partial z}{\partial s}$ is obtained by holding t fixed.

This can be generalized to a function of any number of variables; the variables themselves also being function of other variables. Thus, if

$$z = f(x_1, x_2, ..., x_n) \quad x_r = g_r(t_1, t_2, ..., t_n), (r = 1, 2, 3, ...n)$$

then one can obtain the generalized partial derivative as:

$$\frac{\partial z}{\partial t_s} = \frac{\partial z}{\partial x_1}\bullet\frac{\partial x_1}{\partial t_s} + \frac{\partial z}{\partial x_2}\bullet\frac{\partial x_2}{\partial t_s} + \cdots\cdots + \frac{\partial z}{\partial x_n}\bullet\frac{\partial x_n}{\partial t_s} \quad (s = 1, 2, 3, \cdots n)$$

These results are called the chain rules for transforming derivatives from one set of variables to another.

Example. Given that $z = 3x^2 - y^2$ where $x = 2s + 7t$ and $y = 5st$. Find $\frac{\partial z}{\partial t}$ and express it in terms of s and t.

Solution: $\frac{\partial z}{\partial x} = 6x, \quad \frac{\partial z}{\partial y} = -2y, \quad \frac{\partial x}{\partial t} = 7 \quad \text{and} \quad \frac{\partial y}{\partial t} = 5s$

Using $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \bullet \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \bullet \frac{\partial y}{\partial t}$

$$= (6x)(7) + (-2y)(5s)$$
$$= 42(2s + 7t) + -10s(5st)$$
$$= 84s + 294t - 50s^2t$$

Example. If $w = x^2 + y^2 + z^2 + xy$ where $x = st,\ y = s - t$ and $z = s + 2t$. Find $\frac{\partial w}{\partial t}$ and express the answer in terms of s and t.

Solution: $\frac{\partial w}{\partial x} = 2x + y, \quad \frac{\partial w}{\partial y} = 2y + x, \quad \frac{\partial w}{\partial z} = 2z \quad \text{and}$

$$\frac{\partial x}{\partial t} = s, \quad \frac{\partial y}{\partial t} = -1, \quad \frac{\partial z}{\partial t} = 2$$

Now, $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \bullet \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \bullet \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \bullet \frac{\partial z}{\partial t}$

$$= (2x + y)(s) + (2y + x)(-1) + (2z)(2)$$
$$= 2xs + ys - 2y - x + 4z$$
$$= 2(st)s + (s - t)s - 2(s - t) - st + 4(s + 2t)$$
$$= 2s^2t + s^2 - st - 2s + 2t - st + 4s + 8t$$
$$= 2s^2t + s^2 - 2st + 2s + 10t$$

4.5 HIGHER ORDER DERIVATIVES OF TOTAL AND PARTIAL DERIVATIVES

Higher order derivatives of total and partial derivatives of functions of several variables are obtained by repeated application of the chain rules. If z = f (x, y) and x and y are functions of the single variable t, then by differentiating (the function) total derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \bullet \frac{dx}{dt} + \frac{\partial z}{\partial y} \bullet \frac{dy}{dt}, \quad \text{we obtain}$$

$$\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{d}{dt}\left[\frac{\partial z}{\partial x}\bullet\frac{dx}{dt}\right]+\frac{d}{dt}\left[\frac{\partial z}{\partial y}\bullet\frac{dy}{dt}\right]$$

$$\Rightarrow\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\bullet\frac{dx}{dt}+\frac{\partial z}{\partial x}\bullet\frac{d^2x}{dt^2}+\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\bullet\frac{dy}{dt}+\frac{\partial z}{\partial y}\bullet\frac{d^2y}{dt^2}\ldots\ldots\ldots\ldots(1)$$

But, $\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)=\frac{d}{dx}\left(\frac{\partial z}{\partial x}\right)\bullet\frac{dx}{dt}+\frac{d}{dy}\left(\frac{\partial z}{\partial x}\right)\bullet\frac{dy}{dt}$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2z}{\partial x^2}\bullet\frac{dx}{dt}+\frac{\partial^2z}{\partial y\partial x}\bullet\frac{dy}{dt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)=\frac{d}{dx}\left(\frac{\partial z}{\partial y}\right).\frac{dx}{dt}+\frac{d}{dy}\left(\frac{\partial z}{\partial y}\right).\frac{dy}{dt}$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2z}{\partial x\partial y}.\frac{dx}{dt}+\frac{\partial^2z}{\partial y^2}.\frac{dy}{dt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Substituting equations (2) and (3) into equation (1), we obtain.

