MATH 180A Review

Taanish Reja and Kevin Jacob

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1 Sample Spaces and Probabilities 1.1

1.1 Probability space

Sample space Ω: Collection of all possible outcomes
$\mathcal{F}$: Collection of events. An event $A \subset \Omega$, and $A \in \mathcal{F}$
Probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$
A probability measure satisfies three axioms:

- $0 \leq \mathbb{P}(A) \leq 1$
- $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\omega) = 1$
- If $A_1, A_2, \dots$ is a sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$
- Note: Disjoint means $A_i \cap A_j = \emptyset$ (share no common outcome)

Example: A fair die is rolled twice
$\Omega = \{1, 2, 3, 4, 5, 6\}^2$
Calculate the probability that the sum of the two rolls is 8
$\mathbb{P}(sum8) = \mathbb{P}(\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\} = \frac{5}{36}$

2 Random Sampling 1.2

$S = \{r, w, b\}$, $k = 2$

2.1 With Replacement, Order Matters

$\Omega = \{r, w, b\} \times \{r, w, b\}$
$= \{(r, r), (r, w), (r, b), (w, r), (w, w), (w, b), (w, b), (b, r), (b, w), (b, b)$
$\Omega = s^k$
$\#\Omega = n^k$
Example: Flip a fair coin 6 times:
Describe the sample space:
$\Omega = \{H, T\}^6 = \{(C_1, C_2, C_3, C_4, C_5, C_6) : C_i \in \{H, S\}\}$ Calculate $\mathbb{P}$(every even flip is heads)
$= \frac{\text{Every even flip is H}}{\#\Omega} =$
$= \frac{\#\{(C_1, H, C_3, H, C_5, H) : C_1, C_3, C_5 \in \{H, T\}\}}{\#\Omega}$
$= \frac{2^3}{2^6} = \frac{1}{2^3} = \frac{1}{8}$

2.2 Without Replacement, Order Matters

$\Omega =$ set of distinct k-tuples from S
$\Omega = \{(r, w), (r, b), (w, r), (w, b), (b, r), (b, w)\}$
$\#\Omega = n \cdot (n - 1) \cdot \cdots \cdot (n - k + 1) = \frac{n!}{(n-k)!}$ **Example:** There are 6 numbered balls in an urn. 3 are removed w/o replacement and lined up in ordered. If 5 balls are drawn what is the probability that the 1st two numbers removed are (3,6)
$\mathbb{P}(firstTwo(3, 6)) = \frac{\#\{(3,6,n_1,n_2,n_3) : n_1 \in \{1,2,4,5\}, n_2 \in \{1,2,4,5\} \setminus \{n_1\}, n_3 \in \{1,2,4,5\} \setminus \{n_1,n_2\}\}}{(6)_5} = \frac{4 \cdot 3 \cdot 2}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$

2.3 Without Replacement, Order Doesn't Matter

$\Omega =$ set of unordered distinct k tuples from s = collection of subsets of S w/ cardinality k
$\Omega = \{\{r, w\}, \{r, b\}, \{w, b\}$
$\#\Omega = \binom{n}{k} = \frac{n!}{k!(n-k!)}$ (n choose k)

Example: An urn contains 10 marbles: 5 are red, 2 are white, 3 are blue. Three are chosen out uniformly at random and taken out
Calculate the probability we drew 2 blue and 1 red $\mathbb{P}(2b, 1r) = \frac{\binom{3}{2}\cdot\binom{5}{1}}{\binom{10}{3}} = \frac{1}{8}$

3 Infinitely Many Outcomes 1.3

Choosing over a uniform interval/area
$\mathbb{P}(A) = \frac{length(A)}{length(\Omega)}$ or $\mathbb{P}(A) = \frac{area(A)}{area(\Omega)}$
$\mathcal{F}$ = all subsets of interval/area which can reasonably be given a length/area
note: The probability of a single point is 0 because there are infinitely many outcomes in the interval/area
Example 1:A number X is chosen uniformly at random from the interval $[0, 2]$
$\Omega = [0, 2]$
$\mathcal{F}$ =all subsets of $[0, 2]$ which can "reasonably" be given a length
$\mathbb{P}(X \geq 0.5) = \mathbb{P}([\frac{1}{2}, 2]) = \frac{length([\frac{1}{2},2])}{length([0,2])} = \frac{2-\frac{1}{2}}{2-0} = \frac{3}{4}$
$\mathbb{P}(X = 1) = \mathbb{P}([1, 1]) = \frac{length([1,1])}{length([0,2])} = \frac{1-1}{2-0} = 0$
Example 2: An archery target is a 50cm disk containing a middle disk of 25cm and a bullseye of 5cm. You shoot an arrow which hits a point uniformly at a random target.

