1. Objective

The goal of this study is to approximate the Runge function using a neural network:

$$f(x) = \frac{1}{1 + 25x^2}, \qquad x \in [-1, 1].$$

The Runge function is known to cause oscillations in classical polynomial interpolation (Runge's phenomenon). Here, we examine whether a neural network can provide a stable and accurate approximation.

2. Methodology

Dataset

- 200 evenly spaced points in $[-1, 1]$ are used as the training set.
- 50 additional points are used as the validation set.

Neural Network Architecture

- Feedforward neural network with structure: $1 \to 64 \to 64 \to 1$.
- Hidden layers use ReLU activation; output layer is linear.

Training Setup

- Loss function: Mean Squared Error (MSE)
- Optimizer: Adam with learning rate 0.001
- Training epochs: 3000

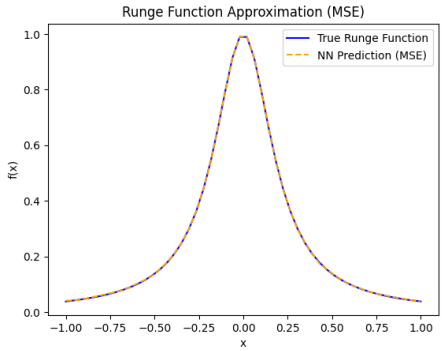
Evaluation

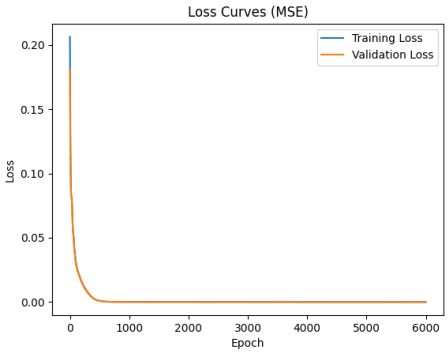
- Compare the true function with the neural network prediction.
- Plot training and validation loss curves.
- Compute validation errors: MSE and maximum absolute error.

3. Results

Function Approximation

- Blue solid line: true Runge function
- Orange dashed line: neural network prediction
- The network successfully captures the overall shape, especially in the central region.





Loss Curves

- Training and validation losses decrease steadily and converge, showing stable learning without overfitting.

Result

- Training with MSE Loss

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Epoch:[200/3000], Train Loss: 0.000669, Val Loss: 0.000653
Epoch:[400/3000], Train Loss: 0.000041, Val Loss: 0.000041
Epoch:[600/3000], Train Loss: 0.000012, Val Loss: 0.000012
Epoch:[800/3000], Train Loss: 0.000006, Val Loss: 0.000006
Epoch:[1000/3000], Train Loss: 0.000003, Val Loss: 0.000003
Epoch:[1200/3000], Train Loss: 0.000002, Val Loss: 0.000002
Epoch:[1400/3000], Train Loss: 0.000002, Val Loss: 0.000002
Epoch:[1600/3000], Train Loss: 0.000005, Val Loss: 0.000008
Epoch:[1800/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[2000/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[2200/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[2400/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[2600/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[2800/3000], Train Loss: 0.000001, Val Loss: 0.000001
Epoch:[3000/3000], Train Loss: 0.000001, Val Loss: 0.000009
```

- Error Metrics

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Validation MSE: 8.813443e-06 Max error: 0.004483726
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