$$\frac{d^2z}{dt^2}=\left[\frac{\partial^2z}{\partial x^2}\bullet\frac{dx}{dt}+\frac{\partial^2z}{\partial y\partial x}\bullet\frac{dy}{dt}\right]\frac{dx}{dt}+\frac{\partial z}{\partial x}\bullet\frac{d^2x}{dt^2}+\left[\frac{\partial^2z}{\partial x\partial y}\bullet\frac{dx}{dt}+\frac{\partial^2z}{\partial y^2}\bullet\frac{dy}{dt}\right]\frac{dy}{dt}+\frac{\partial z}{\partial y}\bullet\frac{d^2y}{dt^2}$$

$$=\frac{\partial^2z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+\frac{\partial^2z}{\partial y\partial x}\bullet\frac{dy}{dt}\bullet\frac{dx}{dt}+\frac{\partial z}{\partial x}\bullet\frac{d^2x}{dt^2}+\frac{\partial^2z}{\partial x\partial y}\bullet\frac{dx}{dt}\bullet\frac{dy}{dt}+\frac{\partial^2z}{\partial y^2}\bullet\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial y}\bullet\frac{d^2y}{dt^2}$$

$$=\frac{\partial^2z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2z}{\partial x\partial y}\bullet\frac{dx}{dt}\bullet\frac{dy}{dt}+\frac{\partial^2z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial x}\bullet\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\bullet\frac{d^2y}{dt^2}\ldots\ldots\ldots(4)$$

Equation (4) above is the derivative of the function with respect to the variable t in its second order.

Similarly, if x and y are functions of s and t and z is a function of x and y, then from the first order partial derivatives in the second version, we obtain the following.

$$\frac{\partial^2z}{\partial s^2}=\frac{\partial^2z}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2+2\frac{\partial^2z}{\partial x\partial y}\bullet\frac{\partial x}{\partial s}\bullet\frac{\partial y}{\partial s}+\frac{\partial^2z}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2+\frac{\partial z}{\partial x}\bullet\frac{\partial^2x}{\partial s^2}+\frac{\partial z}{\partial y}\bullet\frac{\partial^2y}{\partial s^2}$$

$$\frac{\partial^2z}{\partial t^2}=\frac{\partial^2z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+2\frac{\partial^2z}{\partial x\partial y}\bullet\frac{\partial x}{\partial t}\bullet\frac{\partial y}{\partial t}+\frac{\partial^2z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2+\frac{\partial z}{\partial x}\bullet\frac{\partial^2x}{\partial t^2}+\frac{\partial z}{\partial y}\bullet\frac{\partial^2y}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial s\partial t}=\frac{\partial^2 z}{\partial x^2}\bullet\frac{\partial x}{\partial s}\bullet\frac{\partial x}{\partial t}+\frac{\partial^2 z}{\partial x\partial y}\left(\frac{\partial x}{\partial s}\bullet\frac{\partial y}{\partial t}+\frac{\partial x}{\partial t}\bullet\frac{\partial y}{\partial s}\right)+\frac{\partial^2 z}{\partial y^2}\bullet\frac{\partial x}{\partial s}\bullet\frac{\partial y}{\partial t}+\frac{\partial z}{\partial x}\bullet\frac{\partial^2 x}{\partial s\partial t}+\frac{\partial z}{\partial y}\bullet\frac{\partial^2 y}{\partial s\partial t}$$

In similar manner, expression for third and higher order derivatives may be found.

Example. If $z=e^{xy^2}$, $x=t\operatorname{Cos}t$ and $y=t\operatorname{Sin}t$ find $\frac{dz}{dt}$ at $t=\frac{\pi}{2}$ and $\frac{d^2z}{dt^2}$

Solution: $z=e^{xy^2}\Rightarrow\frac{\partial z}{\partial x}=y^2e^{xy^2}$ and $\frac{\partial z}{\partial y}=2xye^{xy^2}$

$$\frac{dx}{dt}=-t\operatorname{Sin}t+\operatorname{Cos}t \quad \text{and} \quad \frac{dy}{dt}=t\operatorname{Cos}t+\operatorname{Sin}t$$