- Describe the sample space of this experiment: $\Omega = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 25^2\}$ (The disk has a radius of 25cm. The arrow must hit inside the circle)
- Calculate the probability the arrow hits the middle disk:
  $\mathbb{P}(\text{arrow hits the middle disk}) = \frac{\text{area(middle disk)}}{\pi 25^2} = \frac{\pi \frac{25}{2}^2}{\pi 25^2} = \frac{1}{4}$
- Calculate the probability the arrow hits the bullseye:
  $\mathbb{P}(\text{arrow hits bullseye}) = \frac{\text{area(bullseye)}}{\pi 25^2} = \frac{\pi \frac{5}{2}^2}{\pi 25^2} = \frac{1}{100}$
- Calculate the probability the arrow hits the middle ring
  $\mathbb{P}(\text{middle ring}) = \frac{\text{area(middle ring))}}{\pi 25^2} = \frac{\pi \frac{25}{2}^2 - \pi \frac{5}{2}^2}{\pi 25^2} = \frac{6}{25}$

4 Consequences of Rules of Probability 1.4

4.1 Disjoint Events

For disjoint events $A_1, A_2, \ldots, \mathbb{P}(A_1 \cup A_2 \cup \ldots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots$
Example: A fair coin is flipped 5 times. Calculate the probability that it lands tails at least 3 times
$A = \geq 3$ tails
A_3 = exactly 3 tails A_4 = exactly 4 tails A_5 = exactly 5 tails
A_3, A_4, A_5 Are disjoint and $A = A_3 \cup A_4 \cup A_5$
$\mathbb{P}(A_3) = \frac{\#A_3}{\#\Omega} = \frac{\binom{5}{3}}{2^5} = \frac{10}{32}$
$\mathbb{P}(A_4) = \frac{\#A_4}{\#\Omega} = \frac{\binom{5}{4}}{2^5} = \frac{5}{32}$
$\mathbb{P}(A_5) = \frac{\#A_5}{\#\Omega} = \frac{\binom{5}{5}}{2^5} = \frac{1}{32}$
$\mathbb{P}(A) = \mathbb{P}(A_3) + \mathbb{P}(A_4) + \mathbb{P}(A_4) = \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{1}{2}$

4.2 Complement Rule

$\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$
Example: 4 fair dice are rolled. Calculate the probability that we get at least one pair of doubles

$\mathbb{P}(\text{At least 1 pair doubles}) = 1 - \mathbb{P}(\text{no doubles}) = 1 - \mathbb{P}(\text{4 distinct rolls}) = 1 - \frac{\#\text{4 distinct rolls}}{\#\Omega} = 1 - \frac{(6)_4}{6^4} = \frac{13}{18}$

4.3 Inclusion-Exclusion Principle

Inclusion Exclusion for two events
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$
Generalized Inclusion Exclusion
$\bigcup_{i=1}^{n} A_i = \sum_{i=k}^{n} (-1)^{k+1} (\sum_{1 \leq i_1 < \cdots \leq i_k \leq n} (A_1 \cap \cdots \cap A_{i_k})$
Example: In a country, 20% of the population owns a cat, 25% owns a dog, and 5% owns one of each. What is the probability that a person owns neither?
C = Person Owns Cat $\mathbb{P}(C) = 0.2$
D = Person Owns Dog $\mathbb{P}(D) = 0.25$
$\mathbb{P}(C \cap D) = 0.05$
$\mathbb{P}(\text{Neither cat nor dog}) = \mathbb{P}(C^c \cap D^c) = \mathbb{P}((C \cup D)^c) = 1 - \mathbb{P}(C \cup D) = 1 - [\mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C \cap D)] = 1 - (0.2 + 0.25 - 0.05) = 0.60$

4.4 Monotonicty

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}$
Proof:
$B = A \cup B \backslash A$
$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(A \backslash B) \geq \mathbb{P}(A) + 0 = \mathbb{P}(A)$
Example: Suppose that out of the total North American population, 50% of the people have at some time visited the US, 30% visited Texas and 40% visited California. What is the smallest possible percentage of the North American Population that has visited Texas and California?
U = person has visited US
T = person has visited Texas
C = person has visited California
$0.5 = \mathbb{P}(U) \geq \mathbb{P}(T \cup C) = \mathbb{P}(T) + \mathbb{P}(C) - \mathbb{P}(T \cap C) =$
$0.5 \geq 0.3 + 0.4 - \mathbb{P}(T \cap C)$
$\mathbb{P}(T \cap C) \geq 0.2$ (Largest possible percentage would be min of percentage that has visited California and Texas)

5 Conditional Probability 2.1

5.1 Definition

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a positive-probability event $B \in \mathcal{F}$, we define a new probability measure $\mathbb{P}(\cdot|B)$ on $(\Omega, \mathcal{F})$ defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$\mathbb{P}(\cdot|B)$ satsifies all probabilities of a probability measure

1. $0 \leq \mathbb{P}(A|B) \leq 1$
2. $\mathbb{P}(\Omega, B) = 1$
3. $\mathbb{P}(A_1 \cup A_2 \cup \ldots |B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B) + \ldots$
4. $\mathbb{P}(A^C|B) = 1 - \mathbb{P}(A|B)$

Example: An urn contains 4 red marbles and 6 blue marbles. 3 are drawn w/o replacement

- a) What is the probability that exactly 2 of the marbles are red
  $\mathbb{P}(2 \text{ red}) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = \frac{3}{10}$
- b) Suppose we know that at least 1 of the marbles is red. Given this information, what is the probability that exactly 2 are red?
  $\mathbb{P}(2\text{red}|\text{at least 1 red}) = \frac{\mathbb{P}(2\text{red}\cap\text{at least 1 red})}{\mathbb{P}(\text{at least 1 red})} = \frac{\mathbb{P}(2\text{red})}{1-\mathbb{P}(0\text{red})} = \frac{\frac{3}{10}}{1-\frac{1}{6}} = \frac{9}{25}$

5.2 Multiplication Rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

5.3 Two-Stage Experiments

1. First, an experiment with a random outcome is performed
2. Second, another experiment is performed whose setup depends on the outcome of the first experiment

Example Urn I contains 1 red marble and 2 blue marbles. Urn II contains 3 white marbles and 2 red marbles. First one of urns is chosen uniformly at random, and then one marble is drawn uniformly at random from the chosen urn. What is the probability that the marble drawn is red.
Let R be the event the marble drawn is red, I be the event that urn I is selected, and II be the event urn II is selected. $\mathbb{P}(R) = \mathbb{P}(R \cap I) + \mathbb{P}(R \cap II) =$
$\mathbb{P}(R) = \mathbb{P}(R|I)\mathbb{P}(I) + \mathbb{P}(R|II)\mathbb{P}(II) =$
$\frac{1}{3}\frac{1}{2} + \frac{2}{5}\frac{1}{2} = \frac{11}{30}$

5.4 Law of Total Probability

If $B_1, B_2, \dots \in \mathcal{F}$ is a finite or infinite sequence of events partitioning Ω, then for any event $A \in \mathcal{F}$,

- $\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap B_i) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \dots$
- $\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i)\mathbb{P}(B_i) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \dots$

Example: 90% of coins are fair. 9% of coins land on heads 60% of the time, and 1% of the coins land heads 80% of the time. You pick a random coin off the street. How likely is it land heads?
$\mathbb{P}(H) = \mathbb{P}(H|C_1)\mathbb{P}(C_1) + \mathbb{P}(H|C_2)\mathbb{P}(C_2) + \mathbb{P}(H|C_3)\mathbb{P}(C_3) =$
$0.5 \cdot 0.9 + 0.6 \cdot 0.09 + 0.8 \cdot 0.01 = 0.512$