Now $\frac{dz}{dt}=\frac{\partial z}{\partial x}.\frac{dx}{dt}+\frac{\partial z}{\partial y}.\frac{dy}{dt}$

$$=y^2e^{xy^2}\left[-t\operatorname{Sin}t+\operatorname{Cos}t\right]+2xye^{xy^2}\left[t\operatorname{Cos}t+\operatorname{Sin}t\right]$$

$$=\left(t\operatorname{Sin}t\right)^2e^{(t\operatorname{Cos}t)(t\operatorname{Sin}t)^2}\left[-t\operatorname{Sin}t+\operatorname{Cos}t\right]+2\left(t\operatorname{Cos}t\right)\left(t\operatorname{Sin}t\right)e^{(t\operatorname{Cos}t)(t\operatorname{Sin}t)^2}\left[t\operatorname{Cos}t+\operatorname{Sin}t\right]$$

$$=t^2\operatorname{Sin}^2te^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[-t\operatorname{Sin}t+\operatorname{Cos}t\right]+2t^2\operatorname{Cos}t\operatorname{Sin}te^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[t\operatorname{Cos}t+\operatorname{Sin}t\right]$$

$$=t^2\operatorname{Sin}^2te^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[-t\operatorname{Sin}t+\operatorname{Cos}t\right]+t^2\operatorname{Sin}2te^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[t\operatorname{Cos}t+\operatorname{Sin}t\right]$$

$$=e^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[-t^3\operatorname{Sin}^3t+t^2\operatorname{Sin}^2t\operatorname{Cos}t+t^3\operatorname{Sin}2t\operatorname{Cos}t+t^2\operatorname{Sin}2t\operatorname{Sin}t\right]$$

$$=t^2\operatorname{Sin}te^{t^3\operatorname{Cos}t\operatorname{Sin}^2t}\left[-t\operatorname{Sin}^2t+\operatorname{Sin}t\operatorname{Cos}t+2t\operatorname{Cos}^2t+\operatorname{Sin}2t\right]$$

$$=t^2\sin t\left[3\operatorname{Sin}t\operatorname{Cos}t-t\operatorname{Sin}^2t+2t\operatorname{Cos}^2t\right]e^{t^3\operatorname{Cos}t\sin^2t}$$

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}}=\left(\frac{\pi}{2}\right)^2\operatorname{Sin}\left(\frac{\pi}{2}\right)\left[3\operatorname{Sin}\frac{\pi}{2}\operatorname{Cos}\frac{\pi}{2}-\frac{\pi}{2}\operatorname{Sin}^2\frac{\pi}{2}+2\frac{\pi}{2}\operatorname{Cos}^2\frac{\pi}{2}\right]e^{\left(\frac{\pi}{8}\right)^3\operatorname{Cos}\frac{\pi}{2}\operatorname{Sin}^2\frac{\pi}{2}}$$

$$=\frac{\pi^2}{4}\bullet1\left(3(1)(0)-\frac{\pi}{2}(1)+\pi(0)\right)e^0$$

$$=\frac{\pi^2}{4}\left(-\frac{\pi}{2}\right)$$

$$=-\frac{\pi^3}{8}$$

$$\frac{d^2z}{dt^2}=\frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\bullet\frac{dx}{dt}\bullet\frac{dy}{dt}+\frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial x}\bullet\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\bullet\frac{d^2y}{dt^2}$$

$$\frac{\partial z}{\partial x}=y^2e^{xy^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}=y^4e^{xy^2}$$

$$\frac{\partial z}{\partial y}=2xye^{xy^2}\Rightarrow\frac{\partial^2 z}{\partial y^2}=4x^2y^2e^{xy^2}+2xe^{xy^2}=2x\left(2xy^2+1\right)e^{xy^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(2xye^{xy^2}\right)=2ye^{xy^2}+2xy^3e^{xy^2}=2y\left(1+xy^2\right)e^{xy^2}$$

$$\frac{dx}{dt}=-t\mathrm{Sin}\,t+\mathrm{Cos}\,t\Rightarrow\frac{d^2x}{dt^2}=-t\mathrm{Cos}t-\mathrm{Sin}t-\mathrm{Sin}t=-t\mathrm{Cos}t-2\mathrm{Sin}t$$

$$\frac{dy}{dt}=t\mathrm{Cos}\,t+\mathrm{Sin}\,t\Rightarrow\frac{d^2y}{dt^2}=-t\mathrm{Sin}t+\mathrm{Cos}t+\mathrm{Cos}t=-t\mathrm{Sin}t+2\mathrm{Cos}t$$