6 Bayes' Formula 2.2

$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$
Example: A flu test is 99% accurate, meaning that the false positive rate is 1% and the false negative rate is 1%. Currently, 0.33% of the US population has the flu. A randomly chosen person from the US population tests positive for the flu. What is the probability that this person actually has the flu, rounded to the nearest percent?
$F =$ Person has flu $P =$ Person tests positive
$\mathbb{P}(P|F) = 1 - \mathbb{P}(P^c|F)$
$\mathbb{P}(P) = \mathbb{P}(P|F)\mathbb{P}(F) + \mathbb{P}(P|F^c)\mathbb{P}(F^c)$
$\mathbb{P}(F|P) = \frac{\mathbb{P}(P|F)\mathbb{P}(F)}{\mathbb{P}(P)} = \frac{0.99(0.0033)}{0.99(0.0033)+0.01(0.9967)} = 0.247$

7 Independence 2.3

Two events are independent if:

- $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
- $\mathbb{P}(A|B) = \mathbb{P}(A)$
- $\mathbb{P}(B|A) = \mathbb{P}(B)$

Example: Consider the probability space where the sample space is $[0, 1]$ and the probability measure is uniform. For which value of b in $(0, 1)$ are the events $[\frac{1}{4}, \frac{1}{3}]$ and $[0, b]$ independent? (Note the requirement $0 < b < 1$, so $b = 0, 1$ are not allowed).
$A = [\frac{1}{4}, \frac{1}{3}]$ $\mathbb{P}(A) = \frac{\frac{1}{12}}{1} = \frac{1}{12}$
$B = [0, b]$ $\mathbb{P}(b) = \frac{b}{1} = b$
$\mathbb{P}(A \cap B) = \frac{b - \frac{1}{4}}{1} = b - \frac{1}{4}$ want to set equal to $\mathbb{P}(A)\mathbb{P}(B) = \frac{b}{12}$
$b = \frac{3}{11}$

7.1 Mutual Independence

A sequence of events $A_1, A_2, \dots$ in a probability space are mutually independent if any sequence of events $A_{i_1}, A_{i_2}, A_{i_n}$ chosen from $A_1, A_2, \dots$ satisfy:
$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots A_{i_n}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \dots \mathbb{P}(A_{i_n})$
Example: Toss a fair coin 3 times. Consider the events. Are A, B, C mutually independent?

- $A = \{\text{exactly 1 tails in 1st and 2nd toss}\} = \{THH, THT, HTH, HTT\}$
- $B = \{\text{exactly 1 tails in 2nd and 3rd toss}\} = \{HTH, TTH, HHT, THT\}$
- $C = \{\text{exactly 1 tails in 1st and 3rd toss}\} = \{THH, TTH, HHT, HTT\}$

$A \cap B = \{THT, HTH\}$ $B \cap C = \{TTH, HHT\}$ $A \cap C = \{THH, HTT\}$ $A \cap B \cap C = \emptyset$
$\mathbb{P}(A \cap B) = \frac{2}{8} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(B)$
$\mathbb{P}(B \cap C) = \frac{2}{8} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(B)\mathbb{P}(C)$
$\mathbb{P}(A \cap C) = \frac{2}{8} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(C)$
$\mathbb{P}(A \cap B \cap C) = 0 \neq \frac{1}{8} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$
Shows A,B,C are not mutually independent but are pairwise independent

8 Random Variables=: a first look 1.5

A (real-valued) random variable is a function $X : \Omega \to \mathbb{R}$ where:

- Ω is the sample space for some probability measure
- X is measurable ($\{X \in (a, b)\} \in \mathcal{F}$ for all intervals (a, b) (alternatively, $\{w \in \Omega : X(w) \in (a, b)\}$)

8.1 Probability Distribution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure space and $X : \Omega \to \mathbb{R}$ be a random variable. The probability distribution of X is the probability measure on $\mathbb{R}$ defined by:
$\mathbb{P}_X(A)$ $(A \subseteq \mathbb{R}) = \mathbb{P}(X \in A) = \mathbb{P}(\{w \in \Omega : X(w) \in A\})$
Example 1 Fair coin toss: $\Omega = \{H, T\}$. Define $X : \Omega \to \mathbb{R}$ by $X(H) = \pi, X(T) = -\sqrt{2}$. What is the distribution of X?
$\mathbb{P}_X([0, \infty]) = \mathbb{P}(X \in [0, \infty) = \mathbb{P}(\{\omega \in \{H, T\} : X(\omega) \in [0, \infty)\}) = \mathbb{P}(\{H\}) = \frac{1}{2}$
$\mathbb{P}((-2, 2)) = \mathbb{P}(\{T\}) = \frac{1}{2}$

$$\mathbb{P}(A) = \begin{cases} 0 & -\sqrt{2}, \pi \notin A \\ \frac{1}{2} & \text{exactly one of } -\sqrt{2}, \pi \text{ is in A} \\ 1 & -\sqrt{2}, \pi, \in A \end{cases}$$

Example 2 Roll a fair die twice: $\Omega = \{1,2,3,4,5,6\}^2$. Define $S : \Omega \to \mathbb{R}$ by $S((i,j)) = i + j$. What is the distribution of S?

$\mathbb{P}_s(2) = \mathbb{P}(s \in \{2\}) = \mathbb{P}(S = 2) = \mathbb{P}(\{(1,1)\}) = \frac{\#\{(1,1)\}}{\#\Omega} = \frac{1}{36}$

$\mathbb{P}(\{(3,3)\}) = \cdots = \frac{\#(1,2),(2,1)}{\#\Omega} = \frac{2}{36}$

$\mathbb{P}(\{4\}) = \frac{\#\{(1,3),(2,2),(3,1)\}}{\#\Omega} = \frac{3}{36}$

$\mathbb{P}(\{5\}) = \frac{4}{36}$

$\mathbb{P}(\{6\}) = \frac{5}{36}$

$\mathbb{P}(\{7\}) = \frac{6}{36}$

$\mathbb{P}(\{8\}) = \frac{5}{36}$

$\mathbb{P}(\{9\}) = \frac{4}{36}$

$\mathbb{P}(\{10\}) = \frac{3}{36}$

$\mathbb{P}(\{11\}) = \frac{2}{36}$

$\mathbb{P}(\{1\}) = \frac{1}{36}$

Example 3 Choose a point uniformly on the unit disk: $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Define $X : \Omega \to \mathbb{R}$ by $X((x,y)) = \sqrt{x^2 + y^2}$. What is the distribution of X

$$F_x(t) = \mathbb{P}(X \leq t) = \frac{\text{area}(\{X \leq t\})}{\pi 1^2} = \begin{cases} 0 & t < 0 \\ \frac{\pi(t)^2}{\pi} & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases} = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases}$$

9 Cumulative Distribution Functions 3.2

Let X be a random variable. The cumulative distribution function (CDF) of X is the function $F_x : \mathbb{R} \to \mathbb{R}$ defined by:

$F_x(t) := \mathbb{P}(A \leq t) = \mathbb{P}(X \in [-\infty, t]) = \mathbb{P}_X([-\infty, t])$

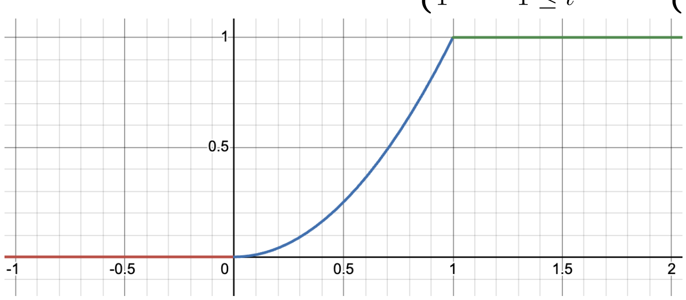
Crucial property: The CDF completely determines the distribution. Alternatively, if $F_x(t) = F_y(t)$, then $\mathbb{P}_X(A) = \mathbb{P}_Y(A)$ for every $A \subseteq \mathbb{R}$

Example 1: A fair coin is tossed 3 times. Let $X =$ "number of heads". Find the CDF.

$$F_x(t) = \mathbb{P}(X \leq t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{8} & 0 \leq t \leq 1 \\ \frac{4}{8} & 1 \leq t \leq 2 \\ \frac{7}{8} & 2 \leq t \leq 3 \\ 1 & 3 \leq t \end{cases}$$