Therefore

$$\frac{d^2z}{dt^2}=y^4e^{xy^2}\left[-tSint+Cost\right]^2+2\left[2y\left(1+xy^2\right)e^{xy^2}\bullet\left(-tSint+Cost\right)\left(tCost+Sint\right)\right]$$
$$+2x\left(2xy^2+1\right)e^{xy^2}\left(tCost+Sint\right)^2+y^2e^{xy^2}\left(-tCos\,t-2\,Sin\,t\right)+2xye^{xy^2}\left(-tSin\,t+2Cost\right)$$

$$=y^4e^{xy^2}\left(t^2Sin^2t-2tSintCost+Cos^2t\right)+4y\left(1+xy^2\right)e^{xy^2}\left(-t^2SintCost-tSin^2t+tCos^2t+SintCost\right)+$$
$$\left(4x^2y^2e^{xy^2}+2xe^{xy^2}\right)\left(t^2Cos^2t+2tSintCost+Sin^2t\right)-y^2e^{xy^2}tCost-2y^2e^{xy^2}Sint-2xye^{xy^2}tSint+8xye^{xy^2}Cost$$

$$=y^4e^{xy^2}t^2Sin^2t-2y^4e^{xy^2}tSintCost+y^4e^{xy^2}Cos^2t-4ye^{xy^2}t^2SintCost-4ye^{xy^2}tSin^2t+4ye^{xy^2}tCos^2t+$$
$$4ye^{xy^2}SintCost-4xy^3e^{xy^2}t^2SintCost-4xy^3e^{xy^2}tSin^2t+4xy^3e^{xy^2}tCos^2t+4xy^3e^{xy^2}SintCost+$$
$$4x^2y^2e^{xy^2}t^2Cos^2t+8x^2y^2e^{xy^2}tSintCost+4x^2y^2e^{xy^2}Sin^2t+2xe^{xy^2}t^2Cos^2t+4xe^{xy^2}tSintCost+$$
$$2xe^{xy^2}Sin^2t-y^2e^{xy^2}tCost-2y^2e^{xy^2}Sint-2xye^{xy^2}tSint+8xye^{xy^2}Cost$$

$$=\begin{bmatrix}y^4tSin^2t-2y^4tSintCost+y^4Cos^2t-4yt^2SintCost-4ytSin^2t+4ytCos^2t+4ySintCost-\\4xy^3t^2SintCost-4xy^3tSin^2t+4xy^3tCos^2t+4xy^3SintCost+4x^2y^2t^2Cos^2t+8x^2y^2tSintCost+\\4x^2y^2Sin^2t+2xt^2Cos^2t+4xtSintCost+2xSin^2t-y^2tCost-2y^2Sint-2xytSint+8xyCost\end{bmatrix}e^{xy^2}$$

$$=\begin{bmatrix}t^6Sin^6t-2t^5Sin^5tCost+t^4Sin^4tCos^2t-4t^3Sin^2tCost-4t^2Sin^3t+4t^2SintCos^2t+\\4tSin^2tCost-4t^6Sin^4tCos^2t-4t^5Sin^5tCost+4t^5Sin^3tCos^3t+4t^4Sin^4tCos^2t+\\4t^6Sin^2tCos^4t+8t^5Sin^3tCos^3t+4t^4Sin^4tCos^2t+2t^3Cos^3t+4t^2SintCos^2t+\\2tSin^2tCost-t^3Sin^2tCost-2t^2Sin^3t-2t^3Sin^2tCost+8t^2SintCos^2t\end{bmatrix}e^{t^3Sin^2tCost}$$

$$\therefore \frac{d^2z}{dt^2}=\begin{bmatrix} t^6Sin^6t-6t^5Sin^5tCost+9t^4Sin^4tCos^2t-7t^3Sin^2tCost-6t^2Sin^3t+16t^2S\,\mathrm{int}\,Cos^2t+ \\ 6tSin^2tCost-4t^6Sin^4tCos^2t+4t^6Sin^2tCos^4t+12t^5Sin^3tCos^3t+2t^3Cos^3t \end{bmatrix}e^{t^3Sin^2tCost}$$