Example 2: A point is chosen uniformly at random from the unit disk. Let $X =$ "distance of the point from the center". Find the cdf of X and sketch its graph

$$F_X(t) = \mathbb{P}(X \leq t) = \frac{\text{area}(\{X \leq t\})}{\pi 1^2} = \begin{cases} 0 & t < 0 \\ \frac{\pi(t)^2}{\pi} & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases} = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases}$$



10 Probability Distributions of Random Variables 3.1

10.1 Discrete Probability Measure

A probability measure $\mathbb{P}$ on $\mathbb{R}$ is discrete if there exists a finite or infinite sequence of numbers $a_1, a_2, \cdots \in \mathbb{R}$ such that

$$\sum_{i=1}^{\infty} \mathbb{P}(\{a_i\}) = 1 \text{ or } \mathbb{P}(\mathbb{R}\backslash\{a_1, a_2, \dots\}) = 0$$

A random variable X is discrete if its distribution $\mathbb{P}_x$ is discrete
The function $p_x : \mathbb{R} \to \mathbb{R}$ defined by $p_x(a) := \mathbb{P}(x = a)$ is the **probability mass function** of X
Crucial fact: The pmf completely determines the distribution of a discrete random variable because $p_x(A) = \mathbb{P}(x \in A) = \sum_{a \in A} \mathbb{P}(x = a) = \sum_{a \in A} p_x(a)$
Example: A fair coin is tossed 3 times. Let X = number of heads. X is a discrete random variable. Find its pmf.
$0, 1, 2, 3$ are the possible values of X
$p_x(a) = \mathbb{P}(X = a)$
$p_x(0) = \mathbb{P}(X = 0) = \frac{1}{8}$
$p_x(1) = \mathbb{P}(X = 1) = \frac{3}{8}$
$p_x(2) = \mathbb{P}(X = 2) = \frac{3}{8}$
$p_x(3) = \mathbb{P}(X = 3) = \frac{1}{8}$
$p_x(a) = 0$ for all other a **Make sure to add default case!**

10.2 Continuous Probability Measure

A probability measure $\mathbb{P}$ on $\mathbb{R}$ is continuous if there exists an integrable function $f : \mathbb{R} \to \mathbb{R}$ s.t. $\mathbb{P}(A) = \int_A f(x)dx$
A random variable is continuous if its distribution $\mathbb{P}_X$ is continuous
f_x is the **probability density function**.
Critical Fact: The pdf completely determines the distribution of X, since $\mathbb{P}(A) = \int_A f_x(t)dt$
Example: A point is chosen uniformly at random from the unit disk. Let X = distance of the point from the center. X is a continuous random variable. Find its pdf.
$A = (-\infty, s]$
$F_x(s) = \mathbb{P}(x \leq s) = \mathbb{P}(x \in (-\infty, s]) = \int_{-\infty}^{s} f_x(t)dt$
$F_x(s) = \in (-\infty, s]) = \int_{-\infty}^{s} f_x(t)dt \iff F_x'(s) = f_x(s)$ *take* $\frac{d}{ds}$ *each side based on fundamental theorem of calculus*

$$f_x(s) = \frac{d}{ds}\begin{cases} 0 & s < 0 \\ s^2 & 0 \leq s \leq 1 \\ 1 & 1 < s \end{cases} = \begin{cases} 0 & s < 0 \\ 2s & 0 < s < 1 \\ 0 & 1 < s \end{cases}$$

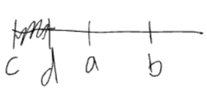
note:changing pdf at finitely many points doesn't affect it

10.3 Uniform Probability Measure

A random variable X is uniform on $[a, b]$ ($X \sim unif[a, b]$) if X is continuous and its pdf is given by:

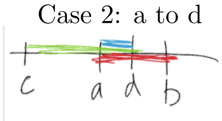
$$f_X(t) = \begin{cases} \frac{1}{b-a} & t \in [a, b] \\ 0 & t \notin [a, b] \end{cases}$$

Example Suppose $X \sim$ Unif$[a, b]$. Calculate $\mathbb{P}(c \leq X \leq d)$
$\mathbb{P}(c \leq X \leq d) = \int_c^d f_X(t)dt$ **Cases:**
Case 1: $d < a$



$\int_c^d 0dt = 0$

Case 2: a to d



$\int_a^d \frac{1}{b-a} dt = \frac{d-a}{b-a}$

Case 3: c to d (between a and b)

$\int_c^d \frac{1}{b-a} dt = \frac{d-c}{b-a}$ Case 4: c to b

$\int_c^b \frac{1}{b-a} dt = \frac{b-c}{b-a}$

10.4 Going from pmf/pdf to cdf

10.4.1 X is discrete

$F_x(t) = \mathbb{P}(X \leq t) = \sum_{a \leq t} p_x(a)$

Example: Suppose X is discrete w/ pmf $p_x(-\pi) = \frac{1}{3}, p_x(\sqrt{2}) = \frac{2}{3}$. Find its cdf.