$$\left.\frac{d^2z}{dt^2}\right|_{t=\frac{\pi}{2}}=\begin{bmatrix} \left(\frac{\pi}{2}\right)^6\mathrm{Sin}^6\left(\frac{\pi}{2}\right)-6\left(\frac{\pi}{2}\right)^5\mathrm{Sin}^5\left(\frac{\pi}{2}\right)\mathrm{Cos}\left(\frac{\pi}{2}\right)+9\left(\frac{\pi}{2}\right)^4\mathrm{Sin}^4\left(\frac{\pi}{2}\right)\mathrm{Cos}^2\left(\frac{\pi}{2}\right)- \\ 7\left(\frac{\pi}{2}\right)^3\mathrm{Sin}^2\left(\frac{\pi}{2}\right)\mathrm{Cos}\left(\frac{\pi}{2}\right)-6\left(\frac{\pi}{2}\right)^2\mathrm{Sin}^3\left(\frac{\pi}{2}\right)+16\left(\frac{\pi}{2}\right)^2\mathrm{Sin}\left(\frac{\pi}{2}\right)\mathrm{Cos}^2\left(\frac{\pi}{2}\right)+ \\ 6\left(\frac{\pi}{2}\right)\mathrm{Sin}^2\left(\frac{\pi}{2}\right)\mathrm{Cos}\left(\frac{\pi}{2}\right)-4\left(\frac{\pi}{2}\right)^6\mathrm{Sin}^4\left(\frac{\pi}{2}\right)\mathrm{Cos}^2\left(\frac{\pi}{2}\right)+4\left(\frac{\pi}{2}\right)^6\mathrm{Sin}^2\left(\frac{\pi}{2}\right)\mathrm{Cos}^4\left(\frac{\pi}{2}\right)+ \\ 12\left(\frac{\pi}{2}\right)^5\mathrm{Sin}^3\left(\frac{\pi}{2}\right)\mathrm{Cos}^3\left(\frac{\pi}{2}\right)+2\left(\frac{\pi}{2}\right)^3\mathrm{Cos}^3\left(\frac{\pi}{2}\right) \end{bmatrix}e^{\frac{\pi^3}{8}\mathrm{Sin}^2\frac{\pi}{2}\mathrm{Cos}\frac{\pi}{2}}$$

$$=\left(\frac{\pi^6}{64}-6\frac{\pi^2}{4}\right)e^0=\frac{\pi^6}{64}-\frac{3\pi^2}{2}$$

$$=\frac{\pi^6-96\pi^2}{64}$$

4.6 GRADIENT, DIVERGENCE AND CURL

A *vector field F* is a function that takes any point in space and assigns a vector to it:

$$F:\ \textit{From points in the space} \to \textit{To vectors in the space}$$

$$(x,\ y,\ z)\to\left[F_1(x,\ y,\ z),\ F_2(x,\ y,\ z),\ F_3(x,\ y,\ z)\right]$$

A *scalar field f* is a function that takes a point in space and assigns a number to it:

$$f:\ \textit{From points in the space} \to \textit{To real numbers}$$

$$(x,\ y,\ z)\to\ f(x,\ y,\ z)$$

The gradient, divergence and curl are first order differential operators acting on fields. The easiest way to describe them is via a vector **del** or **nabla** whose components are partial derivatives with respect to the three principal directions in space:

$$\nabla=\frac{\partial}{\partial x}\mathrm{i}+\frac{\partial}{\partial x}\mathrm{j}+\frac{\partial}{\partial z}\mathrm{k}$$

If $\phi(x,y,z)$ and $A(x,y,z)$ are a scalar and vector functions respectively and these functions have continuous first partial derivatives in a region; a condition which is not only necessary but sufficient, then we can define the following;

4.6.1 Gradient

The gradient of a scalar function $\phi(x,y,z)$ denoted by $\nabla\phi$ is defined by

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\phi$$

$$= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

For a given surface with equation $f(x,y,z)$, the normal to the surface is given by $\nabla\phi$. Thus, $\nabla\phi$ is the normal to the curve with equation $\phi(x,y,z)$.

Example. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(1,-2,-1)$.

Solution: By definition

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(3x^2y - y^3z^2\right)$$

$$= \left[\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right)i + \frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right)j + \frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)k\right]$$

$$= 6xy\,i + \left(3x^2 - 3y^2z^2\right)j + \left(-2y^3z\right)k$$

$$\therefore \nabla\phi = 6xy\,i + 3\left(x^2 - y^2z^2\right)j - 2y^3z\,k$$

Now, $$\nabla\phi\big|_{(1,-2,-1)} = \left[6(1)(-2)\right]i + 3\left[(1)^2 - (-2)^2(-1)^2\right]j - \left[2(-2)^3(-1)\right]k$$

$$= -12i - 9j + -16k$$

4.6.2 Divergence

The divergence of a vector function $A(x,y,z)$ denoted by $\nabla \bullet A$ is defined by:

$$\operatorname{div} A = \nabla \bullet A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(A_1 i + A_2 j + A_3 k\right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Example. If $A = x^2 z\, i - 2y^3 z^2\, j + xy^2 z\, k$, find div A at the point $(1,-1,1)$

Solution: By definition:

$$\operatorname{div} A = \nabla \bullet A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(x^2 z\, i - 2y^3 z^2\, j + xy^2 z\, k\right)$$

$$= \frac{\partial}{\partial x}\left(x^2 z\right) + \frac{\partial}{\partial y}\left(-2y^3 z^2\right) + \frac{\partial}{\partial z}\left(xy^2 z\right)$$

$$\therefore \nabla \bullet A = 2xz - 6y^2 z^2 + xy^2$$

Now $\nabla \bullet A\big|_{(1,-1,1)} = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2$

$$= 2 - 6 + 1$$

$$= -3$$

4.6.3 Curl

The curl of a vector function $A(x,y,z)$ denoted by $\nabla \times A$ is defined by:

$$\operatorname{Curl} A = \nabla \times A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(A_1 i + A_2 j + A_3 k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 & A_3 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_1 & A_3 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_1 & A_2 \end{vmatrix} k$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) k$$

Example. If $A = xz^3\, i - 2x^2yz\, j + 2yz^2\, k$, find $\nabla \times A$ at the point $(1, -1, 1)$

Solution: By definition

$$\text{Curl}\, A = \nabla \times A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \times \left(xz^3\, i - 2x^2yz\, j + 2yz^2\, k \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2yz & 2yz^2 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz^3 & 2yz^2 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz^3 & -2x^2yz \end{vmatrix} k$$

$$= \left(\frac{\partial}{\partial y}\left(2yz^2\right) - \frac{\partial}{\partial z}\left(-2x^2yz\right) \right) i + \left(\frac{\partial}{\partial z}\left(xz^3\right) - \frac{\partial}{\partial x}\left(2yz^2\right) \right) j + \left(\frac{\partial}{\partial x}\left(-2x^2yz\right) - \frac{\partial}{\partial y}\left(xz^3\right) \right) k$$

$$= \left(2z^2 + 2x^2y\right) i + \left(3xz^2 - 0\right) j + \left(-4xyz - 0\right) k$$

$$= \left(2z^2 + 2x^2y\right) i + 3xz^2\, j - 4xyz\, k$$

Now $\nabla \times A\big|_{(1,-1,1)} = \left[2(1)^2 + 2(1)^2(-1)\right] i + \left[3(1)(1)^2\right] j - \left[4(1)(-1)(1)\right] k$

$$= (2-2) i + 3j + 4k$$

$$= 3j + 4k$$

Exercise

1. If $f(x, y) = x^3y + e^{xy^2}$, find

(i) $\frac{\partial f}{\partial x}$ (iii) $\frac{\partial^2 f}{\partial x^2}$ (v) $\frac{\partial^2 f}{\partial y \partial x}$

(ii) $\frac{\partial f}{\partial y}$ (iv) $\frac{\partial^2 f}{\partial x \partial y}$ (vi) $\frac{\partial^2 f}{\partial y^2}$

2. If $f(x,y,z) = x\cos(y-z)$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. Find also all the second partial derivatives of the function.

3. If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{d^2 z}{dxdy}$ at the point (1, 2).

4. If $F(x,y) = x^4 y^2 Sin^{-1}\left(\frac{x}{y}\right)$. Show that $x\frac{df}{dx} + y\frac{df}{dy} = 6F$

5. If $z^3 - xz - y = 0$, show that $\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2 + x}{\left(3z^2 - x\right)^3}$

6. If $x = 2r - s$ and $y = r + 2s$. Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{25}\left[2\frac{\partial^2 u}{\partial r^2} + 3\frac{\partial^2 u}{\partial r \partial s} - 2\frac{\partial^2 u}{\partial s^2}\right]$

7. If $\phi(x,y,z) = 2x^2 + 3y^2 + z^2$, find $\nabla\phi$ at the point (1, 0, -2)

8. If $A = \left(x^3 + y^3\right)i + 3x^2 j + 3y^2 zk$, find div A

9. If $A = e^x Cosyi + e^x Sinyj + 2k$, find Curl A

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