$F_x(t) = \mathbb{P}(X \leq t) = \sum_{a \leq t} p_x(a)$

$$F_x(t) = \begin{cases} 0 & t < -\pi \\ \frac{1}{3} & -\pi \leq t < \sqrt{2} \\ 1 & \sqrt{2} \leq 2 \end{cases}$$

Be Careful With Interval Equalities!

10.4.2 X is continous

$F_X(t) = \mathbb{P}(X \leq T) = \int_{\infty}^{t} f_X(s) ds$

Example: Suppose $X\ Unif[-1, 2]$. Find its cdf

$$F_x(t) = \mathbb{P}(x \leq t) = \in_{-\infty}^{t} f_x(s) ds = \begin{cases} 0 & t < -1 \\ \frac{t+1}{3} & -1 \leq t < 2 \textit{ Based on:} \in_{-1}^{t} \frac{1}{3} ds = \frac{1}{3}(t - (-1)) \\ 1 & 2 \leq t \textit{ Based on:} \in_{-1} 2 \frac{1}{3} ds = \frac{1}{3}(2 - (-1)) \end{cases}$$

10.5 Going from cdf to pmf/pdf

10.5.1 CDF to PMF, discrete

Let X be piecwise constant and F_x be its CDF

If F_x is piecewise constant, then X is discrete and the pmf is found by calculating the magnitude of the jumps at points of discontinuity

Example: Suppose X has cdf given by $F_X(s) = \begin{cases} 0 & s < -\pi \\ \frac{1}{3} & -\pi \leq s < \sqrt{2} \\ 1\sqrt{2} \leq s \end{cases}$

Is X discrete or continous? If so, find its pmf/pdf.

X is discrete because it is piecewise constant

$P_X(-\pi) = \frac{1}{3} - 0 = \frac{1}{3}$

$P_X(\sqrt{2}) = 1 - \frac{1}{3} = \frac{2}{3}$

10.5.2 CDF to PDF, continuous

If F_x is continuous and piecewise continuously differentiable, then X is a continuous random variable and its pdf is given by $f_X(t) = F'_X(t)$

Example: Suppose X has cdf given by $F_X(s) = \begin{cases} 0 & s < 0 \\ \sin s & 0 \le s < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le s \end{cases}$

Is X discrete or continuous? If so, find its pmf/pdf.
F_X is continuous and piecewise continuously differentiable. Thus, X is a continuous random variable.

$f_X(s) = F'_X(S) = \begin{cases} 0 & s < 0 \\ \cos s & 0 < s < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < s \end{cases}$

10.6 Properties of cdf, pmf, and pdf

10.6.1 CDF Properties

A function $F : \mathbb{R} \to \mathbb{R}$ is the cdf of a random variable X if and only if:

- $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to\infty} F_x(t) = 1$
- $F_x(t) \ge F_x(s)$ if $t \ge s$ (nondecreasing)
- right-continous ($\lim_{t\to A^+} F_x(t) = F_x(a)$

10.6.2 PMF Properties

A function $p : \mathbb{R} \to \mathbb{R}$ if and only if there exists a finite or infinite sequence of numbers $a_1, a_2, \cdots \in \mathbb{R}$ such that:

- $0 \le p(a) \le 1$ for all $a \in \mathbb{R}$
- $\sum_{i=0}^{\infty} p(a_i) = 1$
- $p(b) = 0$ for all $b \in \{a_1, a_2, \dots\}$

10.6.3 PDF Properties

A function $f : \mathbb{R} \to \mathbb{R}$ is the pdf of a continuous random variable if and only if f is integrable and

- $\int_{-\infty}^{\infty} f(t) = 1$
- $f(t) \ge 0$ for all t